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Type IIB at eight derivatives: insights from Superstrings, Superfields and Superparticles

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Abstract: We study the non-linear structure of Type IIB eight-derivative couplings involving the metric and the complexified three-form G_3. We show that, at the level of five-point string amplitudes, the kinematics in the maximally R-symmetry-violating sector is fully matched by standard superspace integrals and by superparticle amplitudes in M-theory on a two-torus. The latter approach is used to determine the complete effective action in this sector and to verify its invariance under $\mathrm{SL}(2,\mathbb{Z})$ duality. We further comment on the general structure of the higher-point kinematics. Compactifications to lower dimensions provide both tests for our results and the arena for their applications. We verify that K3 reductions are fully consistent with the constraints of six-dimensional supersymmetry, and derive the four-dimensional flux scalar potential and axion kinetic terms at order $(\alpha')^3$ in Calabi-Yau threefold reductions.

Keywords: Field Theories in Higher Dimensions, String Duality, Flux Compactifications, M-Theory

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Contents

1 Introduction and summary

Despite the long history of studying perturbative higher-derivative corrections to the ten-dimensional effective actions [1, 2], understanding their complete structure, notably non-linear completions arising form higher-point functions, remains a key challenge for string theory. While recent years were marked by conceptual and computational breakthroughs in our ability to compute string amplitudes, a unifying approach or a guiding principle towards completing the effective action at higher order in the α' and loop expansion has not emerged yet. The main outstanding issue is the construction of local effective actions reproducing the precise amplitude, where a plethora of kinematical structures, intricate pole-subtraction procedures and field redefinitions complicate any bootstrapping attempt.

In the NSNS sector, a partial set of quintic and some sextic higher-derivative terms in the Type II effective action have been identified [3–5]. For the Type IIB effective actions, the authors of [6, 7] completed the quartic couplings for all fields but the four-form tensor field with anti-self-dual tensor field strength F_5. In the absence of the dilaton and the complex 3-form on the Type IIB side, the full action has been determined in [8, 9]. Here, the supersymmetric completion of R^4 including the F_5 field strength [8, 10, 11] has been inferred from the $\mathcal{N} = 2$ superspace approach of [12]. Particularly noteworthy are applications of these results to multi D3-brane backgrounds [8] and black-hole solutions with $\mathrm{AdS}_5 \times S^5$ asymptotics in AdS/CFT [9, 13]. Recently a full completion of the NSNS-sector of the effective action at tree level has been obtained in [14] using constraints imposed by T-duality invariance, if not directly string theory. However, given that the choice of field basis is rather unnatural from the point of view of tensor structures like t_8 or ϵ_{10} appearing in string vertex operators or superspace integrals, it is hard to make a straightforward comparison, let alone extend the results to the RR-sector or to 1-loop order.

Up to now, these different developments have not been put together in a systematic fashion, and no unifying approach towards completing string effective actions has been proposed. One would hope that a proper framing of effective actions with quantum corrections in terms of some generalised or super-geometry should emerge and eventually be helpful in constraining if not predicting the higher-order interactions. So far, the usefulness of connections with torsion given by the NSNS three-form H_3, $\Omega_\pm = \Omega^{\mathrm{LC}} \pm \frac{1}{2}H$ in terms of the Levi-Civita connection Ω^{LC}, observed at the linearised level [2], has been confirmed at the non-linear level as well [4, 5]. However it has also been shown that a simple replacement of Ω^{LC} by a connection with torsion fails to capture the full kinematics of eight-derivative terms at one-loop and especially at tree-level.

In this paper, we make progress towards developing such an approach by scrutinising the kinematical structures discovered in [5]. We compare the results of conventional ten-dimensional (10D) superstring amplitudes with eleven-dimensional (11D) superparticle amplitudes compactified on a two-torus [15–17], and superspace approaches to Type IIB $\mathcal{N} = 2$ supergravity [12]. In the effective action, we find that couplings in the maximally U(1)-violating (MUV) sector of U(1) (or R-symmetry) charge $|Q_{\max}| = 2(P-4)$ at the level of P-point amplitudes are captured by simple fundamental higher-dimensional index structures generalising the well-known t_8 or ϵ_D tensors. Our results for the 10D action at

5-point level including the metric and complexified three-form flux G_3 (up to ∇G_3 terms) can then be summarised as

$$\mathcal{L} = f_0(\tau,\bar\tau)t_{16}R^4 + \frac{3}{2}\left(f_1(\tau,\bar\tau)t_{18}G_3^2R^3 + f_{-1}(\tau,\bar\tau)t_{18}\overline{G}_3^2R^3\right) + f_0(\tau,\bar\tau)\left[T(\epsilon_{10},t_8) - t_{18}\right]|G_3|^2R^3\,. \tag{1.1}$$

The first line encodes the MUV couplings which are nicely repackaged into a single index structure t_N (with $N = 16 + 2w$ for couplings of the form $G_3^{2w}R^{4-w}$). In comparison to [5], this means an extraordinary simplification of the 5-point results.

We have chosen to split the non-MUV sector in (1.1), and write separately the kinematical structure $T(\epsilon_{10},t_8)$, which is a function of the standard tensors t_8 and ϵ_{10} and a factor of $-t_{18}$. The latter cannot be written in terms of t_8 or ϵ_{10}, as observed in [5] where these terms were simply given by the expansion in the full basis of H^2R^3 terms. Stated differently, t_{18} is not directly seen in string amplitudes, given that it is not directly built into the vertex operators. As we will see, an a posteriori justification of such a split in the non-MUV sector comes from the fact that the t_{18} piece plays an important role in Calabi-Yau threefold reductions to four dimensions.

In Type IIB, $\mathrm{SL}(2,\mathbb{Z})$ invariance dictates that the coefficients of higher-derivative terms are written in terms of modular forms. The appearance of the modular function f_0, the non-holomorphic Eisenstein series of weight 3/2, was first observed in [15, 18], while other non-holomorphic modular forms f_w first appeared in [16, 19], see also [20] for a more recent discussion. For R^4, leading order D-instanton calculations [21, 22] confirmed this result. Moreover, the amount of supersymmetry in 10D is a powerful tool to relate various higher-derivative terms in the α' expansion [23]. While linearised SUSY is powerful enough to predict the existence of higher-derivative terms, it is incapable of explaining either the presence of the coefficient functions f_w or the tensor structures in the non-MUV sector. These coefficient functions can be derived, instead, by studying their origin in M-theory, integrating out towers of winding modes on T^2 of vanishing volume, in a light-cone worldline formalism for the 11D superparticle [15–17]. We compute such amplitudes in 11D and, besides deriving explicitly the axio-dilaton dependent coefficients, we are able to reproduce exactly the kinematics in the MUV sector of (1.1), as expected from the superspace approach.

Going beyond 5 points, we prove that MUV couplings (as well as a specific subset of non-MUV couplings) of the 3-form, the 5-form, and the metric in the effective action are given by

$$\mathcal{L}^{\mathrm{MUV}} = \sum_{w=0}^{4} C_w\, f_w(\tau,\bar\tau)\, t_{24}G_3^{2w}\mathcal{R}^{4-w} + \text{c.c.} \tag{1.2}$$

in terms of numerical coefficients C_w and the 6-index tensor $\mathcal{R}$

$$\mathcal{R} = R + \mathrm{i}\nabla F_5 + F_5^2 + |G_3|^2\,. \tag{1.3}$$

The tensor $\mathcal{R}$ is tightly constrained by non-linear supersymmetry by appearing at Θ^4 in a (non-linear) scalar superfield. The derivation of (1.2) is based on all the three aforementioned approaches:

- *superstring amplitudes*: up to 5-points, (1.2) essentially reduces to the full first and a subset of terms in the second line of (1.1). At 6-points, [24] provides the coefficient C_2 from tree-level pure spinor amplitudes. Higher amplitudes are in principle available, but determining the structure of contact terms in (1.2) is currently out of reach.

- *superfields*: the string kinematics is easily determined from 16-fermion integrals, thereby making the existence of a single unifying index structure obvious. In principle, the coefficients C_w can be determined from supersymmetry/geometry following e.g. [23], but we do not follow this approach here.

- *superparticles*: the structure of MUV amplitudes of M-theory compactified on T^2 is actually simple enough not only to reproduce (1.2) kinematically, but also to derive the $f_w(\tau, \bar{\tau})$ alongside the C_w from first principles. This proves the higher power of this approach in determining the MUV effective couplings at any order.

The most significant takeaway message from (1.2) is that it unifies 46 individual tensor structures in such a way that they are kinematically captured by *a single index structure* t_{24}. Indeed, t_{24} is the largest structure from which all other tensors t_N with $N < 24$ relevant for this paper can be constructed upon suitable metric contractions. For a given weight w, we find

1. $w = 0$: the generalised R^4 term corresponding to $f_0 t_{24} \mathcal{R}^4$ was inferred in [8–11] in the absence of the $|G_3|^2$ term. At the level of 5-point contact terms, we found evidence for the coupling $|G_3|^2$ inside $\mathcal{R}$, which is again obtained from $f_0 t_{24} \mathcal{R}^4$ upon expanding to quadratic order in the 3-form. Given that the relative coefficients inside $\mathcal{R}$ are determined by supersymmetry, we provide further contact with a supersymmetric completion of the R^4 coupling in the presence of a non-trivial G_3 background.

2. $w = 1$: the part $f_1 t_{24} G_3^2 \mathcal{R}^3$ reduces at 5 points to the second term in the first line of (1.1). From the superfield perspective, the replacement $R \to \mathcal{R}$ is completely justified by supersymmetry, even though there might be further higher-order contributions in the non-MUV sector, just as for $|G_3|^2 R^3$.

3. $w \geq 2$: we utilise the 11D superparticle to predict the string coefficients of MUV amplitudes beyond five points. The $w = 2$ coefficient C_2 matches the predictions of [24] at the level of six-point pure spinor amplitudes. Moreover, the higher order coefficients are in agreement with expectations from modular invariance of the Type IIB superstring.

Our results are reminiscent of the MUV amplitudes computed in [24] based on the spinor helicity formalism of [25, 26]. It was argued there that general MUV amplitudes appear without any poles (i.e. they *are* contact terms), are represented by a superfield which matches the linearised on-shell superfield of Type IIB supergravity, and violate the U(1) by $2(P-4)$ units of charge (see also [27–29]).

We conclude our analysis by two basic lower-dimensional consistency checks of our findings. When compactifying our proposed 10D action on a K3 to six dimensions, we show how several non-trivial cancellations among the various 5-point index structures ensure consistency with the constraints imposed by $\mathcal{N} = (2,0)$ supersymmetry in 6D. When reducing the 10D action to four dimensions on a Calabi-Yau threefold we show that constraints on the $(\alpha')^3$-corrected flux scalar potential from 4D supersymmetry are also perfectly matched. Furthermore, we derive the 4D kinetic terms for the hypermultiplet scalars, in particular the C_2/B_2-axions, at order $(\alpha')^3$ at string tree and 1-loop level.

The paper is organised as follows. In section 2 we review the systematics of 8-derivative terms in the Type IIB effective action with a particular focus on $\mathrm{SL}(2,\mathbb{Z})$-invariance and sixteen fermionic integrals giving rise to higher-dimensional index structures. Subsequently, in section 3, we demonstrate that such structures play an outstanding role also for 5-point contact terms of the form $|G_3|^2R^3, G_3^2R^3$ and $\overline{G}_3^2R^3$. In section 4, we argue that the entire eight-derivative action for couplings of the form $G_3^m\overline{G}_3^n R^{4-w}$, $w = (m+n)/2$, in the MUV sector (i.e. for $m \cdot n = 0$) is determined by a single index structure obtained from a sixteen fermion integral. Using supersymmetry, these terms can be partially generalised to include also some non-MUV couplings through replacing $R \to \mathcal{R}$. In section 5 we apply our proposal for the 10D effective action to compactifications to 6D and to 4D. Finally we list a number of open question that should hopefully be addressed in the near future in section 6. Some technical material is collected in five appendices.

2 Type IIB supergravity and its α'-expansion

The classical 10D effective action reads in Einstein frame

$$S^{(0)} = \frac{1}{2\kappa_{10}^2}\int \left(R - 2\mathcal{P}_M\overline{\mathcal{P}}^M - \frac{|G_3|^2}{2\cdot 3!} - \frac{|F_5|^2}{4\cdot 5!}\right)\star_{10}\mathbb{1} + \frac{1}{8\mathrm{i}\kappa_{10}^2}\int C_4\wedge G_3\wedge\overline{G}_3 \tag{2.1}$$

in terms of the complexified fields

$$\tau = C_0 + \mathrm{i}e^{-\phi}\,, \quad \mathcal{P}_M = \frac{\mathrm{i}\nabla_M\tau}{2\mathrm{Im}(\tau)}\,, \quad G_3 = \frac{1}{\sqrt{\mathrm{Im}(\tau)}}(F_3 - \tau H_3) = \frac{\tilde{G}_3}{\sqrt{\mathrm{Im}(\tau)}}\,. \tag{2.2}$$

where the p-form field strengths are defined as

$$H_3 = \mathrm{d}B_2\,, \qquad F_1 = \mathrm{d}C_0\,, \qquad F_3 = \mathrm{d}C_2\,, \qquad F_5 = \mathrm{d}C_4 - \frac{1}{2}H_3\wedge C_2 + \frac{1}{2}F_3\wedge B_2\,. \tag{2.3}$$

In addition to the standard equations of motion, the 5-form flux must satisfy the self-duality condition $F_5 = \star_{10}F_5$.

The Type IIB fields form representations under $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$ with the first being a global, the second being a local symmetry denoted by Q_{IIB}. The various fields have in our convention U(1)-charges

$$Q_{\mathrm{IIB}}(\mathcal{P}) = +2\,, \quad Q_{\mathrm{IIB}}(G_3) = +1\,, \quad Q_{\mathrm{IIB}}(g_{MN}) = Q_{\mathrm{IIB}}(F_5) = 0 \tag{2.4}$$

with the opposite charges for the complex conjugates. The complexified scalars parametrise the coset (or moduli) space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)\cong\mathrm{SU}(1,1)/\mathrm{U}(1)$ [12, 30–33]. In string theory, the U(1) subgroup of $SL(2,\mathbb{R})$ rotating the two supercharges into each other does not leave the superstring invariant: the S-duality $SL(2,\mathbb{Z})$ group survives [15, 18]. Under $SL(2,\mathbb{Z})$, the axio-dilaton transforms according to

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \qquad ad - bc = 1\,. \tag{2.5}$$

This implies that

$$\tilde{G}_3 \to \frac{\tilde{G}_3}{c\tau + d}\,, \qquad \tau_2 \to \frac{\tau_2}{|c\tau + d|^2} \tag{2.6}$$

as well as

$$\mathcal{P} \to \frac{c\bar{\tau} + d}{c\tau + d}\mathcal{P}\,, \qquad G_3 \to \left(\frac{c\bar{\tau} + d}{c\tau + d}\right)^{\frac{1}{2}} G_3\,. \tag{2.7}$$

More generally, a combination Φ of fields with U(1)-charge $Q_{\text{IIB}} = 2k$ transforms with weight k so that

$$\Phi \to \left(\frac{c\bar{\tau} + d}{c\tau + d}\right)^{k} \Phi\,. \tag{2.8}$$

The individual contact terms in the effective action must be invariant under these $SL(2,\mathbb{Z})$ transformations. For higher derivative terms, this highly constrains the coefficient functions to be appropriate $SL(2,\mathbb{Z})$-covariant modular forms.

2.1 Perturbative corrections in 10D at order $(\alpha')^3$

The effective action enjoys a double expansion in terms of g_s (worldsheet topologies / loops in the spacetime theory) and α' (loops in the worldsheet theory / higher-derivative terms). Given that α' parametrises the way string theory deviates from a theory of point-like objects, it is of critical importance for our understand of quantum gravity.

The low-energy description of superstrings is traditionally obtained from string scattering amplitudes of massless string excitations giving rise to an effective field theory description in the limit $\alpha' \to 0$. Below, we argue that other approaches can be equally effective by employing duality considerations to M-theory and a superspace formalism.

Throughout this paper, we focus on 8-derivative couplings where the bosonic action can schematically be written as

$$\begin{aligned} S^{(3)} \sim \int \star_{10}\mathbb{1}\Big\{ & R^4 + R^3\left(G_3^2 + |G_3|^2 + \overline{G}_3^2 + F_5^2 + \dots\right) \\ & + R^2\left(|\nabla G_3|^2 + (\nabla F_5)^2 + G_3^4 + \dots\right) + R\left(G_3^6 + \dots\right) + \left(G_3^8 + |\nabla G_3|^4 + \dots\right)\Big\}. \end{aligned} \tag{2.9}$$

To ensure invariance of the individual terms in (2.9) under $SL(2,\mathbb{Z}) \times U(1)$, each of the contact terms must be multiplied by an appropriate $SL(2,\mathbb{Z})$ covariant function of opposite charge. For our purposes, it suffices to consider the *modular functions*[1]

$$f_w(\tau,\bar{\tau}) = \sum_{(\hat{l}_1,\hat{l}_2)\neq(0,0)} \frac{\text{Im}(\tau)^{\frac{3}{2}}}{\left(\hat{l}_1 + \tau\hat{l}_2\right)^{\frac{3}{2}+k}\left(\hat{l}_1 + \bar{\tau}\hat{l}_2\right)^{\frac{3}{2}-k}}\,, \qquad Q_{\text{IIB}}(f_w) = -2w\,. \tag{2.10}$$

[1]At higher orders in the α' expansion, more general modular forms have to be introduced, see [24] for a recent discussions and for further references.

These forms have special properties collected in appendix A.1 which are quintessential for our investigations. By counting the total U(1) charge, we can determine which modular function needs to be supplemented to each of the terms in (2.9). For instance, the uncharged term R^4 is multiplied by f_0 which is the non-holomorphic Eisenstein series of weight 3/2 [18, 34–37]. Further constraints on the structure of (2.9) arise from supersymmetry [23, 38].

2.2 The quartic effective action

A complete assessment of string 4-point amplitudes [6, 7] leads to the quartic action[2]

$$\mathcal{L}^{(3)}_{4-\mathrm{pt}} = \alpha\, f_0\,(\tau,\bar{\tau}) \Big\{ \mathcal{J}_0 + \left(t_8 t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right) \Big[6R^2 \left(4|\nabla\mathcal{P}|^2 + |\nabla G_3|^2\right) + 24|\nabla\mathcal{P}|^2|\nabla G_3|^2 + 12R\left(\nabla\mathcal{P}\left(\nabla\overline{G}_3\right)^2 + \nabla\overline{\mathcal{P}}\left(\nabla G_3\right)^2\right)\Big] + \mathcal{O}_1\left(\left(|\nabla\mathcal{P}|^2\right)^2\right) + \mathcal{O}_2\left(\left(|\nabla G_3|^2\right)^2\right)\Big\} \tag{2.11}$$

where

$$\alpha = \frac{(\alpha')^3}{3\cdot 2^{12}}\,. \tag{2.12}$$

For details concerning the definition of the operators $\mathcal{O}_1$ and $\mathcal{O}_2$, we refer the reader to [5, 7] which can also be recovered from an effective 12D lift [39].

The well-known R^4 structure is defined as [2, 40, 41]

$$\mathcal{J}_0 = \left(t_8 t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right) R^4\,. \tag{2.13}$$

It is obtained from four closed-string scattering or directly from the worldsheet σ-model. While at the level of the 4-point amplitude only the $t_8 t_8 \tilde{R}^4$ part in terms of the linearised Riemann tensor $\tilde{R}_{\mu\nu\rho\sigma} = -2\partial_{[\mu}h_{\nu][\rho,\sigma]}$ is non-vanishing, the additional $\epsilon_8\epsilon_8$ piece can already be inferred from the structure of the 4-point amplitude and directly verified by computing the odd-odd 5-point function, cf. [4] for a summary. General covariance dictates that in the purely gravitational sector higher-point graviton amplitudes replace the linearised Riemann tensor $\tilde{R}$ by the full Riemann tensor.

One can show by computing the r.h.s. of (2.13) explicitly that $\mathcal{J}_0$ can be written in terms of the Weyl tensor C_{MNPQ} as [2, 42]

$$\frac{\mathcal{J}_0}{3\cdot 2^8} = -\frac{1}{4} C^{MNPQ} C_{MN}{}^{RS} C_{PR}{}^{TU} C_{QSTU} + C^{MNPQ} C_M{}^R{}_P{}^S C_R{}^T{}_N{}^U C_{STQU}\,. \tag{2.14}$$

The fact that only the Weyl tensor appears as part of (2.13) is due to the symmetries in the linearised scalar superfield constructed in appendix B.

[2]Notice that the last term in the first line of eq. (3.3) in [7] is in fact wrong due to U(1) violation, see in particular [5] for the corrected result.

2.3 Going beyond four points

Beyond four points, the gravitational part of the action is fixed by general covariance. The full completion including the anti-symmetric tensors and the axio-dilaton remains however an open task. Generalised geometry provides a hint in this direction by introducing a torsionful connection involving the B_2-field, cf. section 3.1. However, it is confirmed that such an approach does not capture the complete string-theoretic result as verified by 1-loop 5-point [3, 43] and 6-point [4] function computations as well as more recently tree level 5-point results [5]. It is therefore desirable to introduce a new ordering principle that incorporates ideas from supersymmetry and $\mathrm{SL}(2,\mathbb{Z})$-invariance.

The first crucial feature that appears in the higher-point effective action is the presence of non-U(1)-invariant terms. Indeed, for a given number of P fields, the maximal U(1) charge satisfies the bound [24]

$$|Q_{\mathrm{IIB}}| \leq 2(P-4)\,, \tag{2.15}$$

which is compatible with the fact that the quartic action (2.11) contains only U(1)-preserving terms.

Moreover, although the tensor structures t_8 and ϵ_{10} in (2.13), which are very natural from the perspective of string amplitudes, seem to be appropriate representations for the kinematics at 4 points, when one goes to higher points, more fundamental higher-dimensional tensors, generalising t_8, make their appearance in the kinematics.

The higher-derivative action can be constructed from fundamental superspace integrals as established in the 1980s by the seminal works [12, 44–46]. We provide a review of this approach in appendix B. The general outcome of this formalism is that the couplings in the effective action are obtained from a single superspace integral

$$t_{3n+2m} = \int \mathrm{d}^{16}\Theta (\Theta\Gamma^{(3)}\Theta)^n (\Theta\Gamma^{(2)}\Theta)^m\,, \quad 2n+2m=16\,. \tag{2.16}$$

Here, $\Gamma^{(3)}$ denotes the anti-symmetric product of 3 $\mathrm{SO}(1,9)$ Γ-matrices and

$$\left(\Theta\Gamma^{(2)}\Theta\right)^2 = \left(\Theta\Gamma^{M_1M_2k}\Theta\right)\left(\Theta\Gamma^{N_1N_2}{}_k\Theta\right) \tag{2.17}$$

so that t_{3n+2m} carries $3n+2m$ indices. The role of such index structures has previously been discussed in [3, 6, 8, 9, 42, 47, 48], though their direct manifestation for non-trivial G_3 and $\nabla\tau$ backgrounds in the string effective action remains largely unexplored.

Before we get to that, let us review some established aspects of the linearised superspace approach. Interestingly, in this formalism the complete kinematics for R^4 is obtained from a single elementary superspace integral

$$t_{16}R^4 = \int \mathrm{d}^{16}\Theta \left[\left(\Theta\Gamma^{M_1M_2k}\Theta\right)\left(\Theta\Gamma_k{}^{M_3M_4}\Theta\right) R_{M_1M_2M_3M_4}\right]^4 . \tag{2.18}$$

We find indeed that

$$\mathcal{J}_0 = t_{16}R^4 \tag{2.19}$$

which is consistent with appendix B.2 of [47]. One arrives at a similar result by studying 4-point functions of light-cone supermembrane vertex operators [49] or in the Green-Schwarz formalism [18]. Crucially, the above result cannot be extended to the torsionful Riemann tensor as we show in more detail in section 3.3.

The formalism of [12] already allows to infer additional non-linear couplings. In fact, the authors of [8, 10] constructed the entire effective action at order $(\alpha')^3$ for vanishing G_3 and $\nabla\tau$ from a *single superspace integral*. To this end, one defines the 6-index tensor

$$\tilde{\mathcal{R}}_{M_1M_2M_3M_4M_5M_6} = \frac{g_{M_3M_6}}{8} C_{M_1M_2M_4M_5} + \frac{\mathrm{i}}{48} \nabla_{M_1} F_{M_2M_3M_4M_5M_6} + \frac{1}{768} \left(F_{M_1M_2M_3kl} F_{M_4M_5M_6}{}^{kl} - 3 F_{M_1M_2M_6kl} F_{M_4M_5M_3}{}^{kl} \right) \tag{2.20}$$

associated with the Θ^4-term in the superfield language.[3] This term enters the superfield Φ at order Θ^4 in such a way that

$$\Phi \supset \left(\Theta\Gamma^{M_1M_2M_3}\Theta\right) \left(\Theta\Gamma^{M_4M_5M_6}\Theta\right) \left(\tilde{\mathcal{R}}_{M_1M_2M_3M_4M_5M_6} + \dots\right) \tag{2.21}$$

where ... denotes further non-linear terms $\sim |G_3|^2$ or $\sim |\mathcal{P}|^2$. This clearly implies that $\tilde{\mathcal{R}}$ as defined in (2.20) is symmetric under the exchange $(M_1, M_2, M_3) \leftrightarrow (M_4, M_5, M_6)$, anti-symmetric in (M_1, M_2, M_3) and (M_4, M_5, M_6) and enjoys additional symmetries collected in section B.2.

After having identified the non-linear piece at order Θ^4 in (2.21), it is straightforward to determine the contribution to the effective action. It is encoded in the integral

$$t_{24}\tilde{\mathcal{R}}^4 = \int \mathrm{d}^{16}\Theta \left[\left(\Theta\Gamma^{M_1M_2M_3}\Theta\right) \left(\Theta\Gamma^{M_4M_5M_6}\Theta\right) \tilde{\mathcal{R}}_{M_1M_2M_3M_4M_5M_6} \right]^4 \subset \int \mathrm{d}^{16}\Theta\, \Phi^4 \tag{2.22}$$

which was explicitly computed[4] in [9] using the results of [42]. For instance, applying the results of [42], we can show that[5]

$$t_{24}\tilde{\mathcal{R}}^4\big|_{gR} = \frac{1}{3^2 \cdot 2^5}\, \mathcal{J}_0\,. \tag{2.23}$$

By expanding (2.22) to higher order in F_5, one finds schematically [9, 13]

$$t_{24}\tilde{\mathcal{R}}^4 = \mathcal{J}_0 + F_5^2 R^3 + \left(\nabla F_5\right)^2 R^2 + F_5^4 R^2 + F_5^6 R + \left(\nabla F_5\right)^2 F_5^2 R + \left(\nabla F_5\right)^4 + F_5^8\,. \tag{2.24}$$

The odd powers of F_5 have to be absent because the action is necessarily real.

With regard to 8-derivative terms involving G_3 or τ, much less is known about the structure of the action (2.9). Partial one-loop results for terms like $H_3^2 R^3$ at the 5-point level have been computed [3, 4, 43] with the tree-level counterparts obtained in [5]. The authors of [51] succeeded in restricting terms of the form $(\nabla\phi)^2 R^3$ through consistency with supersymmetry in 4D which is equivalent to the earlier work [52].

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[3]To arrive at (2.20), one uses non-linear SUSY constraints [8, 10] and applies straightforward rules for the decomposition of Γ-matrices for which we employed the `Gamma` software package [50]. Afterwards, one utilises the corresponding projection operators as detailed in appendix B.2. These methods can be employed in a similar fashion to construct the corresponding expression for $|G_3|^2$ which we leave for the future.

[4]We refer the reader to [13] for the corrected results of [9].

[5]Notice that our normalisation differs from [42] where they found $3^4 \cdot 2^{24}$ on the r.h.s.

At tree level, the complete 8-derivative action in the NSNS-sector was inferred in [14, 53, 54] upon using constraints of T-duality and double geometry. However, the exorbitant use of field redefinitions and the missing representation of the final result in terms of fundamental index structures makes it virtually impossible to compare the results to the other literature on this subject. This motivates initiating a more unifying approach.

Although RR-sector couplings can be partially inferred from NSNS-sector results at tree and 1-loop level [5], a concise definition of manifestly $\mathrm{SL}(2,\mathbb{Z})$-invariant quintic vertices demands a more unifying approach. For instance, one expects further contributions to (2.20) of the schematic for $|G_3|^2$ and $|\mathcal{P}|^2$ which would relate a subset of higher derivative terms $|G_3|^{2m}|\mathcal{P}|^{2n}R^{4-m-n}$ to the famous R^4 structure by means of a single superspace integral (2.22). We will have more to say about this in section 3.3.

3 Eight-derivative couplings at five points

Throughout this paper, we are particularly interested in the 5-point structure $G_3^2R^3$ together with its variants $|G_3|^2R^3$ and $\overline{G}_3^2R^3$ which can be treated in a similar fashion. These couplings contribute e.g. to the leading order $(\alpha')^3$-correction to the 4D F-term scalar potential [52, 55, 56]. In this section we discuss them, and in general the full structure arising at five point, according to the behavior of the various terms under the U(1) R-symmetry.

3.1 Couplings from superstring amplitudes

Up to 5-points and including only R and G_3, the effective action up to quadratic[6] order in the flux may be written as [5]

$$\mathcal{L}\left(R,G_3,\overline{G}_3\right) = \mathcal{L}_{R(\Omega_+)^4} + \mathcal{L}_{|G_3|^2R^3} + \mathcal{L}_{G_3^2R^3+\text{c.c.}} + \mathcal{L}_{\text{CP-odd}} \tag{3.1}$$

where

$$\mathcal{L}_{|G_3|^2R^3} = \alpha f_0 \left\{ -\frac{1}{2}t_8t_8|G_3|^2R^3 - \frac{7}{24}\epsilon_9\epsilon_9|G_3|^2R^3 + 2\cdot 4! \sum_{i=1}^{8} \tilde{d}_i|G_3|^2\tilde{Q}^i \right\} , \tag{3.2}$$

$$\mathcal{L}_{G_3^2R^3+\text{c.c.}} = \alpha f_1 \left\{ \frac{3}{4}t_8t_8G_3^2R^3 - \frac{1}{16}\epsilon_9\epsilon_9G_3^2R^3 - 3\cdot 4! \sum_{i=1}^{8} \tilde{d}_iG_3^2\tilde{Q}^i \right\} + \text{c.c.} , \tag{3.3}$$

$$\mathcal{L}_{\text{CP-odd}} = 3^2\cdot 2^4\alpha \left\{ G_3\wedge\left(f_0X_7\left(\Omega,\overline{G}_3\right) + f_1X_7\left(\Omega,G_3\right)\right) + \text{c.c.} \right\} \tag{3.4}$$

in terms of

$$\left(\tilde{d}_1,\ldots,\tilde{d}_8\right) = 4\left(1,-\frac{1}{4},0,\frac{1}{3},1,\frac{1}{4},-2,\frac{1}{8}\right) . \tag{3.5}$$

[6] The NSNS-sector couplings $H_3^2(\nabla H_3)^2R$ have been completely specified at 1-loop where, in addition to the piece coming from expanding $R(\Omega_+)^4$, one finds an additional contribution $4/9\,\epsilon_9\epsilon_9H_3^2(\nabla H_3)^2R$. The tree-level counterparts could in principle be determined following the procedures outlined in [5]. Obtaining the equivalent expressions in terms of $G_3,\overline{G}_3$ is slightly more complicated given that the pure NSNS-sector terms do not fully determine cross terms with the RR 3-form F_3 beyond quadratic order. This becomes already evident at 4-points in [7] which led to a new operator $\mathcal{O}_2((|\nabla G_3|^2)^2)$ reducing to t_8t_8 only in the pure NSNS- or RR-sector.

We suppressed indices on the objects $\tilde{Q}^i$ corresponding to certain 6-index elements of a basis for R^3 to be introduced below, see also appendix A.2 for definitions. The index structure in the even-even sector is

$$t_8t_8G_3^2R^3 = t_{M_1\dots M_8}t^{N_1\dots N_8}G^{M_1M_2P}G_{N_1N_2P}R^{M_3M_4}{}_{N_3N_4}\dots R^{M_7M_8}{}_{N_7N_8} \tag{3.6}$$

and the odd-odd sector couplings are [3–5]

$$\epsilon_9\epsilon_9G_3^2R^3 = -\epsilon_{PM_0\dots M_8}\epsilon^{PN_0\dots N_8}G^{M_1M_2}{}_{N_0}\,G_{N_1N_2}{}^{M_0}\,R^{M_3M_4}{}_{N_3N_4}\,R^{M_5M_6}{}_{N_5N_6}\,R^{M_7M_8}{}_{N_7N_8}\,. \tag{3.7}$$

The expressions (3.2) and (3.3) has the rather surprising feature that some contractions cannot be repacked into the conventional t_8t_8 or $\epsilon_9\epsilon_9$ structures. We argue in the following that this should not come as a surprise, but rather as a clear indication that more fundamental index structures are prerequisites to encode the full string kinematics.

The remaining piece $\mathcal{L}_{R(\Omega_+)^4}$ in (3.1) originates from a connection with torsion given by[7]

$$(\Omega_\pm)_M{}^{KL} = \Omega_M{}^{KL} \pm \frac{1}{2}\mathrm{e}^{-\phi/2}H_M{}^{KL} \tag{3.8}$$

resulting in the 4-index tensor

$$R(\Omega_\pm)_{MN}{}^{KL} = R_{MN}{}^{KL} \pm \mathrm{e}^{-\phi/2}\nabla_{[M}H_{N]}{}^{KL} + \frac{\mathrm{e}^{-\phi}}{2}H_{[M}{}^{KP}H_{N]P}{}^{L}\,. \tag{3.9}$$

In Einstein frame, the R^4 contribution is replaced by

$$\mathcal{L}_{R(\Omega_+)^4} = \alpha\, f_0(\tau,\bar\tau)\,\left(t_8t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)R(\Omega_+)^4\,. \tag{3.10}$$

Up to 5-point contact terms, we may expand $R(\Omega_+)^4$ as usual

$$\mathcal{L}_{R(\Omega_+)^4} = \alpha\, f_0\,\left(t_8t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)\Big\{R^4 + \mathrm{e}^{-2\phi}(\nabla H_3)^4 + 6\mathrm{e}^{-\phi}(\nabla H_3)^2R^2 \\ + 2\mathrm{e}^{-\phi}H_3^2R^3 + 6\mathrm{e}^{-2\phi}H_3^2(\nabla H_3)^2R + \dots\Big\} \tag{3.11}$$

where the appropriate (anti-)symmetrisation of indices on H_3 and ∇H_3 is implied. The identity $R_{M_1M_2N_1N_2}(\Omega_+) = R_{N_1N_2M_1M_2}(\Omega_-)$ due to closure of H_3 implies the absence of odd powers of H_3 in the above expansion. Given that all terms are multiplied by f_0, i.e. the string kinematics is equivalent at tree and 1-loop level, we can replace H_3^2 and $(\nabla H_3)^2$ by the corresponding U(1)-preserving combinations[8] such that

$$\mathcal{L}_{R(\Omega_+)^4} = \mathcal{L}^{(3)}_{4-\mathrm{pt}}\big|_{\mathcal{P},\overline{\mathcal{P}}=0} + 2\alpha\, f_0\,\left(\tilde t_8\tilde t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)|G_3|^2R^3 + \dots\,. \tag{3.12}$$

[7]In Einstein frame, the torsionful connection and likewise $R(\Omega_+)$ include additional derivatives with respect to the dilaton. For the purposes of this paper, we may ignore such terms by treating ϕ as a constant.

[8]One needs to take special care of $(|\nabla G_3|^2)^2$ since the index structure is $\mathcal{O}_2$ rather than $t_8t_8 - \epsilon_8\epsilon_8/4$ in (2.11). Further, we expect that $H_3^2(\nabla H_3)^2R \to |G_3|^2|\nabla G_3|^2R$ based on the structure of superparticle amplitudes. One again runs into the aforementioned issue that crossterms between F_3 and H_3 are not fully kinematically determined by the pure NSNS expressions. We leave the study of such terms for future works.

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	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$\frac{-t_8 t_8 \|G_3\|^2 R^3}{16 \cdot 4!}$	0	0	0	0	0	$\frac{-1}{32}$	$\frac{1}{2}$	$\frac{-1}{16}$	0	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	1	-1	$\frac{1}{4}$	0	0	0	0	0	0	0	0	0
$\frac{-\tilde{t}_8 \tilde{t}_8 \|G_3\|^2 R^3}{16 \cdot 4!}$	0	0	0	0	0	$\frac{1}{64}$	$\frac{-1}{4}$	$\frac{1}{32}$	0	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-1}{4}$	0	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{-1}{4}$	0	0	0	0	0	0	0	0
$\frac{\epsilon_9 \epsilon_9 \|G_3\|^2 R^3}{192 \cdot 4!}$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{2}$	$\frac{1}{32}$	$\frac{-1}{2}$	$\frac{1}{16}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	0	-1	$\frac{1}{4}$	$\frac{-1}{2}$	0	$\frac{-1}{4}$	$\frac{-1}{2}$	$\frac{-1}{3}$	-1	$\frac{-1}{4}$	2	$\frac{-1}{8}$
$\frac{-t_{18} \|G_3\|^2 R^3}{8 \cdot 4!}$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{2}$	0	0	0	0	0	0	0	1	-2	$\frac{1}{2}$	$\frac{-1}{2}$	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	0	0	0	0
$\frac{-\epsilon_8 \epsilon_8 \|G_3\|^2 R^3}{96 \cdot 4!}$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{1}{3}$	$\frac{1}{96}$	$\frac{-1}{6}$	$\frac{1}{48}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{-1}{6}$	0	$\frac{-1}{3}$	$\frac{1}{12}$	$\frac{-1}{6}$	0	0	0	0	0	0	0	0

Table 1: Decomposition of index structures for $|G_3|^2 R^3$ in the 24 component basis for R^3 defined in appendix A.2.

The quartic terms clearly reproduce (2.11) which was already observed in [6, 7]. Further, we defined a second[9] $t_8 t_8$ and $\epsilon_8 \epsilon_8$ index structure

$$\tilde{t}_8 \tilde{t}_8 |G_3|^2 R^3 = t_{M_1 \ldots M_8} t^{N_1 \ldots N_8} G^{[M_1}{}_{N_1 P} \overline{G}^{M_2]P}{}_{N_2} R^{M_3 M_4}{}_{N_3 N_4} \ldots R^{M_7 M_8}{}_{N_7 N_8} \,, \tag{3.13}$$

$$\epsilon_8 \epsilon_8 |G_3|^2 R^3 = \epsilon^{M_1 \ldots M_8} \epsilon^{N_1 \ldots N_8} G_{[M_1|N_1 k} \overline{G}_{|M_2]}{}^{k}{}_{N_2} R_{M_3 M_4 N_3 N_4} \ldots R_{M_7 M_8 N_7 N_8} \,. \tag{3.14}$$

Before we proceed, let us comment on the role of the object $R(\Omega_+)$. From the perspective of generalised geometry, it seems natural to introduce torsion in the form of H_3 in order to capture a big part of the NSNS-sector kinematics. As is evident from the additional contributions in (3.1), this is clearly not sufficient to specify the complete effective action, see also section 3.4 and [5]. Furthermore, this approach is not manifestly $SL(2, \mathbb{Z})$ invariant and might even fail when working with sixteen fermion integrals. Below, we argue that the terms obtained from (3.10) are actually highly non-trivial from the perspective of the superfield approach in the sense that such contributions seem to be (at least partially) associated with non-linear terms at order Θ^6 rather than Θ^4.

For later convenience, we expand the above kinematical structures into independent Lorentz singlets that can be built from $|G_3|^2 R^3$. This can be easily determined by utilising the software package `LiE` [58, 59] looking for all singlets under $SO(1, 9)$. In appendix A.2, we define a 24-dimensional basis for R^3 based on the conventions in [5] in order to write

$$\mathcal{L}_{|G_3|^2 R^3} = \sum_{i=1}^{2} a_i \, |G_3|^2 \, \tilde{S}^i + \sum_{i=1}^{3} b_i \, G^{M_1}{}_{N_1 N_2} \overline{G}^{M_2 N_1 N_2} \, \tilde{W}^i_{M_1 M_2} \tag{3.15}$$
$$+ \sum_{i=1}^{11} c_i \, G^{M_1 M_2}{}_{N_1} \overline{G}^{M_3 M_4 N_1} \, \tilde{X}^i_{M_1 M_2 M_3 M_4} + \sum_{i=1}^{8} d_i \, G^{M_1 M_2 M_3} \overline{G}^{M_4 M_5 M_6} \, \tilde{Q}^i_{M_1 M_2 M_3 M_4 M_5 M_6}$$

[9] In the NSNS sector, the $t_8 t_8$-combination (3.6) was proposed in [57] to recover supersymmetry in Type IIA Calabi-Yau compactifications to 4 dimensions, but was also previously obtained in [3] from a covariant RNS calculation at the 1-loop level. Similarly, (3.13) was found by computing one-loop string amplitudes at 5-points in the light-cone gauge GS formalism in [43] and utilised in [4, 5].

where we implicitly symmetrise G_3 and $\overline{G}_3$. In this basis, we may expand the kinematical structures appearing in (3.6), (3.7), (3.13) and (3.14) as summarised in table 1.

3.2 The U(1)-violating sector

In general, for maximally U(1)-violating (MUV) processes, substantial progress has been made in [24, 27, 60] at the level of amplitudes and modular forms. This makes this sector of the effective action particularly well-behaved. Moreover, one is led to a similar conclusion by studying superparticle/supermembrane amplitudes in 11 dimensions on a T^2 [15–17, 48] as we argue in section 3.2.2.

3.2.1 The role of higher-dimensional index structures

In section 2.3, we argued that the notion of higher-dimensional index tensors is natural from the effective-action point of view, though unnatural from the string amplitude calculus. In this section, we show that the 5-point action can be dramatically simplified with the use of these tensors, thereby explaining a subset of relative coefficients. Subsequently, we provide a microscopic derivation of these results from M-theory loop amplitudes which we show to agree in the U(1)-violating sector with the linearised superfield expectation.

Specifically, the linearised superfield suggests that the following tensor is expected to play an outstanding role in the string kinematics at 5 points

$$t_{18}G_3^2R^3 = \int \mathrm{d}^{16}\Theta \left[(\Theta\Gamma^{M_1M_2M_3}\Theta)G_{M_1M_2M_3}\right]^2 \left[(\Theta\Gamma^{M_1M_2}{}_k\Theta)(\Theta\Gamma^{M_3M_4k}\Theta)R_{M_1M_2M_3M_4}\right]^3 . \tag{3.16}$$

In principle, this expression can be computed via the methods of [42]. Instead, we make use of two independent results available in the literature. First, the authors of [48] expanded t_{18} in a basis of 26 Lorentz singlets under SO(9). Equivalently, a second way to compute (3.16) is using the tensor t_{24} computed in [42] as a generating object for lower-order tensors t_{24-2n} upon appropriate contraction with n metric factors. We discuss this second option in more detail in section 3.3. Utilising the results of [48], we may simplify (3.3) drastically by writing

$$\mathcal{L}_{G_3^2R^3+\text{c.c.}} = \frac{3}{2}\alpha\left\{f_1 t_{18}G_3^2R^3 + f_{-1}t_{18}\overline{G}_3^2R^3\right\} \tag{3.17}$$

where in our normalisation

$$t_{18}G_3^2R^3 = \frac{1}{2}t_8t_8G_3^2R^3 - \frac{1}{24}\epsilon_9\epsilon_9G_3^2R^3 - 2\cdot 4!\sum_{i=1}^{8}\tilde{d}_iG_3^2\tilde{Q}^i \tag{3.18}$$

in terms of the $\tilde{d}_i$ coefficients (3.5) (and equivalently for $\overline{G}_3^2R^3$). The overall coefficient 3/2 in eq. (3.17) is expected from tree level 5-point scattering in the pure spinor formalism [24].

The result (3.17) has many striking implications. The definition of $t_{18}G_3^2R^3$ in (3.18) resolves the apparent puzzle that some terms $\sim \tilde{Q}^i$ in (3.1) do not repackage nicely into t_8t_8 or $\epsilon_n\epsilon_n$. Even more interestingly, it captures the entire string kinematics in the MUV sector. In this sense, the t_{18} structure (3.16) is arguably a more suitable representation of the string kinematics at the level of the 5-point effective action. While this has clearly been known for many years (at least implicitly in the linearised superfield approach [3]),[10] the above provides the first direct proof that such higher-dimensional index structures appear in both the MUV and non-MUV (see the next subsection) sectors at the level of 5-point string amplitudes.

For the MUV terms, we can be even more precise with regard to the overall coefficient. In fact, the modular forms satisfy (see eq. (A.4))

$$\mathcal{D}_0 f_0 = \frac{3}{4} f_1\,, \quad \overline{\mathcal{D}}_0 f_0 = \frac{3}{4} f_{-1} \tag{3.19}$$

which allows us to write

$$\mathcal{L}_{G_3^2 R^3 + \text{c.c.}} = 2\alpha \Big\{ (\mathcal{D}_0 f_0) t_{18} G_3^2 R^3 + \left(\overline{\mathcal{D}}_0 f_0\right) t_{18} \overline{G}_3^2 R^3 \Big\}\,. \tag{3.20}$$

3.2.2 A derivation from superparticles

While the index structure t_{18} naturally appears in the context of the linearised superfield approximation [3], it is not obvious at all from standard string amplitudes (see however [24] for MUV amplitudes and references therein). In this part, we show that the t_{18} structure appears naturally in superparticle amplitudes in M-theory on 2-tori T^2 [15–17]. The calculation proceeds similar to the famous derivation of R^4 in [15] which matched not only the well-known 4-graviton kinematics at tree and 1-loop level, but also provided evidence for modular functions in the Type IIB effective action [18, 35].

In the MUV sector, the only non-vanishing 5-point superparticle amplitude involving two 3-forms and three gravitons in 9D is given by[11]

$$\begin{aligned}\mathcal{A}^{(\text{SP})}_{G_3^2 R^3 + \text{c.c.}} = \frac{1}{2^6 \pi^{9/2}\, \Gamma\left(\frac{3}{2}\right) v_0} \int \frac{\mathrm{d}t}{t} \int \mathrm{d}^9 \mathbf{p} \sum_{l_1, l_2 \in \mathbb{Z}} \mathrm{e}^{-t\left(\mathbf{p}^2 + g^{ab} l_a l_b\right)} \\ t^5 \mathrm{Tr}\left(\left[2 h_{ij} \mathcal{R}^{il} \mathcal{R}^{jm} k_l k_m \right]^3 \left[-\sqrt{2} G_3 P^z \mathcal{R}^{lmn} \right]^2 \right)\end{aligned} \tag{3.21}$$

in terms of $v_0 = \text{Vol}(T^2)$. This contribution is associated with superparticles running in the loop carrying non-trivial KK-charges on the T^2 compensating for the U(1)-charges of G_3^2 to give a real expression.[12] Looking at the trace over fermions, we clearly notice the resemblance with (3.16) in terms of the linearised Riemann tensor in 9 dimensions which allows us to simplify the expression to

$$\mathcal{A}^{(\text{SP})}_{G_3^2 R^3 + \text{c.c.}} = \frac{t_{18} G_3^2 R^3}{2^2\, \Gamma\left(\frac{3}{2}\right) v_0} \int \frac{\mathrm{d}t}{\sqrt{t}} \sum_{l_1, l_2 \in \mathbb{Z}} P_z^2 \mathrm{e}^{-t g^{ab} l_a l_b} + \text{c.c.}\,. \tag{3.22}$$

[10] In essence, the authors of [3] argued that the full kinematics of the NSNS-sector coupling $H_3^2 R^3$ obtained from string scattering amplitudes is **not** captured by only the index structure t_{18} appearing in the linearised superfield calculus. Our results demonstrate that t_{18} nonetheless plays an important role for the tree level $H_3^2 R^3$ couplings, cf. eq. (3.40) below.

[11] The normalisation of these amplitudes will be discussed further in section 4.1.

[12] In general, in the context of superparticle amplitudes, the higher-dimensional index structures t_N for the G-flux naturally arise (even in the non-MUV sector) whenever the superparticles in the loop carry non-trivial KK-momentum.

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In appendix C.2, we show that

$$\int \frac{\mathrm{d}t}{\sqrt{t}} \sum_{l_1,l_2\in\mathbb{Z}} P_z^2 \mathrm{e}^{-tg^{ab}l_a l_b} = \frac{4\Gamma\left(\frac{5}{2}\right)}{\sqrt{v_0}}\, f_1(\tau,\bar\tau)\,. \tag{3.23}$$

The volume scaling implies that the amplitude vanishes in the decompactification limit $v_0 \to \infty$. As opposed to U(1)-preserving amplitudes like $t_{16}R^4$ [15], U(1)-violating effects such as the above are not present in 11D supergravity [16, 17].

Taking the limit to Type IIB, we find

$$v_0 \mathcal{A}^{(\mathrm{SP})}_{G_3^2R^3+\mathrm{c.c.}} \xrightarrow{v_0\to 0} \mathcal{L}^{(\mathrm{SP})}_{G_3^2R^3+\mathrm{c.c.}} \tag{3.24}$$

in terms of

$$\mathcal{L}^{(\mathrm{SP})}_{G_3^2R^3+\mathrm{c.c.}} = \frac{3}{2}\left(f_1(\tau,\bar\tau)\, t_{18}G_3^2R^3 + f_{-1}(\tau,\bar\tau)\, t_{18}\overline{G}_3^2R^3\right) \tag{3.25}$$

in agreement with (3.3). We stress that, while the linearised superfield and perturbative superstring amplitudes typically only see a small subset of terms of the full modular forms f_w, a single superparticle amplitude derives the full f_w from first principles. Critically, this involves also non-perturbative D-instanton contributions which have only recently been derived for R^4 from string field theory [21, 22].

3.3 The U(1)-preserving sector — evidence for non-linear superfields

Let us now move our attention to the U(1)-preserving sector of the five-point effective action, whose structure turns out to be much more involved. As described in section 3.1, there is a contribution originating from the torsionful Riemann tensor (3.12) as well as a remainder given by (3.2). Inspecting the latter, we notice a close resemblance to the MUV contact terms in (3.3). Indeed, (3.18) is equivalently defined for $|G_3|^2R^3$ allowing us to recast (3.2) together with (3.12) in the form

$$\mathcal{L}_{R(\Omega_+)^4} + \mathcal{L}_{|G_3|^2R^3}\bigg|_{\text{5-point}} = \alpha f_0\left(-t_{18} - \frac{1}{3}\epsilon_9\epsilon_9 + 2\tilde t_8\tilde t_8 - \frac{1}{2}\epsilon_8\epsilon_8\right)|G_3|^2R^3\,. \tag{3.26}$$

Contrary to above, there is an odd-odd structure remaining which can in fact be traced back to the 1-loop NSNS strcture $-1/3\epsilon_9\epsilon_9 H_3^2R^3$ of [4, 43], while $t_{18}H_3^2R^3$ can only be seen at string tree level [5]. We will have more to say about this in the next section.

The fact that $\mathcal{L}_{|G_3|^2R^3}$ contains another t_{18} piece with an overall factor of -1 will in fact play quite a crucial role in the reductions to 4D in section 5.2. While in the MUV sector such a term is completely specified by the linearised superfield, we stress that (3.26) can only be obtained from non-linear terms in the superfield such as through contributions $\sim |G_3|^2$ to (2.20). In this way, we expect a subset of the terms in (3.26) to be related to R^4 through (2.22) as we now demonstrate. Ultimately, we will arrive at a similar conclusion as [3] in the NSNS sector, namely that generating the odd-odd contribution $\epsilon_9\epsilon_9|G_3|^2R^3$ from a non-linear superfield requires corrections at order Θ^6.

To recapitulate, we observed that the linearised superfield indeed captures the complete string-theory result in the MUV sector which is encoded by a single superspace integral giving rise to the tensor structure t_{18}. Even more importantly, we obtained evidence that the same tensor also enters in the non-MUV sector of the action in such a way that it cancels out at 1-loop for $H_3^2 R^3$. We expect this to be a clear hint at potential non-linear couplings in the superfield. Non-linear completions of the superfield (B.6) are given by (schematically)

$$\Delta \supset \Theta^2 G_3 + \Theta^4 \left(R + |G_3|^2 + \dots\right) + \Theta^6 \left(\nabla^2 \overline{G}_3 + R\overline{G}_3 + \dots\right) + \Theta^8 \left(R\overline{G}_3^2 + \dots\right) . \tag{3.27}$$

The terms entering at order Θ^4 were discussed in section 2.3 for F_5^2. Clearly, one similarly expects terms of the form $|G_3|^2$ also to enter at this order [8, 10]. They can indeed be obtained utilising the results of [12]. For now, we work with a general parametrisation modifying (2.20) in such a way that

$$\begin{aligned}\mathcal{R}_{M_1 \dots M_6} = \tilde{\mathcal{R}}_{M_1 \dots M_6} + \frac{1}{768}\Big(&\lambda_1\, G_{M_1 M_2 M_3}\overline{G}_{M_4 M_5 M_6} + \lambda_2\, G_{M_1 M_2 M_6}\overline{G}_{M_4 M_5 M_3} \\ &+ \lambda_3\, g_{M_3 M_6} G_{M_1 M_2 k}\overline{G}_{M_4 M_5}{}^{k} + \lambda_4\, g_{M_3 M_6} G_{M_1 M_5 k}\overline{G}_{M_4 M_2}{}^{k} \\ &+ \lambda_5\, \epsilon_{k_1 \dots k_5 M_2 \dots M_6}\left(G^{k_1 k_2}{}_{M_1}\overline{G}^{k_3 k_4 k_5} + \overline{G}^{k_1 k_2}{}_{M_1} G^{k_3 k_4 k_5}\right)\Big) .\end{aligned} \tag{3.28}$$

We generically expect $\lambda_i \neq 0$ for all λ_i. The symmetries of $\mathcal{R}$ are determined e.g. by Fierz identities implying the absence of double traces, cf. appendix B.2.

The contribution to the effective action may be written as

$$\mathcal{L}_{\mathcal{R}^4} = c \int \mathrm{d}^{16}\Theta \left[\left(\Theta\Gamma^{M_1 M_2 M_3}\Theta\right)\left(\Theta\Gamma^{M_4 M_5 M_6}\Theta\right)\mathcal{R}_{M_1 M_2 M_3 M_4 M_5 M_6}\right]^4 \tag{3.29}$$

The normalisation constant c is fixed such that we recover $\mathcal{J}_0$ at order R^4 as defined in (2.13). We find that (recall (2.23))

$$t_{24}\mathcal{R}^4\big|_{gR} = \frac{1}{2^5 \cdot 3^2}\mathcal{J}_0 \quad \Rightarrow \quad c = 2^5 \cdot 3^2 \tag{3.30}$$

We are mainly interested in the terms arising to linear order in λ_i where we find that the CP-even part is given by

$$\begin{aligned} c\, t_{24}\mathcal{R}^4 = \frac{\lambda_1}{2^5} t_{18}|G_3|^2 R^3 - 2\lambda_2\, T_{18}^{(1)}|G_3|^2 R^3 - 2\left(2\lambda_3 + \lambda_4\right) T_{16}^{(1)}|G_3|^2 R^3 \\ + \frac{36\lambda_5}{5}\, T_{18}^{(2)}|G_3|^2 R^3 \end{aligned} \tag{3.31}$$

where $T_N^{(i)}$ are certain tensor structures carrying N indices. We summarised the decomposition of the individual kinematical structures in table 2. We find the following relationships among the different terms

$$\begin{aligned} T_{18}^{(1)}|G_3|^2 R^3 &= \frac{-t_{18}|G_3|^2 R^3}{8 \cdot 4!} - 2T_{18}^{(2)}|G_3|^2 R^3 , \\ T_{16}^{(1)}|G_3|^2 R^3 &= \frac{-t_{18}|G_3|^2 R^3}{8 \cdot 4!} - T_{18}^{(2)}|G_3|^2 R^3 , \\ T_{18}^{(1)}|G_3|^2 R^3 &= T_{16}^{(1)}|G_3|^2 R^3 - T_{18}^{(2)}|G_3|^2 R^3 . \end{aligned} \tag{3.32}$$

	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
$T_{18}^{(1)}\|G_3\|^2R^3$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{2}$	0	0	0	0	0	$\frac{2}{3}$	0	$\frac{1}{3}$	-2	$\frac{1}{6}$	$\frac{-1}{6}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0
$T_{16}^{(1)}\|G_3\|^2R^3$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{3}$	0	$\frac{2}{3}$	-2	$\frac{1}{3}$	$\frac{-1}{3}$	0	0	0	0	0	0	0	0
$T_{18}^{(2)}\|G_3\|^2R^3$	0	0	0	0	0	0	0	0	0	0	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{-1}{6}$	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	0	0	0	0

Table 2: Decomposition of leading order flux terms obtained from $t_{24}\mathcal{R}^4$ in the 24 component basis for R^3 defined in appendix A.2.

Before we continue, we highlight the following caveat. The λ_i in (3.28) are not the only non-linear modifications contributing at the level of the 5-point contact terms in the non-MUV sector. Terms at order $\Theta^6\overline{G}_3R$ in (3.27) contribute at the same level. This means we cannot simply expect the U(1)-neutral sector of eq. (3.1) to be constructable from $t_{24}\mathcal{R}^4$ alone. Hence, we can only compare the value of the λ_i to the string amplitude result and make a prediction about contributions from higher-order non-linear terms.

From (3.17), we expect to find a term of the form $t_{18}|G_3|^2R^3$ upon expanding $t_{24}\mathcal{R}^4$ which, according to (3.31), is trivially achieved by setting $\lambda_1 = -2^5$ and $\lambda_i = 0$ for all other coefficients. However, we generically expect that all λ_i are non-vanishing and, given the identities (3.32), a non-trivial combination of values for the λ_i can also do the job. In fact, it turns out that the equation $ct_{24}\mathcal{R}^4 = -t_{18}|G_3|^2R^3$ has the solution

$$\lambda_5 = \frac{5}{36}\left(24 + \frac{3}{4}\lambda_1 - \lambda_2\right)\,, \qquad \lambda_4 = -24 - \frac{3}{4}\lambda_1 - \lambda_2 - 2\lambda_3\,. \tag{3.33}$$

Initially, one might hope that by modifying (3.28) accordingly the additional piece $\sim T(\epsilon_{10}, t_8)|G_3|^2R^3$ (see (3.37) below) can also be reabsorbed into the definition of $t_{24}\mathcal{R}^4$. However, looking at the coefficients collected in table 1, this seems not to be the case. For instance, $\epsilon_9\epsilon_9|G_3|^2R^3$ involves terms with $d_i \neq 0$ for $i \geq 4$ which cannot arise from t_{24} as observed in table 2. In principle, a natural way to extend the linearised superfield (B.6) would be terms of the form $\Theta^6R\overline{G}_3$ with the symmetry properties conjectured in [3]. According to these arguments, one would need terms of the form[13]

$$\begin{aligned}\Theta^6R\overline{G} \supset \Theta\Gamma^{M_1N_1P_1}\Theta\,\Theta\Gamma^{M_2N_2P_2}\Theta\,\Theta\Gamma^{M_3N_3P_3}\Theta\Big(&a_1g_{P_1P_2}R_{M_1N_1[M_2N_2}\overline{G}_{M_3N_3P_3]}\\ &+a_2g_{P_1P_2}g_{M_1P_3}R^p{}_{(N_1|\,[M_2N_2}\overline{G}_{|M_3)\,N_3]\,p}\Big)\end{aligned} \tag{3.34}$$

which would appear as

$$\Delta^4 \supset \left(\Theta^2G_3\right)\left(\Theta^6R\overline{G}_3\right)\left(\Theta^4R\right)^2\,. \tag{3.35}$$

However, this is beyond the scope of the present work.

[13]One can argue e.g. for the existence of the former based on the structure of terms found in [12] following the derivation of Θ^4 terms in [10].

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3.4 A new perspective on tree and 1-loop kinematics

To summarise, the structure of 5-point contact terms built from the complex 3-form and the Riemann tensor in (3.2) and (3.3) is dramatically simplified by introducing t_{18}. Altogether, we showed that up to five points (3.1), the effective action can be summarised as (ignoring the terms involving ∇G_3)

$$\mathcal{L} = \alpha\Big\{ f_0(\tau,\bar\tau)\, t_{16} R^4 + \frac{3}{2}\left(f_1(\tau,\bar\tau)\, t_{18} G_3^2 R^3 + f_{-1}(\tau,\bar\tau)\, t_{18} \overline{G}_3^2 R^3\right) \\ + f_0(\tau,\bar\tau)\, \left(T(\epsilon_{10},t_8) - t_{18}\right) |G_3|^2 R^3 \Big\} \tag{3.36}$$

where we defined

$$T(\epsilon_{10},t_8) = -\frac{1}{3}\epsilon_9\epsilon_9 + 2\tilde t_8 \tilde t_8 - \frac{1}{2}\epsilon_8\epsilon_8 \tag{3.37}$$

and the appropriate (anti-)symmetrisation and contraction of indices is implied.

For later purposes and to make contact with previous work [4, 5, 43], we now extract the tree and 1-loop kinematics in the respective sectors. In the NSNS-sector one finds after using the large $\mathrm{Im}(\tau)$ expansion of f_w in eq. (A.6)

$$f_0 t_{18} |G_3|^2 R^3 - \frac{3}{2}\left(f_1 t_{18} G_3^2 R^3 + f_{-1} t_{18} \overline{G}_3^2 R^3\right)\Big|_{\text{NSNS}} = \mathrm{e}^{-\phi}\left(f_0 + \frac{3}{2}(f_1 + f_{-1})\right) t_{18} H_3^2 R^3 \\ = 4 a_T \mathrm{e}^{-\phi}\, t_{18} H_3^2 R^3\,, \tag{3.38}$$

while the corresponding RR-sector expression reads

$$f_0 t_{18} |G_3|^2 R^3 - \frac{3}{2}\left(f_1 t_{18} G_3^2 R^3 + f_{-1} t_{18} \overline{G}_3^2 R^3\right)\Big|_{\text{RR}} = \left(f_0 - \frac{3}{2}(f_1 + f_{-1})\right) t_{18} F_3^2 R^3 \\ = -2\mathrm{e}^{\phi}(a_T - a_L) t_{18} F_3^2 R^3\,. \tag{3.39}$$

Hence, the structure of terms in (3.17) together with (3.26) is such that in the NSNS-sector

$$\mathcal{L}_{H_3^2 R^3}\Big|_{\text{tree}} = a_T \alpha \mathrm{e}^{-\phi}\left(-4\, t_{18} + T(\epsilon_{10},t_8)\right) H_3^2 R^3\,, \tag{3.40}$$

$$\mathcal{L}_{H_3^2 R^3}\Big|_{\text{1-loop}} = a_L \alpha \mathrm{e}^{-\phi}\, T(\epsilon_{10},t_8) H_3^2 R^3 \tag{3.41}$$

and in the RR-sector

$$\mathcal{L}_{F_3^2 R^3}\Big|_{\text{tree}} = a_T \alpha \mathrm{e}^{\phi}\left(2\, t_{18} + T(\epsilon_{10},t_8)\right) F_3^2 R^3\,, \tag{3.42}$$

$$\mathcal{L}_{F_3^2 R^3}\Big|_{\text{1-loop}} = a_L \alpha \mathrm{e}^{\phi}\left(-2\, t_{18} + T(\epsilon_{10},t_8)\right) F_3^2 R^3\,. \tag{3.43}$$

The combination of index structures (3.37) appears universally in all contributions since it is associated with the U(1)-neutral part of (3.1). Let us further stress that, while t_{18} plays a role in both tree and 1-loop kinematics for processes $F_3^2 R^3$, it only appears at tree level in the NSNS sector $H_3^2R^3$. The absence of $t_{18}H_3^2R^3$ at 1-loop has already been observed in [3]. We showed that this is due to an important interplay of modular forms and the relative coefficients in the MUV and, in particular, the non-MUV sector. In fact, it turns out that the difference of tree and 1-loop kinematics at the level of the effective action (and equally of amplitudes since the pole structure from additional exchange of massless states are removed) is determined by t_{18} only since

$$\Delta\mathcal{L}_{H_3^2R^3} = \frac{1}{a_T}\mathcal{L}_{H_3^2R^3}\Big|_{\text{tree}} - \frac{1}{a_L}\mathcal{L}_{H_3^2R^3}\Big|_{\text{1-loop}} \sim t_{18}H_3^2R^3 \tag{3.44}$$

which agrees with the first line of table 2 in [5] by comparing to the corresponding line for t_{18} in table 1. We expect this to be special about 5-point amplitudes since the non-MUV sector is unique in the sense that it consists of terms of vanishing U(1)-charge only.

In the superfield language, the above observation is actually a highly non-trivial cancellation between linear effects (MUV terms) and non-linear contributions (non-MUV terms). Indeed, the latter are encoded by

$$T_{\text{non-lin.}} = -t_{18} + T(\epsilon_{10}, t_8)\,, \tag{3.45}$$

but the linear superfield contributes another t_{18} such that it precisely cancels out in (3.41). Hence, separating eqs. (3.40)–(3.43) into linear and non-linear superfield contributions, the tree (1-loop) kinematics is $\mp 3\, t_{18} + T_{\text{non-lin.}}$ ($\pm t_{18} + T_{\text{non-lin.}}$) with the upper (lower) sign for H_3 (F_3). We see that precisely at 1-loop in the NSNS sector the coefficients conspire to cancel t_{18}.

In 10 dimensions, t_{18} contains two CP-odd pieces which we ignored throughout this section. These CP-odd couplings enter at NSNS tree level (3.40), but not at NSNS 1-loop (3.41). This is precisely opposite to the expectations of [5] and the terms summarised in (3.4). From the string world-sheet point of view, the absence of CP-odd couplings at NSNS tree level is due to missing ϵ_{10} contributions from only NSNS emission vertex operators [5].

This apparent issue is resolved by adding an additional CP-odd piece in the non-MUV sector as in (3.37) $\sim \vartheta\, t_8\epsilon_{10}$ which must be such that the CP-odd terms in (3.40) cancel, i.e.,

$$\left(-4t_{18}\big|_{\text{CP-odd}} + \vartheta\, t_8\epsilon_{10}\right) H_3^2R^3 = 0\,. \tag{3.46}$$

Further, agreement with (3.4) demands

$$\left(-t_{18}\big|_{\text{CP-odd}} + \vartheta\, t_8\epsilon_{10}\right) |G_3|^2R^3 = 3^2\cdot 2^4\, G_3\wedge X_7\left(\Omega,\overline{G}_3\right)\big|_{\text{lin. in }\overline{G}_3} \tag{3.47}$$

which leads us to conclude

$$t_{18}|G_3|^2R^3\big|_{\text{CP-odd}} = 3\cdot 2^4\, G_3\wedge X_7\left(\Omega,\overline{G}_3\right)\big|_{\text{lin. in }\overline{G}_3}\,. \tag{3.48}$$

We leave a more thorough investigation of CP-odd couplings for the future.

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4 Effective action beyond five points

4.1 The maximally U(1)-violating couplings

Restricting our attention to terms involving G_3 and R, we conjecture that maximally U(1)-violating terms are kinematically captured by the linearised superfield in the sense that

$$\mathcal{L}^{\text{max.}}_{G_3,R} = \alpha \sum_{w=0}^{4} C_w \, f_w(\tau,\bar{\tau}) \, t_{16+2w} G_3^{2w} R^{4-w} + \text{c.c.} \tag{4.1}$$

or more explicitly

$$\begin{aligned}\mathcal{L}^{\text{max.}}_{G_3,R} = \alpha \Big(& C_0 f_0 \, t_{16} R^4 + C_1 f_1 \, t_{18} G_3^2 R^3 + C_2 f_2 \, t_{20} G_3^4 R^2 \\ & + C_3 f_3 \, t_{22} G_3^6 R + C_4 f_4 \, t_{24} G_3^8 + \text{c.c.} \Big) \,.\end{aligned} \tag{4.2}$$

The numerical coefficients C_w will be discussed below and α was already defined in (2.12). This is supported by observations made in [27] and confirmed explicitly for the 5-point structure $G_3^2 R^3 + \text{c.c.}$ in [5]. The special role of MUV amplitudes is further discussed in [24]. Further evidence is provided by the 11D superparticle calculus for which the relevant vertex operator contributions lead to 9D kinematical structures of the form

$$\begin{aligned}\tilde{K}_{G_3^{2n} R^{4-n}} &= \int \mathrm{d}^{16}\theta \left(\left(\theta \Gamma^{M_1 M_2 M_3} \theta\right) G_{M_1 M_2 M_3} \right)^{2n} \left[\left(\theta \Gamma^{M_1 M_2} \theta\right) \left(\theta \Gamma^{M_3 M_4} \theta\right) R_{M_1 M_2 M_3 M_4} \right]^{4-n} \\ &= t_{16+2n} G_3^{2n} R^{4-n}\end{aligned} \tag{4.3}$$

and equivalently for the complex conjugates. One might therefore formulate the conjecture:

> *The effective action for maximally* U(1)*-violating tensor structures involving* G_3*,* $\overline{G}_3$ *and* R *is fully and equivalently determined by either* 11D *superparticle amplitudes or the linearised superfield approximation.*

The equivalence to proper string amplitudes to all loop orders has only been confirmed at 5-points, but we expect this to be true up to 8-points where we conjecture $f_4 t_{24} G_3^8 + \text{c.c.}$. Below, we provide further evidence for higher-point coefficients which would appear at the level of string 7- and 8-point amplitudes. In contrast, amplitudes with less U(1) charge receive further contributions from A) other components of the superparticle vertex operators[14] or from B) non-linear completions of the superfield.

The coefficients C_w are such that the pre-factor for MUV terms satisfy [24]

$$C_w f_w = 2^w \mathcal{D}_{w-1} \ldots \mathcal{D}_0 f_0 \,. \tag{4.4}$$

As discussed in section 4.4 of [24], one expects up to 6-point tree level closed-string amplitudes

$$C_0 = 1 \,, \qquad C_1 = \frac{3}{2} \,, \qquad C_2 = \frac{15}{4} \,. \tag{4.5}$$

[14]One should keep in mind that this calculus might not necessarily capture the full kinematics due to the light-cone gauge fixing condition. It is hence imperative to make a direct comparison to string amplitude results.

One can easily confirm that with this choice and upon applying (A.5) the above identities (4.4) are indeed satisfied. We therefore claim that

$$C_w = \frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2}+w\right) = \frac{\Gamma\left(\frac{3}{2}+w\right)}{\Gamma\left(\frac{3}{2}\right)}\,. \tag{4.6}$$

While determining the coefficients from the string amplitude perspective seems to be quite challenging, we can make progress by investigating again superparticle amplitudes. We limit our attention to amplitudes built from the vertex operator contributions

$$V_h \supset 2h_{ij}\mathcal{R}^{il}\mathcal{R}^{jm}k_l k_m\,, \qquad V_{G_3} \supset -\sqrt{2}G_3 P^z \mathcal{R}^{lmn}\,. \tag{4.7}$$

The left is the vertex operator for the linearised Riemann tensor, while the right is restricted to contributions involving KK states on the T^2. In terms of KK-charges l_i, we defined

$$P^z = \frac{1}{\sqrt{\tau_2 v_0}}\left(l_1 - \tau l_2\right)\,, \qquad P_{\bar{z}} = \overline{P_z}\,. \tag{4.8}$$

The contributions to the Type IIB couplings $G_3^{2w}R^{4-w}$ arise from P-point amplitudes with $P = 4+w$ where

$$v_0\mathcal{A}_{G_3^{2w}R^{4-w}} = \frac{2^{4-w}\,2^w}{2^6\,\Gamma\left(\frac{3}{2}\right)}\,S(P,w,0)\,t_{16+2w}G_3^{2w}R^{4-w} \tag{4.9}$$

in terms of

$$S(P,w,0) = \int \frac{\mathrm{d}t}{t}\frac{t^P}{t^{9/2}}\sum_{l_1,l_2} P_z^{2w}\mathrm{e}^{-tg^{ab}l_a l_b}\,. \tag{4.10}$$

Here, the factor 2^{P-w} arises from the graviton vertex operators and the normalisation $2^6\,\Gamma\left(\frac{3}{2}\right)$ is chosen such that the numerical coefficient of R^4 is set to $C_0 = 1$. In appendix C.2, we find that $S(P,w,0)$ is given by (C.25) which implies

$$v_0\mathcal{A}_{G_3^{2w}R^{4-w}} = \frac{\Gamma\left(\frac{3}{2}+w\right)}{\Gamma\left(\frac{3}{2}\right)\sqrt{v_0}}\,f_w\,t_{16+2w}G_3^{2w}R^{4-w} \xrightarrow{v_0\to 0} \alpha\,C_w\,f_w\,t_{16+2w}G_3^{2w}R^{4-w} \tag{4.11}$$

as expected from (4.1) and (4.6).

4.2 Non-MUV couplings

The non-MUV couplings are more difficult to determine given the significantly involved kinematics. However, we may be able to at least determine a particular class of contributions based on the structure of superparticle amplitudes. Let us define

$$\mathcal{L}^{(P)}_{\text{non-MUV}} = \sum_{w=0}^{2(P-4)-1} C_w^{(P)}\,f_w\,\left(t_{16+2(m-w)} + T_{16+2(m-w)}\right)G_3^m\overline{G}_3^{m-2w}R^{4+w-m} + \text{c.c.} \tag{4.12}$$

in terms of $m = P-4+w$. The kinematics is encoded in some tensor structures $t_{16+2(m-w)} + T_{16+2(m-w)}$ where $T_{16+2(m-w)}$ generalises (3.37). From the superparticle perspective, a

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contribution involving $t_{16+2(m-w)}$ is always guaranteed even at higher points by inspecting the corresponding vertex operators. In the context of the superfield language, a non-linear contribution $\sim |G_3|^2$ at order Θ^4 equally ensures the presence of a contribution $\sim t_{16+m+n}$ upon expanding $t_{24}\mathcal{R}^4$, see the conclusions in section 6 and specifically table 3.

We again use the vertex operator contributions defined in (4.7). However, we now pick up contributions that are non-MUV, namely

$$v_0 \mathcal{A}_{G_3^m \overline{G}^n R^{4-(m+n)/2}} = \frac{2^{4-(m+n)/2}\,(-2)^{m+n}}{2^6\,\Gamma\left(\frac{3}{2}\right)} S(P,m,n)\, t_{16+m+n} G_3^m \overline{G}_3^n R^{4-(m+n)/2} \tag{4.13}$$

in terms of

$$S(P,m,n) = \int \frac{\mathrm{d}t}{t}\frac{t^P}{t^{9/2}} \sum_{l_1,l_2} P_z^m P_{\bar{z}}^n \mathrm{e}^{-tg^{ab}l_a l_b}\,. \tag{4.14}$$

These functions can be computed as in the MUV case of appendix C.2. We compute a total of 10 amplitudes which we summarise in appendix C.3. In the limit $v_0 \to 0$, we recover

$$\mathcal{L}^{(\mathrm{SP})}_{\text{non-MUV}} = \sum_{P=5}^{8} \sum_{w=0}^{2(P-4)-1} C_w^{(P)}\, f_w\, t_{16+2m-w} G_3^m \overline{G}_3^{m-w} R^{4-2m+w} + \text{c.c.}\,, \quad P = 4+2m-w\,. \tag{4.15}$$

in terms of the coefficients

$$C_w^{(P)} = \frac{(2|w|+1)(2|w|-1)C_{P-4}}{(2(P-4)+1)(2(P-4)-1)}\,, \quad |w| \leq P-4\,. \tag{4.16}$$

Notice that for MUV amplitudes $|w| = P-4$ we recover $C_{P-4}^{(P)} = C_{P-4}$ as expected. The fact that the coefficients for 10 distinct amplitudes can be summarised by a single expression (4.16) is quite astonishing and hints at a deeper relationship among the various terms even in the non-MUV sector.

While we are unable to verify the correctness of the above results from string amplitudes, it certainly provides evidence for the appearance of the index structures t_{16+N} even in the non-MUV sector. Again, such effects are sourced by non-linear couplings in superfield language.

5 Five-point contact terms in compactifications

Compactifications allow to test new higher-derivative interactions by checking their consistency with the constraints imposed by lower-dimensional supersymmetry. In turn, these interactions have interesting implications for lower-dimensional physics.

5.1 K3 reductions and $\mathcal{N} = (2,0)$ supersymmetry in six dimensions

A non-trivial test of the five point couplings concerns K3 reduction to 6 dimensions. For NSNS sector couplings, these have been previously studied in [4, 5]. Here, we take the opportunity and provide a further check including the RR-sector by working in terms of G_3 directly. In particular, we highlight several non-trivial cancellations among the various 5-point index structures of section 3 necessary to ensure consistency with $\mathcal{N} = (2,0)$ supersymmetry in 6D.

The reduction of IIB supergravity on K3 results in six-dimensional $\mathcal{N} = (2,0)$ supergravity coupled to 21 tensor multiplets. As shown in [61], supersymmetry restricts the four-derivative couplings to be a F-term interaction that is quartic in the tensor multiplets. In particular, the $\mathcal{N} = (2,0)$ supergravity multiplet receives corrections only starting at the 8-derivative level, just as in the Type II case in 10 dimensions.

The bosonic components of the $\mathcal{N} = (2,0)$ supergravity multiplet are comprised of a graviton and five self-dual tensors. From the IIB perspective, the graviton and two of the self-dual tensors come from the spacetime reduction of the 10-dimensional graviton and G_3 with self-dual projection. The other three self-dual tensors arise from reducing the self-dual F_5 on the three self-dual 2-cycles of K3. The bosonic components of a $\mathcal{N} = (2,0)$ tensor multiplet are comprised of an anti-self dual tensor and five scalars. Of the 21 tensor multiplets from the reduction, 19 come directly from the reduction of G_3 and F_5 on the anti-self-dual 2-cycles of K3 along with the 19×3 K3 moduli. The other two tensor multiplets come from the spacetime reduction of G_3 with anti-self dual projection along with G_3 reduced on the three self-dual 2-cycles of K3, the IIB axio-dilaton, K3 volume modulus, and F_5 reduced fully on K3.

For simplicity, we avoid the fields obtained by reducing on the cohomology of K3. We also disregard six-dimensional scalars since knowledge of the scalar couplings will necessarily be incomplete in the absence of the full 5-point action involving the axio-dilaton. Thus we focus only on couplings of the 6D Riemann tensor to G_3 and its complex conjugate. These fields will provide information on the $\mathcal{N} = (2,0)$ supergravity multiplet along with the two special tensor multiplets.

As in [5], we focus on factorised pieces where a piece $\int_{\mathrm{K3}} R^2$ soaks up four derivatives. This reduces the eight-derivative couplings in ten dimensions to four-derivative couplings in six. Schematically, such couplings will take the form R^2, $G_3^2 R$ and G_3^4 (with possible complex conjugates on some of the fields), corresponding to two-, three- and four-point interactions. Supersymmetry requires the two- and three-point terms to vanish, and restricts the four-point interactions to the tensor multiplets [61]. Restricted to the NSNS fields only (i.e., taking $G_3 \to H_3$), ref. [5] confirmed the vanishing of R^2 and $H_3^2 R$ couplings at both tree and one-loop level and demonstrated that the one-loop H_3^4 coupling is indeed restricted to the tensor multiplet.[15] Decoupling of the gravity sector H_3^4 required the combination of both CP-even and CP-odd four-derivative terms.

Since we only have knowledge of Riemann and G_3 couplings up to five points in ten dimensions, we are unable to probe the quartic G_3^4 couplings in six dimensions. At the same time, it is well established that the quadratic $(\mathrm{Riemann})^2$ couplings automatically vanish for the IIB combination $(t_8 t_8 - \frac{1}{4}\epsilon_8\epsilon_8)R^4$. Hence we restrict the K3 analysis to the three-point couplings $|G_3|^2 R$ and $G_3^2 R$. Here R is shorthand for the Riemann tensor as Ricci terms can be removed by using the leading order equations of motion, thereby introducing quartic dilaton and 3-form terms in 6D that no longer contribute to three-point couplings.

[15] Tree-level H_3^4 was not examined in [5] as that would require knowledge of the six-point $H_3^4 R^2$ coupling. On the other hand, the one-loop test was possible because of heterotic/Type II duality.

The six-dimensional $|G_3|^2R$ and G_3^2R couplings arise from the U(1)-preserving and MUV sectors, respectively. We start with the ten-dimensional MUV couplings, which are given by (3.3), or more elegantly by (3.17). These $G_3^2R^3$ couplings are reduced by taking two of the Riemann tensors to be on K3, leaving G_3^2R in six dimensions. In our choice of basis of R^3 as given in appendix A.2, the only coefficient that we are sensitive to is c_1 multiplying $\tilde{X}^1_{M_1M_2M_3M_4}$ as this is the only term that can yield a factorized form involving G_3 and Riemann. From the decomposition of index structures in table 1, we then deduce immediately that t_{18} does not contribute any factorised terms in K3 reductions since $c_1 = 0$. Thus the three-point MUV couplings G_3^2R vanish trivially as required by supersymmetry.

Turning to the U(1)-preserving sector, we need to consider the following CP-even pieces from (3.36)

$$\mathcal{L}\big|_{K3} = f_0\alpha\left\{t_{16}R^4 + 6\left(t_8t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)|\nabla G_3|^2R^2 + \left(T\left(\epsilon_{10},t_8\right) - t_{18}\right)|G_3|^2R^3\right\}\Big|_{K3}. \tag{5.1}$$

As mentioned above, in 6D the $(\text{Riemann})^2$ term in the factorised part of $t_{16}R^4$ cancel, leaving us with only the $|\nabla G_3|^2R^2$ and $|G_3|^2R^3$ terms to consider. Furthermore, the t_{18} term vanishes on K3 for the same reason that its MUV counterpart vanishes.

From the definition (3.37) of $T(\epsilon_{10},t_8)$, we notice by using the coefficients c_1 collected in table 1 that

$$\left(2\tilde{t}_8\tilde{t}_8 - \frac{1}{2}\epsilon_8\epsilon_8\right)|G_3|^2R^3 = 0 + \cdots, \tag{5.2}$$

where $\cdots$ denotes Ricci and non-factorised terms. Recall that this is the piece obtained from generalised geometry. In addition, resorting to table 1, we find the factorised terms inside $\epsilon_9\epsilon_9|G_3|^2R^3$ to be

$$-\frac{1}{3}\epsilon_9\epsilon_9|G_3|^2R^3 = -48R^{N_1N_2M_1M_2}\,\overline{G}_{N_1N_2}{}^{P}G_{M_1M_2P}(\text{Riemann})^2 + \cdots. \tag{5.3}$$

We are thus left with

$$T(\epsilon_{10},t_8)\,|G_3|^2R^3 = -48R^{N_1N_2M_1M_2}\,\overline{G}_{N_1N_2}{}^{P}G_{M_1M_2P}(\text{Riemann})^2 + \cdots. \tag{5.4}$$

Consistency with the lack of three-point interactions in $\mathcal{N} = (2,0)$ supersymmetry requires that this term vanishes when combined with the factorized $|\nabla G_3|^2R^2$ contribution.

We now determine the factorised piece inside $|\nabla G_3|^2R^2$. After using the Bianchi identity for G_3, i.e., $\mathrm{d}G_3 = 0$ up to axio-dilaton terms, we obtain

$$\begin{aligned}6\left(t_8t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)|\nabla G_3|^2R^2 &= 48\left(\nabla_{M_1}\overline{G}_{M_2M_3M_4}\right)\left(\nabla^{M_2}G^{M_1M_3M_4}\right)(\text{Riemann})^2 + \cdots\\ &= 48R^{M_1M_2M_3M_4}\,\overline{G}_{M_1M_2}{}^{P}G_{M_3M_4P}(\text{Riemann})^2 + \cdots,\end{aligned} \tag{5.5}$$

where we integrated by parts in the second line and ignored terms proportional to the equations of motion for G_3. It is now apparent that the factorized piece inside the U(1)-preserving sector cancels, namely

$$6\left(t_8t_8 - \frac{1}{4}\epsilon_8\epsilon_8\right)|\nabla G_3|^2R^2 + T(\epsilon_{10},t_8)\,|G_3|^2R^3 = 0 + \cdots. \tag{5.6}$$

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In this way, we tested the coefficient $-1/3$ of $\epsilon_9\epsilon_9$ implicit in (5.3) in the non-MUV sector to which the Calabi-Yau threefold reductions are insensitive.

Finally, note that there are CP-odd couplings in both the U(1)-preserving and MUV sector that can potentially lead to three-point couplings. However, as shown in [4] and further discussed in [5], the CP-odd $H_3^2 R$ coupling involves the Ricci tensor, and hence does not contribute to the three-point function. The same argument applies for couplings to the complex three-form G_3.

5.2 Calabi-Yau threefold reductions to four dimensions

We now apply our previous results in the context of Calabi-Yau (CY) threefold reductions to $\mathcal{N}=2$ 4D SUGRA. Given that we identified the relevant 5-point kinematics between the 3-form and the metric, we are in the perfect position to directly derive for the first time from 10D the $(\alpha')^3$-corrected 4D scalar potential in compactifications on CY threefolds X_3 with background fluxes.

In the low-energy 4D effective action, 3-form fluxes induce a non-trivial F-term scalar potential [62]. More precisely, it is determined by a Kähler potential $\mathcal{K}$ and superpotential $\mathcal{W}$ through

$$V_F = \mathrm{e}^{\mathcal{K}}\left(\mathcal{K}^{A\bar{B}}\, D_A\mathcal{W}\, D_{\bar{B}}\overline{\mathcal{W}} - 3|\mathcal{W}|^2\right), \quad D_A\mathcal{W} = \partial_A\mathcal{W} + \mathcal{K}_A\mathcal{W}\,. \tag{5.7}$$

Here, the sum over A includes $h^{1,1}(X_3)$ Kähler moduli T_α, $h^{1,2}(X_3)$ complex structure moduli Z_i and the axio-dilaton τ. We work with the Kähler potential [63]

$$\mathcal{K} = \mathcal{K}^{(0)} - 2\log\left(\mathcal{V} + \frac{\zeta}{4} f_0(\tau,\bar{\tau})\right) \tag{5.8}$$

where $\mathcal{K}^{(0)}$ is the Kähler potential on complex structure moduli space

$$\mathcal{K}^{(0)} = -\ln(-\mathrm{i}(\tau-\bar{\tau})) - \log\left(-\mathrm{i}\int_{X_3}\Omega\wedge\overline{\Omega}\right). \tag{5.9}$$

Moreover, the constant ζ is given by [63]

$$\zeta = -\frac{\chi(X_3)}{2(2\pi)^3} \tag{5.10}$$

in terms of the Euler characteristic $\chi(X_3)$ of X_3. In the case of Type IIB flux compactifications, the superpotential entering (5.7) is the *Gukov-Vafa-Witten superpotential* [62, 64] (recall (2.2))

$$\mathcal{W} \equiv \mathcal{W}_{\mathrm{GVW}}(\tau, Z) = \int_{X_3}\tilde{G}_3\wedge\Omega_3\,. \tag{5.11}$$

This superpotential depends on the axio-dilaton τ through the complexified 3-form flux as well as on the complex structure moduli Z_i, $i=1,\ldots,h^{1,2}(X_3)$, due to the presence of the holomorphic 3-form $\Omega_3 = \Omega_3(Z_i)$.

Despite many efforts, the derivation of $(\alpha')^3$ corrections to the 4D flux scalar potential from first principles is still lacking. It was already noted in [52] that this requires the presence of both non-MUV and MUV higher derivative terms in 10D of the form $|G_3|^2R^3$,

$G_3^2R^3$ and $\overline{G}_3^2R^3$. One might also expect $|\nabla G_3|^2R^2$ to contribute at the same level based on dimensional grounds [55, 56, 65]. Of course, a major bottleneck has been the construction of the 10D higher derivative action for G_3 up to 5-points which has finally been achieved through [4–7, 43] and further concretised in this paper.

Below, we show that the structure of the F-term scalar potential (5.7) beyond leading order depends only on a single 10D kinematical structure, while the relative coefficients between RR-flux and NSNS-flux contributions can already be identified from 10D $\mathrm{SL}(2,\mathbb{Z})$ invariance. The remaining overall coefficient can only be identified upon constructing the corrected flux background which is beyond the scope of the current work, though this should be feasible by employing similar strategies to those of [66, 67]. Instead, we deduce novel relationships between 4D supersymmetry and the kinematical structure of 10D higher derivative terms where once again t_{18} plays a very prominent role. More specifically, we argue that the absence of RR-flux contributions to (5.7) at order $(\alpha')^3$ at string tree level already observed in [52] highly constrains the non-MUV kinematics when put entirely on a CY threefold. The strategy is as follows: by investigating the form of the corrected scalar potential (5.7) obtained from (5.8) and (5.11), we trace constraints from 4D supersymmetry back to the 10D action (3.36).

5.2.1 The 4D perspective

In this first part, we compute (5.7) by plugging in the Kähler potential (5.8) and the superpotential (5.11). Here, it is convenient to expand V_F as follows

$$V_F = \frac{V_{\rm Flux}}{\mathcal{V}^2} + (\alpha')^3\, V^{(3)} + \mathcal{O}\left((\alpha')^4\right) . \tag{5.12}$$

The first term corresponds to the standard no-scale flux scalar potential given by [64, 68–70][16]

$$V_{\rm Flux} = \frac{1}{2\mathrm{Im}(\tau)}\int_{X_3} \tilde{G}_3^+ \wedge \star_6 \overline{\tilde{G}}{}_3^+ = \mathrm{e}^{\mathcal{K}^{(0)}}\left(\mathcal{K}^{i\bar{j}}\, D_i\mathcal{W}\, D_{\bar{j}}\overline{\mathcal{W}} + \mathcal{K}^{\tau\bar{\tau}}\, D_\tau\mathcal{W}\, D_{\bar{\tau}}\overline{\mathcal{W}}\right) \tag{5.13}$$

in terms of (A)ISD flux $\star_6\tilde{G}_3^\pm = \mp\mathrm{i}\tilde{G}_3^\pm$. As explained in [52], the $(\alpha')^3$ corrections encoded in $V^{(3)}$ fall into two classes, i.e.,

$$V^{(3)} = -\frac{\zeta f_0(\tau,\bar{\tau})}{2\mathcal{V}^3}\, V_{\rm Flux} + V_\zeta\, . \tag{5.14}$$

The first term simply originates from a 4D Weyl rescaling $g_4^E = (\mathcal{V}+\zeta f_0/4)g_4$ by expanding to linear order in ζ.

[16] To arrive at the r.h.s., one decomposes $\tilde{G}_3^+$ in a basis $\{\Omega,\overline{\chi}_i\}$ of $H^{(3,0)}\oplus H^{(1,2)}$ (see e.g. appendix B of [71])

$$\tilde{G}_3^+ = -\frac{1}{\int_{X_3}\Omega\wedge\overline{\Omega}}\left(\Omega\int_{X_3}\overline{\Omega}\wedge\tilde{G}_3 + \mathcal{K}^{i\bar{j}}\overline{\chi}_j\int_{X_3}\chi_i\wedge\tilde{G}_3\right) .$$

The other term $\sim\int_{X_3} G_3\wedge\overline{G}_3$ is cancelled through the integrated Bianchi identity for F_5 [62].

The second term V_ζ is more interesting because it is obtained from direct dimensional reduction of (3.36). Using (A.4) for the 4D axio-dilaton, one finds to linear order in ζ

$$V_\zeta = \frac{3\zeta e^{\mathcal{K}^{(0)}}}{8\mathcal{V}^3}\left\{ f_0\left[|\mathcal{W}|^2 - (\tau-\bar{\tau})^2\,|D_\tau\mathcal{W}|^2\right] + (\tau-\bar{\tau})\left[f_{-1}\overline{\mathcal{W}}D_\tau\mathcal{W} - f_1\mathcal{W}\overline{D_\tau\mathcal{W}}\right]\right\} \tag{5.15}$$

where the classical Kähler covariant derivative with respect to τ is given by

$$D_\tau\mathcal{W} = (\partial_\tau + \mathcal{K}^{(0)}_\tau)\mathcal{W} = \frac{-1}{\tau-\bar{\tau}}\int_{X_3}\bar{\tilde{G}}_3\wedge\Omega\,. \tag{5.16}$$

As shown in [52], the coefficient on the right hand side cannot be reproduced by simply considering the flux kinetic term in the corrected 10D background. It was therefore argued that additional higher-derivative terms must contribute in the reduction.

To make contact with expressions obtained from direct dimensional reduction, it is even more instructive to rewrite (5.15) in terms of fundamental integrals using (5.11) and (5.16) such that

$$\begin{aligned} V_\zeta = \frac{3}{8}\,\frac{\zeta e^{\mathcal{K}^{(0)}}}{\mathcal{V}^3}\Bigg\{ f_0\left[\int_{X_3}\tilde{G}_3\wedge\Omega\int_{X_3}\bar{\tilde{G}}_3\wedge\overline{\Omega} + \int_{X_3}\bar{\tilde{G}}_3\wedge\Omega\int_{X_3}\tilde{G}_3\wedge\overline{\Omega}\right] \\ - \left[f_1\int_{X_3}\tilde{G}_3\wedge\Omega\int_{X_3}\tilde{G}_3\wedge\overline{\Omega} + f_{-1}\int_{X_3}\bar{\tilde{G}}_3\wedge\overline{\Omega}\int_{X_3}\bar{\tilde{G}}_3\wedge\Omega\right]\Bigg\}. \end{aligned} \tag{5.17}$$

The above expressions makes clear the way the three different kinematical structures appear in the reduction to 4D. In particular, the 10D U(1)-violating terms appear prominently in the scalar potential as already anticipated in [52]. The U(1)-neutral contribution is obtained from $|G_3|^2R^3$ and $|\nabla G_3|^2R^2$, but is also affected by backreaction effects from warping.

For the subsequent arguments, it is actually instructive to separate the above integrals into their NSNS- and RR-flux contributions at tree and 1-loop level. Plugging in the expansion (A.8) for the modular functions at large $\mathrm{Im}(\tau)$, we find (ignoring non-perturbative terms $\mathcal{O}(e^{-\mathrm{Im}(\tau)})$)

$$\begin{aligned} V_\zeta = \frac{\zeta e^{\mathcal{K}^{(0)}}}{\mathcal{V}^3}\left(-\frac{1}{4}\right)\Bigg\{ (-6a_T - 2a_L)\,e^{-2\phi_0}\int_{X_3}H_3\wedge\Omega\int_{X_3}H_3\wedge\overline{\Omega} \\ + (-4a_L)\int_{X_3}F_3\wedge\Omega\int_{X_3}F_3\wedge\overline{\Omega} + \dots\Bigg\}. \end{aligned} \tag{5.18}$$

Here, the tree-level term only depends on NSNS-flux and is in agreement with [52] after using $\xi = \zeta(3)\,\zeta$. In the subsequent section, we argue that the structure of (5.18) is directly accessible from our 10D expressions for the NSNS- and RR-flux terms (3.40)–(3.43) derived in section 3.4.

5.2.2 Dimensional reduction

Beforehand, let us collect all of the relevant pieces potentially contributing to (5.18). Initially, we write the 10D action as

$$S = S^{(0)} + \alpha S^{(3)} + \mathcal{O}\left((\alpha')^5\right) \tag{5.19}$$

where $S^{(0)}$ is the classical action (2.1) (setting $2\kappa_{10}^2 = 1$ in what follows). After solving the equations of motion to order $(\alpha')^3$, the solutions will be of the form

$$\varphi = \varphi^{(0)} + \alpha \left(f_0 \varphi_0^{(1)} + f_1 \varphi_1^{(1)} + \text{c.c.} \right) + \mathcal{O}\left((\alpha')^5 \right) \tag{5.20}$$

for $\varphi \in \{g, \tau, G_3, F_5, \mathcal{A}\}$. In particular, in the presence of non-trivial G_3, the background becomes warped as parametrised by $\mathcal{A}$. We provide a few more details on the corrected background in appendix D.1. In (5.20), we allow backreaction effects that carry non-trivial U(1)-charge which we expect to appear for G_3 and $\mathcal{P}$, though the latter are irrelevant in our Type IIB background where $\mathcal{P}$ vanishes internally.

The scalar potential. For the subsequent discussion, we collect the relevant terms in (5.19) (after evaluation on the corrected background) contributing to (5.17) in the following action

$$S_{\text{Flux}} = S_{\text{backreact.}} + S_{|\nabla G_3|^2 R^2} + S_{|G_3|^2 R^3} + S_{G_3^2 R^3 + \text{c.c.}} \,. \tag{5.21}$$

Here, $S_{\text{backreact.}}$ arises from evaluating the classical action on the corrected background. Then, the scalar potential can very schematically be written as

$$M_P^4 \int V_F \sqrt{-g^{(4)}}\, \mathrm{d}^4 x = S_{\text{Flux}} \Big|_{X_3} \tag{5.22}$$

where $\ldots|_{X_3}$ indicates that all legs of each tensor are taken along the internal CY directions.

At the level of the above discussion, we may limit our attention to terms that have the same form as $|G_3|^2 R^3$ (at least in the reduction) for which we write

$$S_{\text{backreact.}} + S_{|\nabla G_3|^2 R^2} = \alpha \int \left(f_0\, \delta_0 |G_3|^2 R^3 + f_1\, \delta_1 G_3^2 R^3 + f_{-1}\, \delta_1 \overline{G}_3^2 R^3 \right) \star_{10} 1 \tag{5.23}$$

in terms of some index structures δ_0, δ_1. We comment further on the corrected background and contributions to δ_0, δ_1 in section D.1. We stress however that their actual form is completely irrelevant for our argument. We essentially rely on $\mathcal{N} = 2$ supersymmetry in 4D: we know that in the absence of D-branes and O-planes there are no additional contributions coming from the above reduction. That is, we argue that all such terms must have the form given in (5.23).

We know due to 4D supersymmetry that the total contribution from higher-derivative terms and backreaction effects involving the RR-flux has to vanish at tree level. In particular, this implies that the reduction of the tree level $F_3^2 R^3$ terms (3.42) has to cancel against backreaction effects, i.e.,

$$S_{\text{Flux}} \Big|_{\text{RR, tree, } X_3} = \alpha \int a_T \mathrm{e}^{\phi} \left(2\, t_{18} + T(\epsilon_{10}, t_8) + \delta_0 + 2\delta_1 \right) F_3^2 R^3 \Big|_{X_3} = 0\,. \tag{5.24}$$

Even more importantly, this means that kinematically

$$\int T(\epsilon_{10}, t_8) F_3^2 R^3 \Big|_{X_3} = - \int (2\, t_{18} + \delta_0 + 2\delta_1) F_3^2 R^3 \Big|_{X_3} \tag{5.25}$$

and by trivial extension this also holds for $H_3^2 R^3$ (it does not matter what the 3-form is in our background). It is crucial to notice that this kinematical constraint based on the requirement of 4D $\mathcal{N}=2$ supersymmetry is the main ingredient for our argumentation. It relates the complicated kinematics in the non-MUV sector encoded by $T(\epsilon_{10}, t_8)$ to t_{18} as well as further uknown effects in the reduction through requiring the absence of RR-flux in the 4D scalar potential.[17] Overall, the contribution to the $\mathcal{N}=1$ 4D scalar potential from (3.40)–(3.43) are given by

$$S_{\text{Flux}}\Big|_{\text{NSNS, tree, } X_3} = -6\,\alpha \int a_T \mathrm{e}^{-\phi} \left[t_{18} + \frac{2}{3}\delta_1\right] H_3^2 R^3\Big|_{X_3}\,, \tag{5.26}$$

$$S_{\text{Flux}}\Big|_{\text{NSNS, 1-loop, } X_3} = -2\,\alpha \int a_L \mathrm{e}^{-\phi} \left[t_{18} + \frac{2}{3}\delta_1\right] H_3^2 R^3\Big|_{X_3}\,, \tag{5.27}$$

$$S_{\text{Flux}}\Big|_{\text{RR, tree, } X_3} = 0\,, \tag{5.28}$$

$$S_{\text{Flux}}\Big|_{\text{RR, 1-loop, } X_3} = -4\,\alpha \int a_L \mathrm{e}^{\phi} \left[t_{18} + \frac{2}{3}\delta_1\right] F_3^2 R^3\Big|_{X_3}\,. \tag{5.29}$$

The relative coefficients are *exactly the ones found in 4D* in (5.18). In order to derive also the overall coefficient as well as the structure of terms in (5.18), it remains to show that

$$\alpha \int_{X_3} \mathrm{e}^{-\phi} \left[t_{18} + \frac{2}{3}\delta_1\right] H_3^2 R^3 = -\frac{\zeta \mathrm{e}^{\mathcal{K}^{(0)}}}{4\mathcal{V}} \int_{X_3} H_3 \wedge \Omega \int_{X_3} H_3 \wedge \overline{\Omega} \tag{5.30}$$

and equivalently for $F_3^2 R^3$.

Let us highlight the importance of the above result: imposing only the absence of RR-flux in 4D through (5.24) gave rise to a single relevant kinematical structure depending on t_{18} and possible backreaction effects entering in the MUV sector. Stated differently, only the MUV kinematics is relevant which is a rather unexpected observation from the 10D point of view. The remaining relative coefficients are fixed through $\mathrm{SL}(2,\mathbb{Z})$ invariance. Clearly, these arguments only apply to the scalar potential, though the non-MUV kinematics will be checked at the level of 4D kinetic terms for hypermultiplets below.

Having derived the above results, we may actually come back to the MUV expression in 10D (3.17) which must reduce to the second line of (5.17). Using (5.30) for G_3, one verifies that reducing (3.17) together with MUV backreaction effects leads to

$$\alpha \int_{X_3} \mathrm{e}^{-\phi} \left[\frac{3}{2} t_{18} + \delta_1\right] G_3^2 R^3 = -\frac{3\zeta \mathrm{e}^{\mathcal{K}^{(0)}}}{8\mathcal{V}} \int_{X_3} G_3 \wedge \Omega \int_{X_3} G_3 \wedge \overline{\Omega} \tag{5.31}$$

with $-3/8$ being precisely the coefficient in (5.17). Let us stress that, if we had not gone through the argument for NSNS- and RR-flux separately, the kinematic constraints arising in the reduction would have been totally obscure in the non-MUV sector.

While it will generically be hard to identify δ_1 explicitly, reducing t_{18} should be feasible. Depending on the result, one might be able to identify δ_1 indirectly through (5.31). Naively,

[17] Clearly, it would have been great to prove the absence from first principles via direct dimensional reduction, but this is beyond the scope of the present work.

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given that in the MUV sector the complete kinematics is determined by t_{18} only, one could speculate that δ_1 is kinematically related to t_{18}, i.e., $\delta_1 = a_1 t_{18}$ for some numerical constant a_1.

Clearly, there remain several interesting future directions. For once, the absence of $(2,1)$-form flux in (5.15) (i.e., no couplings involving $D_Z\mathcal{W}$) as already observed in [52] requires a delicate cancellation among higher-derivative terms. We essentially reduced this problem to proving that (5.30) contains no such terms. We leave a derivation of (5.15) as well as (5.30) from direct dimensional reduction for future works.

The kinetic terms. As a final application of our five-point results, we derive the moduli space metrics for the hypermultiplets in Type IIB reductions to 4D. Initially, we recall that the non-chirality of Type IIA implies that both sign combinations of the odd-odd $\epsilon_8\epsilon_8 R^4$ structure appear in the 10D action. In CY threefold reductions to four dimensions, this implies that the Einstein Hilbert term is corrected as $(a_T - a_L)\chi(X_3)R^{(4)}$. Ultimately, this ensures that the vectormultiplets are only corrected at tree level, while hypermultiplets are corrected at 1-loop [72]. In contrast, the hypermultiplets in Type IIB are corrected at both tree and 1-loop level, while the vectormultiplets remain uncorrected, see e.g. [73].

In Type IIB, the hypermultiplet scalars consist of Kähler moduli t^α as well as p-form axions $(c^\alpha, b^\alpha, \rho_\alpha)$. Let $\omega_\alpha \in H^{1,1}(X_3)$ be a basis of $(1,1)$-forms. We express the CY volume in terms of the Kähler moduli as follows

$$\mathcal{V} = \int_{X_3} J\wedge J\wedge J = \frac{1}{3!}k_{\alpha\beta\gamma}\, t^\alpha t^\beta t^\gamma\,, \qquad k_{\alpha\beta\gamma} = \int_{X_3} \omega_\alpha\wedge\omega_\beta\wedge\omega_\gamma \tag{5.32}$$

where we expanded the Kähler form as $J = t^\alpha\omega_\alpha$. Below, we make use of the following

$$G_3 = \omega_\alpha G^\alpha\,, \qquad G^\alpha = \mathrm{d}c^\alpha - \tau\,\mathrm{d}b^\alpha\,, \qquad k_\alpha = \frac{1}{2}k_{\alpha\beta\gamma}\, t^\beta t^\gamma\,, \qquad k_{\alpha\beta} = k_{\alpha\beta\gamma}\, t^\gamma\,. \tag{5.33}$$

In the following, we reduce the relevant higher-derivative terms in (3.36) and (2.11) together with contributions from the corrected background (D.7), see section D.1 for details. We partially use the tree level results of [51, 57] and provide the full reduction in appendix D.2. Overall, we obtain the 4D action

$$\begin{aligned} S^{(4)} = \int\Big\{ &\Big[R^{(4)} - V_{\mathrm{F}}\Big]\star_4 1 - \mathrm{e}^{2\phi}\left(\frac{1}{2} - \frac{3f_0\zeta}{16\mathcal{V}^2}\right)\mathrm{d}\tau\wedge\star_4\mathrm{d}\overline{\tau} \\ &- \mathrm{i}\frac{3\mathrm{e}^\phi\zeta k_\alpha}{4\mathcal{V}^2}\,\mathrm{d}t^\alpha\wedge\star_4(f_1\,\mathrm{d}\tau - f_{-1}\,\mathrm{d}\overline{\tau}) \\ &+ \left(\frac{1}{\mathcal{V}}\left[\frac{1}{2}k_{\alpha\beta} - \frac{k_\alpha k_\beta}{\mathcal{V}}\right] - \frac{f_0\zeta}{8\mathcal{V}^2}\left[k_{\alpha\beta} - 4\frac{k_\alpha k_\beta}{\mathcal{V}}\right]\right)\mathrm{d}t^\alpha\wedge\star_4\mathrm{d}t^\beta \\ &+ \mathrm{e}^\phi\left(\frac{1}{2\mathcal{V}}\left[k_{\alpha\beta} - \frac{k_\alpha k_\beta}{\mathcal{V}}\right] - \frac{f_0\zeta}{8\mathcal{V}^2}\left[k_{\alpha\beta} - \frac{5}{4}\frac{k_\alpha k_\beta}{\mathcal{V}}\right]\right)G^\alpha\wedge\star_4\overline{G}^\beta \\ &- \frac{3\zeta\mathrm{e}^\phi k_\alpha k_\beta}{64\mathcal{V}^3}\Big[f_1\, G^\alpha\wedge\star_4 G^\beta + f_{-1}\,\overline{G}^\alpha\wedge\star_4\overline{G}^\beta\Big]\Big\}\,. \end{aligned} \tag{5.34}$$

Here, V_{F} is the $(\alpha')^3$-corrected scalar potential defined in (5.12). The terms without G^α restricted to tree level are equivalent to [51]. What is left to be done is to find suitable coordinates to bring the above into a canonical form.

To match with 4D SUSY, let us first compute the Kähler metric from the Kähler potential $\mathcal{K}$ defined in (5.8), namely

$$\mathcal{K}_{\alpha\beta} = -\frac{2}{\mathcal{V}}\left[k_{\alpha\beta} - \frac{k_\alpha k_\beta}{\mathcal{V}}\right] + \frac{\zeta f_0}{2\mathcal{V}^2}\left[k_{\alpha\beta} - 2\frac{k_\alpha k_\beta}{\mathcal{V}}\right],$$
$$\mathcal{K}_{\alpha\tau} = -\mathrm{i}\mathrm{e}^{\phi}\frac{3f_1}{8}\frac{k_\alpha \zeta}{\mathcal{V}^2}, \quad \mathcal{K}_{\tau\bar{\tau}} = \frac{\mathrm{e}^{2\phi}}{2}\left(\frac{1}{2} - \frac{3\zeta f_0}{16\mathcal{V}}\right). \tag{5.35}$$

Let us ignore all terms $\sim k_\alpha$ in the above action for a moment (these are affected by field redefinitions). Then we have

$$S^{(4)} = \int \Big\{ \left[R^{(4)} - V_{\mathrm{F}}\right] \star_4 1 - 2\mathcal{K}_{\tau\bar{\tau}}\, \mathrm{d}\tau \wedge \star_4 \mathrm{d}\bar{\tau}$$
$$- \frac{1}{4}\tilde{\mathcal{K}}_{\alpha\beta}\left[\mathrm{d}t^\alpha \wedge \star_4 \mathrm{d}t^\beta + \mathrm{e}^{\phi} G^\alpha \wedge \star_4 \overline{G}^\beta\right] \Big\} \tag{5.36}$$

where

$$\tilde{\mathcal{K}}_{\alpha\beta} = -\frac{2k_{\alpha\beta}}{\mathcal{V}}\left[1 - \frac{\zeta f_0}{4\mathcal{V}^2}\right]. \tag{5.37}$$

Notice that this result is essentially trivial: the terms $k_{\alpha\beta}$ at order $(\alpha')^3$ all come from the 4D Weyl rescaling and as such must all have the same coefficient.

In [57], it was argued that the Type IIA terms $\sim k_\alpha k_\beta\, \mathrm{d}b^\alpha \wedge \star_4 \mathrm{d}b^\beta$ cannot be absorbed into a redefinition of the (universal) hypermultiplet scalar. This led to the prediction of a tree level $-2t_8t_8H_3^2R^3$ operator in Type IIA. To make contact with these results, let us write out (5.34) in terms of NSNS fields at tree and 1-loop level by using (A.6). At tree level NSNS, we obtain

$$S^{(4)} \supset \int \mathrm{e}^{-\phi}\Big\{\frac{1}{2\mathcal{V}}\left[k_{\alpha\beta} - \frac{k_\alpha k_\beta}{\mathcal{V}}\right] - \frac{a_T\zeta}{8\mathcal{V}^2}\left[k_{\alpha\beta} - 2\frac{k_\alpha k_\beta}{\mathcal{V}}\right]\Big\}\mathrm{d}b^\alpha \wedge \star_4 \mathrm{d}b^\beta$$
$$= \frac{-1}{4}\int \mathrm{e}^{-\phi}\,\mathcal{K}_{\alpha\beta}\Big|_{\mathrm{tree}} \mathrm{d}b^\alpha \wedge \star_4 \mathrm{d}b^\beta\,. \tag{5.38}$$

This is in agreement with the Type IIA analysis of [57] as expected. Looking at NSNS 1-loop, we find

$$S^{(4)} \supset \mathrm{e}^{-\phi}\left(-\frac{a_L\zeta}{8\mathcal{V}^2}\left[k_{\alpha\beta} - \frac{k_\alpha k_\beta}{\mathcal{V}}\right]\right)\mathrm{d}b^\alpha \wedge \star_4 \mathrm{d}b^\beta\,. \tag{5.39}$$

This does not match the 4D Kähler metric which is not at all surprising: the 10D NSNS 1-loop action (3.41) does not contain the $-2t_8t_8$ piece which would lead to the 4D replacement $k_\alpha k_\beta \to 2k_\alpha k_\beta$. The above analysis can be repeated for the kinetic terms for RR C_2-axions to find

$$S^{(4)} \supset -\int \frac{\zeta \mathrm{e}^{\phi}}{8\mathcal{V}^2}\left[(a_T + a_L)\,k_{\alpha\beta} - \frac{1}{2}(a_T + 3a_L)\frac{k_\alpha k_\beta}{\mathcal{V}}\right]\mathrm{d}c^\alpha \wedge \star_4 \mathrm{d}c^\beta\,. \tag{5.40}$$

Here, both tree and 1-loop result are in disagreement with the 4D Kähler metric.

There are two ways in which this issue could be alleviated. The first option is a suitable redefinition of the 4D coordinates modifying only the piece $\sim k_\alpha k_\beta$. The second possibility would be additional contributions from a more careful treatment of backreaction effects.

U(1)	0	2	4	6	8
MUV	$t_{24}\mathcal{R}^4$	$t_{24}G_3^2\mathcal{R}^3$	$t_{24}G_3^4\mathcal{R}^2$	$t_{24}G_3^6\mathcal{R}$	$t_{24}G_3^8$
5-point	$t_{18}\|G_3\|^2R^3$				
6-point	$t_{20}(\|G_3\|^2)^2R^2$	$t_{20}G_3^2\|G_3\|^2R^2$			
7-point	$t_{22}(\|G_3\|^2)^3R$	$t_{22}G_3^2(\|G_3\|^2)^2R$	$t_{22}G_3^4\|G_3\|^2R$		
8-point	$t_{24}(\|G_3\|^2)^4$	$t_{24}G_3^2(\|G_3\|^2)^3$	$t_{24}G_3^4(\|G_3\|^2)^2$	$t_{24}G_3^6\|G_3\|^2$	

Table 3: 3-form contact terms that are captured by t_{24} and a suitable definition of $\mathcal{R}$.

6 Open questions and outlook

The main result of this paper is in revealing the structure of the 10D Type IIB effective action involving G_3 and R in the maximally R-symmetry-violating sector. We examined it with two different approaches, the superfield and the 11D superparticle, and compared our findings to the expectations from string-theory amplitudes. There are a number of open questions and venues for further research concerning the ten-dimensional effective actions.

The way how the couplings of the form $|G_3|^2R^3$, $G_3^2R^3$ + c.c. computed in [5] are repackaged using elementary tensor structures should motivate the study of non-linear extensions in the superspace approach of [12].

We have argued that the entire eight-derivative action in the MUV sector for couplings of the form $G_3^{2w}R^{4-w}$ is determined by a single index structure obtained from a sixteen fermion integral. Regarding the non-maximally R-symmetry-violating sectors, we only provided concrete evidence of the existence of certain kinematical structures and of their outstanding role in compactifications to lower dimensions, but much work remains to be done to determine the full effective action. In particular, if we were to replace the curvature tensor $R \to \mathcal{R}$ (defined in (1.3)) as dictated by non-linear SUSY, a whole tower of kinematical structures listed in table 3 will be generated.[18]

In section 3.3, we illustrated the way $t_{24}\mathcal{R}^4$ contributes at the level of 5-point contact terms corresponding to the third line of table 3. Fixing the coefficients will however require a more detailed calculation of the non-linear superfield following [8, 10].

We recall that the tree-level effective action should be T-duality invariant. Imposing this invariance apparently allows to determine the eight-derivative NSNS action to all powers in H_3 [14]. Moreover, the recent work [74] discusses 5-point structures constrained by $\mathrm{O}(d,d)$ invariance. Comparing these results to ours, which requires the extensive use of field redefinitions, and eventual use of T-duality as a way of constraining higher-order interactions, is left to future work.

[18]Note that here we have collected only various t_N as defined by standard fermionic integrals in (2.16). There are however further contributions similar to the ones listed in table 2 at 5-points.

We have moved closer to a full completion of the five-point effective action at order $(\alpha')^3$. Terms of the form $H_3^2(\nabla H_3)^2R$ and their RR counterparts which are in principle obtainable from the results of [5, 43, 75] have not been analysed here. Moreover, the relation of the CP-odd couplings (3.4) to the elementary tensor structures used here needs further clarifications. Finally, the dilaton couplings continue to be a top challenge. In the MUV sector, dilaton amplitudes were systematically analysed in [24]. The non-MUV part however remains largely unexplored. In this context, the F-theory lifts along the lines of [39] could offer a geometric principle underlying these couplings and eventually provide a key to determining scalar couplings also beyond four points.

Even though the notion of the higher-dimensional tensors we use here is quite established at the eight-derivative level through e.g. the linearised superfield [12], their role in the 10D effective action in the presence of a non-trivial background with $\mathcal{P}$ and G_3 remains to large extent unexplored. Also their generalisation to orders $(\alpha')^5$ and higher remains unclear. Given that they correspond to 1/4-BPS and 1/8-BPS interactions as opposed to 1/2-BPS for dimension-8 operators, they are certainly less constrained by supersymmetry. For instance, while the coefficients of 1/2-BPS and 1/4-BPS interactions satisfy Laplace eigenvalue equations, the pre-factors of 1/8-BPS terms satisfy an inhomogeneous Laplace equation instead [76], see also [24] for a more recent discussion. It would be interesting to understand the appearance of potentially novel index structures at higher orders in α'.

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A Definitions and conventions

A.1 $\mathrm{SL}(2,\mathbb{Z})$-covariant modular forms

Throughout this paper, we make heavy use of special modular forms and their properties. The relevant functions are all generalisations of the non-holomorphic Eisenstein series of weight 3/2 denoted as $f_0(\tau,\bar\tau)$. More generally, we define

$$f_w(\tau,\bar\tau) = \sum_{(\hat l_1,\hat l_2)\neq(0,0)} \frac{\mathrm{Im}\,(\tau)^{\frac{3}{2}}}{\left(\hat l_1+\tau\hat l_2\right)^{\frac{3}{2}+w}\left(\hat l_1+\bar\tau\hat l_2\right)^{\frac{3}{2}-w}}\,. \tag{A.1}$$

They transform covariantly under $\mathrm{SL}(2,\mathbb{Z})$

$$f_w\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar\tau+b}{c\bar\tau+d}\right) = \left(\frac{c\tau+d}{c\bar\tau+d}\right)^w f_w(\tau,\bar\tau)\,. \tag{A.2}$$

Further, these functions satisfy

$$(\tau-\bar{\tau})\frac{\partial}{\partial\tau}f_w=\left(w+\frac{3}{2}\right)f_{w+1}-wf_w\,,\qquad(\tau-\bar{\tau})\frac{\partial}{\partial\bar{\tau}}f_w=\left(w-\frac{3}{2}\right)f_{w-1}-wf_w\,.\tag{A.3}$$

More elegantly, this can be written in terms of a covariant derivative $\mathcal{D}_w$ where

$$\mathcal{D}_wf_w=i\left(\tau_2\frac{\partial}{\partial\tau}-i\frac{w}{2}\right)f_w=\frac{3+2w}{4}f_{w+1}\tag{A.4}$$

so that (see (2.11) in [24])

$$f_w=\frac{2^{w-1}\sqrt{\pi}}{\Gamma(\frac{3}{2}+w)}\mathcal{D}_{w-1}\ldots\mathcal{D}_0f_0=\frac{2^{3w+1}\,(w+1)!}{(2(w+1))!}\mathcal{D}_{w-1}\ldots\mathcal{D}_0f_0\,.\tag{A.5}$$

Last but not least, we expand f_w in the large $\mathrm{Im}(\tau)\gg1$ (small string coupling) regime where

$$f_w(\tau,\bar{\tau})=a_T+\frac{a_L}{(1-4w^2)}+\mathcal{O}\left(\mathrm{e}^{-\mathrm{Im}(\tau)}\right)\tag{A.6}$$

in terms of

$$a_T=2\zeta(3)\mathrm{Im}(\tau)^{\frac{3}{2}}\,,\quad a_L=\frac{2\pi^2}{3}\mathrm{Im}(\tau)^{-\frac{1}{2}}\,.\tag{A.7}$$

The first term is associated with closed string tree level [2], whereas the second term with 1-loop effects [77]. The final piece encodes contributions from non-perturbative D-instanton states [18]. For the lowest order modular functions, we can write

$$f_0\,(\tau,\bar{\tau})=a_T+a_L+\mathcal{O}\left(\mathrm{e}^{-\mathrm{Im}(\tau)}\right)\,,\quad f_{\pm1}\,(\tau,\bar{\tau})=a_T-\frac{1}{3}a_L+\mathcal{O}\left(\mathrm{e}^{-\mathrm{Im}(\tau)}\right)\,.\tag{A.8}$$

A.2 A basis for R^3

As shown in [5], the decomposition of $H_3^2R^3$ requires 24 independent Lorentz singlets. We therefore introduce a 24-dimensional basis for contractions of $H_3^2R^3$ built from R^3 invariants $\{\tilde{S}^i,\tilde{W}^i_{M_1M_2},\tilde{X}^i_{M_1M_2M_3M_4},\tilde{Q}^i_{M_1M_2M_3M_4M_5M_6}\}$ transforming *reducibly* under $\mathrm{SO}(1,9)$ (with the exception of $\tilde{S}^i$). First, there are the following two singlets

$$\tilde{S}^1=R_{MN}{}^{RS}R^{MNOP}R_{OPRS}\,,\quad\tilde{S}^2=R^M{}_{NP}{}^QR^R{}_{MQ}{}^SR^N{}_{RS}{}^P\,.\tag{A.9}$$

One infers that in this basis the 6d Euler density may be written as

$$Q=\frac{1}{12}\left(\tilde{S}^1+2\tilde{S}^2\right)\,.\tag{A.10}$$

Furthermore, we define the 2-tensors

$$\begin{aligned}&\tilde{W}^1_{M_1M_2}=R_{M_1NM_2P}R^N{}_{QRS}R^{PQRS}\,,\qquad\tilde{W}^2_{M_1M_2}=R_{M_1NPQ}R_{M_2}{}^{NRS}R^{PQ}{}_{RS}\,,\\&\tilde{W}^3_{M_1M_2}=R_{M_1NPQ}R_{M_1}{}^{RPS}R^N{}_R{}^Q{}_S\,.\end{aligned}\tag{A.11}$$

There are 11 independent 4-index tensors

$$
\begin{aligned}
\tilde{X}^1_{M_1M_2M_3M_4} &= R_{M_1M_2M_3M_4}R_{NPQR}R^{NPQR}\,, &\quad \tilde{X}^2_{M_1M_2M_3M_4} &= R_{M_1M_2M_3N}R_{M_4PQR}R^{NPQR}\,,\\
\tilde{X}^3_{M_1M_2M_3M_4} &= R_{M_1M_2NP}R_{M_3M_4QR}R^{NPQR}\,, &\quad \tilde{X}^4_{M_1M_2M_3M_4} &= R_{M_1M_3NP}R_{M_2M_4QR}R^{NPQR}\,,\\
\tilde{X}^5_{M_1M_2M_3M_4} &= R_{M_1NM_3P}R_{M_2QM_4R}R^{NPQR}\,, &\quad \tilde{X}^6_{M_1M_2M_3M_4} &= R_{M_1M_3NP}R_{M_2}{}^{N}{}_{QR}R_{M_4}{}^{PQR}\,,\\
\tilde{X}^7_{M_1M_2M_3M_4} &= R_{M_1NM_3P}R_{M_2}{}^{N}{}_{QR}R_{M_4}{}^{PQR}\,, &\quad \tilde{X}^8_{M_1M_2M_3M_4} &= R_{M_1M_3NP}R_{M_2Q}{}^{N}{}_{R}R_{M_4}{}^{QPR}\,,\\
\tilde{X}^9_{M_1M_2M_3M_4} &= R_{M_1NM_3P}R_{M_2Q}{}^{N}{}_{R}R_{M_4}{}^{QPR}\,, &\quad \tilde{X}^{10}_{M_1M_2M_3M_4} &= R_{M_1M_2}{}^{NP}R_{M_3NQR}R_{M_4P}{}^{QR}\,,\\
\tilde{X}^{11}_{M_1M_2M_3M_4} &= R_{M_1M_2NP}R_{M_3Q}{}^{N}{}_{R}R_{M_4}{}^{QPR}
\end{aligned} \tag{A.12}
$$

and another 8 combinations of 6-tensors

$$
\begin{aligned}
\tilde{Q}^1_{M_1M_2M_3M_4M_5M_6} &= R_{M_1M_4N}{}^{P}R_{M_2M_5P}{}^{Q}R_{M_3QM_6}{}^{N}\,,\\
\tilde{Q}^2_{M_1M_2M_3M_4M_5M_6} &= R_{M_1M_2N}{}^{P}R_{M_4M_5P}{}^{Q}R_{M_3QM_6}{}^{N}\,,\\
\tilde{Q}^3_{M_1M_2M_3M_4M_5M_6} &= R_{M_1M_2N}{}^{P}R_{M_3M_4P}{}^{Q}R_{M_5QM_6}{}^{N}\,,\\
\tilde{Q}^4_{M_1M_2M_3M_4M_5M_6} &= R_{M_1NM_4}{}^{P}R_{M_2PM_5}{}^{Q}R_{M_3QM_6}{}^{N}\,,\\
\tilde{Q}^5_{M_1M_2M_3M_4M_5M_6} &= R_{M_1NPQ}R_{M_2M_4}{}^{PQ}R_{M_3M_5M_6}{}^{N}\,,\\
\tilde{Q}^6_{M_1M_2M_3M_4M_5M_6} &= R_{M_1NPQ}R_{M_4M_5}{}^{PQ}R_{M_2M_3M_6}{}^{N}\,,\\
\tilde{Q}^7_{M_1M_2M_3M_4M_5M_6} &= R_{M_1NPQ}R_{M_2}{}^{N}{}_{M_4}{}^{Q}R_{M_3M_5M_6}{}^{P}\,,\\
\tilde{Q}^8_{M_1M_2M_3M_4M_5M_6} &= R_{M_1M_2M_4M_5}R_{M_3NPQ}R_{M_6}{}^{NPQ}\,.
\end{aligned} \tag{A.13}
$$

B Superfield calculus

B.1 The linearised description

This section closely follows the conventions in appendix C of [24]. Let θ_i be two sixteen-component chiral spinors of $\mathrm{Spin}(1,9)$ which we combine into the complex supercharge $\Theta = \theta_1 + \mathrm{i}\theta_2$. Linearised effective interactions preserving half of the original 32 supersymmetries are derived from a constrained superfield $\Phi(x^\mu - \mathrm{i}\bar{\Theta}\gamma^\mu\Theta, \Theta)$ which satisfies the holomorphic condition

$$\overline{D}_\Theta\Phi = 0\,, \quad (\overline{D}_\Theta)_A = -\frac{\partial}{\partial\bar{\Theta}^A}\,,\ A = 1,\ldots,16\,. \tag{B.1}$$

It is further constrained by

$$(D_\Theta)^4\Phi = \left(\overline{D}_\Theta\right)^4\overline{\Phi} \tag{B.2}$$

where

$$(D_\Theta)_A = \frac{\partial}{\partial\Theta^A} + 2\mathrm{i}\left(\gamma^\mu\bar{\Theta}\right)_A\partial_\mu\,, \qquad \left(\overline{D}_\Theta\right)_A = -\frac{\partial}{\partial\bar{\Theta}^A}\,. \tag{B.3}$$

The two operators D_Θ and $\overline{D}_\Theta$ are the (anti-)holomorphic covariant derivatives and commute with

$$Q_A = \frac{\partial}{\partial\Theta^A}\,, \qquad \bar{Q}_A = -\frac{\partial}{\partial\bar{\Theta}^A} + 2\mathrm{i}\left(\bar{\Theta}\gamma^\mu\right)_A\partial_\mu \tag{B.4}$$

corresponding to the rigid supersymmetries.

The two conditions (B.1) and (B.2) imply that the expansion of Φ in powers of Θ terminates at Θ^8. The scalar superfield components are completely specified by choosing the lowest order scalar component. It turns out to be convenient to work in a parametrisation of scalar fluctuations as $\hat{\tau} = \mathrm{i}\delta\tau/2\tau_2^0$ for $\delta\tau = \tau - \tau^0$ [24, 33]. The corresponding superfield Φ was previously discussed in [10, 12] and is defined as

$$\Phi = \tau_2^0 + \tau_2^0 \Delta \tag{B.5}$$

where $\tau_2^0\Delta$ parametrises the linearised fluctuations around a constant, purely imaginary flat background $\tau_2^0 = g_s^{-1}$ with (see also eq. (5.26) in [23])

$$\begin{aligned}\Delta &= \sum_{r=0}^{8} \Theta^r \Phi^{(r)} \\ &= \hat{\tau} + \Theta\lambda + \Theta^2 G_3 + \Theta^3 \partial\psi + \Theta^4 (R + \partial F_5) \\ &\quad + \Theta^5 \partial^2 \psi^* + \Theta^6 \partial^2 \bar{G}_3 + \Theta^7 \partial^3 \lambda^* + \Theta^8 \partial^4 \hat{\bar{\tau}} \,.\end{aligned} \tag{B.6}$$

Here, λ and ψ are the complex dilatino and gravitino respectively. For our purposes below, it suffices to note that

$$\Theta^2 G_3 = \left(\Theta\Gamma^{i_1 i_2 i_3}\Theta\right) G_{i_1 i_2 i_3}\,, \quad \Theta^4 R = \left(\Theta\Gamma^{i_1 i_2 k}\Theta\right)\left(\Theta\Gamma_k{}^{i_3 i_4}\Theta\right) R_{i_1 i_2 i_3 i_4}\,. \tag{B.7}$$

Terms encoded in $\Phi^{(r)}$ have U(1) R-symmetry charge

$$q_r = -2 + \frac{r}{2} \tag{B.8}$$

where we assigned charge $-1/2$ to Θ and -2 to Φ. This leads to

$$q_{\hat{\tau}} = -2\,, \quad q_\lambda = -\frac{3}{2}\,, \quad q_{G_3} = -1\,, \quad q_\psi = -\frac{1}{2}\,, \quad q_R = q_{F_5} = 0\,. \tag{B.9}$$

Even though the linearised approximation gives only partial results for the structure of terms in the effective action, it is still useful to find and relate various terms in the weak coupling limit $\tau_2^0 = g_1^{-1} \to \infty$. Generally, interactions are constructed from a function $\mathbf{F}[\Phi]$ of Φ by integrating over the 16 components of Θ, that is,

$$S_{\text{linear}} = \int \mathrm{d}^{10}x\, \mathrm{d}^{16}\Theta\, \det(e)\, \mathbf{F}[\Phi] + \text{c.c.}\,. \tag{B.10}$$

Here, $\det(e)$ is the determinant of the zehnbein and the total expression is invariant under the rigid supersymmetries. In an expansion in powers of Θ, we find

$$\mathbf{F}[\Phi] = F(\tau_2^0) + \sum_{n=1}^{\infty} \frac{1}{n!} \Delta^n \left(\frac{\partial}{\partial\tau_2^0}\right)^n F(\tau_2^0) \tag{B.11}$$

Substituting this expansion into (B.10) and keeping the terms with Θ^{16}, we recover all the interactions at order $(\alpha')^3$

$$\begin{aligned} S^{(3)} = \int \mathrm{d}^{10}x \Big\{ & f^{(12,-12)}\lambda^{16} + f^{(11,-11)} G_3 \lambda^{14} + \ldots + f^{(4,-4)} G_3^8 + \ldots + f^{(1,-1)} G_3^2 R^3 \\ & + f^{(0,0)} \left(|G_3|^2 + |\mathcal{P}|^2\right) R^3 + \ldots + f^{(0,0)} R^4 + \ldots + f^{(-12,12)} (\lambda^*)^{16} \Big\}\,. \end{aligned} \tag{B.12}$$

Here, the modular forms $f_w = f^{(w,-w)}(\tau, \bar{\tau})$ have holomorphic and anti-holomorphic weights $(w, -w)$ and are eigenfunctions of the SL$(2, \mathbb{Z})$-Laplacian [23]. The presence of these coefficient functions is required by SL$(2, \mathbb{Z})$ invariance, see [24] and references therein. The functions $f^{(w,-w)}$ carry U(1) charge $q_{f_w} = 2w$ (in our convention) and appear generically as

$$\int \mathrm{d}^{10}x \det(e)\, f^{(w,-w)} \prod_{n=1}^{P} \Phi^{(r_n)}\,. \tag{B.13}$$

The value w is fixed by the sum of U(1) charges of the $\Phi^{(r_n)}$:

$$\sum_{n=1}^{P} q_{r_n} = 8 - 2P \overset{!}{=} -2w \tag{B.14}$$

where we used that $\sum_n r_n = 16$ to ensure the presence of 16 powers of Θ.

As far as the index structures in (B.12) are concerned, one finds contributions like

$$\begin{aligned} t_{16}R^4 &= \int \mathrm{d}^{16}\Theta \left[\left(\Theta\Gamma^{i_1 i_2 k}\Theta\right)\left(\Theta\Gamma_k{}^{i_3 i_4}\Theta\right) R_{i_1 i_2 i_3 i_4}\right]^4 ,\\ t_{18}G_3^2 R^3 &= \int \mathrm{d}^{16}\Theta \left(\left(\Theta\Gamma^{i_1 i_2 i_3}\Theta\right) G_{i_1 i_2 i_3}\right)^2 \left[\left(\Theta\Gamma^{i_1 i_2 k}\Theta\right)\left(\Theta\Gamma_k{}^{i_3 i_4}\Theta\right) R_{i_1 i_2 i_3 i_4}\right]^3 . \end{aligned} \tag{B.15}$$

Several comments are in order. In the linearised approximation, we work in a regime where we neglect the inhomogeneous part of the modular covariant derivative $D_w = w + 2\mathrm{i}\tau_2^0 \partial_{\tau_2^0}$, that is,

$$2\mathrm{i}\tau_2^0 \partial_{\tau_2^0} f_w \gg w f_w\,. \tag{B.16}$$

This is clearly violated for terms in f_w that are powers of τ_2^0. In contrast, D-instanton contributions $\sim (\tau_2^0)^n \mathrm{e}^{-2\pi|N|\tau_2^0}$ satisfy the above inequality in the limit $\tau_2^0 \to \infty$. Thus, the linearised description contains the exact leading multi-instanton effects. In a non-linearly completed theory, the SL$(2, \mathbb{Z})$ symmetry requires the f_w to become the familiar modular forms. Then, the relative coefficients for interactions of differing U(1) charge can be computed from supersymmetry.

B.2 Non-linear superfield

The non-linear superfield completion is generally cumbersome in the presence of non-trivial $\mathcal{P}$ and G_3 backgrounds. Looking at the Θ^4 term in (B.6), it was already proposed in [9] (and even earlier in [10, 78]) that full graviton and F_5 kinematics is encoded in

$$\Theta^4 \tilde{\mathcal{R}} = \left(\Theta\Gamma^{i_1 i_2 i_3}\Theta\right)\left(\Theta\Gamma^{i_4 i_5 i_6}\Theta\right) \tilde{\mathcal{R}}_{i_1 i_2 i_3 i_4 i_5 i_6} \tag{B.17}$$

where $\tilde{\mathcal{R}}$ was defined in (2.20). Then, the single index structure

$$t_{24}\tilde{\mathcal{R}}^4 = \int \mathrm{d}^{16}\Theta \left[\left(\Theta\Gamma^{i_1 i_2 i_3}\Theta\right)\left(\Theta\Gamma^{i_4 i_5 i_6}\Theta\right) \tilde{\mathcal{R}}_{i_1 i_2 i_3 i_4 i_5 i_6}\right]^4 \tag{B.18}$$

includes tensor contractions of the form $(\nabla F_5)^n F_5^{2m} R^{4-n-m}$ [9] which are in agreement with string amplitude calculations [79].

This clearly implies that $\tilde{\mathcal{R}}$ as defined in (2.20) enjoys the following symmetries to be imposed implicitly further below:

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1. Invariance under the exchange of fermion bilinears in (2.21) implies that only the part of $\tilde{\mathcal{R}}_{i_1 i_2 i_3 i_4 i_5 i_6}$ symmetric under the exchange $(i_1, i_2, i_3) \leftrightarrow (i_4, i_5, i_6)$ contributes.

2. Anti-symmetry of the Γ-matrices in (2.21) means that we have to anti-symmetrise in both (i_1, i_2, i_3) and (i_4, i_5, i_6).

3. The fermion bilinears in (2.21) enjoy further Fierz identities which essentially project onto certain tensor representations. Applying considerations from representation theory, it turns out that [3]

$$(\mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16})_{\text{anti-sym}} = \mathbf{770} \oplus \mathbf{1050}^+ \tag{B.19}$$

where $\mathbf{770} = [0, 2, 0, 0, 0]$ and $\mathbf{1050}^+ = [1, 0, 0, 0, 2]$ in terms of their Dynkin labels under D_5. We define the following two projection operators

$$\begin{aligned} \mathcal{T}_{i_1 i_2 i_3\,,\, i_4 i_5 i_6}\big|_{\mathbf{1050}^+} = \frac{1}{2}\bigg\{ \frac{1}{2}\Big(&\mathcal{T}_{i_1 i_2 i_3\,,\, i_4 i_5 i_6} - 3\,\mathcal{T}_{i_1 i_2 i_6\,,\, i_4 i_5 i_3} - \mathcal{T}_{i_1 i_2 k\,,\, i_4 i_5}{}^{k}\, g_{i_3 i_6} \\ &+ 2\,\mathcal{T}_{i_1 i_5 k\,,\, i_4 i_2}{}^{k}\, g_{i_3 i_6} \Big) \pm \frac{1}{4!} \epsilon_{i_1 i_2 i_3 i_4 i_5}{}^{k_1 k_2 k_3 k_4 k_5} \mathcal{T}_{k_1 k_2 k_3\,,\, k_4 k_5 i_6} \bigg\}\,, \end{aligned} \tag{B.20}$$

$$\mathcal{T}_{i_1 i_2\,,\, i_4 i_5}\big|_{\mathbf{770}} = \frac{2}{3}\left(\mathcal{T}_{i_1 i_2\,,\, i_4 i_5} + \mathcal{T}_{i_1 i_5\,,\, i_4 i_2}\right) - \frac{1}{2} \mathcal{T}_{i_1 k\,,\, i_4}{}^{k}\, g_{i_2 i_5} + \frac{1}{36} \mathcal{T}_{jk\,,}{}^{jk}\, g_{i_1 i_4} g_{i_2 i_5}\,.$$

One easily verifies applying the projector onto $\mathbf{770}$ to the Riemann tensor that

$$R_{i_1 i_2 i_4 i_5}\big|_{\mathbf{770}} = C_{i_1 i_2 i_4 i_5} \tag{B.21}$$

which implies that only the Weyl tensor enters (2.20).

Both F_5 terms in (2.20) do not contain any $\mathbf{770}$ piece, though a $\mathbf{1050}^+$ part. For F_5^2, one uses self duality to remove the ϵ-tensor, thereby finding [9]

$$\left(F_{i_1 i_2 i_3 k l} F_{i_4 i_5 i_6}{}^{kl}\right)\big|_{\mathbf{1050}^+} = \frac{1}{2}\left(F_{i_1 i_2 i_3 k l} F_{i_4 i_5 i_6}{}^{kl} - 3 F_{i_1 i_2 i_6 k l} F_{i_4 i_5 i_3}{}^{kl}\right) \tag{B.22}$$

which is already imposed in (2.20). For ∇F_5, the $\mathbf{1050}^+$ component is obtained by imposing

$$\nabla_k F^k{}_{i_1 i_2 i_3 i_4} = 0\,, \quad F_5 = \star_{10} F_5\,. \tag{B.23}$$

C 11D superparticle amplitudes

We compute 11D amplitudes in the superparticle formalism compactified on a 2-torus [15–17, 35]. We start from the vertex operator

$$V_{G_4} = 4 k_{[I} C_{LMN]} \left(\dot{X}^I - \frac{2}{3} \mathcal{R}^{IJ} k_J\right) \mathcal{R}^{LMN} \mathrm{e}^{-\mathrm{i}k\cdot X} \tag{C.1}$$

for the 3-form C_3 in terms of 11d indices $I, J, K, L, \ldots$ using the conventions of [17] for the fermion bilinears $\mathcal{R}^{IJ}, \mathcal{R}^{LMN}$. Once we compactify the vertex operator on a T^2, we split the 11d indices as $\{I,J,K,L,\ldots\}$ into T^2-indices $\alpha,\beta,\ldots = 1,2$ and 9d indices $i,j,k,l,\ldots = 0,3,\ldots,10$. Schematically, we distinguish the following types of terms in the reduction

$$k_{[i}C_{lmn]} \to F_{ilmn1}\,, \qquad k_{[i}C_{lm]1} \to F_{ilm}\,, \qquad k_{[i}C_{lm]2} \to H_{ilm}\,, \qquad k_{[i}C_{l]1\,2} \to F_{il} \tag{C.2}$$

where $k_{[i}C^{(4)}_{lmn]1} = F_{ilmn1}$ and F_{il} is the field-strength tensor of the 9d Type IIB gravi-photon A_i. We will only be interested in the Type IIB 3-forms F_3 and H_3 in 9D.

For our purposes (on the Type IIB side), it is more convenient to work with a complexified basis for the two T^2 direction for which G_3 is obtained from [17]

$$k_{[l}C_{mn]z} = \frac{1}{\sqrt{v_0\tau_2}}\left(k_{[l}C_{mn]1} - \tau k_{[l}C_{mn]2}\right)\,. \tag{C.3}$$

This amounts to the following set of vertex operators for G_3

$$\begin{aligned} V_{G_3} = 3\sqrt{2}\,k_{[i}C_{mn]z}&\left(\dot{X}^i - \frac{2}{3}\mathcal{R}^{ij}k_j\right)\mathcal{R}^{zmn}\mathrm{e}^{-\mathrm{i}k\cdot X}\,,\\ -\sqrt{2}\,k_{[l}C_{mn]z}&\left(\dot{X}^z - \frac{2}{3}\mathcal{R}^{zj}k_j\right)\mathcal{R}^{lmn}\mathrm{e}^{-\mathrm{i}k\cdot X}\,. \end{aligned} \tag{C.4}$$

Notice that the terms in the first line were not present in [17] which are however important to provide additional contributions in the U(1)-preserving sector at 5-points.

Next, the 11D graviton vertex operator reads

$$V_h = h_{IJ}\left(\dot{X}^I\dot{X}^J - 2\dot{X}^I\mathcal{R}^{JM}k_M + 2\mathcal{R}^{IL}\mathcal{R}^{JM}k_Lk_M\right)\mathrm{e}^{-\mathrm{i}k\cdot X}\,. \tag{C.5}$$

In 9D, we obtain the graviton vertex operator

$$V_h = h_{ij}\left(\dot{X}^i\dot{X}^j - 2\dot{X}^i\mathcal{R}^{jm}k_m + 2\mathcal{R}^{il}\mathcal{R}^{jm}k_lk_m\right)\mathrm{e}^{-\mathrm{i}k\cdot X} \tag{C.6}$$

as well as the axio-dilaton vertex operator

$$V_P = h_{zz}\left(\dot{X}^z\dot{X}^z - 2\dot{X}^z\mathcal{R}^{zm}k_m + 2\mathcal{R}^{zl}\mathcal{R}^{zm}k_lk_m\right)\mathrm{e}^{-\mathrm{i}k\cdot X}\,. \tag{C.7}$$

C.1 General amplitudes

In this section, we discuss the general form of amplitudes to be encounter below. To begin with, we stress that there are essentially two classes of amplitudes. If the fields are neutral under the Type IIB U(1), then the general result can be written as

$$\mathcal{A}_{\text{neutral}} = \mathcal{N}_1\,\tilde{K}_{\text{neutral}}\left(C + \mathcal{N}_2\frac{f_0(\tau,\bar{\tau})}{v_0^{3/2}}\right) \tag{C.8}$$

for some kinematical structure $\tilde{K}$. The constant C generally remains undetermined in this formalism without a proper microscopic description, but can be determined from e.g. duality considerations. In contrast, a U(1)-violating combination of fields results in an amplitude of the form

$$\mathcal{A}^{(w)}_{\text{viol.}} = \mathcal{N}\,\tilde{K}_{\text{viol.}}\,\frac{f_w(\tau,\bar{\tau})}{v_0^{3/2}}\,. \tag{C.9}$$

This is expected simply because there is not associated analogue in 11d and the result must disappear in the limit $v_0 \to \infty$!

The factor of $v_0^{-3/2}$ in (C.8) and (C.9) is reminiscent of the terms coming with $\alpha'/R^2 = \alpha'/(r_A^{(s)})^2$ in [80] since

$$\frac{g_B^{1/2}}{v_0^{3/2}} = (r_B^{(s)})^2 = \frac{1}{(r_A^{(s)})^2} \,. \tag{C.10}$$

Hence, U(1) uncharged amplitudes of the form (C.8) naturally appear with a $1 \pm v_0^{-3/2} \sim 1 \pm \alpha'/R^2$ prefactor, whereas charged amplitudes (C.9) only come with a $v_0^{-3/2} \sim \alpha'/R^2$ term.

More explicitly, we will be interested in the following types of P-point amplitudes

$$\mathcal{A}_P(m,n) = \tilde{K}(P) \int \frac{\mathrm{d}t}{t} t^P \int \mathrm{d}^9\mathbf{p} \sum_{l_1,l_2} P_z^m P_{\bar{z}}^n \mathrm{e}^{-t\left(\mathbf{p}^2 + g^{ab} l_a l_b\right)} \,. \tag{C.11}$$

For the moment, we keep the index structures in $\tilde{K}$ implicit. Clearly, for $m = n$, $\mathcal{A}^{m,n}$ will be of the form (C.8). After integrating out the 9d momenta, we obtain

$$\mathcal{A}_P(m,n) = \pi^{\frac{9}{2}} \tilde{K}(P)\, S(P,m,n) \tag{C.12}$$

in terms of

$$S(P,m,n) = \int \frac{\mathrm{d}t}{t} \frac{t^P}{t^{9/2}} \sum_{l_1,l_2} P_z^m P_{\bar{z}}^n \mathrm{e}^{-t g^{ab} l_a l_b} \,. \tag{C.13}$$

These functions can be computed systematically for any number of points and KK-momenta. Throughout this work, we require only the following explicit results

$$\begin{aligned} S(5,0,0) &= C + 4\sqrt{\pi}\frac{f_0(\tau,\bar{\tau})}{v_0^{3/2}} \,, \qquad & S(5,1,1) &= C - \sqrt{\pi}\frac{f_0(\tau,\bar{\tau})}{v_0^{3/2}} \,, \\ S(5,2,0) &= 3\sqrt{\pi}\frac{f_1(\tau,\bar{\tau})}{v_0^{3/2}} \,, \qquad & S(5,0,2) &= 3\sqrt{\pi}\frac{f_{-1}(\tau,\bar{\tau})}{v_0^{3/2}} \,. \end{aligned} \tag{C.14}$$

Here, C is typically a divergent constant which can be identified through duality considerations [15].

With the above formulas, the open task remains to determine kinematical structures. In contrast to string amplitudes, we do *not* impose a priori that we compute even/even, odd/odd or even/odd sector couplings. This comes about naturally from the higher-dimensional index structures which we define as

$$t_{N+M}^{i_1\ldots i_{N+M}} = \mathrm{Tr}\left(\mathcal{R}^{i_1 i_2} \ldots \mathcal{R}^{i_{N-1} i_N} \mathcal{R}^{i_{N+1} i_{N+2} i_{N+3}} \ldots \mathcal{R}^{i_{M-2} i_{M-1} i_M}\right) \,. \tag{C.15}$$

In 9 (or any number of odd) spacetime dimensions, index structures t_N with N odd are associated with parity-odd couplings. The reason is simple: if N is even, then there is a chance to find terms of the form $t_8 t_8$ or $\epsilon_D \epsilon_D$. However, if N is odd, then there must be always a single ϵ_D of odd dimensions be involved. This is to be contrasted with the situation discussed in [42]. Here, they provide the decomposition of t_{24} in 10 dimensions which involves both parity-even and parity-odd contributions.

Another comment concerns the situation where the index structure carries torus indices, that is

$$t_N^{z\bar{z}} = \mathrm{Tr}\left(\mathcal{R}^{z i_1 i_2}\mathcal{R}^{\bar{z} i_3 i_4}\mathcal{R}^{i_5 i_6}\dots\right) . \tag{C.16}$$

In particular, structures of this type either appear alone as in the case of 1 $b_{\mu 9}$ and 4 $h_{\mu\nu}$ or in combination with t_N. The latter scenario appears frequently in the amplitudes discussed below. These instances can be understood from the 11d perspective where one might find an index structure of the form

$$t_N^{ijklmn\dots} G_{ijka} G_{lmn}{}^a \dots \to \left(t_N^{ijklmn\dots} G_{ijkz} G_{lmn}{}^z + t_N^{zjk\bar{z}mn\dots} G_{zjka} G_{\bar{z}mn}{}^a + \dots\right)\dots \tag{C.17}$$

As we will see below, this happens for instance when studying 9d amplitudes for $|G_3|^2 R^3$.

C.2 Maximally U(1)-violating amplitudes

We derive the coefficients for MUV terms in the Type IIB action involving only the 3-form and the metric. The general 9D superparticle amplitude for such contributions is given by

$$v_0 \mathcal{A}_{G_3^{2w} R^{4-w}} = \frac{2^{4-w}\, 2^w}{2^6\, \Gamma\left(\frac{3}{2}\right)} S(P,w,0)\, t_{16+2w} G_3^{2w} R^{4-w} \tag{C.18}$$

in terms of

$$S(P,w,0) = \int \frac{\mathrm{d}t}{t} \frac{t^P}{t^{9/2}} \sum_{l_1,l_2} P_z^{2w} \mathrm{e}^{-t g^{ab} l_a l_b} . \tag{C.19}$$

One easily verifies that after Poisson resummation

$$\sum_{l_1,l_2} P_z^{2w} \mathrm{e}^{-t g^{ab} l_a l_b} = \frac{v_0^{w+1}}{\tau_2^w (2t)^{2w+1}} \sum_{\hat{l}_1,\hat{l}_2} \left(\hat{l}_1 + \tau \hat{l}_2\right)^{2w} \mathrm{e}^{-g_{ab}\hat{l}_a \hat{l}_b/(4t)} \tag{C.20}$$

For $w > 0$, the zero winding term with $(\hat{l}_1, \hat{l}_2) = (0,0)$ simply drops out. Next, we substitute $t \to (4\tilde{t})^{-1} g_{ab}\hat{l}_a\hat{l}_b$ to find

$$\int \frac{\mathrm{d}t}{t} \frac{t^P}{t^{9/2+2w+1}} = \int t^{P-\frac{13}{2}-2w}\, \mathrm{d}t \to \int \left(\frac{g_{ab}\hat{l}_a\hat{l}_b}{4\tilde{t}}\right)^{P-\frac{11}{2}-2w} \frac{1}{\tilde{t}}\, \mathrm{d}\tilde{t} \tag{C.21}$$

where we used

$$\mathrm{d}t = -\frac{g_{ab}\hat{l}_a\hat{l}_b}{4\tilde{t}^2}\, \mathrm{d}\tilde{t} . \tag{C.22}$$

Putting everything together, we recover

$$\begin{aligned} S(P,w,0) &= \frac{v_0^{w+1}}{\tau_2^w 2^{2w+1}} \left(\frac{4}{v_0}\right)^{\frac{3}{2}+w} \sum_{(\hat{l}_1,\hat{l}_2)\neq(0,0)} (\hat{l}_1 + \tau\hat{l}_2)^{2w} \left(\tilde{g}_{ab}\hat{l}_a\hat{l}_b\right)^{-\frac{3}{2}-w} \int \tilde{t}^{\frac{1}{2}+w} \mathrm{e}^{-\tilde{t}}\, \mathrm{d}\tilde{t} \\ &= \frac{4\Gamma\left(\frac{3}{2}+w\right)}{\tau_2^w \sqrt{v_0}} \sum_{(\hat{l}_1,\hat{l}_2)\neq(0,0)} (\hat{l}_1 + \tau\hat{l}_2)^{2w} \left(\tilde{g}_{ab}\hat{l}_a\hat{l}_b\right)^{-\frac{3}{2}-w} . \end{aligned} \tag{C.23}$$

Since

$$\tilde{g}_{ab}\hat{l}_a\hat{l}_b = \frac{\left(\hat{l}_2 + \tau\hat{l}_1\right)\left(\hat{l}_2 + \bar{\tau}\hat{l}_1\right)}{\tau_2}\,, \tag{C.24}$$

we can use the definition (A.1) for modular forms $f_w(\tau,\bar{\tau})$ to obtain

$$S(P,w,0) = \frac{4\Gamma\left(\frac{3}{2}+w\right)}{\sqrt{v_0}}\, f_w(\tau,\bar{\tau})\,. \tag{C.25}$$

C.3 Special non-MUV amplitudes

The superparticle amplitudes in the non-MUV sector giving rise to contributions involving the higher-dimensional index structures $t_{18}, t_{20}, \ldots$ are given by

$$v_0\mathcal{A}_{G_3^m\overline{G}^n R^{4-(m+n)/2}} = \frac{2^{4-(m+n)/2}\,(-2)^{m+n}}{2^6\,\Gamma\left(\frac{3}{2}\right)}\, S(P,m,n)\, t_{16+m+n} G_3^m \overline{G}_3^n R^{4-(m+n)/2} \tag{C.26}$$

in terms of

$$S(P,m,n) = \int \frac{\mathrm{d}t}{t}\frac{t^P}{t^{9/2}} \sum_{l_1,l_2} P_z^m P_{\bar{z}}^n \mathrm{e}^{-tg^{ab}l_a l_b}\,. \tag{C.27}$$

The objects $S(P,m,n)$ can be computed as before. The final expressions will be of the form

$$v_0\mathcal{A}_{G_3^m\overline{G}^n R^{4-(m+n)/2}} = \left(v_0 C_\infty \delta_{w,0} + C_w^{(P)}\frac{f_w(\tau,\bar{\tau})}{\sqrt{v_0}}\right) t_{16+m+n} G_3^m \overline{G}_3^n R^{4-(m+n)/2} \tag{C.28}$$

where

$$w = \frac{m-n}{2}\,. \tag{C.29}$$

The first zero winding piece only appears at the U(1)-neutral level $m = n$ which contributes in the limit $v_0 \to \infty$ to the 11D M-theory action. However, the constant C_∞ is generically divergent because the superparticle picture does not provide a microscopic description of M-theory. Such constants can be inferred though e.g. via dualities to Type IIA as discussed in [15] for R^4.

The second term in (C.28) encodes as usual the contributions to the Type IIB effective action upon taking the limit $v_0 \to 0$. Overall, the corresponding coefficients can be expressed in the following compact way

$$C_w^{(P)} = \frac{(2|w|+1)(2|w|-1)C_{P-4}}{(2(P-4)+1)(2(P-4)-1)}\,. \tag{C.30}$$

Notice that for MUV amplitudes $|w| = P-4$ we recover $C_{P-4}^{(P)} = C_{P-4}$ as expected.

D Details on the reduction to 4D

D.1 Comment on the corrected background

To give some intuition on effects contributing to δ_0, δ_1, the corrected metric background in Einstein frame involves an overall Weyl rescaling (see e.g. [51]) which in string frame is

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associated with the corrected dilaton [52], that is,

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{\Phi}\left[\mathrm{e}^{2\mathcal{A}}\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{e}^{-2\mathcal{A}}g_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n\right] \tag{D.1}$$

with

$$\Phi = \alpha\Phi^{(1)} + \mathcal{O}\left(\alpha^2\right)\,, \qquad \mathcal{A} = \mathcal{A}^{(0)} + \alpha\mathcal{A}^{(1)} + \mathcal{O}\left(\alpha^2\right)\,, \qquad g_{mn} = g_{mn}^{(0)} + \alpha g_{mn}^{(1)} + \mathcal{O}\left(\alpha^2\right) \tag{D.2}$$

In 10D Einstein frame, neither ϕ nor G_3 are corrected at order $(\alpha')^3$,

$$\phi = \phi_0 + \mathcal{O}\left((\alpha')^4\right)\,, \qquad G_3 = (\alpha')^1\,G_3^{(0)} + (\alpha')\;\alpha G_3^{(1)} + \mathcal{O}\left((\alpha')^7\right)\,. \tag{D.3}$$

From the Bianchi identity for F_5, one deduces that also $F_5 \sim \mathcal{O}((\alpha')^2)$. The remaining leading order solutions to the equations of motion can be determined from the results of [51]. From Einstein's equations, one infers that the internal Ricci tensor receives a correction of the form

$$R_{mn}^{(1)} = -3\cdot 2^9\,f_0(\tau,\bar\tau)\,\nabla_m^{(0)}\nabla_n^{(0)}Q^{(0)}\,. \tag{D.4}$$

Finally, the 10D Weyl rescaling Φ is determined to be

$$\Phi^{(1)} = -3\cdot 2^6\,f_0(\tau,\bar\tau)\,Q^{(0)}\,. \tag{D.5}$$

When reducing the classical action (2.1), we perform the Weyl rescaling

$$g_{MN} = \mathrm{e}^{\alpha\Phi^{(1)}}\tilde g_{MN}\,. \tag{D.6}$$

Then, we obtain

$$S^{(0)}(g) = S^{(0)}(\tilde g) + \frac{\alpha}{2\kappa_{10}^2}\int \Phi^{(1)}\left(4R - 8|\mathcal{P}|^2 - \frac{|G_3|^2}{6}\right)\tilde\star_{10}\mathbb{1} \tag{D.7}$$

where $S^{(0)}(\tilde g)$ is the classical action evaluated on the new metric. The term $\sim \Phi^{(1)}|G_3|^2$ contributes to δ_0 in (5.23).

D.2 Details on the derivation of 4D kinetic terms

In this section, we provide further details on the reduction to 4D. We reduce (D.7) as well as the relevant terms in (3.36) and (2.11). For the moment, we ignore terms involving F_5 which complement the hypermultiplets in 4D as well as contribute warping terms. They are known explicitly by means of (2.22) and will be studied in more detail in the future.

For the Einstein Hilbert term, the reduction of (D.7) to 4D leads to

$$\begin{aligned}\int_{X_3} 4\Phi^{(1)}R\,\tilde\star_{10}1 = -384\Big[&f_0 2(2\pi)^3\chi(X_3)R^{(4)} + f_0\left(\mathscr{R}_{\alpha\beta} + 2\mathscr{I}_{\alpha\beta}\right)\,\mathrm{d}t^\alpha\wedge\star_4\,\mathrm{d}t^\beta\\ &- 6\mathscr{I}_\alpha\,\mathrm{d}t^\alpha\wedge\star_4\left(f_1\mathcal{P} + f_{-1}\overline{\mathcal{P}}\right)\Big]\end{aligned} \tag{D.8}$$

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where we defined

$$\mathscr{I}_\alpha = -\mathrm{i}(2\pi)^3 \int_{X_3} \omega_{\alpha\, i}{}^{i}\, c_3(X_3) = (2\pi)^3 \chi(X_3) \frac{k_\alpha}{\mathcal{V}}\,, \tag{D.9}$$

$$\mathscr{R}_{\alpha\beta} = (2\pi)^3 \int_{X_3} \omega_{\alpha\, i\bar{\jmath}}\, \omega_\beta{}^{\bar{\jmath} i}\, c_3(X_3)\,, \tag{D.10}$$

$$\mathscr{I}_{\alpha\beta} = (2\pi)^3 \int_{X_3} \omega_{\alpha\, i}{}^{i}\, \omega_{\beta\, j}{}^{j}\, c_3(X_3) = -\frac{k_\alpha k_\beta}{\mathcal{V}} (2\pi)^3 \chi(X_3)\,. \tag{D.11}$$

As observed in [57], $\mathscr{R}_{\alpha\beta}$ cancels in the reduction and only integrals of the form $\mathscr{I}_\alpha$, $\mathscr{I}_{\alpha\beta}$ appear which can be evaluated explicitly given that the trace of $(1,1)$-forms is constant.

From R^4, we obtain in the reduction

$$\int_{X_3} \left(t_8 t_8 \pm \frac{1}{4}\epsilon_8\epsilon_8\right) R^4 \tilde{\star}_{10} 1 = \pm 768 (2\pi)^3 \chi(X_3) R^{(4)} + 384 \mathscr{R}_{\alpha\beta}\, \mathrm{d}t^\alpha \wedge \star_4 \mathrm{d}t^\beta\,. \tag{D.12}$$

As a remark, recall that in Type IIA both sign combinations appear giving rise to $(a_T - a_L)\chi(X_3)R^{(4)}$ in the reduction. Ultimately, this ensures that the vectormultiplets are only corrected at tree level, while hypermultiplets are corrected at 1-loop. In contrast, we find in Type IIB only a single sign corresponding to t_{16} defined in (2.19) so that

$$\int_{X_3} \left[4\Phi^{(1)} R + f_0 t_{16} R^4\right] \tilde{\star}_{10} 1 = -1536 (2\pi)^3 \chi(X_3)\, f_0\, R^{(4)} - 768\, f_0 \mathscr{I}_{\alpha\beta}\, \mathrm{d}t^\alpha \wedge \star_4 \mathrm{d}t^\beta$$
$$+ 6\cdot 384 \mathscr{I}_\alpha\, \mathrm{d}t^\alpha \wedge \star_4 \left(f_1 \mathcal{P} + f_{-1}\overline{\mathcal{P}}\right)\,. \tag{D.13}$$

As we will see below, this implies that the hypermultiplets are corrected at both tree and 1-loop level, while the vectormultiplets remain uncorrected.

Next, let us look at the contribution from the 3-form. The backreaction from the metric gives rise to

$$\int_{X_3} \Phi^{(1)} \frac{|G_3|^2}{6} \tilde{\star}_{10} 1 = 384 \mathrm{e}^\phi f_0\, \mathscr{R}_{\alpha\beta} G^\alpha \wedge \star_4 \overline{G}^\beta \tag{D.14}$$

in terms of $G^\alpha = \mathrm{d}c^\alpha - \tau\, \mathrm{d}b^\alpha$. From the higher derivative terms, we find from the torsionful Riemann tensor (essentially equivalent to [57])

$$\int_{X_3} 2 f_0 \tilde{t}_8 \tilde{t}_8 \left(|G_3|^2 R^3 + 3|\nabla G_3|^2 R^2\right) = 384 \mathrm{e}^\phi f_0\, \mathscr{R}_{\alpha\beta} G^\alpha \wedge \star_4 \overline{G}^\beta \tag{D.15}$$

as well as from t_{18} (this is the piece proposed by [57] at NSNS tree level)

$$\int_{X_3} \frac{1}{2} t_8 t_8 |G_3|^2 R^3 = -192 \mathrm{e}^\phi f_0\, \mathscr{I}_{\alpha\beta} G^\alpha \wedge \star_4 \overline{G}^\beta\,. \tag{D.16}$$

Altogether, this amounts to

$$\int_{X_3} \left[-\Phi^{(1)} \frac{|G_3|^2}{6} + f_0 \left(2\tilde{t}_8\tilde{t}_8 - \frac{1}{2} t_8 t_8\right) |G_3|^2 R^3\right] \tilde{\star}_{10} 1 = 192 \mathrm{e}^\phi f_0\, \mathscr{I}_{\alpha\beta} G^\alpha \wedge \star_4 \overline{G}^\beta\,. \tag{D.17}$$

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Notice that $\mathscr{R}_{\alpha\beta}$ cancels out exactly which is actually necessary to perform the remaining integrals explicitly as we will see below. In addition, we also have contributions from the 10D MUV sector which are of the form

$$\int_{X_3}\left[\frac{3f_1}{4}t_8t_8G_3^2R^3+\text{c.c.}\right]\tilde{\star}_{10}1=-288\mathrm{e}^{\phi}f_1\,\mathscr{I}_{\alpha\beta}G^{\alpha}\wedge\star_4G^{\beta}+\text{c.c.}\,. \tag{D.18}$$

To complete the argument, we also have to add terms involving the dilaton. At the 5-point level, contact terms with two dilatons and three gravitons can only be U(1)-preserving.[19] They remain to large extent unspecified, see however [39] for a proposal based on 12D convariance. Here, we make an ansatz similar to [51] by adding a term proportional to the 6D Euler density, namely

$$\int_{X_3}\left[-8\Phi^{(1)}|\mathcal{P}|^2-3\cdot 2^{10}|\mathcal{P}|^2Q\right]\tilde{\star}_{10}1=-1536(2\pi)^3\chi(X_3)\,f_0\,\mathcal{P}\wedge\star_4\overline{\mathcal{P}}\,. \tag{D.19}$$

To summarise, we obtain the 4D action

$$\begin{aligned}
S^{(4)}=\frac{1}{2\kappa_{10}^2}\int\Big\{&\Big(\mathcal{V}-1536\alpha\,(2\pi)^3\,\chi\,(X_3)\,\,f_0\Big)\,R^{(4)}\star_4 1-(V_{\text{Flux}}+V_{\zeta})\star_4 1\\
&-\Big(2\mathcal{V}+1536\alpha\,(2\pi)^3\,\chi\,(X_3)\,\,f_0\Big)\,|\mathcal{P}|^2\star_4 1+6\cdot 384\alpha\mathscr{I}_{\alpha}\,\mathrm{d}t^{\alpha}\wedge\star_4\Big(f_1\mathcal{P}+f_{-1}\overline{\mathcal{P}}\Big)\\
&+\left(\frac{1}{2}\left[k_{\alpha\beta}+\frac{k_\alpha k_\beta}{\mathcal{V}}\right]-768\alpha\,f_0\mathscr{I}_{\alpha\beta}\right)\,\mathrm{d}t^{\alpha}\wedge\star_4\,\mathrm{d}t^{\beta}\\
&+\left(\frac{1}{2}\left[k_{\alpha\beta}-\frac{k_\alpha k_\beta}{\mathcal{V}}\right]+192\alpha\mathrm{e}^{\phi}f_0\,\mathscr{I}_{\alpha\beta}\right)G^{\alpha}\wedge\star_4\overline{G}^{\beta}\\
&-288\alpha\mathrm{e}^{\phi}f_1\,\mathscr{I}_{\alpha\beta}G^{\alpha}\wedge\star_4G^{\beta}+\text{c.c.}\Big\}\,.
\end{aligned} \tag{D.20}$$

Up to this point, we collected all the relevant contributions at the 2-derivative level in 4D. The final step is to perform a Weyl rescaling of the 4D metric to arrive at 4D Einstein frame. To this end, we define

$$g_{\mu\nu}=\mathrm{e}^{\kappa/2}\tilde{g}_{\mu\nu}\,,\qquad \kappa=-2\log\left(\mathcal{Y}\right)\,,\qquad \mathcal{Y}=\mathcal{V}-(2\pi)^3\,\chi\,(X_3)\,\frac{f_0}{8} \tag{D.21}$$

and expand to liner order in χ. In string units, we set

$$\ell_s=2\pi\sqrt{\alpha'}=1\quad\Rightarrow\quad(\alpha')^3=\frac{1}{(2\pi)^6}\,,\quad \zeta=-\frac{\chi\,(X_3)}{2(2\pi)^3} \tag{D.22}$$

to arrive at (dropping the tilde on $\tilde{g}$ again) (5.34).

[19]In fact, as we will see below, we can turn this argument around by arguing that terms like $f_2\mathcal{P}^2R^3$ in 10D are actually forbidden by 4D SUSY.

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