# The cosmological constant as a boundary term

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ABSTRACT: We compare the path integral for transition functions in unimodular gravity and in general relativity. In unimodular gravity the cosmological constant is a property of states that are specified at the boundaries whereas in general relativity the cosmological constant is a parameter of the action. Unimodular gravity with a nondynamical background spacetime volume element has a time variable that is canonically conjugate to the cosmological constant. Wave functions depend on time and satisfy a Schrödinger equation. On the contrary, in the covariant version of unimodular gravity with a 3-form gauge field, proposed by Henneaux and Teitelboim, wave functions are time independent and satisfy a Wheeler-DeWitt equation, as in general relativity. The 3-form gauge field integrated over spacelike hypersurfaces becomes a "cosmic time" only in the semiclassical approximation. In unimodular gravity the smallness of the observed cosmological constant has to be explained as a property of the initial state.

# Contents

## 1   Introduction

The origin and interpretation as well as the observed value of the cosmological constant presents a puzzle of particle physics and cosmology [1]. In particular the seemingly huge contribution of zero-point energies is often considered to be a severe fine-tuning problem. It is therefore suggestive that the cosmological constant is just an integration constant, rather than a fundamental parameter, in a version of Einstein's theory where the volume element $\sqrt{g}$ is fixed. This has been noticed from time to time [2–7] and has led to a canonical theory of quantum gravity [8–10].

Unimodular gravity (UG) can be defined by imposing $\sqrt{g} = \omega$ as a constraint, where $\omega$ is a nondynamic background volume element. One often chooses $\sqrt{g} = 1$, hence the name unimodular gravity. The background volume element breaks the invariance of general relativity (GR) under general diffeomorphisms to the invariance under volume preserving diffeomorphisms. Nevertheless, the classical theory is equivalent to Einstein gravity except for the cosmological constant which now appears as an integration constant. This feature also arises in a generally covariant theory with a 3-form gauge field, which was obtained by Henneaux and Teitelboim in an analysis of unimodular gravity as a constrained Hamiltonian system [8]. Note that 3-form gauge fields can also contribute to the cosmological constant by vacuum expectation values of their 4-form field strengths [11–13]. Introducing further gauge fields also Newton's constant can become an integration constant [14, 15].

In a theory with invariance only under volume preserving diffeomorphisms the conformal factor of the metric, $\sigma = \frac{1}{2} \ln\left(\sqrt{g}\right)$, is an ordinary scalar field that can have arbitrary kinetic term and potential. However, its couplings may be restricted by additional symmetries such as scale invariance. In this way it plays a prominent role in Higgs-dilaton theories; see, for example, [16–19].

During the past years quantum effects in UG have been studied in detail, and there has been a still ongoing debate whether or not UG and GR are equivalent as quantum theories. The investigations include semiclassical calculations [20], the quantum effective action [21], the renormalization group flow [22–24], the quantum equivalence of UG and GR [25], quantum corrections to the cosmological constant [26, 27], the path integral in the Hamiltonian formalism [28–30] and the computation of one-loop divergencies [31]. Recently, significant progress has been made in the BRST quantization of UG as well as GR in the unimodular gauge [32–35]. It is perhaps not surprizing that at present there is no consensus on how to precisely define unimodular quantum gravity, and it is far from clear what the differences to ordinary quantum gravity are.

In the following we shall attempt to compare the quantum theories of GR and the two versions of UG. The comparison will be based on the path integral for transition amplitudes. The main difference is that in UG the cosmological constant enters as a boundary term, i.e., as a property of states, whereas in GR it is a parameter of the action. GR and the Henneaux-Teitelboim version of UG are generally covariant. Hence, there is no notion of time on which wave functions could depend. On the other hand, in UG with a nondynamical background volume element canonical quantization is possible and wave functions do depend on time.

The paper is organized as follows. After a general discussion of the path integral and the Henneaux-Teitelboim action in section 2 we analyze the path integral for unimodular gravity in sections 3 and 4, with emphasis on the boundary terms. Wave functions are briefly considered in section 5. We conlude in section 6. BRST quantization of general relativity in unimodular gauge is discussed in the appendix.

## 2 The path integral in quantum gravity

A natural starting point for quantizing gravity is the path integal (see, for example, [36, 37]). To obtain an expression for the amplitude one has to identify dynamical variables and study their "time evolution". As a first step one introduces a "time function" $t(x)$ that provides a foliation of a hyperbolic spacetime manifold $\mathcal{M}$ into spacelike 3-surfaces $\Sigma_t$. One can then define transition amplitudes between states corresponding to different configurations of the gravitational field on 3-surfaces of different "parametric time" $t$. For simplicity, we shall restrict our discussion to compact 3-surfaces.

Einstein's equations for the gravitational field are obtained from the action[1]

$$S[g] = \int_{\mathcal{M}} R\epsilon + 2 \int_{\partial\mathcal{M}} K\tilde{\epsilon} \,, \tag{2.1}$$

where $g_{\alpha\beta}$ is the metric tensor, $R$ is the Ricci scalar and $K$ is the trace of the extrinsic curvature. For a region bounded by two hypersurfaces $\Sigma_1$ and $\Sigma_2$ the transition amplitude is formally given by

$$\langle g_2; \Sigma_2 | g_1; \Sigma_1 \rangle = \int [Dg] \exp\left(iS[g]\right) \,. \tag{2.2}$$

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[1]The volume form is given by $\epsilon = \frac{1}{4!}\sqrt{g}\epsilon_{\alpha\beta\gamma\delta}dx^\alpha dx^\beta dx^\gamma dx^\delta$, where $g = -\det g_{\alpha\beta}$, and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor density with $\epsilon_{0123} = 1$. $\tilde{\epsilon}$ is the induced volume form on $\partial\mathcal{M}$. We work in units $16\pi G_N = 1$.

Here one integrates over all metric fields $g$ that smoothly interpolate between the boundary fields $g_1$ and $g_2$. If an intermediate 3-surface $\Sigma_3$ is introduced, one has $S[g_{(23)}] + S[g_{(31)}] = S[g_{(21)}]$ where $g_{(ij)}$ interpolates between $g_i$ and $g_j$ on $\Sigma_i$ and $\Sigma_j$, respectively. The quantum-mechanical superposition principle implies

$$\langle g_2; \Sigma_2 | g_1; \Sigma_1 \rangle = \int [Dg_3] \langle g_2; \Sigma_2 | g_3; \Sigma_3 \rangle \langle g_3; \Sigma_3 | g_1; \Sigma_1 \rangle \,. \tag{2.3}$$

The amplitude (2.2) is only a formal expression and its precise physical meaning is not clear since the "times" $t_1$ and $t_2$ are merely coordinate parameters. Despite much effort it has not been possible to decompose the metric field into "true dynamical degrees of freedom" and some "intrinsic time"; for a discussion and references, see [38, 39].

In the following we study the possibility to label the boundary surfaces by values of a 3-form density $A_{\alpha\beta\gamma}$, which is covariantly constant on a 3-surface. Such a 3-form density can be sourced by the gravitational field, which is achieved by equating its field strength to the canonical volume density on $\mathcal{M}$. The corresponding action is obtained from the Einstein-Hilbert action (2.1) by adding a Lagrange multiplier term,

$$S[g, A, \Lambda] = \int_{\mathcal{M}} (R\epsilon + \Lambda(dA - \epsilon)) + 2 \int_{\partial\mathcal{M}} K\tilde{\epsilon} \,, \tag{2.4}$$

where $\Lambda$ is an auxiliary scalar field. Note that the action is invariant under the gauge transformation $A \to A + d\eta$ where $\eta$ is a 2-form field. The equations of motion are obtained by varying the action with respect to $g_{\alpha\beta}$, $A_{\alpha\beta\gamma}$ and $\Lambda$, which yields

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2} \Lambda g_{\alpha\beta} \,, \tag{2.5}$$

$$\partial_\alpha \Lambda = 0 \,, \tag{2.6}$$

$$4\partial_{[\alpha} A_{\beta\gamma\delta]} = \sqrt{g} \epsilon_{\alpha\beta\gamma\delta} \,. \tag{2.7}$$

Eqs. (2.5) are Einstein's equations with a cosmological term, eq. (2.6) implies that the scalar field $\Lambda$ becomes an unspecified cosmological constant $\lambda$, and eq. (2.7) identifies the field strength of $A$ with the canonical volume form. The action (2.4) has been obtained by Henneaux and Teitelboim from a constrained Hamiltonian analysis of a theory where the determinant of the metric is treated as an external field [8]. Instead of the 3-form density $A$ they used the dual vector density, $A_{\alpha\beta\gamma} = \epsilon_{\delta\alpha\beta\gamma} \mathcal{T}^\delta$.

On a 3-surface $\Sigma_t$ the 3-form density $A$ is given by a constant $A(t)$. To study the time evolution one has to specify $g_{\alpha\beta}(t, x)$ and $A(t)$ on some initial 3-surface $\Sigma_1$, together with a constant cosmological term, $\Lambda(t, x) = \lambda$. Einstein's equations then determine the metric at some later time $t_2$, and the integrated 3-form density at $t_2$ is given by

$$\mathcal{A}_2 = \mathcal{A}_1 + \mathcal{V}_{\mathcal{M}}[g] \,, \tag{2.8}$$

with

$$\mathcal{A}_t = A(t) \int_{\Sigma_t} d^3x \sqrt{h} \,, \quad \mathcal{V}_{\mathcal{M}}[g] = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} \,, \tag{2.9}$$

where $h$ is the induced volume density on $\Sigma_t$. By construction, $\mathcal{A}_t$ increases monotonically with the coordinate time $t$. This has motivated the interpretation of $\mathcal{A}_t$ as a "cosmic time" [8–10]; see, however, [40].

Similar to eq. (2.2) we can now consider transition amplitudes where initial and final states depend on the fields $g$ and $A$. On the boundary surfaces $\Sigma_{1,2}$ the 3-form field $A$ is covariantly constant and can therefore be specified in terms of the integrals $\mathcal{A}_{1,2}$. Hence, the transition amplitude takes the form

$$
\begin{aligned}
\langle g_2, &\mathcal{A}_2; \Sigma_2 | g_1, \mathcal{A}_1; \Sigma_1 \rangle \\
&= \int [Dg][DA][D\Lambda] \exp\left(iS[g, A, \Lambda]\right) \\
&= \int [Dg][D\Lambda] \delta(\partial_\alpha \Lambda) \exp\left(i\left(S[g] - \int_{\mathcal{M}} \Lambda\epsilon + \int_{\Sigma_2} d^3x \Lambda A - \int_{\Sigma_1} d^3x \Lambda A\right)\right),
\end{aligned} \quad (2.10)
$$

where $\delta(\partial_\alpha \Lambda) \equiv \prod_{x,\alpha} \delta(\partial_\alpha \Lambda)$. Because of the $\delta$-function the integration over $\Lambda$ is restricted to constant values. Assuming that this constant is fixed by boundary conditions we replace $\delta(\partial_\alpha \Lambda)$ by $\delta(\Lambda - \lambda_0)$, which leads to the transition amplitude

$$
\langle g_2, \mathcal{A}_2; \Sigma_2 | g_1, \mathcal{A}_1; \Sigma_1 \rangle = \exp\left(i\lambda_0(\mathcal{A}_2 - \mathcal{A}_1)\right) \int [Dg] \exp\left(i(S[g] - \lambda_0 \mathcal{V}_{\mathcal{M}}[g])\right). \quad (2.11)
$$

Compared to standard GR the amplitude contains a phase factor that is determined by the boundary conditions, and in the path integral the Einstein-Hilbert action appears with an undetermined cosmological constant $\lambda_0$, which is the characteristic feature of unimodular gravity. Contrary to the classical relation (2.8) the integral includes volumes that are not related to the boundary terms $\mathcal{A}_1$ and $\mathcal{A}_2$. To obtain a better understanding of the boundary conditions we now turn to the Hamiltonian formalism.

## 3 The path integral in the ADM formalism

In the Arnowitt-Deser-Misner (ADM) [41] formalism one starts from a foliation of the manifold $\mathcal{M}$ with spacelike 3-surfaces $\Sigma_t$. An embedding[2] of these 3-surfaces with coordinates $y^a$, $a = 1, .., 3$, into the ambient space $\mathcal{M}$ is given by functions $x^\alpha(t, y^a)$, and the matrix $E_a^\alpha = \partial x^\alpha / \partial y^a \equiv \partial_a x^\alpha$ provides the push-forward for tangent vectors of $\Sigma_t$ to tangent vectors of $\mathcal{M}$. The metric induced on $\Sigma_t$ reads

$$
h_{ab} = g_{\alpha\beta} E_a^\alpha E_b^\beta, \quad (3.1)
$$

and the vectorfield $E_t^\alpha \equiv t^\alpha$ represents the "time flow" that can be decomposed into components normal and tangential to $\Sigma_t$,

$$
t^\alpha = \partial_t x^\alpha = N n^\alpha + E_a^\alpha N^a. \quad (3.2)
$$

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[2]We essentially follow the conventions of the Lecture Notes on General Relativity by M. Blau (http://www.blau.itp.unibe.ch/GRLecturenotes.html, 2021).

Here $n^\alpha$ is a unit normal vector, $n^\alpha n_\alpha = -1$, and $N$ and $N^a$ are the lapse function and the shift vector of the ADM formalism, respectively. The induced metric $h^{\alpha\beta} = E_a^\alpha E_b^\beta h^{ab}$, lapse function and shift vector determine the metric $g^{\alpha\beta}$ of the ambient space as

$$g^{\alpha\beta} = h^{\alpha\beta} - n^\alpha n^\beta = E_a^\alpha E_b^\beta h^{ab} - \frac{1}{N^2}(t^\alpha - E_a^\alpha N^a)(t^\beta - E_b^\beta N^b) \,. \tag{3.3}$$

The extrinsic curvature

$$K_{ab} = E_a^\alpha E_b^\beta K_{\alpha\beta}\,, \quad K_{\alpha\beta} = h_\alpha^\gamma h_\beta^\delta \nabla_\gamma n_\delta\,, \tag{3.4}$$

describes the curvature of $\Sigma_t$ in the ambient space $\mathcal{M}$, with $K = K_\alpha^\alpha = K_a^a = \nabla_\alpha n^\alpha$.

The Hamiltonian formalism for GR with a 3-from field $A_{\alpha\beta\gamma}$, or equivalently the vector density $\mathcal{T}^\alpha$, has previously studied in [21, 28, 40]. In the following discussion the emphasis lies on the effect of the boundary conditions. In terms of the induced metric $h_{ab}$, the lapse function $N$, the extrinsic curvature $K$, the field $\mathcal{T}^\alpha = (\mathcal{T}^t, \mathcal{T}^a)$ and $\Lambda$ the Lagrangian density $\mathcal{L}_g$ corresponding to the action (2.4) reads,

$$\mathcal{L}_g = \sqrt{h}N(\tilde{R} + K_{ab}K^{ab} - K^2) + \Lambda(\partial_t \mathcal{T}^t + \partial_a \mathcal{T}^a - \sqrt{h}N) \,. \tag{3.5}$$

Here $\tilde{R}$ is the Ricci scalar on $\Sigma_t$, which is determined by $h_{ab}$ (see, for example, [38]). The extrinsic curvature depends on the time derivative of the metric $\dot{h}_{ab} = \partial_t h_{ab}$,

$$K_{ab} = \frac{1}{2N}(\dot{h}_{ab} - D_{(a}N_{b)}) \,. \tag{3.6}$$

For the variables $h_{ab}$ and $\mathcal{T}^t$ one obtains the canonical momenta

$$\pi^{ab} = \sqrt{h}(K^{ab} - h^{ab}K)\,, \quad \pi_t = \Lambda \,. \tag{3.7}$$

The canonical momenta $\pi_a$, $\pi_\Lambda$, $\pi_N$ and $\pi_{N^a}$ for the variables $\mathcal{T}^a$, $\Lambda$, $N$ and $N^a$, respectively, all vanish. This leads to the Hamiltonian density

$$\begin{aligned}\mathcal{H}_g &= \pi^{ab}\dot{h}_{ab} + \pi_t \dot{\mathcal{T}}^t - \mathcal{L}_g \\ &= N(\mathcal{H} + \sqrt{h}\Lambda) + N^a \mathcal{H}_a - \Lambda \partial_a \mathcal{T}^a \,, \end{aligned} \tag{3.8}$$

where

$$\mathcal{H} = \sqrt{h}\left(-\tilde{R} + \frac{1}{h}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right)\right)\,, \quad \mathcal{H}_a = -2\sqrt{h}D^b\left(\frac{1}{\sqrt{h}}\pi_{ab}\right) \,. \tag{3.9}$$

The fields $N$, $N^a$, $\mathcal{T}^a$ and $\Lambda$ are Lagrange multipliers. Variation of the Hamiltonian $H_g = \int d^3x \mathcal{H}_g$ with respect to these fields yields the phase space constraints

$$\mathcal{H} + \sqrt{h}\Lambda = 0\,, \quad \mathcal{H}_a = 0\,, \quad \partial_a \Lambda = 0\,, \quad \partial_a \mathcal{T}^a - \sqrt{h}N = 0\,, \tag{3.10}$$

in agreement with the analysis in [21].

Using eqs. (3.7), (3.8) and (3.10) we can now write down the path integral. The third of the constraints (3.10) implies that $\Lambda$ is spatially constant. On the boundary 3-surfaces $\Sigma_{1,2}$ we can therefore specify constants $\lambda_{1,2}$. On each 3-surface $\Sigma_t$ the field $\mathcal{T}^t$ can be split into a zero mode $A(t)$ and a field whose integral over $\Sigma_t$ vanishes, $\mathcal{T}^t = A(t) + \partial_a \omega^a$. We can

therefore fix the gauge symmetry of the Lagrangian (3.5), $\mathcal{T}^t \to \mathcal{T}^t - \partial_a \rho^a$, $\mathcal{T}^a \to \mathcal{T}^a + \partial_t \rho^a$, by the condition $\partial_a \mathcal{T}^t = 0$. On the boundary surfaces $\Sigma_{1,2}$ the 3-metric $h_{ab}$, the constants $\mathcal{A}_t = \int d^3 x \mathcal{T}^t = \int_{\Sigma_t} A$, and $\lambda$ can be independently chosen, and the transition amplitude is given by the functional integral

$$\langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | h_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle$$

$$= \int [Dh_{ab}][D\pi^{ab}][D\mathcal{T}^t][D\pi_t][D\Lambda][DN][DN^a][D\mathcal{T}^a]\delta(\pi_t - \Lambda)\delta(\partial_a \Lambda)\delta(\partial_a \mathcal{T}^t)$$

$$\times \exp\left(i \int_{\mathcal{M}} d^4 x (\pi^{ab}\dot{h}_{ab} + \pi_t \dot{\mathcal{T}}^t - N(\mathcal{H} + \sqrt{h}\Lambda) - N^a \mathcal{H}_a + \Lambda \partial_a \mathcal{T}^a)\right). \qquad (3.11)$$

For spatially constant $\Lambda$ the exponent no longer depends on $\mathcal{T}^a$, and integration over the fields $\mathcal{T}^a$ yields a constant factor. Performing the integration over $\pi_t$ and replacing $\delta(\partial_a \Lambda)$ by $[D\lambda(t)]\delta(\Lambda - \lambda(t))$, the amplitude becomes

$$\langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | h_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle$$

$$= \int [Dh_{ab}][D\pi^{ab}][D\mathcal{A}_t][D\lambda(t)][DN][DN^a]$$

$$\times \exp\left(i \int_{t_1}^{t_2} dt \lambda(t)\dot{\mathcal{A}}_t + i \int_{\mathcal{M}} d^4 x (\pi^{ab}\dot{h}_{ab} - N(\mathcal{H} + \sqrt{h}\lambda(t)) - N^a \mathcal{H}_a)\right). \qquad (3.12)$$

After a partial integration yielding the boundary term $[\lambda(t)\mathcal{A}_t]\big|_1^2$, the integral over $\mathcal{A}_t$ can be performed which leads to a factor $\delta(\dot{\lambda}(t))$ in the functional integral. Since $\lambda(t)$ has to satisfy the boundary conditions $\lambda(t_{1,2}) = \lambda_{1,2}$ we replace $\delta(\dot{\lambda}(t))$ by $\delta(\lambda(t) - \lambda_1)\delta(\lambda(t) - \lambda_2)$. Integrating over the canonical momenta $\pi^{ab}$ we finally obtain,

$$\langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | h_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle$$

$$= \delta(\lambda_2 - \lambda_1) \exp\left(i\lambda_1(\mathcal{A}_2 - \mathcal{A}_1)\right)\mathcal{N}_2\mathcal{N}_1 \int [Dg] \exp\left(iS[g] - \lambda_1 \mathcal{V}_{\mathcal{M}}[g]\right), \qquad (3.13)$$

where $\mathcal{N}_{1,2}$ are normalization factors related to the boundaries. The amplitude essentially agrees with eq. (2.11), with the important difference that instead of an unspecified constant $\lambda_0$ now the boundary values $\lambda_1$ and $\lambda_2$ appear. The result is consistent with the one obtained in [28]. Note that the integral over the metric is not affected by the boundary conditions $\mathcal{A}_{1,2}$. In particular the integration includes metric fields $g$ interpolating between $h_1$ and $h_2$ with volumes of arbitrary size.[3] The phase factor suggests that $\mathcal{A}$ and $\lambda$ are conjugate variables with $\mathcal{A}$ and $\lambda$ playing the role of "time" and "energy", respectively [8]. However, $\mathcal{A}$ can take arbitrary positive and negative values and it does not increase monotonically with the parameter time $t$. Therefore, generically, $\mathcal{A}$ cannot be interpreted as a time parameter.

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[3]The result differs from the path integral obtained in [21] where the integration is restricted to volumes of some fixed size that is introduced via a gauge fixing condition.

The amplitude clearly satisfies the superposition principle. Splitting the manifold $\mathcal{M}_{(21)}$ bounded by $\Sigma_2$ and $\Sigma_1$ into two regions $\mathcal{M}_{(23)}$ and $\mathcal{M}_{(31)}$ separated by $\Sigma_3$, one has

$$\int [Dh_3] d\mathcal{A}_3 d\lambda_3 \langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | h_3, \mathcal{A}_3, \lambda_3; \Sigma_3 \rangle \langle h_3, \mathcal{A}_3, \lambda_3; \Sigma_3 | h_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle$$

$$= \delta(\lambda_2 - \lambda_1) \exp\left(i\lambda_1(\mathcal{A}_2 - \mathcal{A}_1)\right) \mathcal{N}_2 (\mathcal{N}_3)^2 \mathcal{N}_1 \int [Dg_{(23)}][Dh_3][Dg_{(31)}]$$

$$\times \exp\left(i(S[g_{(23)}] + S[g_{(31)}] - \lambda_1(\mathcal{V}_\mathcal{M}[g_{(23)}] + \mathcal{V}_\mathcal{M}[g_{(31)}]))\right)$$

$$= \langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | h_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle \,, \tag{3.14}$$

where the metric $g_{(ij)}$ interpolates between $h_i$ and $h_j$ on $\Sigma_i$ and $\Sigma_j$, respectively, and the boundary normalization factors have been fixed to $(\mathcal{N}_i)^{-2} = \int d\mathcal{A}_i$.

In the semiclassical approximation the exponent in (3.13) is evaluated at a stationary point satisfying Einstein's equations,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2} \lambda_1 g_{\alpha\beta} \,. \tag{3.15}$$

As a simple example consider the case of positve cosmological constant, $\lambda_1 > 0$, for which a solution of Einstein's equations is given by the FLRW metric $g_{\text{FLRW}}$ with an exponentially growing scale factor. For a foliation with 3-spheres one has

$$ds^2 = -N(t)dt^2 + h_{ab}(t, y^a)dy^a dy^b \,, \quad h_{ab}(t, y^a) = a(t)^2 \tilde{h}_{ab} \,, \tag{3.16}$$

where $a(t)$ is the scale factor and $\tilde{h}_{ab}$ is the well known metric on the unit 3-sphere. Volume and Ricci scalar of the 3-sphere are given by $\mathcal{V}_{\Sigma_t} = 2\pi^2 a(t)^3$ and $\tilde{R} = 6/a(t)^2$, respectively. The four-dimensional Ricci scalar is $R = 2\lambda_1$. In eq. (3.16) a comoving time coordinate has been chosen, hence the shift vector $N^a$ is zero. The presence of the lapse function allows for reparametrizations of time.

From eqs. (3.6) and (3.7) one obtains for the extrinsic curvature and the canonical momenta

$$K_{ab} = \frac{\dot{a}a}{N} \tilde{h}_{ab} \,, \quad \pi^{ab} = -2\frac{\dot{a}}{N} \tilde{h}^{ab} \,, \tag{3.17}$$

and using eqs. (3.9) and (3.10) with a cosmological constant $\lambda_1$ one finds for the Hamiltonian constraint

$$\mathcal{H} + \sqrt{h}\lambda_1 = -6\sqrt{\tilde{h}}a^3 \left( \left( \frac{\dot{a}}{Na} \right)^2 + \frac{1}{a^2} - \frac{\lambda_1}{6} \right) = 0 \,, \tag{3.18}$$

which corresponds to Friedmann's equation. Einstein's equations also yield Raychaudhuri's equation for the second time-derivative of the scale factor, and the two equations together have the well-known solution $a(\tau) = \sqrt{6/\lambda_1} \cosh\left(\sqrt{\lambda_1/6}\tau\right)$, where $d\tau = N(t)dt$ determines the proper comoving time $\tau$. Considering for simplicity times $\tau \gg \sqrt{6/\lambda_1}$, one obtains for the total volume ($a_2 \equiv a(\tau_2) \gg a(\tau_1) \equiv a_1$)

$$\mathcal{V}_\mathcal{M} = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} = 2\pi^2 \int_{t_1}^{t_2} dt N(t) a(t)^3 \simeq 2\pi^2 \sqrt{\frac{2}{3\lambda_1}} a_2^3 \,. \tag{3.19}$$

With $h_{ab}(t)$ determined by $a(t)$, the amplitude (3.13) can be written as

$$\langle a_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | a_1, \mathcal{A}_1, \lambda_1; \Sigma_1 \rangle \propto \delta(\lambda_2 - \lambda_1) \exp\left(i\lambda_1(\mathcal{A}_2 - \mathcal{A}_1 + \mathcal{V}_\mathcal{M})\right). \qquad (3.20)$$

Note that the action for the FLRW metric is given by $S[g_{\text{FLRW}}] = \lambda_1 \mathcal{V}_\mathcal{M}$.

# 4  Unimodular gravity

It is instructive to compare covariant UG with a 3-form density and UG with the constraint $\sqrt{g} = \omega$, where $\omega$ is some nondynamic background spacetime volume element. In this case one starts from the Hamiltonian density

$$\mathcal{H}_g = N(\mathcal{H} + \sqrt{h}\Lambda) + N^a \mathcal{H}_a - \Lambda\omega, \qquad (4.1)$$

where $\mathcal{H}$ and $\mathcal{H}_a$ are again given by eq. (3.9) and $\Lambda$ is a Lagrange multiplier field. Variation with respect to $N^a$ and $\Lambda$ yields the constraints

$$\mathcal{H}_a = 0, \quad N\sqrt{h} - \omega = 0. \qquad (4.2)$$

Because $N$ is now fixed to $\omega/\sqrt{h}$ there is no Hamiltonian constraint. However, a tertiary constraint follows from the requirement that the time evolution preserves the momentum constraint. Using the Poisson bracket algebra

$$\begin{aligned}
\{(h^{-1/2}\mathcal{H})(x), \mathcal{H}_a(x')\} &= \partial_a(h^{-1/2}\mathcal{H})(x))\delta(x, x'), \\
\{\mathcal{H}_a(x), \mathcal{H}_b(x')\} &= \mathcal{H}_a(x')\partial_b\delta(x, x') + \mathcal{H}_b(x')\partial_a\delta(x, x'),
\end{aligned} \qquad (4.3)$$

one obtains the constraint

$$\begin{aligned}
0 = \left\{ H_g, \int d^3x\, \xi^a \mathcal{H}_a \right\} &= \left\{ \int d^3x'(\omega h^{-1/2}\mathcal{H} + N^b\mathcal{H}_b), \int d^3x\, \xi^a\mathcal{H}_a \right\} \\
&= \int d^3x\, \xi^a(\omega\partial_a(h^{-1/2}\mathcal{H} - (\partial_a N^b + \partial_c N^c \delta_a^b)\mathcal{H}_b - N^b\partial_b\mathcal{H}_a).
\end{aligned}$$

For arbitary vector fields $N^a$ and $\xi^a$ this implies [8–10]

$$\partial_a\left(\frac{1}{\sqrt{h}}\mathcal{H}\right) = 0. \qquad (4.4)$$

The constraint can be solved by

$$\mathcal{H} + \sqrt{h}\lambda = 0, \qquad (4.5)$$

where $\lambda$ is constant, which has to be satisfied on each 3-surface. Therefore we again have to specify constants $\lambda_{1,2}$ on the boundary surfaces $\Sigma_{1,2}$.

It is now straightforward to write down the path integral for the transition amplitude analogous to eq. (3.12),

$$\begin{aligned}
\langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle = \int [Dh_{ab}][D\pi^{ab}][DN^a][D\lambda(t)]\delta(\mathcal{H} + \sqrt{h}\lambda(t)) \\
\times \exp\left( i\int_\mathcal{M} d^4x(\pi^{ab}\dot{h}_{ab} - h^{-1/2}\omega\mathcal{H} - N^a\mathcal{H}_a) \right),
\end{aligned} \qquad (4.6)$$

Note that the constraint (4.5) has been implemented for each hypersurface $\Sigma_t$ and that the integration is performed over $\lambda(t)$, with the boundary conditions $\lambda(t_{1,2}) = \lambda_{1,2}$. Exponentiating the constraint (4.5) by introducing again a Lagrange multiplier $N$, and shifting $N$ to $N - \omega/\sqrt{h}$ one arrives at

$$\langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle$$
$$= \int [Dh_{ab}][D\pi^{ab}][DN][DN^a][D\lambda(t)]$$
$$\times \exp\left( i \int_{\mathcal{M}} d^4x (\pi^{ab} \dot{h}_{ab} - N(\mathcal{H} + \sqrt{h}\lambda(t)) - N^a \mathcal{H}_a + \lambda(t)\omega) \right). \qquad (4.7)$$

We can now integrate over the canonical momenta $\pi^{ab}$ which yields the amplitude in Lagrangian form,

$$\langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle$$
$$= \int [Dg][D\lambda(t)] \exp\left( iS[g] - i \int_{\mathcal{M}} d^4x \lambda(t)(\sqrt{g} - \omega) \right). \qquad (4.8)$$

Contrary to eq. (3.13) the amplitude does not contain a factor $\delta(\lambda_1 - \lambda_2)$. Instead a Lagrange multiplier appears for the volume of each 3-surface $\Sigma_t$. Correspondingly, integration over $\lambda(t)$ yields a product of $\delta$-functions in the functional integral,

$$\langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle = \int [Dg] \prod_t \delta(N(t)\mathcal{V}_{\Sigma_t} - \Omega(t)) \exp\left( iS[g] \right), \qquad (4.9)$$

with

$$\mathcal{V}_{\Sigma_t} = \int_{\Sigma_t} d^3x \sqrt{h}, \quad \Omega(t) = \int_{\Sigma_t} d^3x \, \omega \equiv \Omega(t). \qquad (4.10)$$

The spatially integrated background volume element $\Omega(t)$ depends on the chosen coordinate system.

The transition amplitude satisfies a Schrödinger equation with respect to the upper end $t_2$ of the time integration. Using the momentum constraint in eq. (4.2) and the constraint (4.5) one obtains for the Hamiltonian appearing in the exponent of (4.7) at the boundary $\Sigma_2$,

$$H_g\big|_{\Sigma_2} = \int_{\Sigma_2} \left( N(\mathcal{H} + \sqrt{h}\lambda(t)) + N^a \mathcal{H}_a - \lambda(t)\omega \right) = -\lambda_2 \Omega(t_2). \qquad (4.11)$$

This yields the Schrödinger equation

$$i\frac{\partial}{\partial t_2} \langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle = \langle h_2, \lambda_2; \Sigma_2 | H_g\big|_{\Sigma_2} | h_1, \lambda_1; \Sigma_1 \rangle$$
$$= -\lambda_2 \Omega(t_2) \langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle. \qquad (4.12)$$

Unruh and Wald obtained this equation for a wave function $\psi(t; h, \lambda)$, with $t$ playing the role of a "Heraclitian time parameter" [10]. Note that $\Omega(t)$ can be absorbed into a redefined time variable.

In the semiclassical approximation the amplitude (4.8) is dominated by the contribution of stationary points that satisfy the field equations

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -\frac{1}{2}\lambda(t)g_{\alpha\beta}\,, \quad \partial_\alpha\lambda(t) = 0\,, \tag{4.13}$$

where the second equation follows from the Bianchi identity. Variation with respect to $\lambda(t)$ yields

$$N(t)\mathcal{V}_{\Sigma_t} = \Omega(t)\,. \tag{4.14}$$

Since for stationary points $\lambda(t)$ is constant, the amplitude is again proportional to $\delta(\lambda_2 - \lambda_1)$. With $R = 2\lambda_1$, one finds

$$\langle h_2, \lambda_2; \Sigma_2 | h_1, \lambda_1; \Sigma_1 \rangle \propto \delta(\lambda_2 - \lambda_1) \exp\left(2i\lambda_1 \int_{t_1}^{t_2} dt\,\Omega(t)\right)\,. \tag{4.15}$$

Consider now the example of the FLRW metric (3.16). From eqs. (4.10) and (4.14) one obtains the constraint

$$2\pi^2 N(t)a(t)^3 = \Omega(t)\,. \tag{4.16}$$

Knowing $a(t)$ from the solution of Friedmann's and Raychaudhuri's equations, this fixes the lapse function, and therefore the time coordinate, to $N(t) = \Omega(t)/(2\pi^2 a(t)^3)$. For an exponential expansion the growth of $a(t)^3$ is compensated by the decrease of $N(t)$ such that the amplitude is still given by eq. (4.15).

Finally, we compare the result of unimodular gravity with standard general relativity. Here the transition amplitude reads

$$\langle h_2; \Sigma_2 | h_1; \Sigma_1 \rangle = \int [Dh_{ab}][D\pi^{ab}][DN][DN^a]$$
$$\times \exp\left(i\left(\int_{\mathcal{M}} d^4x(\pi^{ab}\dot{h}_{ab} - N(\mathcal{H} + \sqrt{h}\lambda) - N^a\mathcal{H}_a)\right)\right)\,, \tag{4.17}$$

where $\lambda$ is now a parameter of the Lagrangian. The characteristic feature of the amplitude is the Hamiltonian and the momentum constraints that follow from the integration over $N$ and $N^a$, respectively,

$$\mathcal{H} + \sqrt{h}\lambda = 0\,, \quad \mathcal{H}_a = 0\,. \tag{4.18}$$

Hence, the amplitude satisfies the differential equation

$$i\frac{\partial}{\partial t_2}\langle h_2; \Sigma_2 | h_1; \Sigma_1 \rangle = \langle h_2; \Sigma_2 | H_g\big|_{\Sigma_2} | h_1; \Sigma_1 \rangle$$
$$= \left(\int_{\Sigma_2} d^3x(N\mathcal{H} + N^a\mathcal{H}_a)\right)\langle h_2; \Sigma_2 | h_1; \Sigma_1 \rangle = 0\,. \tag{4.19}$$

This is the well-known feature of the Wheeler-DeWitt equation that in general relativity wave functions have no time dependence.

In the semiclassical approximation one has to solve Einstein's equation for a given cosmological constant $\lambda$. The solution $g_{\text{cl}}$ yields $R = 2\lambda$ and an exponentially growing scale factor with spacetime volume $\mathcal{V}_{\mathcal{M}} = \simeq \sqrt{2}2\pi^2 a_2^3/\sqrt{3\lambda}$. The corresponding amplitude reads

$$\langle h_2; \Sigma_2 | h_1; \Sigma_1 \rangle \propto \exp\left(iS[g_{\text{cl}}]\right) = \exp\left(i\lambda\mathcal{V}_{\mathcal{M}}\right)\,. \tag{4.20}$$

Unimodular gravity in the Henneaux-Teitelboim form shares features of standard general relativity as well as unimodular gravity with a fixed background volume element. As discussed above, the cosmological term is not a parameter of Lagrangian but appears as a boundary term, i.e., as a property of states. On the other hand, wavefunctions do not depend on time. For the amplitude (3.12) the constraints

$$\mathcal{H} + \sqrt{h}\lambda(t) = 0 \,, \quad \mathcal{H}_a = 0 \tag{4.21}$$

hold on each 3-surface $\Sigma_t$. Hence, as in general relativity, the Hamiltonian on the boundary surface $\Sigma_2$ vanishes, $H_g\big|_{\Sigma_2} = 0$. This implies for the amplitude

$$
\begin{aligned}
i\frac{\partial}{\partial t_2}\langle h_2, \mathcal{A}_2, &\lambda_2; \Sigma_2 | h_1, \mathcal{A}_2, \lambda_1; \Sigma_1\rangle \\
&= \langle h_2, \mathcal{A}_2, \lambda_2; \Sigma_2 | H_g\big|_{\Sigma_2} | h_1, \mathcal{A}_2, \lambda_1; \Sigma_1\rangle = 0 \,.
\end{aligned}
\tag{4.22}
$$

This result is analogous to eq. (4.19), with the only difference that in addition to the metric also the integrated 3-form field $\mathcal{A}$ and a cosmological constant $\lambda$ appear as variables of the boundary states.

## 5 Time (in)dependent wave functions

In quantum gravity there is no intrinsic time and therefore no canonical formalism and no Hilbert space of physical states as in quantum field theory in flat spacetime. One considers wave functions of the form

$$\psi[h; \Sigma] = \int_{\mathcal{C}} [Dg] \exp\left(iS[g]\right), \tag{5.1}$$

where $\mathcal{C}$ denotes a class of spacetimes with only one compact spacelike 3-surface $\Sigma$ as boundary on which $h$ is the induced metric [37, 39]. The scalar product

$$(\psi', \psi) = \int [Dh]\bar{\psi}'[h; \Sigma]\psi[h; \Sigma] = \int_{(\mathcal{C}', \mathcal{C})} [Dg] \exp\left(iS[g]\right) \tag{5.2}$$

has the geometric interpretation as a sum over all histories which lie in class $\mathcal{C}$ to the past of the surface and in the time reversed class $\mathcal{C}'$ to its future [37]. This product cannot be interpreted as a scalar product of physical states in a Hilbert space. Only in the semiclassical approximation the WKB form (4.20) of the transition amplitude is reproduced. But this is just classical physics and it is far from clear how to extend the semiclassical approximation to the quantum regime. For the quantum mechanical system of a homogeneous scalar field in FLRW spacetime the scalar field can be used as a time variable [42].

Since the interpretation of solutions of the Wheeler-DeWitt equation is very challenging (see, for example, [1, 10, 39, 43]), UG appeared as an interesting possibility to achieve a canonical quantization of gravity [9, 10]. Starting from orthogonal eigenstates of the variables $h$ and $\lambda$ one can define time-dependent wave functions

$$\psi[h, \lambda; \Sigma_t] = \int [Dh_1] d\lambda_1 \langle h, \lambda; \Sigma_t | h_1, \lambda_1; \Sigma_1\rangle \phi(h_1, \lambda_1) \tag{5.3}$$

by integrating the transition amplitude over initial-state parameters weighted with some distribution function $\phi$. Like the amplitude (4.8) also the wave functions satisfy a Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi[h, \lambda; \Sigma_t] = -\lambda\Omega(t)\psi[h, \lambda; \Sigma_t] \, . \tag{5.4}$$

For these wave functions one can define a scalar product by integrating over the variables $h$ and $\lambda$,

$$(\psi', \psi) = \int [Dh] d\lambda \bar{\phi}'(h, \lambda)\phi(h, \lambda) \, . \tag{5.5}$$

Hence, normalizable states can be defined such that a probability interpretation of $|\psi(t)|^2 \equiv (\psi[h, \lambda; \Sigma_t], \psi[h, \lambda; \Sigma_t])$ is possible, which is difficult to achieve for solutions of the Wheeler-DeWitt equation. The transition amplitude of the theory is given by eq. (4.8). As discussed in the previous section it has the characteristic feature that the cosmological constant enters as a property of states. On the other hand, the dependence of the time evolution of states on an arbitrary background volume element appears as a weakness of this modification of GR [2].

The Henneaux-Teitelboim version of UG is generally covariant. Hence, as discussed above, wave functions are time independent, as in GR. However, as in UG, the 3-form field $A$, sourced by the metric $g$, leads to the appearance of a cosmological constant as boundary term. From eqs. (3.13) and (5.1) we infer that the wave function has the form

$$\begin{aligned}
\psi[h, \mathcal{A}, \lambda; \Sigma] &= \int_{\mathcal{C}} [D\mu(\Sigma')][Dh'] d\mathcal{A}' d\lambda'[Dg]\langle h, \mathcal{A}, \lambda; \Sigma | h', \mathcal{A}', \lambda'; \Sigma'\rangle \phi(h', \mathcal{A}', \lambda'; \Sigma') \\
&= \int_{\mathcal{C}} [D\mu(\Sigma')][Dh'] d\mathcal{A}' d\lambda'[Dg]\delta(\lambda - \lambda') \exp\left(iS[g]\right) \\
&\quad \times \exp\left(i\lambda'(\mathcal{A} - \mathcal{A}' - \mathcal{V}_{\mathcal{M}}[g])\right)\phi(h', \mathcal{A}', \lambda'; \Sigma') \, ,
\end{aligned} \tag{5.6}$$

where $\mathcal{C}$ again denotes a class of spacetimes with final 3-surface $\Sigma$ and initial 3-surfaces $\Sigma'$ over which one integrates with some measure, $\mathcal{V}_{\mathcal{M}}[g]$ is the volume bounded by $\Sigma$ and $\Sigma'$, and $\phi$ defines the initial states. The wave function satisfies a Schrödinger-type differential equation,

$$i\frac{\partial}{\partial \mathcal{A}}\psi[h, \mathcal{A}, \lambda; \Sigma] = -\lambda\psi[h, \mathcal{A}, \lambda; \Sigma] \, , \tag{5.7}$$

which is a consequence of the particular form of the action (2.4).[4] Note, however, that $\mathcal{A}$ is just the value of the 3-form field $A$ on the 3-surface $\Sigma$, with positive or negative values, which generically cannot be interpreted as a time variable. Only in a stationary-phase approximation the situation changes. Then $S[g] - \lambda\mathcal{V}_{\mathcal{M}}[g]$ is evaluated for solutions $g_{\text{cl}}$ of Einstein's equations with cosmological constant $\lambda$. Moreover, stationarity of the phase

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[4]We could have started from an action where the Lagrange multiplier term $\Lambda dA$ in eq. (2.4) is replaced by $-Ad\Lambda$ [20], without changing the classical equations of motion. In this case the phase factor in eqs. (2.11) and (3.13) disappears, the 3-form field can be completely integrated out, the amplitudes in UR and GR are identical, and the cosmological term is simply a constant determined by initial conditions. This has been pointed out in [25]. However, in this version of the theory the relation (2.8) of the classical theory cannot be obtained in a semiclassical approximation of the quantum theory.

with respect to $\lambda'$ yields the relation (2.8) between the boundary terms $\mathcal{A}$, $\mathcal{A}'$ and the volume $\mathcal{V}_\mathcal{M}[g_{\text{cl}}]$,

$$\mathcal{A}' = \mathcal{A} - \mathcal{V}_\mathcal{M}[g_{\text{cl}}] . \tag{5.8}$$

Hence, in this approximation $\mathcal{A}$ increases monotonically with the parameter time labeling the 3-surfaces of the foliation and can therefore be used as a time variable. A solution of Einstein's equation determines $h'$ as function of $h$ and $\lambda$, and $\mathcal{A}'$ as function of $\mathcal{A}$, $h$ and $\lambda$. Therefore, in the stationary-phase approximation the wave function becomes

$$\psi[h, \mathcal{A}, \lambda; \Sigma] \sim \int_\mathcal{C} [D\mu(\Sigma')] \exp\left(iS[g_{\text{cl}}]\right) \phi(h'[h, \lambda], \mathcal{A} - \mathcal{V}_\mathcal{M}[h, \lambda], \lambda; \Sigma') . \tag{5.9}$$

This means that the wave function at "time" $\mathcal{A}$ is obtained by integrating over initial values at "times" $\mathcal{A}' < \mathcal{A}$.

## 6 Summary and conclusions

In the previous sections we have compared the path integral for transition amplitudes in general relativity with the corresponding amplitudes in the two versions of unimodular gravity, the one with a nondynamical background volume element and the covariant form with a 3-form gauge field. The amplitude (3.13) for covariant UG agrees with the one of GR except for a phase factor that depends on the boundary states and the interpretation of the cosmological constant which is a property of the boundary states rather than a parameter of the action. On the contrary, the amplitude (4.8) for UG with a background volume form explicitly depends on the volume form $\omega$. Hence, the two versions of UG generically lead to different predictions for observables.

As covariant theories wave functions in GR and in covariant UG have no time dependence and satisfy a Wheeler-DeWitt equation, which makes their interpretation challenging, except for cases where a semiclassical approximation applies. On the other hand, UG with a background volume form has a time variable that is canonically conjugate to the cosmological constant. Wave functions do depend on time and satisfy a Schrödinger equation. It is interesting that in covariant UG the 3-form gauge field integrated over spacelike hypersurfaces emerges as a "cosmic time" in the semiclassical approximation.

The change of the cosmological constant from a parameter of the action to a property of states does not solve the cosmological constant problem, but it does change it in a suggestive way [1], from a question of fine-tuning to a question of initial conditions. In general, a cosmological initial state is now a superposition of states with different cosmological constants. It has been suggested that a vanishing or very small cosmological constant today can be explained in such a framework, based on Euclidean quantum gravity [44–46] or, alternatively, on unimodular gravity [21, 47]. It is interesting that the additional fields needed in unimodular gravity occur in higher-dimensional supergravity theories and in string theory [12, 43, 48].

## Acknowledgments

## A BRST quantization

In this appendix we briefly review gauge fixing for the gravitational field, which we have ignored in the previous sections. In covariant theories, such as general relativity or the Hennaux-Teitelboim version of unimodular gravity, this is well known. One may choose, for instance, the harmonic gauge condition, or de Donder gauge,

$$C_\mu(g) = -\frac{1}{\sqrt{g}} g_{\mu\nu} \partial_\lambda(\sqrt{g} g^{\nu\lambda}) = 0 \,, \tag{A.1}$$

together with eight real Faddeev-Popov vector ghosts $u^\mu$ and $\bar{u}^\mu$ for which the BRST invariance and the unitarity of the physical S-matrix have been explicitly demonstrated [49, 50].

In unimodular gravity with a fixed background spacetime volume element (we choose $\sqrt{g} = 1$) one can choose

$$C(g) = \sqrt{g} - 1 = 0 \tag{A.2}$$

as one of four gauge fixing conditions. A complete gauge fixing is achieved by demanding in addition that the vector field $C_\mu$ is the gradient of an auxiliary scalar field [6],

$$C_\mu(g) + \partial_\mu B = 0. \tag{A.3}$$

From eqs. (A.2) and (A.3) one obtains the gauge fixing Lagrangian[5]

$$\mathcal{L}_{\mathrm{GF}} = \frac{1}{2\alpha} \Lambda^\mu \Lambda_\mu - \Lambda^\mu \left( C_\mu + \beta \partial_\mu B \right) + \frac{1}{2\gamma} \Lambda^2 - \Lambda C \,, \tag{A.4}$$

where $\Lambda_\mu$ and $\Lambda$ are additional auxiliary fields. The BRST invariant extension of the gauge fixing Lagrangian requires two scalar ghosts $v$ and $\bar{v}$ in addition to the eight vector ghosts $u^\mu$ and $\bar{u}^\mu$. The ghost lagrangian reads

$$L_{\mathrm{GH}} = -i(\bar{u}^\mu s C_\mu + \beta \bar{u}^\mu \partial_\mu v + \bar{v} s C) \,, \tag{A.5}$$

where $s$ is a real, nilpotent antiderivation, and the BRST transformations of all fields are given by

$$\begin{aligned}
s g_{\mu\nu} &= u^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu u^\lambda g_{\lambda\nu} + \partial_\nu u^\lambda g_{\mu\lambda} \,, \\
s u^\mu &= u^\lambda \partial_\lambda u^\mu \,, \\
s B &= v \,, \qquad\quad s v = 0 \,, \\
s \bar{u}^\mu &= i \Lambda^\mu \,, \qquad s \Lambda^\mu = 0 \,, \\
s \bar{v} &= i \Lambda \,, \qquad\quad s \Lambda = 0 \,.
\end{aligned} \tag{A.6}$$

Recently, the fields $B$, $v$, $\bar{v}$ and $\Lambda$ have been identified as a BRST quartet [32] and the decoupling of BRST quartets in momentum space has been discussed in detail in [33].

Eliminating in eq. (A.4) the Lagrange multiplier fields by their equations of motion one obtains the gauge fixing Lagrangian

$$\mathcal{L}_{\mathrm{GF}} = -\frac{\alpha}{2} \left( C^\mu + \beta \partial^\mu B \right) \left( C_\mu + \beta \partial_\mu B \right) - \frac{\gamma}{2} C^2 \,. \tag{A.7}$$

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[5]Compared to [6] we have rescaled $\beta \to \alpha\beta$; moreover, since $\delta \neq 0$ only leads to an uninteresting variation of the harmonic gauge, we have set $\delta = 0$ for simplicity.

In the linear approximation around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the Green's functions can be written in a compact form [6]. With $\omega = (u_\mu, v)$ one finds for the ghost propagator matrix

$$\langle \omega(x)\bar{\omega}(y) \rangle = - \begin{pmatrix} \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} & \partial_\mu \\ \frac{1}{\beta}\partial_\mu & -\frac{1}{\beta}\Box \end{pmatrix} \frac{1}{\Box} \delta^4(x-y) \,. \tag{A.8}$$

Correspondingly, defining for the gravitational field and the auxiliary scalar field $\hat{h} = (h_{\mu\nu}, B)$, one obtains the propagator matrix

$$\langle \hat{h}(x)\hat{h}(y) \rangle = \begin{pmatrix} D^{(\alpha)}_{\mu\nu\lambda\tau} & -\frac{1}{2\beta}(\eta_{\mu\nu} - \frac{4}{\gamma}(\Box + \gamma)\frac{\partial_\mu \partial_\nu}{\Box}) \\ -\frac{1}{2\beta}(\eta_{\lambda\tau} - \frac{4}{\gamma}(\Box + \gamma)\frac{\partial_\lambda \partial_\tau}{\Box}) & -\frac{1}{\beta^2 \gamma}(\Box + \gamma(\frac{3}{2} - \frac{1}{\alpha})) \end{pmatrix}$$
$$\times \frac{i}{\Box}\delta^4(x-y) \,,$$

with
$$D^{(\alpha)}_{\mu\nu\lambda\tau} = P^{(\alpha)}_{\mu\nu\lambda\tau} + \frac{1}{\Box}(\eta_{\mu\nu}\partial_\lambda \partial_\tau + \eta_{\lambda\tau}\partial_\mu \partial_\nu) \tag{A.9}$$
$$- \frac{4}{\gamma}\left(\Box + \gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right)\right)\frac{1}{\Box^2}\partial_\mu \partial_\nu \partial_\lambda \partial_\tau \,,$$

and
$$P^{(\alpha)}_{\mu\nu\lambda\tau} = \frac{1}{2}(\eta_{\mu\lambda}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\tau})$$
$$- \frac{1}{2}\left(1 - \frac{2}{\alpha}\right)\frac{1}{\Box}(\partial_\mu \partial_\lambda \eta_{\nu\tau} + \partial_\nu \partial_\lambda \eta_{\mu\tau} + \partial_\mu \partial_\tau \eta_{\nu\lambda} + \partial_\nu \partial_\tau \eta_{\mu\lambda}) \,.$$

Note that $\Delta^{(\alpha)}_{\mu\nu\lambda\tau}(x-y) = P^{(\alpha)}_{\mu\nu\lambda\tau}\frac{i}{\Box}\delta^4(x-y)$ is the well-known graviton propagator in harmonic gauge.

The propagators in eqs. (A.8) and (A.9) involve terms with $1/\Box^2$ and $1/\Box^3$. The situation is similar for the propator matrix obtained from the Lagrangian (A.4) for the fields $h_{\mu\nu}$, $B$, $\Lambda_\mu$ and $\Lambda$ [33]. It is a non-trivial task to count the physical states for such a system of propagators. In principle one has to rewrite the Lagrangian in terms of simple-pole fields. An analysis directly in terms of multiple-pole fields leads to the conclusion that the propagator matrix decribes indeed just two physical graviton states with helicities $\pm 2$ [33]. As an alternative, the BRST quantization in unimodular gauge has also been discussed using ghost systems that include antisymmetric tensor fields [34, 35].

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