

RECEIVED: February 1, 2022

REVISED: June 12, 2022

ACCEPTED: June 16, 2022

PUBLISHED: August 4, 2022

Entanglement entropy of gravitational edge modes

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ABSTRACT: We consider the linearised graviton in $4d$ Minkowski space and decompose it into tensor spherical harmonics and fix the gauge. The Gauss law of gravity implies that certain radial components of the Riemann tensor of the graviton on the sphere labels the superselection sectors for the graviton. We show that among these 6 normal components of the Riemann tensor, 2 are related locally to the algebra of gauge-invariant operators in the sphere. From the two-point function of these components of the Riemann tensor on S^2 we compute the logarithmic coefficient of the entanglement entropy of these superselection sectors across a spherical entangling surface. For sectors labelled by each of the two components of the Riemann tensor these coefficients are equal and their total contribution is given by $-\frac{16}{3}$. We observe that this coefficient coincides with that extracted from the edge partition function of the massless spin-2 field on the 4-sphere when written in terms of its Harish-Chandra character. As a preliminary step, we also evaluate the logarithmic coefficient of the entanglement entropy from the superselection sectors labelled by the radial component of the electric field of the $U(1)$ theory in even d dimensions. We show that this agrees with the corresponding coefficient of the edge Harish-Chandra character of the massless spin-1 field on S^d .

KEYWORDS: Gauge Symmetry, Scale and Conformal Symmetries

ARXIV EPRINT: [2201.06043](https://arxiv.org/abs/2201.06043)

Contents

1 Introduction	1
2 U(1) edge modes	4
2.1 Entanglement from correlators on the sphere	4
2.2 The Maxwell theory in $d = 4$	6
2.3 U(1) theory in arbitrary even d	14
2.4 The U(1) theory on spheres and entanglement of edge modes	17
3 Graviton in $d = 4$ dimension	18
3.1 Lagrangian in terms of tensor harmonics	20
3.2 Curvature and the gauge fixed modes	26
3.3 Quantization of the modes	29
3.4 Entanglement entropy of the edge states	30
4 Conclusions	34
A Electric correlator in the Coulomb gauge	35

1 Introduction

Entanglement entropy in theories with local gauge symmetries is difficult to define. In order to define entanglement entropy, the surface at a constant time slice is sub-divided into a region A and its complement \bar{A} . We then require that the Hilbert space of the theory naturally factorizes as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}. \tag{1.1}$$

Scalar and spinor field theories have local physical excitations and their Hilbert space admits such factorization. In this factorized Hilbert space, one defines the reduced density matrix by tracing over $\mathcal{H}_{\bar{A}}$ which we denote by $\rho_A = \text{Tr}_{\bar{A}}\rho$. Then the entanglement entropy of the region A is given by

$$S(\rho_A) = -\text{Tr}_A(\rho_A \log \rho_A). \tag{1.2}$$

Theories with gauge symmetries including quantum gravity do not always admit local gauge-invariant operators and therefore the gauge-invariant Hilbert space does not admit a natural tensor product structure. For gauge theories, this issue was discussed in [1–16] and has been summarized recently in [17]. This lack of factorization of the Hilbert space is due to the presence of a centre of the algebra of operators. The operators belonging to the centre commute with all others and a unique entropy can be assigned to the centre. This centre can be chosen in many ways [3]. A choice which arises naturally when one considers the

extended Hilbert space description of lattice gauge theories is called the electric centre [9]. This non-trivial centre occurs due to the electric Gauss law constraint on physical states. Similarly, the magnetic Gauss law constraint gives rise to the magnetic centre. Such choices of centres also exist for scalar field theories on a lattice in which keeping the field itself fixed on the boundary leads to a non-trivial centre and a contribution to entanglement entropy. It is also possible to choose the algebra on lattice gauge theories such that the centre is trivial and therefore there is no entropy associated with it [3]. In this paper, we will evaluate the entanglement entropy associated with centres that arise naturally from the Gauss law constraints for linearised gravity. At this point, the reader might wonder, why is it that we are focused on a quantity that is non-universal and depends on the choice of the centre. As we will see, the logarithmic coefficient corresponding to the centre obtained naturally from the Gauss law constraints in linearised gravity coincides with that extracted from the edge partition function of the massless spin-2 field on the 4-sphere when written in terms of its Harish-Chandra character. Furthermore, we feel that the techniques developed in this paper are useful and applicable to other theories with local symmetries.

To begin, let us consider the U(1) theory, since physical states obey the Gauss law constraint and gauge-invariant operators do not change the electric flux normal to the entangling surface, the Hilbert space \mathcal{H}_A as well as $\mathcal{H}_{\bar{A}}$ factorizes into superselection sectors labelled by the electric flux normal to the entangling surface. Then the entanglement entropy for a theory with U(1) gauge symmetry is given by

$$S(\rho_A) = - \sum_E p_E \log(p_E) + \sum_E p_E S(\rho_A^E). \quad (1.3)$$

Here p_E is the probability associated with a given superselection sector labelled by the electric flux. The first term in this expression is just the Shannon or classical entropy associated with the superselection sector. This contribution is also referred to as the entanglement entropy of edge modes and its contribution is non-extractable. That is, this entropy cannot be distilled into a number of Bell pairs [16]. This term is the entanglement contribution of the electric centre. In [7, 8, 16], the entanglement entropy of electromagnetic edge modes was evaluated for a spherical entangling surface of a U(1) theory in 4-dimensions and it was shown that it is captured by the partition function of a massless scalar on S^2 .

As mentioned earlier, the main aim of this paper is to evaluate the contribution of the entanglement entropy of the edge modes of the linearised graviton. This theory can be treated as a quantum field theory of spin-2 particles and therefore the question of whether local subsystems exist in the full quantum theory of gravity does not arise. In a certain gauge the linearised graviton $h_{\mu\nu}$, can be algebraically related to the curvature which is gauge-invariant and it generates the algebra of gauge-invariant operators of this theory. In [18], this approach was used to evaluate the logarithmic term in the entanglement entropy of linearised graviton across a spherical entangling surface. The Gauss law of the theory implies that the Hilbert space decomposes as a sum of superselection sectors similar to (1.3). We will show that the superselection sectors in this case are labelled by the normal components of the Riemann tensor on the sphere. Using this we will evaluate the classical non-extractable contribution of the entanglement entropy of the edge modes of the graviton.

A particularly direct method of evaluating the entropy of the edge modes for the Maxwell theory involves constructing the probability distribution p_E of the superselection sectors using the two-point function of the normal component of the electric field on the sphere. This was developed in [16]. We revisit this computation before we proceed to the graviton. In [16], the Maxwell theory was quantised in cartesian coordinates. We find it convenient to expand the U(1) field in vector spherical harmonics, fix gauge and quantise the theory following [14, 18]. Then we show that the radial component of the electric field on the sphere is directly related to one of the two canonical momenta. This allows us to evaluate the two-point function of the electric field on the sphere and compute the contribution of the edge modes. We show that indeed the logarithmic coefficient of the entanglement entropy of the edge modes or the electric centre is obtained from the corresponding coefficient of the partition function of the massless scalar on the sphere S^2 . To demonstrate the utility of this approach, we generalise the computation to the U(1) theory in arbitrary even d -dimensions. We show that the logarithmic coefficient of the non-extractable contribution to the entanglement entropy can be obtained from the corresponding coefficient of the partition function of a massless scalar on S^{d-2} . Recently it has been shown that the partition function of the vector on S^d when written as an integral over the Harish-Chandra character naturally decomposes into a sum of contributions from bulk and edge characters [19]. We observe that the contribution from the edge characters precisely coincides with that non-extractable entanglement entropy corresponding to the superselection sectors determined by the Gauss law.

We then consider the linearised graviton, we first demonstrate that the Gauss law of gravity implies that the certain normal components of the Riemann tensor to the entangling surface labels the superselection sectors. These components have one time direction, one radial direction and the rest arbitrary, there are 6 such components. Then we briefly review the quantization of the graviton developed in [18]. Here the field $h_{\mu\nu}$ is expanded in terms of tensor harmonics and under an appropriate gauge choice leads to a pair of canonical coordinates and momenta. We show that among the 6 components of the Riemann tensor, only two are algebraically or locally related to the canonical coordinates. We choose these components of the curvature tensor to label the superselection sectors and evaluate their two-point function on S^2 . From this we evaluate the contribution of the superselection sectors to the non-extractable entanglement entropy. We see that the logarithmic coefficient of the classical entanglement of the superselection sectors labelled by each of the 2 Riemann tensors are equal and given by $-8/3$. Since the 2 Riemann tensors are independent labels of the superselection sectors, their sum represents the complete contribution of the edge modes due to the centre obtained due to the Gauss law constraints. Then we observe that just as in the U(1) case, we see that the logarithmic term of this contribution of both the superselection sectors of the graviton $-16/3$, agrees with the term from edge mode partition function of the massless spin-2 field on S^4 . Here the edge mode partition function is identified by writing the spin-2 partition function on S^4 as an integral over its Harish-Chandra character.

The organisation of the paper is as follows. In section 2 we re-visit the evaluation of the classical entropy of the superselection sectors for the U(1) theory. This is first done in

$4d$ and then in arbitrary even dimensions. In section 3, we evaluate the contribution of the superselection sectors or in other words the electric centre to the logarithmic coefficient of the entanglement entropy of the linearised graviton across a spherical surface. Section 3.2 contains the evaluation of the 6 components of the curvature which must obey the Gauss law. The explicit computation reveals only 2 of these are locally related to the algebra of gauge-invariant operators in a sphere. Section 4 contains our conclusions. The appendix A, compares the two-point function of the radial components of the electric field on S^2 evaluated in the Coulomb gauge evaluated in [16] with that evaluated using the expansion in vector harmonics and the gauge introduced in [14, 18]. This paper uses the latter gauge which is more suited to the spherical symmetry of the problem and which can be generalised for the graviton.

2 U(1) edge modes

In this section we begin with the review of the approach of [16] to evaluate the contribution of the entanglement entropy of the superselection sectors of the U(1) gauge field in even d dimensions. Since we work with a spherical entangling surface it is convenient to use the methods of [14, 18] to quantize the system.

2.1 Entanglement from correlators on the sphere

Consider the U(1) theory with the action given by

$$S = -\frac{1}{4} \int d^d x F_{\mu\nu} F^{\mu\nu}, \quad (2.1)$$

and the spherical entangling surface S^{d-2} . We restrict our attention to even d since our focus is to obtain the logarithmic coefficient of the entanglement entropy. All physical states satisfy the Gauss law constraint

$$\partial^\mu F_{\mu 0} = 0. \quad (2.2)$$

This implies that the normal component of the electric field F_{0r} has to match across the entangling surface. Gauge-invariant operators like Wilson lines acting inside the entangling region A or those outside in \bar{A} cannot change this electric field. This leads to the factorisation of the density matrix into superselection sectors labelled by the normal component of the electric field. The classical contribution to the entanglement entropy from these superselection sectors is given by

$$S_{\text{edge}}(\rho_A) = - \sum_E p(E) \log p(E), \quad (2.3)$$

where p_E is the probability associated with a given superselection sector.

The U(1) theory is free and therefore $p(E)$ is a Gaussian functional of the normal component of the electric field given by

$$p[E_r] = \mathcal{N} \exp \left[-\frac{1}{2} \int d^d x d^d x' E_r(x) G_{rr'}^{-1}(x, x') E_{r'}(x') \right]. \quad (2.4)$$

Here x, x' are coordinates on the $\hat{d} \equiv d - 2$ sphere, $S^{\hat{d}}$. $G_{rr'}(x, x')$ is the two-point correlator of the radial component of the electric field on the sphere which is defined by

$$G_{rr'}(x, x') = \langle 0 | F_{0r'}(x) F_{0r'}(x') | 0 \rangle, \quad (2.5)$$

and its inverse satisfies the equation

$$\int d^{\hat{d}} x'' G_{rr''}(x, x'') G_{r''r'}^{-1}(x'', x') = \delta^{\hat{d}}(x - x'). \quad (2.6)$$

Note that the two-point function in (2.5) are evaluated on the entangling sphere, the labels r, r' , just refers to the fact the correlator is between the radial components at angular locations x, x' , on the sphere. The integrals in (2.4) and (2.6) are over the sphere $S^{\hat{d}}$

Evaluating the classical contribution we get

$$S_{\text{edge}}(\rho_A) = -\log \mathcal{N} + \int d^{\hat{d}} x d^{\hat{d}} x' G_{rr'}(x, x') G_{r'r}^{-1}(x', x). \quad (2.7)$$

Using (2.6), we see that the term involving the integrals is divergent

$$\int d^{\hat{d}} x d^{\hat{d}} x' G_{rr'}(x, x') G_{r'r}^{-1}(x', x) = \int d^{\hat{d}} x \delta^{\hat{d}}(0). \quad (2.8)$$

We can regulate the delta function by introducing a cut-off ϵ , this cut off is a short distance cut off along the angular directions on surface of the sphere. Therefore we can write

$$\int d^{\hat{d}} x d^{\hat{d}} x' G_{rr'}(x, x') G_{r'r}^{-1}(x', x) = \frac{R^{\hat{d}} \text{Vol}(S^{\hat{d}})}{\epsilon^{\hat{d}}}. \quad (2.9)$$

Here R is the radius of the entangling surface and $\text{Vol}(S^{\hat{d}})$ is the volume of the unit $S^{\hat{d}}$ sphere. This term is proportional to the area and does not contribute to the logarithmic term which is proportional to $\log \frac{R}{\epsilon}$. Let us now study the term \mathcal{N} . From the condition

$$\int \mathcal{D}E_r p[E_r] = 1, \quad (2.10)$$

we obtain

$$\log \mathcal{N} - \frac{1}{2} \log (\det G_{rr'}^{-1}) = 0. \quad (2.11)$$

We have defined the measure of the functional integral by the following integral

$$\int \mathcal{D}E_r \exp \left[-\frac{1}{2} \int d^{\hat{d}} x E_r^2(x) \right] = 1. \quad (2.12)$$

Therefore the contribution to the entanglement of the superselection sectors is given by

$$S_{\text{edge}}(\rho_A) = \frac{1}{2} \log \det G_{rr'} \Big|_{\log \text{ coefficient}}. \quad (2.13)$$

Here we have implicitly assumed that is it only the logarithmic contribution to S_{edge} we are interested in.

The two-point function of the electric field with both the electric fields on the same sphere diverges, as we will see subsequently. To regulate this divergence we consider the correlator where the radial component of the electric fields lie on two spheres of radius r and $r' = r + \delta$

$$G_{rr'}(r, r'; x, x') = \langle 0 | F_{0r}(r, x) F_{0r'}(r', x') | 0 \rangle. \quad (2.14)$$

We have introduced r, r' in the arguments of the Greens function to make it explicit that the electric field correlator involves insertion on spheres of different radii. We need to take $\delta \rightarrow 0$ such that

$$\delta \ll \epsilon, \quad (2.15)$$

where ϵ is the short distance cut-off on the sphere.

Therefore we look for the leading divergence in (2.14) when $\delta \rightarrow 0$ and use its coefficient in (2.13) to obtain the logarithmic term proportional to $\log \frac{r}{\epsilon}$. In the section 2.3 we will evaluate the two-point function of the radial component of the electric field in the angular momentum basis on $S^{\hat{d}}$. We show that in the $\delta \rightarrow 0$ limit, the two-point function admits the expansion

$$\lim_{\delta \rightarrow 0} \langle \ell \lambda | G_{rr'}(r, r + \delta x, x') | \ell' \lambda' \rangle = \frac{\ell(\ell + d - 3)}{4\pi r^d} \left(\log \frac{r^2}{\delta^2} \right) \delta_{\ell, \ell'} \delta_{\lambda, \lambda'} + O(\delta^0). \quad (2.16)$$

Here ℓ labels the eigen value of the Laplacian of scalars, and λ refers to all the other quantum numbers of the scalar harmonics on $S^{\hat{d}}$. Note that $\ell(\ell + d - 3)$ is the eigenvalue of the scalar Laplacian on $S^{\hat{d}}$. Then substituting the coefficient of the leading divergence in (2.16) we see that the coefficient to the entanglement entropy of the superselection sectors is given by coefficient of the logarithmic divergence of the one-loop determinant of the massless scalar on $S^{\hat{d}}$. Since the cut off on the surface of the sphere is ϵ , this divergence is proportional to $\log \frac{R}{\epsilon}$.

2.2 The Maxwell theory in $d = 4$

In this section, we first briefly discuss the method introduced by [18] to quantise the photon in spherical coordinates. We introduce the covariant notation for the vector harmonics which enables evaluation of field strengths easily, this notation will be carried over to the discussion of the graviton. After fixing gauge and setting up the canonical commutation relations for the gauge-invariant conjugate variables, we evaluate the two-point function of the radial component of the electric field

We first expand the vector potential A_μ as follows

$$A_\mu = \sum_{\ell, m} \left(A_{0(\ell, m)}(r, t) Y_{\ell m, \mu}^{\mathbf{0}} + A_{r(\ell, m)}(r, t) Y_{\ell m, \mu}^{\mathbf{r}} + A_{e(\ell, m)}(r, t) Y_{\ell m, \mu}^{\mathbf{e}} + A_{m(\ell, m)}(r, t) Y_{\ell m, \mu}^{\mathbf{m}} \right), \quad (2.17)$$

here the greek subscript refers to the component of the covariant vectors which are defined as follows

$$\begin{aligned}
 Y_{\ell m}^0 &= \{Y_{\ell m}(\theta, \phi), 0, 0, 0\}, \\
 Y_{\ell m}^r &= \{0, Y_{\ell m}(\theta, \phi), 0, 0\}, \\
 Y_{\ell m}^e &= \frac{r}{\sqrt{\ell(\ell+1)}} \left\{ 0, 0, \frac{\partial Y_{\ell m}(\theta, \phi)}{\partial \theta}, \frac{\partial Y_{\ell m}(\theta, \phi)}{\partial \phi} \right\}, \\
 Y_{\ell m}^m &= \frac{r}{\sqrt{\ell(\ell+1)}} \left\{ 0, 0, -\frac{1}{\sin \theta} \frac{\partial Y_{\ell m}(\theta, \phi)}{\partial \phi}, \sin(\theta) \frac{\partial Y_{\ell m}(\theta, \phi)}{\partial \theta} \right\}.
 \end{aligned} \tag{2.18}$$

$Y_{\ell m}(\theta, \phi)$ are scalar spherical harmonics with $\ell = 0, 1, 2, \dots$ and $-\ell \leq m \leq \ell$. In (2.18) wherever derivatives of the spherical harmonics occur it is understood that $\ell = 1, 2, \dots$. We have converted the conventional vector harmonics used in [18] which are vectors whose inner product is defined by the dot product using Krönecker delta to covariant vectors. This makes it easy to apply the methods of covariant tensor calculus rather than vector calculus. The metric is the flat space metric written in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.19}$$

Raising and lowering indices as well as covariant derivatives are defined with respect to this metric. It is also useful to introduce covariant orthonormal vectors in the directions of vectors in (2.18).

$$\begin{aligned}
 \hat{t} &= \{1, 0, 0, 0\}, \\
 \hat{r} &= \{0, 1, 0, 0\}, \\
 \hat{e} &= \frac{Y_{\ell m}^e}{|Y_{\ell m}^e|}, & |Y_{\ell m}^e|^2 &= g^{\mu\nu} Y_{\ell m; \mu}^e Y_{\ell m; \nu}^e, \\
 \hat{m} &= \frac{Y_{\ell m}^m}{|Y_{\ell m}^m|}, & |Y_{\ell m}^m|^2 &= g^{\mu\nu} Y_{\ell m; \mu}^m Y_{\ell m; \nu}^m,
 \end{aligned} \tag{2.20}$$

where the metric $g_{\mu\nu}$ is given in (2.19). Projections of tensors along these directions are defined as follows, consider a covariant tensor $T_{\mu\nu\rho}$, then

$$T_{\hat{e}\hat{m}\hat{r}} = \hat{e}^\mu \hat{m}^\nu \hat{r}^\rho T_{\mu\nu\rho}. \tag{2.21}$$

Similar definitions apply for other projections. The reality of the vector potential implies the following reality property of the coefficients of the expansion in (2.17).

$$\begin{aligned}
 (-1)^m A_{0,(l,m)}^* &= A_{0,(l,-m)}, & (-1)^m A_{r,(l,m)}^* &= A_{r,(l,-m)}, \\
 (-1)^m A_{e,(l,m)}^* &= A_{e,(l,-m)}, & (-1)^m A_{m,(l,m)}^* &= A_{m,(l,-m)}.
 \end{aligned} \tag{2.22}$$

Let us also expand the gauge transformation in terms of spherical Harmonics by

$$\chi = \sum_{\ell, m} \chi_{\ell m}(r, t) Y_{\ell m}(\theta, \phi). \tag{2.23}$$

Then the gauge field transforms as

$$A_\mu = \sum_{\ell,m} \left([A_{0(\ell,m)}(r,t) + \dot{\chi}_{\ell m}(r,t)] Y_{\ell m;\mu}^{\mathbf{0}} + [A_{r(\ell,m)}(r,t) + \partial_r \chi_{\ell m}(r,t)] Y_{\ell m;\mu}^{\mathbf{r}} \right. \quad (2.24)$$

$$\left. \times \left[A_{e(\ell,m)}(r,t) + \frac{\chi_{\ell m}(r,t)}{r} \right] Y_{\ell m;\mu}^{\mathbf{e}} + A_{m(\ell,m)}(r,t) Y_{\ell m;\mu}^{\mathbf{m}} \right).$$

Here the superscript ‘ \cdot ’ refers to partial derivative with respect to time. As in [18] we choose the gauge such that the longitudinal component of the gauge field $A_{e;\ell m}$ vanishes for every ℓ, m . Evaluating the field strengths in this gauge we obtain

$$F_{\hat{t}\hat{r}} = \sum_{\ell,m} (\dot{A}_{r;\ell m} - \partial_r A_{0;\ell m}) Y_{\ell m} \quad (2.25)$$

$$F_{\hat{t}\hat{m}} = \sum_{\ell,m} \dot{A}_{m;\ell m} |Y_{\ell m}^{\mathbf{m}}|,$$

$$F_{\hat{t}\hat{e}} = - \sum_{\ell,m} \frac{\sqrt{\ell(\ell+1)}}{r} A_{0;\ell m} |Y_{\ell m}^{\mathbf{e}}|,$$

$$F_{\hat{r}\hat{e}} = \sum_{\ell,m} \frac{\sqrt{\ell(\ell+1)}}{r} A_{r;\ell m} |Y_{\ell m}^{\mathbf{e}}|,$$

$$F_{\hat{r}\hat{m}} = - \sum_{\ell,m} \frac{\partial_r (r A_{m;\ell m})}{r} |Y_{\ell m}^{\mathbf{m}}|,$$

$$F_{\hat{e}\hat{m}} = \sum_{\ell,m} \frac{\sqrt{\ell(\ell+1)}}{r} A_{m;\ell m} Y_{\ell m}.$$

Expanding the gauge field in the action (2.1) for $d = 4$ and using the orthogonality properties of spherical harmonics we obtain

$$S = -\frac{1}{4} \int r^2 dr d\Omega F_{\mu\nu} F^{\mu\nu} = \sum_{\ell,m=-\ell}^{\ell} \int dr L_{\ell m},$$

$$L_{\ell m} = \frac{1}{2} \left[r^2 \dot{A}_{r(\ell,m)} \dot{A}_{r(\ell,m)}^* + r^2 \dot{A}_{m(\ell,m)} \dot{A}_{m(\ell,m)}^* - \ell(\ell+1) A_{r(\ell,m)} A_{r(\ell,m)}^* \right.$$

$$- \ell(\ell+1) A_{m(\ell,m)} A_{m(\ell,-m)} - |A_{m(\ell,m)} + r \partial_r A_{m(\ell,m)}|^2 + r^2 \partial_r A_{0(\ell,m)} \partial_r A_{0(\ell,m)}^*$$

$$\left. - r^2 \dot{A}_{r(\ell m)} \partial_r A_{0(\ell m)}^* - r^2 \dot{A}_{r(\ell m)}^* \partial_r A_{0(\ell m)} + \ell(\ell+1) A_{0(\ell,m)} A_{0(\ell,m)}^* \right]. \quad (2.26)$$

The sum over ℓ in (2.26) is understood to run from either $\ell = 0, 1 \dots$ or $\ell = 1, 2, \dots$ depending on the whether the mode corresponds to scalar or vector harmonics. The canonical conjugate momenta to A_r and A_m for $m = 0, 1 \dots$ are given by¹

$$\pi_{(\ell,m)}^{r*} = \frac{\partial L_\ell}{\partial \dot{A}_{r(\ell,m)}^*} = r^2 \left[\dot{A}_{r(\ell,m)} - \partial_r A_{0(\ell,m)} \right], \quad \pi_{(\ell,m)}^{m*} = \frac{\partial L_\ell}{\partial \dot{A}_{m(\ell,m)}^*} = r^2 \dot{A}_{m(\ell,m)}. \quad (2.27)$$

As a step towards quantization of the modes A_r and A_m we obtain the wave equations as well as the solutions satisfied by these modes and their conjugate variables.

¹We have taken the contribution of both positive and negative values of m and used the reality properties in writing the canonical momenta. This removes the factor of 1/2. For $m = 0$, the field is real.

The mode A_r . From the action in (2.26), we obtain

$$r^2(\ddot{A}_{r(\ell,m)} - \partial_r \dot{A}_{0(\ell,m)}) + l(l+1)A_{r(\ell,m)} = 0, \quad (2.28)$$

which can also be written as

$$\dot{\pi}_{(\ell,m)}^r + \ell(\ell+1)A_{r(\ell,m)} = 0. \quad (2.29)$$

We can eliminate $A_{0(\ell,m)}$ using the constraint

$$\partial_r \pi_{(\ell,m)}^r + \ell(\ell+1)A_{0(\ell,m)} = 0. \quad (2.30)$$

This leads a closed equation for A_r

$$r^2(\ddot{A}_{r(\ell,m)} - \partial_r^2 A_{r(\ell,m)}) + \ell(\ell+1)A_{r(\ell,m)} = 0. \quad (2.31)$$

We can solve this equation by expanding in Fourier modes in time and then solving the radial equation. The solution for a particular Fourier mode labelled by k which is regular at the origin can be written as

$$A_{r(\ell,m)} = e^{-ikt} a_{r(\ell,m)}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \quad (2.32)$$

Here $a_{\ell,m}$ is the integration constant and $J_{\ell+\frac{1}{2}}$ refers to the Bessel function. The equation $\pi_{(\ell,m)}^r$ can be obtained using the definition in (2.27)

$$\begin{aligned} \pi_{(\ell,m)}^r &= r^2 \left(-ike^{-ikt} a_{\ell,m}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r) - \partial_r A_{0(\ell,m)} \right), \\ &= r^2 \left(-ike^{-ikt} a_{r(\ell,m)}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r) + \frac{1}{\ell(\ell+1)} \partial_r^2 \pi_{(\ell,m)}^r \right). \end{aligned} \quad (2.33)$$

In the second line we have used the constraint in (2.30) to eliminate A_0 . Therefore the equation $\pi_{(\ell,m)}^r$ is an in-homogenous second order equation in the radial coordinate. The general solution is given by

$$\pi_{(\ell,m)}^r = c_1 r^{\ell+1} + c_2 r^{-\ell} + a_{r(\ell,m)}(k) e^{-ikt} \left(\frac{i 2^{-\ell-\frac{1}{2}} \ell(\ell+1) k^\ell}{\sqrt{k} \Gamma\left(\ell+\frac{3}{2}\right)} r^{\ell+1} - \frac{i \ell(\ell+1) \sqrt{r}}{k} J_{\ell+\frac{1}{2}}(|k|r) \right). \quad (2.34)$$

Demanding that the solution be regular at the origin yields $c_2 = 0$ since $\ell \geq 1$,² for these modes. Further demanding that the solution be well defined at infinity fixes c_1 and yields

$$\pi_{(\ell,m)}^r = -\frac{i \ell(\ell+1)}{k} a_{r(\ell,m)}(k) e^{-ikt} \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \quad (2.35)$$

The classical solutions in (2.32) and (2.35) allow us to write the mode expansion of the fields $A_{r(\ell,m)}, \pi_{(\ell,m)}^r$ as follows

$$\begin{aligned} A_{r(\ell,m)}(r,t) &= \frac{1}{\sqrt{2}} \int_0^\infty k dk \left(a_{r(\ell,m)}(k) e^{-ikt} + (-1)^m a_{r(\ell,-m)}^\dagger(k) e^{ikt} \right) \sqrt{r} J_{\ell+\frac{1}{2}}(kr), \\ \pi_{(\ell,m)}^r(r,t) &= \frac{\ell(\ell+1)}{\sqrt{2}} \int_0^\infty dk \left(-i a_{r(\ell,m)}(k) e^{-ikt} + i (-1)^m a_{r(\ell,-m)}^\dagger(k) e^{ikt} \right) \sqrt{r} J_{\ell+\frac{1}{2}}(kr). \end{aligned} \quad (2.36)$$

²For $\ell = 0$, using (2.30) and demanding that the solution vanish at infinity we see $\pi_{(\ell,m)}^r = 0$.

This form for the mode expansion respects the reality condition (2.22). Let us also write the mode expansion

$$\pi_{(\ell,m)}^r(r,t) = \frac{\ell(\ell+1)}{\sqrt{2}} \int_0^\infty dk \left(i a_{r(\ell,m)}^\dagger(k) e^{ikt} - i(-1)^m a_{r(\ell,-m)}(k) e^{-ikt} \right) \sqrt{r} J_{\ell+\frac{1}{2}}(kr). \quad (2.37)$$

We can now promote A_r, π^r to be operators which implies that $a_{\ell,m}, a_{\ell,m}^\dagger$ are operators. The equal time canonical commutation relation of these conjugate variables is given by

$$[A_{r(\ell,m)}(r,t), \pi_{(\ell',m')}^r(r',t)] = i \delta_{\ell,\ell'} \delta_{m,m'} \delta(r-r'). \quad (2.38)$$

Using the mode expansion in (2.36), it can be seen that this commutation relation implies the following commutation relations between the creation and annihilation operators

$$[a_{r(\ell,m)}(k), a_{r(\ell',m')}^\dagger(k')] = \frac{1}{\ell(\ell+1)} \delta(k-k') \delta_{\ell,\ell'} \delta_{m,m'}. \quad (2.39)$$

All other commutation relations are trivial. To show (2.39) holds, we substitute the expansion (2.36) in (2.38) and use the closure relation, see [20], section 11.2,

$$\int_0^\infty k dk J_{\ell+\frac{1}{2}}(kr) J_{\ell+\frac{1}{2}}(kr') = \frac{1}{r} \delta(r-r'). \quad (2.40)$$

The mode A_m . From the action (2.26), the equations of motion for the mode A_m is given by

$$r^2 (\partial_r^2 A_{m(\ell,m)} - \ddot{A}_{m(\ell,m)}) + 2r \partial_r A_{m(\ell,m)} - \ell(\ell+1) A_{m(\ell,m)} = 0. \quad (2.41)$$

The solution for each Fourier mode in time which is regular at the origin is given by

$$A_{m(\ell,m)} = a_{m(\ell,m)}(k) r^{-\frac{1}{2}} e^{-ikt} J_{\ell+\frac{1}{2}}(|k|r). \quad (2.42)$$

From the definition of the canonical conjugate momentum in (2.27), the corresponding Fourier mode is

$$\pi_{(\ell,m)}^{m*} = -i k a_{m(\ell,m)}(k) r^{-\frac{1}{2}} e^{-ikt} J_{\ell+\frac{1}{2}}(|k|r). \quad (2.43)$$

Using the solutions in (2.42) and (2.43) we can write the mode expansion as

$$\begin{aligned} A_{m(\ell,m)}(r,t) &= \frac{1}{\sqrt{2}} \int_0^\infty dk \left(a_{m(\ell,m)}(k) e^{-ikt} + (-1)^m a_{m(\ell,-m)}^\dagger(k) e^{ikt} \right) r^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(kr), \quad (2.44) \\ \pi_{(\ell,m)}^{m*}(r,t) &= \frac{1}{\sqrt{2}} \int_0^\infty k dk \left(-i a_{m(\ell,m)}(k) e^{-ikt} + i(-1)^m a_{m(\ell,-m)}^\dagger(k) e^{ikt} \right) r^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(kr). \end{aligned}$$

We also have

$$\pi_{(\ell,m)}^m(r,t) = \frac{1}{\sqrt{2}} \int_0^\infty k dk \left(i a_{m(\ell,m)}^\dagger(k) e^{ikt} - i(-1)^m a_{m(\ell,-m)}(k) e^{-ikt} \right) r^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(kr). \quad (2.45)$$

We can promote the fields A_m, π^m to operators by promoting $a_{m(\ell,m)}(k), a_{m(\ell,m)}^\dagger(k)$ to operators. Then the canonical commutation relations

$$[A_{m(\ell,m)}(r,t), \pi_{(\ell',m')}^m(r',t)] = i \delta(r-r') \delta_{\ell,\ell'} \delta_{m,m'}, \quad (2.46)$$

imply that the commutation relations

$$[a_{m(\ell,m)}(k), a_{m(\ell',m')}^\dagger(k')] = \delta(k-k') \delta_{\ell,\ell'} \delta_{m,m'}. \quad (2.47)$$

Electric correlator on the sphere and edge entanglement. We proceed to evaluate the two-point function of the radial component of the electric field. Examining the components of the field strengths in (2.25) and using the definition of the canonical momenta (2.27), we see that the electric field $F_{\hat{t}\hat{r}}$ is related to the momentum π^r

$$\begin{aligned} F_{\hat{t}\hat{r}} &= \sum_{\ell,m} (\dot{A}_{r;\ell m} - \partial_r A_{0;\ell m}) Y_{\ell m}(\theta, \phi), \\ &= \sum_{\ell,m} \frac{\pi_{(\ell,m)}^{r*}(r,t)}{r^2} Y_{\ell m}(\theta, \phi). \end{aligned} \quad (2.48)$$

Using the mode expansion in (2.36), we can proceed to evaluate the two-point function of the electric field

$$\begin{aligned} \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}\hat{r}'}(t, r', \theta', \phi') | 0 \rangle &= \\ \frac{1}{2(rr')^{\frac{3}{2}}} \sum_{\ell, \ell', m, m'} \ell(\ell+1)\ell'(\ell'+1) \int_0^\infty dk dk' & \left[J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') \right. \\ & \left. \times (-1)^{m'} \langle 0 | a_{r(\ell,m)}(k) a_{r(\ell',m')}^\dagger(k') | 0 \rangle Y_{\ell,m}(\theta, \phi) Y_{\ell',-m'}(\theta', \phi') \right]. \end{aligned} \quad (2.49)$$

Using the commutation relations in (2.39), we obtain

$$\begin{aligned} \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}\hat{r}'}(t, r', \theta', \phi') | 0 \rangle &= \\ \frac{1}{2(rr')^{\frac{3}{2}}} \sum_{\ell,m} \ell(\ell+1) \int_0^\infty dk & J_{\ell+\frac{1}{2}}(kr) J_{\ell+\frac{1}{2}}(k'r') Y_{\ell,m}(\theta, \phi) Y_{\ell,m}^*(\theta', \phi'). \end{aligned} \quad (2.50)$$

We perform the integral using the identity in [21], see page 696 equation 3 of **6.612**

$$\int_0^\infty dk J_{\ell+\frac{1}{2}}(kr) J_{\ell+\frac{1}{2}}(k'r') = \frac{1}{\pi\sqrt{rr'}} Q_\ell \left(\frac{r^2 + r'^2}{2rr'} \right). \quad (2.51)$$

Where $Q_\nu(z)$ is the Legendre function of second kind which can be written in terms of the hypergeometric function [21], see page 1024 equation 2 of **8.820**

$$Q_\nu(z) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\nu+1) z^{-\nu-1} {}_2F_1\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{1}{2}(2\nu+3); \frac{1}{z^2}\right)}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)}. \quad (2.52)$$

Therefore the correlator is given by

$$\langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}\hat{r}'}(t, r', \theta', \phi') | 0 \rangle = \sum_{\ell,m} \frac{\ell(\ell+1)}{2\pi(rr')^2} Q_\ell \left(\frac{r^2 + r'^2}{2rr'} \right) Y_{\ell,m}(\theta, \phi) Y_{\ell,m}^*(\theta', \phi'). \quad (2.53)$$

In equation (A.12) of [16], this correlator was evaluated in the Coloumb gauge. The authors evaluated the electric field correlator in Cartesian coordinates, transformed to polar coordinates and then decomposed the answer in spherical harmonics. The answer for each harmonic was not obtained in closed form unlike the result in (2.53). In the appendix A, we compare our result with their result and show that our closed form result for each ℓ agrees with the expansion found in [16].

From the discussion in section 2.1, we see that we need the 2-point functions on the sphere, in the limit that the two-points have the same radial position. Therefore let us take

$$r' = r + \delta, \quad \delta \rightarrow 0. \tag{2.54}$$

In this limit, the Legendre function of the second kind can be expanded as

$$\lim_{\delta \rightarrow 0} Q_\ell \left(\frac{r^2 + (r + \delta)^2}{2r(r + \delta)} \right) = \frac{1}{2} \log \left(\frac{r^2}{\delta^2} \right) + \log 2 - H_\ell + O(\delta), \tag{2.55}$$

where H_ℓ refers to the Harmonic number. Note that the leading divergence is independent of ℓ . Substituting this limit in the correlator (2.53), we obtain

$$\lim_{\delta \rightarrow 0} G_{rr}(r, r + \delta; x, y) = \frac{1}{4\pi r^4} \log \left(\frac{r^2}{\delta^2} \right) \sum_{\ell \geq 1, m} \ell(\ell + 1) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'), \tag{2.56}$$

where we retain only the leading term in the $\delta \rightarrow 0$ limit. Thus the correlator is diagonal in the angular momentum basis and the diagonal elements are given by

$$\lim_{\delta \rightarrow 0} \langle \ell, m | G_{rr}(r, r + \delta) | \ell', m' \rangle = \frac{\ell(\ell + 1)}{4\pi r^4} \log \left(\frac{r^2}{\delta^2} \right) \delta_{\ell, \ell'} \delta_{m, m'}. \tag{2.57}$$

As mentioned earlier, we see that the correlator is proportional to the Laplacian of the massless scalar on S^2 . From (2.13), we see that the edge mode contribution to the entanglement is obtained by evaluating the log-determinant of this operator. The coefficient which is proportional to the $\log(R/\epsilon)$ where R is the radius of the entangling sphere and ϵ is a cutoff on the sphere, can be obtained from

$$S_{\text{edge}}(\rho_A) = \frac{1}{2} \sum_{\ell=1}^{\infty} (2\ell + 1) \log [\ell(\ell + 1)]. \tag{2.58}$$

Here we are ignoring all terms which are proportional which grow as the area R^2/ϵ^2 and retained the term which contains the logarithmic coefficient to the entanglement entropy.

We can use the standard methods to evaluate the determinant of the scalar Laplacian to obtain the contribution to the entanglement entropy from the edge modes. The free energy of the of the massless scalar or the 0-form is given by

$$-\frac{1}{2} \log(\det \Delta_0^{S^2}) = -\frac{1}{2} \sum_{\ell=1}^{\infty} (2\ell + 1) \log(\ell(\ell + 1)). \tag{2.59}$$

Then using the methods of [19, 22] we can write the Harish-Chandra character integral representation of the free energy. This is given by

$$-\frac{1}{2} \log \det(\Delta_0^{S^2}) = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \chi_{(1,0)}^{dS}(t), \tag{2.60}$$

where $\chi_{(2,0)}^{dS}(t)$ is the $SO(1, 2)$ Harish-Chandra character of the 0-form in the $\Delta = \frac{1}{2} + i\nu$ representation with $i\nu = \frac{1}{2}$. This character is given by

$$\chi_{(1,0)}^{dS}(t) = \frac{1 + e^{-t}}{1 - e^{-t}}. \tag{2.61}$$

Therefore, the result for the free energy is

$$-\frac{1}{2} \log \det(\Delta_0^{S^2}) = \int_0^\infty dt \frac{1+e^{-t}}{2t} \frac{1+e^{-t}}{1-e^{-t}}. \quad (2.62)$$

In the character integral representation the coefficient of the $\log(\frac{R}{\epsilon})$ term to the free energy is easy to read out. It is given by the coefficient of the t^{-1} term of the integrand in (2.62) which is $\frac{1}{3}$. Therefore, the logarithmic contribution of the edge modes to the entanglement entropy is given by

$$\begin{aligned} S_{\text{edge}}(\rho_A) &= \frac{1}{2} \log \det(\Delta_0^{S^2}), \\ &= -\frac{1}{3} \log \frac{R}{\epsilon}. \end{aligned} \quad (2.63)$$

We see that this result agrees with that evaluated in [16]. Here we emphasize that this entanglement is the contribution of the electric centre of the local algebra as discussed in [3]. This resulted from our choice of labelling the superselection sectors using the electric Gauss law.

Magnetic correlator on the sphere and edge entanglement. As discussed in [3], the entanglement entropy of the centre of the algebra depends on the choice centre. We can also consider the case where the superselection sectors are labelled by the magnetic field, the magnetic centre. The magnetic field satisfies the condition

$$\nabla^i B_i = 0. \quad (2.64)$$

Following the same arguments as that for the electric field, this implies that the radial component of the $B_{\hat{r}}$ must agree across the entangling surface. Therefore the field strength $F_{\hat{e}\hat{m}} = B_{\hat{r}}$ labels superselection sectors. The arguments in section 2.1, then imply that we would need the two-point function of the magnetic field on the sphere to evaluate the entanglement entropy of the edge modes.

From (2.25), we see, that the magnetic flux is given by

$$F_{\hat{e}\hat{m}} = \sum_{\ell, m} \frac{\sqrt{\ell(\ell+1)}}{r} A_{m; \ell m} Y_{\ell m}. \quad (2.65)$$

Using the mode expansion in (2.44), the two-point function of the magnetic field is given by

$$\begin{aligned} \langle 0 | F_{\hat{e}\hat{m}}(t, r, \theta, \phi) F_{\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle &= \\ &= \frac{1}{2(rr')^{\frac{3}{2}}} \sum_{\ell, \ell', m, m'} \sqrt{\ell(\ell+1)\ell'(\ell'+1)} \int_0^\infty dk dk' \left[J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') \right. \\ &\quad \left. \times (-1)^{m'} \langle 0 | a_{m(\ell, m)}(k) a_{m(\ell', m')}^\dagger(k') | 0 \rangle Y_{\ell, m}(\theta, \phi) Y_{\ell', -m'}(\theta', \phi') \right]. \end{aligned} \quad (2.66)$$

Using the commutation relations in (2.47), we get

$$\begin{aligned} \langle 0 | F_{\hat{e}\hat{m}}(t, r, \theta, \phi) F_{\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle &= \\ &= \frac{1}{2(rr')^{\frac{3}{2}}} \sum_{\ell, m} \ell(\ell+1) \int_0^\infty dk J_{\ell+\frac{1}{2}}(kr) J_{\ell+\frac{1}{2}}(k'r') Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \end{aligned} \quad (2.67)$$

Then using the identity (2.51), we obtain

$$\langle 0|F_{\hat{e}\hat{m}}(t, r, \theta, \phi)F_{\hat{e}'\hat{m}'}(t, r', \theta', \phi')|0\rangle = \sum_{\ell, m} \frac{\ell(\ell+1)}{2\pi(r r')^2} Q_\ell \left(\frac{r^2 + r'^2}{2r r'} \right) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \quad (2.68)$$

Comparing (2.53) and (2.68) we see that they are identical. Following the same steps as in the case of the electric superselection sectors, we see that the logarithmic contribution to entanglement entropy when the magnetic field labels superselection sectors is given by

$$\begin{aligned} S_{\text{edge}}(\rho_A)|_{\text{magnetic}} &= \frac{1}{2} \log \det(\Delta_0^{S^2}), \\ &= -\frac{1}{3} \log \frac{R}{\epsilon}. \end{aligned} \quad (2.69)$$

Thus the entanglement entropy of the magnetic centre coincides with that of the electric centre. This is due to the electric magnetic duality of the U(1) theory. In general choice of different centres can result in different entanglement associated with the centre. In [3] it was shown in lattice gauge theory one could choose a trivial centre resulting no entanglement entropy of the centre.

2.3 U(1) theory in arbitrary even d

In this section we generalise the discussion to arbitrary even d . Let us expand the gauge potential as

$$\begin{aligned} A_\mu &= \sum_{\ell, \lambda, m} \left(A_{0(\ell, \lambda, m)}(r, t) Y_{\ell\lambda m, \mu}^0 + A_{r(\ell, \lambda, m)}(r, t) Y_{\ell\lambda m, \mu}^r \right. \\ &\quad \left. + A_{e(\ell, \lambda, m)}(r, t) \hat{Y}_{\ell\lambda m, \mu}^e + A_{\vec{m}(\ell, \lambda, m)}(r, t) Y_{\ell\lambda m, \mu}^{\vec{m}} \right), \end{aligned} \quad (2.70)$$

where the d dimensional covariant vectors are defined as

$$\begin{aligned} Y_{\ell\lambda m}^0 &= \{Y_{\ell\lambda m}(\Omega), 0, \dots, 0\}, \\ Y_{\ell\lambda m}^r &= \{0, Y_{\ell\lambda m}(\Omega), 0, \dots, 0\}, \\ Y_{\ell\lambda m}^e &= \frac{r}{\sqrt{\ell(\ell + \hat{d} - 1)}} \{0, 0, \partial_i Y_{\ell\lambda m}(\Omega)\}, \\ \hat{Y}_{\ell\lambda m}^{\vec{m}} &= \{0, 0, Y_{\ell\lambda m}^{\vec{m}}(\Omega)\}. \end{aligned} \quad (2.71)$$

$Y_{\ell\lambda m}$ are scalar spherical harmonics on $S^{\hat{d}}$, l is the principal quantum number, m the azimuthal quantum number and λ refers to the rest of the $\hat{d} - 2$ quantum numbers. It is important to note that the azimuthal quantum number can take values both in the positive and negative values in the set of integers. These harmonics can be found in [23]. The principal quantum number takes values $\ell = 0, 1, \dots$. The variable Ω refers to all the angles on the sphere $S^{\hat{d}}$. We work with the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad (2.72)$$

where the metric on sphere is the conventional round metric. The partial derivative in the definition of $Y_{\ell\lambda m}^e$ refers to derivative in the angular directions of the sphere. Finally

$Y_{\ell\lambda m}^{\bar{m}}(\Omega)$ refers to the other $\hat{d} - 1$ vector harmonics on $S^{\hat{d}}$. The properties of both the scalar and vector harmonics can be found in [23]. From the construction³ of these harmonics we see that they obey the property

$$Y_{\ell\lambda m}(\Omega)^* = (-1)^m Y_{\ell\lambda -m}(\Omega). \quad (2.73)$$

Then the reality of the vector potential implies that we have the relations

$$\begin{aligned} A_{0(\ell,\lambda,m)}^*(r, t) &= (-1)^m A_{0(\ell,\lambda,m)}(r, t), \\ A_{r(\ell,\lambda,m)}^*(r, t) &= (-1)^m A_{r(\ell,\lambda,m)}(r, t). \end{aligned} \quad (2.74)$$

Following the same discussion as in the case of 4d, we can fix gauge so that $A_e = 0$. It is sufficient to focus on the components A_0 and A_r to obtain the entanglement entropy of the superselection sectors corresponding to the electric field. The action for these components derived from the Maxwell action is given by

$$\begin{aligned} S &= \sum_{\ell\lambda m=-\ell}^{\ell} \int dr L_{\ell\lambda m}, \\ L_{\ell\lambda m} &= \frac{1}{2} \left[r^{\hat{d}} \dot{A}_{r(\ell,\lambda,m)} \dot{A}_{r(\ell,\lambda,m)}^* + r^{\hat{d}} \partial_r A_{0(\ell,\lambda,m)} \partial_r A_{0(\ell,\lambda,m)}^* \right. \\ &\quad \left. - r^{\hat{d}} \dot{A}_{r(\ell,\lambda,m)} \partial_r A_{0(\ell,\lambda,m)}^* - r^{\hat{d}} \dot{A}_{r(\ell,\lambda,m)}^* \partial_r A_{0(\ell,\lambda,m)} + \ell(\ell + \hat{d} - 1) r^{\hat{d}-2} A_{0(\ell\lambda m)} A_{0(\ell,\lambda,m)}^* \right]. \end{aligned} \quad (2.75)$$

The canonical conjugate momentum to A_r is given by

$$\pi_{(\ell,\lambda,m)}^r = r^{\hat{d}} (\dot{A}_{r(\ell,\lambda,m)} - \partial_r A_{0(\ell,\lambda,m)}). \quad (2.76)$$

Using the constraint for A_0 , we can write

$$\partial_r \pi_{(\ell,\lambda,m)}^r + \ell(\ell + \hat{d} - 1) A_{0(\ell,\lambda,m)} = 0. \quad (2.77)$$

Eliminating A_0 , we obtain the equation of motion for the k -th Fourier mode A_r , which is given by

$$\partial_r^2 A_{r(\ell,\lambda,m)} + \frac{\hat{d} - 2}{r} \partial_r A_{r(\ell,\lambda,m)} + \left(k^2 - \frac{\ell(\ell + \hat{d} - 1) + \hat{d} - 2}{r^2} \right) A_{r(\ell,\lambda,m)} = 0. \quad (2.78)$$

The solution which is regular at the origin is given by

$$A_{r(\ell,\lambda,m)} = e^{-ikt} a_{r(\ell,\lambda,m)} r^{-\frac{\hat{d}-3}{2}} J_{\ell+\frac{\hat{d}-1}{2}}(|k|r), \quad (2.79)$$

where $a_{r(\ell,\lambda,m)}(k)$ is the integration constant. The equation for the momenta can be obtained by using the definition in (2.76), eliminating A_0 and substituting for A_r from (2.79). This results in

$$\pi_{(\ell,\lambda,m)}^r = r^{\hat{d}} \left(-ik e^{-ikt} a_{r(\ell,\lambda,m)}(k) r^{-\frac{\hat{d}-3}{2}} J_{\ell+\frac{\hat{d}-1}{2}}(|k|r) + \frac{1}{\ell(\ell + \hat{d} - 1)} \partial_r \left(r^{\hat{d}-2} \partial_r \pi_{(\ell,\lambda,m)}^r \right) \right). \quad (2.80)$$

³The scalar harmonics are products of associated Legendre functions of the first kind and a phase. The phase and one of the associated Legendre functions can be grouped to form the spherical harmonic on S^2 . The property of these functions under conjugations can be obtained from this observation.

The solution to this differential equation is given by

$$\begin{aligned} \pi_{(\ell,\lambda,m)}^{r*} &= c_1 r^{\hat{d}-1+\ell} + c_2 r^{-\ell} \\ &+ a_{r(\ell,\lambda,m)}(k) e^{-ikt} \left(\frac{i 2^{-(\ell+\frac{\hat{d}-1}{2})} \ell(\ell+\hat{d}-1) k^{\ell+\frac{\hat{d}-3}{2}}}{\Gamma(\ell+\frac{\hat{d}+1}{2})} r^{\ell+\hat{d}+1} - i \frac{\ell(\ell+\hat{d}-1) r^{\frac{\hat{d}-1}{2}}}{k} J_{\ell+\frac{\hat{d}-1}{2}}(|k|r) \right). \end{aligned} \quad (2.81)$$

$\ell \geq 1$ for these modes, regularity at the origin results in $c_2 = 0$ and demanding that the solution is well behaved at infinity determines c_1 giving

$$\pi_{(\ell,\lambda,m)}^{r*} = -i \frac{\ell(\ell+\hat{d}-1)}{k} a_{r(\ell,\lambda,m)}(k) e^{-ikt} r^{\frac{\hat{d}-1}{2}} J_{\ell+\frac{\hat{d}-1}{2}}(|k|r). \quad (2.82)$$

Using the classical solutions (2.79) and (2.82) for each Fourier mode, we can write down the mode expansions

$$\begin{aligned} A_{r(\ell,\lambda,m)}(r,t) &= \frac{1}{\sqrt{2}} \int_0^\infty k dk \left(a_{r(\ell,\lambda,m)}(k) e^{-ikt} + (-1)^m a_{r(\ell,\lambda,-m)}^\dagger(k) e^{ikt} \right) r^{-\frac{\hat{d}-3}{2}} J_{\ell+\frac{\hat{d}-1}{2}}(kr), \\ \pi_{(\ell,\lambda,m)}^{r*}(r,t) &= \frac{\ell(\ell+\hat{d}-1)}{\sqrt{2}} \\ &\times \int_0^\infty dk \left(-i a_{r(\ell,\lambda,m)}(k) e^{-ikt} + i (-1)^m a_{r(\ell,\lambda,-m)}^\dagger(k) e^{ikt} \right) r^{\frac{\hat{d}-1}{2}} J_{\ell+\frac{\hat{d}-1}{2}}(kr). \end{aligned} \quad (2.83)$$

This form for the mode expansion respects the reality condition (2.74). Let us also write the mode expansion

$$\pi_{(\ell,m)}^r(r,t) = \frac{\ell(\ell+\hat{d}-1)}{\sqrt{2}} \int_0^\infty dk \left(i a_{r(\ell,\lambda,m)}^\dagger(k) e^{ikt} - i (-1)^m a_{r(\ell,\lambda,-m)}(k) e^{-ikt} \right) r^{\frac{\hat{d}-1}{2}} J_{\ell+\frac{1}{2}}(kr). \quad (2.84)$$

The equal time commutation relation of the fields A_r, π^r is given by

$$[A_{r(\ell,\lambda,m)}(r,t), \pi_{r(\ell',\lambda',m')}^r(r,t)] = i \delta_{\ell,\ell'} \delta_{\lambda,\lambda'} \delta_{m,m'} \delta(r-r'). \quad (2.85)$$

These relations together with the mode expansions (2.83), (2.84) imply the following commutation relations between the creation and annihilation operators

$$[a_{r(\ell,\lambda,m)}(k), a_{r(\ell',\lambda',m')}(k')] = \frac{1}{\ell(\ell+\hat{d}-1)} \delta_{\ell\ell'} \delta_{\lambda,\lambda'} \delta_{m,m'}. \quad (2.86)$$

Electric field correlators on $S^{\hat{d}}$. The electric field is related to the canonical momentum by

$$\begin{aligned} F_{\hat{t}\hat{r}} &= \sum_{\ell,\lambda,m} \left(\dot{A}_{r(\ell,\lambda,m)} - \partial_r A_{0(\ell,\lambda,m)} \right) Y_{\ell\lambda m}(\Omega), \\ &= \sum_{\ell,\lambda,m} \frac{\pi_{(\ell,\lambda,m)}^{r*}}{r^{\hat{d}}} Y_{\ell\lambda m}(\Omega), \end{aligned} \quad (2.87)$$

where we have used (2.76). Proceeding as in the case of $d = 4$, we evaluate the two-point function of the radial electric field

$$\langle 0|F_{\hat{t}\hat{r}}(t, r, \Omega)F_{\hat{t}\hat{r}'}(t, r', \Omega')|0\rangle = \sum_{\ell, \lambda, m} \frac{\ell(\ell + \hat{d} - 1)}{2\pi(r r')^{\frac{d}{2}}} Q_{\ell} \left(\frac{r^2 + r'^2}{2r r'} \right) Y_{\ell, \lambda, m}(\Omega) Y_{\ell, \lambda, m}^*(\Omega'). \quad (2.88)$$

The result is a generalisation of that seen for the case of $d = 4$ in (2.53), we see that the correlator is proportional to the Laplacian of the scalar or the 0-form on the sphere $S^{\hat{d}}$. We can take the coincident limit in the radial direction. From the leading divergence we evaluate the contribution of the edge modes to the entanglement entropy. The logarithmic coefficient is given by

$$S_{\text{edge}}(\rho_A) = \frac{1}{2} \sum_{\ell=1}^{\infty} g_{\ell, d} \log(\ell(\ell + \hat{d} - 1)), \quad g_{\ell, d} = \frac{(2\ell + d - 3)\Gamma(\ell + d - 3)}{\ell! \Gamma(d - 2)}, \quad (2.89)$$

where $g_{\ell, d}$ are the degeneracies of the Laplacian of the 0-form on $S^{\hat{d}}$. Note that the $\ell = 0$ mode is not counted, and therefore the logarithmic coefficient entanglement entropy of the edge modes coincides precisely with that of the free energy of the 0-form. Using the methods of [19, 22], the partition function of the 0-form can be written in terms of its Harish-Chandra characters

$$-\frac{1}{2} \log \det(\Delta_0^{\text{S}^{\hat{d}}}) = \int_0^{\infty} \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \chi_{\hat{d}, 0}. \quad (2.90)$$

Here $\chi_{\hat{d}, 0}$ is the $\text{SO}(\hat{d}, 1)$ Harish-Chandra character of the 0-form in the $\Delta = \frac{\hat{d}-1}{2} + i\nu$ representation with $i\nu = \frac{\hat{d}-1}{2}$ which is given by

$$\chi_{\hat{d}, 0}^{\text{dS}}(t) = \frac{1 + e^{-(\hat{d}-1)t}}{1 - e^{-(\hat{d}-1)t}}. \quad (2.91)$$

Therefore the logarithmic coefficient of the entanglement entropy is obtained from the $1/t$ coefficient of the integrand in

$$\begin{aligned} S_{\text{edge}}(\rho_A) &= \frac{1}{2} \log \det(\Delta_0^{\text{S}^{\hat{d}}}), \\ &= - \int_0^{\infty} \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \frac{1 + e^{-(\hat{d}-1)t}}{1 - e^{-(\hat{d}-1)t}}. \end{aligned} \quad (2.92)$$

2.4 The U(1) theory on spheres and entanglement of edge modes

In this section we consider the representation of the partition function the U(1) theory on the even d dimensional sphere in terms of the Harish-Chandra character as constructed in [19, 22]. We show that the contribution to the partition function from the edge character coincides with the contribution of the superselection sectors to the entanglement entropy. After gauge fixing, the partition function of the U(1) theory on S^d can be written as

$$\log \mathcal{Z}_1[S^d] = -\frac{1}{2} \log(\det_{\text{T}} \Delta_1^{\text{S}^d}) + \frac{1}{2} \log(\det' \Delta_0^{\text{S}^d}). \quad (2.93)$$

Here Δ_1 is the Laplacian on 1-forms and the subscript T refers to the fact that the determinant is evaluated over transverse 1-forms. The prime in the second term is to denote that the determinant ignores the zero mode of the scalar. In [22] it was shown that the partition function of p -forms can be written in terms of Harish-Chandra characters. Using these results we get

$$\begin{aligned}
 -\frac{1}{2} \log(\det_T \Delta_1^{S^d}) &= \int_0^\infty \frac{dt}{2t} \frac{1+e^{-t}}{1-e^{-t}} \left(\chi_{(d-1,1)}^{dS}(t) - \chi_{(d-3,0)}^{dS}(t) \right), & (2.94) \\
 \chi_{(d-1,1)}^{dS}(t) &= (d-1) \frac{e^{-(d-2)t} + e^{-t}}{(1-e^{-t})^{d-1}}, & \chi_{(d-3,0)}^{dS}(t) &= \frac{e^{-(d-3)t} + 1}{(1-e^{-t})^{d-3}}, \\
 -\frac{1}{2} \log(\det' \Delta_0^{S^d}) &= \int_0^\infty \frac{dt}{2t} \frac{1+e^{-t}}{1-e^{-t}} \chi_{(d-1,0)}^{dS}(t), & \chi_{(d-1,0)}^{dS}(t) &= \frac{e^{-(d-1)t} + 1}{(1-e^{-t})^{d-1}}.
 \end{aligned}$$

From this representation of the one loop partition function of the 1-form, we see that it can be written as a contribution from bulk and edge characters as

$$\begin{aligned}
 -\frac{1}{2} \log(\det_T \Delta_1^{S^d}) &= \int_0^\infty \frac{dt}{2t} \frac{1+e^{-t}}{1-e^{-t}} (\chi_{\text{bulk}}(t) - \chi_{\text{edge}}(t)), & (2.95) \\
 \chi_{\text{bulk}}(t) &= \chi_{(d-1,1)}^{dS}(t) - \chi_{(d-1,0)}^{dS}(t), \\
 \chi_{\text{edge}}(t) &= \chi_{(d-3,0)}^{dS}(t).
 \end{aligned}$$

Note that the edge character is the Harish-Chandra character for the 0-form on a sphere of $d-2 = \hat{d}$ dimensions. Comparison with (2.92), we see that its contribution to the logarithmic coefficient of the free energy of the U(1) theory on S^d including the sign agrees precisely with that of the entanglement entropy of the edge modes.

3 Graviton in $d = 4$ dimension

In this section we evaluate the contribution of the edge modes to the logarithmic coefficient of the entanglement entropy of gravitons across a spherical surface in $d = 4$. Our discussion will closely follow that of the U(1) theory. The Lagrangian for the theory of linearized gravitons is given by

$$\mathcal{L} = -\partial_\mu h^{\mu\nu} \partial_\alpha h_\nu^\alpha + \frac{1}{2} \partial^\alpha h_{\mu\nu} \partial_\alpha h^{\mu\nu} + \partial_\mu h^{\mu\nu} \partial_\nu h_\alpha^\alpha - \frac{1}{2} \partial_\alpha h_\mu^\mu \partial^\alpha h_\nu^\nu. \quad (3.1)$$

This Lagrangian admits the gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (3.2)$$

In the linearized theory, the curvature is gauge-invariant. It is given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\mu \partial_\rho h_{\nu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\nu \partial_\sigma h_{\mu\rho}). \quad (3.3)$$

The equations of motion from the action (3.1) are the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = 0. \quad (3.4)$$

In the case of the Maxwell field, the Gauss law resulted in superselection sectors. To obtain a gauge-invariant characterization of the superselection sectors in gravity it is convenient to think of the Riemann curvature (3.3) as a field strength of a U(1) gauge potential. Consider the gauge potential constructed out of the graviton

$$A_{\mu\alpha\beta} = \frac{1}{2}(\partial_\beta h_{\mu\alpha} - \partial_\alpha h_{\mu\beta}). \quad (3.5)$$

Under the gauge symmetry (3.2) the potential transforms as an U(1) gauge field

$$\delta A_{\mu\alpha\beta} = \partial_\mu \lambda_{\alpha\beta}, \quad \lambda_{\alpha\beta} = \frac{1}{2}(\partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta). \quad (3.6)$$

The Riemann curvature can then be written as a field strength of this U(1) gauge field.

$$R_{\mu\nu\alpha\beta} = \partial_\mu A_{\nu\alpha\beta} - \partial_\nu A_{\mu\alpha\beta}. \quad (3.7)$$

Indeed in [24], the components R_{0i0j} where i, j are spatial indices were identified as the electric fields while $B_{ij} = \frac{1}{2}\epsilon_{ilm}R_{0j}^{lm}$ were identified as magnetic fields.

We show that the Riemann curvature $R_{0\mu\alpha\beta}$ obeys a similar Gauss law as that in electrodynamics once the Hamiltonian constraint together with the on shell conditions are satisfied. The Gauss constraint or the Hamiltonian constraint in gravity are the following

$$R_{0i} = 0, \quad (3.8)$$

$$R_{00} + \frac{1}{2}R = 0. \quad (3.9)$$

Consider the Bianchi identity in linearised gravity,

$$\partial_\lambda R_{\mu\nu\alpha\beta} + \partial_\mu R_{\nu\lambda\alpha\beta} + \partial_\nu R_{\lambda\mu\alpha\beta} = 0. \quad (3.10)$$

Contracting μ and α we obtain

$$\partial_\lambda R_{\nu\beta} + \partial^\mu R_{\nu\lambda\mu\beta} - \partial_\nu R_{\lambda\beta} = 0. \quad (3.11)$$

We choose β to be in the time direction and ν, λ to be space like. Then the Hamiltonian constraint (3.8) implies that we obtain the following constraint that must be satisfied by the curvature

$$\partial^\mu R_{0\mu ij} = 0. \quad (3.12)$$

Now choose ν, β in (3.11) to be in the time direction then the Hamiltonian constraint (3.9) together with the on shell conditions imply

$$\partial^\mu R_{0\mu 0i} = 0. \quad (3.13)$$

The equations (3.12) and (3.13) are analogous to the Gauss law constraint of electrodynamics. As we have seen that these fields are field strengths of a U(1) gauge potential. Therefore the same argument that led to the labelling of superselection sectors by the radial component of the electric or magnetic field applies to these curvature components. Thus

we conclude that superselection sectors are labelled by the components $R_{0\hat{r}ij}, R_{0\hat{r}0j}$ of the Riemann tensor.

In the next sub-sections we will quantize the linearise graviton following the methods of [18]. We decompose the graviton in terms of spherical tensor harmonics and fix gauge and obtain the algebra of gauge-invariant observables. We will show that among the 6 components $R_{0\hat{r}ij}, R_{0\hat{r}0i}$, only the components $R_{0\hat{r}0\hat{r}}$ and $R_{0\hat{r}\hat{e}\hat{m}}$ where \hat{e}, \hat{m} are angular directions are locally related to the canonical coordinates. We then choose these components to label the superselection sectors and evaluate their entanglement entropy. Since both $R_{0\hat{r}0\hat{r}}$ and $R_{0\hat{r}\hat{e}\hat{m}}$ label the superselection sectors, we would need to sum their contribution to the entanglement entropy of the edge modes. It is important to note that both these superselection sectors arose from the Hamiltonian constraints (3.8), (3.9) of gravity.

3.1 Lagrangian in terms of tensor harmonics

In this section, we expand the graviton in terms of tensor harmonics and decompose the Lagrangian in (3.1) in terms of these modes and fix gauge. We follow the methods of [18] with the difference that we write the tensor harmonics in terms of covariant tensors rather than cartesian tensors. This allows us to apply the methods of tensor calculus.

The spin-2 field is first written as

$$h_{\mu\nu} = (h_T)_{\mu\nu} + (\hat{h}_v)_{\mu\nu} + (h_s)_{\mu\nu}, \quad (3.14)$$

where h_T is the tensor mode which is purely spatial, h_v is the vector mode which is non-zero when one index is temporal and one spatial, while h_s is non-zero only when both indices are temporal. The gauge transformation parameter is similarly decomposed as

$$\xi_\mu = (\xi_v)_\mu + (\xi_s)_\mu. \quad (3.15)$$

Here ξ_v has only spatial components and ξ_s has only temporal components.

The tensor mode. We use tensor harmonics as a basis of symmetric tensors to expand the tensor mode h_T as follows

$$(h_T)_{\mu\nu} = \sum_{J,s,\ell,m} h_{(\ell,m)}^{J,s} (T_{\ell,m}^{J,s})_{\mu\nu}, \quad (3.16)$$

where $T_{(\ell,m)}^{J,s}$ is the tensor harmonics constructed out of the vector harmonics. The covariant form of the tensor harmonics are given by⁴

$$\begin{aligned} (T_{\ell m}^{0l})_{\mu\nu} &= \hat{r}_\mu \otimes Y_{\ell m, \nu}^r, & (T_{\ell, m}^{0t})_{\mu\nu} &= \frac{Y_{\ell m}(\theta, \phi)}{\sqrt{2}} (\delta_{\mu,0}\delta_{0,\nu} - \delta_{\mu,1}\delta_{1,\nu} + g_{\mu\nu}), \\ (T_{\ell, m}^{1e})_{\mu\nu} &= \sqrt{2}[\hat{r}_\mu \otimes Y_{\ell, m, \nu}^e]^S, & (T_{\ell, m}^{1m})_{\mu\nu} &= \sqrt{2}[\hat{r}_\mu \otimes Y_{\ell, m; \nu}^m]^S, \\ (T_{\ell, m}^{2e})_{\mu\nu} &= \sqrt{\frac{2}{(l-1)(l+2)}} \left[[r\nabla_\mu Y_{\ell, m, \nu}^e]^S + \frac{1}{\sqrt{2}}(T_{\ell, m}^{1e})_{\mu\nu} + \sqrt{\frac{l(l+1)}{2}}(T_{\ell, m}^{0t})_{\mu\nu} \right], \\ (T_{\ell m}^{2m})_{\mu\nu} &= \sqrt{\frac{2}{(l-1)(l+2)}} \left\{ [r\nabla_\mu Y_{\ell, m, \nu}^m]^S + \frac{1}{\sqrt{2}}(T_{\ell, m}^{1m})_{\mu\nu} \right\}. \end{aligned} \quad (3.17)$$

⁴We define $[V_\mu \otimes W_\nu]^S = \frac{1}{2}(V_\mu \otimes W_\nu + V_\nu \otimes W_\mu)$.

The covariant form of the vector harmonics and the unit orthonormal vectors are given in (2.18) and (2.20) respectively. Here for the mode $J = 0$ we have $l \geq 0$, $J = 1$, $l \geq 1$ and $J = 2$, we have $l \geq 2$. Under the gauge transformation the tensor mode of the field transforms as

$$(h_T)_{\mu\nu} = \sum_{J,s,\ell,m} h_{(\ell,m)}^{J,s} (T_{\ell,m}^{J,s})_{\mu\nu} + 2 \sum_{s,\ell,m} \left[\xi_{(\ell,m)}^s \nabla_\mu Y_{\ell m, \nu}^s + Y_{\ell m, \mu}^s \otimes \partial_r \xi_{(\ell,m)}^s \hat{r}_\nu \right]^S, \quad s = \mathbf{r}, \mathbf{e}, \mathbf{m} \quad (3.18)$$

Here we have also decomposed the vector gauge parameter ξ_ν whose index takes values in the spatial directions in terms of vector harmonics.

$$(\xi_\nu)_\mu = \sum_{s,\ell,m} \xi_{(\ell,m)}^s Y_{\ell m, \mu}^s. \quad (3.19)$$

Using the properties of vector and spherical harmonics, one can express the gauge transformed variable h'_T in terms of modes $h_{(\ell,m)}^{J,s}$ and gauge parameters $\xi_{(\ell,m)}^s$.

$$\begin{aligned} (h'_T)_{\mu\nu} = & \sum_{\ell,m} \left(h_{(\ell,m)}^{0l} + 2\partial_r \xi_{(\ell,m)}^{\mathbf{r}} \right) (T_{\ell,m}^{0l})_{\mu\nu} + \left(h_{(\ell,m)}^{0t} + \frac{2\sqrt{2}}{r} \xi_{(\ell,m)}^{\mathbf{r}} - \frac{\sqrt{2\ell(\ell+1)}}{r} \xi_{(\ell,m)}^{\mathbf{e}} \right) (T_{\ell,m}^{0t})_{\mu\nu} \\ & + \left(h_{(\ell,m)}^{1e} + \frac{\sqrt{2\ell(\ell+1)}}{r} \xi_{(\ell,m)}^{\mathbf{r}} + \sqrt{2}\partial_r \xi_{(\ell,m)}^{\mathbf{e}} - \frac{\sqrt{2}}{r} \xi_{(\ell,m)}^{\mathbf{e}} \right) T_{(\ell,m)}^{1e} \\ & + \left(h_{(\ell,m)}^{2e} + \frac{\sqrt{2(\ell-1)(\ell+2)}}{r} \xi_{(\ell,m)}^{\mathbf{e}} \right) T_{(\ell,m)}^{2e} + \left(h_{(\ell,m)}^{2m} + \frac{\sqrt{2(\ell-1)(\ell+2)}}{r} \xi_{(\ell,m)}^{\mathbf{m}} \right) (T_{\ell,m}^{2m})_{\mu\nu} \\ & + \left(h_{(\ell,m)}^{1m} + \sqrt{2}\partial_r \xi_{(\ell,m)}^{\mathbf{m}} - \frac{\sqrt{2}}{r} \xi_{(\ell,m)}^{\mathbf{m}} \right) (T_{\ell,m}^{1m})_{\mu\nu}. \end{aligned} \quad (3.20)$$

Note that $\xi_{(\ell,m)}^{\mathbf{m}}$ allows us to cancel the coefficient of $T_{(\ell,m)}^{2m}$, for all ℓ and m . It is clear from this transformation, that we can also use $\xi^{\mathbf{m}}$ and $\xi^{\mathbf{r}}$ to eliminate two modes out of $h_{(\ell,m)}^{0l}$, $h_{(\ell,m)}^{0t}$ and $h_{(\ell,m)}^{1e}$. Following [18], we fix gauge so that one linear combination of these fields remains. We call this linear combination the mode $h_{(\ell,m)}^{te}$. Thus fixing gauge we can expand the tensor mode as

$$(h_T)_{\mu\nu} = \sum_{\ell,m} h_{(\ell,m)}^{0l} (T_{\ell,m}^{0l})_{\mu\nu} + h_{(\ell,m)}^{te} \left(\alpha (T_{\ell,m}^{0t})_{\mu\nu} + \beta (T_{\ell,m}^{1e})_{\mu\nu} + \gamma (T_{\ell,m}^{2e})_{\mu\nu} \right) + h_{\ell,m}^{1m} (T_{\ell,m}^{1m})_{\mu\nu}, \quad (3.21)$$

where α , β and γ are some constants.

The vector mode. The vector mode $(\hat{h}_\nu)_{\mu\nu}$ is first written as

$$(\hat{h}_\nu)_{\mu\nu} = [(h_\nu)_\mu \otimes \hat{t}_\nu]^S, \quad (3.22)$$

where \hat{t} is the covariant vector given in (2.20). We expand the vector h_ν in terms of vector harmonics given in (2.18).

$$(h_\nu)_\mu = \sum_{s\ell m} h_{(\ell,m)}^{0s}(t, r) Y_{\ell m, \mu}^s(\theta, \phi). \quad (3.23)$$

Let also expand the scalar gauge transformation parameter as

$$(\xi_s)_\mu = \sum_{\ell,m} \xi_{(\ell,m)}^0(t,r) Y_{\ell m}(\theta,\phi) \hat{t}_\mu. \quad (3.24)$$

Under the gauge transformation, the vector mode transforms as

$$\begin{aligned} (h'_v)_\mu &= \sum_{\ell,m} \left(h_{(\ell,m)}^{0r} + \dot{\xi}_{(\ell,m)}^r + \partial_r \xi_{(\ell,m)}^0 \right) Y_{\ell,m,\mu}^r + \left(h_{(\ell,m)}^{0e} + \dot{\xi}_{(\ell,m)}^e + \frac{\xi_{(\ell,m)}^0}{r} \right) Y_{\ell,m,\mu}^e \\ &+ \left(h_{(\ell,m)}^{0m} + \dot{\xi}_{(\ell,m)}^m \right) Y_{\ell,m,\mu}^m. \end{aligned} \quad (3.25)$$

The gauge transformation allows us to choose $\xi_{(\ell,m)}^0$ such that $h_{(\ell,m)}^{0e}$ vanishes for all ℓ and m . Using this gauge choice the vector mode can be written as

$$(h_v)_\mu = \sum_{\ell m} h_{(\ell,m)}^{0r} Y_{\ell,m,\mu}^r + h_{(\ell,m)}^{0m} Y_{\ell,m,\mu}^m. \quad (3.26)$$

The scalar mode. Finally the scalar mode is also expanded in terms of spherical harmonics

$$(h_s)_{\mu\nu} = \sum_{\ell,m} h_{(\ell,m)}^{00} Y_{\ell,m}(\theta,\phi) \hat{t}_\mu \otimes \hat{t}_\nu. \quad (3.27)$$

We now substitute the tensor harmonic expansion of $h_{\mu\nu}$ in the Lagrangian given in (3.1). The Lagrangian can be written in two parts [18].

$$\int d^3x \mathcal{L} = \sum_{\ell,m=-\ell}^{\ell} \int_0^\infty dr \left(\mathcal{L}_{\ell,m}^{(1)} + \mathcal{L}_{\ell,m}^{(2)} \right), \quad (3.28)$$

where $\mathcal{L}^{(1)}$ contains only the fields $h_{(\ell,m)}^{1m}$ and the non-dynamical modes $h_{(\ell,m)}^{0m}$. The second part $\mathcal{L}^{(2)}$ involves the fields $h_{(\ell,m)}^{0l}$ and $h_{(\ell,m)}^{te}$ together with the Lagrange multipliers $h_{(\ell,m)}^{0r}$ and $h_{(\ell,m)}^{00}$. To write the Lagrangian explicitly in terms of the modes we use the following reality property of the tensor harmonics

$$\hat{T}_{\ell,m}^{J,s*} = (-1)^m \hat{T}_{\ell,-m}^{J,s}. \quad (3.29)$$

Then the reality of the tensor $h_{\mu\nu}$ leads to the following reality conditions obeyed by the modes.

$$h_{(\ell,-m)}^{1m} = (-1)^m h_{(\ell,m)}^{1m*}, \quad h_{(\ell,m)}^{te*} = (-1)^m h_{(\ell,m)}^{te}, \quad (3.30)$$

$$h_{(\ell,-m)}^{0l} = (-1)^m h_{(\ell,m)}^{0l*}, \quad h_{(\ell,-m)}^{0t} = (-1)^m h_{(\ell,m)}^{0t*}. \quad (3.31)$$

The Lagrangian $\mathcal{L}^{(1)}$ and its equations of motion. The Lagrangian $\mathcal{L}^{(1)}$ is given by

$$\begin{aligned} \mathcal{L}_{(\ell,m)}^{(1)} &= \frac{r^2}{2} \dot{h}_{(\ell,m)}^{1m} \dot{h}_{(\ell,m)}^{1m*} - \frac{(\ell-1)(\ell+2)}{2} h_{(\ell,m)}^{1m} h_{(\ell,m)}^{1m*} + r^2 \partial_r h_{(\ell,m)}^{0m} \partial_r h_{(\ell,m)}^{0m*} \\ &+ \ell(\ell+1) h_{(\ell,m)}^{0m} h_{(\ell,m)}^{0m*} + \sqrt{2} \dot{h}_{(\ell,m)}^{1m*} \left(r h_{(\ell,m)}^{0m} - r^2 \partial_r h_{(\ell,m)}^{0m} \right). \end{aligned} \quad (3.32)$$

Here ℓ runs from $1, 2, \dots \infty$. Note that though the cross term between h^{0m} and h^{1m} appears complex, the sum over m from $-\ell$ to ℓ in (3.28) ensures the reality of the full Lagrangian. The canonical conjugate momentum to h^{1m} by varying the Lagrangian with respect to $\dot{h}_{(\ell,m)}^{1m*}$

$$\begin{aligned} \pi_{(\ell,m)}^{1m*} &= \frac{\partial \mathcal{L}_{(\ell,m)}^{(1)}}{\partial \dot{h}_{(\ell,m)}^{1m*}} \\ &= r^2 \dot{h}_{(\ell,m)}^{1m} + \sqrt{2}(r h_{(\ell,m)}^{0m} - r^2 \partial_r h_{(\ell,m)}^{0m}). \end{aligned} \quad (3.33)$$

Let us discuss the modes $\ell \geq 2$ first. From the Lagrangian (3.32) we obtain

$$r^2 \ddot{h}_{(\ell,m)}^{1m} + \sqrt{2}(r \dot{h}_{(\ell,m)}^{0m} - r^2 \partial_r \dot{h}_{(\ell,m)}^{0m}) + (\ell - 1)(\ell + 2)h_{(\ell,m)}^{1m} = 0, \quad (3.34)$$

which can also be written as

$$\dot{\pi}_{(\ell,m)}^{1m*} + (\ell - 1)(\ell + 2)h_{(\ell,m)}^{1m} = 0. \quad (3.35)$$

The mode h^{0m} can be eliminated by the constraint equation which is given by

$$\sqrt{2} \left(\partial_r \pi_{(\ell,m)}^{1m*} + \frac{\pi_{(\ell,m)}^{1m*}}{r} \right) = -2(\ell - 1)(\ell + 2)h_{(\ell,m)}^{0m}. \quad (3.36)$$

Finally we get the equation of motion only in $h_{(\ell,m)}^{1m}$ variable

$$r^2 (\ddot{h}_{(\ell,m)}^{1m} - \partial_r^2 h_{(\ell,m)}^{1m}) + \ell(\ell + 1)h_{(\ell,m)}^{1m} = 0. \quad (3.37)$$

We solve this equation of motion by expanding it in Fourier modes in time first and then solve the radial equation. The solution which is regular at the origin is given by

$$h_{(\ell,m)}^{1m}(t, r) = e^{-ikt} a_{1m(\ell,m)}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \quad (3.38)$$

Here $a_{1m(\ell,m)}(k)$ is the integration constant and $J_{\ell+\frac{1}{2}}(|k|r)$ is the Bessel function. The equation of $\pi_{(\ell,m)}^{1m*}$ can be obtained from (3.33)

$$\begin{aligned} \pi_{(\ell,m)}^{1m*} &= r^2 \left(-ike^{-ikt} a_{1m(\ell,m)}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r) \right) + \sqrt{2}(r h_{(\ell,m)}^{0m} - r^2 \partial_r h_{(\ell,m)}^{0m}) \\ &= r^2 \left(-ike^{-ikt} a_{1m(\ell,m)}(k) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r) \right) + \frac{r^2 \partial_r^2 \pi_{(\ell,m)}^{1m*} - 2\pi_{(\ell,m)}^{1m*}}{(\ell - 1)(\ell + 2)}. \end{aligned} \quad (3.39)$$

In the second line we have used the constraint equation (3.36) to eliminate $h_{(\ell,m)}^{0m}$ from the equation of motion. Therefore the equation for π^{1m*} becomes an in-homogenous second order equation in the radial coordinate. The general solution is given by

$$\begin{aligned} \pi_{(\ell,m)}^{1m*}(r, t) &= c_1 r^{\ell+1} + c_2 r^{-\ell} \\ &+ (\ell - 1)(\ell + 2) a_{1m(\ell,m)}(k) e^{-ikt} \left(\frac{ir 2^{-\ell-\frac{1}{2}} (kr)^\ell}{\sqrt{k} \Gamma\left(\ell + \frac{3}{2}\right)} - \frac{i \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r)}{k} \right). \end{aligned} \quad (3.40)$$

Demanding that the solution be regular at the origin yields $c_2 = 0$ since $\ell \geq 2$ for these modes. We further demand the regularity of the solution at infinity which fixes c_1 and we obtain

$$\pi_{(\ell,m)}^{1m*}(r,t) = -\frac{i(\ell-1)(\ell+2)}{k} a_{1m(\ell,m)}(k) e^{-ikt} \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \quad (3.41)$$

For $\ell = 1$, the equations of motion (3.36) leads to the following solution

$$\pi_{(1,m)}^{1m*} = \frac{c}{r}, \quad (3.42)$$

where c is a constant. Regularity at the origin implies that we have $c = 0$ leading to

$$\pi_{(1,m)}^{1m*} = \pi_{(1,m)}^{1m} = 0. \quad (3.43)$$

The Lagrangian $\mathcal{L}^{(2)}$ and its equations of motion. We use the reality conditions to express the Lagrangian of the h^{te} mode in terms of field variables

$$\begin{aligned} \mathcal{L}_{(\ell,m)}^{(2)} = & \frac{r^2}{2} \left(\beta^2 - \alpha^2 + \gamma^2 \right) \dot{h}_{(\ell,m)}^{te} \dot{h}_{(\ell,m)}^{te*} - \sqrt{2} r^2 \alpha \dot{h}_{(\ell,m)}^{0l} \dot{h}_{(\ell,m)}^{te*} + \frac{r^2}{2} \left(\alpha^2 - \gamma^2 \right) \partial_r h_{(\ell,m)}^{te} \partial_r h_{(\ell,m)}^{te*} \\ & + \sqrt{2} \alpha r h_{(\ell,m)}^{te} \partial_r h_{(\ell,m)}^{0l*} + h_{(\ell,m)}^{0l} h_{(\ell,m)}^{0l*} + \left(\beta^2 - \frac{\sqrt{\ell(\ell+1)}}{2} \alpha \beta - \frac{\sqrt{(\ell-1)(\ell+2)}}{2} \beta \gamma \right) h_{(\ell,m)}^{te} h_{(\ell,m)}^{te*} \\ & + \sqrt{2} \left(\frac{\ell(\ell+1)}{2} \alpha - \sqrt{\ell(\ell+1)} \beta + \frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{2} \gamma \right) h_{(\ell,m)}^{0l} h_{(\ell,m)}^{te*} + \ell(\ell+1) h_{(\ell,m)}^{0r} h_{(\ell,m)}^{0r*} \\ & + h_{(\ell,m)}^{0r*} \left[4r \dot{h}_{(\ell,m)}^{0l} - 2\sqrt{2} \alpha r^2 \partial_r \dot{h}_{(\ell,m)}^{te} - \sqrt{2} \left(2\alpha + \sqrt{\ell(\ell+1)} \beta \right) r \dot{h}_{(\ell,m)}^{te} \right] + h_{(\ell,m)}^{00*} \left[-2r \partial_r h_{(\ell,m)}^{0l} \right. \\ & - (\ell(\ell+1) + 2) h_{(\ell,m)}^{0l} + \sqrt{2} \alpha r^2 \partial_r \partial_r h_{(\ell,m)}^{te} + \sqrt{2} \left(3\alpha + \sqrt{\ell(\ell+1)} \beta \right) r \partial_r h_{(\ell,m)}^{te} \\ & \left. + \frac{1}{\sqrt{2}} \left(-(\ell-1)(\ell+2) \alpha + 4\sqrt{\ell(\ell+1)} \beta - \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)} \gamma \right) h_{(\ell,m)}^{te} \right]. \quad (3.44) \end{aligned}$$

Here as shown in [18], $\ell = 2, 3, \dots$, the Lagrangian for the $\ell = 1$ mode vanishes.

Let us use the constraints to simplify the Lagrangian. Varying the action with respect to $h_{(\ell,m)}^{00*}$ we obtain the constraint

$$\begin{aligned} & -2r \partial_r h_{(\ell,m)}^{0l} - (\ell(\ell+1) + 2) h_{(\ell,m)}^{0l} + \sqrt{2} \left(3\alpha + \sqrt{\ell(\ell+1)} \beta \right) r \partial_r h_{(\ell,m)}^{te} + \sqrt{2} \alpha r^2 \partial_r \partial_r h_{(\ell,m)}^{te} \\ & + \frac{1}{\sqrt{2}} \left(-(\ell-1)(\ell+2) \alpha + 4\sqrt{\ell(\ell+1)} \beta - \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)} \gamma \right) h_{(\ell,m)}^{te} = 0. \quad (3.45) \end{aligned}$$

The field $h_{(\ell,m)}^{0r}$ is non-dynamical, we can eliminate it using its equations of motion which is given by

$$\ell(\ell+1) h_{(\ell,m)}^{0r} + \left[4r \dot{h}_{(\ell,m)}^{0l} - 2\sqrt{2} \alpha r^2 \partial_r \dot{h}_{(\ell,m)}^{te} - \sqrt{2} \left(2\alpha + \sqrt{\ell(\ell+1)} \beta \right) r \dot{h}_{(\ell,m)}^{te} \right] = 0. \quad (3.46)$$

Solving for h^{0l} using (3.45) will in general result in non-local terms. To obtain a local Lagrangian we follow [18]. The remaining gauge freedom allows the ansatz

$$h_{(\ell,m)}^{0l} = ah_{(\ell,m)}^{te} + br\partial_r h_{(\ell,m)}^{te}. \quad (3.47)$$

Substituting (3.47) in (3.45) we obtain

$$\begin{aligned} & \sqrt{2}(\alpha - \sqrt{2}b)r^2\partial_r\partial_r h_{(\ell,m)}^{te} + \left(3\sqrt{2}\alpha + \sqrt{2\ell(\ell+1)}\beta - (\ell(\ell+1) + 4)b - 2a\right)r\partial_r h_{(\ell,m)}^{te} \\ & \frac{1}{\sqrt{2}}\left(-(\ell-1)(\ell+2)\alpha + 4\sqrt{\ell(\ell+1)}\beta - \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}\gamma \right. \\ & \left. - a\sqrt{2}(\ell(\ell+1) + 2)\right)h_{(\ell,m)}^{te} = 0. \end{aligned} \quad (3.48)$$

Demanding that the coefficients vanish at each order in derivatives in the above equation we obtain

$$\begin{aligned} a &= \sqrt{\frac{2}{\ell(\ell+1)}}\beta - \sqrt{\frac{(\ell-1)(\ell+2)}{2\ell(\ell+1)}}\gamma, \\ b &= \sqrt{\frac{2}{\ell(\ell+1)}}\beta + \sqrt{\frac{2}{(\ell-1)\ell(\ell+1)(\ell+2)}}\gamma, \\ \alpha &= \frac{2}{\sqrt{\ell(\ell+1)}}\beta + \frac{2}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\gamma. \end{aligned} \quad (3.49)$$

The equation (3.46) and (3.47) together with (3.49) implies that the Lagrangian $\mathcal{L}^{(2)}$ can be written only in terms of h^{te} . This is given by

$$\mathcal{L}_{\ell,m}^{(2)} = \frac{\gamma^2}{2} \left(r^2 \dot{h}_{(\ell,m)}^{te} \dot{h}_{(\ell,m)}^{te*} - r^2 (\partial_r h_{(\ell,m)}^{te}) (\partial_r h_{(\ell,m)}^{te*}) - \ell(\ell+1) h_{(\ell,m)}^{te} h_{(\ell,m)}^{te*} \right). \quad (3.50)$$

The canonical conjugate momentum to h^{te} is given by

$$\pi_{(\ell,m)}^{te*} = \frac{\partial \mathcal{L}^{(2)}}{\partial \dot{h}_{(\ell,m)}^{te*}} = \gamma^2 r^2 \dot{h}_{(\ell,m)}^{te}. \quad (3.51)$$

The equation of motion is then given by

$$r^2 (\ddot{h}_{(\ell,m)}^{te} - \partial_r^2 h_{(\ell,m)}^{te}) - 2r\partial_r h_{(\ell,m)}^{te} + \ell(\ell+1)h_{(\ell,m)}^{te} = 0. \quad (3.52)$$

We solve this equation of motion by expanding it in Fourier modes in time first and then solve the radial equation. The solution which is regular at the origin is given by

$$h_{(\ell,m)}^{te}(t, r) = a_{te(\ell,m)}(k)r^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(|k|r)e^{-ikt}. \quad (3.53)$$

We use (3.51) to write the Fourier modes of the momentum which is given by

$$\pi_{(\ell,m)}^{te*} = -ik\gamma^2 a_{te(\ell,m)}(k)r^{\frac{3}{2}}J_{\ell+\frac{1}{2}}(|k|r)e^{-ikt}. \quad (3.54)$$

Here $a_{te(\ell,m)}(k)$ is the arbitrary integration constant for the classical solution.

3.2 Curvature and the gauge fixed modes

We have shown that the superselection sectors are in principle labelled by the 6 curvature tensors of the form $R_{0\hat{r}ij}, R_{0\hat{r}0i}$, where i, j are spatial directions. In this section by explicitly evaluating these curvature components using the gauge discussed in section 3.1, we will see only 2 are related locally to the canonical coordinates π^{1m} and h^{te} . The gauge choice is adapted to the spherical symmetry of the problem. It is known that though gauge fixing converts a gauge field to a physical quantity, locality depends on the gauge choice [3]. To evaluate the curvatures we use Mathematica. It is first convenient to write the linearized curvature in terms of covariant derivatives. This and the writing the tensor harmonics as covariant tensors (3.17) and covariant vectors (2.18) allows us to use tensor calculus to evaluate the curvature.

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}[\nabla_\nu\nabla_\rho h_{\mu\sigma} - \nabla_\mu\nabla_\rho h_{\nu\sigma} + \nabla_\mu\nabla_\sigma h_{\nu\rho} - \nabla_\nu\nabla_\sigma h_{\mu\rho}]. \quad (3.55)$$

We substitute the tensor, vector and scalar mode decomposition given in (3.14) (3.21), (3.23) and (3.27) respectively in the expression of the curvature tensor (3.55) and use Mathematica to simplify the calculation.⁵

$R_{\hat{t}\hat{r}\hat{e}\hat{m}}$. Consider $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ which is obtained by taking appropriate projections with contravariant unit vectors

$$R_{\hat{t}\hat{r}\hat{e}\hat{m}} = \hat{t}^\mu \hat{r}^\nu \hat{e}^\rho \hat{m}^\sigma R_{\mu\nu\rho\sigma}. \quad (3.56)$$

Substituting the expansions of the metric in terms of its tensor, vector and scalar modes, we obtain

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{e}\hat{m}} &= \sum_{\ell, m=-\ell}^{\ell} \frac{\left(2h_{(\ell, m)}^{0m}(t, r) - 2r\partial_r h_{(\ell, m)}^{0m}(t, r) + \sqrt{2}r\dot{h}_{(\ell, m)}^{1m}(t, r)\right)}{2r^2} \sqrt{\ell(\ell+1)} Y_{\ell m}(\theta, \phi) \\ &= \sum_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{\pi_{(\ell, m)}^{1m*}}{r^3} Y_{\ell m}(\theta, \phi). \end{aligned} \quad (3.57)$$

Here though the sum over ℓ runs from $\ell = 2, \dots, \infty$, since from (3.43) we see that the $\ell = 1$ component $\pi_{(1, m)}^{1m*}$ vanishes. Note that $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ is a gauge-invariant observable and it is related to the canonical momentum of the mode $h_{(\ell, m)}^{1m}$ without any radial derivative. Therefore this relation is local in the radial co-ordinates and the curvature component can be used to label superselection sectors.

$R_{\hat{t}\hat{r}\hat{t}\hat{r}}$. To evaluate $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$, we use the vacuum Einstein equation

$$R_{\hat{r}\hat{r}} = 0. \quad (3.58)$$

⁵The Mathematica notebook can be found in the supplementary material of this paper.

From the equation of motion, we relate the curvature component $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$ to other curvature components which are easy to evaluate

$$\begin{aligned}
 R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= g^{\theta\theta} R_{\theta r \theta r} + g^{\phi\phi} R_{\phi r \phi r} \\
 &= \sum_{\ell m} \left[\frac{2r \partial_r h_{(\ell,m)}^{0l}(t,r) + h_{(\ell,m)}^{0l}(t,r) \ell(\ell+1)}{r^2} - \frac{\sqrt{2}\alpha \left(2\partial_r h_{(\ell,m)}^{te}(t,r) + r \partial_r^2 h_{(\ell,m)}^{te}(t,r) \right)}{r} \right. \\
 &\quad \left. - \beta \sqrt{2\ell(\ell+1)} \frac{\left(h_{(\ell,m)}^{te}(t,r) + \partial_r h_{(\ell,m)}^{te}(t,r) \right)}{r^2} \right] Y_{\ell,m}(\theta, \phi). \tag{3.59}
 \end{aligned}$$

In the last line we substitute the mode expansion of $h_{\mu\nu}$. At this stage, this component of the curvature tensor appears to be non-local in r . But using the gauge choice in (3.47) we relate $h_{(\ell,m)}^{0l}$ as a function of $h_{(\ell,m)}^{te}$. Finally we replace a , b and α in terms of β and γ given in (3.49) to obtain

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = - \sum_{\ell,m} \frac{\gamma \sqrt{(\ell-1)\ell(\ell+1)(\ell+2)} h_{(\ell,m)}^{te}(t,r) Y_{\ell,m}(\theta, \phi)}{\sqrt{2} r^2}. \tag{3.60}$$

Here l runs from $\ell = 2, 3, \dots$. It is interesting to note that, imposing the gauge condition (3.47) removes the apparent non-locality of the curvature tensor $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$. This relation also tells us the canonical coordinate h^{te} is gauge-invariant in our gauge choice. Therefore this curvature component can also be used to label super-selection sectors for a spherical entangling surface.

$R_{\hat{t}\hat{r}\hat{t}\hat{m}}$. Evaluating this curvature component using the same methods we obtain

$$R_{\hat{t}\hat{r}\hat{t}\hat{m}} = \sum_{\ell,m} \frac{1}{\sqrt{2\ell(\ell+1)}} \left(\ddot{h}_{(\ell,m)}^{1m} - \sqrt{2} \partial_r \dot{h}_{(\ell,m)}^{0m} \right) |Y_{\ell,m}^m(\theta, \phi)|. \tag{3.61}$$

Since it involves terms containing time derivatives, we can use the we use the equation of motion and the constraint given in (3.34) and (3.36) to simplify the expression. Note that the curvature component is proportional to the norm of the vector harmonics $Y_{\ell,m}^m(\theta, \phi)$. We finally obtain

$$R_{\hat{t}\hat{r}\hat{t}\hat{m}} = - \sum_{\ell,m} \frac{1}{\sqrt{2\ell(\ell+1)}} \left(\frac{1}{r} \partial_r h_{(\ell,m)}^{1m} + \frac{(\ell-1)(\ell+2)+1}{r^2} h_{(\ell,m)}^{1m} \right) |Y_{\ell,m}^m(\theta, \phi)|. \tag{3.62}$$

We observe that the expression of the curvature tensor involves the radial derivative acting on the field. Therefore it does not belong to the local algebra of observables that contribute to the entanglement entropy of a spherical entangling surface.

$R_{\hat{t}\hat{r}\hat{r}\hat{m}}$. This curvature component is given by

$$\begin{aligned}
 R_{\hat{t}\hat{r}\hat{r}\hat{m}} &= - \sum_{\ell,m} \frac{1}{r^2 \sqrt{2\ell(\ell+1)}} \left[\sqrt{2} h_{(\ell,m)}^{0m} - \sqrt{2} r \partial_r (r \partial_r h_{(\ell,m)}^{0m}) + 2r \dot{h}_{(\ell,m)}^{1m} + r^2 \partial_r \dot{h}_{(\ell,m)}^{1m} \right] |Y_{\ell,m}^m(\theta, \phi)| \\
 &= - \sum_{\ell,m} \frac{1}{r^2 \sqrt{2\ell(\ell+1)}} \partial_r \pi_{(\ell,m)}^{1m*} |Y_{\ell,m}^m(\theta, \phi)|. \tag{3.63}
 \end{aligned}$$

To obtain the second line we have used the definition of the conjugate momentum $\pi_{(\ell,m)}^{1m}$ in (3.33). The radial derivative on the conjugate momentum indicates that this curvature component also does belong the local algebra of observables in a sphere.

$R_{\hat{t}\hat{r}\hat{r}\hat{e}}$. We now evaluate the curvature tensor $R_{\hat{t}\hat{r}\hat{r}\hat{e}}$

$$R_{\hat{t}\hat{r}\hat{r}\hat{e}} = \sum_{\ell,m} \left[-\frac{(-h_{(\ell,m)}^{0r} + r\partial_r h_{(\ell,m)}^{0r})}{r^2} + \frac{\dot{h}_{(\ell,m)}^{0l}}{r} - \frac{\beta}{r\sqrt{2\ell(\ell+1)}}(2\dot{h}_{(\ell,m)}^{te} + r\partial_r \dot{h}^{te}) \right] |Y_{\ell,m}^e(\theta, \phi)|. \quad (3.64)$$

The expression involves the non-dynamical field $h_{(\ell,m)}^{0r}$ which can be replaced by the constraint equation given in (3.46). We also use the gauge condition (3.47) to substitute for $h_{(\ell,m)}^{0l}$ and finally we obtain

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{r}\hat{e}} &= \left[-\frac{\gamma\sqrt{\ell^2 + \ell - 2}}{r\sqrt{2\ell(\ell+1)}}\partial_r h_{(\ell,m)}^{te} - \left(\frac{2\gamma}{\sqrt{2\ell(\ell+1)}(\ell^2 + \ell - 2)} + \frac{\beta}{\sqrt{2\ell(\ell+1)}} \right) \partial_r \dot{h}_{(\ell,m)}^{te} \right] |Y_{\ell,m}^e| \\ &= \left[-\frac{\gamma\sqrt{\ell^2 + \ell - 2}}{r\sqrt{2\ell(\ell+1)}}\partial_r h_{(\ell,m)}^{te} - \left(\frac{2\gamma}{\sqrt{2\ell(\ell+1)}(\ell^2 + \ell - 2)} + \frac{\beta}{\sqrt{2\ell(\ell+1)}} \right) \partial_r \left(\frac{\pi_{(\ell,m)}^{te*}}{\gamma^2 r^2} \right) \right] |Y_{\ell,m}^e|. \end{aligned} \quad (3.65)$$

Again, the curvature component involves the radial derivatives acting on fields even after using the local gauge condition. We do have the freedom to set term containing the derivative of the canonical momentum π^{te*} to zero by choosing a suitable β . However there will still remain the term containing the derivative of h^{te} . Therefore we conclude that this component of the curvature also does not belong to the local algebra of observables in the sphere and cannot be used to label superselection sectors.

$R_{\hat{t}\hat{r}\hat{t}\hat{e}}$. The last of the 6 curvature components is $R_{\hat{t}\hat{r}\hat{t}\hat{e}}$. We use the vacuum Einstein equation $R_{ri} = 0$ to evaluate it, where i denotes the angular coordinates on S^2 .

$$R_{\hat{t}\hat{r}\hat{t}\hat{e}} = g^{ab} R_{\hat{a}\hat{r}\hat{b}\hat{e}}. \quad (3.66)$$

Therefore we write

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{t}\hat{e}} &= \hat{e}^k R_{\hat{t}\hat{r}\hat{t}\hat{k}} \\ &= e^\theta g^{\phi\phi} R_{\phi r \phi \theta} + e^\phi g^{\theta\theta} R_{\theta r \theta \phi}. \end{aligned} \quad (3.67)$$

Substituting the expansion of $h_{\mu\nu}$ in (3.17), we obtain

$$R_{\hat{t}\hat{r}\hat{t}\hat{e}} = \left(\frac{h_{(\ell,m)}^{0l}(t, r)}{r^2} - \alpha \frac{\partial_r h_{(\ell,m)}^{te}}{\sqrt{2}r} - \sqrt{\frac{2}{\ell(\ell+1)}} \frac{\beta h_{(\ell,m)}^{te}}{r^2} - \frac{(\ell+1)(\ell-2)}{\sqrt{\ell(\ell+1)}} \gamma \frac{\partial_r h_{(\ell,m)}^{te}(t, r)}{2r} \right) |Y_{\ell,m}^e|. \quad (3.68)$$

We now we use the relation (3.49) to eliminate h^{0l}

$$R_{\hat{r}\hat{t}\hat{e}} = \left[-\frac{\gamma(\ell^2 + \ell - 2)}{2r\sqrt{\ell(\ell + 1)}} \partial_r h_{(\ell,m)}^{te}(t, r) - \gamma \sqrt{\frac{(\ell - 1)(\ell + 2)}{\ell(\ell + 1)}} \frac{h_{(\ell,m)}^{te}(t, r)}{\sqrt{2}r^2} \right] |Y_{\ell,m}^e|. \quad (3.69)$$

This is again contains a radial derivative which cannot be eliminated by further choice of β and γ . Therefore this component does not belong to the algebra of local observables in a sphere and cannot be used to label superselection sectors.

The explicit evaluation of the curvature components leads us to conclude that our of the 6 components that satisfy the Gauss law, only 2 of them, $R_{\hat{r}\hat{e}\hat{m}}$ and $R_{\hat{r}\hat{t}\hat{r}}$ are related to the algebra of local observables in a sphere. We therefore proceed to evaluate the two-point functions of these components on the sphere to obtain the contribution of superselection sectors to the entanglement entropy.

3.3 Quantization of the modes

We first need to quantize the canonical coordinates (h^{1m}, π^{1m}) and (h^{te}, π^{te}) . We have found the solutions to the wave equations of these modes in section 3.1. We use these solutions to promote these coordinates to operators and impose canonical commutation relations.

The mode $h_{(\ell,m)}^{1m}$. The classical solution obtained for the Fourier mode of $h_{(\ell,m)}^{1m}$ and $\pi_{(\ell,m)}^{1m}$ from the equation of motion in (3.38) and (3.41) respectively implies the following mode expansion of these fields.

$$\begin{aligned} h_{(\ell,m)}^{1m}(t, r) &= \frac{1}{\sqrt{2}} \int_0^\infty k dk \left(a_{1m(\ell,m)}(k) e^{-ikt} + (-1)^m a_{1m(\ell,-m)}^\dagger(k) e^{ikt} \right) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r) \\ \pi_{(\ell,m)}^{1m*}(t, r) &= \frac{(\ell - 1)(\ell + 2)}{\sqrt{2}} \int_0^\infty dk \left(-i a_{1m(\ell,m)}(k) e^{-ikt} + i(-1)^m a_{1m(\ell,-m)}^\dagger(k) e^{ikt} \right) \\ &\quad \times \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \end{aligned} \quad (3.70)$$

Here $\ell \geq 2$, the mode expansion obeys the reality condition given in (3.30). Let us also give the mode expansion of $\pi_{(\ell,m)}^{1m}$

$$\pi_{\ell m}^{1m}(t, r) = \frac{(\ell - 1)(\ell + 2)}{\sqrt{2}} \int_0^\infty dk \left(i a_{1m(\ell,m)}^\dagger(k) e^{ikt} - i(-1)^m a_{1m(\ell,-m)} e^{-ikt} \right) \sqrt{r} J_{\ell+\frac{1}{2}}(|k|r). \quad (3.71)$$

We now promote the variables $h_{(\ell,m)}^{1m}$ and $\pi_{(\ell,m)}^{1m}$ to the operators which implies $a_{(\ell,m)}^{1m}$ and $a_{(\ell,m)}^{\dagger 1m}$ are also operators. The equal time commutation relation of the conjugate operators are given by

$$[h_{(\ell,m)}^{1m}(t, r), \pi_{(\ell',m')}^{1m}(t, r')] = i\delta(r - r')\delta_{\ell\ell'}\delta_{m,m'}. \quad (3.72)$$

From the mode expansions given in (3.70) one can see the equal time commutation relation yields the commutation relation of creation and annihilation operators.

$$[a_{1m(\ell,m)}(k), a_{1m(\ell',m')}^\dagger(k')] = \frac{\delta(k - k')\delta_{\ell\ell'}\delta_{m,m'}}{(\ell + 2)(\ell - 1)}. \quad (3.73)$$

All other commutation relations are trivial. To show the commutation relation of creation and annihilation operator we substitute the mode expansion (3.70) in (3.72) and use the closure relation of the Bessel function given in (2.40).

The mode $h_{(\ell,m)}^{te}$. From the classical solution of the equation of motion (3.53) and (3.54), we write the mode expansion of $h_{(\ell,m)}^{te}$ and $\pi_{(\ell,m)}^{te}$

$$h_{(\ell,m)}^{te}(t,r) = \frac{r^{-\frac{1}{2}}}{\sqrt{2}} \int_0^\infty dk \left(a_{te(\ell,m)}(k) e^{-ikt} + (-1)^m a_{te(\ell,-m)}^\dagger(k) e^{ikt} \right) J_{\ell+\frac{1}{2}}(|k|r) \quad (3.74)$$

$$\pi_{(\ell,m)}^{te*}(t,r) = \frac{\gamma^2 r^{\frac{3}{2}}}{\sqrt{2}} \int_0^\infty k dk \left(-i a_{te(\ell,m)}(k) e^{-ikt} + (-1)^m i a_{te(\ell,-m)}^\dagger(k) e^{ikt} \right) J_{\ell+\frac{1}{2}}(|k|r).$$

We have used $\pi_{(\ell,m)}^{*te} = \gamma^2 r^2 \dot{h}_{(\ell,m)}^{te}$ to write the momentum mode expansion. The mode expansion of $\pi_{(\ell,m)}^{te}$

$$\pi_{(\ell,m)}^{te} = \frac{\gamma^2 r^{\frac{3}{2}}}{\sqrt{2}} \int_0^\infty k dk \left(i a_{te(\ell,m)}^\dagger(k) e^{ikt} - (-1)^m i a_{te(\ell,-m)}(k) e^{-ikt} \right) J_{\ell+\frac{1}{2}}(|k|r). \quad (3.75)$$

Now one promotes the variables $h_{(\ell,m)}^{te}$ $\pi_{(\ell,m)}^{te}$ to operators and imposes the equal time commutation relation.

$$[h_{(\ell,m)}^{te}(t,r), \pi_{\ell'}^{te}(t,r')] = i\delta(r-r')\delta_{\ell\ell'}\delta_{mm'}. \quad (3.76)$$

This implies the commutation relation of creation and annihilation operators.

$$[a_{te(\ell m)}(k), a_{te(\ell',m')}^\dagger(k')] = \frac{\delta(k-k')\delta_{\ell\ell'}\delta_{m,m'}}{\gamma^2}. \quad (3.77)$$

3.4 Entanglement entropy of the edge states

We compute the two-point function of the normal components of the curvature tensors which label the super-selection sector. As we have seen in section 3.2, only the curvature components $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ and $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$ are related locally to the algebra of gauge-invariant operators in a sphere.

Two-point function of $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$. From (3.57) we see that $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ is related to the momentum field π^{1m*}

$$R_{\hat{t}\hat{r}\hat{e}\hat{m}} = \sum_{\ell,m=-\ell}^{\ell} \sqrt{\frac{\ell(\ell+1)}{2}} \frac{\pi_{(\ell,m)}^{1m*}}{r^3} Y_{\ell m}(\theta, \phi). \quad (3.78)$$

The sum over ℓ runs from $\ell = 2, 3, \dots$. Substituting the mode expansion of π^{1m*} from (3.70) and using the canonical computation relations (3.73) we compute the two-point function of $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{e}\hat{m}}(t,r,\theta,\phi) R_{\hat{t}\hat{r}\hat{e}\hat{m}'}(t,r',\theta',\phi') | 0 \rangle &= \quad (3.79) \\ \frac{1}{2(rr')^{\frac{5}{2}}} \sum_{\ell,\ell',m,m'} (\ell+2)(\ell-1)(\ell'+2)(\ell'-1) (\ell(\ell+1)\ell'(\ell'+1))^{\frac{1}{2}} \\ &\times \int_0^\infty dk dk' \left[J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') \right. \\ &\quad \left. \times (-1)^{m'} \langle 0 | a_{1m(\ell,m)}(k) a_{1m(\ell',m')}^\dagger(k') | 0 \rangle Y_{\ell,m}(\theta,\phi) Y_{\ell',-m'}(\theta',\phi') \right]. \end{aligned}$$

Using the commutation relations in (3.73), we obtain

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{e}\hat{m}}(t, r, \theta, \phi) R_{\hat{t}'\hat{r}'\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle = & \quad (3.80) \\ & \frac{1}{2(rr')^{\frac{5}{2}}} \sum_{\ell, m} (\ell + 2)(\ell - 1)\ell(\ell + 1) \int_0^\infty dk J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \end{aligned}$$

We now use the identity (2.51) to perform the integral over Bessel function and obtain

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{e}\hat{m}}(t, r, \theta, \phi) R_{\hat{t}'\hat{r}'\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle = & \quad (3.81) \\ & \frac{1}{2\pi r^3 r'^3} \sum_{\ell, m} (\ell + 2)(\ell - 1)\ell(\ell + 1) Q_\ell \left(\frac{r^2 + r'^2}{2rr'} \right) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \end{aligned}$$

We need the two-point functions at the same radial point. In this limit, the expansion of the Legendre function of the second kind is given in (2.55). Substituting that in the expression of the two-point function and keeping only the leading term we obtain

$$\lim_{\delta \rightarrow 0} G_{rr}(r, r + \delta; x, y) = \frac{1}{4\pi r^6} \log \left(\frac{r^2}{\delta^2} \right) \sum_{\ell \geq 2, m} \ell(\ell + 1)(\ell - 1)(\ell + 2) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \quad (3.82)$$

Two-point function of $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$. The second curvature component locally related to the canonical coordinate is $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$ which is given by (3.60)

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = - \sum_{\ell, m=-\ell}^{\ell} \frac{\gamma \sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)} h_{(\ell, m)}^{te}(t, r) Y_{\ell, m}(\theta, \phi)}{\sqrt{2} r^2}. \quad (3.83)$$

The sum over ℓ again runs from $\ell = 2, 3, \dots$. Using the mode expansion of h^{te} in (2.53), we compute the two-point function of $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{t}\hat{r}}(t, r, \theta, \phi) R_{\hat{t}'\hat{r}'\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle = & \quad (3.84) \\ & \frac{\gamma^2}{2(rr')^{\frac{5}{2}}} \sum_{\ell, \ell', m, m'} (\ell(\ell + 1)(\ell - 1)(\ell + 2)\ell'(\ell' + 1)(\ell' - 1)(\ell' + 2))^{\frac{1}{2}} \\ & \times \int_0^\infty dk dk' \left[J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') \right. \\ & \quad \left. \times (-1)^{m'} \langle 0 | a_{te(\ell, m)}(k) a_{te(\ell', m')}^\dagger(k') | 0 \rangle Y_{\ell, m}(\theta, \phi) Y_{\ell', -m'}(\theta', \phi') \right]. \end{aligned}$$

Using the commutation relations in (3.77), we obtain

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{t}\hat{r}}(t, r, \theta, \phi) R_{\hat{t}'\hat{r}'\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle = & \quad (3.85) \\ & \frac{1}{2(rr')^{\frac{5}{2}}} \sum_{\ell, m} (\ell + 2)(\ell - 1)\ell(\ell + 1) \int_0^\infty dk J_{\ell+\frac{1}{2}}(kr) J_{\ell'+\frac{1}{2}}(k'r') Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \end{aligned}$$

We now use the identity (2.51) to perform the integral over Bessel function and obtain

$$\begin{aligned} \langle 0 | R_{\hat{t}\hat{r}\hat{t}\hat{r}}(t, r, \theta, \phi) R_{\hat{t}'\hat{r}'\hat{e}'\hat{m}'}(t, r', \theta', \phi') | 0 \rangle = & \quad (3.86) \\ & \frac{1}{2\pi (rr')^3} \sum_{\ell, m} (\ell + 2)(\ell - 1)\ell(\ell + 1) Q_\ell \left(\frac{r^2 + r'^2}{2rr'} \right) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \end{aligned}$$

We need the two-point functions at the same radial point. In this limit, the expansion of the Legendre function of the second kind is given in (2.55). Substituting that in the expression of the two-point function, we obtain the leading contribution to be given by

$$\lim_{\delta \rightarrow 0} G_{rr}(r, r + \delta; x, y) = \frac{1}{4\pi r^6} \log\left(\frac{r^2}{\delta^2}\right) \sum_{\ell \geq 2, m} \ell(\ell + 1)(\ell - 1)(\ell + 2) Y_{\ell, m}(\theta, \phi) Y_{\ell, m}^*(\theta', \phi'). \quad (3.87)$$

From (3.81) and (3.86) we see that the correlators coincide. This is expected since the theory of linearised graviton, satisfying vacuum Einstein equations are invariant under the ‘electric-magnetic’ duality [24, 25]

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\alpha\beta} R_{\mu\nu}{}^{\alpha\beta}. \quad (3.88)$$

Entanglement of the superselection sectors. From the discussion in section (2.13), to determine the logarithmic coefficient of the entanglement entropy of the superselection sectors we see that we need to evaluate the log-determinant of the leading contribution of the radial coincident Green’s function. Therefore, from (2.13) and (3.82) or (3.87), the entanglement entropy of the superselections sectors determined by the curvature components $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ or $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$ is given by

$$S_{\text{edge}}(\rho_A) = \frac{1}{2} \sum_{\ell=2}^{\infty} (2\ell + 1) \log [\ell(\ell + 1)(\ell - 1)(\ell + 2)]. \quad (3.89)$$

Note that the correlators in (3.82) or (3.87) are diagonal in scalar spherical harmonic basis. The diagonal elements are independent of the quantum number m , this implies that each eigen value contributes with a multiplicity of $(2\ell + 1)$ to the log-determinant.

To evaluate the coefficient of the logarithmic divergence we open up the logarithm and write each term using the identity⁶

$$-\log(y) = \int_0^{\infty} \frac{dt}{t} (e^{-yt} - e^{-t}). \quad (3.90)$$

Using this representation of the logarithm, we obtain

$$S_{\text{edge}}(\rho_A) = - \int_0^{\infty} \frac{dt}{2t} \sum_{\ell=2}^{\infty} (2\ell + 1) (e^{-(\ell-1)t} + e^{-\ell t} + e^{-(\ell+1)t} + e^{-(\ell+2)t} - 4e^{-t}). \quad (3.91)$$

The last term involves the sum over degeneracies

$$g = \sum_{\ell=2}^{\infty} (2\ell + 1). \quad (3.92)$$

To perform this sum we resort to ‘dimensional regularization’ which was introduced in [26], for more details see around equation 2.7 of [22]. Here one performs the sum of degeneracies

⁶This method of studying the sums involved in the log-determinant can also be used to obtain the logarithmic contribution of the U(1) theory.

of scalar harmonics in sufficiently negative dimensions for which the sum is convergent and then continues the result analytically to positive dimensions. This results in

$$\sum_{\ell=2}^{\infty} (2\ell + 1) = -4. \tag{3.93}$$

Substituting this result and performing the rest of the sums in (3.91), we obtain

$$S_{\text{edge}}(\rho_A) = - \int_0^{\infty} \frac{dt}{2t} \left(\frac{e^{-t}(1 + e^{-t})(1 + e^{-2t})(5 - 3e^{-t})}{(1 - e^{-t})^2} + 16 \right). \tag{3.94}$$

One can easily extract the coefficient of the logarithmic divergence from this representation by examining the coefficient of the $1/t$ term in the integrand. We obtain⁷

$$S_{\text{edge}}(\rho_A)|_{\log \text{ coefficient}} = -\frac{8}{3}. \tag{3.95}$$

As we have discussed earlier the Gauss of gravity or the Hamiltonian constraints (3.8), (3.9) results in the equations (3.12), (3.13) which determine the superselection sectors. Note that these constraints are on the same footing as the physical state condition (2.2) in the U(1) theory. We need to sum the contributions arising from all possible superselection sectors arising out of the conditions (3.12), (3.13). We have demonstrated that only 2 of these curvatures can be written locally in the gauge invariant observables on the sphere. Therefore we need to sum the contributions to the entanglement entropy from the Gauss law which constrains the curvature components $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ and $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$. This leads to the conclusion that the logarithmic coefficient of the gravitational edge modes for a spherical entangling surface is given by

$$S_{\text{gravitational edgemodes}}(\rho_A) = -\frac{16}{3} \log \frac{R}{\epsilon}. \tag{3.96}$$

Again we emphasize that this contribution resulted from our choice of the centre in which the superselection sectors are labelled by the curvature components $R_{\hat{t}\hat{r}\hat{e}\hat{m}}$ and $R_{\hat{t}\hat{r}\hat{t}\hat{r}}$. It will be interesting to study the linearised graviton theory in more detail, so that we can prove a trivial centre can be chosen just as in the case of the U(1) theory [3].

We compare the coefficient of the edge modes of the linearized graviton to the edge partition function of the massless spin-2 theory on S^4 . This was evaluated in [19]⁸ and is given by the following integral over the Harish-Chandra character

$$\begin{aligned} \log \mathcal{Z}_2[S^4] &= \int_0^{\infty} \frac{dt}{2t} \frac{1+q}{1-q} ([\hat{\chi}_{\text{bulk},2}]_+ - [\hat{\chi}_{\text{edge},2}]_+), \\ [\hat{\chi}_{\text{bulk},2}]_+ &= \frac{10q^3 - 6q^4}{(1-q)^3}, \quad q = e^{-t}, \\ [\hat{\chi}_{\text{edge},2}]_+ &= \frac{10q^2 - 2q^3}{(1-q)}. \end{aligned} \tag{3.97}$$

⁷We have also performed this computation using the methods of [19] and obtained the same result.

⁸These can be read out from equations 5.8 and 5.11 of [19]. Note that the last term in 5.11 does not contribute to the logarithmic coefficient.

Extracting the contribution of the edge character to the logarithmic coefficient we find

$$\log \mathcal{Z}_2[S^4] \Big|_{\log \text{ coefficient, edge}} = -\frac{16}{3}. \quad (3.98)$$

Just as in the U(1) case, for the graviton, the logarithmic coefficient of edge partition function of the graviton on the sphere agrees with that of the superselection sectors of the graviton. It will be interesting to understand the relationship between the contribution of superselection to the entanglement entropy to that of the edge partition function on spheres further. One direction would be to extend the methods in this paper to higher spin fields. The coefficient from the edge partition function obtained from the Harish-Chandra character of a massless spin- s field in $d = 4$ dimension is given by $-\frac{s^4}{3}$ [27].

4 Conclusions

The extractable entanglement or the bulk entanglement of the linearised graviton across a spherical surface was first evaluated in [18]. Decomposing the spin-2 field into tensor harmonics it was shown that the algebra of gauge-invariant operators is equivalent to two scalar fields with their $\ell = 0$ and $\ell = 1$ modes removed. The logarithmic coefficient is given by $-\frac{61}{45}$. Furthermore, considering the spin-2 field on hyperbolic cylinders and evaluating the entanglement entropy also reproduces this coefficient [27].⁹ However an evaluation of the contribution of the superselection sectors to the entanglement entropy of the linearized graviton was missing in the literature. In this paper we have used the method of decomposing the spin-2 fields into tensor harmonics and fixing a gauge which respects the spherical symmetry of the problem developed in [18] to evaluate the logarithmic coefficient of the edge modes or the superselection sectors resulting from the Gauss law of gravity. One crucial ingredient in the calculation was to determine which among the curvature components satisfying the Gauss law of gravity were locally related to the algebra of gauge-invariant operators in the sphere.

Our choice of superselection sectors resulting from the Gauss law picks out a centre for the algebra of local operators for the graviton. It would be interesting to develop the extended Hilbert space definition of entanglement entropy defined for gauge theories in [1, 2, 6, 9, 10] in detail for the linearized graviton to determine which choice of centre results from the extended Hilbert space definition. Here the general methods developed in gravity to define subregions by [28–30] would be useful.

The methods developed in this paper can be extended to other fields and to other dimensions. In particular it would be interesting to study the contribution of the edge modes of p -forms in arbitrary even dimensions using this approach and verify if their contribution to the logarithmic coefficient agrees with that from the edge Harish-Chandra character of the sphere partition function. As we have seen in this paper this agreement is true for the U(1) fields in arbitrary even dimensions. Similar questions can be addressed for higher spin fields as well.

⁹Indeed, we have verified that even the Rényi entropies of two scalar fields with their $\ell = 0$ and $\ell = 1$ modes removed also precisely coincides with that evaluated from considering the spin-2 field on the hyperbolic cylinder.

Finally, the entanglement entropy of non-abelian gauge fields contains an additional edge term. This additional term is tied to the fact that irreducible representations of the superselection sectors in the non-Abelian theories have dimensions greater than unity [6, 12]. Recently, the authors of [31] studied the Hayward term in gravity and suggested that the Hayward term corresponds to the edge entanglement associated with the above additional contribution in the graviton theory. This occurs in the full non-linear theory. It would be interesting to study this further using the methods of [28–30].

A Electric correlator in the Coulomb gauge

The two-point functions of the electric field on sphere gauge-invariant and therefore it should not depend on gauge choices. In [16], the two-point function of the electric field was evaluated in the Coulomb gauge, i.e, $A_0 = 0$ and $\nabla \cdot \vec{A} = 0$. In this paper we have chosen a different gauge which is $A'_{(\ell,m)} = 0$ for all ℓ and m . Here A^e is the component of the vector field obtained by taking the projection

$$A^e_{(\ell,m)} = e^\mu A_{\mu,(\ell,m)}. \tag{A.1}$$

In this appendix we would like to compare the two-point obtained in equation A.12 of [16] with what we obtain in (2.53). Equation A.12 [16] reads

$$G_{rr'} = \frac{1}{\pi^2(r^2 + r'^2)^2} \sum_{\ell=0}^{\infty} (4\ell + 1) \left\{ \sum_{n=0}^{\infty} \left(\frac{1 - \alpha^2}{\alpha} n - \alpha \right) \frac{2n!!}{(2n - 2\ell)!!} \frac{(2n + 1)!!}{(2n + 2\ell + 1)!!} \alpha^{2n} \right\} P_{2\ell} \\ + \sum_{\ell=0}^{\infty} (4\ell + 3) \left\{ \sum_{n=0}^{\infty} \left(\frac{1 - \alpha^2}{\alpha} n - \frac{3\alpha^2 - 1}{2\alpha} \right) \frac{(2n + 2)!!}{(2n - 2\ell)!!} \frac{(2n + 1)!!}{(2n + 2\ell + 3)!!} \alpha^{2n+1} \right\} P_{2\ell+1}. \tag{A.2}$$

Here P_ℓ is the Legendre polynomial related to the spherical harmonics by the following relation

$$P_\ell(\cos \gamma) = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell + 1} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'), \tag{A.3}$$

where γ is the angle between the unit vectors determined by the angular coordinates on the sphere. α is related to the radial coordinates r and r' by the following relation

$$\frac{2rr'}{r^2 + (r')^2} = \alpha. \tag{A.4}$$

Note that equation (A.2) is a series in α for every ℓ , while the expression in (2.53) is a closer function of α .

To compare the two correlators we expand (2.53) around $\alpha = 0$ and compare it with (A.2) for each ℓ values. Expanding the two pint function given in (2.53) around $\alpha = 0$,

we obtain

$$\begin{aligned}
 & \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle = \\
 & \sum_{\ell, m} \frac{Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') \alpha^\ell}{\sqrt{\pi} (rr')^2} \left[\frac{\alpha 2^{-\ell-2} \ell(\ell+1) \Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})} + \frac{\alpha^3 2^{-\ell-3} \ell(\ell+1)^2(\ell+2) \Gamma(\ell+1)}{(2\ell+3) \Gamma(\ell + \frac{3}{2})} \right. \\
 & + \frac{\alpha^5 2^{-\ell-5} \ell(\ell+1)^2(\ell+2)(\ell+3)(\ell+4) \Gamma(\ell+1)}{(2\ell+3)(2\ell+5) \Gamma(\ell + \frac{3}{2})} \\
 & \left. + \frac{\alpha^7 2^{-\ell-6} \ell(\ell+1)^2(\ell+2)(\ell+3)(\ell+4)(\ell+5)(\ell+6) \Gamma(\ell+1)}{3(2\ell+3)(2\ell+5)(2\ell+7) \Gamma(\ell + \frac{3}{2})} + \dots \right]. \tag{A.5}
 \end{aligned}$$

Now we are in a position to compare the series in α for each value of ℓ with (A.2).

$\ell = 0$. Both correlators vanish at $\ell = 0$.

$\ell = 1$. We substitute $\ell = 1$ in (A.5) and obtain

$$\langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle_{\ell=1} = \frac{Y_{1m}(\theta, \phi) Y_{1m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{\alpha^2}{3} + \frac{\alpha^4}{5} + \frac{\alpha^6}{7} + \frac{\alpha^8}{9} + \dots \right). \tag{A.6}$$

From (A.2) we obtain

$$G_{rr'}|_{\ell=1} = \frac{Y_{1m}(\theta, \phi) Y_{1m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{\alpha^2}{3} + \frac{\alpha^4}{5} + \frac{\alpha^6}{7} + \frac{\alpha^8}{9} + \dots \right). \tag{A.7}$$

Similarly we have checked the expansion for $\ell = 2$ to $\ell = 5$

$$\begin{aligned}
 & \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle_{\ell=2} \\
 & = \frac{Y_{2m}(\theta, \phi) Y_{2m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{2\alpha^3}{5} + \frac{12\alpha^5}{35} + \frac{2\alpha^7}{7} + \frac{8\alpha^9}{33} + \dots \right) \\
 & = G_{rr'}|_{\ell=2}. \tag{A.8}
 \end{aligned}$$

$$\begin{aligned}
 & \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle_{\ell=3} \\
 & = \frac{Y_{3m}(\theta, \phi) Y_{3m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{12\alpha^4}{35} + \frac{8\alpha^6}{21} + \frac{4\alpha^8}{11} + \frac{48\alpha^{10}}{143} + \dots \right) \\
 & = G_{rr'}|_{\ell=3}. \tag{A.9}
 \end{aligned}$$

$$\begin{aligned}
 & \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle_{\ell=4} \\
 & = \frac{Y_{4m}(\theta, \phi) Y_{4m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{16\alpha^5}{63} + \frac{80\alpha^7}{231} + \frac{160\alpha^9}{429} + \frac{160\alpha^{11}}{429} + \dots \right) \\
 & = G_{rr'}|_{\ell=4}. \tag{A.10}
 \end{aligned}$$

$$\begin{aligned}
 & \langle 0 | F_{\hat{t}\hat{r}}(t, r, \theta, \phi) F_{\hat{t}'\hat{r}'}(t, r', \theta', \phi') | 0 \rangle_{\ell=5} \\
 & = \frac{Y_{5m}(\theta, \phi) Y_{5m}^*(\theta', \phi')}{\pi (rr')^2} \left(\frac{40\alpha^6}{231} + \frac{40\alpha^8}{143} + \frac{48\alpha^{10}}{143} + \frac{80\alpha^{12}}{221} + \dots \right) \\
 & = G_{rr'}|_{\ell=5}. \tag{A.11}
 \end{aligned}$$

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