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Strong coupling expansion of free energy and BPS Wilson loop in $\mathcal{N}=2$ superconformal models with fundamental hypermultiplets

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Abstract: As a continuation of the study (in arXiv:2102.07696 and arXiv:2104.12625) of strong-coupling expansion of non-planar corrections in $\mathcal{N}=2$ 4d superconformal models we consider two special theories with gauge groups SU(N) and Sp($2N$). They contain N-independent numbers of hypermultiplets in rank 2 antisymmetric and fundamental representations and are planar-equivalent to the corresponding $\mathcal{N}=4$ SYM theories. These $\mathcal{N}=2$ theories can be realised on a system of N D3-branes with a finite number of D7-branes and O7-plane; the dual string theories should be particular orientifolds of $\mathrm{AdS}_5\times S^5$ superstring. Starting with the localization matrix model representation for the $\mathcal{N}=2$ partition function on S^4 we find exact differential relations between the $1/N$ terms in the corresponding free energy F and the $\frac{1}{2}$-BPS Wilson loop expectation value $\langle\mathcal{W}\rangle$ and also compute their large 't Hooft coupling ($\lambda\gg1$) expansions. The structure of these expansions is different from the previously studied models without fundamental hypermultiplets. In the more tractable Sp($2N$) case we find an exact resummed expression for the leading strong coupling terms at each order in the $1/N$ expansion. We also determine the exponentially suppressed at large λ contributions to the non-planar corrections to F and $\langle\mathcal{W}\rangle$ and comment on their resurgence properties. We discuss dual string theory interpretation of these strong coupling expansions.

Keywords: AdS-CFT Correspondence, $1/N$ Expansion

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1 Introduction and summary

An important problem in understanding detailed workings of AdS/CFT duality is to study $1/N$ corrections to superconformal gauge theory observables and their matching to string loop corrections. BPS Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory provides a remarkable

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example when its expectation value $\langle\mathcal{W}\rangle$ as a function of N and $\lambda = g^2_{\text{YM}} N$ can be found exactly [1]. Expanding first at large N and then at large λ one finds in the SU(N) theory

$$\begin{aligned}\langle\mathcal{W}\rangle^{\mathcal{N}=4}_{\text{SU(N)}} = e^{\frac{\lambda}{8N}(1-1/N)} L^{(1)}_{N-1}\left(-\frac{\lambda}{4N}\right) &= N\, e^{\sqrt{\lambda}} \sum_{p=0}^{\infty} c_p \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p}} \left[1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right] \\ &= e^{2\pi T} \sum_{p=0}^{\infty} c'_p \left(\frac{g_{\text{s}}}{\sqrt{T}}\right)^{2p-1} \left[1+\mathcal{O}\left(T^{-1}\right)\right], \end{aligned} \tag{1.1}$$

where we expressed the result in terms of the string coupling and tension of the dual $\text{AdS}_5 \times S^5$ string theory

$$g_{\text{s}} = \frac{\lambda}{4\pi N}, \qquad T = \frac{L^2}{2\pi a'} = \frac{\sqrt{\lambda}}{2\pi}, \qquad \frac{1}{N} = \frac{g_{\text{s}}}{\pi T^2}. \tag{1.2}$$

As was argued in [2], the particular structure (1.1) of the small g_{s}, large T expansion of $\langle\mathcal{W}\rangle$ is indeed expected on the string-theory side and may apply also to other closely related theories with less supersymmetry.

Indeed, the same expansion (1.1) was found recently for two special $\mathcal{N} = 2$ 4d superconformal models — SU(N) × SU(N) "orbifold" [3] and SU(N) "orientifold" [4] that are planar-equivalent to $\mathcal{N} = 4$ SYM theory. Here the localization approach [5, 6] allows one to expresses the expectation value $\langle\mathcal{W}\rangle$ in terms of a non-trivial matrix model integral. One is then able to extract the large λ behaviour of the leading non-planar $1/N$ correction, finding that it scales as $\lambda^{3/2}$ relative to the planar (i.e. $\mathcal{N} = 4$ SYM) term, in agreement with (1.1).

The aim of the present paper is to consider two other (SU(N) and Sp($2N$)) examples of $\mathcal{N} = 2$ "orientifold" superconformal models for which $\langle\mathcal{W}\rangle$ can be also computed using the localization matrix model of [5] (see also [7–9]). These models are still planar-equivalent to $\mathcal{N} = 4$ SYM but in contrast to the "orientifold" model studied in [4] ($\mathcal{N} = 2$ vector multiplet coupled to hypermultiplets in symmetric and in antisymmetric SU(N) representation) will contain a finite (N-independent) number n_{F} of hypermultiplets in the fundamental representation. The later are effectively related to the presence of (a finite number of) D7-branes in the dual string theory description and thus to a different type of the orbifold/orientifold of $\text{AdS}_5 \times S^5$ string theory than in the previous case of $n_{\text{F}} = 0$ [10–13]. We shall find that here the structure of the large N, large λ expansion of the BPS Wilson loop expectation value $\langle\mathcal{W}\rangle$ will be *different* from (1.1), raising an interesting question of how to explain this on the dual string theory side.

1.1 Review of $\mathcal{N} = 2$ models

Let us first review 4d $\mathcal{N} = 2$ superconformal gauge theories we are interested in. The condition of conformal invariance of an SU(N) model with a number of hypermultiplets in the adjoint, fundamental, rank-2 symmetric, and rank-2 antisymmetric representations is [14, 15]

$$\text{SU}(N): \qquad \beta_1 = 2N - 2N\, n_{\text{Adj}} - n_{\text{F}} - (N+2)\, n_{\text{S}} - (N-2)\, n_{\text{A}} = 0\,. \tag{1.3}$$

The non-zero number of adjoints can only be $n_{\rm Adj} = 1$ when we find the $\mathcal{N} = 4$ SYM ($n_{\rm F} = n_{\rm A} = n_{\rm S} = 0$). For $n_{\rm Adj} = 0$ we get $\mathcal{N} = 2$ superconformal models with $n_{\rm F} = 2N - (N+2)\, n_{\rm S} - (N-2)\, n_{\rm A}$. To have planar equivalence with $\mathcal{N} = 4$ SYM (and thus a relatively simple AdS dual) the number $n_{\rm F}$ should not depend on N. This implies that $n_{\rm S} + n_{\rm A} = 2$ and thus there are only two non-trivial solutions that we shall refer to as "SA" (symmetric+antisymmetric) and "FA" (fundamental+antisymmetric) models

$$\text{SU}(N): \qquad \text{SA}: \quad (n_{\rm F}, n_{\rm S}, n_{\rm A}) = (0,1,1)\,, \qquad \text{FA}: \quad (n_{\rm F}, n_{\rm S}, n_{\rm A}) = (4,0,2)\,. \tag{1.4}$$

Both $\mathcal{N} = 2$ theories are dual to certain orbifold/orientifold projections of $\text{AdS}_5 \times S^5$ superstring [13] and for that reason we shall refer to them respectively as the "SA-orientifold" and the "FA-orientifold". It is the SA-orientifold model that was discussed in [4] and here we shall study the FA-orientifold model.

For completeness, let us recall that the 4d conformal anomaly a and c coefficients of an $\mathcal{N} = 2$ superconformal model are determined by the free-theory values, i.e. in terms of the total number of the vector multiplets and hypermultiplets (counting also dimensions of their representations): $\text{a} = \frac{5}{24}\, \text{n}_\text{v} + \frac{1}{24}\, \text{n}_\text{h}$, $\text{c} = \frac{1}{6}\, \text{n}_\text{v} + \frac{1}{12}\, \text{n}_\text{h}$. The resulting explicit values are given below

SU(N)	a	c
$\mathcal{N} = 4$ SYM	$\frac{1}{4}N^2 - \frac{1}{4}$	$\frac{1}{4}N^2 - \frac{1}{4}$
$\mathcal{N} = 2$ SA	$\frac{1}{4}N^2 - \frac{5}{24}$	$\frac{1}{4}N^2 - \frac{1}{6}$
$\mathcal{N} = 2$ FA	$\frac{1}{4}N^2 + \frac{1}{8}N - \frac{5}{24}$	$\frac{1}{4}N^2 + \frac{1}{4}N - \frac{1}{6}$

Similarly, in the case of the Sp($2N$) gauge group the condition of conformal invariance of the $\mathcal{N} = 2$ model containing the adjoint, fundamental and antisymmetric hypermultiplets reads [14] (cf. (1.3))[1]

$$\text{Sp}(2N): \qquad \beta_1 = 2N + 2 - (2N+2)\, n_{\rm Adj} - n_{\rm F} - (2N-2)\, n_{\rm A} = 0\,. \tag{1.5}$$

The Sp($2N$) $\mathcal{N} = 4$ SYM theory corresponds to $n_{\rm Adj} = 1$, $n_{\rm F} = n_{\rm A} = 0$. For $n_{\rm Adj} = 0$ demanding planar equivalence to $\mathcal{N} = 4$ SYM implies that $n_{\rm F}$ should be independent of N and thus the only solution is the FA-orientifold model with $n_{\rm F} = 4$, $n_{\rm A} = 1$

$$\text{Sp}(2N): \qquad \text{FA}: \quad (n_{\rm F}, n_{\rm A}) = (4,1)\,. \tag{1.6}$$

[1] In this paper we shall denote by Sp($2N$) the compact symplectic group USp($2N$) = U($2N$) $\cap$ Sp($2N, C$) (sometimes also denoted as Sp(N)) so that Sp(2) = SU(2). The dimensions of its adjoint, fundamental and antisymmetric representations are, respectively, $\dim \text{Adj} = \dim[\text{Sp}(2N)] = N(2N+1)$, $\dim \text{F} = 2N$, $\dim \text{A} = N(2N-1) - 1$. Note while the groups Sp($2N$) and SO($2N$) and their representations are formally related by $N \to -N$ [16], the index of a representation that enters the 1-loop beta-function is always positive (i.e. its sign is changed at the same time with taking $N \to -N$). Thus the conformal invariance condition is not invariant and has different solutions for the two groups. For example, the antisymmetric representation of Sp($2N$) is mapped to the symmetric traceless representation of SO($2N$) with the index $2N + 2$ which is larger than the index of the adjoint SO($2N$) representation $2N - 2$. Thus there are no SO($2N$) conformal theories with hypermultiplets in the symmetric traceless representation [14].

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The corresponding conformal anomaly coefficients are given below:

Sp($2N$)	a	c
$\mathcal{N}=4$ SYM	$\frac{1}{2}N^2+\frac{1}{4}N$	$\frac{1}{2}N^2+\frac{1}{4}N$
$\mathcal{N}=2$ FA	$\frac{1}{2}N^2+\frac{1}{2}N-\frac{1}{24}$	$\frac{1}{2}N^2+\frac{3}{4}N-\frac{1}{12}$

1.2 Summary of the results

Let us now summarise the main results of this paper starting with the SU(N) case. As in the case of the SA-orientifold [4] the structure of the localization matrix model implies that the leading $1/N$ corrections to the Wilson loop expectation value can be expressed in terms of the corresponding corrections to the gauge theory free energy $F(\lambda, N) = -\log Z$ on 4-sphere. For that reason the main effort goes into the study for the large N expansion of F.

To recall, in the case of the SU(N) $\mathcal{N}=4$ SYM theory where the partition function Z is given by the Gaussian matrix model [1, 5] one finds (after subtracting the "trivial" UV divergence in a particular scheme, see also appendix A) [17]

$$\mathrm{SU}(N): \qquad F^{\mathcal{N}=4}(\lambda) = -\frac{1}{2}(N^2-1)\log\lambda\,. \tag{1.7}$$

The large N expansion of the free energy of the $\mathcal{N}=2$ FA-orientifold model which is planar-equivalent to the $\mathcal{N}=4$ SYM may be represented as

$$\mathrm{SU}(N): \qquad F(\lambda) = F^{\mathcal{N}=4}(\lambda) + N\,F_1(\lambda) + F_2(\lambda) + \mathcal{O}\left(\frac{1}{N}\right). \tag{1.8}$$

The F_1 term was absent in the case of the SA-orientifold in [4] (it is related to the presence of the fundamental hypermultiplets in the spectrum of this $\mathcal{N}=2$ model). F_1 admits an explicit integral representation in terms of Bessel functions (3.14) allowing to find its strong coupling expansion

$$F_1 \stackrel{\lambda\gg 1}{=} f_1\lambda + f_2\log\lambda + f_3 + f_4\,\lambda^{-1} + \mathcal{O}\left(e^{-\sqrt{\lambda}}\right), \tag{1.9}$$

$$f_1 = \frac{\log 2}{4\pi^2}, \quad f_2 = -\frac{1}{4}, \quad f_3 = \frac{1}{2}\log\pi + \frac{7}{6}\log 2 + \frac{3}{4} - 6\log\mathsf{A}, \quad f_4 = -\frac{\pi^2}{4}, \tag{1.10}$$

where A is the Gleisher's constant.[2] There is just a finite number of "polynomial" in large λ corrections and an infinite number of exponential $e^{-(2n+1)\sqrt{\lambda}}$ corrections reflecting the asymptotic nature of the strong coupling expansion (see (6.19); here we omit the $\lambda^{-1/4}$ prefactor of $e^{-\sqrt{\lambda}}$).

F_2 may be written as the sum of the two different contributions: a simpler one $\widetilde{F}_2$ which is related to F_1 by a differential relation and a more complicated one $\bar{F}_2$ which turns out to be the same as the leading $1/N^2$ correction to F in the SA-orientifold case in [4]

$$F_2(\lambda) = \widetilde{F}_2(\lambda) + \bar{F}_2(\lambda)\,, \qquad \widetilde{F}_2' = -\frac{\lambda}{2}\left[(\lambda\,F_1)''\right]^2, \tag{1.11}$$

[2]Note that $\log 2$ in f_1 originates from the Dirichlet η-function value $\eta(1) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = \log 2$ (see (6.3)).

where $(\dots)' = \frac{d}{d\lambda}(\dots)$. As a result,[3]

$$\widetilde{F}_2 \overset{\lambda\gg 1}{=} p_1\lambda^2 + p_2\,\lambda + p_3\log\lambda + p_4 + \mathcal{O}\left(e^{-\sqrt{\lambda}}\right), \tag{1.12}$$

$$\bar{F}_2 \overset{\lambda\gg 1}{=} k_1\lambda^{1/2} + k_2\log\lambda + k_3 + \mathcal{O}(\lambda^{-1/2}), \tag{1.13}$$

$$p_1 = -f_1^2\,, \qquad p_2 = -2f_1 f_2\,, \qquad p_3 = -\frac{1}{2}f_2^2\,, \quad \dots\,, \qquad k_1 = \frac{1}{2\pi}\,, \dots\,. \tag{1.14}$$

where the values of f_i were given in (1.10). The form of the exponential corrections in $\widetilde{F}_2$ follows from those in F_1 and the relation in (1.11), and similar corrections are expected in $\bar{F}_2$.

The large N expansion of the circular $\frac{1}{2}$-BPS Wilson loop expectation value in this $\mathcal{N}=2$ theory can be written as

$$\mathrm{SU}(N): \qquad \langle\mathcal{W}\rangle = N\,W_0(\lambda) + W_1(\lambda) + \frac{1}{N}\left[W_{0,2}(\lambda) + W_2(\lambda)\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{1.15}$$

where W_0 and $W_{0,2}$ are the leading $\mathcal{N}=4$ SYM contributions following from (1.1) [1, 18]

$$W_0 = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\,, \qquad\qquad W_{0,2} = \frac{1}{48}\left[-12\sqrt{\lambda}\,I_1(\sqrt{\lambda}) + \lambda\,I_2(\sqrt{\lambda})\right], \tag{1.16}$$

while W_1 and W_2 are the genuine $\mathcal{N}=2$ corrections. As we will show, they can be expressed in terms of the $1/N$ corrections F_1 and F_2 to the free energy (1.8) by the following remarkable differential relations (cf. (1.11))

$$W_1' = -\frac{\lambda}{4}\,W_0\,(\lambda\,F_1)''\,, \qquad\qquad W_2 = -\frac{\lambda^2}{4}\,W_0\,F_2'\,. \tag{1.17}$$

Using (1.9)–(1.14) in (1.17) and normalizing to the leading planar value

$$W_0 \overset{\lambda\gg 1}{=} \sqrt{\frac{2}{\pi}}\lambda^{-3/4}e^{\sqrt{\lambda}}\left[1 + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big)\right], \tag{1.18}$$

we then find for the strong coupling expansions of W_1 and W_2

$$\frac{W_1}{W_0} \overset{\lambda\gg 1}{=} -\,f_1\,\lambda^{3/2} + \frac{3}{2}f_1\lambda - \left(\frac{3}{8}f_1 + \frac{1}{2}f_2\right)\lambda^{1/2} + \mathcal{O}(\lambda^0)\,, \tag{1.19}$$

$$\frac{W_2}{W_0} \overset{\lambda\gg 1}{=} \frac{1}{2}f_1^2\,\lambda^3 + \frac{1}{2}f_1 f_2\,\lambda^2 - \frac{1}{8}k_1\lambda^{3/2} + \mathcal{O}(\lambda)\,. \tag{1.20}$$

Like F_1 in (1.8), the W_1 term in (1.15) was absent in the case of the SA-orientifold in [4] (where there were no odd powers in $1/N$ series). Also, in the SA-orientifold case the expansion of W_2/W_0 started with the $k_1\lambda^{3/2}$ term that originated from the $\bar{F}_2$ term in (1.13)

[3] The analysis in [4] showed that the leading large λ term in $\bar{F}_2$ is definitely $\lambda^{1/2}$. The derivation of its coefficient $k_1 = \frac{1}{2\pi}$ was based on partially heuristic analysis of the determinant of an infinite matrix, whose matrix elements admit an asymptotic expansion for large λ. A comparison with Padé resummation of the determinant revealed that $\frac{1}{2\pi}$ may actually be a lower estimate of the exact value of k_1. This issue will not be relevant for the large λ expansion in the models considered here where $\widetilde{F}_2$ is dominant over $\bar{F}_2$ at large coupling.

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in view of (1.17). The expressions (1.19), (1.20) also contain exponential corrections as follows from (1.9), (1.12) and (1.17).

Similar results are found in the case of the Sp$(2N)$ FA-orientifold model (1.6) which is more tractable as the corresponding localization $\mathcal{N}=2$ matrix model is simpler than in the SU(N) case. Here[4]

$$\text{Sp}(2N):\qquad F = F^{\mathcal{N}=4} + N\,\mathsf{F}_1(\lambda) + \mathsf{F}_2(\lambda) + \frac{1}{N}\mathsf{F}_3(\lambda) + \frac{1}{N^2}\mathsf{F}_4(\lambda) + \mathcal{O}\left(\frac{1}{N^3}\right), \tag{1.21}$$

$$F^{\mathcal{N}=4} = -\frac{1}{2}N(2N+1)\log\lambda\,. \tag{1.22}$$

It turns out that the structure of the corresponding matrix model implies that F_1, F_2 and F_3 can be expressed in terms of the function F_1 in (1.8) (and its integral $\widetilde{F}_2$ in (1.11)) that appeared in the SU(N) case

$$\mathsf{F}_1 = 2F_1\,, \qquad \mathsf{F}_2 = \frac{1}{2}(\lambda F_1)' + 2\,\widetilde{F}_2\,, \qquad \widetilde{F}_2' = -\frac{\lambda}{2}\left[(\lambda\, F_1)''\right]^2, \tag{1.23}$$

$$\mathsf{F}_3 = \frac{\lambda^2}{24}(\lambda F_1)''' - \frac{\lambda^2}{4}\Big[(\lambda F_1)''\Big]^2 + \frac{\lambda^3}{3}\Big[(\lambda F_1)''\Big]^3, \qquad \mathsf{F}_4 = -\frac{2\lambda^2}{4!}\Big(\lambda^3\left[(\lambda F_1)''\right]^4\Big)' + \dots\,. \tag{1.24}$$

Similar expressions in terms of derivatives of F_1 appear to exist also for higher F_n terms in (1.21).

Computing the strong-coupling expansion of F_n we find that (cf. (1.8), (1.9), (1.12))

$$F = F^{\mathcal{N}=4} + \Delta F \overset{\lambda\gg 1}{=} \Delta F_{\rm pol} - \left(N^2 + N - \frac{3}{16}\right)\log\lambda - \frac{\pi^2}{2}\frac{N}{\lambda} + \mathcal{O}(e^{-\sqrt{\lambda}})\,, \tag{1.25}$$

where $\Delta F_{\rm pol}$ stands for the polynomial in λ part of the strong coupling expansion. Note that $\log\lambda$ term in (1.25) receives contributions only at orders N^2, N and N^0 while the λ^{-1} term appears only at order N.

Remarkably, the sum of the leading large λ terms in $\Delta F_{\rm pol}$ at each order in $1/N$ appears to have a closed log expression ($f_1 = \frac{\log 2}{4\pi^2}$ as in (1.10))

$$\begin{aligned}\Delta F_{\rm pol} &= N\Big[2f_1\lambda + \mathcal{O}(\lambda^0)\Big] + \Big[2f_1^2\lambda^2 + \mathcal{O}(\lambda)\Big] + \frac{1}{N}\Big[\frac{8}{3}f_1^3\lambda^3 + \mathcal{O}(\lambda^2)\Big] + \mathcal{O}\left(\frac{1}{N^2}\right)\\ &= N^2\mathcal{F}\left(\frac{\lambda}{N}\right) + \dots\,, \qquad \mathcal{F}\left(\frac{\lambda}{N}\right) = \log\left(1 + 2f_1\frac{\lambda}{N}\right).\end{aligned} \tag{1.26}$$

Combined with the $N^2\log\lambda$ term in (1.25) the leading strong-coupling expression for F is then

$$F \overset{\lambda\gg 1}{=} -N^2\log\lambda + N^2\mathcal{F}\left(\frac{\lambda}{N}\right) + \dots = N^2\log\left(\lambda^{-1} + 2f_1N^{-1}\right) + \dots = N^2\log\Big[N^{-1}\left(g_{_{\rm YM}}^{-2} + 2f_1\right)\Big] + \dots\,, \tag{1.27}$$

suggesting possible role of a finite redefinition of the inverse coupling constant.

[4] Here we shall use the same definition for λ as in the SU(N) case, i.e. $\lambda = g_{_{\rm YM}}^2 N$ (i.e. without extra factor of 2 as, e.g., in [19]).

The large N expansion of the Wilson loop expectation value here can be written as (cf. (1.15))

$$\mathrm{Sp}(2N): \qquad \langle \mathcal{W}\rangle = \langle \mathcal{W}\rangle^{\mathcal{N}=4} + \langle \mathcal{W}\rangle\,, \qquad \Delta\langle \mathcal{W}\rangle = \mathsf{W}_1 + \frac{1}{N}\mathsf{W}_2 + \frac{1}{N^2}\mathsf{W}_3 + \mathcal{O}\left(\frac{1}{N^3}\right), \tag{1.28}$$

where the $\mathcal{N}=4$ Sp($2N$) SYM contribution is [7, 19] (cf. (1.1))

$$\langle \mathcal{W}\rangle^{\mathcal{N}=4} = 2\,e^{\frac{\lambda}{16N}} \sum_{k=0}^{N-1} L_{2k+1}\left(-\frac{\lambda}{8N}\right) = N\,\mathsf{W}_0 + \mathsf{W}_{0,1} + \frac{1}{N}\mathsf{W}_{0,2} + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{1.29}$$

$$\mathsf{W}_0 = \frac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = 2W_0\,, \qquad \mathsf{W}_{0,1} = \frac{1}{2} I_0(\sqrt{\lambda}) - \frac{1}{2}\,, \qquad \mathsf{W}_{0,2} = \frac{\lambda}{96}\, I_2(\sqrt{\lambda}). \tag{1.30}$$

As in the SU(N) case, one finds that the $\mathcal{N}=2$ corrections W_1 and W_2 are expressed in terms of $\mathsf{F}_1 = 2F_1$ and F_2 as in (1.17) so that

$$\mathsf{W}_1' = -\frac{\lambda}{4}\mathsf{W}_0\,(\lambda F_1)''\,, \qquad \mathsf{W}_2 = -\frac{\lambda^2}{8}\mathsf{W}_0\,\mathsf{F}_2' = -\frac{\lambda^2}{8}\mathsf{W}_0\left[\frac{1}{2}\,(\lambda F_1)'' - \lambda\left[(\lambda F_1)''\right]^2\right]. \tag{1.31}$$

Comparing W_1 and W_0 with W_1 and W_0 in the SU(N) case in (1.16), (1.17) we conclude that their ratio is the same for any λ. The analog of the strong-coupling expansions in (1.19), (1.20) is[5]

$$\frac{\mathsf{W}_1}{\mathsf{W}_0} \overset{\lambda\gg 1}{=} \frac{W_1}{W_0} = -f_1\,\lambda^{3/2} + \frac{3}{2}\,f_1\,\lambda - \left(\frac{3}{8}f_1 + \frac{1}{2}f_2\right)\lambda^{1/2} + \mathcal{O}(\lambda^0)\,, \tag{1.32}$$

$$\frac{\mathsf{W}_2}{\mathsf{W}_0} \overset{\lambda\gg 1}{=} \frac{1}{2}f_1^2\lambda^3 - \frac{1}{8}f_1(1-4f_2)\lambda^2 - \frac{1}{16}f_2(1-2f_2)\lambda + \mathcal{O}(e^{-\sqrt{\lambda}})\,. \tag{1.33}$$

Similar relations between higher order $1/N$ terms F_n in free energy (1.21) and W_n in (1.28) are expected also in general, with the dominant large λ term in F_n determining the strong coupling asymptotics of W_n. In particular,

$$\mathsf{W}_3 = -\frac{\lambda^{3/2}}{4!}\mathsf{W}_0\left[\lambda(\lambda F_1)''\right]^3 + \dots\,, \qquad \frac{\mathsf{W}_3}{\mathsf{W}_0} \overset{\lambda\gg 1}{=} -\frac{1}{3!}f_1^3\lambda^{9/2} + \mathcal{O}(\lambda^4)\,. \tag{1.34}$$

Combining the leading terms in (1.32), (1.33) and (1.34) suggests that the dominant (at each order in $1/N$) strong coupling terms in $\Delta\langle \mathcal{W}\rangle$ in (1.28) exponentiate as

$$\langle \mathcal{W}\rangle = (N\mathsf{W}_0 + \dots) + \Delta\langle \mathcal{W}\rangle \overset{\lambda\gg 1}{=} N\mathsf{W}_0 \exp\left[-f_1\frac{\lambda^{3/2}}{N}\right] + \dots\,. \tag{1.35}$$

This may be compared with similar exponentiation of the leading large λ terms in the $\mathcal{N}=4$ SYM case: as one finds from (1.1) in SU(N) case [1] and from (1.29) in the Sp($2N$)

[5] Note that the leading terms in (1.33) and (1.20) are the same but subleading terms have different structure.

case (see appendix C)

$$\mathrm{SU}(N): \qquad \langle \mathcal{W} \rangle^{\mathcal{N}=4} \overset{\lambda \gg 1}{=} N W_0 \exp\left[\frac{\lambda^{3/2}}{96N^2}\right] + \dots, \tag{1.36}$$

$$\mathrm{Sp}(2N): \qquad \langle \mathcal{W} \rangle^{\mathcal{N}=4} \overset{\lambda \gg 1}{=} 2N W_0 \left(1 + \frac{\lambda^{1/2}}{8N}\right) \exp\left[\frac{\lambda^{3/2}}{96\,(2N)^2}\right] + \dots, \tag{1.37}$$

where W_0 is given by (1.18). Note that the $\left(1 + \frac{\lambda^{1/2}}{8N}\right)$ prefactor that generates odd powers of $1/N$ in the expansion of $\langle \mathcal{W} \rangle^{\mathcal{N}=4}$ in $\mathrm{Sp}(2N)$ case in (1.37) can be absorbed into $e^{\sqrt{\lambda}}$ in W_0 by shifting $N \to N + \frac{1}{4}$ in the definition of $\lambda = g^2_{\text{YM}} N$ (assuming one keeps only the leading large λ term at each order in $1/N$).[6]

1.3 Comments on dual string theory interpretation

Let us now discuss string theory interpretation of these strong-coupling expansions derived on the gauge theory side. The $\mathrm{SU}(N)$ FA-orientifold (i.e. the $\mathcal{N} = 2$ $\mathrm{SU}(N)$ superconformal model with $n_{\text{F}} = 4$ and $n_{\text{A}} = 2$) may be engineered in flat-space type IIB superstring as a low-energy limit of the worldvolume theory on a stack of coincident N D3-branes in the presence of four D7-branes and one O7-plane (see [13] and references there).[7] Taking the large-N near-horizon limit of the underlying brane configuration one concludes that the dual string theory should be a projection $\mathrm{AdS}_5 \times S'^5$, $S'^5 = S^5/G_{\text{ori}}$, of the original $\mathrm{AdS}_5 \times S^5$ type IIB theory [13]. Here $\mathbb{Z}_{2,\text{orb}}$ of $G_{\text{ori}} = \mathbb{Z}_{2,\text{orb}} \times \mathbb{Z}_{2,\text{ori}}$ acts as $\varphi_1 \to \varphi_1 + \pi$, $\varphi_2 \to \varphi_2 + \pi$ and $\mathbb{Z}_{2,\text{ori}}$ acts as $\varphi_3 \to \varphi_3 + \pi$ on the coordinates of S^5 with the metric $ds^2_5 = d\theta_1^2 + \cos^2\theta_1\,(d\theta_2^2 + \cos^2\theta_2\,d\varphi_1^2 + \sin^2\theta_2\,d\varphi_2^2) + \sin^2\theta_1\,d\varphi_3^2$.

Similarly, the dual string theory for the $\mathrm{Sp}(2N)$ FA-orientifold (i.e. the $\mathcal{N} = 2$ $\mathrm{Sp}(2N)$ superconformal model with $n_{\text{F}} = 4$ and $n_{\text{A}} = 1$) corresponds [10, 11] to the near-horizon limit of N D3-branes with 8 D7-branes stuck on one O7-plane, i.e. is the type IIB superstring on $\mathrm{AdS}_5 \times S'^5$, $S'^5 = S^5/\mathbb{Z}_{2,\text{ori}}$ (D7 is wrapped on $\mathrm{AdS}_5 \times S^3$ where S^3 is fixed-point locus of $\mathbb{Z}_{2,\text{ori}}$).

In both $\mathrm{SU}(N)$ and $\mathrm{Sp}(2N)$ cases, the presence of D7-branes introduces the new D3-D7 open string sector (with massless modes being related to the fundamental hypermultiplets in the corresponding gauge theory). That means, in particular, that the dual string theory perturbation theory will involve both closed-string and open-string world-sheet topologies, i.e. corrections of both even and odd powers in g_{s}, corresponding to even and odd powers of $1/N$ on the gauge theory side.

While in the $\mathrm{SU}(N)$ $\mathcal{N} = 2$ model one expects contributions from only orientable surfaces (with topologies of 2-sphere with holes and handles) in the $\mathrm{Sp}(2N)$ case there should be additional contributions with non-orientable crosscups (as is also suggested by

[6] We thank S. Giombi for this observation.

[7] This implies modding out by the orientifold group $G_{\text{ori}} = \mathbb{Z}_{2,\text{orb}} \times \mathbb{Z}_{2,\text{ori}}$, where $\mathbb{Z}_{2,\text{orb}} = \{1, I_{6789}\}$ and $\mathbb{Z}_{2,\text{ori}} = \{1, I_{45}\,\Omega\,(-1)^{F_L}\}$. The inversions $I_{n_1 \dots n_r}$ act on the $\mathbb{R}^6$ (with directions $4, \dots, 9$) transverse to the D3-branes as $\mathbb{Z}_{2,\text{orb}}$: $x_{6,7,8,9} \to -x_{6,7,8,9}$ and $\mathbb{Z}_{2,\text{ori}}$: $x_{4,5} \to -x_{4,5}$. The fixed-point set of $\mathbb{Z}_{2,\text{ori}}$ is the hyperplane $x_{4,5} = 0$, which corresponds to the position of the O7-plane and four D7-branes, while the fixed set of $\mathbb{Z}_{2,\text{orb}}$ is the hyperplane $x_{6,7,8,9} = 0$.

the structure of the $1/N$ expansion of perturbative gauge theory diagrams, cf. [7]). In the $Sp(2N)$ $\mathcal{N}=4$ SYM case all odd-power $1/N$ contributions should come from crosscups [20], while in the $Sp(2N)$ $\mathcal{N}=2$ FA-orientifold model there should be additional contributions from world sheets with boundaries introduced due to the presence of D7-branes (and related to the presence of fundamental hypermultiplets on the gauge theory side), see also [21].

Accounting for the open string (type I, or disc) term in the dual string theory effective action that here may be interpreted as the D7-brane world-volume action allowed to give [22, 23] the holographic interpretation of the order N term in the (super)conformal anomalies of the $Sp(2N)$ FA-orientifold (cf. table below eq. (1.6)).

The AdS/CFT duality suggests that the conformal gauge theory free energy F on S^4 should be matched with the string partition function $Z_{\rm str}$ in $AdS_5 \times S'^5$. The leading 2-sphere topology contribution to the (properly defined) $Z_{\rm str}$ is approximated by the type IIB supergravity action (plus a'-corrections). In particular, in the maximally supersymmetric $\mathcal{N}=4$ $SU(N)$ SYM case one can match the leading N^2 term in the free energy $F = 4\mathrm{a}\log(\Lambda\,\mathrm{r}) + \mathrm{f}_0,\ \mathrm{a} = \frac{1}{4}(N^2-1)$, with the leading term in the supergravity action proportional to the (IR divergent) volume of AdS_5 (reproducing, in particular, the conformal anomaly [24, 25]). Here Λ is a UV cutoff, r is the radius of S^4 and f_0 is a regularization scheme dependent constant (cf. (A.2)). In the particular scheme selected by the localization matrix model representation for the gauge-theory partition function $Z = e^{-F}$ (with the λ-independent measure) one finds that $F^{\mathcal{N}=4} = -\frac{1}{2}(N^2-1)\log\lambda$. Then the leading N^2 term in $F^{\mathcal{N}=4}$ can be matched [17] with the on-shell value of the supergravity term in the string effective action in $AdS_5 \times S^5$ (assuming particular IR cutoff in the AdS_5 volume).[8] The subleading $\frac{1}{2}\log\lambda$ term should come from the 1-loop (torus) contribution to $Z_{\rm str}$, which is again proportional to the regularized AdS_5 volume and receives contributions only from short multiplets, i.e. is the same as the 1-loop supergravity correction [26].

The localization matrix model result for the large N, large λ expansion of the free energy of the $SU(N)$ FA-orientifold model in (1.8)–(1.14) may be written as

$$\begin{aligned} F(\lambda;N) &\overset{\lambda\gg 1}{=} -\frac{1}{2}N^2\log\lambda + N\left(f_1\,\lambda + f_2\,\log\lambda + f_3 + \dots\right) \\ &\quad + \Big(p_1\,\lambda^2 + p_2\lambda + k_1\lambda^{1/2} + k_2'\,\log\lambda + k_3' + \dots\Big) + \mathcal{O}\left(\frac{1}{N}\right), \end{aligned} \tag{1.38}$$

where $k_2' = k_2 + p_3,\ k_3' = k_3 + p_4$. The leading $1/N$ terms in the $Sp(2N)$ FA-orientifold case are similar (see (1.25), (1.26), (1.27)).

Let us note that in the $SU(N)$ case the $-2\log\lambda$ term in (1.38) has the coefficient $\frac{1}{4}N^2 + \frac{1}{8}N - \frac{1}{2}k_2'$. In the $Sp(2N)$ case the analog of this coefficient in (1.25) is $\frac{1}{2}N^2 + \frac{1}{2}N - \frac{3}{16}$. Thus in both cases not only the N^2 term (as expected from the planar equivalence)[9] but also the order N term is the same as in the a-anomaly coefficients of the two theories (see

[8] On the AdS_5 side the IR cutoff ℓ is measured in units of the AdS_5 radius L and is related to the product of the radius r of S^4 and UV cutoff Λ as $\mathrm{r}\Lambda = \frac{L\ell}{\mathrm{a}'} = \sqrt{\lambda}\frac{\ell}{L}$ [17]. Then the regularized AdS_5 volume (with power ℓ^n divergences dropped) scales as $\log\frac{\ell}{L} \to -\log\sqrt{\lambda} + \log(\Lambda\mathrm{r})$, suggesting that $F = 4\mathrm{a}\log(\Lambda\mathrm{r}) + \dots \to -2\mathrm{a}\log\lambda + \dots$.

[9] In the case of the $\mathcal{N}=4$ SYM theory with the group $Sp(2N)$ which may be viewed as an orientifold projection of $U(2N)$ theory and which is dual to type IIB string on $AdS_5 \times \mathbb{RP}^5$ [20] the presence of the

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the tables below eq. (1.4)). At the same time, the order N^0 coefficient of $\log\lambda$ in the Sp($2N$) case does not match the one in the conformal anomaly. This is not surprising: as discussed in appendix A below, in contrast to what happens in the $\mathcal{N}=4$ SYM case, in the $\mathcal{N}=2$ theory cases there is no a priori reason why the $\log\lambda$ term in the strong-coupling limit of the free energy derived from the localization matrix model should have the conformal a-anomaly as its coefficient.

Rewriting (1.38) in terms of the dual string theory coupling and string tension as defined in (1.2) we get (renaming coefficients to absorb factors of 2 and π)[10]

$$F(T,g_s) \stackrel{T\gg 1}{=} -\frac{\pi^2T^4}{g_s^2}\log(2\pi T)+\frac{\pi T^2}{g_s}\Big(f_1' T^2+f_2'\log T+f_3'+\dots\Big)$$
$$+\Big(p_1' T^4+p_2' T^2+k_1' T+k_2''\log T+k_3''+\dots\Big)+\mathcal{O}(g_s)\,. \qquad (1.39)$$

The leading (2-sphere) term in the tree-level string theory effective action $\frac{1}{g_s^2\mathrm{a}'^4}\int d^{10}x\sqrt{g}(R+\dots)$ evaluated on the $\mathrm{AdS}_5\times S'^5$ background is expected to match the $\frac{1}{g_s^2}$ term in (1.39) (after using, as in the $\mathcal{N}=4$ SYM case [17], the IR cutoff related to T in the AdS volume).

The $\frac{1}{g_s}$ term in (1.39) should come from the disc contribution, and, in the Sp($2N$) case, also from the crosscup topology. In particular, one may expect the $\frac{T^2}{g_s}\log T$ term to originate from the curvature squared term $\frac{1}{g_s\mathrm{a}'^2}\int d^8x\sqrt{g}\,RR$ in the D7-brane action (with D7-brane wrapping AdS_5 and S^3 from S'^5). The background value of this term is proportional to the AdS_5 volume and thus after the same IR regularization it should give the $\frac{T^2}{g_s}\log T$ contribution. In [23] the $\frac{1}{g_s\mathrm{a}'^2}\int d^8x\sqrt{g}\,RR$ term was shown to reproduce the order N term in the conformal anomaly of the Sp($2N$) FA-orientifold model. This is consistent with the above observation that the order N term in the coefficient of the $\log\lambda$ in (1.38) or $\log T$ in (1.39) is the same as in the a-anomaly coefficient of the corresponding $\mathcal{N}=2$ superconformal model.

The interpretation of the $\frac{T^4}{g_s}$ term in (1.39) is not immediately clear. Naively, such term could come from the D7-brane tension, i.e. $\frac{1}{g_s\mathrm{a}'^4}\int d^8x\sqrt{g}$ but this term should cancel against the orientifold (crosscup) contribution (cf. [27]), so that the leading term in the D7-brane action should be the above curvature-squared term. The order g_s^0 terms in (1.39) should come from the closed-string (torus) and open-string (annulus or disc with crosscup)

O3-plane (carrying RR charge of $\frac14$) leads to the effective shift of N by $\frac14$ and thus to the expression $L^4=4\pi g_s\left(2N+\frac12\right)\mathrm{a}'^2$ for the AdS radius. As a result, one reproduces both leading N^2 and N terms in the conformal anomaly from the on-shell value of the 10d supergravity action [19, 23]. For example, the $\mathcal{N}=4$ Sp($2N$) SYM free energy in (1.22) may be written as $F=-N^2\log\lambda-\frac12 N\log\lambda$ or as $F=-\frac14\left[\left(2N+\frac12\right)^2-\frac14\right]\log\lambda$. From the flat space perspective, the shift $N\to N+\frac14$ may be equivalently attributed to the crosscup contributions (cf. [23]). One may also interpret the odd-power $1/N$ terms in the Wilson loop expectation value of the $\mathcal{N}=4$ Sp($2N$) theory [7] (see (7.36), (7.37)) as coming from the crosscup contributions, but they can also be formally generated (at least in the large λ expansion) by shifting $N\to N+\frac14$ in the semiclassical string tension prefactor $e^{2\pi T}$ ($2\pi T=\sqrt{\lambda}=\frac{L^2}{\mathrm{a}'}$ with $g_{\rm YM}^2=2\times 4\pi g_s$, $\lambda=g_{\rm YM}^2 N$) of the even-power $1/N$ terms in (1.37) (we thank S. Giombi for a discussion of this issue).

[10] In contrast to the $\mathcal{N}=4$ SYM case, in the $\mathcal{N}=2$ Sp($2N$) case we shall assume that N is not shifted in the definition of AdS_5 radius and string tension and will also ignore possible extra factor of 2 in the relation between g_s and $g_{\rm YM}^2$.

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1-loop corrections. Since the compact S'^5 part of the background is not smooth (orbifold action has fixed points) they may originate from "localized" contributions (rather than "extensive" contributions proportional to the volume of $\mathrm{AdS}^5 \times S'^5$ like terms in the local part of the string effective action).

The resummed expression for leading strong coupling terms in the free energy of the Sp(2N) theory (1.25), (1.26) written in terms of the string coupling and string tension in (1.2) is (we use that $f_1 = \frac{\log 2}{4\pi^2}$)

$$F \stackrel{T\gg 1}{=} \frac{\pi^2 T^4}{g_s^2}\left[\log\left(1+\frac{2\log 2}{\pi}g_s\right)+\dots\right]-\left(\frac{\pi^2T^4}{g_s^2}+\frac{\pi T^2}{g_s}-\frac{3}{16}\right)\log(2\pi T)-\frac{\pi}{8g_s}+\mathcal{O}(e^{-2\pi T})\,. \tag{1.40}$$

Remarkably, the leading log term (dots stand for terms that are subleading in $1/T$ at each order in g_s) has non-trivial dependence only on the string coupling. The special $-\frac{\pi}{8g_s}$ term (that also depends only on g_s) should be a particular crosscup contribution. The exponential corrections should have a world-sheet instanton interpretation, i.e. should be related to world sheets wrapping compact S^2 parts of S'^5 that are non-contractable and thus stable due to orbifolding (see also discussion in section 6.3).

The large N, large λ expansion of the Wilson loop expectation values in the SU(N) and Sp(2N) FA-orientifold models may be written as (see (1.15), (1.19), (1.20), (1.18) and (1.28), (1.32), (1.33))

$$\begin{aligned}\langle \mathcal{W}\rangle \stackrel{\lambda\gg 1}{=} e^{\sqrt{\lambda}}\Big[&N(b_0\lambda^{-3/4}+b_{01}\lambda^{-1/4}+\dots)+(b_1\lambda^{3/4}+b_{12}\lambda^{1/4}+\dots)\\ &+\frac{1}{N}\,(b_2\lambda^{9/4}+b_{21}\lambda^{5/4}+\dots)+\mathcal{O}\left(\frac{1}{N^2}\right)\Big]\,.\end{aligned} \tag{1.41}$$

Expressed in terms of the string coupling and tension in (1.2) the leading strong coupling terms in (1.41) become

$$\langle \mathcal{W}\rangle \stackrel{T\gg 1}{=} e^{2\pi T}\left(b_0'\frac{T^{1/2}}{g_s}+b_1' T^{3/2}+b_2' g_s T^{5/2}+\dots\right)=\frac{T^{1/2}}{g_s}e^{2\pi T}\left(b_0'+b_1' g_s T+b_2' g_s^2 T^2+\dots\right). \tag{1.42}$$

The computation of $\langle \mathcal{W}\rangle$ on the string side should proceed in a similar way as for the circular loop in the $\mathrm{AdS}_5 \times S^5$ case [2, 28] (the minimal surface ending on a circle at the boundary of AdS_5 is the same AdS_2 one). The crucial difference is the presence of a new open-string sector and thus extra "disc with holes" and also (in the Sp(2N) case) "disc with crosscups" diagrams, in addition to the "disc with handles" ones. In the SU(N) case the structure of subleading terms in (1.41), (1.42) is different compared to the $\mathcal{N} = 4$ SYM case in (1.1). In particular, the order g_s^0 term in (1.42) should correspond to the annulus contribution (with one boundary with Dirichlet and one — with Neumann boundary conditions).

The prediction (1.35) for the resummation of the leading large λ terms in the Sp(2N) theory is the following specification of (1.42)

$$\langle \mathcal{W}\rangle \stackrel{T\gg 1}{=} \frac{T^{1/2}}{\pi\, g_s}\, e^{2\pi T}\, e^{-8\pi^2 f_1 g_s T}+\dots=\frac{T^{1/2}}{\pi\, g_s}\,\exp\left[2\pi T\left(1-\frac{\log 2}{\pi}\,g_s\right)\right]+\dots\,, \tag{1.43}$$

where we used (1.18) and $f_1 = \frac{\log 2}{4\pi^2}$ from (1.10). Note that the structure in the exponent that involves a function of $1 + c\, g_s$ is similar to the one of the first log term in the free energy in (1.40). The expression (1.43) may be compared with the leading-order one in the case of, e.g., SU(N) $\mathcal{N} = 4$ SYM theory (1.36) (the Sp($2N$) result (1.37) is similar, cf. footnote 9)

$$\langle \mathcal{W} \rangle \overset{T \gg 1}{=} \frac{T^{1/2}}{2\pi\, g_s} \exp\left[2\pi T + \frac{\pi}{12} \frac{g_s^2}{T}\right] + \dots, \tag{1.44}$$

that should represent the sum of handle insertions on the disc [2]. Similarly, (1.43) should be summing up the leading crosscup insertions.

Finally, let us note that the exact in λ differential relations like (1.17), (1.31) between the $1/N$ corrections to the free energy and the Wilson loop expectation value that we find from the localization matrix model representation on the gauge theory side appear to be very non-trivial on the dual string theory side where F and $\langle \mathcal{W} \rangle$ are computed using quite different procedures. It would be interesting to uncover their string theory interpretation.

The rest of this paper is organized as follows. We shall first discuss the SU(N) case. In section 2 we shall review the structure of the matrix model representation for the partition function of the $\mathcal{N} = 2$ superconformal FA-orientifold theory. In section 3 we shall find the explicit representations for the leading non-planar corrections F_1 and F_2 to its free energy.

In section 4 we shall discuss the matrix model representation for the Wilson loop expectation value $\langle \mathcal{W} \rangle$ and in section 5 find the general relations between the $1/N$ terms in $\langle \mathcal{W} \rangle$ and the free energy F. Section 6 will contain the results of the strong-coupling expansion of the $1/N$ terms in $\langle \mathcal{W} \rangle$ and F. In particular, in section 6.3 we shall discuss the structure of exponentially small $e^{-n\sqrt{\lambda}}$ corrections to the leading non-planar term in F, their resurgence properties and comment on their possible string theory interpretation.

Section 7 will be devoted to a similar analysis in the Sp($2N$) FA-orientifold model: matrix model representation, structure of $1/N$ corrections to the free energy and $\langle \mathcal{W} \rangle$ and strong-coupling expansions. This case turns out be much simpler than the SU(N) one and we are able to determine the structure of the large λ asymptotics of free energy in rather explicit way.

In aection A we will review the general structure of the partition function of $\mathcal{N} = 2$ models as described by the localization matrix model and explain how it encodes the information about the value of the conformal anomaly a-coefficient of the $\mathcal{N} = 2$ model. Appendix B will contain some details of derivation of the strong-coupling expansion of F_1 using Mellin transform. In appendix C we will discuss the relation between the $1/N$ coefficients in the Wilson loop and in the free energy in the case of the Sp($2N$) theory and their large λ asymptotics.

2 Matrix model representation for $\mathcal{N} = 2$ SU(N) theory

Using supersymmetric localization, the partition function of an $\mathcal{N} = 2$ gauge theory on a sphere S^4 of unit radius may be written as a matrix integral over the eigenvalues $\{m\}_{r=1}^N$

of a $N \times N$ hermitian traceless matrix m [5] (see also appendix A)

$$\hat{Z} \equiv e^{-F} = \mathcal{N} \int \mathcal{D}m\, e^{-S_0(m)-S_{\rm int}(m)}\,, \qquad S_0(m) = \frac{8\pi^2 N}{\lambda}\,\mathrm{tr}\, m^2, \qquad \lambda = g^2_{\rm YM}\, N\,, \tag{2.1}$$

$$\mathcal{D}m \equiv \prod_{r=1}^{N} dm_r\, \delta\left(\sum_{s=1}^{N} m_s\right) [\Delta(m)]^2\,, \qquad \Delta(m) = \prod_{1\le r<s\le N} (m_s - m_s). \tag{2.2}$$

The "interacting action" $S_{\rm int}(m)$ that vanishes in the $\mathcal{N} = 4$ theory is non-trivial for the $\mathcal{N} = 2$ theories. We will neglect the instanton contribution since we are going to consider the $1/N$ expansion. In the case of the $\mathcal{N} = 2$ model containing hypermultiplets in the fundamental, symmetric and antisymmetric representations of $SU(N)$ (with numbers subject to the conformal invariance condition (1.3)) one finds (see e.g. [29])

$$\begin{aligned} S_{\rm int}(m) &= \sum_{r=1}^{N} \left[n_{\rm F} \log H(m_r) + n_{\rm S} \log H(2m_r)\right] \\ &\quad + \sum_{r<s=1}^{N} \left[(n_{\rm S} + n_{\rm A}) \log H(m_r + m_s) - 2 \log H(m_r - m_s)\right], \end{aligned} \tag{2.3}$$

where H is given in terms of the Barnes G-function[11]

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} = e^{-(1+\gamma_{\rm E})\, x^2}\, \mathrm{G}(1 + ix)\, \mathrm{G}(1 - ix)\,. \tag{2.4}$$

We will normalize the $\mathcal{N} = 2$ partition function (2.1) to its $\mathcal{N} = 4$ SYM value. After scaling the matrix $m \to a$ according to

$$a = \sqrt{\frac{8\pi^2 N}{\lambda}}\, m\,, \tag{2.5}$$

the normalized partition function of the FA-orientifold in (1.4) ($n_{\rm F} = 4$, $n_{\rm S} = 0$, $n_{\rm A} = 2$) may be written as

$$Z = \langle e^{-S_{\rm int}(a)} \rangle = \int Da\, e^{-\mathrm{tr}\, a^2}\; e^{-S_{\rm int}(a)}\,, \qquad \int Da\, e^{-\mathrm{tr}\, a^2} = 1\,, \tag{2.6}$$

$$S_{\rm int}(a) \equiv \mathrm{S}_1 + \mathrm{S}_2 = \sum_{i=1}^{\infty} B_i(\lambda)\; \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2} + \sum_{i,j=1}^{\infty} C_{ij}(\lambda)\; \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+1} \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2j+1}, \tag{2.7}$$

$$B_i(\lambda) = 4\left(\frac{\lambda}{8\pi^2}\right)^{i+1} \frac{(-1)^i}{i+1} \zeta_{2i+1}(1 - 2^{2i})\,, \tag{2.8}$$

$$C_{ij}(\lambda) = 4\left(\frac{\lambda}{8\pi^2}\right)^{i+j+1} (-1)^{i+j}\, \zeta_{2i+2j+1} \frac{\Gamma(2i + 2j + 2)}{\Gamma(2i + 2)\,\Gamma(2j + 2)}\,, \tag{2.9}$$

where $\zeta_{2i+1} \equiv \zeta(2i + 1)$ are the Riemann ζ-function values.

[11]Note that the exponential prefactor in the r.h.s. of (2.4) cancels in $S_{\rm int}$ in superconformal models (with $n_{\rm F}$ satisfying (1.3)).

Z in (2.6) is related to the free energy as

$$Z = e^{-\Delta F}, \qquad \Delta F = F^{\mathcal{N}=2} - F^{\mathcal{N}=4}, \qquad F^{\mathcal{N}=4} = -\frac{1}{2}(N^2-1)\log\lambda\,. \tag{2.10}$$

Expanding ΔF at large N we find that the leading N^2 term cancels due to planar equivalence[12] so that

$$\Delta F(\lambda) = N\,F_1(\lambda) + F_2(\lambda) + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.11}$$

The order N term was absent in the case of the SA-orientifold [4] where $n_{\rm F} = 0$.

The weak coupling expansions of F_1 and F_2 are readily computed by doing the matrix model integrals in (2.6) (here we set $\hat\lambda = \frac{\lambda}{8\pi^2}$)

$$\begin{aligned} F_1 =\,& 3\zeta_3\hat\lambda^2 - \frac{25}{2}\zeta_5\hat\lambda^3 + \frac{441}{8}\zeta_7\hat\lambda^4 - \frac{1071}{4}\zeta_9\hat\lambda^5 + \frac{11253}{8}\zeta_{11}\hat\lambda^6 - \frac{250965}{32}\zeta_{13}\hat\lambda^7 \\ & + \frac{11713845}{256}\zeta_{15}\hat\lambda^8 - \frac{53105195}{192}\zeta_{17}\hat\lambda^9 + \frac{1100738457}{640}\zeta_{19}\hat\lambda^{10} + \cdots, \end{aligned} \tag{2.12}$$

$$\begin{aligned} F_2 =\,& 5\zeta_5\hat\lambda^3 - \left(\frac{81}{2}\zeta_3^2 + \frac{105}{2}\zeta_7\right)\hat\lambda^4 + (540\zeta_3\zeta_5 + 441\zeta_9)\hat\lambda^5 - \left(1900\zeta_5^2 + \frac{6615}{2}\zeta_3\zeta_7 + 3465\zeta_{11}\right)\hat\lambda^6 \\ & + \left(24150\zeta_5\zeta_7 + 20655\zeta_3\zeta_9 + \frac{212355}{8}\zeta_{13}\right)\hat\lambda^7 - \left(\frac{5044305}{64}\zeta_7^2 + \frac{1238895}{8}\zeta_5\zeta_9 + \frac{2126817}{16}\zeta_3\zeta_{11}\right. \\ & \left. + \frac{6441435}{32}\zeta_{15}\right)\hat\lambda^8 + \left(\frac{500}{3}\zeta_5^3 + \frac{4125555}{4}\zeta_7\zeta_9 + 1016400\zeta_5\zeta_{11} + \frac{1756755}{2}\zeta_3\zeta_{13} + \frac{12167155}{8}\zeta_{17}\right)\hat\lambda^9 \\ & - \left(5250\zeta_5^2\zeta_7 + \frac{54846477}{16}\zeta_9^2 + \frac{110007513}{16}\zeta_7\zeta_{11} + \frac{13635765}{2}\zeta_5\zeta_{13}\right. \\ & \left. + \frac{189764289}{32}\zeta_3\zeta_{15} + \frac{91869921}{8}\zeta_{19}\right)\hat\lambda^{10} + \cdots. \end{aligned} \tag{2.13}$$

We shall see that as in the case of the SA-orientifold in [4], the large N expansion of the BPS Wilson loop expectation value can be expressed in terms of F, so it is important to study the latter first.

3 Explicit representation for free energy corrections F_1 and F_2

Following the same strategy as in [4] we can find the explicit representations of the leading and next-to-leading terms in the $1/N$ expansion of the free energy (2.11). To this aim, let us introduce the generating function

$$X(\eta,\chi) = \int Da\, e^{-\operatorname{tr} a^2}\, e^{V(\eta,\chi,a)} \equiv \langle e^V \rangle\,, \tag{3.1}$$

$$V(\eta,\chi,a) = \sum_{i=1}^{\infty} \eta_i \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+1} + \sum_{i=1}^{\infty} \chi_i \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}. \tag{3.2}$$

[12] Note, in particular, that at large N the number of hypers in 2 antisymmetric representations $2\times\frac{N(N-1)}{2} \approx N^2$ is the same as in the adjoint representation $N^2-1 \approx N^2$.

Expanding in powers of the "sources" η_i, χ_i and evaluating the integrals over a gives

$$\begin{aligned}\log X(\eta,\chi) &= N\left(\frac{1}{2}\chi_1+\frac{5}{8}\chi_2+\cdots\right)+\left(\frac{3}{16}\eta_1^2+\frac{15}{16}\eta_1\eta_2+\frac{5}{4}\eta_2^2+\frac{63}{32}\eta_1\eta_3+\frac{175}{32}\eta_2\eta_3+\frac{1575}{256}\eta_3^2+\cdots\right)\\ &\quad+\left(\frac{9}{8}\chi_1^2+\frac{9}{2}\chi_1\chi_2+\frac{75}{16}\chi_2^2+\cdots\right)+\mathcal{O}\left(\frac{1}{N}\right)\\ &= N\,R_i\chi_i+Q_{ij}\eta_i\eta_j+\widetilde{Q}_{ij}\chi_i\chi_j+\mathcal{O}\left(\frac{1}{N}\right),\end{aligned} \tag{3.3}$$

where we assume summation over $i,j=1,\ldots,\infty$. The linear in χ terms in (3.3) have the following general form

$$R_i\chi_i = N^{-1}\sum_{i=1}^{\infty}\chi_i\left\langle \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}\right\rangle = \sum_{i=1}^{\infty}\chi_i\,\frac{1}{2^{i+1}(i+2)}\binom{2i+2}{i+1}, \tag{3.4}$$

where the coefficient R_i may be written as

$$R_i = \frac{2^{i+1}\,\Gamma\left(i+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(i+3)}\,. \tag{3.5}$$

The infinite-dimensional matrices Q and $\widetilde{Q}$ in (3.3) can be expressed in terms of the connected correlators of $\mathrm{tr}\,a^n$ (see e.g. [30]; here $\langle AB\rangle_c \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$)

$$\langle \mathrm{tr}\,a^{2k_1+1}\ \mathrm{tr}\,a^{2k_2+1}\rangle = N^{k_1+k_2+1}\frac{2^{k_1+k_2+1}\,k_1\,k_2\Gamma\left(k_1+\frac{3}{2}\right)\Gamma\left(k_2+\frac{3}{2}\right)}{\pi\,(k_1+k_2+1)\Gamma(k_1+2)\Gamma(k_2+2)}, \tag{3.6}$$

$$\langle \mathrm{tr}\,a^{2k_1}\ \mathrm{tr}\,a^{2k_2}\rangle_c = N^{k_1+k_2}\frac{2^{k_1+k_2}\Gamma\left(k_1+\frac{1}{2}\right)\Gamma\left(k_2+\frac{1}{2}\right)}{\pi\,(k_1+k_2)\Gamma(k_1)\Gamma(k_2)}. \tag{3.7}$$

The matrix Q_{ij} is same as the one that appeared in the case of the SA-orientifold in [4]

$$Q_{ij} = \frac{1}{\pi}\frac{2^{i+j}\,i\,j\,\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{(i+j+1)\,\Gamma(i+2)\,\Gamma(j+2)}\,, \tag{3.8}$$

while for $\widetilde{Q}_{ij}$ we find

$$\widetilde{Q}_{ij} = \frac{1}{\pi}\,\frac{2^{i+j+1}\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{(i+j+2)\,\Gamma(i+1)\Gamma(j+1)} = \frac{2\,(i+1)(j+1)(i+j+1)}{i\,j\,(i+j+2)}\,Q_{ij}. \tag{3.9}$$

Using (2.7), the leading terms in the large N expansion of the free energy ΔF in (2.11) may then be represented as

$$\begin{aligned} e^{-NF_1-F_2} &= e^{-C_{ij}\frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j}-B_i\frac{\partial}{\partial\chi_i}}\,X(\eta,\chi)\Big|_{\eta=\chi=0}\\ &= e^{-C_{ij}\frac{\partial}{\partial\eta_i}\frac{\partial}{\partial\eta_j}-B_i\frac{\partial}{\partial\chi_i}}\,e^{N\,R_i\chi_i+Q_{ij}\eta_i\eta_j+\widetilde{Q}_{ij}\chi_i\chi_j}\Big|_{\eta=\chi=0}, \end{aligned} \tag{3.10}$$

where $B_i(\lambda)$ and $C_{ij}(\lambda)$ were defined in (2.8), (2.9). To compute (3.10) we may use that

$$e^{-B_i\partial_i}f(\chi_i) = f(\chi_i-B_i)\,, \qquad e^{-C_{ij}\partial_i\partial_j} = \int dy\,e^{-\frac{1}{4}C^{-1}_{ij}y_iy_j+y_i\partial_i}\,. \tag{3.11}$$

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This leads to an explicit weak coupling expansion of the leading large N correction to the free energy:

$$F_1 = \sum_{i=1}^{\infty} R_i\, B_i = -\frac{1}{\sqrt{\pi}} \sum_{i=1}^{\infty} \frac{(-1)^i}{(i+1)} \frac{\Gamma\left(i+\frac{3}{2}\right)}{\Gamma(i+3)} \left(1-2^{-2i}\right) \zeta_{2i+1} \left(\frac{\lambda}{\pi^2}\right)^{i+1} . \tag{3.12}$$

This weak coupling expansion is clearly convergent, with radius of convergence π^2. It can be summed up into an integral representation using the identity:

$$\left(1-2^{-2i}\right) \zeta_{2i+1} = \frac{1}{\Gamma(2i+1)} \int_0^{\infty} dt\, \frac{t^{2i}}{e^t+1} . \tag{3.13}$$

This leads to the compact expression

$$F_1(\lambda) = \frac{2}{\sqrt{\lambda}} \int_0^{\infty} dt\, \frac{e^{2\pi t}}{(e^{2\pi t}+1)^2} \left[\frac{J_1(2t\sqrt{\lambda}) - t\sqrt{\lambda} + \frac{1}{2}(t\sqrt{\lambda})^3}{t^2}\right] . \tag{3.14}$$

It is straightforward to check that the expansion of the Bessel J_1 function, combined with the identity (3.13), leads to the weak coupling expansion in (2.12) and (3.12). However, the integral representation (3.14) can also be used to analyze the strong coupling expansion, which is an asymptotic expansion, in contrast to the convergent weak coupling expansion (3.12). The strong coupling expansion is discussed below in section 6.

The next subleading correction to the free energy, the $O(N^0)$ term F_2 in (2.11), may be naturally split as

$$F_2(\lambda) = \bar{F}_2(\lambda) + \widetilde{F}_2(\lambda) , \tag{3.15}$$

where $\bar{F}_2$ comes from the $Q_{ij}\eta_i\eta_j$ part of (3.10) (i.e. depends on C_{ij} and Q_{ij}). This $\bar{F}_2$ part is identical to the one for the SA-orientifold found in [4] and can be written as

$$\bar{F}_2(\lambda) = \frac{1}{2} \log\det(1+4CQ) = \frac{1}{2}\log\det(1+M) , \tag{3.16}$$

$$M_{ij} = 8\sqrt{2i+1}\sqrt{2j+1} \sum_{k=0}^{\infty} (-1)^k\, c_{ijk}\, \zeta_{2i+2j+2k+1} \left(\frac{\lambda}{16\pi^2}\right)^{i+j+k+1} , \tag{3.17}$$

$$c_{ijk} = \sum_{m=0}^{k} \frac{\Gamma(2i+2j+2k+2)}{\Gamma(m+1)\,\Gamma(2i+m+2)\,\Gamma(k-m+1)\,\Gamma(2j+k-m+2)} . \tag{3.18}$$

The properties of the weak coupling and strong coupling expansions of $\bar{F}_2(\lambda)$ have been studied in detail in [4].

The second term in (3.15), denoted $\widetilde{F}_2(\lambda)$, comes from the $\widetilde{Q}_{ij}\chi_i\chi_j$ part of (3.10) (cf. (3.11)) $e^{-B_i \frac{\partial}{\partial \chi_i}}\, e^{\widetilde{Q}_{ij}\chi_i\chi_j}\big|_{\chi=0} = e^{\widetilde{Q}_{ij}B_iB_j}$. It can therefore be written as a double sum:

$$\widetilde{F}_2(\lambda) = -\sum_{i,j=1}^{\infty} \widetilde{Q}_{ij} B_i B_j , \tag{3.19}$$

where the function $B_i(\lambda)$ was defined in (2.8) and the coefficients $\widetilde{Q}_{ij}$ in (3.9) and we explicitly indicated summation over i, j. Thus, the weak coupling series representation for

$\widetilde{F}_2(\lambda)$ is (cf. (3.12))

$$\widetilde{F}_2(\lambda) = \frac{1}{\pi}\sum_{i,j=1}^{\infty} \frac{(-1)^{i+j+1}\,(1-2^{-2i})(1-2^{-2j})\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{(i+j+2)\Gamma(i+2)\Gamma(j+2)}\zeta_{2i+1}\zeta_{2j+1}\left(\frac{\lambda}{\pi^2}\right)^{i+j+2}. \tag{3.20}$$

Note that $\widetilde{F}_2(\lambda)$ is simpler than $\bar{F}_2(\lambda)$, being only quadratic in the zeta factors ζ_{2k+1}, while $\bar{F}_2(\lambda)$ involves sums over products of zetas to all orders. The weak-coupling expansion of the total $F_2(\lambda)$ (3.15) of course agrees with the direct expansion of $F_2(\lambda)$ at weak coupling in (2.13).

Remarkably, there is a direct differential relation between $\widetilde{F}_2(\lambda)$ and $F_1(\lambda)$. Indeed, differentiating $\widetilde{F}_2(\lambda)$ in (3.20) with respect to λ we observe that the double sum factorizes in terms of the second derivative of the product $\lambda\, F_1(\lambda)$ with respect to λ, implying that

$$\frac{d}{d\lambda}\widetilde{F}_2 = -\frac{\lambda}{2}\left[\frac{d^2}{d\lambda^2}(\lambda\, F_1)\right]^2. \tag{3.21}$$

Thus the form of $\widetilde{F}_2(\lambda)$ is determined by that of $F_1(\lambda)$. Using (3.14) we then get also

$$\frac{d}{d\lambda}\widetilde{F}_2 = 2\left(\int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}+1)^2}\left[J_1(2t\sqrt{\lambda}) - t\sqrt{\lambda}\right]\right)^2. \tag{3.22}$$

This integral representation also permits a direct access to the strong coupling expansion of $\widetilde{F}_2(\lambda)$.

4 Wilson loop expectation value

The $\mathcal{N}=2$ vector multiplet of the $\mathcal{N}=2$ theories contains the gauge vector A_μ, a complex scalar φ, and two Weyl fermions. The $\frac{1}{2}$-BPS Wilson loop depends only on the fields of the vector multiplet and is defined as

$$\mathcal{W} = \operatorname{tr}\mathcal{P}\,\exp\oint\left[i\,A_\mu(x)dx^\mu + \frac{1}{\sqrt{2}}\Big(\varphi(x)+\varphi^+(x)\Big)\,ds\right], \tag{4.1}$$

where the contour $x^\mu(s)$ represents a circle of unit radius and the trace is taken in the fundamental representation. The expectation value of $\mathcal{W}$ may be computed in the matrix model as (cf. (2.6))

$$\langle\mathcal{W}\rangle = \langle \operatorname{tr} e^{2\pi m}\rangle = \left\langle \operatorname{tr} e^{\sqrt{\frac{\lambda}{2N}}\,a}\right\rangle. \tag{4.2}$$

Its large N expansion may be written as

$$\langle\mathcal{W}\rangle = N\,W_0(\lambda) + W_1(\lambda) + \frac{1}{N}\left(W_{0,2}(\lambda)+W_2(\lambda)\right) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{4.3}$$

where we separated the $\mathcal{N}=4$ SYM parts

$$W_0 \equiv \langle\mathcal{W}\rangle_0^{\mathcal{N}=4} = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}), \qquad W_{0,2} \equiv \langle\mathcal{W}\rangle_2^{\mathcal{N}=4} = \frac{1}{48}\left[-12\sqrt{\lambda}\,I_1(\sqrt{\lambda}) + \lambda\,I_2(\sqrt{\lambda})\right]. \tag{4.4}$$

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The leading terms in the weak-coupling expansions of the $\mathcal{N} = 2$ parts W_1 and W_2 are found to be

$$W_1 \equiv \langle \mathcal{W} \rangle_1^{\mathcal{N}=2}, \qquad W_2 \equiv \langle \mathcal{W} \rangle_2^{\mathcal{N}=2}, \tag{4.5}$$

$$\begin{aligned} W_1 =& -\zeta_3 \frac{3\lambda^3}{2\,(8\pi^2)^2}\left(1+\frac{3\lambda}{32}+\frac{\lambda^2}{320}+\frac{\lambda^3}{18432}+\frac{\lambda^4}{1720320}+\frac{\lambda^5}{235929600}+\frac{\lambda^6}{44590694400}+\cdots\right) \\ &+\zeta_5 \frac{75\lambda^4}{8\,(8\pi^2)^3}\left(1+\frac{\lambda}{10}+\frac{\lambda^2}{288}+\frac{\lambda^3}{16128}+\frac{\lambda^4}{1474560}+\frac{\lambda^5}{199065600}+\frac{\lambda^6}{37158912000}+\cdots\right) \\ &-\zeta_7 \frac{441\lambda^5}{8\,(8\pi^2)^4}\left(1+\frac{5\lambda}{48}+\frac{5\lambda^2}{1344}+\frac{5\lambda^{\ 3}}{73728}+\frac{\lambda^4}{1327104}+\frac{\lambda^5}{176947200}+\cdots\right)+\cdots, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \frac{W_2}{\pi^2 W_0} =& -30\zeta_5\hat{\lambda}^4+(324\zeta_3^2+420\zeta_7)\hat{\lambda}^5-(5400\zeta_3\zeta_5+4410\zeta_9)\hat{\lambda}^6 \\ &+(22800\zeta_5^2+39690\zeta_3\zeta_7+41580\zeta_{11})\hat{\lambda}^7-\left(338100\zeta_5\zeta_7+289170\zeta_3\zeta_9+\frac{1486485}{4}\zeta_{13}\right)\hat{\lambda}^8 \\ &+\left(\frac{5044305}{4}\zeta_7^2+2477790\zeta_5\zeta_9+2126817\zeta_3\zeta_{11}+\frac{6441435}{2}\zeta_{15}\right)\hat{\lambda}^9+\cdots. \end{aligned} \tag{4.7}$$

Let us find the closed form of the series for the simpler W_1 term that is linear in ζ_{2n+1}. W_1 gets contributions from the single-trace term in (2.7) that were absent in the case of the SA-orientifold in [4]. If we write $S_{\rm int}$ in (2.7) as $\mathrm{S}_1+\mathrm{S}_2$ where $\mathrm{S}_1 = \sum_{i=1}^{\infty} B_i(\lambda)\ \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}$ and S_2 is the double-trace term, then expanding (4.2) to linear order in S_1 we get

$$\langle \mathcal{W} \rangle = \frac{\int Da\, e^{-\mathrm{tr}\, a^2}\, e^{-\mathrm{S}_1-\mathrm{S}_2}\ \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}a}}{\int Da\, e^{-\mathrm{tr}\, a^2}\, e^{-\mathrm{S}_1-\mathrm{S}_2}} \quad \to \quad \frac{\left\langle (1-\mathrm{S}_1)\ \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}a}\right\rangle}{\langle 1-\mathrm{S}_1\rangle}. \tag{4.8}$$

Picking up the part linear in S_1 gives

$$\begin{aligned} W_1 =& -\left\langle \mathrm{tr}\, \mathrm{S}_1\, e^{\sqrt{\frac{\lambda}{2N}}a}\right\rangle + \langle \mathrm{S}_1\rangle \left\langle \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}a}\right\rangle = -\left\langle \mathrm{S}_1\ \mathrm{tr}\, e^{\sqrt{\frac{\lambda}{2N}}a}\right\rangle_c \\ =& -\sum_{p=0}^{\infty}\frac{1}{(2p)!}\left(\frac{\lambda}{2N}\right)^p \langle \mathrm{tr}\, a^{2p}\ \mathrm{S}_1\rangle_c = -\sum_{p=0}^{\infty}\frac{1}{(2p)!}\left(\frac{\lambda}{2N}\right)^p \sum_{i=1}^{\infty} B_i \left\langle \mathrm{tr}\, a^{2p}\ \ \mathrm{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}\right\rangle_c \\ =& -4\sum_{p=0}^{\infty}\frac{1}{(2p)!}\left(\frac{\lambda}{2N}\right)^p \sum_{n=1}^{\infty}\left(\frac{\lambda}{8\pi^2 N}\right)^{n+1}\frac{(-1)^n}{n+1}\zeta_{2n+1}(1-2^{2n})\left\langle \mathrm{tr}\, a^{2p}\ \ \mathrm{tr}\, a^{2n+2}\right\rangle_c. \end{aligned} \tag{4.9}$$

Using (3.6), we then find

$$W_1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{(4\pi^2)^p}{(2p)!}\,\frac{(-1)^n}{n+1}\zeta_{2n+1}(2^{2n}-1)\frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{(p+n+1)\Gamma(p)\Gamma(n+1)}\left(\frac{\lambda}{4\pi^2}\right)^{n+p+1}, \tag{4.10}$$

which agrees with (4.6).[13]

[13] Let us note that doing the sum over p for each n we obtain the exact form of the coefficients of all ζ_{2n+1} terms

$$W_1 = -\zeta_3 \frac{6\lambda^2}{2\,(8\pi^2)^2}[2I_2(\sqrt{\lambda})+I_4(\sqrt{\lambda})]+\zeta_5\frac{15\lambda^3}{(8\pi^2)^3}\left[5I_2(\sqrt{\lambda})+4I_4(\sqrt{\lambda})+I_6(\sqrt{\lambda})\right]+\cdots$$

matching eq. (3.29) of [29].

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Using the identity (3.13) we can resum this double series expansion into an explicit integral representation

$$W_1(\lambda) = 2\sqrt{\lambda} I_2(\sqrt{\lambda}) \int_0^\infty dt \, \frac{e^{2\pi t}}{(e^{2\pi t}+1)^2} \left[\frac{J_1(2t\sqrt{\lambda})}{4t^2+1} - t\sqrt{\lambda} \right]$$
$$+ 4\sqrt{\lambda} I_1(\sqrt{\lambda}) \int_0^\infty dt \, \frac{e^{2\pi t}}{(e^{2\pi t}+1)^2} \frac{t\, J_2(2t\sqrt{\lambda})}{4t^2+1} \,. \tag{4.11}$$

It is straightforward to verify that the expansion of the Bessel functions, combined with the identity (3.13), leads to the weak coupling expansion in (4.6).

A closed expression for $W_2(\lambda)$ in (4.3) will be given in the next section after relating it to the corresponding terms in the free energy.

5 General relations between the $1/N$ terms in $\langle \mathcal{W} \rangle$ and F

The coefficients W_1 and W_2 in the large N expansion (4.3) of the Wilson loop expectation value turn out to have close relation with the F_1 and F_2 in the free energy expansion (2.11) (see also appendix C).

To relate W_1 to F_1 let us first write (4.10) as

$$W_1 = -\frac{1}{\pi} \sum_{p=0}^{\infty} \frac{\lambda^p}{(2p)!} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p)} Y_p(\lambda) = -\frac{1}{\sqrt{\pi}} \sum_{p=0}^{\infty} \frac{\lambda^p}{4^p\, \Gamma(p)\Gamma(p+1)} Y_p(\lambda) \,, \tag{5.1}$$

$$Y_p(\lambda) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i+1} (1-2^{-2i}) \frac{\Gamma\left(i+\frac{3}{2}\right)}{(p+i+1)\Gamma(i+1)} \zeta_{2i+1} \left(\frac{\lambda}{\pi^2}\right)^{i+1} . \tag{5.2}$$

We notice that differentiating (5.1) over λ leads to the expression where the double sum factorizes. Using the expression for F_1 in (3.12) we then obtain

$$\frac{d}{d\lambda} W_1 = \left[-\frac{1}{\sqrt{\pi}} \sum_{p=0}^{\infty} \frac{\lambda^p}{4^p\, \Gamma(p)\Gamma(p+1)} \right] \times \sqrt{\pi} \, \frac{d^2}{d\lambda^2} (\lambda F_1) \,. \tag{5.3}$$

This relation may be written as

$$\frac{d}{d\lambda} W_1 = -\frac{1}{2} \sqrt{\lambda}\, I_1(\sqrt{\lambda}) \, \frac{d^2}{d\lambda^2} (\lambda F_1) \,. \tag{5.4}$$

Using also the expression for W_0 in (4.4) we conclude that

$$\frac{d}{d\lambda} W_1 = -\frac{\lambda}{4} \, W_0 \, \frac{d^2}{d\lambda^2} (\lambda\, F_1) \,. \tag{5.5}$$

The term W_2 in (4.3) turns out to be related to F_2 in (2.11), (3.15) by

$$W_2 = -\frac{\lambda^2}{4} \, W_0 \frac{d}{d\lambda} F_2 \,. \tag{5.6}$$

This can be proved in the same way as in [4][14] by expanding the Wilson loop factor to leading order, using the large N factorization of correlators and observing that the insertion of $\operatorname{tr} a^2$ is the same as the insertion of the Gaussian "action" which, in turn, can be obtained by differentiating the matrix model integral over λ.

Using that in $F_2 = \widetilde{F}_2 + \bar{F}_2$ and (3.21) we may represent (5.6) as

$$W_2 = \frac{\lambda^3}{8}\, W_0 \left[\frac{d^2}{d\lambda^2}(\lambda\, F_1)\right]^2 - \frac{\lambda^2}{4}\, W_0 \frac{d}{d\lambda}\bar{F}_2\,. \tag{5.7}$$

In view of (5.5) the first term here is thus related to the square of $\frac{dW_1}{d\lambda}$.

6 Strong coupling expansions of the $\mathcal{N} = 2$ SU(N) free energy and Wilson loop

In this section we present results for the large λ expansions of the terms $F_1(\lambda)$ and $F_2(\lambda)$ in the large N expansion (2.11) of the free energy. Using the relations (5.5), (5.6) these will also determine the expansion of the terms $W_1(\lambda)$ and $W_2(\lambda)$ in the large N expansion (4.3) of the Wilson loop.

6.1 Large λ expansion of F_1 and F_2

The large λ expansion of the first subleading large N correction $F_1(\lambda)$ in (2.11) for the free energy can be derived in several different but complementary ways. The simplest way is to use the representation

$$(1-2^{-2i})\,\zeta_{2i+1} = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2i+1}} \equiv \eta(2i+1)\,. \tag{6.1}$$

where $\eta(2i+1)$ is the value of the Dirichlet η-function. Then the expansion (3.12) for F_1 yields

$$F_1(\lambda) = \sum_{k=1}^{\infty}\frac{(-1)^k}{4k}\left[-\frac{\lambda}{\pi^2}+\frac{8k^4\pi^2}{\lambda}\left(\sqrt{1+\frac{\lambda}{\pi^2k^2}}-1\right)-4k^2+8k^2\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{\lambda}{\pi^2k^2}}\right)\right]. \tag{6.2}$$

Expanding at large λ gives an expansion that can be evaluated using ζ-function regularization

$$F_1 \overset{\lambda\gg 1}{=} \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\lambda}{4\pi^2 k}+k\left[1+2\log(2\pi k)-\log\lambda\right]-\frac{4\pi k^2}{\sqrt{\lambda}}+\frac{2\pi^2k^3}{\lambda}+\cdots\right). \tag{6.3}$$

Using the η-function values

$$\eta(1)=\log 2\,,\quad \eta(-1)=\frac{1}{4}\,,\quad \eta'(-1)=-\frac{1}{4}-\frac{\log(2)}{3}+3\log A\,,\quad \eta(-2)=0,\quad \eta(-3)=-\frac{1}{8}\,, \tag{6.4}$$

[14] In [4] we used the notation $\frac{W_2}{W_0}=\Delta q$ and $F_2=\Delta F$.

where A is Glaisher's constant, we thus obtain the strong coupling expansion

$$F_1(\lambda) \overset{\lambda\gg 1}{=} f_1\lambda + f_2 \log\lambda + f_3 + f_4\,\lambda^{-1} + \mathcal{O}\left(\lambda^{-1/4}\,e^{-\sqrt{\lambda}}\right), \tag{6.5}$$

$$f_1 = \frac{\log 2}{4\pi^2}, \qquad f_2 = -\frac{1}{4}, \qquad f_3 = \frac{3}{4} + \frac{7}{6}\log 2 + \frac{1}{2}\log\pi - 6\log \mathsf{A}, \qquad f_4 = -\frac{\pi^2}{4}. \tag{6.6}$$

Here we indicated that there is only a finite number of power-law corrections: as will be discussed below in section 6.3 and appendix B, all further corrections turn out to be exponentially small as $\lambda \to +\infty$. An indication of this is that all higher order corrections in (6.3) have coefficients that are expressed in terms of η-function values that vanish.

The strong coupling expansion (6.5)–(6.6) for $F_1(\lambda)$ can be also obtained from the integral representation (3.14) using the Mellin transform method (see appendix B), or by expanding the $\frac{e^{2\pi t}}{(e^{2\pi t}+1)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n\, e^{-(2n-1)\pi t}$ factor in the integral representation (3.14) and integrating.

The $\bar{F}_2$ part (3.16) of F_2 in (3.15) is same as in the SA-orientifold and thus [4]

$$\bar{F}_2 \overset{\lambda\gg 1}{=} k_1\lambda^{1/2} + k_2\log\lambda + k_3 + O(\lambda^{-1/2}), \qquad k_1 = \frac{1}{2\pi}, \quad \ldots. \tag{6.7}$$

The strong coupling expansion of $\widetilde{F}_2$ in (3.20) may be derived directly from (3.21) using (6.5)[15]

$$\widetilde{F}_2 \overset{\lambda\gg 1}{=} p_1\lambda^2 + p_2\,\lambda + p_3\log\lambda + p_4 + \mathcal{O}\left(\lambda^{5/4}\,e^{-\sqrt{\lambda}}\right), \tag{6.8}$$

$$p_1 = -f_1^2, \qquad p_2 = -2f_1f_2, \qquad p_3 = -\frac{1}{2}f_2^2, \qquad \ldots, \tag{6.9}$$

where f_i have the values listed in (6.6). Notice that, as for $F_1(\lambda)$ in (6.5), there is only a finite number of power law corrections, followed by exponentially suppressed terms, whose origin is discussed below in section 6.3.

6.2 Large λ expansion of W_1 and W_2

Using the relations (5.5), (5.6), (5.7) allows us to find the strong coupling expansions of W_1 and W_2 from those of F_1 and F_2. In particular, from (6.5) and the expansion of W_0 in (4.4)

$$W_0 \overset{\lambda\gg 1}{=} \sqrt{\frac{2}{\pi}}\lambda^{-3/4}e^{\sqrt{\lambda}}\left(1 - \frac{3}{8\sqrt{\lambda}} - \frac{15}{128\lambda} + \ldots\right) - i\sqrt{\frac{2}{\pi}}\lambda^{-3/4}e^{-\sqrt{\lambda}}\left(1 + \frac{3}{8\sqrt{\lambda}} - \frac{15}{128\lambda} + \ldots\right), \tag{6.10}$$

we find (dropping exponentially suppressed parts, cf. (6.5))

$$\frac{W_1}{W_0} = -f_1\,\lambda^{3/2} + \frac{3}{2}f_1\lambda - \frac{1}{8}(3f_1 + 4f_2)\lambda^{1/2} + \mathcal{O}(\lambda^0). \tag{6.11}$$

[15] Note that the value of the constant term p_4 can not be deduced from the differential relation (3.21) and requires separate derivation using the method of appendix B that gives $p_4 = \frac{1}{16} + \frac{\log 2}{12} + \frac{\log\pi}{16} - \frac{3}{4}\log \mathsf{A}$.

Comparing (6.7) and (6.8) we observe that $\widetilde{F}_2$ dominates over $\bar{F}_2$ at the first two leading orders of expansion in $\lambda \gg 1$. As a result, the dominant contribution to W_2 comes from the first term in (5.7)

$$\left[\frac{W_2}{W_0}\right]_1 \equiv \frac{\lambda^3}{8}\left[\frac{d^2}{d\lambda^2}(\lambda F_1)\right]^2 \overset{\lambda\gg 1}{=} \frac{1}{2}f_1^2\lambda^3 + \frac{1}{2}f_1 f_2\lambda^2 + \mathcal{O}(\lambda)\,, \tag{6.12}$$

where we used (6.5). The contribution to (6.12) coming from $\bar{F}_2$ term in (5.7) is

$$\left[\frac{W_2}{W_0}\right]_2 \equiv -\frac{\lambda^2}{4}\frac{d}{d\lambda}\bar{F}_2 \overset{\lambda\gg 1}{=} -\frac{1}{8}k_1\lambda^{3/2} - \frac{1}{4}k_2\lambda + \mathcal{O}(\lambda^{1/2})\,, \tag{6.13}$$

so that in total

$$\frac{W_2}{W_0} = \left[\frac{W_2}{W_0}\right]_1 + \left[\frac{W_2}{W_0}\right]_2 \overset{\lambda\gg 1}{=} \frac{1}{2}f_1^2\,\lambda^3 + \frac{1}{2}f_1 f_2\,\lambda^2 - \frac{1}{8}k_1\lambda^{3/2} + \mathcal{O}(\lambda)\,, \tag{6.14}$$

where the values of f_1, f_2 and k_1 are given in (6.6), (6.7).

6.3 Exponentially suppressed corrections at large $\boldsymbol{\lambda}$

The leading large N correction to the free energy $F_1(\lambda)$ has, in addition to the "perturbative" terms in (6.5), also exponentially suppressed corrections in the large λ limit. These can be computed directly from the integral representation (3.14). It is actually slightly simpler to begin with the combination $\frac{d^2}{d\lambda^2}(\lambda F_1)$ which appears in the relation to W_1 as in (5.4). From the integral representation (3.14) we deduce that

$$\begin{aligned}\frac{d^2}{d\lambda^2}(\lambda F_1) &= -\frac{2}{\sqrt{\lambda}}\int_0^\infty dt\,\frac{e^{2\pi t}}{(e^{2\pi t}+1)^2}\left[J_1(2t\sqrt{\lambda}) - t\sqrt{\lambda}\right] \\ &= \frac{\log 2}{2\pi^2} - \frac{1}{4\lambda} + \frac{2}{\pi^2}\sum_{n=0}^\infty\left[K_0\left((2n+1)\sqrt{\lambda}\right) + \frac{K_1\left((2n+1)\sqrt{\lambda}\right)}{(2n+1)\sqrt{\lambda}}\right].\end{aligned} \tag{6.15}$$

Both these expressions are exact, but the first expression in terms of Bessel J-functions is well suited to a small λ expansion, while the second expression in terms of Bessel K-functions is well suited to a large λ expansion. As $\lambda \to +\infty$ each Bessel K-function in (6.15) is given by the exponentially small factor $e^{-(2n+1)\sqrt{\lambda}}$, multiplied by an asymptotic series in $\frac{1}{\sqrt{\lambda}}$. Thus we obtain an expansion in the form of an "instanton sum", with each exponential multiplied by a "fluctuation expansion" in inverse powers of $\sqrt{\lambda}$:

$$\frac{d^2}{d\lambda^2}(\lambda F_1) \overset{\lambda\gg 1}{=} \frac{\log 2}{2\pi^2} - \frac{1}{4\lambda} + \frac{\sqrt{2}}{\pi^{5/2}}\sum_{n=0}^\infty \frac{e^{-(2n+1)\sqrt{\lambda}}}{\sqrt{(2n+1)\sqrt{\lambda}}}\sum_{k=0}^\infty \frac{(-1)^k\left(k^2+\frac{3}{4}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k-\frac{3}{2}\right)}{2^k\,\Gamma(k+1)\left[(2n+1)\sqrt{\lambda}\right]^k}. \tag{6.16}$$

The reconstruction of $F_1(\lambda)$ from this expansion requires two integrations, and the integration constants are easily fixed by the comparison with (6.5), (6.6). As a result, we find that F_1 in (6.5) may be represented as

$$F_1 \overset{\lambda\gg 1}{=} F_1^{\text{pol}} + F_1^{\text{exp}}\,, \qquad F_1^{\text{pol}} = f_1\lambda + f_2\log\lambda + f_3 + f_4\,\lambda^{-1}\,, \tag{6.17}$$

Here $F_1^{\rm pol}$ is the "polynomial" in $\lambda \gg 1$ part, with a finite number of nonzero coefficients f_j as in (6.5)–(6.6), and $F_1^{\rm exp}$ is the exponentially small contribution given by

$$F_1^{\rm exp}(\lambda) \overset{\lambda \gg 1}{=} -\frac{1}{\pi}\left(\frac{2}{\pi}\right)^{3/2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(k^2+\frac{3}{4}\right) \Gamma\left(k+\frac{1}{2}\right) \Gamma\left(k-\frac{3}{2}\right)}{2^k\, \Gamma(k+1)} \times \left\{ \Gamma\left(\frac{3}{2}-k, (2n+1)\sqrt{\lambda}\right) - \frac{\Gamma\left(\frac{7}{2}-k, (2n+1)\sqrt{\lambda}\right)}{(2n+1)^2 \lambda} \right\}. \tag{6.18}$$

Here the sum over n looks like an "instanton" expansion: for each n and k the incomplete Γ-function terms in (6.18) are proportional to $e^{-(2n+1)\sqrt{\lambda}}$ when $\lambda \to +\infty$. Using the expansions of these Γ-functions we find explicitly that

$$F_1^{\rm exp}(\lambda) \overset{\lambda \gg 1}{=} 2\left(\frac{2}{\pi}\right)^{3/2} \lambda^{-1/4} \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\sqrt{\lambda}}}{(2n+1)^{5/2}} \sum_{l=0}^{\infty} \frac{(-1)^l \left[4l(l+4)+3\right] \Gamma\left(l+\frac{1}{2}\right) \Gamma\left(l-\frac{3}{2}\right)}{\pi\, 2^{l+2}\, \Gamma(l+1) \left[(2n+1)\sqrt{\lambda}\right]^l}. \tag{6.19}$$

For each n, the fluctuation series is factorially divergent, but it is resurgent in the sense that the large l behaviour is encoded in the low l terms. To see this explicitly, let us define the "fluctuation" coefficients from (6.19):

$$c_l = \frac{(-1)^l \left[4l(l+4)+3\right] \Gamma\left(l+\frac{1}{2}\right) \Gamma\left(l-\frac{3}{2}\right)}{\pi\, 2^{l+2}\, \Gamma(l+1)}. \tag{6.20}$$

The first few *low-order* values of c_l are given by

$$c_l = \left\{1, \frac{23}{8}, \frac{153}{128}, -\frac{435}{1024}, \frac{13755}{32768}, -\frac{172935}{262144}, \frac{5893965}{4194304}, -\frac{126080955}{33554432}, \dots \right\}. \tag{6.21}$$

At large order, $l \to \infty$, these coefficients are alternating in sign and factorially divergent, and including the subleading corrections the large order behaviour can be written as:

$$c_l \overset{l\to\infty}{=} \frac{(-1)^l}{\pi} \frac{\Gamma(l)}{2^l} \left[1 + \frac{2 \cdot \frac{23}{8}}{(l-1)} + \frac{2^2 \cdot \frac{153}{128}}{(l-1)(l-2)} + \frac{2^3 \cdot \left(-\frac{435}{1024}\right)}{(l-1)(l-2)(l-3)} + \dots \right]. \tag{6.22}$$

Notice that the numerators of the subleading corrections correspond precisely to the low order coefficients in (6.21). The powers of 2 correspond to the difference between the two Bessel function saddles (e^{-x} vs. e^{+x}) whose ratio is e^{-2x}. Thus we see that the subleading corrections to the *large-order* growth of the fluctuation coefficients are directly encoded in the *low-order* fluctuation coefficients.

This behaviour in (6.22) is the typical low-order/large-order resurgence relation [31–33]. These resurgence properties are inherited from the large argument expansion of the Bessel function term in square brackets in the r.h.s. of (6.15). Furthermore, this resurgent behaviour of $F_1(\lambda)$ is inherited by the exponentially small corrections to the Wilson loop ratio $W_1(\lambda)/W_0(\lambda)$ in (6.11), due to the expression (5.5) relating $W_1(\lambda)$ to $F_1(\lambda)$. Similar

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exponential terms will appear in the strong coupling expansion of F_2 and W_2 and also in the corresponding terms in the $\mathrm{Sp}(2N)$ theory case discussed in the next section.

The exponential $e^{-c\sqrt{\lambda}}$ corrections found here in the $1/N$ term in $\mathcal{N}=2$ free energy are generally expected in observables in conformal gauge theory with an AdS string dual. The perturbative expansion (in inverse string tension) in 2d string sigma model is expected to be asymptotic and such corrections may have a world-sheet theory origin (which may be different in different observables). Similar terms appear, e.g., in the $\mathcal{N}=4$ SYM theory in the large λ expansion of the cusp anomalous dimension (see [34, 35] and also [36, 37] for their relation to resurgence).

One may conjecture that the $e^{-(2k+1)\sqrt{\lambda}}$ terms in F_1 have a string instanton interpretation in terms of world sheets wrapping part of the compact internal space S'^5 that has fixed points under the orientifold/orbifold action on S^5 (see discussion in the Introduction).

It is useful to compare this with what happens in the case of the Wilson loop expectation in $\mathcal{N}=4$ SYM theory (see (1.1), (4.3), (4.4)). The large λ expansion of the Bessel I_1 function in W_0 in (4.4) leads to just two exponential terms in (6.10), with the subleading one being imaginary (the same pattern is found also for higher $1/N$ terms in $\langle\mathcal{W}\rangle$ in (1.1)). While the leading $e^{\sqrt{\lambda}}$ term in (6.10) represents the expansion near the minimal AdS_2 surface embedded in AdS_5, the second term may be interpreted[16] [38, 39] as the contribution of an unstable surface wrapping S^2 of S^5.[17] Note that higher order terms $\sim e^{-n\sqrt{\lambda}}$ do not appear, as multiple wrappings would correspond to multiply wrapped Wilson loop.

In contrast, in the case of $F_1(\lambda)$ in the $\mathcal{N}=2$ theory we get an infinite series of exponential terms as here multiple wrappings should be allowed[18] and they have real coefficients as the corresponding world-sheet solutions should be stable due to orbifolding of S^5.

Note that the appearance of the imaginary term in the formal large λ expansion of W_0 is related to fact that the asymptotic expansion of the Bessel I_1 function about the dominant $e^{\sqrt{\lambda}}$ term is non Borel summable: the coefficients of the expansion about $e^{\sqrt{\lambda}}$ are factorially divergent and non-alternating in sign and then the naive Borel summation integral has an imaginary contribution, and this must be cancelled against the $ie^{-\sqrt{\lambda}}$ term as total W_0 should be real. At the same time, the exponentially small factors $e^{-(2k+1)\sqrt{\lambda}}$ in F_1 are multiplied by asymptotic series that are Borel summable (note that the c_l coefficients in (6.20) are factorially divergent but alternate in sign) and therefore, one finds only real exponentially suppressed contributions.

In view of the relation (5.5) between W_1 and F_1 and the expansion of W_0 in (6.10) the resulting expression for the $1/N$ correction W_1 to the Wilson loop in the $\mathcal{N}=2$ theory will thus contain two different sources of the subleading exponential corrections since

$$\frac{d}{d\lambda}W_1=-\frac{1}{4}\lambda W_0\frac{d^2}{d\lambda^2}(\lambda F_1)\sim\Big[w(\sqrt{\lambda})\,e^{\sqrt{\lambda}}+iw(-\sqrt{\lambda})\,e^{-\sqrt{\lambda}}\Big]\sum_{k=0}^{\infty}u_k(\sqrt{\lambda})\,e^{-(2k+1)\sqrt{\lambda}}\,. \tag{6.23}$$

[16] An instanton interpretation of this second term was originally conjectured in [1].

[17] This may be viewed as a limit of the result found in the case of $\frac{1}{4}$-BPS "latitude" Wilson loop where there are two solutions of disc topology covering (in addition to AdS_2) the smaller or bigger part of S^2 in S^5.

[18] To recall, the $1/N$ correction F_1 should be given by string path integral over surfaces of disc topology with free boundary.

Thus, $\frac{d}{d\lambda} W_1$ has a trans-series expansion involving an overall $e^{\sqrt{\lambda}}$ factor, multiplied by even powers of $e^{-\sqrt{\lambda}}$. These alternate between being real and imaginary,[19] in such a way that the full trans-series is well-defined and real (as W_1 should be when λ is real and positive). The same structure also survives the λ- integration that gives W_1. The resurgence properties of this final trans-series for W_1 would be interesting to study in more detail.[20]

7 $\mathcal{N} = 2$ superconformal Sp(2N) theory

Let us now repeat similar analysis in the case of the FA-orientifold model (1.6) with the gauge group Sp(2N).

7.1 Matrix model formulation

The structure of the matrix model here is the same as in (2.1). For the model with $n_{\text{Adj}}, n_{\text{A}}$ and n_{F} expressed in terms of them using the finiteness condition (1.5) the interacting action in (2.1) reads [9] (cf. (2.7) and also appendix A)

$$S_{\text{int}}(a) = \sum_{i=1}^{\infty} \left(\frac{\lambda}{8\pi^2}\right)^{i+1} \frac{(-1)^i}{i+1} \zeta_{2i+1} \Bigg\{ 2\,(2^{2i}-1)(n_{\text{Adj}} - n_{\text{A}} - 1)\ \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2} + \frac{1}{2}\,(n_{\text{Adj}} + n_{\text{A}} - 1) \sum_{k=1}^{i} \binom{2i+2}{2k}\ \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i-2k+2} \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2k} \Bigg\}, \tag{7.1}$$

where the matrix a is in the $2N$-dimensional fundamental representation of Sp(2N). The expression (7.1) greatly simplifies for the FA-orientifold where $n_{\text{Adj}} = 0$, $n_{\text{A}} = 1$ (and $n_{\text{F}} = 4$): only the single-trace term survives so that (cf. (2.7))[21]

$$S_{\text{int}}(a) = \sum_{i=1}^{\infty} B_i(\lambda)\ \text{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}, \tag{7.2}$$

where B_i is same as in (2.8).

Perturbative calculations are most efficiently performed by the same methods as in [40] in the SU(N) case. The matrix model variable is written in a basis of $\mathfrak{sp}(2N)$ generators in the fundamental representation with the following normalization

$$a = \sum_{r=1}^{N(2N+1)} a^r\, T_F^r\,, \qquad \text{tr}\,(T_F^r T_F^s) = \frac{1}{2}\delta^{rs}\,. \tag{7.3}$$

[19] From the string theory point of view, W_1 comes from contributions of world sheets with annulus topology (with one boundary being fixed by the Wilson loop circle and the other being free). Then the argument about stability of all wrappings of subspace in S'^5 (given above for F_1 case) should no longer apply.

[20] An alternative approach is to start directly with the integral representation for W_1 in (4.11) and perform the large λ expansion, getting both perturbative and non-perturbative contributions.

[21] There is no similar simplification with no double-trace terms in S_{int} in the SU(N) case (2.3) (apart from "trivial" $\mathcal{N} = 4$ SYM case where $S_{\text{int}} = 0$).

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Then the matrix model measure is simply

$$Da = \mathcal{N} \prod_{r=1}^{N(2N+1)} da^r \,. \tag{7.4}$$

Integration is done with respect to the Gaussian weight $e^{-\operatorname{tr} a^2}$ (cf. (2.6)), i.e. it reduces to repeated Wick contractions using $\langle a^r a^s \rangle = \delta^{rs}$ and the Sp($2N$) fusion/fission relations [41, 42]

$$\operatorname{tr}(T^a\, M_1\, T^a\, M_2) = \frac{1}{4} \operatorname{tr} M_1\ \operatorname{tr} M_2 + \frac{1}{4}(-1)^{n_2} \operatorname{tr}(M_1 \overline{M}_2), \tag{7.5}$$

$$\operatorname{tr}(T^a\, M_1)\ \operatorname{tr}(T^a\, M_2) = \frac{1}{4} \operatorname{tr}(M_1\, M_2) - \frac{1}{4}(-1)^{n_2} \operatorname{tr}(M_1 \overline{M}_2), \tag{7.6}$$

where M_1 and M_2 are products of generators, n_2 is the number of factors in M_2, and $\overline{M}_2$ is the product in reverse order. In particular, one finds the following useful correlators[22]

$$\begin{aligned} \langle \operatorname{tr} a^{2n} \rangle &= N^{n+1}\, \frac{2^{1+n}\Gamma\left(\frac{1}{2}+n\right)}{\sqrt{\pi}\,\Gamma(2+n)} \\ &\quad \times \left[1 + \frac{n+1}{4N} + \frac{n(n^2-1)}{48N^2} + \frac{n(n^2-1)(n-2)}{192N^3} + \cdots\right], \end{aligned} \tag{7.7}$$

$$\begin{aligned} \langle \operatorname{tr} a^{2n}\ \operatorname{tr} a^{2m} \rangle_c &= N^{n+m}\, \frac{2^{n+m+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\pi\,(n+m)\,\Gamma(n)\Gamma(m)} \\ &\quad \times \left[1 + \frac{n+m}{4N} + \frac{(n+m)(1-2n-2m+n^2+nm+m^2)}{48N^2} + \cdots\right], \end{aligned} \tag{7.8}$$

$$\begin{aligned} \langle \operatorname{tr} a^{2n}\ \operatorname{tr} a^{2m}\ \operatorname{tr} a^{2k} \rangle_c &= N^{n+m+k-1}\, \frac{2^{n+m+k+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi^{3/2}\,\Gamma(n)\Gamma(m)\Gamma(k)} \\ &\quad \times \left(1 + \frac{n+m+k-1}{4N} + \cdots\right), \end{aligned} \tag{7.9}$$

$$\begin{aligned} \langle \operatorname{tr} a^{2n}\ \operatorname{tr} a^{2m}\ \operatorname{tr} a^{2k}\ \operatorname{tr} a^{2\ell} \rangle_c &= N^{n+m+k+\ell-2}\, \frac{2^{n+m+k+\ell+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)}{\pi^2\,\Gamma(n)\Gamma(m)\Gamma(k)\Gamma(\ell)} \\ &\quad \times (n+m+k+\ell-1) + \cdots, \end{aligned} \tag{7.10}$$

$$\begin{aligned} \langle \operatorname{tr} a^{2n}\ \operatorname{tr} a^{2m}\ \operatorname{tr} a^{2k}\ \operatorname{tr} a^{2\ell}\ \operatorname{tr} a^{2s} \rangle_c &= N^{n+m+k+\ell+s-3} \\ &\quad \times \frac{2^{n+m+k+\ell+s+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{\pi^2\,\Gamma(n)\Gamma(m)\Gamma(k)\Gamma(\ell)\Gamma(s)} \\ &\quad \times (n+m+k+\ell+s-1)(n+m+k+\ell+s-2) + \cdots. \end{aligned} \tag{7.11}$$

7.2 Free energy

The free energy of the Sp($2N$) FA-orientifold has the same structure of the $1/N$ expansion as in (2.11), i.e. after the subtraction of the $\mathcal{N}=4$ SYM free energy we have (see (1.22))

$$\Delta F(\lambda) = N\,\mathsf{F}_1(\lambda) + \mathsf{F}_2(\lambda) + \frac{1}{N}\mathsf{F}_3(\lambda) + \frac{1}{N^2}\mathsf{F}_4(\lambda) + \frac{1}{N^3}\mathsf{F}_5(\lambda) + \mathcal{O}\left(\frac{1}{N^4}\right), \tag{7.12}$$

[22] Note that $\langle ABC\rangle_c = \langle ABC\rangle - \langle A\rangle\langle BC\rangle - \langle B\rangle\langle AC\rangle - \langle C\rangle\langle AB\rangle + 2\langle A\rangle\langle B\rangle\langle C\rangle$, etc.

where we included two more terms, compared to (2.11). To get the explicit expressions for the terms $\mathsf{F}_1(\lambda), \mathsf{F}_2(\lambda)$, and $\mathsf{F}_3(\lambda)$ we repeat the analysis in section 3 (the computation of F_4 follows similar steps). In this case we need to consider the analog of the generating function (3.2) containing only χ-part

$$X(\chi) = \int Da\, e^{-\operatorname{tr} a^2}\, e^{V(\chi,a)}\,, \qquad V(\chi,a) = \sum_{i=1}^{\infty} \chi_i \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}. \tag{7.13}$$

Evaluating the integrals gives

$$\begin{aligned} \log X(\chi) = &\sum_{i=1}^{\infty}\left\langle \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}\right\rangle \chi_i + \frac{1}{2}\sum_{i,j=1}^{\infty}\left\langle \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2} \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2j+2}\right\rangle_c \chi_i\chi_j \\ &+ \frac{1}{3!}\sum_{i,j,k=1}^{\infty}\left\langle \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2} \operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2j+2}\left(\frac{a}{\sqrt{N}}\right)^{2k+2}\right\rangle_c \chi_i\chi_j\chi_k + \cdots . \end{aligned} \tag{7.14}$$

Using (7.7)–(7.11), this may be written as

$$\log X(\chi) = \mathsf{R}_i\chi_i + \mathsf{U}_{ij}\chi_i\chi_j + \mathsf{T}_{ijk}\chi_i\chi_j\chi_k + O\left(\frac{1}{N^2}\right), \tag{7.15}$$

where

$$\begin{aligned} &\mathsf{R}_i = N\,\mathsf{R}_i^{(0)} + \mathsf{R}_i^{(1)} + \frac{1}{N}\mathsf{R}_i^{(2)} + \mathcal{O}\left(\frac{1}{N^2}\right), \qquad \mathsf{U}_{ij} = \mathsf{U}_{ij}^{(0)} + \frac{1}{N}\mathsf{U}_{ij}^{(1)} + \mathcal{O}\left(\frac{1}{N^2}\right), \\ &\mathsf{T}_{ijk} = \frac{1}{N}\,\mathsf{T}_{ijk}^{(0)} + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} &\mathsf{R}_i^{(0)} = \frac{2^{i+2}\Gamma\left(i+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(i+3)} = 2\,R_i, \qquad \mathsf{R}_i^{(1)} = \frac{i+2}{2}\,R_i, \qquad \mathsf{R}_i^{(2)} = \frac{i(i+1)(i+2)}{24}\,R_i, \\ &\mathsf{U}_{ij}^{(0)} = \frac{2^{i+j+2}\,\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}{\pi(i+j+2)\,\Gamma(i+1)\,\Gamma(j+1)} = 2\,\widetilde{Q}_{ij}, \qquad \mathsf{U}_{ij}^{(1)} = \frac{i+j+2}{2}\,\widetilde{Q}_{ij}, \\ &\mathsf{T}_{ijk}^{(0)} = \frac{2^{i+j+k+4}\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{6\pi^{3/2}\,\Gamma(i+1)\Gamma(j+1)\Gamma(k+1)} \\ &\qquad = \frac{1}{3}(i+1)(i+2)(j+1)(j+2)(k+1)(k+2)\,R_iR_jR_k, \end{aligned} \tag{7.17}$$

with R_i and $\widetilde{Q}_{ij}$ being the same as in (3.5) and (3.9).

The free energy ΔF in (7.12) is then obtained by acting on $-\log X$ with the operator $\exp\left(-B_i\frac{\partial}{\partial\chi_i}\right)$ and setting $\chi_i \to 0$.[23] This replaces $\chi_i \to -B_i$ (cf. (3.11)) and thus

$$\Delta F(\lambda) = \sum_{i=1}^{\infty}\mathsf{R}_iB_i - \sum_{i,j=1}^{\infty}\mathsf{U}_{ij}\,B_iB_j + \sum_{i,j,k=1}^{\infty}\mathsf{T}_{ijk}\,B_iB_jB_k + O\left(\frac{1}{N^2}\right). \tag{7.18}$$

[23]Equivalently, we just start with $\exp\left[-\sum_{i=1}^{\infty}B_i(\lambda)\operatorname{tr}\left(\frac{a}{\sqrt{N}}\right)^{2i+2}\right]$ (cf. (7.2)), compute its expectation value expanding in powers of B_i terms using the connected correlators in (7.7)–(7.11) and then rewrite the result as $e^{-\Delta F}$.

The F_1 term in (7.12) is then simply

$$\mathsf{F}_1(\lambda) = \sum_{i=1}^{\infty} \mathsf{R}_i^{(0)} B_i = 2\sum_{i=1}^{\infty} R_i\, B_i = 2F_1(\lambda), \tag{7.19}$$

where $F_1(\lambda)$ is the corresponding SU(N) term in (3.12). Thus, $\mathsf{F}_1(\lambda)$ for the Sp($2N$) model also has an exact integral representation of the form in (3.14) multiplied by factor of 2.

For the F_2 term we obtain

$$\begin{aligned}\mathsf{F}_2(\lambda) &= \sum_{i=1}^{\infty} \mathsf{R}_i^{(1)} B_i \; - \sum_{i,j=1}^{\infty} \mathsf{U}_{ij}^{(0)} B_i B_j = \frac{1}{2}\sum_{i=1}^{\infty}(i+2)R_i\, B_i - 2\sum_{i,j=1}^{\infty} \widetilde{Q}_{ij} B_i B_j \\ &= \frac{1}{2}\frac{d}{d\lambda}\left[\lambda F_1(\lambda)\right] + 2\,\widetilde{F}_2(\lambda)\,, \qquad\qquad \frac{d}{d\lambda}\widetilde{F}_2 = -\frac{\lambda}{2}\left[\frac{d^2}{d\lambda^2}(\lambda\, F_1)\right]^2, \end{aligned} \tag{7.20}$$

where $\widetilde{F}_2(\lambda)$ is the same as in (3.19), (3.20), (3.21).

We conclude that in this Sp($2N$) model the F_2 term is much simpler than in the SU(N) case in (3.15) — it does not contain the analog of the $\bar{F}_2$ term (3.16). In (7.20), the first term is linear in the ζ_{2n+1}-values, while the second is quadratic. The presence of this first term is related to the different structure of the large N expansion in (7.7) that contains the $1/N$ term which was absent in the SU(N) case.[24]

Furthermore, since $F_1(\lambda)$ has a simple integral representation (3.14), and $\widetilde{F}_2(\lambda)$ is directly related to $F_1(\lambda)$ as in (3.21), we see from (7.19) and (7.20) that in the Sp($2N$) model both $\mathsf{F}_1(\lambda)$ and $\mathsf{F}_2(\lambda)$ have explicit integral representations that permit precise analysis of both the convergent weak coupling expansion and the asymptotic strong coupling expansion. This carries over to the Wilson loop corrections, as discussed in the next subsections.

Finally, from (7.18) we conclude that the $1/N$ term $\mathsf{F}_3(\lambda)$ in (7.12) is given by

$$\begin{aligned}\mathsf{F}_3(\lambda) &= \sum_{i=1}^{\infty} \mathsf{R}_i^{(2)} B_i \; - \sum_{i,j=1}^{\infty} \mathsf{U}_{ij}^{(1)} B_i B_j \; + \sum_{i,j,k=1}^{\infty} \mathsf{T}_{ijk}^{(0)}\, B_i B_j B_k \\ &= \frac{1}{24}\sum_{i=1}^{\infty} i(i+1)(i+2)\, R_i\, B_i - \frac{1}{2}\sum_{i,j=1}^{\infty}(i+j+2)\,\widetilde{Q}_{ij} B_i B_j + \frac{1}{3}\left[\sum_{i=1}^{\infty}(i+1)(i+2)\, R_i\, B_i\right]^3. \end{aligned} \tag{7.21}$$

Using that according to (2.8) we have $B_i \sim \lambda^{i+1}$ and also the relation in (3.21), the expression for F_3 may be written as (cf. (7.20))

$$\begin{aligned}\mathsf{F}_3(\lambda) &= \frac{\lambda^2}{24}\left[\lambda F_1(\lambda)\right]''' + \frac{\lambda}{2}\,\widetilde{F}_2'(\lambda) + \frac{\lambda^3}{3}\Big(\left[\lambda F_1(\lambda)\right]''\Big)^3 \\ &= \frac{\lambda^2}{24}\,(\lambda F_1)''' - \frac{\lambda^2}{4}\Big[(\lambda F_1)''\Big]^2 + \frac{2\lambda^3}{3!}\Big[(\lambda F_1)''\Big]^3, \end{aligned} \tag{7.22}$$

where $f'(\lambda) \equiv \frac{d}{d\lambda} f(\lambda)$.

[24] Note, for example, that

$$\langle \mathrm{tr}\, a^6 \rangle = \begin{cases} \frac{5}{8N^2}(N^2-1)(3-3N^2+N^4) = \frac{5N^4}{8} + 0\times N^3 - \frac{5N^2}{2} + \cdots\,, & \mathrm{SU}(N) \\ \frac{5}{32}N(1+2N)(1+2N+4N^2) = \frac{5N^4}{4} + \frac{5N^3}{4} + \frac{5N^2}{8} + \cdots\,, & \mathrm{Sp}(2N)\,. \end{cases}$$

It is possible to generalize the above computation of F_3 to the case of the next terms F_4 and F_5 in (7.12). The analog of the last term in (7.22) with highest number of powers of derivatives over λ or of highest power in $(\lambda F_1)''$ turns out to be (cf. (7.21), (7.10), (7.11), (7.17))

$$\mathsf{F}_4(\lambda)=-\frac{1}{4!}\sum_{i,j,k,\ell=1}^{\infty} c_{ijk\ell}R_iR_jR_kR_\ell B_iB_jB_kB_\ell+\dots=-\frac{2\lambda^2}{4!}\left(\lambda^3\left[(\lambda F_1)''\right]^4\right)'+\dots\,, \tag{7.23}$$

$$\mathsf{F}_5(\lambda)=\frac{1}{5!}\sum_{i,j,k,\ell,s=1}^{\infty} c_{ijk\ell s}R_iR_jR_kR_\ell R_s B_iB_jB_kB_\ell B_s+\dots=\frac{2\lambda^2}{5!}\left[\lambda^2\left(\lambda^3\left[(\lambda F_1)''\right]^5\right)'\right]'+\dots\,, \tag{7.24}$$

where we used that, as follows from (7.10), (7.11),

$$\begin{aligned}
c_{ijk\ell} &\equiv 2(i+j+k+\ell+3)(i+1)(i+2)(j+1)(j+2)(k+1)(k+2)(\ell+1)(\ell+2)\,,\\
c_{ijk\ell s} &\equiv 2(i+j+k+\ell+3)(i+j+k+\ell+s+4)(i+1)(i+2)(j+1)(j+2)(k+1)(k+2)(\ell+1)(\ell+2)\,.
\end{aligned} \tag{7.25}$$

These terms provide the dominant contributions in F_4 and F_5 at strong coupling: $\mathsf{F}_4\sim\lambda^4$, $\mathsf{F}_5\sim\lambda^5$ (see below). Comparing the last term in (7.22) with (7.23) and (7.24) we observe a definite pattern for generalization.

Thus it is natural to expect that all higher order $1/N$ corrections in the free energy in (7.12) will be expressed in terms of derivatives of $F_1(\lambda)$. The integral representation for F_1 (3.14) will then imply a similar representation not only for F_2 (cf. (7.20), (3.22)) and F_3 (7.22) but also for all F_n.

7.3 Strong coupling expansion of free energy

Given the relations (7.19), (7.20) and (7.22) the strong coupling expansions of the free energy terms F_1, F_2 and F_3 in (7.12) follow from the SU(N) results for F_1 and $\widetilde{F}_2$ in (6.5), (6.6) and (6.8) and the leading terms in F_4 and F_5 from (7.23), (7.24)

$$\begin{aligned}
\mathsf{F}_1 &= 2f_1\lambda+2f_2\log\lambda+2f_3+2f_4\lambda^{-1}+\mathcal{O}(e^{-\sqrt{\lambda}})\\
&=\frac{\log 2}{2\pi^2}\lambda-\frac{1}{2}\log\lambda+\text{const.}-\frac{\pi^2}{2\lambda}+\mathcal{O}(e^{-\sqrt{\lambda}})\,,
\end{aligned} \tag{7.26}$$

$$\mathsf{F}_2=-2f_1^2\lambda^2+f_1(1-4f_2)\lambda+\frac{1}{2}f_2(1-2f_2)\log\lambda+\frac{1}{2}(f_2+f_3+4p_4)+\mathcal{O}(e^{-\sqrt{\lambda}})\,, \tag{7.27}$$

$$\mathsf{F}_3=\frac{8}{3}f_1^3\lambda^3-f_1^2(1-4f_2)\lambda^2-f_1f_2(1-2f_2)\lambda-\frac{1}{24}f_2(1+6f_2-8f_2^2)+\mathcal{O}(e^{-\sqrt{\lambda}})\,, \tag{7.28}$$

$$\mathsf{F}_4=-4f_1^4\lambda^4+\mathcal{O}(\lambda^3)\,, \tag{7.29}$$

$$\mathsf{F}_5=\frac{32}{5}f_1^5\lambda^5+\mathcal{O}(\lambda^4)\,, \tag{7.30}$$

Here $\mathcal{O}(e^{-\sqrt{\lambda}})$ stands for the corresponding exponentially suppressed corrections $\sim\lambda^{-k/4}e^{-n\sqrt{\lambda}}$ that follow from the ones in F_1 in (6.17), (6.19).[25]

[25]While F_1 has exponentials that are odd powers of $e^{-\sqrt{\lambda}}$, F_2 (that contains squares of derivatives of F_1 and cross-terms, cf. (7.20)) has both even and odd powers of $e^{-\sqrt{\lambda}}$. Similarly, for F_3 in (7.22) one also finds both odd and even powers of $e^{-\sqrt{\lambda}}$.

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We observe that the leading large λ asymptotics of F_n appears to be λ^n. Note also that F_3 has no $\log\lambda$ term while the order λ^{-1} term appears only in F_1. Assuming that all higher F_n terms are expressed in terms of derivatives of λF_1 as in (7.20), (7.22), (7.23), (7.24) the only $\log\lambda$ corrections will come from F_1 and F_2, i.e. the coefficient of the $\log\lambda$ term in F receives contributions only from the N^2, N and N^0 orders in the $1/N$ expansion while the λ^{-1} term in F is exactly captured by (7.26).

Including also the $\mathcal{N}=4$ SYM contribution in (1.22) the full expression for the free energy expanded at large λ may be written as

$$F = F^{\mathcal{N}=4} + \Delta F \overset{\lambda\gg 1}{=} \Delta F_{\text{pol}} - \left(N^2+N-\frac{3}{16}\right)\log\lambda - \frac{\pi^2}{2}\frac{N}{\lambda} + \mathcal{O}(e^{-\sqrt{\lambda}})\,, \tag{7.31}$$

$$\begin{aligned}\Delta F_{\text{pol}} &= N\lambda\left[2f_1+\mathcal{O}(\lambda^{-1})\right]+\lambda^2\left[2f_1^2+\mathcal{O}(\lambda^{-1})\right]+\frac{1}{N}\lambda^3\left[\frac{8}{3}f_1^3+\mathcal{O}(\lambda^{-1})\right]+\mathcal{O}\left(\frac{1}{N^2}\right)\\ &= N^2\mathcal{F}\left(\frac{\lambda}{N}\right)+\dots\,,\end{aligned} \tag{7.32}$$

$$\mathcal{F}\left(\frac{\lambda}{N}\right) = 2f_1\frac{\lambda}{N} + 2f_1^2\left(\frac{\lambda}{N}\right)^2 + \frac{8}{3}f_1^3\left(\frac{\lambda}{N}\right)^3 - 4f_1^4\left(\frac{\lambda}{N}\right)^4 + \frac{32}{5}f_1^5\left(\frac{\lambda}{N}\right)^5 + \dots\,, \tag{7.33}$$

where ΔF_{pol} represents the polynomial in $\lambda \gg 1$ contributions with $\mathcal{F}\left(\frac{\lambda}{N}\right)$ being the sum of the leading λ^n terms at each order in $1/N$.

Remarkably, the coefficients in (7.33) suggest that $\mathcal{F}$ has the following exact form

$$\mathcal{F}\left(\frac{\lambda}{N}\right) = \log\left(1+2f_1\frac{\lambda}{N}\right). \tag{7.34}$$

Using that according to (1.2) we have $\frac{\lambda}{N} = 4\pi g_{\text{s}}$ we conclude that this leading order term expressed in terms of string parameters non-trivially depends just on string coupling ($8\pi f_1 = \frac{2}{\pi}\log 2$)

$$F = N^2\mathcal{F}\left(\frac{\lambda}{N}\right) + \dots = \frac{\pi^2 T^4}{g_{\text{s}}^2}\log\left(1+8\pi f_1 g_{\text{s}}\right) + \dots\,. \tag{7.35}$$

This term should be summing the leading large string tension contributions from each order in string topological expansion

The term $-\frac{\pi^2}{2}\frac{N}{\lambda} = -\frac{\pi}{8}\frac{1}{g_{\text{s}}}$ in (7.31) should also have a special origin on the string side, coming from a particular crosscup or disc contribution not involving (in contrast to the $\frac{1}{g_{\text{s}}}$ term in (7.35)) extra powers of string tension (and thus subleading compared to (7.35) at large T).

7.4 Wilson loop

The $\frac{1}{2}$-BPS Wilson loop is again defined as in (4.1). In the $\text{Sp}(2N)$ $\mathcal{N}=4$ SYM theory its expectation value (exact in N and λ defined still as $\lambda = N g_{\text{YM}}^2$) is given by the sum of the

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Laguerre polynomials [7] (cf. (1.1)[26])

$$\langle \mathcal{W}\rangle^{\mathcal{N}=4} = 2\, e^{\frac{\lambda}{16N}} \sum_{k=0}^{N-1} L_{2k+1}\left(-\frac{\lambda}{8N}\right). \tag{7.36}$$

The resulting $1/N$ expansion is

$$\langle \mathcal{W}\rangle^{\mathcal{N}=4} = N\,\frac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{1}{2}\left[I_0(\sqrt{\lambda}) - 1\right] + \frac{1}{N}\,\frac{\lambda}{96}\, I_2(\sqrt{\lambda}) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{7.37}$$

Then the $\mathcal{N}=2$ expectation value may be written as in (1.28)[27]

$$\langle \mathcal{W}\rangle = N\,\mathsf{W}_0(\lambda) + \mathsf{W}_{0,1}(\lambda) + \mathsf{W}_1(\lambda) + \frac{1}{N}\left[\mathsf{W}_{0,2}(\lambda) + \mathsf{W}_2(\lambda)\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{7.38}$$

where the $\mathcal{N}=4$ parts $\mathsf{W}_{0,n}$ are given by (7.37)

$$\mathsf{W}_0 \equiv \langle \mathcal{W}\rangle_0^{\mathcal{N}=4} = \frac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = 2W_0, \qquad \mathsf{W}_{0,1} \equiv \langle \mathcal{W}\rangle_1^{\mathcal{N}=4} = \frac{1}{2}\left[I_0(\sqrt{\lambda}) - 1\right], \tag{7.39}$$

$$\mathsf{W}_{0,2} \equiv \langle \mathcal{W}\rangle_2^{\mathcal{N}=4} = \frac{\lambda}{96}\, I_2(\sqrt{\lambda}). \tag{7.40}$$

The relation between the genuine $\mathcal{N}=2$ parts W_1 and W_2 in (7.38) and the free energy terms in (7.12) is the same (up to factor of 1/2) as in SU(N) case in (5.5), (5.6) (see appendix C)

$$\mathsf{W}_1' = -\frac{\lambda}{8}\mathsf{W}_0\,(\lambda \mathsf{F}_1)'', \qquad \mathsf{W}_2 = -\frac{\lambda^2}{8}\mathsf{W}_0\,\mathsf{F}_2'. \tag{7.41}$$

We thus find using (5.5) and (7.20) (cf. (5.7))

$$\mathsf{W}_1(\lambda) = 2W_1(\lambda), \qquad \mathsf{W}_2(\lambda) = -\frac{\lambda^2}{8}\mathsf{W}_0\left(\frac{1}{2}(\lambda F_1)'' - \lambda\left[(\lambda\, F_1)''\right]^2\right). \tag{7.42}$$

Like for the free energy in (7.22)–(7.24), these relations can be extended also to higher $1/N$ orders.

Using (7.41), (7.42) we find for the strong-coupling expansion of the coefficients in (7.38)

$$\frac{\mathsf{W}_1}{\mathsf{W}_0} = \frac{W_1}{W_0} = -f_1\,\lambda^{3/2} + \frac{3}{2}\,f_1\,\lambda - \left(\frac{3}{8}f_1 + \frac{1}{2}f_2\right)\lambda^{1/2} + \mathcal{O}(\lambda^0) = -\frac{\log 2}{4\pi^2}\lambda^{3/2} + \mathcal{O}(\lambda), \tag{7.43}$$

$$\frac{\mathsf{W}_2}{\mathsf{W}_0} = \frac{1}{2}f_1^2\lambda^3 - \frac{1}{8}f_1(1-4f_2)\lambda^2 - \frac{1}{16}f_2(1-2f_2)\lambda + \mathcal{O}(e^{-\sqrt{\lambda}}) = -\frac{\log^2 2}{32\pi^4}\,\lambda^3 + \mathcal{O}(\lambda^2). \tag{7.44}$$

[26] The Laguerre polynomials in (7.36) are the basic ones, while in the SU(N) case in (1.1) we have the associated Laguerre polynomial arising from the sum in (7.36) without parity restriction on the index, i.e. from the identity $L_N^{(1)}(x) = \sum_{k=0}^{N} L_k(x)$.

[27] To recall, we define $\langle \mathcal{W}\rangle$ so that $\langle 1\rangle = 1$, i.e. we divide over the matrix model partition function $Z = e^{-F}$.

Note that like F_n in free energy the Wilson loop coefficients W_n have additional exponentially suppressed corrections $\sim e^{-\sqrt{\lambda}}$ at strong coupling, which are resurgent, and which follow directly from the exponentially suppressed corrections to $F_1(\lambda)$ derived in section 6.3.

Similar relations between higher order $1/N$ terms F_n in free energy (1.21) and W_n in (1.28) are expected also in general, with the dominant large λ term in F_n determining the strong coupling asymptotics of W_n (see appendix C). In particular,

$$\mathsf{W}_3 = -\frac{\lambda^{3/2}}{4!}\mathsf{W}_0\left[\lambda(\lambda F_1)''\right]^3 + \dots, \qquad \frac{\mathsf{W}_3}{\mathsf{W}_0} \overset{\lambda\gg 1}{=} -\frac{1}{6}f_1^3\lambda^{9/2} + \mathcal{O}(\lambda^4)\,. \tag{7.45}$$

Comparing to (7.43), (7.44) thus suggests that the leading (at each order in $1/N$) strong coupling terms in $\Delta\langle\mathcal{W}\rangle$ in (1.28) exponentiate as

$$\langle\mathcal{W}\rangle = (N\mathsf{W}_0 + \dots) + \Delta\langle\mathcal{W}\rangle \overset{\lambda\gg 1}{=} N\mathsf{W}_0 \exp\left[-f_1\frac{\lambda^{3/2}}{N}\right] + \dots\,. \tag{7.46}$$

This may be compared with similar exponentiation [1] of the leading large λ terms in the $\mathcal{N}=4$ SYM case in (1.36), (1.37) that on string side may be interpreted as representing sum of separated handle insertions into the disc diagram [2]. Similarly, (7.46) may be interpreted as a sum of crosscup insertions into the disc.

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A Partition function of $\mathcal{N}=2$ matrix model and conformal anomaly

Let us first recall that the conformal anomaly coefficients a and c in $\mathcal{N}=2$ superconformal models are not renormalized, i.e. are given just by their free-theory values found by summing up contributions of particular fields (see, e.g., [43]). In a model with $\mathrm{n_v}$ vector multiplets and $\mathrm{n_h}$ hypermultiplets one finds

$$\mathrm{a} = \frac{5}{24}\mathrm{n_v} + \frac{1}{24}\mathrm{n_h}\,. \tag{A.1}$$

In particular, in the $\mathcal{N}=4$ SYM theory ($\mathrm{n_v}=\mathrm{n_h}$) with group G we get $\mathrm{a}=\frac{1}{4}\dim G$. The free energy of a massless superconformal model on S^4 of radius r may be written as

$$\hat{F} = -\log\hat{Z} = 4\mathrm{a}\log(\Lambda\,\mathrm{r}) + F_{\rm fin}(\lambda, N)\,, \tag{A.2}$$

where Λ is a UV cutoff, i.e. the r dependence is controlled by the a-coefficient. The free energy thus depends on a subtraction scheme and below we shall denote by F its regularized value.

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The localization matrix model expression for the partition function Z of $\mathcal{N}=2$ gauge theory on S^4 is [5]

$$Z = e^{-F} = \int Da\, e^{-\frac{8\pi^2 N\, \mathrm{r}^2}{\lambda}\, \mathrm{tr}\, a^2}\, \mathcal{Z}_{\text{1-loop}}(a)\,, \qquad \mathcal{Z}_{\text{1-loop}}(a) = e^{-S_{\rm int}(a)}\,. \tag{A.3}$$

In $\mathcal{N}=4$ SYM case $\mathcal{Z}_{\text{1-loop}}(a)=1$ and doing the Gaussian integral we get

$$Z^{\mathcal{N}=4} = C\left(\frac{N\mathrm{r}^2}{\lambda}\right)^{-\frac{1}{2}\dim G}, \qquad F^{\mathcal{N}=4} = 4\mathrm{a}\log \mathrm{r} - 2\mathrm{a}\log\lambda + \text{const}\,, \qquad \mathrm{a} = \frac{1}{4}\dim G\,. \tag{A.4}$$

Setting $\mathrm{r}=1$ we conclude that in the subtraction scheme assumed in the localization approach $F^{\mathcal{N}=4} = -2\mathrm{a}\log\lambda$ (up to a constant). In particular, in the SU(N) case $F^{\mathcal{N}=4} = -\frac{1}{2}(N^2-1)\log\lambda$. This was noted in [17] and an AdS/CFT interpretation of this result was suggested.

One may wonder what happens in other $\mathcal{N}=2$ superconformal models, in particular, if the conformal anomaly a-coefficient is also encoded the $\log\lambda$ term of the large λ expansion of the free energy F on S^4. For the models that are planar-equivalent to $\mathcal{N}=4$ SYM this is certainly the case at the leading N^2 order but as we shall see below this does not need to be true at subleading orders in $1/N$.

For an $\mathcal{N}=2$ model with a collection of hypermultiplets in representation $R=\oplus R_i$ of a group G with algebra $\mathfrak{g}$ one finds [5][28]

$$\hat{\mathcal{Z}}_{\text{1-loop}}(a,\mathrm{r}) = \prod_{n=1}^{\infty}\left(\frac{\prod_{\alpha\in\mathrm{roots}(\mathfrak{g})}\left[\mathrm{r}^{-2}n^2 + (\alpha\cdot a)^2\right]}{\prod_{w\in\mathrm{weights}(R)}\left[\mathrm{r}^{-2}n^2+(w\cdot a)^2\right]}\right)^n. \tag{A.5}$$

$\hat{\mathcal{Z}}_{\text{1-loop}}$ coming from the ratio of 1-loop determinants on S^4 in a constant scalar a background does not depend on λ but does depend on r. Note that the product over roots here includes also the "massless" contributions of the zero roots corresponding to Cartan directions for which $\alpha\cdot a=0$ (same also applies to the product over weights in the case of the adjoint representation).

The regularized value of $\hat{\mathcal{Z}}$ in (A.5) used in [5] was

$$\mathcal{Z}_{\text{1-loop}}(a\,\mathrm{r}) = \frac{\prod_{\alpha\in\mathrm{roots}(\mathfrak{g})}\mathrm{H}(i\,\alpha\cdot a\,\mathrm{r})}{\prod_{w\in\mathrm{weights}(R)}\mathrm{H}(i\,w\cdot a\,\mathrm{r})}\,, \tag{A.6}$$

where $\mathrm{H}(x)\equiv \mathrm{G}(1+x)\,\mathrm{G}(1-x)$ is the product of the Barnes G-functions. Notice that here the contribution of the "massless" terms present in (A.5) is trivial as $H(0)=1$. As a result, the contribution of (A.6) to the $\log\mathrm{r}$ term in F or to the conformal anomaly is trivial — the r dependence can be absorbed into the rescaling of the integration variable a in (A.3) and this the resulting Z will depend on r in the same way (A.4) as in the $\mathcal{N}=4$ SYM case.

[28]We ignore the instanton contribution since it is exponentially suppressed in the $1/N$ expansion we are interested in here.

To properly account for the conformal anomaly of the $\mathcal{N} = 2$ model we need to go back to the original unregularized expression (A.5) and compute its dependence on the radius r. Rearranging (A.5) using that

$$\prod_{n=1}^{\infty}\left(\mathrm{r}^{-2}n^2+\mu^2\right)^n=\prod_{n=1}^{\infty}\mathrm{r}^{-2n}\prod_{n=1}^{\infty}\left(n^2+\mathrm{r}^2\mu^2\right)^n, \tag{A.7}$$

where μ stands for $\alpha\cdot a$ or $w\cdot a$, we conclude that the non-trivial dependence on r (that cannot be absorbed into a) is captured by the infinite product factor that can be defined using the standard Riemann ζ-function regularization as

$$\prod_{n=1}^{\infty}\mathrm{r}^{-2n}=e^{-2\zeta(-1)\log\mathrm{r}}=e^{\frac{1}{6}\log\mathrm{r}}. \tag{A.8}$$

As a result, we find from (A.5)[29]

$$\hat{\mathcal{Z}}_{\text{1-loop}}(a,\mathrm{r})\quad\to\quad e^{\frac{1}{6}(\dim G-\dim R)\log\mathrm{r}}\,\mathcal{Z}_{\text{1-loop}}(a\,\mathrm{r}). \tag{A.9}$$

Redefining $\mathrm{r}a\to a$ to account for the dependence on r in the free action in (A.3) and in $\mathcal{Z}_{\text{1-loop}}(a\,\mathrm{r})$ we need also to include the contribution of the Gaussian measure or the $\mathcal{N} = 4$ term in (A.4), so that the total r dependence of the $\mathcal{N} = 2$ free energy is (cf. (A.2))

$$F=\left[\dim G-\frac{1}{6}(\dim G-\dim R)\right]\log\mathrm{r}+\cdots=4\mathrm{a}\log\mathrm{r}+\ldots,\qquad \mathrm{a}=\frac{5}{24}\dim G+\frac{1}{24}\dim R, \tag{A.10}$$

in agreement with the general expression for the a-anomaly in (A.1).

We have thus shown that it is the "bare" expression for the matrix model integral (A.3) using (A.5) that correctly includes the conformal a-anomaly term in free energy. It is clear that the direct correlation between the dependence on r and on λ is a feature of only the Gaussian part of the integral in (A.3). In particular, the dependence of the $\mathcal{N} = 2$ free energy on $\log\lambda$ beyond the leading planar limit need not be controlled by the a-anomaly coefficient as that happened in the $\mathcal{N} = 4$ SYM case in (A.4).

Nevertheless, we have found (see discussion below (1.38)) that not only the order N^2 but also the order N coefficient of the $\log\lambda$ term in the large λ limit of the free energies of the SU(N) and Sp($2N$) FA-orientifold theories computed in this paper do agree with the corresponding terms in the conformal a-anomalies. We suspect that the matching of the order N term should be also related to the fact that these models are planar-equivalent to $\mathcal{N} = 4$ SYM theory.

B Derivation of large λ expansion of F_1 using Mellin transform

In the main text, we computed the large λ expansion of F_1 using the approach described in (6.1)–(6.3). Here we shall compute the large λ expansion of F_1 given by the integral

[29] Here we use that the total number of roots counting also the trivial Cartan ones is the same as $\dim G$.

representation (3.14) by applying the Mellin transform method (see e.g. [44, 45]). The first step is to rewrite (3.14) in the form of a Mellin convolution

$$h(x) \equiv (f \star g)(x) = \int_0^\infty dt\, f(t\,x)\, g(t)\,, \qquad x = \sqrt{\lambda}\,. \tag{B.1}$$

The Mellin transform is $\widetilde{h}(s) = \mathcal{M}[h](s) = \int_0^\infty dx\, x^{s-1}\, h(x) = \widetilde{f}(s)\, \widetilde{g}(1-s)$. If $\alpha < s < \beta$ is the fundamental strip of analyticity of $\widetilde{h}(s)$, the asymptotic expansion of $h(x)$ for $x \to \infty$ is obtained from the poles of its Mellin transform in the region $s \geq \beta$. In particular, the pole $\frac{1}{(s-s_0)^n}$ gives a term $\frac{(-1)^n}{(n-1)!}\,\frac{1}{x^{s_0}}\,\log^{n-1} x$ in the asymptotic expansion of $h(x)$.

Explicitly, let us first put (3.14) in the equivalent form[30]

$$\begin{aligned} F_1(\lambda) &= \frac{2}{\sqrt{\lambda}} \int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\, \frac{3t\sqrt{\lambda} - 8 J_1(t\sqrt{\lambda}) + J_1(2t\sqrt{\lambda})}{t^2} \\ &= 2\sqrt{\lambda} \int_0^\infty dt\, f(t\sqrt{\lambda})\, g(t) = 2\sqrt{\lambda}\, (f\star g)(\sqrt{\lambda})\,, \end{aligned} \tag{B.2}$$

where

$$f(t) = \frac{3t - 8J_1(t) + J_1(2t)}{t^2}\,, \qquad g(t) = \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,. \tag{B.3}$$

The Mellin transform of $g(t)$ is

$$\begin{aligned} \mathcal{M}\left[\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\right](s) &= -\frac{1}{2\pi}\mathcal{M}\left[\frac{d}{dt}\frac{1}{e^{2\pi t}-1}\right](s) = \frac{1}{2\pi}(s-1)\mathcal{M}\left[\frac{1}{e^{2\pi t}-1}\right](s-1) \\ &= (2\pi)^{-s}\,\Gamma(s)\,\zeta(s-1)\,. \end{aligned} \tag{B.4}$$

Computing the Mellin transform of f, then using $\widetilde{f\star g} = \widetilde{f}(s)\,\widetilde{g}(1-s)$, and finally evaluating the residues gives

$$F_1 \overset{\lambda\gg 1}{=} \frac{\log 2}{4}\,\frac{\lambda}{\pi^2} - \frac{1}{4}\log\frac{\lambda}{\pi^2} + \left(\frac{7}{6}\log 2 + \frac{3}{4} - 6\log \mathsf{A}\right) - \frac{\pi^2}{4}\left(\frac{\lambda}{\pi^2}\right)^{-1} + \dots\,, \tag{B.5}$$

where A is Glaisher's constant. There are no additional pole contributions beyond those giving (B.5). This implies that dots in (B.5) stand for the exponentially suppressed corrections (discussed in section 6.3).

C Strong coupling expansion of Wilson loop in Sp($2N$) theory

Let us first consider the expectation value of the BPS Wilson loop (defined in fundamental representation) in the $\mathcal{N}=4$ Sp($2N$) SYM theory [7] (see also [19])

$$\langle \mathcal{W}\rangle^{\mathcal{N}=4} = 2\, e^{\frac{\lambda}{16N}} \sum_{i=0}^{N-1} L_{2i+1}\left(-\frac{\lambda}{8N}\right). \tag{C.1}$$

[30]For an odd function $\hat{f}(t)$, we have the identity $\int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}+1)^2}\hat{f}(t) = \int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2} f(t)$ with $f(t) = \hat{f}(t) - 2\hat{f}\left(\frac{t}{2}\right)$ and the inversion relation $\hat{f}(t) = \sum_{k=0}^\infty 2^k f(2^{-k}t)$.

Using the integral representation of Laguerre polynomials $L_n(x) = \frac{1}{2\pi i}\oint \frac{dt}{t^{n+1}}(t+x)^n e^{-t}$, we can write

$$\langle \mathcal{W} \rangle^{\mathcal{N}=4} = \frac{1}{2\pi i}\oint dt\, \frac{8\,N\,e^{-t+\frac{\lambda}{16N}}\left(1-\frac{\lambda}{8tN}\right)}{\lambda\left(1-\frac{\lambda}{16tN}\right)}\left[1-\left(1-\frac{\lambda}{8Nt}\right)^{2N}\right]. \tag{C.2}$$

Expanding at large N and observing that

$$\frac{1}{2\pi i}\int \frac{du}{u^n}\, e^{-x(u+u^{-1})} = (-1)^{n-1}\, I_{n-1}(2x)\,, \tag{C.3}$$

we obtain for the leading terms [19]

$$\langle \mathcal{W} \rangle^{\mathcal{N}=4} = 4N\,\frac{I_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{1}{2}\Big[I_0(\sqrt{\lambda})-1\Big] + \frac{\lambda I_2(\sqrt{\lambda})}{96N} + \frac{1}{N^2}\left[-\frac{\lambda I_0(\sqrt{\lambda})}{192} + \frac{\sqrt{\lambda}(\lambda+8)I_1(\sqrt{\lambda})}{768}\right] + \cdots. \tag{C.4}$$

Let us denote the leading large N term here as $\langle \mathcal{W}\rangle_0 = N\mathsf{W}_0 = 4N\,\frac{I_1(\sqrt{\lambda})}{\sqrt{\lambda}}$ (cf. (1.29), (1.30)). Expanding at large λ and keeping only the dominant term at each order in $1/N$ we find

$$\frac{\langle \mathcal{W} \rangle^{\mathcal{N}=4}}{\langle \mathcal{W}\rangle_0} \overset{\lambda\gg 1}{=} 1 + \frac{\lambda^{1/2}}{8N} + \frac{\lambda^{3/2}}{384N^2} + \frac{\lambda^2}{3072N^3} + \frac{\lambda^3}{294912N^4} + \frac{\lambda^{7/2}}{2359296N^5} + \frac{\lambda^{9/2}}{339738624N^6} + \frac{\lambda^5}{2717908992N^7} + \cdots. \tag{C.5}$$

A natural guess for the sum of this expansion is

$$\frac{\langle \mathcal{W} \rangle^{\mathcal{N}=4}}{\langle \mathcal{W}\rangle_0} \overset{\lambda\gg 1}{=} \left(1+\frac{\lambda^{1/2}}{8N}\right)\exp\left(\frac{\lambda^{3/2}}{384\,N^2}\right). \tag{C.6}$$

This expression can be proved rigorously starting from the exact relations between $\langle \mathcal{W} \rangle^{\mathcal{N}=4}$ in $\mathrm{U}(N)$ and $\mathrm{Sp}(2N)$ theories given in [7]

$$\langle \mathcal{W} \rangle^{\mathcal{N}=4}_{\mathrm{Sp(2N)}}(\lambda) = \langle \mathcal{W} \rangle^{\mathcal{N}=4}_{\mathrm{U(2N)}}(\lambda) + \frac{1}{16N}\int_0^{\lambda} d\lambda'\,\langle \mathcal{W} \rangle^{\mathcal{N}=4}_{\mathrm{U(2N)}}(\lambda')\,, \tag{C.7}$$

and taking the large λ limit.[31]

Let us now turn to the Wilson loop expectation value in the $\mathcal{N}=2$ $\mathrm{Sp}(2N)$ theory given by the matrix model expectation value as in (2.6), (4.2) with the single-trace interaction action in (7.2)

$$S_{\rm int} = B_i(\lambda)\ \mathrm{tr}\,\hat{a}^{2i+2}\,, \qquad \hat{a} \equiv \frac{a}{\sqrt{N}}\,, \tag{C.8}$$

where here and below we assume summation over $i = 1,\ldots,\infty$ and $B_i(\lambda)$ is given by (2.8). Denoting as in (2.6) by $\langle\ldots\rangle$ the normalized expectation value in the Gaussian theory (i.e.

[31] The Wilson loop in the $\mathcal{N}=4$ $\mathrm{U}(N)$ theory is given by $\langle \mathcal{W} \rangle^{\mathcal{N}=4}_{\mathrm{U(N)}}(\lambda) = e^{\frac{\lambda}{8N}}\,L^{(1)}_{N-1}\left(-\frac{\lambda}{4N}\right)$ (cf. (1.1)).

in $\mathcal{N}=4$ SYM case) then

$$\begin{aligned}\langle \mathcal{W}\rangle &= \frac{\langle \operatorname{tr} e^{\sqrt{\frac{\lambda}{2}}\hat a}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}{\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle} = \sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{\lambda}{2}\right)^k \frac{\langle \operatorname{tr}\hat a^{2k}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}{\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}\\
&= 2N + \frac{\lambda}{4}\frac{\langle \operatorname{tr}\hat a^{2}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}{\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle} + \sum_{k=1}^{\infty}\frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1}\frac{\langle \operatorname{tr}\hat a^{2k+2}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}{\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}\\
&= 2N - \frac{1}{4N}\partial_{\lambda^{-1}}\log\hat Z + \sum_{k=1}^{\infty}\frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1}\frac{\langle \operatorname{tr}\hat a^{2k+2}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}{\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle}\\
&= 2N - \frac{\lambda^2}{4N}\partial_\lambda \Delta F + \lambda\frac{N(2N+1)}{8N} + \sum_{k=1}^{\infty}\frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1}\partial_{B_k}\Delta F\,. \end{aligned} \tag{C.9}$$

Here $\hat Z = e^{-F} = \int Da'\; e^{-\frac{1}{\lambda}\operatorname{tr} a'^2 - S_{\rm int}(a')}$ is the total partition function as in (2.1) before rescaling of integration variable by $\lambda^{1/2}$ in (2.5) and the total free energy $F = F^{\mathcal{N}=4}+\Delta F$ as in (2.10) with $F^{\mathcal{N}=4}$ given by (1.22). We used that differentiating $\hat Z$ over λ puts down the factor $\sim \operatorname{tr} a^2$. The third term in (C.9) comes from

$$\log\hat Z = -\Delta F + \log\int Da'\, e^{-\frac{1}{\lambda}\operatorname{tr} a'^2} = -\Delta F + \frac{1}{2}N(2N+1)\log\lambda + \text{const.} \tag{C.10}$$

We also used the formal notation $\partial_{B_k}\Delta F$ for the normalized $\partial_{B_k}\langle e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle = \langle \operatorname{tr}\hat a^{2k+2}\; e^{-B_i \operatorname{tr}\hat a^{2i+2}}\rangle$. Here (see (7.19), (7.20))

$$\Delta F = N\mathsf{F}_1 + \mathsf{F}_2 + \frac{1}{N}\mathsf{F}_3 + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{C.11}$$

$$\mathsf{F}_1 = 2\sum_i R_i B_i\,, \qquad \mathsf{F}_2 = \frac{1}{2}\sum_{i=1}^{\infty}(i+2)\,R_i B_i - 2\sum_{i,j=1}^{\infty}\widetilde Q_{ij}B_iB_j\,. \tag{C.12}$$

where numerical R_i and $\widetilde Q$ are given by (3.5), (3.9) and λ-dependence is contained in B_i. Defining W_n corrections to the $\mathcal{N}=4$ SYM value $\langle\mathcal{W}\rangle^{\mathcal{N}=4} = \langle \operatorname{tr} e^{\sqrt{\frac{\lambda}{2}}\hat a}\rangle$ as in (7.38), i.e.

$$\langle\mathcal{W}\rangle = \langle\mathcal{W}\rangle^{\mathcal{N}=4} + \mathsf{W}_1 + \frac{1}{N}\mathsf{W}_2 + \frac{1}{N^2}\mathsf{W}_3 + \mathcal{O}\left(\frac{1}{N^3}\right), \tag{C.13}$$

we see that derivatives of both F_n and F_{n+1} terms in ΔF in (C.9) contribute to W_n. In particular, $\partial_{B_k}\mathsf{F}_1 = 2R_k$ contributes to the order N (planar) part of $\langle\mathcal{W}\rangle$ while for W_1 we find

$$\mathsf{W}_1 = -\frac{\lambda^2}{4}\mathsf{F}_1' + \sum_{k=1}^{\infty}\frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1}\partial_{B_k}\mathsf{F}_2(B) = -\frac{\lambda^2}{4}\mathsf{F}_1' - 4\sum_{j,k=1}^{\infty}\frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1}\widetilde Q_{kj}B_j\,, \tag{C.14}$$

where $(\dots)' \equiv \partial_\lambda(\dots)$. Since $B_j \sim \lambda^{j+1}$, differentiating W_1 over λ gives

$$\begin{aligned}\mathsf{W}_1' &= -\frac{1}{4}(\lambda^2\mathsf{F}_1')' - \sum_{j,k=1}^{\infty}\frac{2(j+k+2)}{(2k+2)!}\left(\frac{\lambda}{2}\right)^k\widetilde Q_{kj}B_j\\
&= -\frac{1}{4}(\lambda^2\mathsf{F}_1')' - \sum_{j,k=1}^{\infty}\frac{2}{(2k+2)!}\left(\frac{\lambda}{2}\right)^k\frac{2^{j+k+1}\Gamma\left(j+\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{\pi\Gamma(j+1)\Gamma(k+1)}B_j\end{aligned}$$

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$$
\begin{aligned}
&= -\frac{1}{4}(\lambda^2 \mathsf{F}_1')' + \frac{\sqrt{\pi}}{2\sqrt{\lambda}}(\sqrt{\lambda} - 2I_1(\sqrt{\lambda})) \sum_{j=1}^{\infty} \frac{1}{\pi} \frac{2^{j+1}\Gamma\left(j+\frac{3}{2}\right)}{\Gamma(j+1)} B_j \\
&= -\frac{1}{4}(\lambda^2 \mathsf{F}_1')' + \frac{1}{2\sqrt{\lambda}}(\sqrt{\lambda} - 2I_1(\sqrt{\lambda})) \sum_{j=1}^{\infty}(j+1)(j+2)R_j B_j \\
&= -\frac{1}{4}\,\mathsf{W}_0 \sum_{j=1}^{\infty}(j+1)(j+2)R_j B_j = -\frac{\lambda}{8}\mathsf{W}_0(\lambda \mathsf{F}_1)'' \,,
\end{aligned} \tag{C.15}
$$

where $\mathsf{W}_0 = \frac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$ as in (7.39). This demonstrates the relation in (7.41). Similarly one can show also that $\mathsf{W}_2 = -\frac{\lambda^2}{8}\mathsf{W}_0 \mathsf{F}_2'$.

The example of W_2 suggests that the dominant at large λ term in W_n comes from the dominant term in the corresponding F_n. Indeed, from (C.9) and the expression for the dominant term in F_3 in (7.21) we get for the leading order large λ contribution

$$
\begin{aligned}
\mathsf{W}_2 &\overset{\lambda \gg 1}{=} \sum_{k=1}^{\infty} \frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1} \partial_{B_k} \frac{1}{3}\left[\sum_{i=1}^{\infty}(i+1)(i+2)R_i B_i\right]^3 + \dots \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k+2)!}\left(\frac{\lambda}{2}\right)^{k+1} (k+1)(k+2)R_k \left[\sum_{i=1}^{\infty}(i+1)(i+2)R_i B_i\right]^2 + \dots \\
&= -\frac{1}{16}\lambda\left[1 - \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})\right]\left[\lambda(\lambda \mathsf{F}_1)''\right]^2 + \dots = \mathsf{W}_0 \frac{\lambda}{32}[\lambda(\lambda \mathsf{F}_1)'']^2 + \dots \,.
\end{aligned} \tag{C.16}
$$

This is indeed the leading at large λ term in the exact expression for W_2 in terms of $\mathsf{F}_1 = 2F_1$ in (7.42).

Applying the same logic to find the large λ contribution in W_3 we use the expression for the dominant term in F_4 in (7.23)

$$
\begin{aligned}
\mathsf{W}_3 &\overset{\lambda \gg 1}{=} \sum_{m=1}^{\infty} \frac{1}{(2m+2)!}\left(\frac{\lambda}{2}\right)^{m+1} \partial_{B_m}\left(-\frac{1}{4!}\sum_{i,j,k,\ell=1}^{\infty} c_{ijk\ell} R_i R_j R_k R_\ell B_i B_j B_k B_\ell\right) + \dots \\
&= -\frac{1}{3!}\sum_{m=1}^{\infty} \frac{1}{(2m+2)!}\left(\frac{\lambda}{2}\right)^{m+1} R_m \sum_{i,j,k=1}^{\infty} c_{ijkm} R_i R_j R_k B_i B_j B_k + \dots \,,
\end{aligned} \tag{C.17}
$$

where c_{ijkm} is given in (7.25). Summing over m and keeping only leading $e^{\sqrt{\lambda}}$ terms (i.e. terms proportional to $\mathsf{W}_0 = 2\sqrt{\frac{2}{\pi}}\lambda^{-3/4}e^{\sqrt{\lambda}} + \dots$) we get

$$
\begin{aligned}
\mathsf{W}_3 &\overset{\lambda \gg 1}{=} -\frac{1}{3!}\frac{\lambda^{3/2}}{8}\mathsf{W}_0 \sum_{ijk}^{\infty}(i+1)(i+2)(j+1)(j+2)(k+1)(k+2)R_i R_j R_k B_i B_j B_k + \dots \\
&= -\frac{1}{3!}\frac{\lambda^{3/2}}{8}\mathsf{W}_0\left[\sum_{i=1}^{\infty}(i+1)(i+2)R_i B_i\right]^3 + \dots = -\frac{1}{3!}\frac{\lambda^{3/2}}{64}\mathsf{W}_0\left[\lambda(\lambda \mathsf{F}_1)''\right]^3 + \dots \,.
\end{aligned} \tag{C.18}
$$

Then $\mathsf{F}_1 \overset{\lambda \gg 1}{=} 2f_1\lambda + \dots$ (see (7.26)) gives

$$
\frac{\mathsf{W}_3}{\mathsf{W}_0} \overset{\lambda \gg 1}{=} -\frac{1}{6}f_1^3\lambda^{9/2} + \dots \,. \tag{C.19}
$$

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