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Modular invariant models of leptons at level 7

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Abstract: We consider for the first time level 7 modular invariant flavour models where the lepton mixing originates from the breaking of modular symmetry and couplings responsible for lepton masses are modular forms. The latter are decomposed into irreducible multiplets of the finite modular group Γ_7, which is isomorphic to PSL$(2, Z_7)$, the projective special linear group of two dimensional matrices over the finite Galois field of seven elements, containing 168 elements, sometimes written as PSL$_2(7)$ or $\Sigma(168)$. At weight 2, there are 26 linearly independent modular forms, organised into a triplet, a septet and two octets of Γ_7. A full list of modular forms up to weight 8 are provided. Assuming the absence of flavons, the simplest modular-invariant models based on Γ_7 are constructed, in which neutrinos gain masses via either the Weinberg operator or the type-I seesaw mechanism, and their predictions compared to experiment.

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Contents

1 Introduction

The puzzle of quark and lepton masses and mixing, left unanswered by the Standard Model (SM), may be addressed by introducing some family symmetry, which is generally non-Abelian and may be associated with a finite discrete group. Modular symmetry has been suggested as the origin of such a flavour symmetry, with neutrino masses as complex analytic functions called modular forms [1]. Finite non-Abelian discrete family symmetry emerges from the modular symmetry at various positive integer *levels*, with each level associated with a particular flavour group. At each level, the physical fields carry various *modular weights* which do not have to add up to zero in the coupling terms of the effective Lagrangian since the effective Yukawa couplings may be modular forms, which are holomorphic functions of a complex modulus field τ [1]. This may allow flavon fields to be removed, with higher-dimensional operators in the superpotential being completely determined by modular invariance and supersymmetry. The neutrino masses and mixing parameters may be predicted in terms of a few input parameters, although the predictive power of this framework may be reduced by the Kähler potential which is less constrained by modular symmetry [2].

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The finite modular groups $\Gamma_2 \cong S_3$ [3–6], $\Gamma_3 \cong A_4$ [1, 3, 4, 7–14], $\Gamma_4 \cong S_4$ [13, 15–19] and $\Gamma_5 \cong A_5$ [18, 20, 21] have been considered. For example, simple A_4 modular models can reproduce the measured neutrino masses and mixing angles [1, 8, 12]. The quark masses and mixing angles may also be included together with leptons in an A_4 modular invariant model [22], and it has been shown how natural fermion mass hierarchies can arise as a result of a weighton field [23]. The modular invariance approach has been extended to include odd weight modular forms which can be decomposed into irreducible representations of the homogeneous finite modular group Γ'_N [24], and the modular symmetry $\Gamma'_3 \cong T'$ has been discussed, including the new possibility of texture zeroes [25]. Also modular symmetry may be combined with generalized CP symmetry, where the modulus transforms as $\tau \to -\tau^*$ under the CP transformation [26–30]. The formalism of the single modulus has been generalized to the case of a direct product of multiple moduli [31, 32], which is motivated by the additional extra dimensions in superstring theory, assuming toroidal compactification. Indeed, from a top-down perspective, modular symmetry naturally appears in string constructions [27, 33–36]. It has been realised that, if the VEV of the modulus τ takes some special value, a residual subgroup of the finite modular symmetry group Γ_N would be preserved. The phenomenological implications of the residual modular symmetry have been discussed in the context of modular A_4 [10, 13], S_4 [13, 16] and A_5 [20] symmetries. If the modular symmetry is broken down to a residual Z_3 (or Z_5) subgroup in charged lepton sector and to a Z_2 subgroup in the neutrino sector, the trimaximal TM1 and TM2 mixing patterns can be obtained [10, 16]. It has been shown that the complex modulus could possibly be stabilized at some Z_2 fixed points in string compactifications [37]. Moreover, the dynamics of modular symmetry could potentially be tested in present and forthcoming neutrino oscillation experiments [38].

In this paper we consider the level 7 finite modular group $\Gamma_7 \cong \mathrm{PSL}(2, Z_7)$, the projective special linear group of two dimensional matrices over the finite Galois field of seven elements. This is the smallest simple discrete group which contains complex triplets and sextet representations. It contains 168 elements and is sometimes written as $\mathrm{PSL}_2(7)$ or $\Sigma(168)$ [39, 40]. The relationship of this group to some other family symmetries that have been used in the literature is discussed in [41–45]. It has been proposed as a finite modular group in [46], whose notations we follow.

We emphasise that there are good motivations to study level 7 modular symmetry. It is well known that interesting lepton mixing patterns can arise from groups in the sequence $S^2 = 1$, $(ST)^3 = 1$, $T^N = 1$ for $N = 3, 4, 5$ but this sequence leads to an infinite group for $N > 5$ [46]. If we require that the finite modular group possesses three-dimensional irreducible representations to which the left-handed lepton doublets are assigned, modular symmetry approach with Γ_N allows to have $N = 3, 4, 5, 7, 8, 16$ so in this paper we are considering the smallest example with $N > 5$, which has interesting properties, as follows. Level 7 is a non-geometrical group with complex representations, being a subgroup of SU(3) but not SO(3), so opens up new model building possibilities not available for the levels 3, 4, 5 (since A_4, S_4, A_5 are subgroups of SO(3)). In particular, there are several advantages to using a family symmetry with complex triplet and sextet representations [44]. Such models allow a very nice extension to grand unified schemes such as $\Sigma(168) \times \mathrm{SO}(10)$, in which all fermions are in the $(3,16)$ representation and the top quark Yukawa coupling arises from a renormalizable sextet term. Moreover, with complex triplets, the trivial unit mass matrix (which would destroy the mass hierarchies) is forbidden (unlike A_4, S_4, A_5, where it would be allowed). Although the study of such grand unified models is beyond the scope of the present paper, the formal results presented here are useful since they pave the way for such future studies. Moreover, having complex representations allows new possibilities for CP violation, which could be studied in the future, using the results in this paper.

Thus, we consider for the first time level 7 modular invariant flavour models based on Γ_7 where the lepton mixing originates from the breaking of modular symmetry and couplings responsible for lepton masses are modular forms. The latter are decomposed into irreducible multiplets of the finite modular group Γ_7. At weight 2, there are 26 linearly independent modular forms, organised into a triplet, a septet and two octets of Γ_7. A full list of modular forms up to weight 8 are provided. Assuming the absence of flavons, the simplest modular-invariant models are constructed in which neutrinos gain masses via either the Weinberg operator or the type-I seesaw mechanism, and their predictions compared to experiment.

We organise the rest of this paper in the following. In section 2, we describe properties of the modular group Γ and its finite subgroup Γ_7. A full list of modular forms of level 7 and weight up to 10 are derived in section 3, based on the SageMath algebra system [47]. We construct a class of flavon-less lepton flavour models and study their experimental constraints in section 4. Summary is given in section 5. The group theory of Γ_7 is listed in appendix A. We refer to appendices B and C for alternative methods based on the Dedekind eta function method [1] and the theta function method [20], respectively. How to find an independent set of higher weight modular forms from hundreds of constraints is discussed in appendix D.

2 Modular symmetry and modular forms of level $N = 7$

In the following, we briefly review the modular symmetry and its principal congruence subgroups. The special linear group $\mathrm{SL}(2,Z)$ is constituted by 2×2 matrices with integer entries and determinant equal to one [48, 49]:

$$\mathrm{SL}(2,Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\}. \tag{2.1}$$

The $\mathrm{SL}(2,Z)$ group acts on the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$ as the linear fractional transformation,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{H} \to \mathcal{H}, \quad \tau \mapsto \gamma\tau = \gamma(\tau) = \frac{a\tau+b}{c\tau+d}. \tag{2.2}$$

It is easy to see the following identity

$$\Im(\gamma(\tau)) = \frac{\Im\tau}{|c\tau+d|^2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,Z). \tag{2.3}$$

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Consequently the image $\gamma(z) \in \mathcal{H}$ for any $\gamma \in \mathrm{SL}(2, Z)$ and $\tau \in \mathcal{H}$. It is obvious that

$$\frac{a\tau + b}{c\tau + d} \quad \text{is identical with} \quad \frac{-a\tau - b}{-c\tau - d}\,, \tag{2.4}$$

and therefore we identify

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is identical with} \quad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}. \tag{2.5}$$

Hence the actions of γ and $-\gamma$ on the complex modulus τ are exactly the same, and it is sufficient to consider the projective special linear group $\mathrm{PSL}(2, Z) = \mathrm{SL}(2, Z)/\{I, -I\}$, which is the quotient of $\mathrm{SL}(2, Z)$ by $\pm I$. The group $\mathrm{PSL}(2, Z)$ is also called the modular group in the literature, it is a discrete group with infinite elements and it can be generated by two transformations S and T [48]

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

which fulfill the relations

$$S^2 = (ST)^3 = 1\,. \tag{2.7}$$

The actions of S and T on $\mathcal{H}$ are given by

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad T : \tau \mapsto \tau + 1\,. \tag{2.8}$$

For a positive integer N, the principal congruence subgroup of level N is defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \quad a \equiv d \equiv 1 \ (\mathtt{mod}\ N), \quad b \equiv c \equiv 0 \ (\mathtt{mod}\ N) \right\}, \tag{2.9}$$

which is a normal subgroup of the special linear group $\mathrm{SL}(2, Z)$. Obviously $\Gamma(1) \cong \mathrm{SL}(2, \mathbb{Z})$ is the special linear group. It is easy to obtain

$$T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \tag{2.10}$$

which implies $T^N \in \Gamma(N)$, i.e., T^N is an element of $\Gamma(N)$. Taking the quotient of $\Gamma(1)$ and $\Gamma(2)$ by $\{I, -I\}$, we obtain the projective principal congruence subgroups $\overline{\Gamma}(N) = \Gamma(N)/\{I, -I\}$ for $N = 1, 2$, and $\overline{\Gamma}(N > 2) = \Gamma(N)$ since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$. The quotient groups $\Gamma_N = \overline{\Gamma}(1)/\overline{\Gamma}(N)$ are usually called inhomogeneous finite modular groups, and the homogeneous finite modular group is defined as $\Gamma'_N = \mathrm{SL}(2, Z)/\Gamma(N)$ which is the double covering of Γ_N [24]. The finite modular group Γ_N for $N \leq 5$ can be obtained from $\overline{\Gamma}(1)$ by imposing the condition $T^N = 1$. Consequently the generators S and T of Γ_N obey the relations

$$S^2 = (ST)^3 = T^N = 1\,. \tag{2.11}$$

The groups Γ_N with $N = 2, 3, 4, 5$ are isomorphic to the permutation groups S_3, A_4, S_4 and A_5 respectively [46]. Note again that for this group, as for all groups with $N > 5$, at least one additional relation is necessary in order to render the group finite. It is easy to calculate

$$(ST^3)^4 = \begin{pmatrix} -8 & -21 \\ 21 & 55 \end{pmatrix}, \tag{2.12}$$

which implies

$$-(ST^3)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 7\,. \tag{2.13}$$

Hence the element $-(ST^3)^4$ is belong to $\overline{\Gamma}(7)$. Notice that $-(ST^3)^4$ and $(ST^3)^4$ are identified as the same element of the $\overline{\Gamma}(1)$ group, since they lead to the same linear fraction transformations. Therefore the finite modular group Γ_7 of level $N = 7$ can be generated by two generators S and T which satisfy the following multiplication rules[1]

$$S^2 = (ST)^3 = T^7 = (ST^3)^4 = 1\,. \tag{2.14}$$

The crucial element of the modular invariance approach is the modular forms $f(\tau)$ of weight k and level N. They are holomorphic functions of the complex modulus τ with well-defined transformation properties under the group $\Gamma(N)$,

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)\,. \tag{2.15}$$

The modular forms of weight k and level N form a linear space $\mathcal{M}_k(\Gamma(N))$ of finite dimension. In the present work, we shall focus on even weight modular forms, i.e., k being an even number. Then it is always possible to choose a basis of $\mathcal{M}_k(\Gamma(N))$ such that the modular forms transform according to a unitary irreducible representation $\mathbf{r}$ of Γ_N [1],

$$f_i(\gamma\tau) = (c\tau + d)^k (\rho_{\mathbf{r}}(\gamma))_{ij} f_j(\tau)\,, \tag{2.16}$$

where γ is a representative element of Γ_N, and $\rho_{\mathbf{r}}(\gamma)$ is the representation matrix of γ in the irreducible representation $\mathbf{r}$. If the modular weight k is odd, the modular forms can be decomposed into irreducible representations of the homogeneous finite modular group Γ'_N [24]. In order to determine the proper basis, it is sufficient to apply eq. (2.16) to the generators S and T which can generate all elements of Γ_N.

3 Modular forms of level $N = 7$

The general dimension formula for the linear space of modular forms of level N and weight k is given by

$$\mathtt{dim}\mathcal{M}_k(\Gamma(N)) = \frac{(k-1)N+6}{24} N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right), \qquad N > 2,\, k \geq 1\,. \tag{3.1}$$

For $N = 7$, we have

$$\mathtt{dim}\mathcal{M}_k(\Gamma(7)) = \frac{(k-1)\times 7+6}{24} \times 7^2 \times \left(1 - \frac{1}{7^2}\right) = 14k - 2\,. \tag{3.2}$$

Hence the linear space of modular forms of level 7 and weight 2 has dimension $14\times 2 - 2 = 26$.

[1] The multiplication rule of Γ_7 is $S^2 = (ST)^3 = T^7 = (ST^{-1}ST)^4 = \mathbb{1}$ in [46].

One can obtain q-expansions for a basis b_i of the space of lowest weight modular forms for Γ_7 from the SageMath algebra system [47]. They are given by

$$
\begin{aligned}
b_1(\tau) &= q^{1/7}(1-3q+4q^3+2q^4+3q^5-12q^6-5q^7+7q^9)+\dots\,,\\
b_2(\tau) &= q^{2/7}(1-3q-q^2+8q^3-6q^5-4q^6+2q^8)+\dots\,,\\
b_3(\tau) &= q^{4/7}(1-4q+3q^2+5q^3-5q^4-8q^6+10q^7-4q^9)+\dots\,,\\
b_4(\tau) &= 1+252q^5-840q^6+1344q^7-840q^8-420q^9+588q^{10}+\dots\,,\\
b_5(\tau) &= q^{1/7}\left(1+\frac{15}{2}q^3+30q^4-41q^5+44q^6+5q^8-8q^9\right)+\dots\,,\\
b_6(\tau) &= q^{2/7}\left(1+\frac{73}{4}q^3-13q^4+\frac{119}{4}q^5-\frac{43}{4}q^6-\frac{15}{4}q^7+30q^8+\frac{97}{2}q^9\right)+\dots\,,\\
b_7(\tau) &= q^{3/7}(1+15q^3-q^4-3q^5+15q^6+11q^7-3q^8+27q^9)+\dots\,,\\
b_8(\tau) &= q^{4/7}\left(1+\frac{62}{5}q^3+9q^4-\frac{176}{5}q^5+51q^6+\frac{66}{5}q^7-\frac{213}{5}q^8+\frac{88}{5}q^9\right)+\dots\,,\\
b_9(\tau) &= q^{5/7}(1+17q^3-18q^4+19q^5-2q^6+16q^7-3q^8+17q^9)+\dots\,,\\
b_{10}(\tau) &= q^{6/7}(1+15q^3-13q^4+29q^6-q^7-27q^8+43q^9)+\dots\,,\\
b_{11}(\tau) &= q+69q^5-234q^6+385q^7-231q^8-130q^9+168q^{10}+\dots\,,\\
b_{12}(\tau) &= q^{8/7}\left(1+\frac{7}{2}q^2+4q^4+11q^6-3q^7+9q^8\right)+\dots\,,\\
b_{13}(\tau) &= q^{9/7}(1+3q^2-q^3+5q^4+7q^6+6q^8)+\dots\,,\\
b_{14}(\tau) &= q^{10/7}(1+2q^2+3q^4+2q^5+4q^6+5q^8)+\dots\,,\\
b_{15}(\tau) &= q^{11/7}\left(1+\frac{6}{5}q^2+\frac{22}{5}q^4-q^5+\frac{23}{5}q^6-\frac{4}{5}q^7+\frac{29}{5}q^8\right)+\dots\,,\\
b_{16}(\tau) &= q^{12/7}(1+2q^3+2q^4+3q^6+3q^8)+\dots\,,\\
b_{17}(\tau) &= q^{13/7}(1-q^2+3q^3+3q^4-4q^5+3q^6+6q^7-3q^8)+\dots\,,\\
b_{18}(\tau) &= q^2-8q^5+39q^6-62q^7+38q^8+28q^9-28q^{10}+\dots\,,\\
b_{19}(\tau) &= q^{15/7}(1-q+3q^3-3q^5+5q^6)+\dots\,,\\
b_{20}(\tau) &= q^{16/7}\left(1-\frac{9}{4}q+4q^2-\frac{15}{4}q^3+\frac{15}{4}q^4-\frac{1}{4}q^5-\frac{9}{2}q^7\right)+\dots\,,\\
b_{21}(\tau) &= q^{17/7}(1-2q+2q^2+q^3-q^4-q^5+4q^6-3q^7)+\dots\,,\\
b_{22}(\tau) &= q^{18/7}\left(1-\frac{9}{5}q+\frac{32}{5}q^3-7q^4-\frac{12}{5}q^5+\frac{61}{5}q^6-\frac{16}{5}q^7\right)+\dots\,,\\
b_{23}(\tau) &= q^{19/7}(1-3q+5q^2-4q^3+3q^4-3q^5+4q^6-3q^7)+\dots\,,\\
b_{24}(\tau) &= q^{20/7}(1-3q+4q^2-4q^4+q^5+8q^6-9q^7)+\dots\,,\\
b_{25}(\tau) &= q^3-15q^5+55q^6-88q^7+54q^8+34q^9-41q^{10}+\dots\,,\\
b_{26}(\tau) &= q^4-6q^5+17q^6-26q^7+18q^8+4q^9-7q^{10}+\dots\,,
\end{aligned}
\tag{3.3}
$$

with $q = e^{2\pi i\tau}$ and where fractional powers $q^{n/7}$ should be read as $q^{n/7} = e^{2n\pi i\tau/7}$. The above lowest weight modular forms can be organized into a triplet transforming in the representation **3** of Γ_7, a septet transforming in the representation **7** of Γ_7, and two octets in the **8** of Γ_7. To be more explicit, we have

$$Y_{\mathbf{3}}^{(2)}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \\ -b_3(\tau) \end{pmatrix}, \tag{3.4}$$

$$Y_{\mathbf{7}}^{(2)}(\tau) \equiv \begin{pmatrix} Y_4(\tau) \\ Y_5(\tau) \\ Y_6(\tau) \\ Y_7(\tau) \\ Y_8(\tau) \\ Y_9(\tau) \\ Y_{10}(\tau) \end{pmatrix} = \begin{pmatrix} b_4(\tau)+4b_{11}(\tau)+12b_{18}(\tau)+16b_{25}(\tau)+28b_{26}(\tau) \\ -\sqrt{2}\,[b_5(\tau)+15b_{12}(\tau)+24b_{19}(\tau)] \\ -\sqrt{2}\,[3b_6(\tau)+13b_{13}(\tau)+31b_{20}(\tau)] \\ -2\sqrt{2}\,[2b_7(\tau)+9b_{14}(\tau)+9b_{21}(\tau)] \\ -\sqrt{2}\,[7b_8(\tau)+12b_{15}(\tau)+39b_{22}(\tau)] \\ -2\sqrt{2}\,[3b_9(\tau)+14b_{16}(\tau)+10b_{23}(\tau)] \\ -2\sqrt{2}\,[6b_{10}(\tau)+7b_{17}(\tau)+21b_{24}(\tau)] \end{pmatrix}, \tag{3.5}$$

$$Y_{\mathbf{8}a}^{(2)}(\tau) \equiv \begin{pmatrix} Y_{11}(\tau) \\ Y_{12}(\tau) \\ Y_{13}(\tau) \\ Y_{14}(\tau) \\ Y_{15}(\tau) \\ Y_{16}(\tau) \\ Y_{17}(\tau) \\ Y_{18}(\tau) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2b_4(\tau)+2b_{11}(\tau)-24b_{18}(\tau)+56b_{25}(\tau)-b_{26}(\tau) \\ -\sqrt{3}\,[22b_{11}(\tau)+30b_{18}(\tau)+56b_{25}(\tau)+59b_{26}(\tau)] \\ -2\sqrt{2}\,[2b_5(\tau)+15b_{12}(\tau)+21b_{19}(\tau)] \\ \sqrt{2}\,[3b_6(\tau)-16b_{13}(\tau)+11b_{20}(\tau)] \\ 7\sqrt{2}\,[2b_7(\tau)+3b_{14}(\tau)+12b_{21}(\tau)] \\ -\sqrt{2}\,[11b_8(\tau)+54b_{15}(\tau)+54b_{22}(\tau)] \\ 7\sqrt{2}\,[b_{16}(\tau)+2b_{23}(\tau)] \\ 7\sqrt{2}\,[3b_{10}(\tau)+8b_{17}(\tau)+9b_{24}(\tau)] \end{pmatrix}, \tag{3.6}$$

$$Y_{\mathbf{8}b}^{(2)}(\tau) \equiv \begin{pmatrix} Y_{19}(\tau) \\ Y_{20}(\tau) \\ Y_{21}(\tau) \\ Y_{22}(\tau) \\ Y_{23}(\tau) \\ Y_{24}(\tau) \\ Y_{25}(\tau) \\ Y_{26}(\tau) \end{pmatrix} = \frac{1}{60}\begin{pmatrix} 3\,[20b_{11}(\tau)+40b_{18}(\tau)+42b_{25}(\tau)+53b_{26}(\tau)] \\ -\sqrt{3}\,[2b_4(\tau)-12b_{11}(\tau)+18b_{18}(\tau)-42b_{25}(\tau)+27b_{26}(\tau)] \\ 6\sqrt{2}\,[b_5(\tau)+4b_{12}(\tau)+14b_{19}(\tau)] \\ -3\sqrt{2}\,[5b_6(\tau)+20b_{13}(\tau)+37b_{20}(\tau)] \\ 21\sqrt{2}\,[3b_{14}(\tau)+2b_{21}(\tau)] \\ -3\sqrt{2}\,[5b_8(\tau)+8b_{15}(\tau)+22b_{22}(\tau)] \\ 21\sqrt{2}\,[2b_9(\tau)+3b_{16}(\tau)+8b_{23}(\tau)] \\ -21\sqrt{2}\,[b_{10}(\tau)+4b_{17}(\tau)+b_{24}(\tau)] \end{pmatrix}. \tag{3.7}$$

Notice that the weight two modular multiplets $Y_{\mathbf{8}a}^{(2)}(\tau)$ and $Y_{\mathbf{8}b}^{(2)}(\tau)$ are not unique, and in principle they can taken to be any two linearly independent combinations of $Y_{\mathbf{8}a}^{(2)}(\tau)$ and $Y_{\mathbf{8}b}^{(2)}(\tau)$. Higher weight modular multiplets can be obtained from tensor products of the lowest weight multiplets $Y_{\mathbf{3}}^{(2)}$, $Y_{\mathbf{7}}^{(2)}$, $Y_{\mathbf{8}a}^{(2)}$ and $Y_{\mathbf{8}b}^{(2)}$. The missing **1** and **6** representations arise at weight 4. Even though one can form 351 products Y_iY_j where some vanishing modular forms easily seen from the Clebsch-Gordan coefficients are not counted, the space of modular forms of weight 4 (and level 7) has dimension $14k-2=54$. Therefore, there are 297 constraints between the Y_iY_j, which we list in appendix D. The 54 linearly independent modular forms of weight 4 can be arranged into the following multiplets of Γ_7:

$$Y_{\mathbf{1}a}^{(4)} = \left(Y_{\mathbf{7}}^{(2)}Y_{\mathbf{7}}^{(2)}\right)_{\mathbf{1}} = 2Y_{10}Y_5+Y_4^2+2Y_6Y_9+2Y_7Y_8 \equiv Y_1^{(4)}(\tau)\,. \tag{3.8}$$

$$Y^{(4)}_{\mathbf{3}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{3}}=\begin{pmatrix}\sqrt{3}Y_1Y_{11}+Y_1Y_{12}-\sqrt{6}Y_3Y_{16}-\sqrt{6}Y_2Y_{18}\\ \sqrt{6}Y_1Y_{13}-\sqrt{3}Y_2Y_{11}+Y_2Y_{12}-\sqrt{6}Y_3Y_{17}\\ \sqrt{6}Y_1Y_{15}-2Y_3Y_{12}+\sqrt{6}Y_2Y_{14}\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_2(\tau)\\ Y^{(4)}_3(\tau)\\ Y^{(4)}_4(\tau)\end{pmatrix}. \tag{3.9}$$

$$Y^{(4)}_{\mathbf{6}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{3}}\right)_{\mathbf{6}}=\begin{pmatrix}\sqrt{2}Y_3^2\\ \sqrt{2}Y_1^2\\ 2Y_1Y_2\\ \sqrt{2}Y_2^2\\ 2Y_1Y_3\\ 2Y_2Y_3\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_5(\tau)\\ Y^{(4)}_6(\tau)\\ Y^{(4)}_7(\tau)\\ Y^{(4)}_8(\tau)\\ Y^{(4)}_9(\tau)\\ Y^{(4)}_{10}(\tau)\end{pmatrix}, \tag{3.10}$$

$$Y^{(4)}_{\mathbf{6}b}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{6}}=\begin{pmatrix}2Y_1Y_4-2\sqrt{2}Y_{10}Y_2-\sqrt{2}Y_3Y_8\\ -\sqrt{2}Y_1Y_5+2Y_2Y_4-2\sqrt{2}Y_3Y_9\\ -Y_1Y_6-2Y_{10}Y_3+3Y_2Y_5\\ -2\sqrt{2}Y_1Y_7-\sqrt{2}Y_2Y_6+2Y_3Y_4\\ 3Y_1Y_8-2Y_2Y_7-Y_3Y_5\\ -2Y_1Y_9-Y_2Y_8+3Y_3Y_6\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_{11}(\tau)\\ Y^{(4)}_{12}(\tau)\\ Y^{(4)}_{13}(\tau)\\ Y^{(4)}_{14}(\tau)\\ Y^{(4)}_{15}(\tau)\\ Y^{(4)}_{16}(\tau)\end{pmatrix}. \tag{3.11}$$

$$Y^{(4)}_{\mathbf{7}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{7}}=\begin{pmatrix}\sqrt{2}Y_1Y_{10}+\sqrt{2}Y_2Y_9+\sqrt{2}Y_3Y_7\\ -\sqrt{2}Y_1Y_4-2Y_3Y_8\\ -\sqrt{2}Y_2Y_4-2Y_1Y_5\\ -Y_1Y_6+2Y_{10}Y_3+Y_2Y_5\\ -\sqrt{2}Y_3Y_4-2Y_2Y_6\\ Y_1Y_8+2Y_2Y_7-Y_3Y_5\\ 2Y_1Y_9-Y_2Y_8+Y_3Y_6\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_{17}(\tau)\\ Y^{(4)}_{18}(\tau)\\ Y^{(4)}_{19}(\tau)\\ Y^{(4)}_{20}(\tau)\\ Y^{(4)}_{21}(\tau)\\ Y^{(4)}_{22}(\tau)\\ Y^{(4)}_{23}(\tau)\end{pmatrix}, \tag{3.12}$$

$$Y^{(4)}_{\mathbf{7}b}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{7_S}}=\begin{pmatrix}-2Y_{10}Y_5+6Y_4^2-2Y_6Y_9-2Y_7Y_8\\ 4\sqrt{2}Y_{10}Y_6-2Y_4Y_5+4\sqrt{2}Y_7Y_9+2\sqrt{2}Y_8^2\\ 4\sqrt{2}Y_{10}Y_7-2Y_4Y_6+2\sqrt{2}Y_5^2+4\sqrt{2}Y_8Y_9\\ 4\sqrt{2}Y_{10}Y_8-2Y_4Y_7+4\sqrt{2}Y_5Y_6+2\sqrt{2}Y_9^2\\ 4\sqrt{2}Y_{10}Y_9-2Y_4Y_8+4\sqrt{2}Y_5Y_7+2\sqrt{2}Y_6^2\\ 2\sqrt{2}Y_{10}^2-2Y_4Y_9+4\sqrt{2}Y_5Y_8+4\sqrt{2}Y_6Y_7\\ -2Y_{10}Y_4+4\sqrt{2}Y_5Y_9+4\sqrt{2}Y_6Y_8+2\sqrt{2}Y_7^2\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_{24}(\tau)\\ Y^{(4)}_{25}(\tau)\\ Y^{(4)}_{26}(\tau)\\ Y^{(4)}_{27}(\tau)\\ Y^{(4)}_{28}(\tau)\\ Y^{(4)}_{29}(\tau)\\ Y^{(4)}_{30}(\tau)\end{pmatrix}. \tag{3.13}$$

$$Y^{(4)}_{\mathbf{8}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{8}}=\begin{pmatrix}5Y_1Y_{10}-Y_2Y_9-4Y_3Y_7\\ -\sqrt{3}Y_1Y_{10}+3\sqrt{3}Y_2Y_9-2\sqrt{3}Y_3Y_7\\ -4Y_1Y_4-3\sqrt{2}Y_{10}Y_2+2\sqrt{2}Y_3Y_8\\ 2\sqrt{2}Y_1Y_5-4Y_2Y_4-3\sqrt{2}Y_3Y_9\\ 4\sqrt{2}Y_1Y_6+\sqrt{2}Y_{10}Y_3+2\sqrt{2}Y_2Y_5\\ 3\sqrt{2}Y_1Y_7-2\sqrt{2}Y_2Y_6+4Y_3Y_4\\ -2\sqrt{2}Y_1Y_8-\sqrt{2}Y_2Y_7-4\sqrt{2}Y_3Y_5\\ -\sqrt{2}Y_1Y_9-4\sqrt{2}Y_2Y_8-2\sqrt{2}Y_3Y_6\end{pmatrix}\equiv\begin{pmatrix}Y^{(4)}_{31}(\tau)\\ Y^{(4)}_{32}(\tau)\\ Y^{(4)}_{33}(\tau)\\ Y^{(4)}_{34}(\tau)\\ Y^{(4)}_{35}(\tau)\\ Y^{(4)}_{36}(\tau)\\ Y^{(4)}_{37}(\tau)\\ Y^{(4)}_{38}(\tau)\end{pmatrix}, \tag{3.14}$$

	Modular form $Y_{\mathbf{r}}^{(k)}$
$k=2$	$Y_{\mathbf{3}}^{(2)}$, $Y_{\mathbf{7}}^{(2)}$, $Y_{\mathbf{8}a}^{(2)}$, $Y_{\mathbf{8}b}^{(2)}$
$k=4$	$Y_{\mathbf{1}a}^{(4)}$, $Y_{\mathbf{3}a}^{(4)}$, $Y_{\mathbf{6}a}^{(4)}$, $Y_{\mathbf{6}b}^{(4)}$, $Y_{\mathbf{7}a}^{(4)}$, $Y_{\mathbf{7}b}^{(4)}$, $Y_{\mathbf{8}a}^{(4)}$, $Y_{\mathbf{8}b}^{(4)}$, $Y_{\mathbf{8}c}^{(4)}$
$k=6$	$Y_{\mathbf{1}}^{(6)}$, $Y_{\mathbf{3}a}^{(6)}$, $Y_{\mathbf{3}b}^{(6)}$, $Y_{\bar{\mathbf{3}}}^{(6)}$, $Y_{\mathbf{6}a}^{(6)}$, $Y_{\mathbf{6}b}^{(6)}$, $Y_{\mathbf{7}a}^{(6)}$, $Y_{\mathbf{7}b}^{(6)}$, $Y_{\mathbf{7}c}^{(6)}$, $Y_{\mathbf{7}d}^{(6)}$, $Y_{\mathbf{8}a}^{(6)}$, $Y_{\mathbf{8}b}^{(6)}$, $Y_{\mathbf{8}c}^{(6)}$, $Y_{\mathbf{8}d}^{(6)}$
$k=8$	$Y_{\mathbf{1}}^{(8)}$, $Y_{\mathbf{3}a}^{(8)}$, $Y_{\mathbf{3}b}^{(8)}$, $Y_{\bar{\mathbf{3}}}^{(8)}$, $Y_{\mathbf{6}a}^{(8)}$, $Y_{\mathbf{6}b}^{(8)}$, $Y_{\mathbf{6}c}^{(8)}$, $Y_{\mathbf{6}d}^{(8)}$, $Y_{\mathbf{7}a}^{(8)}$, $Y_{\mathbf{7}b}^{(8)}$, $Y_{\mathbf{7}c}^{(8)}$, $Y_{\mathbf{7}d}^{(8)}$, $Y_{\mathbf{8}a}^{(8)}$, $Y_{\mathbf{8}b}^{(8)}$, $Y_{\mathbf{8}c}^{(8)}$, $Y_{\mathbf{8}d}^{(8)}$, $Y_{\mathbf{8}e}^{(8)}$, $Y_{\mathbf{8}f}^{(8)}$
$k=10$	$Y_{\mathbf{1}}^{(10)}$, $Y_{\mathbf{3}a}^{(10)}$, $Y_{\mathbf{3}b}^{(10)}$, $Y_{\mathbf{3}c}^{(10)}$, $Y_{\bar{\mathbf{3}}a}^{(10)}$, $Y_{\bar{\mathbf{3}}b}^{(10)}$, $Y_{\mathbf{6}a}^{(10)}$, $Y_{\mathbf{6}b}^{(10)}$, $Y_{\mathbf{6}c}^{(10)}$, $Y_{\mathbf{6}d}^{(10)}$, $Y_{\mathbf{7}a}^{(10)}$, $Y_{\mathbf{7}b}^{(10)}$, $Y_{\mathbf{7}c}^{(10)}$, $Y_{\mathbf{7}d}^{(10)}$, $Y_{\mathbf{7}e}^{(10)}$, $Y_{\mathbf{7}f}^{(10)}$, $Y_{\mathbf{8}a}^{(10)}$, $Y_{\mathbf{8}b}^{(10)}$, $Y_{\mathbf{8}c}^{(10)}$, $Y_{\mathbf{8}d}^{(10)}$, $Y_{\mathbf{8}e}^{(10)}$, $Y_{\mathbf{8}f}^{(10)}$, $Y_{\mathbf{8}g}^{(10)}$

Table 1. Summary of modular forms of level 7 up to weight 8, the subscript $\mathbf{r}$ denotes the transformation property under Γ_7 modular symmetry. Here $Y_{\mathbf{8}a}^{(2)}$ and $Y_{\mathbf{8}b}^{(2)}$ stand for two weight 2 modular forms transforming in the representation $\mathbf{8}$ of Γ_7. The same convention is adopted for other modular forms.

$$Y_{\mathbf{8}b}^{(4)}=\left(Y_{\mathbf{8}a}^{(2)}Y_{\mathbf{8}a}^{(2)}\right)_{\mathbf{8_{S,1}}}$$
$$=\begin{pmatrix}3\sqrt{3}Y_{11}^2-2Y_{11}Y_{12}-3\sqrt{3}Y_{12}^2+4\sqrt{3}Y_{13}Y_{18}-4\sqrt{3}Y_{15}Y_{16}\\ -Y_{11}^2-6\sqrt{3}Y_{11}Y_{12}+Y_{12}^2-4Y_{13}Y_{18}+8Y_{14}Y_{17}-4Y_{15}Y_{16}\\ -4\sqrt{3}Y_{11}Y_{13}+4Y_{12}Y_{13}-4\sqrt{6}Y_{14}Y_{18}+2\sqrt{6}Y_{16}^2\\ -8Y_{12}Y_{14}+2\sqrt{6}Y_{13}^2+4\sqrt{6}Y_{16}Y_{17}\\ 4\sqrt{3}Y_{11}Y_{15}+4Y_{12}Y_{15}+4\sqrt{6}Y_{16}Y_{18}+2\sqrt{6}Y_{17}^2\\ 4\sqrt{3}Y_{11}Y_{16}+4Y_{12}Y_{16}-4\sqrt{6}Y_{13}Y_{15}-2\sqrt{6}Y_{14}^2\\ -8Y_{12}Y_{17}-4\sqrt{6}Y_{14}Y_{15}-2\sqrt{6}Y_{18}^2\\ -4\sqrt{3}Y_{11}Y_{18}+4Y_{12}Y_{18}+4\sqrt{6}Y_{13}Y_{17}-2\sqrt{6}Y_{15}^2\end{pmatrix}\equiv\begin{pmatrix}Y_{39}^{(4)}(\tau)\\ Y_{40}^{(4)}(\tau)\\ Y_{41}^{(4)}(\tau)\\ Y_{42}^{(4)}(\tau)\\ Y_{43}^{(4)}(\tau)\\ Y_{44}^{(4)}(\tau)\\ Y_{45}^{(4)}(\tau)\\ Y_{46}^{(4)}(\tau)\end{pmatrix}, \tag{3.15}$$

$$Y_{\mathbf{8}c}^{(4)}=\left(Y_{\mathbf{8}b}^{(2)}Y_{\mathbf{8}b}^{(2)}\right)_{\mathbf{8_{S,2}}}=\begin{pmatrix}4Y_{19}Y_{20}+2\sqrt{3}Y_{22}Y_{25}-2\sqrt{3}Y_{23}Y_{24}\\ 2Y_{19}^2-2Y_{20}^2-4Y_{21}Y_{26}+2Y_{22}Y_{25}+2Y_{23}Y_{24}\\ 4Y_{20}Y_{21}-2\sqrt{6}Y_{23}Y_{25}\\ -2\sqrt{3}Y_{19}Y_{22}-2Y_{20}Y_{22}-2\sqrt{6}Y_{23}Y_{26}\\ 2\sqrt{3}Y_{19}Y_{23}-2Y_{20}Y_{23}+2\sqrt{6}Y_{21}Y_{22}\\ 2\sqrt{3}Y_{19}Y_{24}-2Y_{20}Y_{24}-2\sqrt{6}Y_{25}Y_{26}\\ -2\sqrt{3}Y_{19}Y_{25}-2Y_{20}Y_{25}+2\sqrt{6}Y_{21}Y_{24}\\ 4Y_{20}Y_{26}+2\sqrt{6}Y_{22}Y_{24}\end{pmatrix}\equiv\begin{pmatrix}Y_{47}^{(4)}(\tau)\\ Y_{48}^{(4)}(\tau)\\ Y_{49}^{(4)}(\tau)\\ Y_{50}^{(4)}(\tau)\\ Y_{51}^{(4)}(\tau)\\ Y_{52}^{(4)}(\tau)\\ Y_{53}^{(4)}(\tau)\\ Y_{54}^{(4)}(\tau)\end{pmatrix}. \tag{3.16}$$

We summarize the modular forms of level $N=7$ up to weight 10 in table 1, their explicit forms are given in the appendix D. At weight 10, only the modular forms in the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ are presented.

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4 Lepton models based on Γ_7 modular symmetry

In this section, we shall construct some typical models for neutrino masses and mixing based on the Γ_7 modular symmetry. We will not introduce any flavon field, the flavour symmetry is broken when the complex modulus τ obtains a vacuum expectation value. The Higgs doublets fields $H_{u,d}$ are assumed to be singlets of Γ_7 with vanishing modular weights. The three right-handed (RH) charged leptons $E^c_{1,2,3}$ transform as singlet $\mathbf{1}$ under Γ_7 modular group nevertheless they are distinguished by the different modular weights $k_{1,2,3}$. We assign the three generations of left-handed (LH) lepton doublets L and the three right-handed neutrinos N^c to two triplets $\mathbf{3}$ and $\bar{\mathbf{3}}$ with the weights k_L and k_N. We shall employ potentially the lowest weight modular forms as much as possible in order to reduce free parameters.

4.1 Charged lepton sector

If the left-handed lepton fields L are embedded into the triplet $\mathbf{3}$, modular forms in the representation $\bar{\mathbf{3}}$ should be invoked in the charged lepton mass terms. The superpotential for the charged lepton Yukawa coupling reads as

$$\mathcal{W}_e = \alpha\left(E^c_1 L Y^{(6)}_{\bar{\mathbf{3}}} H_d\right)_{\mathbf{1}} + \beta\left(E^c_2 L Y^{(8)}_{\bar{\mathbf{3}}}\right)_{\mathbf{1}} H_d + \gamma_1\left(E^c_3 L Y^{(10)}_{\bar{\mathbf{3}}a} H_d\right)_{\mathbf{1}} + \gamma_2\left(E^c_3 L Y^{(10)}_{\bar{\mathbf{3}}b} H_d\right)_{\mathbf{1}} . \tag{4.1}$$

Notice that there are two linearly independent weight ten modular forms $Y^{(10)}_{\bar{\mathbf{3}}a}$ and $Y^{(10)}_{\bar{\mathbf{3}}b}$ transforming as $\bar{\mathbf{3}}$ at level $N = 7$. The phases of the coupling constants α, β and γ_1 can be absorbed into the lepton fields while γ_2 is generally a complex parameter. Modular invariance of the superpotential $\mathcal{W}_e$ in eq. (4.1) requires the modular weights should fulfill the conditions

$$k_1 = k_2 - 2 = k_3 - 4 = 6 - k_L . \tag{4.2}$$

After the value of τ is fixed by certain modulus stabilization mechanism, the charged lepton mass matrix takes the following form

$$M_e = \begin{pmatrix} \alpha Y^{(6)}_{\bar{\mathbf{3}},1} & \alpha Y^{(6)}_{\bar{\mathbf{3}},2} & \alpha Y^{(6)}_{\bar{\mathbf{3}},3} \\ \beta Y^{(8)}_{\bar{\mathbf{3}},1} & \beta Y^{(8)}_{\bar{\mathbf{3}},2} & \beta Y^{(8)}_{\bar{\mathbf{3}},3} \\ \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,1} + \gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,1} & \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,2} + \gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,2} & \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,3} + \gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,3} \end{pmatrix} v_d , \tag{4.3}$$

where we denote $Y^{(6)}_{\bar{\mathbf{3}}} \equiv (Y^{(6)}_{\bar{\mathbf{3}},1}, Y^{(6)}_{\bar{\mathbf{3}},2}, Y^{(6)}_{\bar{\mathbf{3}},3})^T$, and the expressions of $Y^{(6)}_{\bar{\mathbf{3}},1}$, $Y^{(6)}_{\bar{\mathbf{3}},2}$ and $Y^{(6)}_{\bar{\mathbf{3}},3}$ are given in appendix D. Similar notations are adopted for $Y^{(8)}_{\bar{\mathbf{3}}}$, $Y^{(10)}_{\bar{\mathbf{3}}a}$, $Y^{(10)}_{\bar{\mathbf{3}}b}$ and other modular forms in the following. Similarly if the LH leptons L are assigned to the triplet $\bar{\mathbf{3}}$, the charged lepton mass terms are

$$\mathcal{W}_e = \alpha\left(E^c_1 L Y^{(2)}_{\mathbf{3}} H_d\right)_{\mathbf{1}} + \beta\left(E^c_2 L Y^{(4)}_{\mathbf{3}a} H_d\right)_{\mathbf{1}} + \gamma_1\left(E^c_3 L Y^{(6)}_{\mathbf{3}a} H_d\right)_{\mathbf{1}} + \gamma_2\left(E^c_3 L Y^{(6)}_{\mathbf{3}b} H_d\right)_{\mathbf{1}} , \tag{4.4}$$

for $k_1 = k_2 - 2 = k_3 - 4 = 2 - k_L$, and the charged lepton mass matrix is given by

$$M_e = \begin{pmatrix} \alpha Y^{(2)}_{\mathbf{3},1} & \alpha Y^{(2)}_{\mathbf{3},2} & \alpha Y^{(2)}_{\mathbf{3},3} \\ \beta Y^{(4)}_{\mathbf{3}a,1} & \beta Y^{(4)}_{\mathbf{3}a,2} & \beta Y^{(4)}_{\mathbf{3}a,3} \\ \gamma_1 Y^{(6)}_{\mathbf{3}a,1} + \gamma_2 Y^{(6)}_{\mathbf{3}b,1} & \gamma_1 Y^{(6)}_{\mathbf{3}a,2} + \gamma_2 Y^{(6)}_{\mathbf{3}b,2} & \gamma_1 Y^{(6)}_{\mathbf{3}a,3} + \gamma_2 Y^{(6)}_{\mathbf{3}b,3} \end{pmatrix} v_d . \tag{4.5}$$

	ρ_L	$k_{1,2,3}+k_L$	Charged lepton mass matrices
C_1	$\mathbf{3}$	6, 8, 10	$M_e=\begin{pmatrix} \alpha Y^{(6)}_{\bar{\mathbf{3}},1} & \alpha Y^{(6)}_{\bar{\mathbf{3}},2} & \alpha Y^{(6)}_{\bar{\mathbf{3}},3} \\ \beta Y^{(8)}_{\bar{\mathbf{3}},1} & \beta Y^{(8)}_{\bar{\mathbf{3}},2} & \beta Y^{(8)}_{\bar{\mathbf{3}},3} \\ \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,1}+\gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,1} & \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,2}+\gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,2} & \gamma_1 Y^{(10)}_{\bar{\mathbf{3}}a,3}+\gamma_2 Y^{(10)}_{\bar{\mathbf{3}}b,3} \end{pmatrix} v_d$
C_2	$\bar{\mathbf{3}}$	2, 4, 6	$M_e=\begin{pmatrix} \alpha Y^{(2)}_{\mathbf{3},1} & \alpha Y^{(2)}_{\mathbf{3},2} & \alpha Y^{(2)}_{\mathbf{3},3} \\ \beta Y^{(4)}_{\mathbf{3}a,1} & \beta Y^{(4)}_{\mathbf{3}a,2} & \beta Y^{(4)}_{\mathbf{3}a,3} \\ \gamma_1 Y^{(6)}_{\mathbf{3}a,1}+\gamma_2 Y^{(6)}_{\mathbf{3}b,1} & \gamma_1 Y^{(6)}_{\mathbf{3}a,2}+\gamma_2 Y^{(6)}_{\mathbf{3}b,2} & \gamma_1 Y^{(6)}_{\mathbf{3}a,3}+\gamma_2 Y^{(6)}_{\mathbf{3}b,3} \end{pmatrix} v_d$

Table 2. The charged lepton mass matrices for different possible assignments of the left-handed lepton fields L, where the charged lepton mass matrix M_e is given in the right-left basis $E^c\, M_e\, L$ with $v_d = \langle H^0_d \rangle$.

Since there are two triplet modular forms $Y^{(8)}_{\mathbf{3}a}$ and $Y^{(8)}_{\mathbf{3}b}$ at weight 8, the superpotential $\mathcal{W}_e$ also comprises four independent terms for the weight assignment $k_1 = k_2 - 2 = k_3 - 6 = 2 - k_L$,

$$\mathcal{W}_e = \alpha\left(E^c_1 L Y^{(2)}_{\mathbf{3}} H_d\right)_{\mathbf{1}} + \beta\left(E^c_2 L Y^{(4)}_{\mathbf{3}a} H_d\right)_{\mathbf{1}} + \gamma_1\left(E^c_3 L Y^{(8)}_{\mathbf{3}a} H_d\right)_{\mathbf{1}} + \gamma_2\left(E^c_3 L Y^{(8)}_{\mathbf{3}b} H_d\right)_{\mathbf{1}}, \quad (4.6)$$

which leads to

$$M_e = \begin{pmatrix} \alpha Y^{(2)}_{\mathbf{3},1} & \alpha Y^{(2)}_{\mathbf{3},2} & \alpha Y^{(2)}_{\mathbf{3},3} \\ \beta Y^{(4)}_{\mathbf{3}a,1} & \beta Y^{(4)}_{\mathbf{3}a,2} & \beta Y^{(4)}_{\mathbf{3}a,3} \\ \gamma_1 Y^{(8)}_{\mathbf{3}a,1} + \gamma_2 Y^{(8)}_{\mathbf{3}b,1} & \gamma_1 Y^{(8)}_{\mathbf{3}a,2} + \gamma_2 Y^{(8)}_{\mathbf{3}b,2} & \gamma_1 Y^{(8)}_{\mathbf{3}a,3} + \gamma_2 Y^{(8)}_{\mathbf{3}b,3} \end{pmatrix} v_d\,. \quad (4.7)$$

We find that the triplet modular forms $Y^{(2)}_{\mathbf{3}}$, $Y^{(4)}_{\mathbf{3}}$, $Y^{(8)}_{\mathbf{3}a}$ and $Y^{(8)}_{\mathbf{3}b}$ satisfy the following identities

$$Y^{(8)}_{\mathbf{3}a}(\tau) = Y^{(4)}_{\mathbf{1}}(\tau) Y^{(4)}_{\mathbf{3}a}(\tau), \quad Y^{(8)}_{\mathbf{3}b}(\tau) = -\frac{\sqrt{2}}{3} Y^{(6)}_{\mathbf{1}}(\tau) Y^{(2)}_{\mathbf{3}}(\tau) + 2\sqrt{\frac{2}{3}}\, Y^{(4)}_{\mathbf{1}}(\tau) Y^{(4)}_{\mathbf{3}a}(\tau)\,. \quad (4.8)$$

Therefore the third row of the above mass matrix can be written as a linear combination of the first and the second rows, and the rank of this mass matrix is 2. As a consequence, the charged lepton mass matrix in eq. (4.7) would lead to massless electron. This is obviously not compatible with the present observation, therefore we shall not discuss this case in the following. The resulting charged lepton mass matrices for the rest possible models considered above are summarized in table 2.

4.2 Neutrino sector

In the present paper, we assume neutrinos are Majorana particles, and the neutrino masses are described by the effective Weinberg operator or the type I seesaw mechanism. From the multiplication rules $\mathbf{3}\otimes\mathbf{3} = \bar{\mathbf{3}}_{\mathbf{A}} \oplus \mathbf{6}_{\mathbf{S}}$ and $\bar{\mathbf{3}}\otimes\bar{\mathbf{3}} = \mathbf{3}_{\mathbf{A}} \oplus \mathbf{6}_{\mathbf{S}}$, we see that the sextet modular forms $Y_{\mathbf{6}}$ are necessary when neutrino masses originate from the Weinberg operator $\mathcal{W}_W$. We can uniquely determine the form of $\mathcal{W}_W$ as follow,

$$\mathcal{W}_W = \begin{cases} \frac{g_1}{2\Lambda}\left(LLY^{(4)}_{\mathbf{6}a}\right)_{\mathbf{1}} H_uH_u + \frac{g_2}{2\Lambda}\left(LLY^{(4)}_{\mathbf{6}b}\right)_{\mathbf{1}} H_uH_u, & k_L = 2\,, \\ \frac{g_1}{2\Lambda}\left(LLY^{(6)}_{\mathbf{6}a}\right)_{\mathbf{1}} H_uH_u + \frac{g_2}{2\Lambda}\left(LLY^{(6)}_{\mathbf{6}b}\right)_{\mathbf{1}} H_uH_u, & k_L = 3\,. \end{cases} \tag{4.9}$$

Applying the decomposition rules of Γ_7, we obtain the light neutrino mass matrix as

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,5}+g_2Y^{(2k_L)}_{\mathbf{6}b,5}) & g_1Y^{(2k_L)}_{\mathbf{6}a,4}+g_2Y^{(2k_L)}_{\mathbf{6}b,4} & g_1Y^{(2k_L)}_{\mathbf{6}a,2}+g_2Y^{(2k_L)}_{\mathbf{6}b,2} \\ g_1Y^{(2k_L)}_{\mathbf{6}a,4}+g_2Y^{(2k_L)}_{\mathbf{6}b,4} & \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,3}+g_2Y^{(2k_L)}_{\mathbf{6}b,3}) & g_1Y^{(2k_L)}_{\mathbf{6}a,1}+g_2Y^{(2k_L)}_{\mathbf{6}b,1} \\ g_1Y^{(2k_L)}_{\mathbf{6}a,2}+g_2Y^{(2k_L)}_{\mathbf{6}b,2} & g_1Y^{(2k_L)}_{\mathbf{6}a,1}+g_2Y^{(2k_L)}_{\mathbf{6}b,1} & \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,6}+g_2Y^{(2k_L)}_{\mathbf{6}b,6}) \end{pmatrix}, \text{ for } L\sim\mathbf{3}, \tag{4.10}$$

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,2}+g_2Y^{(2k_L)}_{\mathbf{6}b,2}) & g_1Y^{(2k_L)}_{\mathbf{6}a,3}+g_2Y^{(2k_L)}_{\mathbf{6}b,3} & g_1Y^{(2k_L)}_{\mathbf{6}a,5}+g_2Y^{(2k_L)}_{\mathbf{6}b,5} \\ g_1Y^{(2k_L)}_{\mathbf{6}a,3}+g_2Y^{(2k_L)}_{\mathbf{6}b,3} & \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,4}+g_2Y^{(2k_L)}_{\mathbf{6}b,4}) & g_1Y^{(2k_L)}_{\mathbf{6}a,6}+g_2Y^{(2k_L)}_{\mathbf{6}b,6} \\ g_1Y^{(2k_L)}_{\mathbf{6}a,5}+g_2Y^{(2k_L)}_{\mathbf{6}b,5} & g_1Y^{(2k_L)}_{\mathbf{6}a,6}+g_2Y^{(2k_L)}_{\mathbf{6}b,6} & \sqrt{2}(g_1Y^{(2k_L)}_{\mathbf{6}a,1}+g_2Y^{(2k_L)}_{\mathbf{6}b,1}) \end{pmatrix}, \text{ for } L\sim\bar{\mathbf{3}}, \tag{4.11}$$

where $Y^{(2k_L)}_{\mathbf{6}a,\mathbf{6}b}$ stands for $Y^{(4)}_{\mathbf{6}a,\mathbf{6}b}$ and $Y^{(6)}_{\mathbf{6}a,\mathbf{6}b}$ for $k_L = 2$ and $k_L = 3$, respectively.

If the light neutrino masses are generated by the type I seesaw mechanism, the superpotential for the neutrino masses can be generally written as

$$\mathcal{W}_\nu = g(N^cLY_DH_u)_{\mathbf{1}} + \frac{1}{2}\Lambda(N^cN^cY_N)_{\mathbf{1}}\,, \tag{4.12}$$

where Y_D and Y_N denote the modular form multiplets, and they are required to ensure modular invariance. The explicit forms of Y_D and Y_N are determined by the weight and representation assignments for L and N^c. The Majorana mass term for the heavy neutrinos N^c is similar to the Weinberg operator in eq. (4.9), and Y_N should be modular form multiplets transforming as $\mathbf{6}$ under Γ_7. The mass matrix for the Majorana neutrinos N^c reads as

$$m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,5}+g_2Y^{(2k_N)}_{\mathbf{6}b,5}) & g_1Y^{(2k_N)}_{\mathbf{6}a,4}+g_2Y^{(2k_N)}_{\mathbf{6}b,4} & g_1Y^{(2k_N)}_{\mathbf{6}a,2}+g_2Y^{(2k_N)}_{\mathbf{6}b,2} \\ g_1Y^{(2k_N)}_{\mathbf{6}a,4}+g_2Y^{(2k_N)}_{\mathbf{6}b,4} & \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,3}+g_2Y^{(2k_N)}_{\mathbf{6}b,3}) & g_1Y^{(2k_N)}_{\mathbf{6}a,1}+g_2Y^{(2k_N)}_{\mathbf{6}b,1} \\ g_1Y^{(2k_N)}_{\mathbf{6}a,2}+g_2Y^{(2k_N)}_{\mathbf{6}b,2} & g_1Y^{(2k_N)}_{\mathbf{6}a,1}+g_2Y^{(2k_N)}_{\mathbf{6}b,1} & \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,6}+g_2Y^{(2k_N)}_{\mathbf{6}b,6}) \end{pmatrix}, \text{ for } N^c\sim\mathbf{3}, \tag{4.13}$$

$$m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,2}+g_2Y^{(2k_N)}_{\mathbf{6}b,2}) & g_1Y^{(2k_N)}_{\mathbf{6}a,3}+g_2Y^{(2k_N)}_{\mathbf{6}b,3} & g_1Y^{(2k_N)}_{\mathbf{6}a,5}+g_2Y^{(2k_N)}_{\mathbf{6}b,5} \\ g_1Y^{(2k_N)}_{\mathbf{6}a,3}+g_2Y^{(2k_N)}_{\mathbf{6}b,3} & \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,4}+g_2Y^{(2k_N)}_{\mathbf{6}b,4}) & g_1Y^{(2k_N)}_{\mathbf{6}a,6}+g_2Y^{(2k_N)}_{\mathbf{6}b,6} \\ g_1Y^{(2k_N)}_{\mathbf{6}a,5}+g_2Y^{(2k_N)}_{\mathbf{6}b,5} & g_1Y^{(2k_N)}_{\mathbf{6}a,6}+g_2Y^{(2k_N)}_{\mathbf{6}b,6} & \sqrt{2}(g_1Y^{(2k_N)}_{\mathbf{6}a,1}+g_2Y^{(2k_N)}_{\mathbf{6}b,1}) \end{pmatrix}, \text{ for } N^c\sim\bar{\mathbf{3}}, \tag{4.14}$$

where $Y^{(2k_N)}_{\mathbf{6}a,\mathbf{6}b} = Y^{(4)}_{\mathbf{6}a,\mathbf{6}b}$ for $k_N = 2$ and $Y^{(2k_N)}_{\mathbf{6}a,\mathbf{6}b} = Y^{(6)}_{\mathbf{6}a,\mathbf{6}b}$ for $k_N = 3$. Now we analyze the neutrino Yukawa couplings.

- $L \sim \mathbf{3}$, $N^c \sim \mathbf{3}$

  In this case, modular invariance requires Y_D should be transform in $\mathbf{3}$ or $\mathbf{6}$. In the case of $k_L + k_N = 2$, and the Dirac neutrino mass matrix take the following form

$$m_D = g \begin{pmatrix} 0 & Y^{(2)}_{\mathbf{3},3} & -Y^{(2)}_{\mathbf{3},2} \\ -Y^{(2)}_{\mathbf{3},3} & 0 & Y^{(2)}_{\mathbf{3},1} \\ Y^{(2)}_{\mathbf{3},2} & -Y^{(2)}_{\mathbf{3},1} & 0 \end{pmatrix} v_u \,. \tag{4.15}$$

  Since this m_D is a 3×3 anti-symmetric matrix with zero determinant, the light neutrino mass matrix given by seesaw formula is at most of rank 2, such that at least one light neutrino is massless. If $k_L + k_N = 4$, the neutrino Yukawa couplings would involve three independent terms, and we have

$$m_D = \begin{pmatrix} \sqrt{2}(g_2 Y^{(4)}_{\mathbf{6}a,5} + g_3 Y^{(4)}_{\mathbf{6}b,5}) & g_1 Y^{(4)}_{\mathbf{3},3} + g_2 Y^{(4)}_{\mathbf{6}a,4} + g_3 Y^{(4)}_{\mathbf{6}b,4} & -g_1 Y^{(4)}_{\mathbf{3},2} + g_2 Y^{(4)}_{\mathbf{6}a,2} + g_3 Y^{(4)}_{\mathbf{6}b,2} \\ -g_1 Y^{(4)}_{\mathbf{3},3} + g_2 Y^{(4)}_{\mathbf{6}a,4} + g_3 Y^{(4)}_{\mathbf{6}b,4} & \sqrt{2}(g_2 Y^{(4)}_{\mathbf{6}a,3} + g_3 Y^{(4)}_{\mathbf{6}b,3}) & g_1 Y^{(4)}_{\mathbf{3},1} + g_2 Y^{(4)}_{\mathbf{6}a,1} + g_3 Y^{(4)}_{\mathbf{6}b,1} \\ g_1 Y^{(4)}_{\mathbf{3},2} + g_2 Y^{(4)}_{\mathbf{6}a,2} + g_3 Y^{(4)}_{\mathbf{6}b,2} & -g_1 Y^{(4)}_{\mathbf{3},1} + g_2 Y^{(4)}_{\mathbf{6}a,1} + g_3 Y^{(4)}_{\mathbf{6}b,1} & \sqrt{2}(g_2 Y^{(4)}_{\mathbf{6}a,6} + g_3 Y^{(4)}_{\mathbf{6}b,6}) \end{pmatrix} v_u \,. \tag{4.16}$$

  The Dirac neutrino mass matrices for $k_L + k_N \geq 4$ contain more free parameters than $k_L + k_N = 2$. We shall not consider these cases in the present work.

- $L \sim \bar{\mathbf{3}}$, $N^c \sim \bar{\mathbf{3}}$

  In this case, the Dirac neutrino mass matrix for the modular weights $k_L + k_N = 4$ contains a minimum number of input parameters. Here the modular form Y_D can be $Y^{(4)}_{\mathbf{6}a}$ and $Y^{(4)}_{\mathbf{6}b}$. We can read out the Dirac neutrino mass matrix as follow

$$m_D = \begin{pmatrix} \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,2} + h_2 Y^{(4)}_{\mathbf{6}b,2}) & h_1 Y^{(4)}_{\mathbf{6}a,3} + h_2 Y^{(4)}_{\mathbf{6}b,3} & h_1 Y^{(4)}_{\mathbf{6}a,5} + h_2 Y^{(4)}_{\mathbf{6}b,5} \\ h_1 Y^{(4)}_{\mathbf{6}a,3} + h_2 Y^{(4)}_{\mathbf{6}b,3} & \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,4} + h_2 Y^{(4)}_{\mathbf{6}b,4}) & h_1 Y^{(4)}_{\mathbf{6}a,6} + h_2 Y^{(4)}_{\mathbf{6}b,6} \\ h_1 Y^{(4)}_{\mathbf{6}a,5} + h_2 Y^{(4)}_{\mathbf{6}b,5} & h_1 Y^{(4)}_{\mathbf{6}a,6} + h_2 Y^{(4)}_{\mathbf{6}b,6} & \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,1} + h_2 Y^{(4)}_{\mathbf{6}b,1}) \end{pmatrix} v_u \,. \tag{4.17}$$

  Similar in the case of $k_L + k_N = 6$, we have

$$m_D = \begin{pmatrix} \sqrt{2}(g_2 Y^{(6)}_{\mathbf{6}a,2} + g_3 Y^{(6)}_{\mathbf{6}b,2}) & g_1 Y^{(6)}_{\bar{\mathbf{3}},3} + g_2 Y^{(6)}_{\mathbf{6}a,3} + g_3 Y^{(6)}_{\mathbf{6}b,3} & -g_1 Y^{(6)}_{\bar{\mathbf{3}},2} + g_2 Y^{(6)}_{\mathbf{6}a,5} + g_3 Y^{(6)}_{\mathbf{6}b,5} \\ -g_1 Y^{(6)}_{\bar{\mathbf{3}},3} + g_2 Y^{(6)}_{\mathbf{6}a,3} + g_3 Y^{(6)}_{\mathbf{6}b,3} & \sqrt{2}(g_2 Y^{(6)}_{\mathbf{6}a,4} + g_3 Y^{(6)}_{\mathbf{6}b,4}) & g_1 Y^{(6)}_{\bar{\mathbf{3}},1} + g_2 Y^{(6)}_{\mathbf{6}a,6} + g_3 Y^{(6)}_{\mathbf{6}b,6} \\ g_1 Y^{(6)}_{\bar{\mathbf{3}},2} + g_2 Y^{(6)}_{\mathbf{6}a,5} + g_3 Y^{(6)}_{\mathbf{6}b,5} & -g_1 Y^{(6)}_{\bar{\mathbf{3}},1} + g_2 Y^{(6)}_{\mathbf{6}a,6} + g_3 Y^{(6)}_{\mathbf{6}b,6} & \sqrt{2}(g_2 Y^{(6)}_{\mathbf{6}a,1} + g_3 Y^{(6)}_{\mathbf{6}b,1}) \end{pmatrix} v_u \,. \tag{4.18}$$

  It contains three complex input parameters.

- $L \sim \bar{\mathbf{3}}$, $N^c \sim \mathbf{3}$

  For this type of assignment, L and N^c can form an invariant singlet for $k_L + k_N = 0$, and consequently m_D has a simple structure,

$$m_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_u \,. \tag{4.19}$$

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The lowest non-vanishing weight $k_L+k_N=2$ gives rise to the following neutrino Yukawa coupling

$$h_1(N^cLY^{(2)}_{\mathbf{8}a}H_u)_{\mathbf{1}}+h_2(N^cLY^{(2)}_{\mathbf{8}b}H_u)_{\mathbf{1}}\,. \tag{4.20}$$

The Dirac neutrino mass matrix is given by

$$m_D=\begin{pmatrix} \begin{array}{c}\sqrt{3}(g_1Y^{(2)}_{\mathbf{8}a,1}+g_2Y^{(2)}_{\mathbf{8}b,1})\\ +g_1Y^{(2)}_{\mathbf{8}a,2}+g_2Y^{(2)}_{\mathbf{8}b,2}\end{array} & \sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,3}+g_2Y^{(2)}_{\mathbf{8}b,3}) & \sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,5}+g_2Y^{(2)}_{\mathbf{8}b,5}) \\ -\sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,8}+g_2Y^{(2)}_{\mathbf{8}b,8}) & \begin{array}{c}-\sqrt{3}(g_1Y^{(2)}_{\mathbf{8}a,1}+g_2Y^{(2)}_{\mathbf{8}b,1})\\ +g_1Y^{(2)}_{\mathbf{8}a,2}+g_2Y^{(2)}_{\mathbf{8}b,2}\end{array} & \sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,4}+g_2Y^{(2)}_{\mathbf{8}b,4}) \\ -\sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,6}+g_2Y^{(2)}_{\mathbf{8}b,6}) & -\sqrt{6}(g_1Y^{(2)}_{\mathbf{8}a,7}+g_2Y^{(2)}_{\mathbf{8}b,7}) & -2(g_1Y^{(2)}_{\mathbf{8}a,2}+g_2Y^{(2)}_{\mathbf{8}b,2}) \end{pmatrix}v_u\,, \tag{4.21}$$

which contains one more complex parameter than the above case of $k_L+k_N=0$. A number of free coupling constants are introduced for higher modular weights, the expression of m_D would be too lengthy to display. For the assignment $L\sim\mathbf{3}$ and $N^c\sim\bar{\mathbf{3}}$, the matrix m_D can be obtained from eq. (4.19) by performing transposition. The different possible forms of the neutrino mass matrices for Weinberg operator and type I seesaw mechanism are summarized in table 3. We see that the light neutrino mass matrix of S_2 contains more free parameters than other possible cases. Hence we will not perform numerical analysis for the S_2 model in the following.

4.3 Benchmark models

In the following, we consider two scenarios: the modular symmetry only acts on the neutrino sector and the charged lepton matrix is diagonal in the first scenario and the modular symmetry acts on both charged lepton and neutrino sector in the second scenario.

4.3.1 Modular symmetry on neutrino sector

The possibility that the dynamics of flavour in the charged lepton and neutrino sectors are different can not be excluded [18]. For instance, certain flavon may be involved in the charged lepton sector and neutrino sector is dictated by modular symmetry [18]. For simplicity, we assume that charged lepton sector is diagonal. In this case, there are six different models which are shown in table 3. We find that the model S_1 can not accommodate the experimental data, the model S_2 contains more input parameters than other models, consequently we will not discuss it. Then the rest four models in table 3 only depend on the following four inputs

$$\Re\tau,\quad \Im\tau,\quad |g_2/g_1|,\quad \arg{(g_2/g_1)}\,, \tag{4.22}$$

and an overall parameter which can be fixed by the mass squared difference $\Delta m^2_{21}=m^2_2-m^2_1$. The five dimensionless observable quantities:

$$\sin^2\theta_{12},\quad \sin^2\theta_{13},\quad \sin^2\theta_{23},\quad \delta_{CP},\quad \Delta m^2_{21}/\Delta m^2_{3\ell}\,, \tag{4.23}$$

	$\rho_L,\, \rho_{N^c}$	$k_L,\, k_N$	Neutrino mass matrices
W_1	$\mathbf{3}$, —	2(3), —	$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,5} + g_2 Y^{(2k_L)}_{\mathbf{6}b,5}) & g_1 Y^{(2k_L)}_{\mathbf{6}a,4} + g_2 Y^{(2k_L)}_{\mathbf{6}b,4} & g_1 Y^{(2k_L)}_{\mathbf{6}a,2} + g_2 Y^{(2k_L)}_{\mathbf{6}b,2} \\ g_1 Y^{(2k_L)}_{\mathbf{6}a,4} + g_2 Y^{(2k_L)}_{\mathbf{6}b,4} & \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,3} + g_2 Y^{(2k_L)}_{\mathbf{6}b,3}) & g_1 Y^{(2k_L)}_{\mathbf{6}a,1} + g_2 Y^{(2k_L)}_{\mathbf{6}b,1} \\ g_1 Y^{(2k_L)}_{\mathbf{6}a,2} + g_2 Y^{(2k_L)}_{\mathbf{6}b,2} & g_1 Y^{(2k_L)}_{\mathbf{6}a,1} + g_2 Y^{(2k_L)}_{\mathbf{6}b,1} & \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,6} + g_2 Y^{(2k_L)}_{\mathbf{6}b,6}) \end{pmatrix}$
W_2	$\bar{\mathbf{3}}$, —	2(3), —	$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,2} + g_2 Y^{(2k_L)}_{\mathbf{6}b,2}) & g_1 Y^{(2k_L)}_{\mathbf{6}a,3} + g_2 Y^{(2k_L)}_{\mathbf{6}b,3} & g_1 Y^{(2k_L)}_{\mathbf{6}a,5} + g_2 Y^{(2k_L)}_{\mathbf{6}b,5} \\ g_1 Y^{(2k_L)}_{\mathbf{6}a,3} + g_2 Y^{(2k_L)}_{\mathbf{6}b,3} & \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,4} + g_2 Y^{(2k_L)}_{\mathbf{6}b,4}) & g_1 Y^{(2k_L)}_{\mathbf{6}a,6} + g_2 Y^{(2k_L)}_{\mathbf{6}b,6} \\ g_1 Y^{(2k_L)}_{\mathbf{6}a,5} + g_2 Y^{(2k_L)}_{\mathbf{6}b,5} & g_1 Y^{(2k_L)}_{\mathbf{6}a,6} + g_2 Y^{(2k_L)}_{\mathbf{6}b,6} & \sqrt{2}(g_1 Y^{(2k_L)}_{\mathbf{6}a,1} + g_2 Y^{(2k_L)}_{\mathbf{6}b,1}) \end{pmatrix}$
S_1	$\mathbf{3}$, $\mathbf{3}$	$0(-1)$, 2(3)	$m_D = g\begin{pmatrix} 0 & Y^{(2)}_{\mathbf{3},3} & -Y^{(2)}_{\mathbf{3},2} \\ -Y^{(2)}_{\mathbf{3},3} & 0 & Y^{(2)}_{\mathbf{3},1} \\ Y^{(2)}_{\mathbf{3},2} & -Y^{(2)}_{\mathbf{3},1} & 0 \end{pmatrix} v_u,$ $m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4} & g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2} & g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6}) \end{pmatrix}$
S_2	$\bar{\mathbf{3}}$, $\bar{\mathbf{3}}$	2(1), 2(3)	$m_D = \begin{pmatrix} \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,2} + h_2 Y^{(4)}_{\mathbf{6}b,2}) & h_1 Y^{(4)}_{\mathbf{6}a,3} + h_2 Y^{(4)}_{\mathbf{6}b,3} & h_1 Y^{(4)}_{\mathbf{6}a,5} + h_2 Y^{(4)}_{\mathbf{6}b,5} \\ h_1 Y^{(4)}_{\mathbf{6}a,3} + h_2 Y^{(4)}_{\mathbf{6}b,3} & \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,4} + h_2 Y^{(4)}_{\mathbf{6}b,4}) & h_1 Y^{(4)}_{\mathbf{6}a,6} + h_2 Y^{(4)}_{\mathbf{6}b,6} \\ h_1 Y^{(4)}_{\mathbf{6}a,5} + h_2 Y^{(4)}_{\mathbf{6}b,5} & h_1 Y^{(4)}_{\mathbf{6}a,6} + h_2 Y^{(4)}_{\mathbf{6}b,6} & \sqrt{2}(h_1 Y^{(4)}_{\mathbf{6}a,1} + h_2 Y^{(4)}_{\mathbf{6}b,1}) \end{pmatrix} v_u,$ $m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3} & g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5} & g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1}) \end{pmatrix}$
S_3	$\bar{\mathbf{3}}$, $\mathbf{3}$	$-2(-3)$, 2(3)	$m_D = g\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_u,$ $m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4} & g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2} & g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6}) \end{pmatrix}$
S_4	$\mathbf{3}$, $\bar{\mathbf{3}}$	$-2(-3)$, 2(3)	$m_D = g\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_u,$ $m_N = \Lambda\begin{pmatrix} \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,2} + g_2 Y^{(2k_N)}_{\mathbf{6}b,2}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3} & g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,3} + g_2 Y^{(2k_N)}_{\mathbf{6}b,3} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,4} + g_2 Y^{(2k_N)}_{\mathbf{6}b,4}) & g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6} \\ g_1 Y^{(2k_N)}_{\mathbf{6}a,5} + g_2 Y^{(2k_N)}_{\mathbf{6}b,5} & g_1 Y^{(2k_N)}_{\mathbf{6}a,6} + g_2 Y^{(2k_N)}_{\mathbf{6}b,6} & \sqrt{2}(g_1 Y^{(2k_N)}_{\mathbf{6}a,1} + g_2 Y^{(2k_N)}_{\mathbf{6}b,1}) \end{pmatrix}$

Table 3. The predictions for the neutrino mass matrices, where only lower modular weight assignments are displayed. The models in which neutrino masses are generated through Weinberg operator and seesaw mechanism are denoted as $W_{1,2}$ and $S_{1,2,3,4}$ respectively.

only depend on the four input parameters in eq. (4.22), where $\Delta m^2_{3\ell} = m^2_3 - m^2_1 > 0$ for normal ordering (NO) and $\Delta m^2_{3\ell} = m^2_3 - m^2_2 < 0$ for inverted ordering (IO) [50]. Moreover, our models can also predict the unknown values of absolute neutrino masses, the Majorana CP violation phase and the effective masses in beta decay and neutrinoless double beta decay, as shown in the following. In order to quantitatively assess how well a model can describe the experimental data on the five dimensionless observable quantities in eq. (4.23). We define a χ^2 function to estimate the goodness-of-fit of a set of chosen values of the input parameters,

$$\chi^2 = \sum_{i=1}^{5} \left(\frac{P_i - O_i}{\sigma_i} \right)^2 , \tag{4.24}$$

where O_i denote the global best fit values of the five observable quantities in eq. (4.23), and σ_i refer to the 1σ deviations of the corresponding quantities, and P_i are the theoretical predictions for the five physical observable quantities for the input parameters taking certain values. Here the contribution of the Dirac phase δ_{CP} is also included in the χ^2 function. For each value of the input parameters, one can obtain the predicted values P_i and the corresponding χ^2, then one can find out the lowest χ^2. After performing a detailed numerical analysis for the three mixing angles, Dirac CP phase and $\Delta m^2_{21}/\Delta m^2_{3\ell}$, we find that only models W_1 with $k_L = 3$ and S_4 with $k_N = 3$ for NO case and models W_2 with $k_L = 3$ and S_3 with $k_N = 3$ for IO case can give results in agreement with the experimental data. As an example, we only show the results of NO case. For model W_1 with $k_L = 3$, we find the minimum of χ^2 is $\chi^2_{\text{min}} = 1.937$, and the best fit values of the free parameters are

$$\Re\tau = 0.448, \quad \Im\tau = 0.915, \quad |g_2/g_1| = 0.129, \quad \arg(g_2/g_1) = 0.566\pi, \quad g_1^2 v_u^2/\Lambda = 56.981\,\text{meV}\,. \tag{4.25}$$

The predictions for various observable quantities obtained at the best fit point are

$$\begin{aligned}
&\sin^2\theta_{13} = 0.02236, \quad \sin^2\theta_{12} = 0.311, \quad \sin^2\theta_{23} = 0.556, \quad \delta_{CP} = -0.984\pi\,,\\
&\alpha_{21} = -0.594\pi, \quad \alpha_{31} = 0.0814\pi, \quad m_1 = 63.167\,\text{meV}, \quad m_2 = 63.749\,\text{meV}\,,\\
&m_3 = 80.733\,\text{meV}, \quad m_\beta = 63.789\,\text{meV}\,, \quad m_{\beta\beta} = 42.787\,\text{meV}\,,
\end{aligned} \tag{4.26}$$

where m_β is the effective mass probed by direct kinematic search in beta decay and $m_{\beta\beta}$ refers to the effective Majorana mass in neutrinoless double beta decay. The latest result from KATRIN is $m_\beta < 1.1\,\text{eV}$ at 90% CL [51]. The combined results from KamLAND-Zen and EXO-200 give a Majorana neutrino mass limit of $m_{\beta\beta} < (120 - 250)$ meV [52]. Our above predictions for both m_β and $m_{\beta\beta}$ are compatible with these latest experimental bounds. The experimentally measured values of lepton mixing angles, Dirac CP phase and neutrino masses can also be accommodated well in model S_4 with $k_N = 3$. The values of input parameters and predictions for mixing parameters and neutrino masses at the best fit point are given by

$$\begin{aligned}
&\Re\tau = -0.491, \quad \Im\tau = 1.125, \quad |g_2/g_1| = 0.103, \quad \arg(g_2/g_1) = 0.0267\pi, \quad g_1^2 v_u^2/\Lambda = 101.093\,\text{meV}\,,\\
&\sin^2\theta_{13} = 0.02237, \quad \sin^2\theta_{12} = 0.310, \quad \sin^2\theta_{23} = 0.563, \quad \delta_{CP} = -0.814\pi\,,\\
&\alpha_{21} = 0.466\pi, \quad \alpha_{31} = 0.352\pi, \quad m_1 = 122.204\,\text{meV}, \quad m_2 = 122.506\,\text{meV}\,,\\
&m_3 = 132.143\,\text{meV}, \quad m_\beta = 122.527\,\text{meV}\,, \quad m_{\beta\beta} = 96.646\,\text{meV}\,.
\end{aligned} \tag{4.27}$$

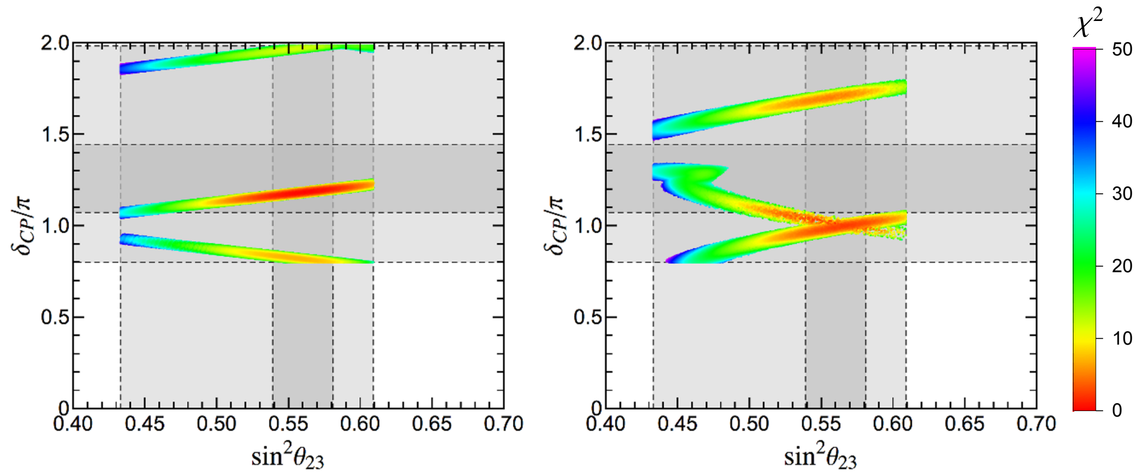

Figure 1. The correlation between δ_{CP} and $\sin^2\theta_{23}$, the left and right panels are for the models S_4 with $k_N = 3$ and W_1 with $k_L = 3$ respectively in the charged lepton diagonal basis. The gray bands denote the experimentally preferred 1σ and 3σ ranges of δ_{CP} and $\sin^2\theta_{23}$ adapted from [50].

Accordingly the global minimum of the χ^2 function is $\chi^2_{\text{min}} = 0.0732$. The predicted values of m_β and $m_{\beta\beta}$ are compatible with the latest results of KATRIN [51] and KamLAND-Zen and EXO-200 [52], and they would potentially be tested in next generation experiments. As regards the experimental bound on neutrino mass sum, the result sensitively depends on the cosmological model and the experimental data considered. Combining the Planck TT, TE, EE, lowE polarization spectra, baryon acoustic oscillation (BAO) data with the CMB lensing reconstruction power spectrum, the Planck collaboration gives $\sum_i m_i < 120$ meV at 95% confidence level [53]. However, if only the BAO data and the CMB lensing reconstruction power spectrum are taken into account in the data analysis, this bound becomes $\sum_i m_i < 600$ meV [53]. For the above two models, we find the neutrino mass sum $\sum_i m_i$ is 207.649 meV and 376.853 meV respectively which are consistent with the Planck's looser constraint $\sum_i m_i < 600$ meV.

We perform a comprehensive numerical scan over the free parameters of the above two models. We find that the three mixing angles can take any values in their 3σ ranges. The two Majorana CP phases are restricted to the ranges $\alpha_{21}/\pi \in [0.562, 0.638] \cup [1.353, 1.446]$ and $\alpha_{31}/\pi \in [0, 0.159] \cup [1.353, 1.118] \cup [1.495, 2)$ in the model W_1 with $k_L = 3$, and they are $\alpha_{21}/\pi \in [0.442, 0.497] \cup [1.503, 1.557]$ and $\alpha_{31}/\pi \in [0.278, 0.389] \cup [1.611, 1.709)$ for the model S_4 with $k_N = 3$. The Dirac CP phase δ_{CP} and θ_{23} are strongly correlated in the two models, as shown in figure 1. The allowed values of the effective Majorana mass $m_{\beta\beta}$ for the model W_1 with $k_L = 3$ are displayed in figure 2. We see that there is portion of parameter space where all the bounds from neutrino oscillation experiments and neutrino mass bound $\sum_i m_i < 120$ meV from Planck are fulfilled. For the model S_4 with $k_N = 3$, the neutrino masses and $m_{\beta\beta}$ lie in quite small regions around the best fit values in eq. (4.27), consequently the corresponding figure is not shown here.

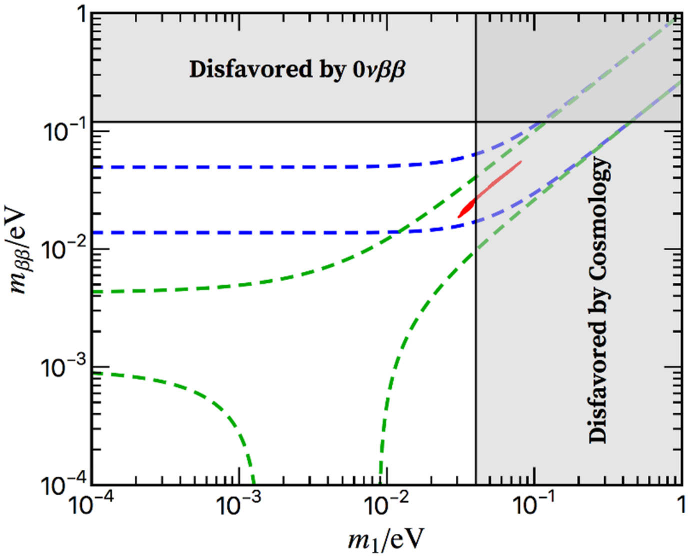

Figure 2. The allowed values of the effective Majorana neutrino mass $m_{\beta\beta}$ with respect to the lightest neutrino mass m_1 in the model W_1 with $k_L = 3$, where we consider the case of NO neutrino masses. The blue (green) dashed lines represent the most general allowed regions for IO (NO) where the neutrino oscillation parameters are varied within their 3σ ranges [50]. The horizontal grey band denote the current experimental bound $m_{\beta\beta} < (120-250)$ meV from KamLAND-Zen [52]. The vertical grey exclusion band denotes the bound on the lightest neutrino mass extracted from $\sum_i m_i < 120$ meV by the Planck collaboration [53].

Models	mass matrices	$\rho_L,\ \rho_{N^c}$	modular weights	
			$k_{1,2,3}{+}k_L$	$k_L,\ k_N$
$\mathcal{M}_1$	C_1, W_1	$\mathbf{3}$, —	$6, 8, 10$	2 (3), —
$\mathcal{M}_2$	C_2, W_2	$\bar{\mathbf{3}}$, —	$2, 4, 6$	2 (3), —
$\mathcal{M}_3$	C_2, S_3	$\bar{\mathbf{3}}$, $\mathbf{3}$	$2, 4, 6$	$-2(-3)$, $2(3)$
$\mathcal{M}_4$	C_1, S_4	$\mathbf{3}$, $\bar{\mathbf{3}}$	$6, 8, 10$	$-2(-3)$, $2(3)$

Table 4. Summary of the "minimal" neutrino mass models with the Γ_7 modular symmetry.

4.3.2 Modular symmetry on both neutrino and charged lepton sector sectors

Combining the different possible constructions in the charged lepton and neutrino sectors, we can easily get all possible models based on Γ_7 modular symmetry. Focusing on the cases with lower weight modular forms and free parameters as few as possible, we find four different types of models named as $\mathcal{M}_{1,2,3,4}$. The assignments of the weights and representations for the leptonic fields are listed in table 4. From the superpotential of C_1 and C_2, we see that the phases of the parameters α, β and γ_1 can be absorbed into the right-handed lepton fields. Then the coupling constants α, β and γ_1 in the charged lepton mass matrix can be taken to be real and positive and γ_2 is a complex parameters for both C_1 and C_2. As a consequence, the charged lepton sector only contain four real dimensionless parameters β/α, γ_1/α, $|\gamma_2/\alpha|$, $\arg(\gamma_2/\alpha)$ and an overall scale αv_d. In the

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neutrino sector, it is easy to check that light neutrino mass matrices for all the four cases W_1, W_2, S_1 and S_2 depend on two positive dimensionless parameters $|g_2/g_1|$, $\arg(g_2/g_1)$ and an overall neutrino mass $g_1^2 v_u^2/\Lambda$ or $g^2 v_u^2/(g_1\Lambda)$. We perform a numerical analysis for each model, the complex modulus τ is restricted to lie in the fundamental domain $\{\tau|\Im\tau > 0, |\Re\tau| \leq \frac{1}{2}, |\tau| \geq 1\}$ of the modular group. The other dimensionless parameters randomly vary in the following regions

$$\arg(\gamma_2/\alpha),\, \arg(g_2/g_1) \in [0, 2\pi)\,, \qquad \beta/\alpha,\, \gamma/\alpha,\, \gamma_1/\alpha,\, |\gamma_2/\alpha|,\, |g_2/g_1| \in [0, 10^4]\,. \tag{4.28}$$

The overall parameters of charged lepton and neutrino mass matrices are fixed by the electron mass and the solar neutrino mass squared difference Δm^2_{21}. Then one can obtained all the predictions for six lepton masses and three lepton mixing angles as well as three CP violating phases. We find that good agreement with experimental data can be achieved for certain values of input parameters for all these four models in both NO and IO cases. Since NO is slightly preferred by present data, we only present the numerical results of the four models for NO. The predictions for lepton mixing parameters and neutrino masses are listed in table 5. The charged lepton mass matrix depends on four parameters α, β, γ_1 and γ_2, the measured values of the three charged lepton masses can be reproduced exactly, hence we do not show the results of charged lepton masses in table 5. We find that the global best fit values of neutrino mixing angles and mass squared differences Δm^2_{21}, Δm^2_{31} from NuFIT v4.1 can be obtained, as shown in table 5. Moreover, the most stringent neutrino mass bound $\sum_i m_i < 120$ meV [53] is saturated in models $\mathcal{M}_2$ and $\mathcal{M}_3$, and the less stringent bound $\sum_i m_i < 600$ meV [53] is fulfilled in models $\mathcal{M}_1$ and $\mathcal{M}_4$. The effective Majorana mass $m_{\beta\beta}$ is predicted to be around 10 meV or few tens of meV which is within the reach of forthcoming $0\nu\beta\beta$ decay experiments such as nEXO [54].

5 Conclusion

We have considered the finite modular group $\Gamma_7 \simeq \mathrm{PSL}(2, Z_7)$ in the framework of the modular invariance approach to lepton flavour. Γ_7 is a quotient group of the infinite modular group Γ which is achieved by imposing the generator condition $T^7 = \mathbb{1}$. An additional condition $(ST^3)^4 = \mathbb{1}$ should also be satisfied, which is essential to make Γ_7 finite.

One crucial ingredient of modular-invariant theories is the introduction of modular forms, which involves the modulus field τ. Given level 7 and an even weight k, there are $14k - 2$ linearly independent modular forms, which can all be decomposed into irreducible representations (irreps) of Γ_7. At weight $k = 2$, we constructed all 26 modular forms with the help of the SageMath algebra system [47]. They are decomposed into a triplet $\mathbf{3}$, a septet $\mathbf{7}$ and two octets $\mathbf{8}$ of Γ_7. A full list of linearly independent modular forms of weight up to 8 is provided in appendix D. We have also considered two alternative ways to derive modular forms of level 7, the Dedekind eta function method proposed in [1] and the theta function method proposed in [20], discussed in appendices B and C, respectively. Results from these methods are consistent with the former one, but incomplete: the Dedekind eta function method gives only 7 modular forms of weight 2, forming the septet of Γ_7; and the theta function method gives 23 modular forms of weight 2, decomposed as $\mathbf{7} + \mathbf{8} + \mathbf{8}$.

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	Best fit values for NO							
	$\mathcal{M}_1$		$\mathcal{M}_2$		$\mathcal{M}_3$		$\mathcal{M}_4$	
	$k_L=2$	$k_L=3$	$k_L=2$	$k_L=3$	$k_N=2$	$k_N=3$	$k_N=2$	$k_N=3$
$\Re\langle\tau\rangle$	-0.219	-0.299	0.248	0.235	-0.206	-0.0712	0.446	-0.437
$\Im\langle\tau\rangle$	0.962	1.435	1.315	1.881	1.343	1.962	0.836	1.285
β/α	0.00579	337.346	75.905	29.100	49.078	51.055	197.311	392.758
γ_1/α	5.641	1760.614	1702.628	692.861	1706.880	326.593	3858.297	2119.228
$\|\gamma_2/\alpha\|$	0.632	328.957	898.143	363.564	477.413	817.142	84.991	389.175
$\arg(\gamma_2/\alpha)/\pi$	0.213	1.022	0.0762	0.530	1.611	0.251	0.945	0.819
$\|g_2/g_1\|$	0.394	0.0344	0.274	0.00880	0.353	0.0410	1.780	0.0781
$\arg(g_2/g_1)/\pi$	0.538	1.404	1.050	1.474	0.290	0.304	0.326	0.584
$(g_1^2 v_u^2/\Lambda)/\mathrm{eV}$	0.104	0.225	0.221	0.392	0.00384	0.00125	0.212	0.0143
$\sin^2\theta_{13}$	0.0224	0.0224	0.0224	0.0224	0.0224	0.0224	0.0224	0.0224
$\sin^2\theta_{12}$	0.310	0.310	0.310	0.310	0.310	0.310	0.310	0.310
$\sin^2\theta_{23}$	0.563	0.563	0.563	0.563	0.563	0.563	0.563	0.563
δ_{CP}/π	-0.772	-0.772	-0.771	-0.773	-0.773	-0.772	-0.772	-0.770
α_{21}/π	0.0188	-0.937	-0.225	0.304	0.0732	-0.591	-0.490	0.415
α_{31}/π	0.381	-0.279	0.360	0.0342	0.0629	-0.375	0.615	-0.321
m_1/meV	39.606	40.078	15.843	10.324	10.841	9.072	77.446	31.159
m_2/meV	40.528	40.989	18.025	13.434	13.836	12.498	77.922	32.323
m_3/meV	64.005	64.298	52.717	51.331	51.434	51.091	92.336	59.151
m_β/meV	40.591	41.052	18.167	13.624	14.020	12.701	77.955	32.402
$m_{\beta\beta}/\mathrm{meV}$	40.373	14.317	16.434	9.852	11.782	5.856	59.281	24.172

Table 5. The predictions for lepton mixing parameters and neutrino masses for the four models given in table 4, where we assume neutrino mass spectrum is NO, and similar results can be obtained for IO.

We also considered flavon-free lepton flavour models, constructed by assigning couplings and matter superfields transforming in irreps of Γ_7. In the considered models, charged leptons gain masses via renormalisable Yukawa couplings, neutrinos gain masses via either the Weinberg operator or the type-I seesaw mechanism. Flavour textures arise after the modulus field τ gains a vacuum expectation value. Lists of charged lepton (C_1, C_2) and neutrino mass matrices (W_1, W_2 and S_1, …, S_4) involving τ are given in tables 2 and 3, respectively.

In the numerical studies, we have considered two scenarios: the modular symmetry acting on *only* the neutrino sector, or *both* charged lepton and neutrino sectors. We have performed a χ^2 analysis with experimental data of neutrino oscillation parameters in the 1σ range taken into account. In the *first* scenario, we found that, in the normal neutrino mass ordering, only models W_1 with lepton doublet weight $k_L = 3$ and S_4 with right-handed neutrino weight $k_N = 3$, give results that agree with the experimental data. Similarly, in the inverted mass ordering, only models W_2 with $k_L = 3$ and S_3 with $k_N = 3$ are allowed by data. The effective neutrino masses m_β and $m_{\beta\beta}$, which control the beta decay and neutrinoless double beta decay rates, respectively, are predicted to be compatible with current data. However, the prediction of the sum of neutrino masses is disfavoured by the current cosmological constraint $\sum_i m_i < 120$ meV [53] although the less stringent bound $\sum_i m_i < 600$ meV [53] is satisfied. In the *second* scenario, eight benchmark models ($\mathcal{M}_1$, ..., $\mathcal{M}_4$ with two sets of modular weights), listed in table 4, have been studied. While all of them are compatible with oscillation data, the models $\mathcal{M}_2$ and $\mathcal{M}_3$ are also favoured by the cosmological constraint on the sum of neutrino masses $\sum_i m_i < 120$ meV, while the prediction of $m_{\beta\beta}$ is within the sensitivity of the next generation of neutrinoless double beta decay experiments.

In conclusion, although the level 7 models that we have considered are no more or less predictive than level 3, 4, 5 models, they do allow new possibilities due to the complex $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations which are different from those of other models leading to new predictions. Although it is not possible to fix the underlying parameters in models of any level, without further theoretical input, modular symmetry at level 7 offers the promise of new types of SO(10) unification closer in spirit to the SU(3) models due to the complex triplet and sextet representations.

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A Group theory of $\Gamma_7 \cong \mathrm{PSL}(2, Z_7)$

The group $\Gamma_7 \cong \mathrm{PSL}(2, Z_7)$ is a non-Abelian finite subgroup of SU(3) of order 168. Γ_7 group can be generated by two generators S and T which satisfy the multiplication rules:

$$S^2 = (ST)^3 = T^7 = (ST^3)^4 = 1. \tag{A.1}$$

The 168 elements of Γ_7 group are divided into 6 conjugacy classes:

$$\begin{aligned}
1C_1 :& \{1\}\,, \\
21C_2 :& \{S, T^3ST^5ST^3, T^2ST^4ST^2, T^6ST, ST^3ST^4S, ST^5ST^2S, ST^4ST^3S, TST^6, \\
& ST^2ST^5S, T^5ST^2, TST^5ST^5, (ST^4)^2, T^2ST^5, (ST^3)^2, T^5ST^5ST, T^4ST^3, \\
& ST^5ST^6, T^2ST^3ST, ST^2ST, T^3ST^4, TST^3ST^2\}\,,
\end{aligned} \tag{A.2}$$

	Conjugacy Classes					
	$1C_1$	$21C_2$	$56C_3$	$42C_4$	$24C_7$	$24C_7'$
1	1	1	1	1	1	1
3	3	-1	0	1	b_7	$\bar{b}_7$
$\bar{\mathbf{3}}$	3	-1	0	1	$\bar{b}_7$	b_7
6	6	2	0	0	-1	-1
7	7	-1	1	-1	0	0
8	8	0	-1	0	1	1

Table 6. The character table of the Γ_7 group with $b_7 = (-1+i\sqrt{7})/2$ and $\bar{b}_7 = b_7^* = -(1+i\sqrt{7})/2$.

$$
\begin{aligned}
56C_3 : \{&ST, TS, ST^3ST^5ST^3, ST^2ST^4ST^2, ST^2ST^4ST^3, ST^3ST^5ST^4, ST^3ST^5ST^2,\\
&ST^2ST^4ST, ST^5ST^3S, ST^3ST^5ST^5, ST^3ST^5ST, ST^2ST^4S, ST^2ST^4ST^5,\\
&ST^4ST^2S, ST^2ST^4ST^6, ST^3ST^5S, T^6S, ST^4ST^2, ST^3ST^5, ST^5ST^3,\\
&ST^2ST^4, T^5ST^5ST^4, T^2ST^3ST^6, T^5ST, ST^2ST^5, T^3ST^4ST^3, ST^6,\\
&TST^5ST^2, TST^3ST^4, ST^5ST^2, T^2ST^6, TST^4ST, T^5ST^3S, T^4ST^5ST^5,\\
&T^4ST^2, T^5ST^5ST^5, T^6ST^2, TST^3S, T^2ST^4S, T^2ST^5ST, T^3ST^5, TST^5ST,\\
&T^6ST^4S, TST^5, T^4ST^2S, TST^4ST^5, T^5ST^3, T^5ST^2S, ST^3ST, T^3ST^3, T^3ST^5S,\\
&T^2ST^4, T^3ST^4ST^6, T^2ST^5S, ST^4ST^6, T^4ST^4\}\,,\\
42C_4 : \{&ST^3, T^3S, TST^5ST^3, T^5ST^5ST^3, T^4S, T^3ST^4ST^2, TST^4ST^2, T^2ST^4ST^3,\\
&T^4ST^5ST^4, T^3ST, ST^4ST^3, ST^5ST^4, T^2ST, T^2ST^4ST, T^2ST^5ST^2, T^5ST^6,\\
&ST^2ST^3, ST^3ST^4, T^4ST^6, T^3ST^5ST^5, TST^2, T^2ST^3S, T^4ST^3S, ST^2ST^6, T^2ST^2,\\
&T^3ST^5ST, ST^5ST, T^3ST^4S, T^6ST^5, T^5ST^4S, T^5ST^5, T^6ST^2S, TST^3, T^3ST^2S,\\
&ST^4ST^5, ST^5ST^5S, ST^4, ST^3ST^2, T^4ST^5S, T^6ST^4, TST^5S, ST^2ST^2S\}\,,\\
24C_7 : \{&T, T^2, T^4, TST^3ST^5, T^2ST^5ST^3, T^5S, ST^3ST^6, T^4ST, ST^2ST^3S, T^3ST^5ST^2,\\
&STS, T^2ST^3ST^4, T^6ST^3S, (ST^5)^2, T^3ST^2, ST^5, ST^4S, ST^3ST^2S, T^2ST^4ST^5,\\
&ST^2S, T^2ST^3, TST^4ST^6, TST^4, (T^5S)^2\}\,,\\
24C_7' : \{&T^3, T^5, T^6, T^2ST^3ST^5, T^4ST^5ST^3, T^2S, T^3ST^5ST^4, ST^6S, TST^3ST^6, ST^4ST,\\
&T^3ST^6, ST^5ST^4S, ST^2, ST^3S, ST^4ST^5S, TST^4S, (ST^2)^2, T^4ST^5, T^3ST^4ST^5,\\
&T^6ST^3, (T^2S)^2, T^2ST^4ST^6, ST^5S, T^5ST^4\}\,,
\end{aligned}
$$

where nC_k denotes a class with n elements which is of order k. The character table of Γ_7 group is given in table 6. Following the convention of ref. [46], we find that Γ_7 group has ninety-two Abelian subgroups in total: twenty-one Z_2 subgroups, twenty-eight Z_3 subgroups, fourteen K_4 subgroups, twenty-one Z_4 subgroups and eight Z_7 subgroups. In terms of the generators S and T, these Abelian subgroups are given as follows:

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- Z_2 subgroups

$$
\begin{array}{lll}
Z_2^{S}=\{1,S\}, & Z_2^{T^3ST^5ST^3}=\{1,T^3ST^5ST^3\}, & Z_2^{T^2ST^4ST^2}=\{1,T^2ST^4ST^2\},\\
Z_2^{T^6ST}=\{1,T^6ST\}, & Z_2^{ST^3ST^4S}=\{1,ST^3ST^4S\}, & Z_2^{ST^5ST^2S}=\{1,ST^5ST^2S\},\\
Z_2^{ST^4ST^3S}=\{1,ST^4ST^3S\}, & Z_2^{TST^6}=\{1,TST^6\}, & Z_2^{ST^2ST^5S}=\{1,ST^2ST^5S\},\\
Z_2^{T^5ST^2}=\{1,T^5ST^2\}, & Z_2^{TST^5ST^5}=\{1,TST^5ST^5\}, & Z_2^{(ST^4)^2}=\{1,(ST^4)^2\},\\
Z_2^{T^2ST^5}=\{1,T^2ST^5\}, & Z_2^{(ST^3)^2}=\{1,(ST^3)^2\}, & Z_2^{T^5ST^5ST}=\{1,T^5ST^5ST\},\\
Z_2^{T^4ST^3}=\{1,T^4ST^3\}, & Z_2^{ST^5ST^6}=\{1,ST^5ST^6\}, & Z_2^{T^2ST^3ST}=\{1,T^2ST^3ST\},\\
Z_2^{ST^2ST}=\{1,ST^2ST\}, & Z_2^{T^3ST^4}=\{1,T^3ST^4\}, & Z_2^{TST^3ST^2}=\{1,TST^3ST^2\}.
\end{array}
$$

All the above twenty-one Z_2 subgroups are conjugate to each other.

- Z_3 subgroups

$$
\begin{array}{ll}
Z_3^{ST}=\{1,ST,T^6S\}, & Z_3^{TS}=\{1,TS,ST^6\},\\
Z_3^{ST^2ST^4ST^2}=\{1,ST^2ST^4ST^2,ST^3ST^5ST^3\}, & Z_3^{ST^2ST^4ST^3}=\{1,ST^2ST^4ST^3,ST^3ST^5ST\},\\
Z_3^{ST^3ST^5ST^4}=\{1,ST^3ST^5ST^4,ST^2ST^4ST^5\}, & Z_3^{ST^2ST^4ST^6}=\{1,ST^2ST^4ST^6,ST^3ST^5ST^2\},\\
Z_3^{ST^2ST^4ST}=\{1,ST^2ST^4ST,ST^3ST^5ST^5\} & Z_3^{ST^4ST^2S}=\{1,ST^4ST^2S,ST^5ST^3S\},\\
Z_3^{ST^2ST^4S}=\{1,ST^2ST^4S,ST^3ST^5S\}, & Z_3^{ST^4ST^2}=\{1,ST^4ST^2,T^5ST^3S\},\\
Z_3^{ST^3ST^5}=\{1,ST^3ST^5,T^2ST^4S\}, & Z_3^{ST^5ST^3}=\{1,ST^5ST^3,T^4ST^2S\},\\
Z_3^{ST^2ST^4}=\{1,ST^2ST^4,T^3ST^5S\}, & Z_3^{T^2ST^5ST}=\{1,T^2ST^5ST,T^5ST^5ST^4\},\\
Z_3^{TST^4ST^5}=\{1,TST^4ST^5,T^2ST^3ST^6\}, & Z_3^{T^5ST}=\{1,T^5ST,T^6ST^2\},\\
Z_3^{T^2ST^5S}=\{1,T^2ST^5S,ST^2ST^5\}, & Z_3^{TST^4ST}=\{1,TST^4ST,T^3ST^4ST^3\},\\
Z_3^{TST^5ST^2}=\{1,TST^5ST^2,T^4ST^5ST^5\}, & Z_3^{TST^3ST^4}=\{1,TST^3ST^4,T^3ST^4ST^6\},\\
Z_3^{T^5ST^2S}=\{1,T^5ST^2S,ST^5ST^2\}, & Z_3^{TST^5}=\{1,TST^5,T^2ST^6\},\\
Z_3^{T^4ST^2}=\{1,T^4ST^2,T^5ST^3\}, & Z_3^{TST^5ST}=\{1,TST^5ST,T^5ST^5ST^5\},\\
Z_3^{TST^3S}=\{1,TST^3S,ST^4ST^6\}, & Z_3^{T^2ST^4}=\{1,T^2ST^4,T^3ST^5\},\\
Z_3^{ST^3ST}=\{1,ST^3ST,T^6ST^4S\}, & Z_3^{T^3ST^3}=\{1,T^3ST^3,T^4ST^4\}.
\end{array}
$$

The twenty-eight Z_3 subgroups are related with each other by group conjugation.

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- K_4 subgroups

$$
\begin{aligned}
K_4^{(S,TST^5ST^5)} &\equiv Z_2^{S}\times Z_2^{TST^5ST^5}=\{1,S,TST^5ST^5,T^2ST^3ST\},\\
K_4^{(T^2ST^5,ST^3ST^4S)} &\equiv Z_2^{T^2ST^5}\times Z_2^{ST^3ST^4S}=\{1,T^2ST^5,ST^3ST^4S,T^3ST^5ST^3\},\\
K_4^{(ST^2ST^5S,T^3ST^4)} &\equiv Z_2^{ST^2ST^5S}\times Z_2^{T^3ST^4}=\{1,ST^2ST^5S,T^3ST^4,T^2ST^4ST^2\},\\
K_4^{(T^6ST,ST^5ST^6)} &\equiv Z_2^{T^6ST}\times Z_2^{ST^5ST^6}=\{1,T^6ST,ST^5ST^6,TST^3ST^2\},\\
K_4^{(TST^6,(ST^4)^2)} &\equiv Z_2^{TST^6}\times Z_2^{(ST^4)^2}=\{1,TST^6,(ST^4)^2,ST^5ST^2S\},\\
K_4^{(T^5ST^5ST,T^4ST^3)} &\equiv Z_2^{T^5ST^5ST}\times Z_2^{T^4ST^3}=\{1,T^5ST^5ST,T^4ST^3,ST^4ST^3S\},\\
K_4^{(ST^2ST,T^5ST^2)} &\equiv Z_2^{ST^2ST}\times Z_2^{T^5ST^2}=\{1,ST^2ST,T^5ST^2,(ST^3)^2\}\\
K_4^{(S,T^5ST^5ST)} &\equiv Z_2^{S}\times Z_2^{T^5ST^5ST}=\{1,S,T^5ST^5ST,TST^3ST^2\},\\
K_4^{(T^5ST^2,ST^4ST^3S)} &\equiv Z_2^{T^5ST^2}\times Z_2^{ST^4ST^3S}=\{1,T^5ST^2,ST^4ST^3S,T^3ST^5ST^3\}\\
K_4^{(ST^5ST^2S,T^4ST^3)} &\equiv Z_2^{ST^5ST^2S}\times Z_2^{T^4ST^3}=\{1,ST^5ST^2S,T^4ST^3,T^2ST^4ST^2\},\\
K_4^{(T^6ST,ST^2ST^5S)} &\equiv Z_2^{T^6ST}\times Z_2^{ST^2ST^5S}=\{1,T^6ST,ST^2ST^5S,(ST^3)^2\}\\
K_4^{(TST^5ST^5,T^3ST^4)} &\equiv Z_2^{TST^5ST^5}\times Z_2^{T^3ST^4}=\{1,TST^5ST^5,T^3ST^4,ST^3ST^4S\},\\
K_4^{(TST^6,ST^2ST)} &\equiv Z_2^{TST^6}\times Z_2^{ST^2ST}=\{1,TST^6,ST^2ST,T^2ST^3ST\}\\
K_4^{(ST^5ST^6,T^2ST^5)} &\equiv Z_2^{ST^5ST^6}\times Z_2^{T^2ST^5}=\{1,ST^5ST^6,T^2ST^5,(ST^4)^2\}.
\end{aligned}
$$

All the fourteen K_4 subgroups are conjugate as well.

- Z_4 subgroups

$$
\begin{aligned}
Z_4^{ST^3} &= \{1,ST^3,(ST^3)^2,T^4S\}, & Z_4^{T^3S} &= \{1,T^3S,(ST^4)^2,ST^4\},\\
Z_4^{T^4ST^5ST^4} &= \{1,T^4ST^5ST^4,S,T^2ST^5ST^2\}, & Z_4^{ST^2ST^2S} &= \{1,ST^2ST^2S,T^3ST^5ST^3,ST^5ST^5S\},\\
Z_4^{T^2ST^2} &= \{1,T^2ST^2,T^2ST^4ST^2,T^5ST^5\}, & Z_4^{TST^5ST^3} &= \{1,TST^5ST^3,T^6ST,T^3ST^5ST^5\},\\
Z_4^{T^3ST^5ST} &= \{1,T^3ST^5ST,TST^6,T^5ST^5ST^3\}, & Z_4^{T^2ST^4ST} &= \{1,T^2ST^4ST,ST^5ST^6,T^3ST^4ST^2\},\\
Z_4^{TST^4ST^2} &= \{1,TST^4ST^2,ST^2ST,T^2ST^4ST^3\}, & Z_4^{TST^3} &= \{1,TST^3,ST^3ST^4S,T^4ST^6\},\\
Z_4^{ST^3ST^2} &= \{1,ST^3ST^2,ST^5ST^2S,T^5ST^4S\}, & Z_4^{T^3ST} &= \{1,T^3ST,ST^4ST^3S,T^6ST^4\},\\
Z_4^{T^4ST^3S} &= \{1,ST^4ST^3,T^5ST^5ST,T^4ST^3S\}, & Z_4^{ST^5ST^4} &= \{1,ST^5ST^4,T^5ST^2,T^3ST^2S\},\\
Z_4^{T^2ST} &= \{1,T^2ST,T^2ST^3ST,T^6ST^5\}, & Z_4^{TST^2} &= \{1,TST^2,TST^3ST^2,T^5ST^6\},\\
Z_4^{ST^2ST^3} &= \{1,ST^2ST^3,T^2ST^5,T^4ST^5S\}, & Z_4^{T^3ST^4S} &= \{1,T^3ST^4S,TST^5ST^5,ST^3ST^4\},\\
Z_4^{ST^4ST^5} &= \{1,ST^4ST^5,ST^2ST^5S,T^2ST^3S\}, & Z_4^{TST^5S} &= \{1,ST^2ST^6,T^4ST^3,TST^5S\},\\
Z_4^{ST^5ST} &= \{1,ST^5ST,T^3ST^4,T^6ST^2S\}.
\end{aligned}
$$

All the twenty-one Z_4 subgroups are related to each other under group conjugation.

- Z_7 subgroups

$$
\begin{aligned}
Z_7^T &= \{1, T, T^2, T^3, T^4, T^5, T^6\}, \\
Z_7^{T^4ST} &= \{1, T^4ST, ST^3ST^2S, T^2ST^4ST^6, TST^3ST^5, ST^5ST^4S, T^6ST^3\}, \\
Z_7^{TST^4} &= \{1, TST^4, ST^2ST^3S, T^2ST^3ST^5, T^2ST^4ST^5, ST^4ST^5S, T^3ST^6\}, \\
Z_7^{T^2ST^3} &= \{1, T^2ST^3, T^6ST^3S, T^3ST^5ST^4, T^2ST^5ST^3, T^4ST^5, ST^4ST\}, \\
Z_7^{T^3ST^2} &= \{1, T^3ST^2, TST^4S, T^5ST^4, T^3ST^5ST^2, T^4ST^5ST^3, ST^3ST^6\}, \\
Z_7^{T^2S} &= \{1, T^2S, ST^5, (ST^5)^2, TST^3ST^6, TST^4ST^6, (T^2S)^2\}, \\
Z_7^{ST^2} &= \{1, ST^2, (ST^2)^2, T^2ST^3ST^4, T^3ST^4ST^5, (T^5S)^2, T^5S\}, \\
Z_7^{STS} &= \{1, STS, ST^2S, ST^3S, ST^4S, ST^5S, ST^6S\}.
\end{aligned}
$$

All the eight Z_7 subgroups are related to each other as well under group conjugation.

The Γ_7 group has six irreducible representations: one singlet representation **1**, two three-dimensional representations **3** and $\bar{\mathbf{3}}$, one six-dimensional representation **6**, one seven-dimensional representation **7** and one eight-dimensional representation **8**. The explicit forms of the generators S and T in the five irreducible representations are chosen as follows[2]

$$
\mathbf{3}\ S = \frac{2}{\sqrt{7}}\begin{pmatrix} -s_2 & -s_1 & s_3 \\ -s_1 & s_3 & -s_2 \\ s_3 & -s_2 & -s_1 \end{pmatrix}, \qquad \bar{\mathbf{3}}:\ S = \frac{2}{\sqrt{7}}\begin{pmatrix} -s_2 & -s_1 & s_3 \\ -s_1 & s_3 & -s_2 \\ s_3 & -s_2 & -s_1 \end{pmatrix},
$$

$$
\mathbf{6}:\ S = \frac{2\sqrt{2}}{7}\begin{pmatrix}
\frac{1-c_2}{\sqrt{2}} & \frac{1-c_1}{\sqrt{2}} & c_2-c_1 & \frac{1-c_3}{\sqrt{2}} & c_3-c_2 & c_1-c_3 \\
\frac{1-c_1}{\sqrt{2}} & \frac{1-c_3}{\sqrt{2}} & c_1-c_3 & \frac{1-c_2}{\sqrt{2}} & c_2-c_1 & c_3-c_2 \\
c_2-c_1 & c_1-c_3 & \frac{1-c_1}{\sqrt{2}} & c_3-c_2 & \frac{1-c_2}{\sqrt{2}} & \frac{1-c_3}{\sqrt{2}} \\
\frac{1-c_3}{\sqrt{2}} & \frac{1-c_2}{\sqrt{2}} & c_3-c_2 & \frac{1-c_1}{\sqrt{2}} & c_1-c_3 & c_2-c_1 \\
c_3-c_2 & c_2-c_1 & \frac{1-c_2}{\sqrt{2}} & c_1-c_3 & \frac{1-c_3}{\sqrt{2}} & \frac{1-c_1}{\sqrt{2}} \\
c_1-c_3 & c_3-c_2 & \frac{1-c_3}{\sqrt{2}} & c_2-c_1 & \frac{1-c_1}{\sqrt{2}} & \frac{1-c_2}{\sqrt{2}}
\end{pmatrix},
$$

$$
\mathbf{7}:\ S = \frac{2}{7}\begin{pmatrix}
-\frac{1}{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
\sqrt{2} & \frac{s2+4s3}{\sqrt{7}} & \frac{s1-4s_2}{\sqrt{7}} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} & \frac{-4s_1-s_3}{\sqrt{7}} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} \\
\sqrt{2} & \frac{s_1-4s_2}{\sqrt{7}} & \frac{-4s_1-s_3}{\sqrt{7}} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} & \frac{s_2+4s_3}{\sqrt{7}} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} \\
\sqrt{2} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} & \frac{s_1-4s_2}{\sqrt{7}} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} & \frac{s_2+4s_3}{\sqrt{7}} & \frac{-4s_1-s_3}{\sqrt{7}} \\
\sqrt{2} & \frac{-4s_1-s_3}{\sqrt{7}} & \frac{s_2+4s_3}{\sqrt{7}} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} & \frac{s_1-4s_2}{\sqrt{7}} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} \\
\sqrt{2} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} & \frac{s_2+4s_3}{\sqrt{7}} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} & \frac{-4s_1-s_3}{\sqrt{7}} & \frac{s_1-4s_2}{\sqrt{7}} \\
\sqrt{2} & \frac{-2s_1+4s_2-2s_3}{\sqrt{7}} & \frac{4s_1+2s_2+2s_3}{\sqrt{7}} & \frac{-4s_1-s_3}{\sqrt{7}} & \frac{2s_1-2s_2-4s_3}{\sqrt{7}} & \frac{s_1-4s_2}{\sqrt{7}} & \frac{s_2+4s_3}{\sqrt{7}}
\end{pmatrix},
$$

[2] In the basis of [44] the triplet representation was given in terms of the standard generators A, B in [42] and may be related to four generators S, T, U, V, with $\Gamma_7 \supset S_4 \supset A_4$ corresponding to the respective generators being $S, T, U, V \supset S, T, U \supset S, T$. However for our purposes here we shall use the representation theory for Γ_7 developed for finite modular groups in [46]. Thus the notation we use for the generators S, T corresponds to [46] rather than [44].

$$
\mathbf{8}: S=\frac{2\sqrt{6}}{7}\begin{pmatrix}
\frac{2c_2-c_1-c_3}{2\sqrt{6}} & \frac{c_1-c_3}{2\sqrt{2}} & \frac{c_1+c_2-2c_3}{2\sqrt{3}} & \frac{c_1-2c_2+c_3}{2\sqrt{3}} & \frac{c_2+c_3-2c_1}{2\sqrt{3}} & \frac{2c_1-c_2-c_3}{2\sqrt{3}} & \frac{2c_2-c_1-c_3}{2\sqrt{3}} & \frac{2c_3-c_1-c_2}{2\sqrt{3}} \\
\frac{c_1-c_3}{2\sqrt{2}} & \frac{c_1-2c_2+c_3}{2\sqrt{6}} & \frac{c_1-c_2}{2} & \frac{c_3-c_1}{2} & \frac{c_2-c_3}{2} & \frac{c_3-c_2}{2} & \frac{c_1-c_3}{2} & \frac{c_2-c_1}{2} \\
\frac{c_1+c_2-2c_3}{2\sqrt{3}} & \frac{c_1-c_2}{2} & \frac{c_2-c_1}{\sqrt{6}} & \frac{c_1-c_3}{\sqrt{6}} & \frac{1-c_3}{\sqrt{6}} & \frac{c_2-c_3}{\sqrt{6}} & \frac{c_1-1}{\sqrt{6}} & \frac{c_2-1}{\sqrt{6}} \\
\frac{c_1-2c_2+c_3}{2\sqrt{3}} & \frac{c_3-c_1}{2} & \frac{c_1-c_3}{\sqrt{6}} & \frac{c_3-c_2}{\sqrt{6}} & \frac{1-c_2}{\sqrt{6}} & \frac{c_1-c_2}{\sqrt{6}} & \frac{c_3-1}{\sqrt{6}} & \frac{c_1-1}{\sqrt{6}} \\
\frac{c_2+c_3-2c_1}{2\sqrt{3}} & \frac{c_2-c_3}{2} & \frac{1-c_3}{\sqrt{6}} & \frac{1-c_2}{\sqrt{6}} & \frac{c_1-c_3}{\sqrt{6}} & \frac{c_1-1}{\sqrt{6}} & \frac{c_1-c_2}{\sqrt{6}} & \frac{c_2-c_3}{\sqrt{6}} \\
\frac{2c_1-c_2-c_3}{2\sqrt{3}} & \frac{c_3-c_2}{2} & \frac{c_2-c_3}{\sqrt{6}} & \frac{c_1-c_2}{\sqrt{6}} & \frac{c_1-1}{\sqrt{6}} & \frac{c_1-c_3}{\sqrt{6}} & \frac{1-c_2}{\sqrt{6}} & \frac{1-c_3}{\sqrt{6}} \\
\frac{2c_2-c_1-c_3}{2\sqrt{3}} & \frac{c_1-c_3}{2} & \frac{c_1-1}{\sqrt{6}} & \frac{c_3-1}{\sqrt{6}} & \frac{c_1-c_2}{\sqrt{6}} & \frac{1-c_2}{\sqrt{6}} & \frac{c_3-c_2}{\sqrt{6}} & \frac{c_1-c_3}{\sqrt{6}} \\
\frac{2c_3-c_1-c_2}{2\sqrt{3}} & \frac{c_2-c_1}{2} & \frac{c_2-1}{\sqrt{6}} & \frac{c_1-1}{\sqrt{6}} & \frac{c_2-c_3}{\sqrt{6}} & \frac{1-c_3}{\sqrt{6}} & \frac{c_1-c_3}{\sqrt{6}} & \frac{c_2-c_1}{\sqrt{6}}
\end{pmatrix},
$$

$$
\mathbf{3}: T=\mathrm{diag}(\rho,\rho^2,\rho^4)\,, \qquad \bar{\mathbf{3}}: T=\mathrm{diag}(\rho^6,\rho^5,\rho^3)\,, \qquad \mathbf{6}: T=\mathrm{diag}(\rho^1,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6)\,,
$$
$$
\mathbf{7}: T=\mathrm{diag}(1,\rho,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6)\,, \qquad \mathbf{8}: T=\mathrm{diag}(1,1,\rho,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6)\,,
$$

where the parameter ρ is the seventh unit root $\rho=e^{2\pi i/7}$, $s_n=\sin\frac{2n\pi}{7}$ and $c_n=\cos\frac{2n\pi}{7}$ with $n=1,2,3$. We can straightforwardly obtain the Kronecker products between various representations:

$$
\begin{aligned}
&\mathbf{1}\otimes\mathbf{r}=\mathbf{r}\otimes\mathbf{1}=\mathbf{r}, \quad \mathbf{3}\otimes\mathbf{3}=\bar{\mathbf{3}}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S}}, \quad \mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}, \quad \mathbf{3}\times\mathbf{6}=\bar{\mathbf{3}}\oplus\mathbf{7}\oplus\mathbf{8},\\
&\mathbf{3}\otimes\mathbf{7}=\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}, \quad \mathbf{3}\otimes\mathbf{8}=\mathbf{3}\oplus\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}, \quad \bar{\mathbf{3}}\otimes\bar{\mathbf{3}}=\mathbf{3}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S}},\\
&\bar{\mathbf{3}}\otimes\mathbf{6}=\mathbf{3}\oplus\mathbf{7}\oplus\mathbf{8}, \quad \bar{\mathbf{3}}\otimes\mathbf{7}=\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}, \quad \bar{\mathbf{3}}\otimes\mathbf{8}=\bar{\mathbf{3}}\oplus\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8},\\
&\mathbf{6}\otimes\mathbf{6}=\mathbf{1}_{\mathbf{S}}\oplus\mathbf{6}_{\mathbf{S},\mathbf{1}}\oplus\mathbf{6}_{\mathbf{S},\mathbf{2}}\oplus\mathbf{7}_{\mathbf{A}}\oplus\mathbf{8}_{\mathbf{S}}\oplus\mathbf{8}_{\mathbf{A}}, \quad \mathbf{6}\otimes\mathbf{7}=\mathbf{3}\oplus\bar{\mathbf{3}}\oplus\mathbf{6}\oplus\mathbf{7}_{\mathbf{1}}\oplus\mathbf{7}_{\mathbf{2}}\oplus\mathbf{8}_{\mathbf{1}}\oplus\mathbf{8}_{\mathbf{2}}\,,\\
&\mathbf{6}\otimes\mathbf{8}=\mathbf{3}\oplus\bar{\mathbf{3}}\oplus\mathbf{6}_{\mathbf{1}}\oplus\mathbf{6}_{\mathbf{2}}\oplus\mathbf{7}_{\mathbf{1}}\oplus\mathbf{7}_{\mathbf{2}}\oplus\mathbf{8}_{\mathbf{1}}\oplus\mathbf{8}_{\mathbf{2}}\,,\\
&\mathbf{7}\otimes\mathbf{7}=\mathbf{1}_{\mathbf{S}}\oplus\mathbf{3}_{\mathbf{A}}\oplus\bar{\mathbf{3}}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S},1}\oplus\mathbf{6}_{\mathbf{S},2}\oplus\mathbf{7}_{\mathbf{S}}\oplus\mathbf{7}_{\mathbf{A}}\oplus\mathbf{8}_{\mathbf{S}}\oplus\mathbf{8}_{\mathbf{A}}\,,\\
&\mathbf{7}\otimes\mathbf{8}=\mathbf{3}\oplus\bar{\mathbf{3}}\oplus\mathbf{6}_{\mathbf{1}}\oplus\mathbf{6}_{\mathbf{2}}\oplus\mathbf{7}_{\mathbf{1}}\oplus\mathbf{7}_{\mathbf{2}}\oplus\mathbf{8}_{\mathbf{1}}\oplus\mathbf{8}_{\mathbf{2}}\oplus\mathbf{8}_{\mathbf{3}}\,,\\
&\mathbf{8}\otimes\mathbf{8}=\mathbf{1}_{\mathbf{S}}\oplus\mathbf{3}_{\mathbf{A}}\oplus\bar{\mathbf{3}}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S},1}\oplus\mathbf{6}_{\mathbf{S},2}\oplus\mathbf{7}_{\mathbf{S}}\oplus\mathbf{7}_{\mathbf{A},1}\oplus\mathbf{7}_{\mathbf{A},\mathbf{2}}\oplus\mathbf{8}_{\mathbf{S},\mathbf{1}}\oplus\mathbf{8}_{\mathbf{S},2}\oplus\mathbf{8}_{\mathbf{A}}\,,
\end{aligned}
\tag{A.3}
$$

where $\mathbf{r}$ denotes any irreducible representation of Γ_7, and $\mathbf{6_1}$, $\mathbf{6_2}$, $\mathbf{7_1}$, $\mathbf{7_2}$, $\mathbf{8_1}$, $\mathbf{8_2}$ and $\mathbf{8_3}$ stand for the two $\mathbf{6}$, $\mathbf{7}$ and three $\mathbf{8}$ representations which appear in the Kronecker products. The subscript "$\mathbf{S}$" ("$\mathbf{A}$") refers to symmetric (antisymmetric) combinations. We now list the Clebsch-Gordan coefficients in our basis. We use the notation α_i (β_i) to denote the elements of the first (second) representation.

$\mathbf{3}\otimes\mathbf{3}=\bar{\mathbf{3}}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S}}$	$\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}=\mathbf{3}_{\mathbf{A}}\oplus\mathbf{6}_{\mathbf{S}}$	$\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}$
$\bar{\mathbf{3}}_{\mathbf{A}}: \begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_3\beta_1-\alpha_1\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}$ $\mathbf{6}_{\mathbf{S}}: \begin{pmatrix}\sqrt{2}\alpha_3\beta_3\\ \sqrt{2}\alpha_1\beta_1\\ \alpha_1\beta_2+\alpha_2\beta_1\\ \sqrt{2}\alpha_2\beta_2\\ \alpha_1\beta_3+\alpha_3\beta_1\\ \alpha_2\beta_3+\alpha_3\beta_2\end{pmatrix}$	$\mathbf{3}_{\mathbf{A}}: \begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_3\beta_1-\alpha_1\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}$ $\mathbf{6}_{\mathbf{S}}: \begin{pmatrix}\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_3+\alpha_3\beta_1\\ \sqrt{2}\alpha_2\beta_2\\ \alpha_1\beta_2+\alpha_2\beta_1\\ \sqrt{2}\alpha_1\beta_1\\ \sqrt{2}\alpha_3\beta_3\end{pmatrix}$	$\mathbf{1}: \alpha_\mathbf{1}\beta_\mathbf{1}+\alpha_\mathbf{2}\beta_\mathbf{2}+\alpha_\mathbf{3}\beta_\mathbf{3}$ $\mathbf{8}: \begin{pmatrix}\sqrt{3}\,(\alpha_1\beta_1-\alpha_2\beta_2)\\ \alpha_1\beta_1+\alpha_2\beta_2-2\alpha_3\beta_3\\ \sqrt{6}\alpha_2\beta_1\\ \sqrt{6}\alpha_3\beta_2\\ \sqrt{6}\alpha_3\beta_1\\ -\sqrt{6}\alpha_1\beta_3\\ -\sqrt{6}\alpha_2\beta_3\\ -\sqrt{6}\alpha_1\beta_2\end{pmatrix}$

$\mathbf{3}\otimes\mathbf{6}=\bar{\mathbf{3}}\oplus\mathbf{7}\oplus\mathbf{8}$	$\bar{\mathbf{3}}\otimes\mathbf{6}=\mathbf{3}\oplus\mathbf{7}\oplus\mathbf{8}$
$\bar{\mathbf{3}}:\begin{pmatrix}\alpha_2\beta_4+\alpha_3\beta_2+\sqrt{2}\alpha_1\beta_5\\ \alpha_1\beta_4+\alpha_3\beta_1+\sqrt{2}\alpha_2\beta_3\\ \alpha_1\beta_2+\alpha_2\beta_1+\sqrt{2}\alpha_3\beta_6\end{pmatrix}$	$\mathbf{3}:\begin{pmatrix}\sqrt{2}\alpha_1\beta_2+\alpha_2\beta_3+\alpha_3\beta_5\\ \sqrt{2}\alpha_2\beta_4+\alpha_1\beta_3+\alpha_3\beta_6\\ \sqrt{2}\alpha_3\beta_1+\alpha_1\beta_5+\alpha_2\beta_6\end{pmatrix}$
$\mathbf{7}:\begin{pmatrix}2\left(\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_3\right)\\ -2\left(\sqrt{2}\alpha_2\beta_6+\alpha_3\beta_4\right)\\ -2\left(\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_5\right)\\ 3\alpha_1\beta_2-\alpha_2\beta_1-\sqrt{2}\alpha_3\beta_6\\ -2\left(\sqrt{2}\alpha_1\beta_3+\alpha_2\beta_1\right)\\ -\alpha_1\beta_4-\sqrt{2}\alpha_2\beta_3+3\alpha_3\beta_1\\ -\sqrt{2}\alpha_1\beta_5+3\alpha_2\beta_4-\alpha_3\beta_2\end{pmatrix}$	$\mathbf{7}:\begin{pmatrix}2\left(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_4\right)\\ -\sqrt{2}\alpha_1\beta_2+3\alpha_2\beta_3-\alpha_3\beta_5\\ -\sqrt{2}\alpha_2\beta_4-\alpha_1\beta_3+3\alpha_3\beta_6\\ -2\left(\sqrt{2}\alpha_1\beta_4+\alpha_2\beta_5\right)\\ -\sqrt{2}\alpha_3\beta_1+3\alpha_1\beta_5-\alpha_2\beta_6\\ -2\left(\sqrt{2}\alpha_3\beta_2+\alpha_1\beta_6\right)\\ -2\left(\sqrt{2}\alpha_2\beta_1+\alpha_3\beta_3\right)\end{pmatrix}$
$\mathbf{8}:\begin{pmatrix}\alpha_1\beta_6+\alpha_2\beta_5-2\alpha_3\beta_3\\ \sqrt{3}\left(\alpha_2\beta_5-\alpha_1\beta_6\right)\\ \sqrt{2}\alpha_2\beta_6-2\alpha_3\beta_4\\ \sqrt{2}\alpha_3\beta_5-2\alpha_1\beta_1\\ 2\alpha_2\beta_1-\sqrt{2}\alpha_3\beta_6\\ 2\alpha_2\beta_2-\sqrt{2}\alpha_1\beta_3\\ \sqrt{2}\alpha_2\beta_3-2\alpha_1\beta_4\\ \sqrt{2}\alpha_1\beta_5-2\alpha_3\beta_2\end{pmatrix}$	$\mathbf{8}:\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_2-2\alpha_3\beta_4\\ -\sqrt{3}\left(\alpha_1\beta_1-\alpha_2\beta_2\right)\\ -\sqrt{2}\alpha_1\beta_2+2\alpha_3\beta_5\\ 2\alpha_1\beta_3-\sqrt{2}\alpha_2\beta_4\\ \sqrt{2}\alpha_1\beta_4-2\alpha_2\beta_5\\ \sqrt{2}\alpha_3\beta_1-2\alpha_2\beta_6\\ 2\alpha_1\beta_6-\sqrt{2}\alpha_3\beta_2\\ 2\alpha_3\beta_3-\sqrt{2}\alpha_2\beta_1\end{pmatrix}$

$\mathbf{3}\otimes\mathbf{7}=\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}$	$\bar{\mathbf{3}}\otimes\mathbf{7}=\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}$
$\mathbf{6}:\begin{pmatrix}2\alpha_1\beta_1-\sqrt{2}\left(\alpha_3\beta_5+2\alpha_2\beta_7\right)\\ 2\alpha_2\beta_1-\sqrt{2}\left(\alpha_1\beta_2+2\alpha_3\beta_6\right)\\ 3\alpha_2\beta_2-\alpha_1\beta_3-2\alpha_3\beta_7\\ 2\alpha_3\beta_1-\sqrt{2}\left(\alpha_2\beta_3+2\alpha_1\beta_4\right)\\ -\alpha_3\beta_2-2\alpha_2\beta_4+3\alpha_1\beta_5\\ 3\alpha_3\beta_3-\alpha_2\beta_5-2\alpha_1\beta_6\end{pmatrix}$	$\mathbf{6}:\begin{pmatrix}2\alpha_1\beta_3+\alpha_2\beta_4-3\alpha_3\beta_6\\ -3\alpha_1\beta_4+2\alpha_2\beta_5+\alpha_3\beta_7\\ \sqrt{2}\left(2\alpha_1\beta_5+\alpha_2\beta_6\right)-2\alpha_3\beta_1\\ 2\alpha_3\beta_2+\alpha_1\beta_6-3\alpha_2\beta_7\\ \sqrt{2}\left(2\alpha_3\beta_3+\alpha_1\beta_7\right)-2\alpha_2\beta_1\\ \sqrt{2}\left(2\alpha_2\beta_2+\alpha_3\beta_4\right)-2\alpha_1\beta_1\end{pmatrix}$
$\mathbf{7}:\begin{pmatrix}\sqrt{2}\left(\alpha_3\beta_4+\alpha_2\beta_6+\alpha_1\beta_7\right)\\ -\sqrt{2}\left(\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_5\right)\\ -\sqrt{2}\left(\alpha_2\beta_1+\sqrt{2}\alpha_1\beta_2\right)\\ \alpha_2\beta_2-\alpha_1\beta_3+2\alpha_3\beta_7\\ -\sqrt{2}\left(\alpha_3\beta_1+\sqrt{2}\alpha_2\beta_3\right)\\ -\alpha_3\beta_2+2\alpha_2\beta_4+\alpha_1\beta_5\\ \alpha_3\beta_3-\alpha_2\beta_5+2\alpha_1\beta_6\end{pmatrix}$	$\mathbf{7}:\begin{pmatrix}\sqrt{2}\left(\alpha_1\beta_2+\alpha_2\beta_3+\alpha_3\beta_5\right)\\ 2\alpha_1\beta_3-\alpha_2\beta_4+\alpha_3\beta_6\\ \alpha_1\beta_4+2\alpha_2\beta_5-\alpha_3\beta_7\\ -\sqrt{2}\left(\alpha_3\beta_1+\sqrt{2}\alpha_2\beta_6\right)\\ 2\alpha_3\beta_2-\alpha_1\beta_6+\alpha_2\beta_7\\ -\sqrt{2}\left(\alpha_2\beta_1+\sqrt{2}\alpha_1\beta_7\right)\\ -\sqrt{2}\left(\alpha_1\beta_1+\sqrt{2}\alpha_3\beta_4\right)\end{pmatrix}$
$\mathbf{8}:\begin{pmatrix}-4\alpha_3\beta_4-\alpha_2\beta_6+5\alpha_1\beta_7\\ -\sqrt{3}\left(2\alpha_3\beta_4-3\alpha_2\beta_6+\alpha_1\beta_7\right)\\ \sqrt{2}\left(2\alpha_3\beta_5-3\alpha_2\beta_7\right)-4\alpha_1\beta_1\\ \sqrt{2}\left(2\alpha_1\beta_2-3\alpha_3\beta_6\right)-4\alpha_2\beta_1\\ \sqrt{2}\left(2\alpha_2\beta_2+4\alpha_1\beta_3+\alpha_3\beta_7\right)\\ 4\alpha_3\beta_1+\sqrt{2}\left(3\alpha_1\beta_4-2\alpha_2\beta_3\right)\\ -\sqrt{2}\left(4\alpha_3\beta_2+\alpha_2\beta_4+2\alpha_1\beta_5\right)\\ -\sqrt{2}\left(2\alpha_3\beta_3+4\alpha_2\beta_5+\alpha_1\beta_6\right)\end{pmatrix}$	$\mathbf{8}:\begin{pmatrix}5\alpha_1\beta_2-\alpha_2\beta_3-4\alpha_3\beta_5\\ -\sqrt{3}\left(\alpha_1\beta_2-3\alpha_2\beta_3+2\alpha_3\beta_5\right)\\ \sqrt{2}\left(\alpha_1\beta_3+4\alpha_2\beta_4+2\alpha_3\beta_6\right)\\ \sqrt{2}\left(2\alpha_1\beta_4+\alpha_2\beta_5+4\alpha_3\beta_7\right)\\ \sqrt{2}\left(2\alpha_2\beta_6-3\alpha_1\beta_5\right)-4\alpha_3\beta_1\\ -\sqrt{2}\left(\alpha_3\beta_2+4\alpha_1\beta_6+2\alpha_2\beta_7\right)\\ 4\alpha_2\beta_1+\sqrt{2}\left(3\alpha_3\beta_3-2\alpha_1\beta_7\right)\\ 4\alpha_1\beta_1+\sqrt{2}\left(3\alpha_2\beta_2-2\alpha_3\beta_4\right)\end{pmatrix}$

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$\mathbf{3}\otimes\mathbf{8}=\mathbf{3}\oplus\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}$	$\bar{\mathbf{3}}\otimes\mathbf{8}=\bar{\mathbf{3}}\oplus\mathbf{6}\oplus\mathbf{7}\oplus\mathbf{8}$
$\mathbf{3}:\begin{pmatrix}\sqrt{3}\alpha_1\beta_1+\alpha_1\beta_2-\sqrt{6}(\alpha_3\beta_6+\alpha_2\beta_8)\\ -\sqrt{3}\alpha_2\beta_1+\alpha_2\beta_2+\sqrt{6}(\alpha_1\beta_3-\alpha_3\beta_7)\\ -2\alpha_3\beta_2+\sqrt{6}(\alpha_2\beta_4+\alpha_1\beta_5)\end{pmatrix}$	$\bar{\mathbf{3}}:\begin{pmatrix}\sqrt{3}\alpha_1\beta_1+\alpha_1\beta_2+\sqrt{6}(\alpha_2\beta_3+\alpha_3\beta_5)\\ -\sqrt{3}\alpha_2\beta_1+\alpha_2\beta_2+\sqrt{6}(\alpha_3\beta_4-\alpha_1\beta_8)\\ -2\alpha_3\beta_2-\sqrt{6}(\alpha_1\beta_6+\alpha_2\beta_7)\end{pmatrix}$
$\mathbf{6}:\begin{pmatrix}\alpha_1\beta_1-\sqrt{3}\alpha_1\beta_2+\sqrt{2}\,(\alpha_3\beta_6-\alpha_2\beta_8)\\ \alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2-\sqrt{2}\,(\alpha_1\beta_3+\alpha_3\beta_7)\\ 2\,(\alpha_1\beta_4+\alpha_3\beta_8)\\ \sqrt{2}\,(\alpha_1\beta_5-\alpha_2\beta_4)-2\alpha_3\beta_1\\ 2\,(\alpha_3\beta_3-\alpha_2\beta_5)\\ 2\,(\alpha_1\beta_7-\alpha_2\beta_6)\end{pmatrix}$	$\mathbf{6}:\begin{pmatrix}2\alpha_1\beta_4-2\alpha_2\beta_5\\ 2\alpha_3\beta_8-2\alpha_2\beta_6\\ 2\alpha_3\beta_1+\sqrt{2}\,(\alpha_1\beta_6-\alpha_2\beta_7)\\ 2\,(\alpha_3\beta_3+\alpha_1\beta_7)\\ -\alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2-\sqrt{2}\,(\alpha_3\beta_4+\alpha_1\beta_8)\\ \sqrt{3}\alpha_1\beta_2-\alpha_1\beta_1+\sqrt{2}\,(\alpha_3\beta_5-\alpha_2\beta_3)\end{pmatrix}$
$\mathbf{7}:\begin{pmatrix}4\,(-\alpha_3\beta_5+\alpha_2\beta_7+\alpha_1\beta_8)\\ 5\alpha_1\beta_1-\sqrt{3}\alpha_1\beta_2+\sqrt{2}\,(3\alpha_2\beta_8-\alpha_3\beta_6)\\ \sqrt{2}\,(\alpha_1\beta_3+3\alpha_3\beta_7)-\alpha_2\beta_1+3\sqrt{3}\alpha_2\beta_2\\ 2\sqrt{2}\,(2\alpha_2\beta_3+\alpha_1\beta_4-\alpha_3\beta_8)\\ \sqrt{2}\,(\alpha_2\beta_4-3\alpha_1\beta_5)-2\left(2\alpha_3\beta_1+\sqrt{3}\alpha_3\beta_2\right)\\ 2\sqrt{2}\,(\alpha_3\beta_3+\alpha_2\beta_5-2\alpha_1\beta_6)\\ 2\sqrt{2}\,(2\alpha_3\beta_4-\alpha_2\beta_6-\alpha_1\beta_7)\end{pmatrix}$	$\mathbf{7}:\begin{pmatrix}4\,(\alpha_1\beta_3+\alpha_2\beta_4-\alpha_3\beta_6)\\ -2\sqrt{2}\,(\alpha_1\beta_4+\alpha_2\beta_5-2\alpha_3\beta_7)\\ 2\sqrt{2}\,(-2\alpha_1\beta_5+\alpha_2\beta_6+\alpha_3\beta_8)\\ 2\left(2\alpha_3\beta_1+\sqrt{3}\alpha_3\beta_2\right)+\sqrt{2}\,(\alpha_2\beta_7-3\alpha_1\beta_6)\\ 2\sqrt{2}\,(-\alpha_3\beta_3+\alpha_1\beta_7+2\alpha_2\beta_8)\\ \alpha_2\beta_1-3\sqrt{3}\alpha_2\beta_2+\sqrt{2}\,(3\alpha_3\beta_4+\alpha_1\beta_8)\\ \sqrt{3}\alpha_1\beta_2-5\alpha_1\beta_1+\sqrt{2}\,(3\alpha_2\beta_3-\alpha_3\beta_5)\end{pmatrix}$
$\mathbf{8}:\begin{pmatrix}-\alpha_3\beta_5-2\alpha_2\beta_7+\alpha_1\beta_8\\ \sqrt{3}\,(\alpha_3\beta_5+\alpha_1\beta_8)\\ \alpha_1\beta_1+\sqrt{3}\alpha_1\beta_2+\sqrt{2}\alpha_3\beta_6\\ -2\alpha_2\beta_1-\sqrt{2}\alpha_1\beta_3\\ -\sqrt{2}\,(\alpha_2\beta_3-\alpha_1\beta_4+\alpha_3\beta_8)\\ \sqrt{3}\alpha_3\beta_2-\alpha_3\beta_1+\sqrt{2}\alpha_2\beta_4\\ -\sqrt{2}\,(\alpha_3\beta_3+\alpha_2\beta_5+\alpha_1\beta_6)\\ \sqrt{2}\,(\alpha_3\beta_4+\alpha_2\beta_6+\alpha_1\beta_7)\end{pmatrix}$	$\mathbf{8}:\begin{pmatrix}\alpha_1\beta_3-2\alpha_2\beta_4-\alpha_3\beta_6\\ \sqrt{3}\,(\alpha_1\beta_3+\alpha_3\beta_6)\\ -\sqrt{2}\,(\alpha_1\beta_4+\alpha_2\beta_5+\alpha_3\beta_7)\\ \sqrt{2}\,(\alpha_1\beta_5+\alpha_2\beta_6+\alpha_3\beta_8)\\ \sqrt{3}\alpha_3\beta_2-\alpha_3\beta_1-\sqrt{2}\alpha_2\beta_7\\ \sqrt{2}\,(\alpha_3\beta_3-\alpha_1\beta_7+\alpha_2\beta_8)\\ \sqrt{2}\alpha_1\beta_8-2\alpha_2\beta_1\\ \alpha_1\beta_1+\sqrt{3}\alpha_1\beta_2-\sqrt{2}\alpha_3\beta_5\end{pmatrix}$

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$\mathbf{6} \otimes \mathbf{6} = \mathbf{1_S} \oplus \mathbf{6_{S,1}} \oplus \mathbf{6_{S,2}} \oplus \mathbf{7_A} \oplus \mathbf{8_S} \oplus \mathbf{8_A}$
$\mathbf{1_S} : \alpha_6\beta_1 + \alpha_5\beta_2 + \alpha_4\beta_3 + \alpha_3\beta_4 + \alpha_2\beta_5 + \alpha_1\beta_6$
$\mathbf{6_{S,1}} : \begin{pmatrix} \sqrt{2}\left(\alpha_5\beta_3 - \alpha_4\beta_4 + \alpha_3\beta_5\right) \\ \sqrt{2}\left(-\alpha_1\beta_1 + \alpha_6\beta_3 + \alpha_3\beta_6\right) \\ \alpha_2\beta_1 + \alpha_1\beta_2 - \sqrt{2}\left(\alpha_6\beta_4 + \alpha_4\beta_6\right) \\ \sqrt{2}\left(-\alpha_2\beta_2 + \alpha_6\beta_5 + \alpha_5\beta_6\right) \\ \alpha_4\beta_1 - \sqrt{2}\alpha_3\beta_2 - \sqrt{2}\alpha_2\beta_3 + \alpha_1\beta_4 \\ -\sqrt{2}\alpha_5\beta_1 + \alpha_4\beta_2 + \alpha_2\beta_4 - \sqrt{2}\alpha_1\beta_5 \end{pmatrix}$
$\mathbf{6_{S,2}} : \begin{pmatrix} \sqrt{2}\alpha_6\beta_2 - \alpha_5\beta_3 - \alpha_3\beta_5 + \sqrt{2}\alpha_2\beta_6 \\ -\alpha_6\beta_3 + \sqrt{2}\alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 - \alpha_3\beta_6 \\ -\sqrt{2}\left(\alpha_2\beta_1 + \alpha_1\beta_2 - \alpha_5\beta_5\right) \\ \sqrt{2}\alpha_3\beta_1 + \sqrt{2}\alpha_1\beta_3 - \alpha_6\beta_5 - \alpha_5\beta_6 \\ -\sqrt{2}\left(\alpha_4\beta_1 + \alpha_1\beta_4 - \alpha_6\beta_6\right) \\ -\sqrt{2}\left(\alpha_4\beta_2 - \alpha_3\beta_3 + \alpha_2\beta_4\right) \end{pmatrix}$
$\mathbf{7_A} : \begin{pmatrix} \alpha_6\beta_1 + \alpha_5\beta_2 - \alpha_4\beta_3 + \alpha_3\beta_4 - \alpha_2\beta_5 - \alpha_1\beta_6 \\ \sqrt{2}\alpha_6\beta_2 + \alpha_5\beta_3 - \alpha_3\beta_5 - \sqrt{2}\alpha_2\beta_6 \\ -\alpha_6\beta_3 + \sqrt{2}\alpha_5\beta_4 - \sqrt{2}\alpha_4\beta_5 + \alpha_3\beta_6 \\ \alpha_2\beta_1 - \alpha_1\beta_2 + \sqrt{2}\left(\alpha_6\beta_4 - \alpha_4\beta_6\right) \\ \sqrt{2}\alpha_3\beta_1 - \sqrt{2}\alpha_1\beta_3 + \alpha_6\beta_5 - \alpha_5\beta_6 \\ -\alpha_4\beta_1 + \sqrt{2}\alpha_3\beta_2 - \sqrt{2}\alpha_2\beta_3 + \alpha_1\beta_4 \\ \sqrt{2}\alpha_5\beta_1 + \alpha_4\beta_2 - \alpha_2\beta_4 - \sqrt{2}\alpha_1\beta_5 \end{pmatrix}$
$\mathbf{8_S} : \begin{pmatrix} \sqrt{3}\left(\alpha_6\beta_1 - 3\alpha_5\beta_2 + 2\alpha_4\beta_3 + 2\alpha_3\beta_4 - 3\alpha_2\beta_5 + \alpha_1\beta_6\right) \\ 5\alpha_6\beta_1 - \alpha_5\beta_2 - 4\alpha_4\beta_3 - 4\alpha_3\beta_4 - \alpha_2\beta_5 + 5\alpha_1\beta_6 \\ -\sqrt{3}\left(\sqrt{2}\alpha_6\beta_2 + 2\alpha_5\beta_3 + 4\alpha_4\beta_4 + 2\alpha_3\beta_5 + \sqrt{2}\alpha_2\beta_6\right) \\ -\sqrt{3}\left(4\alpha_1\beta_1 + 2\alpha_6\beta_3 + \sqrt{2}\alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 + 2\alpha_3\beta_6\right) \\ -\sqrt{3}\left(2\alpha_2\beta_1 + 2\alpha_1\beta_2 + \sqrt{2}\alpha_6\beta_4 + 4\alpha_5\beta_5 + \sqrt{2}\alpha_4\beta_6\right) \\ \sqrt{3}\left(\sqrt{2}\alpha_3\beta_1 + 4\alpha_2\beta_2 + \sqrt{2}\alpha_1\beta_3 + 2\alpha_6\beta_5 + 2\alpha_5\beta_6\right) \\ \sqrt{3}\left(2\alpha_4\beta_1 + \sqrt{2}\alpha_3\beta_2 + \sqrt{2}\alpha_2\beta_3 + 2\alpha_1\beta_4 + 4\alpha_6\beta_6\right) \\ \sqrt{3}\left(\sqrt{2}\alpha_5\beta_1 + 2\alpha_4\beta_2 + 4\alpha_3\beta_3 + 2\alpha_2\beta_4 + \sqrt{2}\alpha_1\beta_5\right) \end{pmatrix}$
$\mathbf{8_A} : \begin{pmatrix} -\alpha_6\beta_1 - \alpha_5\beta_2 - 2\alpha_4\beta_3 + 2\alpha_3\beta_4 + \alpha_2\beta_5 + \alpha_1\beta_6 \\ \sqrt{3}\left(\alpha_6\beta_1 - \alpha_5\beta_2 + \alpha_2\beta_5 - \alpha_1\beta_6\right) \\ \sqrt{2}\alpha_6\beta_2 - 2\alpha_5\beta_3 + 2\alpha_3\beta_5 - \sqrt{2}\alpha_2\beta_6 \\ 2\alpha_6\beta_3 + \sqrt{2}\alpha_5\beta_4 - \sqrt{2}\alpha_4\beta_5 - 2\alpha_3\beta_6 \\ -2\alpha_2\beta_1 + 2\alpha_1\beta_2 + \sqrt{2}\left(\alpha_6\beta_4 - \alpha_4\beta_6\right) \\ -\sqrt{2}\alpha_3\beta_1 + \sqrt{2}\alpha_1\beta_3 + 2\alpha_6\beta_5 - 2\alpha_5\beta_6 \\ -2\alpha_4\beta_1 - \sqrt{2}\alpha_3\beta_2 + \sqrt{2}\alpha_2\beta_3 + 2\alpha_1\beta_4 \\ -\sqrt{2}\alpha_5\beta_1 + 2\alpha_4\beta_2 - 2\alpha_2\beta_4 + \sqrt{2}\alpha_1\beta_5 \end{pmatrix}.$

$\mathbf{6} \otimes \mathbf{7} = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \mathbf{7_1} \oplus \mathbf{7_2} \oplus \mathbf{8_1} \oplus \mathbf{8_2}$
$\mathbf{3}: \begin{pmatrix} 2\alpha_1\beta_1 - 2\alpha_6\beta_3 + 3\alpha_5\beta_4 - 2\sqrt{2}\alpha_4\beta_5 - \alpha_3\beta_6 - \sqrt{2}\alpha_2\beta_7 \\ 2\alpha_2\beta_1 - 2\sqrt{2}\alpha_1\beta_2 - \alpha_6\beta_4 - 2\alpha_5\beta_5 - \sqrt{2}\alpha_4\beta_6 + 3\alpha_3\beta_7 \\ 2\alpha_4\beta_1 - 2\alpha_3\beta_2 - 2\sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_1\beta_4 + 3\alpha_6\beta_6 - \alpha_5\beta_7 \end{pmatrix}$
$\bar{\mathbf{3}}: \begin{pmatrix} 2\alpha_6\beta_1 - \sqrt{2}\alpha_5\beta_2 - \alpha_4\beta_3 - 2\sqrt{2}\alpha_3\beta_4 + 3\alpha_2\beta_5 - 2\alpha_1\beta_6 \\ 2\alpha_5\beta_1 + 3\alpha_4\beta_2 - \sqrt{2}\alpha_3\beta_3 - 2\alpha_2\beta_4 - \alpha_1\beta_5 - 2\sqrt{2}\alpha_6\beta_7 \\ 2\alpha_3\beta_1 - \alpha_2\beta_2 + 3\alpha_1\beta_3 - \sqrt{2}\alpha_6\beta_5 - 2\sqrt{2}\alpha_5\beta_6 - 2\alpha_4\beta_7 \end{pmatrix}$
$\mathbf{6}: \begin{pmatrix} \alpha_1\beta_1 + \alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 - \alpha_3\beta_6 + \sqrt{2}\alpha_2\beta_7 \\ \alpha_2\beta_1 + \sqrt{2}\alpha_1\beta_2 - \alpha_6\beta_4 + \sqrt{2}\alpha_4\beta_6 + \alpha_3\beta_7 \\ -\alpha_3\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_3 - \sqrt{2}\alpha_6\beta_5 - \sqrt{2}\alpha_5\beta_6 \\ \alpha_4\beta_1 + \sqrt{2}\alpha_2\beta_3 + \sqrt{2}\alpha_1\beta_4 + \alpha_6\beta_6 - \alpha_5\beta_7 \\ -\alpha_5\beta_1 - \alpha_4\beta_2 - \sqrt{2}\alpha_3\beta_3 + \alpha_1\beta_5 - \sqrt{2}\alpha_6\beta_7 \\ -\alpha_6\beta_1 - \sqrt{2}\alpha_5\beta_2 + \alpha_4\beta_3 - \sqrt{2}\alpha_3\beta_4 - \alpha_2\beta_5 \end{pmatrix}$
$\mathbf{7_1}: \begin{pmatrix} 2\left(\alpha_6\beta_2 + \alpha_5\beta_3 + \alpha_3\beta_5\right) \\ \sqrt{2}\alpha_5\beta_4 + 2\alpha_4\beta_5 + \sqrt{2}\alpha_3\beta_6 - 2\alpha_2\beta_7 \\ 2\alpha_1\beta_2 + \sqrt{2}\alpha_6\beta_4 - 2\alpha_4\beta_6 + \sqrt{2}\alpha_3\beta_7 \\ 2\alpha_3\beta_1 - \alpha_2\beta_2 - \alpha_1\beta_3 - \sqrt{2}\alpha_6\beta_5 + 2\alpha_4\beta_7 \\ 2\alpha_2\beta_3 - 2\alpha_1\beta_4 + \sqrt{2}\left(\alpha_6\beta_6 + \alpha_5\beta_7\right) \\ 2\alpha_5\beta_1 - \alpha_4\beta_2 - \sqrt{2}\alpha_3\beta_3 + 2\alpha_2\beta_4 - \alpha_1\beta_5 \\ 2\alpha_6\beta_1 - \sqrt{2}\alpha_5\beta_2 - \alpha_4\beta_3 - \alpha_2\beta_5 + 2\alpha_1\beta_6 \end{pmatrix}$
$\mathbf{7_2}: \begin{pmatrix} 2\left(\alpha_4\beta_4 + \alpha_2\beta_6 + \alpha_1\beta_7\right) \\ 2\alpha_1\beta_1 + 2\alpha_6\beta_3 - \alpha_5\beta_4 - \alpha_3\beta_6 - \sqrt{2}\alpha_2\beta_7 \\ 2\alpha_2\beta_1 - \alpha_6\beta_4 + 2\alpha_5\beta_5 - \sqrt{2}\alpha_4\beta_6 - \alpha_3\beta_7 \\ \sqrt{2}\alpha_2\beta_2 + \sqrt{2}\alpha_1\beta_3 - 2\alpha_6\beta_5 + 2\alpha_5\beta_6 \\ 2\alpha_4\beta_1 + 2\alpha_3\beta_2 - \sqrt{2}\alpha_1\beta_4 - \alpha_6\beta_6 - \alpha_5\beta_7 \\ \sqrt{2}\alpha_4\beta_2 - 2\alpha_3\beta_3 + \sqrt{2}\alpha_1\beta_5 + 2\alpha_6\beta_7 \\ -2\alpha_5\beta_2 + \sqrt{2}\alpha_4\beta_3 + 2\alpha_3\beta_4 + \sqrt{2}\alpha_2\beta_5 \end{pmatrix}$
$\mathbf{8_1}: \begin{pmatrix} 3\sqrt{2}\alpha_6\beta_2 - 3\sqrt{2}\alpha_5\beta_3 + 2\alpha_4\beta_4 - \alpha_2\beta_6 - \alpha_1\beta_7 \\ \sqrt{3}\left(\sqrt{2}\alpha_6\beta_2 + \sqrt{2}\alpha_5\beta_3 - 2\sqrt{2}\alpha_3\beta_5 - \alpha_2\beta_6 + \alpha_1\beta_7\right) \\ 4\alpha_1\beta_1 - 2\alpha_6\beta_3 - 2\alpha_5\beta_4 + 4\alpha_3\beta_6 + \sqrt{2}\alpha_2\beta_7 \\ 4\alpha_2\beta_1 + 4\alpha_6\beta_4 - 2\alpha_5\beta_5 + \sqrt{2}\alpha_4\beta_6 - 2\alpha_3\beta_7 \\ 2\sqrt{2}\alpha_2\beta_2 + 2\sqrt{2}\alpha_1\beta_3 + 2\alpha_6\beta_5 - 2\alpha_5\beta_6 + 3\sqrt{2}\alpha_4\beta_7 \\ -4\alpha_4\beta_1 + 2\alpha_3\beta_2 - \sqrt{2}\alpha_1\beta_4 + 2\alpha_6\beta_6 - 4\alpha_5\beta_7 \\ -2\sqrt{2}\alpha_4\beta_2 - 2\alpha_3\beta_3 - 3\sqrt{2}\alpha_2\beta_4 - 2\sqrt{2}\alpha_1\beta_5 + 2\alpha_6\beta_7 \\ -2\alpha_5\beta_2 - 2\sqrt{2}\alpha_4\beta_3 + 2\alpha_3\beta_4 - 2\sqrt{2}\alpha_2\beta_5 - 3\sqrt{2}\alpha_1\beta_6 \end{pmatrix}$
$\mathbf{8_2}: \begin{pmatrix} \alpha_6\beta_2 + \alpha_5\beta_3 - 2\alpha_3\beta_5 + 3\sqrt{2}\left(\alpha_2\beta_6 - \alpha_1\beta_7\right) \\ \sqrt{3}\left(-\alpha_6\beta_2 + \alpha_5\beta_3 + 2\sqrt{2}\alpha_4\beta_4 - \sqrt{2}\alpha_2\beta_6 - \sqrt{2}\alpha_1\beta_7\right) \\ \sqrt{2}\left(-3\alpha_6\beta_3 - 2\alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 - 2\alpha_3\beta_6 - \sqrt{2}\alpha_2\beta_7\right) \\ 2\alpha_1\beta_2 - \sqrt{2}\left(2\alpha_6\beta_4 + 3\alpha_5\beta_5 + \sqrt{2}\alpha_4\beta_6 + 2\alpha_3\beta_7\right) \\ -4\alpha_3\beta_1 - 4\alpha_2\beta_2 + 2\alpha_1\beta_3 - \sqrt{2}\alpha_6\beta_5 + 2\alpha_4\beta_7 \\ 3\sqrt{2}\alpha_3\beta_2 - 2\alpha_2\beta_3 + 2\alpha_1\beta_4 + 2\sqrt{2}\left(\alpha_6\beta_6 + \alpha_5\beta_7\right) \\ 4\alpha_5\beta_1 - 2\alpha_4\beta_2 + \sqrt{2}\alpha_3\beta_3 - 2\alpha_2\beta_4 + 4\alpha_1\beta_5 \\ 4\alpha_6\beta_1 + \sqrt{2}\alpha_5\beta_2 + 4\alpha_4\beta_3 - 2\alpha_2\beta_5 - 2\alpha_1\beta_6 \end{pmatrix}.$

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$\mathbf{6}\otimes\mathbf{8}=\mathbf{3}\oplus\bar{\mathbf{3}}\oplus\mathbf{6_1}\oplus\mathbf{6_2}\oplus\mathbf{7_1}\oplus\mathbf{7_2}\oplus\mathbf{8_1}\oplus\mathbf{8_2}$

$$\mathbf{3}:\begin{pmatrix}\alpha_1\beta_1-\sqrt{3}\alpha_1\beta_2-2\alpha_6\beta_4-\sqrt{2}\alpha_4\beta_6-2\alpha_3\beta_7+\sqrt{2}\alpha_2\beta_8\\ \alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2+\sqrt{2}\alpha_1\beta_3+2\alpha_6\beta_5+2\alpha_5\beta_6+\sqrt{2}\alpha_4\beta_7\\ -2\alpha_4\beta_1-2\alpha_3\beta_3+\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_1\beta_5-2\alpha_5\beta_8\end{pmatrix}$$

$$\bar{\mathbf{3}}:\begin{pmatrix}\alpha_6\beta_1-\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3+2\alpha_4\beta_4+\sqrt{2}\alpha_3\beta_5+2\alpha_1\beta_7\\ \alpha_5\beta_1+\sqrt{3}\alpha_5\beta_2-\sqrt{2}\alpha_3\beta_4-2\alpha_2\beta_5-2\alpha_1\beta_6-\sqrt{2}\alpha_6\beta_8\\ -2\alpha_3\beta_1+2\alpha_2\beta_3+\sqrt{2}\alpha_6\beta_6-\sqrt{2}\alpha_5\beta_7+2\alpha_4\beta_8\end{pmatrix}$$

$$\mathbf{6_1}:\begin{pmatrix}4\alpha_1\beta_2-2\sqrt{3}\alpha_6\beta_4-2\sqrt{3}\alpha_5\beta_5\\ -2\sqrt{3}\alpha_2\beta_1-2\alpha_2\beta_2+2\sqrt{3}\alpha_5\beta_6+2\sqrt{3}\alpha_3\beta_8\\ -2\alpha_3\beta_2-2\sqrt{3}\alpha_2\beta_3+\sqrt{6}\alpha_6\beta_6+\sqrt{6}\alpha_5\beta_7+2\sqrt{3}\alpha_4\beta_8\\ 2\sqrt{3}\alpha_4\beta_1-2\alpha_4\beta_2-2\sqrt{3}\alpha_3\beta_3+2\sqrt{3}\alpha_6\beta_7\\ -\sqrt{3}\alpha_5\beta_1+\alpha_5\beta_2-\sqrt{6}\alpha_3\beta_4-2\sqrt{3}\alpha_2\beta_5+2\sqrt{3}\alpha_1\beta_6+\sqrt{6}\alpha_6\beta_8\\ \sqrt{3}\alpha_6\beta_1+\alpha_6\beta_2-\sqrt{6}\alpha_5\beta_3-2\sqrt{3}\alpha_4\beta_4-\sqrt{6}\alpha_3\beta_5+2\sqrt{3}\alpha_1\beta_7\end{pmatrix}$$

$$\mathbf{6_2}:\begin{pmatrix}\alpha_1\beta_1-\sqrt{3}\alpha_1\beta_2+2\alpha_5\beta_5+\sqrt{2}\alpha_4\beta_6+2\alpha_3\beta_7+\sqrt{2}\alpha_2\beta_8\\ \alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2-\sqrt{2}\alpha_1\beta_3-2\alpha_6\beta_5+\sqrt{2}\alpha_4\beta_7-2\alpha_3\beta_8\\ 2\alpha_3\beta_1+2\alpha_2\beta_3-2\alpha_1\beta_4-\sqrt{2}\alpha_6\beta_6-\sqrt{2}\alpha_5\beta_7\\ -2\alpha_4\beta_1-\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_1\beta_5-2\alpha_6\beta_7+2\alpha_5\beta_8\\ -\alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2-2\alpha_4\beta_3+\sqrt{2}\alpha_3\beta_4-2\alpha_1\beta_6-\sqrt{2}\alpha_6\beta_8\\ -\alpha_6\beta_1+\sqrt{3}\alpha_6\beta_2+\sqrt{2}\alpha_5\beta_3+2\alpha_4\beta_4+\sqrt{2}\alpha_3\beta_5+2\alpha_2\beta_6\end{pmatrix}$$

$$\mathbf{7_1}:\begin{pmatrix}4\left(-\alpha_4\beta_5+\alpha_2\beta_7+\alpha_1\beta_8\right)\\ \alpha_1\beta_1-\sqrt{3}\alpha_1\beta_2+2\alpha_6\beta_4-4\alpha_5\beta_5+3\sqrt{2}\alpha_4\beta_6-2\alpha_3\beta_7+\sqrt{2}\alpha_2\beta_8\\ \alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2-3\sqrt{2}\alpha_1\beta_3+2\alpha_6\beta_5-2\alpha_5\beta_6+\sqrt{2}\alpha_4\beta_7+4\alpha_3\beta_8\\ 2\sqrt{6}\alpha_3\beta_2-2\sqrt{2}\alpha_2\beta_3-2\sqrt{2}\alpha_1\beta_4+2\alpha_6\beta_6-2\alpha_5\beta_7\\ -2\alpha_4\beta_1+2\alpha_3\beta_3-3\sqrt{2}\alpha_2\beta_4-\sqrt{2}\alpha_1\beta_5+4\alpha_6\beta_7-2\alpha_5\beta_8\\ 3\sqrt{2}\alpha_5\beta_1-\sqrt{6}\alpha_5\beta_2-2\sqrt{2}\alpha_4\beta_3-2\alpha_3\beta_4+2\sqrt{2}\alpha_1\beta_6-2\alpha_6\beta_8\\ -3\sqrt{2}\alpha_6\beta_1-\sqrt{6}\alpha_6\beta_2-2\alpha_5\beta_3-2\sqrt{2}\alpha_4\beta_4+2\alpha_3\beta_5+2\sqrt{2}\alpha_2\beta_6\end{pmatrix}$$

$$\mathbf{7_2}:\begin{pmatrix}-4\left(\alpha_6\beta_3+\alpha_5\beta_4-\alpha_3\beta_6\right)\\ -3\sqrt{2}\alpha_1\beta_1-\sqrt{6}\alpha_1\beta_2-2\sqrt{2}\alpha_5\beta_5-2\alpha_4\beta_6+2\sqrt{2}\alpha_3\beta_7+2\alpha_2\beta_8\\ 3\sqrt{2}\alpha_2\beta_1-\sqrt{6}\alpha_2\beta_2+2\alpha_1\beta_3-2\sqrt{2}\alpha_6\beta_5+2\alpha_4\beta_7+2\sqrt{2}\alpha_3\beta_8\\ -2\alpha_3\beta_1+2\alpha_2\beta_3-4\alpha_1\beta_4+\sqrt{2}\alpha_6\beta_6+3\sqrt{2}\alpha_5\beta_7-2\alpha_4\beta_8\\ 2\sqrt{6}\alpha_4\beta_2+2\alpha_2\beta_4-2\alpha_1\beta_5+2\sqrt{2}\alpha_6\beta_7+2\sqrt{2}\alpha_5\beta_8\\ \alpha_5\beta_1+\sqrt{3}\alpha_5\beta_2-4\alpha_4\beta_3-\sqrt{2}\alpha_3\beta_4+2\alpha_2\beta_5-2\alpha_1\beta_6+3\sqrt{2}\alpha_6\beta_8\\ \alpha_6\beta_1-\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3+2\alpha_4\beta_4-3\sqrt{2}\alpha_3\beta_5+4\alpha_2\beta_6-2\alpha_1\beta_7\end{pmatrix}$$

$$\mathbf{8_1}:\begin{pmatrix}\sqrt{2}\alpha_6\beta_3+\sqrt{2}\alpha_5\beta_4+3\alpha_4\beta_5+2\sqrt{2}\alpha_3\beta_6+3\alpha_1\beta_8\\ -\sqrt{6}\alpha_6\beta_3+\sqrt{6}\alpha_5\beta_4+\sqrt{3}\alpha_4\beta_5+2\sqrt{3}\alpha_2\beta_7-\sqrt{3}\alpha_1\beta_8\\ -3\alpha_1\beta_1+\sqrt{3}\alpha_1\beta_2+2\alpha_5\beta_5+\sqrt{2}\alpha_4\beta_6-2\alpha_3\beta_7+2\sqrt{2}\alpha_2\beta_8\\ -2\sqrt{3}\alpha_2\beta_2-\sqrt{2}\alpha_1\beta_3+2\alpha_6\beta_5+2\sqrt{2}\alpha_4\beta_7-2\alpha_3\beta_8\\ -2\sqrt{2}\alpha_3\beta_1-\sqrt{2}\alpha_2\beta_3-\sqrt{2}\alpha_1\beta_4-4\alpha_6\beta_6+\sqrt{2}\alpha_4\beta_8\\ -3\alpha_4\beta_1-\sqrt{3}\alpha_4\beta_2+\sqrt{2}\alpha_2\beta_4+2\sqrt{2}\alpha_1\beta_5+2\alpha_6\beta_7+2\alpha_5\beta_8\\ -\sqrt{2}\alpha_5\beta_1-\sqrt{6}\alpha_5\beta_2+\sqrt{2}\alpha_4\beta_3-4\alpha_3\beta_4+\sqrt{2}\alpha_2\beta_5-\sqrt{2}\alpha_1\beta_6\\ -\sqrt{2}\alpha_6\beta_1+\sqrt{6}\alpha_6\beta_2-4\alpha_5\beta_3+\sqrt{2}\alpha_4\beta_4-\sqrt{2}\alpha_2\beta_6-\sqrt{2}\alpha_1\beta_7\end{pmatrix}$$

$$\mathbf{8_2}:\begin{pmatrix}3\alpha_6\beta_3+2\sqrt{2}\alpha_4\beta_5+3\alpha_3\beta_6+\sqrt{2}\alpha_2\beta_7+\sqrt{2}\alpha_1\beta_8\\ -\sqrt{3}\alpha_6\beta_3+2\sqrt{3}\alpha_5\beta_4+\sqrt{3}\alpha_3\beta_6+\sqrt{6}\alpha_2\beta_7-\sqrt{6}\alpha_1\beta_8\\ -\sqrt{2}\alpha_1\beta_1+\sqrt{6}\alpha_1\beta_2+\sqrt{2}\alpha_6\beta_4+\sqrt{2}\alpha_5\beta_5-\sqrt{2}\alpha_3\beta_7+4\alpha_2\beta_8\\ -\sqrt{2}\alpha_2\beta_1-\sqrt{6}\alpha_2\beta_2+\sqrt{2}\alpha_6\beta_5-\sqrt{2}\alpha_5\beta_6+4\alpha_4\beta_7-\sqrt{2}\alpha_3\beta_8\\ -3\alpha_3\beta_1-\sqrt{3}\alpha_3\beta_2-2\alpha_2\beta_3-2\alpha_1\beta_4-2\sqrt{2}\alpha_6\beta_6-\sqrt{2}\alpha_5\beta_7\\ -2\sqrt{2}\alpha_4\beta_1-\sqrt{2}\alpha_3\beta_3+4\alpha_1\beta_5+\sqrt{2}\alpha_6\beta_7+\sqrt{2}\alpha_5\beta_8\\ -2\sqrt{3}\alpha_5\beta_2+2\alpha_4\beta_3-2\sqrt{2}\alpha_3\beta_4-2\alpha_1\beta_6+\sqrt{2}\alpha_6\beta_8\\ -3\alpha_6\beta_1+\sqrt{3}\alpha_6\beta_2-2\sqrt{2}\alpha_5\beta_3+2\alpha_4\beta_4-\sqrt{2}\alpha_3\beta_5-2\alpha_2\beta_6\end{pmatrix}.$$

$$\mathbf{7} \otimes \mathbf{7} = \mathbf{1_S} \oplus \mathbf{3_A} \oplus \mathbf{\bar{3}_A} \oplus \mathbf{6_{S,1}} \oplus \mathbf{6_{S,2}} \oplus \mathbf{7_S} \oplus \mathbf{7_A} \oplus \mathbf{8_S} \oplus \mathbf{8_A}$$

$$\mathbf{1_S} : \alpha_1\beta_1 + \alpha_7\beta_2 + \alpha_6\beta_3 + \alpha_5\beta_4 + \alpha_4\beta_5 + \alpha_3\beta_6 + \alpha_2\beta_7$$

$$\mathbf{3_A} : \begin{pmatrix} -\sqrt{2}\alpha_2\beta_1 + \sqrt{2}\alpha_1\beta_2 + 2\alpha_7\beta_3 + \alpha_6\beta_4 - \alpha_4\beta_6 - 2\alpha_3\beta_7 \\ -\sqrt{2}\alpha_3\beta_1 + \sqrt{2}\alpha_1\beta_3 - \alpha_7\beta_4 + 2\alpha_6\beta_5 - 2\alpha_5\beta_6 + \alpha_4\beta_7 \\ -\sqrt{2}\alpha_5\beta_1 + 2\alpha_4\beta_2 - 2\alpha_2\beta_4 + \sqrt{2}\alpha_1\beta_5 + \alpha_7\beta_6 - \alpha_6\beta_7 \end{pmatrix}$$

$$\mathbf{\bar{3}_A} : \begin{pmatrix} -\sqrt{2}\alpha_7\beta_1 - 2\alpha_6\beta_2 - \alpha_5\beta_3 + \alpha_3\beta_5 + 2\alpha_2\beta_6 + \sqrt{2}\alpha_1\beta_7 \\ -\sqrt{2}\alpha_6\beta_1 + \alpha_5\beta_2 - 2\alpha_4\beta_3 + 2\alpha_3\beta_4 - \alpha_2\beta_5 + \sqrt{2}\alpha_1\beta_6 \\ -\sqrt{2}\alpha_4\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 + \sqrt{2}\alpha_1\beta_4 - 2\alpha_7\beta_5 + 2\alpha_5\beta_7 \end{pmatrix}$$

$$\mathbf{6_{S,1}} : \begin{pmatrix} 2\alpha_2\beta_1 + 2\alpha_1\beta_2 + \sqrt{2}\left(\alpha_6\beta_4 - \alpha_5\beta_5 + \alpha_4\beta_6\right) \\ 2\alpha_3\beta_1 - \sqrt{2}\alpha_2\beta_2 + 2\alpha_1\beta_3 + \sqrt{2}\alpha_7\beta_4 + \sqrt{2}\alpha_4\beta_7 \\ -\alpha_3\beta_2 - \alpha_2\beta_3 + 2\alpha_7\beta_5 - 2\alpha_6\beta_6 + 2\alpha_5\beta_7 \\ 2\alpha_5\beta_1 - \sqrt{2}\alpha_3\beta_3 + 2\alpha_1\beta_5 + \sqrt{2}\alpha_7\beta_6 + \sqrt{2}\alpha_6\beta_7 \\ -\alpha_5\beta_2 + 2\alpha_4\beta_3 + 2\alpha_3\beta_4 - \alpha_2\beta_5 - 2\alpha_7\beta_7 \\ 2\alpha_6\beta_2 - \alpha_5\beta_3 - 2\alpha_4\beta_4 - \alpha_3\beta_5 + 2\alpha_2\beta_6 \end{pmatrix}$$

$$\mathbf{6_{S,2}} : \begin{pmatrix} 2\alpha_7\beta_3 - \alpha_6\beta_4 - 2\alpha_5\beta_5 - \alpha_4\beta_6 + 2\alpha_3\beta_7 \\ -2\alpha_2\beta_2 - \alpha_7\beta_4 + 2\alpha_6\beta_5 + 2\alpha_5\beta_6 - \alpha_4\beta_7 \\ 2\alpha_4\beta_1 + \sqrt{2}\alpha_3\beta_2 + \sqrt{2}\alpha_2\beta_3 + 2\alpha_1\beta_4 - \sqrt{2}\alpha_6\beta_6 \\ 2\alpha_4\beta_2 - 2\alpha_3\beta_3 + 2\alpha_2\beta_4 - \alpha_7\beta_6 - \alpha_6\beta_7 \\ 2\alpha_6\beta_1 + \sqrt{2}\alpha_5\beta_2 + \sqrt{2}\alpha_2\beta_5 + 2\alpha_1\beta_6 - \sqrt{2}\alpha_7\beta_7 \\ 2\alpha_7\beta_1 + \sqrt{2}\alpha_5\beta_3 - \sqrt{2}\alpha_4\beta_4 + \sqrt{2}\alpha_3\beta_5 + 2\alpha_1\beta_7 \end{pmatrix}$$

$$\mathbf{7_S} : \begin{pmatrix} 6\alpha_1\beta_1 - \alpha_7\beta_2 - \alpha_6\beta_3 - \alpha_5\beta_4 - \alpha_4\beta_5 - \alpha_3\beta_6 - \alpha_2\beta_7 \\ -\alpha_2\beta_1 - \alpha_1\beta_2 + 2\sqrt{2}\alpha_7\beta_3 + 2\sqrt{2}\alpha_6\beta_4 + 2\sqrt{2}\alpha_5\beta_5 + 2\sqrt{2}\alpha_4\beta_6 + 2\sqrt{2}\alpha_3\beta_7 \\ -\alpha_3\beta_1 + 2\sqrt{2}\alpha_2\beta_2 - \alpha_1\beta_3 + 2\sqrt{2}\alpha_7\beta_4 + 2\sqrt{2}\alpha_6\beta_5 + 2\sqrt{2}\alpha_5\beta_6 + 2\sqrt{2}\alpha_4\beta_7 \\ -\alpha_4\beta_1 + 2\sqrt{2}\alpha_3\beta_2 + 2\sqrt{2}\alpha_2\beta_3 - \alpha_1\beta_4 + 2\sqrt{2}\alpha_7\beta_5 + 2\sqrt{2}\alpha_6\beta_6 + 2\sqrt{2}\alpha_5\beta_7 \\ -\alpha_5\beta_1 + 2\sqrt{2}\alpha_4\beta_2 + 2\sqrt{2}\alpha_3\beta_3 + 2\sqrt{2}\alpha_2\beta_4 - \alpha_1\beta_5 + 2\sqrt{2}\alpha_7\beta_6 + 2\sqrt{2}\alpha_6\beta_7 \\ -\alpha_6\beta_1 + 2\sqrt{2}\alpha_5\beta_2 + 2\sqrt{2}\alpha_4\beta_3 + 2\sqrt{2}\alpha_3\beta_4 + 2\sqrt{2}\alpha_2\beta_5 - \alpha_1\beta_6 + 2\sqrt{2}\alpha_7\beta_7 \\ -\alpha_7\beta_1 + 2\sqrt{2}\alpha_6\beta_2 + 2\sqrt{2}\alpha_5\beta_3 + 2\sqrt{2}\alpha_4\beta_4 + 2\sqrt{2}\alpha_3\beta_5 + 2\sqrt{2}\alpha_2\beta_6 - \alpha_1\beta_7 \end{pmatrix}$$

$$\mathbf{7_A} : \begin{pmatrix} -\alpha_7\beta_2 - \alpha_6\beta_3 + \alpha_5\beta_4 - \alpha_4\beta_5 + \alpha_3\beta_6 + \alpha_2\beta_7 \\ -\alpha_2\beta_1 + \alpha_1\beta_2 - \sqrt{2}\alpha_6\beta_4 + \sqrt{2}\alpha_4\beta_6 \\ -\alpha_3\beta_1 + \alpha_1\beta_3 + \sqrt{2}\alpha_7\beta_4 - \sqrt{2}\alpha_4\beta_7 \\ \alpha_4\beta_1 - \sqrt{2}\alpha_3\beta_2 + \sqrt{2}\alpha_2\beta_3 - \alpha_1\beta_4 \\ -\alpha_5\beta_1 + \alpha_1\beta_5 - \sqrt{2}\alpha_7\beta_6 + \sqrt{2}\alpha_6\beta_7 \\ \alpha_6\beta_1 + \sqrt{2}\alpha_5\beta_2 - \sqrt{2}\alpha_2\beta_5 - \alpha_1\beta_6 \\ \alpha_7\beta_1 - \sqrt{2}\alpha_5\beta_3 + \sqrt{2}\alpha_3\beta_5 - \alpha_1\beta_7 \end{pmatrix}$$

$$\mathbf{8_S} : \begin{pmatrix} 5\alpha_7\beta_2 - \alpha_6\beta_3 - 4\alpha_5\beta_4 - 4\alpha_4\beta_5 - \alpha_3\beta_6 + 5\alpha_2\beta_7 \\ -\sqrt{3}\left(\alpha_7\beta_2 - 3\alpha_6\beta_3 + 2\alpha_5\beta_4 + 2\alpha_4\beta_5 - 3\alpha_3\beta_6 + \alpha_2\beta_7\right) \\ 4\alpha_2\beta_1 + 4\alpha_1\beta_2 + \sqrt{2}\left(\alpha_7\beta_3 - 2\alpha_6\beta_4 + 4\alpha_5\beta_5 - 2\alpha_4\beta_6 + \alpha_3\beta_7\right) \\ 4\alpha_3\beta_1 + 4\sqrt{2}\alpha_2\beta_2 + 4\alpha_1\beta_3 - 2\sqrt{2}\alpha_7\beta_4 + \sqrt{2}\alpha_6\beta_5 + \sqrt{2}\alpha_5\beta_6 - 2\sqrt{2}\alpha_4\beta_7 \\ 4\alpha_4\beta_1 - 2\sqrt{2}\alpha_3\beta_2 - 2\sqrt{2}\alpha_2\beta_3 + 4\alpha_1\beta_4 + \sqrt{2}\alpha_7\beta_5 + 4\sqrt{2}\alpha_6\beta_6 + \sqrt{2}\alpha_5\beta_7 \\ -4\alpha_5\beta_1 - \sqrt{2}\alpha_4\beta_2 - 4\sqrt{2}\alpha_3\beta_3 - \sqrt{2}\alpha_2\beta_4 - 4\alpha_1\beta_5 + 2\sqrt{2}\alpha_7\beta_6 + 2\sqrt{2}\alpha_6\beta_7 \\ -4\alpha_6\beta_1 + 2\sqrt{2}\alpha_5\beta_2 - \sqrt{2}\alpha_4\beta_3 - \sqrt{2}\alpha_3\beta_4 + 2\sqrt{2}\alpha_2\beta_5 - 4\alpha_1\beta_6 - 4\sqrt{2}\alpha_7\beta_7 \\ -4\alpha_7\beta_1 - \sqrt{2}\alpha_6\beta_2 + 2\sqrt{2}\alpha_5\beta_3 - 4\sqrt{2}\alpha_4\beta_4 + 2\sqrt{2}\alpha_3\beta_5 - \sqrt{2}\alpha_2\beta_6 - 4\alpha_1\beta_7 \end{pmatrix}$$

$$\mathbf{8_A} : \begin{pmatrix} \alpha_7\beta_2 - 5\alpha_6\beta_3 - 4\alpha_5\beta_4 + 4\alpha_4\beta_5 + 5\alpha_3\beta_6 - \alpha_2\beta_7 \\ \sqrt{3}\left(3\alpha_7\beta_2 - \alpha_6\beta_3 + 2\alpha_5\beta_4 - 2\alpha_4\beta_5 + \alpha_3\beta_6 - 3\alpha_2\beta_7\right) \\ -4\alpha_2\beta_1 + 4\alpha_1\beta_2 + \sqrt{2}\left(-3\alpha_7\beta_3 + 2\alpha_6\beta_4 - 2\alpha_4\beta_6 + 3\alpha_3\beta_7\right) \\ -4\alpha_3\beta_1 + 4\alpha_1\beta_3 + \sqrt{2}\left(-2\alpha_7\beta_4 - 3\alpha_6\beta_5 + 3\alpha_5\beta_6 + 2\alpha_4\beta_7\right) \\ 4\alpha_4\beta_1 + 2\sqrt{2}\alpha_3\beta_2 - 2\sqrt{2}\alpha_2\beta_3 - 4\alpha_1\beta_4 + 3\sqrt{2}\left(\alpha_5\beta_7 - \alpha_7\beta_5\right) \\ 4\alpha_5\beta_1 + 3\sqrt{2}\alpha_4\beta_2 - 3\sqrt{2}\alpha_2\beta_4 - 4\alpha_1\beta_5 + 2\sqrt{2}\left(\alpha_6\beta_7 - \alpha_7\beta_6\right) \\ -4\alpha_6\beta_1 + 2\sqrt{2}\alpha_5\beta_2 + 3\sqrt{2}\alpha_4\beta_3 - 3\sqrt{2}\alpha_3\beta_4 - 2\sqrt{2}\alpha_2\beta_5 + 4\alpha_1\beta_6 \\ -4\alpha_7\beta_1 + 3\sqrt{2}\alpha_6\beta_2 - 2\sqrt{2}\alpha_5\beta_3 + 2\sqrt{2}\alpha_3\beta_5 - 3\sqrt{2}\alpha_2\beta_6 + 4\alpha_1\beta_7 \end{pmatrix}.$$

$\mathbf{7} \otimes \mathbf{8} = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{6_1} \oplus \mathbf{6_2} \oplus \mathbf{7_1} \oplus \mathbf{7_2} \oplus \mathbf{8_1} \oplus \mathbf{8_2} \oplus \mathbf{8_3}$
$\mathbf{3}: \begin{pmatrix} -5\alpha_2\beta_1 + \sqrt{3}\alpha_2\beta_2 + 4\alpha_1\beta_3 - 2\sqrt{2}\alpha_7\beta_4 - 4\sqrt{2}\alpha_6\beta_5 - 3\sqrt{2}\alpha_5\beta_6 + 2\sqrt{2}\alpha_4\beta_7 + \sqrt{2}\alpha_3\beta_8 \\ \alpha_3\beta_1 - 3\sqrt{3}\alpha_3\beta_2 + 3\sqrt{2}\alpha_2\beta_3 + 4\alpha_1\beta_4 - 2\sqrt{2}\alpha_7\beta_5 + 2\sqrt{2}\alpha_6\beta_6 + \sqrt{2}\alpha_5\beta_7 + 4\sqrt{2}\alpha_4\beta_8 \\ 4\alpha_5\beta_1 + 2\sqrt{3}\alpha_5\beta_2 - 2\sqrt{2}\alpha_4\beta_3 + 3\sqrt{2}\alpha_3\beta_4 - \sqrt{2}\alpha_2\beta_5 - 4\alpha_1\beta_6 + 4\sqrt{2}\alpha_7\beta_7 + 2\sqrt{2}\alpha_6\beta_8 \end{pmatrix}$
$\bar{\mathbf{3}}: \begin{pmatrix} 5\alpha_7\beta_1 - \sqrt{3}\alpha_7\beta_2 + \sqrt{2}\alpha_6\beta_3 + 2\sqrt{2}\alpha_5\beta_4 - 3\sqrt{2}\alpha_4\beta_5 - 4\sqrt{2}\alpha_3\beta_6 - 2\sqrt{2}\alpha_2\beta_7 + 4\alpha_1\beta_8 \\ -\alpha_6\beta_1 + 3\sqrt{3}\alpha_6\beta_2 + 4\sqrt{2}\alpha_5\beta_3 + \sqrt{2}\alpha_4\beta_4 + 2\sqrt{2}\alpha_3\beta_5 - 2\sqrt{2}\alpha_2\beta_6 + 4\alpha_1\beta_7 + 3\sqrt{2}\alpha_7\beta_8 \\ -4\alpha_4\beta_1 - 2\sqrt{3}\alpha_4\beta_2 + 2\sqrt{2}\alpha_3\beta_3 + 4\sqrt{2}\alpha_2\beta_4 - 4\alpha_1\beta_5 - \sqrt{2}\alpha_7\beta_6 + 3\sqrt{2}\alpha_6\beta_7 - 2\sqrt{2}\alpha_5\beta_8 \end{pmatrix}$
$\mathbf{6_1}: \begin{pmatrix} -4\sqrt{2}\alpha_2\beta_1 + 4\sqrt{2}\alpha_1\beta_3 + 2\alpha_7\beta_4 + 6\alpha_6\beta_5 - 8\alpha_3\beta_8 \\ 2\sqrt{2}\alpha_3\beta_1 - 2\sqrt{6}\alpha_3\beta_2 + 4\sqrt{2}\alpha_1\beta_4 - 2\alpha_6\beta_6 - 8\alpha_5\beta_7 - 6\alpha_4\beta_8 \\ -2\alpha_4\beta_1 - 4\sqrt{3}\alpha_4\beta_2 + 2\sqrt{2}\alpha_3\beta_3 - 4\sqrt{2}\alpha_2\beta_4 - 4\alpha_1\beta_5 + 5\sqrt{2}\alpha_7\beta_6 - \sqrt{2}\alpha_6\beta_7 + 2\sqrt{2}\alpha_5\beta_8 \\ 2\sqrt{2}\alpha_5\beta_1 + 2\sqrt{6}\alpha_5\beta_2 + 2\alpha_4\beta_3 + 8\alpha_2\beta_5 - 4\sqrt{2}\alpha_1\beta_6 - 6\alpha_7\beta_7 \\ -5\alpha_6\beta_1 + 3\sqrt{3}\alpha_6\beta_2 - 4\sqrt{2}\alpha_5\beta_3 - 5\sqrt{2}\alpha_4\beta_4 - 2\sqrt{2}\alpha_3\beta_5 - 2\sqrt{2}\alpha_2\beta_6 + 4\alpha_1\beta_7 - \sqrt{2}\alpha_7\beta_8 \\ 7\alpha_7\beta_1 + \sqrt{3}\alpha_7\beta_2 - 5\sqrt{2}\alpha_6\beta_3 + 2\sqrt{2}\alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 + 4\sqrt{2}\alpha_3\beta_6 + 2\sqrt{2}\alpha_2\beta_7 + 4\alpha_1\beta_8 \end{pmatrix}$
$\mathbf{6_2}: \begin{pmatrix} \sqrt{6}\left(\sqrt{3}\alpha_2\beta_1 + \alpha_2\beta_2\right) - 2\left(\alpha_7\beta_4 + \alpha_6\beta_5 + \alpha_5\beta_6 + 2\alpha_4\beta_7 - \alpha_3\beta_8\right) \\ \sqrt{6}\left(\alpha_3\beta_2 - \sqrt{3}\alpha_3\beta_1\right) + 2\left(\alpha_2\beta_3 + 2\alpha_7\beta_5 + \alpha_6\beta_6 + \alpha_5\beta_7 + \alpha_4\beta_8\right) \\ 2\alpha_4\beta_1 + \sqrt{2}\left(2\alpha_3\beta_3 + 2\alpha_2\beta_4 + 2\sqrt{2}\alpha_1\beta_5 - 3\alpha_7\beta_6 - \alpha_6\beta_7\right) \\ -2\left(\sqrt{6}\alpha_5\beta_2 + \alpha_4\beta_3 - \alpha_3\beta_4 + \alpha_2\beta_5 - \alpha_7\beta_7 + 2\alpha_6\beta_8\right) \\ -\left(\alpha_6\beta_1 + \sqrt{3}\alpha_6\beta_2\right) - \sqrt{2}\left(-2\alpha_5\beta_3 - 3\alpha_4\beta_4 + 2\alpha_2\beta_6 + 2\sqrt{2}\alpha_1\beta_7 + \alpha_7\beta_8\right) \\ \sqrt{3}\alpha_7\beta_2 - \alpha_7\beta_1 + \sqrt{2}\left(3\alpha_6\beta_3 + 2\alpha_5\beta_4 + \alpha_4\beta_5 - 2\alpha_3\beta_6 - 2\sqrt{2}\alpha_1\beta_8\right) \end{pmatrix}$
$\mathbf{7_1}: \begin{pmatrix} 4\left(-\alpha_5\beta_5 + \alpha_3\beta_7 + \alpha_2\beta_8\right) \\ -3\alpha_2\beta_1 - \sqrt{3}\alpha_2\beta_2 - 4\alpha_1\beta_3 - 2\sqrt{2}\alpha_7\beta_4 - \sqrt{2}\alpha_5\beta_6 - 2\sqrt{2}\alpha_4\beta_7 - \sqrt{2}\alpha_3\beta_8 \\ 3\alpha_3\beta_1 - \sqrt{3}\alpha_3\beta_2 + \sqrt{2}\alpha_2\beta_3 - 4\alpha_1\beta_4 + 2\sqrt{2}\alpha_7\beta_5 + 2\sqrt{2}\alpha_6\beta_6 - \sqrt{2}\alpha_5\beta_7 \\ 4\alpha_4\beta_1 + 2\sqrt{2}\alpha_2\beta_4 + 2\sqrt{2}\alpha_7\beta_6 + 2\sqrt{2}\alpha_6\beta_7 + 2\sqrt{2}\alpha_5\beta_8 \\ 2\sqrt{3}\alpha_5\beta_2 - 2\sqrt{2}\alpha_4\beta_3 + \sqrt{2}\alpha_3\beta_4 + \sqrt{2}\alpha_2\beta_5 + 4\alpha_1\beta_6 - 2\sqrt{2}\alpha_6\beta_8 \\ -2\alpha_6\beta_1 - 2\sqrt{3}\alpha_6\beta_2 + 2\sqrt{2}\alpha_5\beta_3 - 2\sqrt{2}\alpha_4\beta_4 - 2\sqrt{2}\alpha_3\beta_5 + 2\sqrt{2}\alpha_7\beta_8 \\ -2\alpha_7\beta_1 + 2\sqrt{3}\alpha_7\beta_2 - 2\sqrt{2}\alpha_6\beta_3 - 2\sqrt{2}\alpha_4\beta_5 - 2\sqrt{2}\alpha_3\beta_6 + 2\sqrt{2}\alpha_2\beta_7 \end{pmatrix}$
$\mathbf{7_2}: \begin{pmatrix} -4\left(\alpha_7\beta_3 + \alpha_6\beta_4 - \alpha_4\beta_6\right) \\ -2\alpha_2\beta_1 + 2\sqrt{3}\alpha_2\beta_2 - 2\sqrt{2}\alpha_7\beta_4 + 2\sqrt{2}\alpha_6\beta_5 + 2\sqrt{2}\alpha_5\beta_6 + 2\sqrt{2}\alpha_3\beta_8 \\ -2\alpha_3\beta_1 - 2\sqrt{3}\alpha_3\beta_2 - 2\sqrt{2}\alpha_2\beta_3 + 2\sqrt{2}\alpha_6\beta_6 + 2\sqrt{2}\alpha_5\beta_7 - 2\sqrt{2}\alpha_4\beta_8 \\ 2\sqrt{3}\alpha_4\beta_2 + 2\sqrt{2}\alpha_3\beta_3 - 4\alpha_1\beta_5 - \sqrt{2}\alpha_7\beta_6 - \sqrt{2}\alpha_6\beta_7 + 2\sqrt{2}\alpha_5\beta_8 \\ 4\alpha_5\beta_1 - 2\sqrt{2}\alpha_4\beta_3 - 2\sqrt{2}\alpha_3\beta_4 - 2\sqrt{2}\alpha_2\beta_5 - 2\sqrt{2}\alpha_7\beta_7 \\ 3\alpha_6\beta_1 - \sqrt{3}\alpha_6\beta_2 + \sqrt{2}\alpha_4\beta_4 - 2\sqrt{2}\alpha_3\beta_5 - 2\sqrt{2}\alpha_2\beta_6 + 4\alpha_1\beta_7 - \sqrt{2}\alpha_7\beta_8 \\ -3\alpha_7\beta_1 - \sqrt{3}\alpha_7\beta_2 + \sqrt{2}\alpha_6\beta_3 + 2\sqrt{2}\alpha_5\beta_4 + \sqrt{2}\alpha_4\beta_5 + 2\sqrt{2}\alpha_2\beta_7 + 4\alpha_1\beta_8 \end{pmatrix}$
$\mathbf{8_1}: \begin{pmatrix} 3\sqrt{3}\alpha_1\beta_1 + 4\alpha_1\beta_2 - 2\sqrt{3}\alpha_6\beta_4 + 2\sqrt{3}\alpha_5\beta_5 - 2\sqrt{3}\alpha_4\beta_6 + 2\sqrt{3}\alpha_3\beta_7 \\ -4\alpha_1\beta_1 + 3\sqrt{3}\alpha_1\beta_2 + 4\alpha_7\beta_3 - 2\alpha_6\beta_4 - 2\alpha_5\beta_5 + 2\alpha_4\beta_6 + 2\alpha_3\beta_7 - 4\alpha_2\beta_8 \\ 2\sqrt{3}\alpha_2\beta_1 + 2\alpha_2\beta_2 - \sqrt{3}\alpha_1\beta_3 - 2\sqrt{6}\alpha_7\beta_4 + 2\sqrt{6}\alpha_4\beta_7 - 2\sqrt{6}\alpha_3\beta_8 \\ -2\sqrt{3}\alpha_3\beta_1 + 2\alpha_3\beta_2 - \sqrt{3}\alpha_1\beta_4 - 2\sqrt{6}\alpha_7\beta_5 + 2\sqrt{6}\alpha_6\beta_6 - 2\sqrt{6}\alpha_5\beta_7 \\ -4\alpha_4\beta_2 - 2\sqrt{6}\alpha_3\beta_3 - \sqrt{3}\alpha_1\beta_5 - 2\sqrt{6}\alpha_7\beta_6 + 2\sqrt{6}\alpha_5\beta_8 \\ 4\alpha_5\beta_2 + 2\sqrt{6}\alpha_4\beta_3 - 2\sqrt{6}\alpha_2\beta_5 - \sqrt{3}\alpha_1\beta_6 - 2\sqrt{6}\alpha_6\beta_8 \\ 2\sqrt{3}\alpha_6\beta_1 - 2\alpha_6\beta_2 - 2\sqrt{6}\alpha_4\beta_4 + 2\sqrt{6}\alpha_3\beta_5 - 2\sqrt{6}\alpha_2\beta_6 - \sqrt{3}\alpha_1\beta_7 \\ -2\sqrt{3}\alpha_7\beta_1 - 2\alpha_7\beta_2 - 2\sqrt{6}\alpha_6\beta_3 + 2\sqrt{6}\alpha_5\beta_4 - 2\sqrt{6}\alpha_2\beta_7 - \sqrt{3}\alpha_1\beta_8 \end{pmatrix}$

$$\mathbf{8_2}:\begin{pmatrix} 2\sqrt{3}\alpha_1\beta_2-2\alpha_7\beta_3-2\alpha_6\beta_4-\alpha_5\beta_5-4\alpha_4\beta_6-2\alpha_3\beta_7+\alpha_2\beta_8 \\ -2\sqrt{3}\alpha_1\beta_1+2\sqrt{3}\alpha_7\beta_3-2\sqrt{3}\alpha_6\beta_4+\sqrt{3}\alpha_5\beta_5+\sqrt{3}\alpha_2\beta_8 \\ \alpha_2\beta_1+\sqrt{3}\alpha_2\beta_2-2\alpha_1\beta_3+2\sqrt{2}\alpha_6\beta_5-3\sqrt{2}\alpha_5\beta_6+2\sqrt{2}\alpha_4\beta_7 \\ -2\alpha_3\beta_1+3\sqrt{2}\alpha_2\beta_3-2\alpha_1\beta_4-2\sqrt{2}\alpha_7\beta_5-2\sqrt{2}\alpha_4\beta_8 \\ -4\alpha_4\beta_1-\sqrt{2}\alpha_3\beta_3+\sqrt{2}\alpha_2\beta_4+2\alpha_1\beta_5+3\sqrt{2}\alpha_5\beta_8 \\ -\alpha_5\beta_1+\sqrt{3}\alpha_5\beta_2-3\sqrt{2}\alpha_3\beta_4-2\alpha_1\beta_6+2\sqrt{2}\alpha_7\beta_7-2\sqrt{2}\alpha_6\beta_8 \\ -2\alpha_6\beta_1-2\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3+3\sqrt{2}\alpha_3\beta_5-\sqrt{2}\alpha_2\beta_6+2\alpha_1\beta_7 \\ -2\alpha_7\beta_1+2\sqrt{3}\alpha_7\beta_2+\sqrt{2}\alpha_5\beta_4+\sqrt{2}\alpha_3\beta_6-3\sqrt{2}\alpha_2\beta_7+2\alpha_1\beta_8 \end{pmatrix}$$

$$\mathbf{8_3}:\begin{pmatrix} \alpha_7\beta_3+4\alpha_6\beta_4+5\alpha_5\beta_5+5\alpha_4\beta_6+4\alpha_3\beta_7+\alpha_2\beta_8 \\ -3\sqrt{3}\alpha_7\beta_3+2\sqrt{3}\alpha_6\beta_4-\sqrt{3}\alpha_5\beta_5-\sqrt{3}\alpha_4\beta_6+2\sqrt{3}\alpha_3\beta_7-3\sqrt{3}\alpha_2\beta_8 \\ \alpha_2\beta_1-3\sqrt{3}\alpha_2\beta_2+4\alpha_1\beta_3-3\sqrt{2}\alpha_7\beta_4-\sqrt{2}\alpha_6\beta_5+3\sqrt{2}\alpha_5\beta_6-\sqrt{2}\alpha_4\beta_7 \\ 4\alpha_3\beta_1+2\sqrt{3}\alpha_3\beta_2-3\sqrt{2}\alpha_2\beta_3+4\alpha_1\beta_4+\sqrt{2}\alpha_7\beta_5+3\sqrt{2}\alpha_6\beta_6+\sqrt{2}\alpha_4\beta_8 \\ 5\alpha_4\beta_1-\sqrt{3}\alpha_4\beta_2-\sqrt{2}\alpha_3\beta_3+\sqrt{2}\alpha_2\beta_4-4\alpha_1\beta_5-3\sqrt{2}\alpha_6\beta_7-3\sqrt{2}\alpha_5\beta_8 \\ 5\alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2+3\sqrt{2}\alpha_4\beta_3+3\sqrt{2}\alpha_3\beta_4+4\alpha_1\beta_6-\sqrt{2}\alpha_7\beta_7+\sqrt{2}\alpha_6\beta_8 \\ 4\alpha_6\beta_1+2\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3-3\sqrt{2}\alpha_3\beta_5-\sqrt{2}\alpha_2\beta_6-4\alpha_1\beta_7+3\sqrt{2}\alpha_7\beta_8 \\ \alpha_7\beta_1-3\sqrt{3}\alpha_7\beta_2+\sqrt{2}\alpha_5\beta_4-3\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_3\beta_6+3\sqrt{2}\alpha_2\beta_7-4\alpha_1\beta_8 \end{pmatrix}.$$

$$\mathbf{8}\otimes\mathbf{8}=\mathbf{1_S}\oplus\mathbf{3_A}\oplus\mathbf{\bar{3}_A}\oplus\mathbf{6_{S,1}}\oplus\mathbf{6_{S,2}}\oplus\mathbf{7_S}\oplus\mathbf{7_{A,1}}\oplus\mathbf{7_{A,2}}\oplus\mathbf{8_{S,1}}\oplus\mathbf{8_{S,2}}\oplus\mathbf{8_A}$$

$$\mathbf{1_S}:\alpha_1\beta_1+\alpha_2\beta_2-\alpha_8\beta_3-\alpha_7\beta_4-\alpha_6\beta_5-\alpha_5\beta_6-\alpha_4\beta_7-\alpha_3\beta_8$$

$$\mathbf{3_A}:\begin{pmatrix} -\alpha_3\beta_1-\sqrt{3}\alpha_3\beta_2+\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_3+\sqrt{2}\alpha_8\beta_4-\sqrt{2}\alpha_7\beta_5+\sqrt{2}\alpha_5\beta_7-\sqrt{2}\alpha_4\beta_8 \\ 2\alpha_4\beta_1-2\alpha_1\beta_4+\sqrt{2}\alpha_8\beta_5-\sqrt{2}\alpha_7\beta_6+\sqrt{2}\alpha_6\beta_7-\sqrt{2}\alpha_5\beta_8 \\ \alpha_6\beta_1-\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3+\sqrt{2}\alpha_3\beta_5-\alpha_1\beta_6+\sqrt{3}\alpha_2\beta_6+\sqrt{2}\alpha_8\beta_7-\sqrt{2}\alpha_7\beta_8 \end{pmatrix}$$

$$\mathbf{\bar{3}_A}:\begin{pmatrix} -\alpha_8\beta_1-\sqrt{3}\alpha_8\beta_2+\sqrt{2}\alpha_7\beta_3-\sqrt{2}\alpha_6\beta_4+\sqrt{2}\alpha_4\beta_6-\sqrt{2}\alpha_3\beta_7+\alpha_1\beta_8+\sqrt{3}\alpha_2\beta_8 \\ 2\alpha_7\beta_1+\sqrt{2}\alpha_6\beta_3-\sqrt{2}\alpha_5\beta_4+\sqrt{2}\alpha_4\beta_5-\sqrt{2}\alpha_3\beta_6-2\alpha_1\beta_7 \\ \alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2+\sqrt{2}\alpha_4\beta_3-\sqrt{2}\alpha_3\beta_4-\alpha_1\beta_5+\sqrt{3}\alpha_2\beta_5-\sqrt{2}\alpha_8\beta_6+\sqrt{2}\alpha_6\beta_8 \end{pmatrix}$$

$$\mathbf{6_{S,1}}:\begin{pmatrix} 2\sqrt{2}\alpha_3\beta_1+2\sqrt{2}\alpha_1\beta_3-2\alpha_8\beta_4-2\alpha_4\beta_8 \\ -\sqrt{2}\alpha_4\beta_1+\sqrt{6}\alpha_4\beta_2-\sqrt{2}\alpha_1\beta_4+\sqrt{6}\alpha_2\beta_4+2\alpha_7\beta_6+2\alpha_6\beta_7 \\ \alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2-\sqrt{2}\alpha_4\beta_3-\sqrt{2}\alpha_3\beta_4+\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_5+\sqrt{2}\alpha_8\beta_6-2\sqrt{2}\alpha_7\beta_7+\sqrt{2}\alpha_6\beta_8 \\ \sqrt{2}\alpha_6\beta_1+\sqrt{6}\alpha_6\beta_2+2\alpha_5\beta_3+2\alpha_3\beta_5+\sqrt{2}\alpha_1\beta_6+\sqrt{6}\alpha_2\beta_6 \\ 2\alpha_7\beta_1+\sqrt{2}\alpha_6\beta_3+\sqrt{2}\alpha_5\beta_4+\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_3\beta_6+2\alpha_1\beta_7-2\sqrt{2}\alpha_8\beta_8 \\ -\alpha_8\beta_1-\sqrt{3}\alpha_8\beta_2-\sqrt{2}\alpha_7\beta_3+\sqrt{2}\alpha_6\beta_4-2\sqrt{2}\alpha_5\beta_5+\sqrt{2}\alpha_4\beta_6-\sqrt{2}\alpha_3\beta_7-\alpha_1\beta_8-\sqrt{3}\alpha_2\beta_8 \end{pmatrix}$$

$$\mathbf{6_{S,2}}:\begin{pmatrix} \alpha_3\beta_1+\sqrt{3}\alpha_3\beta_2+\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_3-\sqrt{2}\alpha_8\beta_4+\sqrt{2}\alpha_7\beta_5-2\sqrt{2}\alpha_6\beta_6+\sqrt{2}\alpha_5\beta_7-\sqrt{2}\alpha_4\beta_8 \\ -2\alpha_4\beta_1-2\sqrt{2}\alpha_3\beta_3-2\alpha_1\beta_4+\sqrt{2}\alpha_8\beta_5+\sqrt{2}\alpha_7\beta_6+\sqrt{2}\alpha_6\beta_7+\sqrt{2}\alpha_5\beta_8 \\ -\sqrt{2}\alpha_5\beta_1-\sqrt{6}\alpha_5\beta_2-\sqrt{2}\alpha_1\beta_5-\sqrt{6}\alpha_2\beta_5+2\alpha_8\beta_6+2\alpha_6\beta_8 \\ -\alpha_6\beta_1+\sqrt{3}\alpha_6\beta_2+\sqrt{2}\alpha_5\beta_3-2\sqrt{2}\alpha_4\beta_4+\sqrt{2}\alpha_3\beta_5-\alpha_1\beta_6+\sqrt{3}\alpha_2\beta_6-\sqrt{2}\alpha_8\beta_7-\sqrt{2}\alpha_7\beta_8 \\ \sqrt{2}\alpha_7\beta_1-\sqrt{6}\alpha_7\beta_2+2\alpha_5\beta_4+2\alpha_4\beta_5+\sqrt{2}\alpha_1\beta_7-\sqrt{6}\alpha_2\beta_7 \\ -2\sqrt{2}\alpha_8\beta_1-2\alpha_7\beta_3-2\alpha_3\beta_7-2\sqrt{2}\alpha_1\beta_8 \end{pmatrix}$$

$$\mathbf{7_S}:\begin{pmatrix} 3\alpha_1\beta_1+3\alpha_2\beta_2+\alpha_8\beta_3+\alpha_7\beta_4+\alpha_6\beta_5+\alpha_5\beta_6+\alpha_4\beta_7+\alpha_3\beta_8 \\ \alpha_3\beta_1+\sqrt{3}\alpha_3\beta_2+\alpha_1\beta_3+\sqrt{3}\alpha_2\beta_3+\sqrt{2}\alpha_8\beta_4+\sqrt{2}\alpha_7\beta_5+2\sqrt{2}\alpha_6\beta_6+\sqrt{2}\alpha_5\beta_7+\sqrt{2}\alpha_4\beta_8 \\ -2\alpha_4\beta_1+2\sqrt{2}\alpha_3\beta_3-2\alpha_1\beta_4+\sqrt{2}\alpha_8\beta_5-\sqrt{2}\alpha_7\beta_6-\sqrt{2}\alpha_6\beta_7+\sqrt{2}\alpha_5\beta_8 \\ \alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2-\sqrt{2}\alpha_4\beta_3-\sqrt{2}\alpha_3\beta_4+\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_5-\sqrt{2}\alpha_8\beta_6+2\sqrt{2}\alpha_7\beta_7-\sqrt{2}\alpha_6\beta_8 \\ -\alpha_6\beta_1+\sqrt{3}\alpha_6\beta_2-\sqrt{2}\alpha_5\beta_3+2\sqrt{2}\alpha_4\beta_4-\sqrt{2}\alpha_3\beta_5-\alpha_1\beta_6+\sqrt{3}\alpha_2\beta_6-\sqrt{2}\alpha_8\beta_7-\sqrt{2}\alpha_7\beta_8 \\ 2\alpha_7\beta_1+\sqrt{2}\alpha_6\beta_3-\sqrt{2}\alpha_5\beta_4-\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_3\beta_6+2\alpha_1\beta_7+2\sqrt{2}\alpha_8\beta_8 \\ -\alpha_8\beta_1-\sqrt{3}\alpha_8\beta_2+\sqrt{2}\alpha_7\beta_3+\sqrt{2}\alpha_6\beta_4+2\sqrt{2}\alpha_5\beta_5+\sqrt{2}\alpha_4\beta_6+\sqrt{2}\alpha_3\beta_7-\alpha_1\beta_8-\sqrt{3}\alpha_2\beta_8 \end{pmatrix}$$

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$$
\mathbf{7_{A,1}}:\begin{pmatrix}
4\alpha_1\beta_2-4\alpha_2\beta_1\\
\sqrt{3}\alpha_3\beta_1-\alpha_3\beta_2-\sqrt{3}\alpha_1\beta_3+\alpha_2\beta_3+\sqrt{6}\alpha_8\beta_4+\sqrt{6}\alpha_7\beta_5-\sqrt{6}\alpha_5\beta_7-\sqrt{6}\alpha_4\beta_8\\
2\alpha_4\beta_2-2\alpha_2\beta_4-\sqrt{6}\alpha_8\beta_5-\sqrt{6}\alpha_7\beta_6+\sqrt{6}\alpha_6\beta_7+\sqrt{6}\alpha_5\beta_8\\
-\sqrt{3}\alpha_5\beta_1-\alpha_5\beta_2+\sqrt{6}\alpha_4\beta_3-\sqrt{6}\alpha_3\beta_4+\sqrt{3}\alpha_1\beta_5+\alpha_2\beta_5+\sqrt{6}\alpha_8\beta_6-\sqrt{6}\alpha_6\beta_8\\
\sqrt{3}\alpha_6\beta_1+\alpha_6\beta_2-\sqrt{6}\alpha_5\beta_3+\sqrt{6}\alpha_3\beta_5-\sqrt{3}\alpha_1\beta_6-\alpha_2\beta_6-\sqrt{6}\alpha_8\beta_7+\sqrt{6}\alpha_7\beta_8\\
-2\alpha_7\beta_2+\sqrt{6}\alpha_6\beta_3+\sqrt{6}\alpha_5\beta_4-\sqrt{6}\alpha_4\beta_5-\sqrt{6}\alpha_3\beta_6+2\alpha_2\beta_7\\
-\sqrt{3}\alpha_8\beta_1+\alpha_8\beta_2-\sqrt{6}\alpha_7\beta_3-\sqrt{6}\alpha_6\beta_4+\sqrt{6}\alpha_4\beta_6+\sqrt{6}\alpha_3\beta_7+\sqrt{3}\alpha_1\beta_8-\alpha_2\beta_8
\end{pmatrix}
$$

$$
\mathbf{7_{A,2}}:\begin{pmatrix}
4(\alpha_8\beta_3+\alpha_7\beta_4-\alpha_6\beta_5+\alpha_5\beta_6-\alpha_4\beta_7-\alpha_3\beta_8)\\
\alpha_3\beta_1-3\sqrt{3}\alpha_3\beta_2-\alpha_1\beta_3+3\sqrt{3}\alpha_2\beta_3-3\sqrt{2}\alpha_8\beta_4+\sqrt{2}\alpha_7\beta_5-\sqrt{2}\alpha_5\beta_7+3\sqrt{2}\alpha_4\beta_8\\
4\alpha_4\beta_1+2\sqrt{3}\alpha_4\beta_2-4\alpha_1\beta_4-2\sqrt{3}\alpha_2\beta_4-\sqrt{2}\alpha_8\beta_5+3\sqrt{2}\alpha_7\beta_6-3\sqrt{2}\alpha_6\beta_7+\sqrt{2}\alpha_5\beta_8\\
5\alpha_5\beta_1-\sqrt{3}\alpha_5\beta_2-\sqrt{2}\alpha_4\beta_3+\sqrt{2}\alpha_3\beta_4-5\alpha_1\beta_5+\sqrt{3}\alpha_2\beta_5+3\sqrt{2}\alpha_8\beta_6-3\sqrt{2}\alpha_6\beta_8\\
5\alpha_6\beta_1-\sqrt{3}\alpha_6\beta_2+3\sqrt{2}\alpha_5\beta_3-3\sqrt{2}\alpha_3\beta_5-5\alpha_1\beta_6+\sqrt{3}\alpha_2\beta_6-\sqrt{2}\alpha_8\beta_7+\sqrt{2}\alpha_7\beta_8\\
4\alpha_7\beta_1+2\sqrt{3}\alpha_7\beta_2-\sqrt{2}\alpha_6\beta_3+3\sqrt{2}\alpha_5\beta_4-3\sqrt{2}\alpha_4\beta_5+\sqrt{2}\alpha_3\beta_6-4\alpha_1\beta_7-2\sqrt{3}\alpha_2\beta_7\\
\alpha_8\beta_1-3\sqrt{3}\alpha_8\beta_2-3\sqrt{2}\alpha_7\beta_3+\sqrt{2}\alpha_6\beta_4-\sqrt{2}\alpha_4\beta_6+3\sqrt{2}\alpha_3\beta_7-\alpha_1\beta_8+3\sqrt{3}\alpha_2\beta_8
\end{pmatrix}
$$

$$
\mathbf{8_{S,1}}:\begin{pmatrix}
3\sqrt{3}\alpha_1\beta_1-\alpha_2\beta_1-\alpha_1\beta_2-3\sqrt{3}\alpha_2\beta_2+2\sqrt{3}\alpha_8\beta_3-2\sqrt{3}\alpha_6\beta_5-2\sqrt{3}\alpha_5\beta_6+2\sqrt{3}\alpha_3\beta_8\\
-\alpha_1\beta_1-3\sqrt{3}\alpha_2\beta_1-3\sqrt{3}\alpha_1\beta_2+\alpha_2\beta_2-2\alpha_8\beta_3+4\alpha_7\beta_4-2\alpha_6\beta_5-2\alpha_5\beta_6+4\alpha_4\beta_7-2\alpha_3\beta_8\\
-2\sqrt{3}\alpha_3\beta_1+2\alpha_3\beta_2-2\sqrt{3}\alpha_1\beta_3+2\alpha_2\beta_3-2\sqrt{6}\alpha_8\beta_4+2\sqrt{6}\alpha_6\beta_6-2\sqrt{6}\alpha_4\beta_8\\
-4\alpha_4\beta_2+2\sqrt{6}\alpha_3\beta_3-4\alpha_2\beta_4+2\sqrt{6}\alpha_7\beta_6+2\sqrt{6}\alpha_6\beta_7\\
2\sqrt{3}\alpha_5\beta_1+2\alpha_5\beta_2+2\sqrt{3}\alpha_1\beta_5+2\alpha_2\beta_5+2\sqrt{6}\alpha_8\beta_6+2\sqrt{6}\alpha_7\beta_7+2\sqrt{6}\alpha_6\beta_8\\
2\sqrt{3}\alpha_6\beta_1+2\alpha_6\beta_2-2\sqrt{6}\alpha_5\beta_3-2\sqrt{6}\alpha_4\beta_4-2\sqrt{6}\alpha_3\beta_5+2\sqrt{3}\alpha_1\beta_6+2\alpha_2\beta_6\\
-4\alpha_7\beta_2-2\sqrt{6}\alpha_5\beta_4-2\sqrt{6}\alpha_4\beta_5-4\alpha_2\beta_7-2\sqrt{6}\alpha_8\beta_8\\
-2\sqrt{3}\alpha_8\beta_1+2\alpha_8\beta_2+2\sqrt{6}\alpha_7\beta_3-2\sqrt{6}\alpha_5\beta_5+2\sqrt{6}\alpha_3\beta_7-2\sqrt{3}\alpha_1\beta_8+2\alpha_2\beta_8
\end{pmatrix}
$$

$$
\mathbf{8_{S,2}}:\begin{pmatrix}
2\alpha_2\beta_1+2\alpha_1\beta_2+\sqrt{3}\alpha_7\beta_4-\sqrt{3}\alpha_6\beta_5-\sqrt{3}\alpha_5\beta_6+\sqrt{3}\alpha_4\beta_7\\
2\alpha_1\beta_1-2\alpha_2\beta_2-2\alpha_8\beta_3+\alpha_7\beta_4+\alpha_6\beta_5+\alpha_5\beta_6+\alpha_4\beta_7-2\alpha_3\beta_8\\
2\alpha_3\beta_2+2\alpha_2\beta_3-\sqrt{6}\alpha_7\beta_5-\sqrt{6}\alpha_5\beta_7\\
-\sqrt{3}\alpha_4\beta_1-\alpha_4\beta_2-\sqrt{3}\alpha_1\beta_4-\alpha_2\beta_4-\sqrt{6}\alpha_8\beta_5-\sqrt{6}\alpha_5\beta_8\\
\sqrt{3}\alpha_5\beta_1-\alpha_5\beta_2+\sqrt{6}\alpha_4\beta_3+\sqrt{6}\alpha_3\beta_4+\sqrt{3}\alpha_1\beta_5-\alpha_2\beta_5\\
\sqrt{3}\alpha_6\beta_1-\alpha_6\beta_2+\sqrt{3}\alpha_1\beta_6-\alpha_2\beta_6-\sqrt{6}\alpha_8\beta_7-\sqrt{6}\alpha_7\beta_8\\
-\sqrt{3}\alpha_7\beta_1-\alpha_7\beta_2+\sqrt{6}\alpha_6\beta_3+\sqrt{6}\alpha_3\beta_6-\sqrt{3}\alpha_1\beta_7-\alpha_2\beta_7\\
2\alpha_8\beta_2+\sqrt{6}\alpha_6\beta_4+\sqrt{6}\alpha_4\beta_6+2\alpha_2\beta_8
\end{pmatrix}
$$

$$
\mathbf{8_{A}}:\begin{pmatrix}
-2\alpha_8\beta_3+\alpha_7\beta_4-\alpha_6\beta_5+\alpha_5\beta_6-\alpha_4\beta_7+2\alpha_3\beta_8\\
-\sqrt{3}\alpha_7\beta_4-\sqrt{3}\alpha_6\beta_5+\sqrt{3}\alpha_5\beta_6+\sqrt{3}\alpha_4\beta_7\\
2\alpha_3\beta_1-2\alpha_1\beta_3-\sqrt{2}\alpha_7\beta_5+\sqrt{2}\alpha_5\beta_7\\
-\alpha_4\beta_1+\sqrt{3}\alpha_4\beta_2+\alpha_1\beta_4-\sqrt{3}\alpha_2\beta_4+\sqrt{2}\alpha_8\beta_5-\sqrt{2}\alpha_5\beta_8\\
\alpha_5\beta_1+\sqrt{3}\alpha_5\beta_2+\sqrt{2}\alpha_4\beta_3-\sqrt{2}\alpha_3\beta_4-\alpha_1\beta_5-\sqrt{3}\alpha_2\beta_5\\
-\alpha_6\beta_1-\sqrt{3}\alpha_6\beta_2+\alpha_1\beta_6+\sqrt{3}\alpha_2\beta_6-\sqrt{2}\alpha_8\beta_7+\sqrt{2}\alpha_7\beta_8\\
\alpha_7\beta_1-\sqrt{3}\alpha_7\beta_2-\sqrt{2}\alpha_6\beta_3+\sqrt{2}\alpha_3\beta_6-\alpha_1\beta_7+\sqrt{3}\alpha_2\beta_7\\
-2\alpha_8\beta_1+\sqrt{2}\alpha_6\beta_4-\sqrt{2}\alpha_4\beta_6+2\alpha_1\beta_8
\end{pmatrix}.
$$

B Constructing weight 2 modular forms of $\Gamma(7)$ by derivative of Dedekind eta function

For any complex number τ with $\Im\tau > 0$, the Dedekind eta-function $\eta(\tau)$ is defined as

$$
\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \qquad q \equiv e^{i2\pi\tau}\,. \tag{B.1}
$$

The η function satisfies the following identities

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\ \eta(\tau)\,, \tag{B.2}$$

which implies $\eta(\tau)$ is a modular function of weight half. Moreover, we see that the set of functions $\eta(7\tau)$, $\eta(\tau/7)$, $\eta((\tau+1)/7)$, $\eta((\tau+2)/7)$, $\eta((\tau+3)/7)$, $\eta((\tau+4)/7)$, $\eta((\tau+5)/7)$ and $\eta((\tau+6)/7)$ are closed under the action of the generators S and T. To be more specific, we have the following transformation rules under T,

$$\begin{aligned}
&\eta\left(7\tau\right) \to e^{i\frac{7\pi}{12}}\eta\left(7\tau\right)\,, && \eta\left(\frac{\tau}{7}\right) \to \eta\left(\frac{\tau+1}{7}\right)\,,\\
&\eta\left(\frac{\tau+1}{7}\right) \to \eta\left(\frac{\tau+2}{7}\right)\,, && \eta\left(\frac{\tau+2}{7}\right) \to \eta\left(\frac{\tau+3}{7}\right)\,,\\
&\eta\left(\frac{\tau+3}{7}\right) \to \eta\left(\frac{\tau+4}{7}\right)\,, && \eta\left(\frac{\tau+4}{7}\right) \to \eta\left(\frac{\tau+5}{7}\right)\,,\\
&\eta\left(\frac{\tau+5}{7}\right) \to \eta\left(\frac{\tau+6}{7}\right)\,, && \eta\left(\frac{\tau+6}{7}\right) \to e^{i\frac{\pi}{12}}\eta\left(\frac{\tau}{7}\right)\,.
\end{aligned} \tag{B.3}$$

Moreover, we find the following transformation behaviors under S

$$\begin{aligned}
&\eta\left(7\tau\right) \to \sqrt{\frac{-i\tau}{7}}\,\eta\left(\frac{\tau}{7}\right)\,, && \eta\left(\frac{\tau}{7}\right) \to \sqrt{-7i\tau}\,\eta\left(7\tau\right)\,,\\
&\eta\left(\frac{\tau+1}{7}\right) \to e^{-5i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+6}{7}\right)\,, && \eta\left(\frac{\tau+2}{7}\right) \to e^{-i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+3}{7}\right)\,,\\
&\eta\left(\frac{\tau+3}{7}\right) \to e^{i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+2}{7}\right)\,, && \eta\left(\frac{\tau+4}{7}\right) \to e^{-i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+5}{7}\right)\,,\\
&\eta\left(\frac{\tau+5}{7}\right) \to e^{i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+4}{7}\right)\,, && \eta\left(\frac{\tau+6}{7}\right) \to e^{5i\pi/12}\sqrt{-i\tau}\,\eta\left(\frac{\tau+1}{7}\right)
\end{aligned} \tag{B.4}$$

Following the approach proposed in [1], we can construct the weight 2 modular form by linear combination of the logarithmic derivative of above mentioned complete set of η functions,

$$\begin{aligned}
&Y(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8|\tau)\\
&\quad=\frac{d}{d\tau}\Bigg[x_1\ln\eta\left(7\tau\right)+x_2\ln\eta\left(\frac{\tau}{7}\right)+x_3\ln\eta\left(\frac{\tau+1}{7}\right)+x_4\ln\eta\left(\frac{\tau+2}{7}\right)+x_5\ln\eta\left(\frac{\tau+3}{7}\right)\\
&\qquad+x_6\ln\eta\left(\frac{\tau+4}{7}\right)+x_7\ln\eta\left(\frac{\tau+5}{7}\right)+x_8\ln\eta\left(\frac{\tau+6}{7}\right)\Bigg]\,,\\
&\quad=7x_1\frac{\eta'\left(7\tau\right)}{\eta\left(7\tau\right)}+\frac{1}{7}\Bigg[x_2\frac{\eta'\left(\tau/7\right)}{\eta\left(\tau/7\right)}+x_3\frac{\eta'\left((\tau+1)/7\right)}{\eta\left((\tau+1)/7\right)}+x_4\frac{\eta'\left((\tau+2)/7\right)}{\eta\left((\tau+2)/7\right)}+x_5\frac{\eta'\left((\tau+3)/7\right)}{\eta\left((\tau+3)/7\right)}\\
&\qquad+x_6\frac{\eta'\left((\tau+4)/7\right)}{\eta\left((\tau+4)/7\right)}+x_7\frac{\eta'\left((\tau+5)/7\right)}{\eta\left((\tau+5)/7\right)}+x_8\frac{\eta'\left((\tau+6)/7\right)}{\eta\left((\tau+6)/7\right)}\Bigg]\,,
\end{aligned} \tag{B.5}$$

with $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8=0$. Notice that $12\eta'(\tau)/\eta(\tau) \equiv i\pi E_2(\tau)$, where $E_2(\tau)$ is the well-known Eisenstein series of weight 2. Under the action of the generators

S and T, this function transforms as

$$
\begin{aligned}
&Y(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8|\tau) \xrightarrow{T} Y(x_1,x_8,x_2,x_3,x_4,x_5,x_6,x_7|\tau)\,,\\
&Y(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8|\tau) \xrightarrow{S} \tau^2 Y(x_2,x_1,x_8,x_5,x_4,x_7,x_6,x_3|\tau)\,.
\end{aligned}
\tag{B.6}
$$

We can construct a septet $Y_{\mathbf{7}}(\tau)=(Y'_1,Y'_2,Y'_3,Y'_4,Y'_5,Y'_6,Y'_7)^T$ by the modular function $Y(x_1,\,x_2,\,x_3,\,x_4,x_5,x_6,x_7,x_8|\tau)$, and $Y_{\mathbf{7}}(\tau)$ transforms as $\mathbf{7}$ under $\Gamma_7\cong\Sigma(168)$,

$$
Y_{\mathbf{7}}(-1/\tau)=\tau^2\rho_{\mathbf{7}}(S)Y_{\mathbf{7}}(\tau),\qquad Y_{\mathbf{7}}(\tau+1)=\rho_{\mathbf{7}}(T)Y_{\mathbf{7}}(\tau)\,.
\tag{B.7}
$$

We can then straightforwardly find the solutions for Y'_i $(i=1,2,\ldots,7)$ are given by

$$
\begin{aligned}
Y'_1(\tau)&=\frac{c}{2\sqrt{2}}\,Y\left(7,-1,-1,-1,-1,-1,-1,-1|\tau\right)\,,\\
Y'_2(\tau)&=cY\left(0,1,\rho^{-1},\rho^{-2},\rho^{-3},\rho^{3},\rho^{2},\rho|\tau\right)\,,\\
Y'_3(\tau)&=cY\left(0,1,\rho^{-2},\rho^{3},\rho,\rho^{-1},\rho^{-3},\rho^{2}|\tau\right)\,,\\
Y'_4(\tau)&=cY\left(0,1,\rho^{-3},\rho,\rho^{-2},\rho^{2},\rho^{-1},\rho^{3}|\tau\right)\,,\\
Y'_5(\tau)&=cY\left(0,1,\rho^{3},\rho^{-1},\rho^{2},\rho^{-2},\rho,\rho^{-3}|\tau\right)\,,\\
Y'_6(\tau)&=cY\left(0,1,\rho^{2},\rho^{-3},\rho^{-1},\rho,\rho^{3},\rho^{-2}|\tau\right)\,,\\
Y'_7(\tau)&=cY\left(0,1,\rho,\rho^{2},\rho^{3},\rho^{-3},\rho^{-2},\rho^{-1}|\tau\right)\,,
\end{aligned}
\tag{B.8}
$$

up to the overall constant c. We shall choose $c=-\frac{i}{\sqrt{2}\,\pi}$ for convenience. The q-expansion of Y'_i reads

$$
\begin{aligned}
Y'_1(\tau)&=1+4q+12q^2+16q^3+28q^4+24q^5+\ldots\,,\\
Y'_2(\tau)&=-\sqrt{2}q^{1/7}\left(1+15q+24q^2+36q^3+30q^4+91q^5+\ldots\right)\,,\\
Y'_3(\tau)&=-\sqrt{2}q^{2/7}\left(3+13q+31q^2+24q^3+72q^4+38q^5+\ldots\right)\,,\\
Y'_4(\tau)&=-2\sqrt{2}q^{3/7}\left(2+9q+9q^2+30q^3+16q^4+30q^5+\ldots\right)\,,\\
Y'_5(\tau)&=-\sqrt{2}q^{4/7}\left(7+12q+39q^2+31q^3+63q^4+56q^5+\ldots\right)\,,\\
Y'_6(\tau)&=-2\sqrt{2}q^{5/7}\left(3+14q+10q^2+21q^3+24q^4+45q^5+\ldots\right)\,,\\
Y'_7(\tau)&=-2\sqrt{2}q^{6/7}\left(6+7q+21q^2+20q^3+27q^4+21q^5+\ldots\right)\,.
\end{aligned}
\tag{B.9}
$$

Further we find that the above modular forms can be expressed in terms of Miller-like basis vectors in eq. (3.3) as follows,

$$
\begin{aligned}
Y'_1(\tau)&=b_4(\tau)+4b_{11}(\tau)+12b_{18}(\tau)+16b_{25}(\tau)+28b_{26}(\tau)\,,\\
Y'_2(\tau)&=-\sqrt{2}\left[b_5(\tau)+15b_{12}(\tau)+24b_{19}(\tau)\right]\,,\\
Y'_3(\tau)&=-\sqrt{2}\left[3b_6(\tau)+13b_{13}(\tau)+31b_{20}(\tau)\right]\,,\\
Y'_4(\tau)&=-2\sqrt{2}\left[2b_7(\tau)+9b_{14}(\tau)+9b_{21}(\tau)\right]\,,\\
Y'_5(\tau)&=-\sqrt{2}\left[7b_8(\tau)+12b_{15}(\tau)+39b_{22}(\tau)\right]\,,\\
Y'_6(\tau)&=-2\sqrt{2}\left[3b_9(\tau)+14b_{16}(\tau)+10b_{23}(\tau)\right]\,,\\
Y'_7(\tau)&=-2\sqrt{2}\left[6b_{10}(\tau)+7b_{17}(\tau)+21b_{24}(\tau)\right]\,.
\end{aligned}
\tag{B.10}
$$

We see that the modular form $Y_{\mathbf{7}}(\tau)$ is exactly the same as $Y^{(2)}_{\mathbf{7}}(\tau)$ in eq. (3.5).

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C Constructing weight 2 modular forms of $\Gamma(7)$ by theta function method

The modular forms of level 5 can be constructed from the Jacobi theta functions [20]. In the following, we proceed to construct the weight 2 modular forms of level $N = 7$ by using the Jacobi theta functions $\theta_3(u,\tau)$ which is defined as

$$\theta_3(u,\tau) = \sum_{k\in\mathbb{Z}} q^{k^2/2} e^{2\pi i k u} \,. \tag{C.1}$$

It can also be expressed as the following infinite product,

$$\theta_3(u,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}e^{2\pi i u})(1+q^{n-1/2}e^{-2\pi i u}) \,. \tag{C.2}$$

The theta function $\theta_3(u,\tau)$ has the following properties [55]:

$$\begin{aligned}
&\theta_3(-u,\tau) = \theta_3(u,\tau), \quad \theta_3(u+m,\tau+2n) = \theta_3(u,\tau), \quad \theta_3\left(u+\frac{1}{2}+m,\tau+1+2n\right) = \theta_3(u,\tau)\,, \\
&\theta_3(\tau-u,\tau) = e^{(2u-\tau)\pi i}\theta_3(u,\tau), \quad \theta_3(u,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{-\pi i u^2/\tau}\theta_3\left(u/\tau,-1/\tau\right)\,,
\end{aligned} \tag{C.3}$$

with $m, n \in \mathbb{Z}$. The lowest weight modular form with $k = 2$ can be expressed as linear combinations of the logarithmic derivatives of some "seed" functions $\alpha_{i,j}(\tau)$. We choose the closed set of the seed functions $\alpha_{i,j}(\tau)$ as follows,

$$\begin{aligned}
&\alpha_{1,-1}(\tau) = \theta_3\left(\frac{\tau+1}{2},7\tau\right), && \alpha_{2,-1}(\tau) = e^{\frac{2\pi i\tau}{7}}\theta_3\left(\frac{3\tau+1}{2},7\tau\right), && \alpha_{3,-1}(\tau) = e^{\frac{6\pi i\tau}{7}}\theta_3\left(\frac{5\tau+1}{2},7\tau\right), \\
&\alpha_{1,0}(\tau) = \theta_3\left(\frac{\tau+1}{14},\frac{\tau}{7}\right), && \alpha_{2,0}(\tau) = \theta_3\left(\frac{\tau+3}{14},\frac{\tau}{7}\right), && \alpha_{3,0}(\tau) = \theta_3\left(\frac{\tau+5}{14},\frac{\tau}{7}\right), \\
&\alpha_{1,1}(\tau) = \theta_3\left(\frac{\tau+2}{14},\frac{\tau+1}{7}\right), && \alpha_{2,1}(\tau) = \theta_3\left(\frac{\tau+4}{14},\frac{\tau+1}{7}\right), && \alpha_{3,1}(\tau) = \theta_3\left(\frac{\tau+6}{14},\frac{\tau+1}{7}\right), \\
&\alpha_{1,2}(\tau) = \theta_3\left(\frac{\tau+3}{14},\frac{\tau+2}{7}\right), && \alpha_{2,2}(\tau) = \theta_3\left(\frac{\tau+5}{14},\frac{\tau+2}{7}\right), && \alpha_{3,2}(\tau) = \theta_3\left(\frac{\tau+7}{14},\frac{\tau+2}{7}\right), \\
&\alpha_{1,3}(\tau) = \theta_3\left(\frac{\tau+4}{14},\frac{\tau+3}{7}\right), && \alpha_{2,3}(\tau) = \theta_3\left(\frac{\tau+6}{14},\frac{\tau+3}{7}\right), && \alpha_{3,3}(\tau) = \theta_3\left(\frac{\tau+8}{14},\frac{\tau+3}{7}\right), \\
&\alpha_{1,4}(\tau) = \theta_3\left(\frac{\tau+5}{14},\frac{\tau+4}{7}\right), && \alpha_{2,4}(\tau) = \theta_3\left(\frac{\tau+7}{14},\frac{\tau+4}{7}\right), && \alpha_{3,4}(\tau) = \theta_3\left(\frac{\tau+9}{14},\frac{\tau+4}{7}\right), \\
&\alpha_{1,5}(\tau) = \theta_3\left(\frac{\tau+6}{14},\frac{\tau+5}{7}\right), && \alpha_{2,5}(\tau) = \theta_3\left(\frac{\tau+8}{14},\frac{\tau+5}{7}\right), && \alpha_{3,5}(\tau) = \theta_3\left(\frac{\tau+10}{14},\frac{\tau+5}{7}\right), \\
&\alpha_{1,6}(\tau) = \theta_3\left(\frac{\tau+7}{14},\frac{\tau+6}{7}\right), && \alpha_{2,6}(\tau) = \theta_3\left(\frac{\tau+9}{14},\frac{\tau+6}{7}\right), && \alpha_{3,6}(\tau) = \theta_3\left(\frac{\tau+11}{14},\frac{\tau+6}{7}\right).
\end{aligned} \tag{C.4}$$

Note the set of seed functions is not unique although the same results for modular forms are obtained. Under the action of the generators S and T, we can check that each of these seed functions $\alpha_{i,j}(\tau)$ is mapped to another, up to some τ-dependent multiplicative factor. The transformation properties of $\alpha_{i,j}(\tau)$ under S and T are shown in figure 3. Hence we can start from any seed function $\alpha_{i,j}(\tau)$ (e.g. $\alpha_{1,-1}(\tau)$) to generate all the others. Moreover, we find that each seed function is mapped into itself under the actions of the

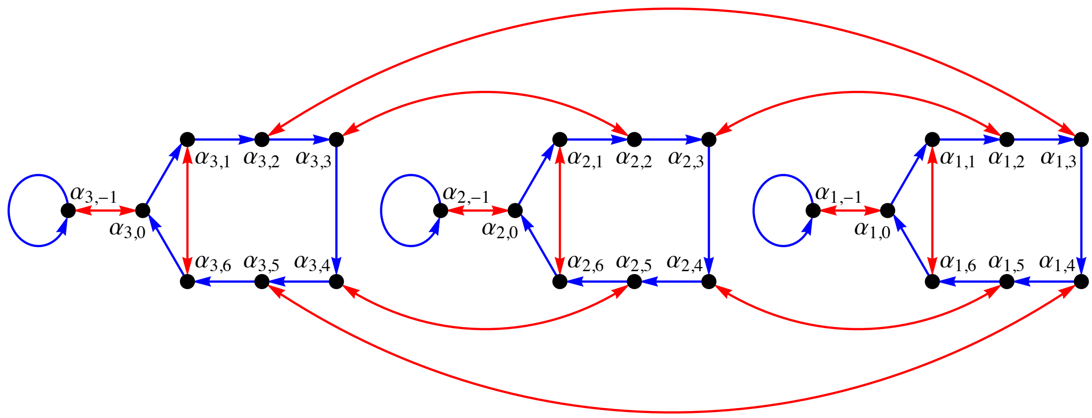

Figure 3. The graph illustrating the transformations of the set of seed functions $\alpha_{i,j}(\tau)$, defined in eq. (C.4), under the actions of $\Gamma_7 \cong \mathrm{PSL}(2, Z_7)$ generators S and T. The red and blue solid lines denote the transformations of S and T respectively.

modular transformations S^2, $(ST)^3$, $(ST^3)^4$ and T^7 up to some τ relevant factors. Taking logarithmic derivatives, we find

$$\frac{d}{d\tau}\log\alpha_{i,j}(-1/\tau) = \frac{i\pi}{28}\left(1-\frac{1}{\tau^2}\right)+\frac{1}{2\tau}+\frac{d}{d\tau}\log\alpha^S_{i,j}(\tau)\,, \tag{C.5}$$

$$\frac{d}{d\tau}\log\alpha_{i,j}(\tau+1) = \frac{d}{d\tau}\log\alpha^T_{i,j}(\tau)\,, \tag{C.6}$$

where $\alpha^S_{i,j}$ and $\alpha^T_{i,j}$ are the images of $\alpha_{i,j}$ under the maps of S and T shown in figure 3, respectively. As a consequence, the modular functions

$$Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots,x_{2,6};x_{3,-1},\ldots,x_{3,6}|\tau) \equiv \sum_{i,j} x_{i,j}\frac{d}{d\tau}\log\alpha_{i,j}(\tau)\,,\quad \text{with } \sum_{i,j}x_{i,j}=0\,, \tag{C.7}$$

span a 24-dimensional linear space of weight two modular forms of level $N = 7$. Under S and T, the modular function Y transforms as follows,

$$\begin{aligned}
S:&Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}|\tau)\xrightarrow{S}Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}|-1/\tau)\\
&=\tau^2Y(x_{1,0},x_{1,-1},x_{1,6},x_{2,3},x_{3,2},x_{3,5},x_{2,4},x_{1,1};x_{2,0},x_{2,-1},x_{2,6},x_{3,3},x_{1,2},x_{1,5},x_{3,4},x_{2,1};\\
&\qquad x_{3,0},x_{3,-1},x_{3,6},x_{1,3},x_{2,2},x_{2,5},x_{1,4},x_{3,1}|\tau)\,,\\
T:&Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}|\tau)\xrightarrow{T}Y(x_{1,-1},\ldots,x_{1,6};x_{2,-1},\ldots x_{2,6};x_{3,-1},\ldots x_{3,6}|\tau+1)\\
&=Y(x_{1,-1},x_{1,6},x_{1,0},x_{1,1},x_{1,2},x_{1,3},x_{1,4},x_{1,5};x_{2,-1},x_{2,6},x_{2,0},x_{2,1},x_{2,2},x_{2,3},x_{2,4},x_{2,5};\\
&\qquad x_{3,-1},x_{3,6},x_{3,0},x_{3,1},x_{3,2},x_{3,3},x_{3,4},x_{3,5}|\tau)\,.
\end{aligned} \tag{C.8}$$

As shown in eq. (2.16), we can always choose a basis such that the modular forms can be organized into different multiplets of Γ_7:

$$\widetilde{Y}_{\mathbf{7}}^{(2)}(\tau)=\frac{-i}{2\sqrt{2}\pi}\begin{pmatrix}\frac{1}{2\sqrt{2}}Y\left(7,-\mathbf{v}_0;7,-\mathbf{v}_0;7,-\mathbf{v}_0|\tau\right)\\ Y(0,\mathbf{v}_1;\,0,\mathbf{v}_1;0,\mathbf{v}_1|\,\tau)\\ Y(0,\mathbf{v}_2;\,0,\mathbf{v}_2;0,\mathbf{v}_2|\,\tau)\\ Y(0,\mathbf{v}_3;\,0,\mathbf{v}_3;0,\mathbf{v}_3|\,\tau)\\ Y(0,\mathbf{v}_4;\,0,\mathbf{v}_4;0,\mathbf{v}_4|\,\tau)\\ Y(0,\mathbf{v}_5;\,0,\mathbf{v}_5;0,\mathbf{v}_5|\,\tau)\\ Y(0,\mathbf{v}_6;\,0,\mathbf{v}_6;0,\mathbf{v}_6|\,\tau)\end{pmatrix}, \tag{C.9}$$

$$\widetilde{Y}_{\mathbf{8}a}^{(2)}(\tau)=\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(2(c_1-c_3),\mathbf{0};2(c_2-c_1),\mathbf{v}_0;2(c_3-c_2),-\mathbf{v}_0|\tau\right)\\ -\frac{1}{\sqrt{6}}Y\left(1+6c_2,-2\mathbf{v}_0;1+6c_3,\mathbf{v}_0;1+6c_1,\mathbf{v}_0|\tau\right)\\ Y\left(0,\mathbf{v}_1;0,\mathbf{0};0,-\mathbf{v}_1|\tau\right)\\ Y\left(0,\mathbf{0};0,-\mathbf{v}_2;0,\mathbf{v}_2|\tau\right)\\ Y\left(0,-\mathbf{v}_3;0,\mathbf{v}_3;0,\mathbf{0}|\tau\right)\\ Y\left(0,\mathbf{v}_4;0,-\mathbf{v}_4;0,\mathbf{0}|\tau\right)\\ Y\left(0,\mathbf{0};0,\mathbf{v}_5;0,-\mathbf{v}_5|\tau\right)\\ Y\left(0,-\mathbf{v}_6;0,\mathbf{0};0,\mathbf{v}_6|\tau\right)\end{pmatrix}, \tag{C.10}$$

$$\widetilde{Y}_{\mathbf{8}b}^{(2)}(\tau)=\begin{pmatrix}\frac{1}{\sqrt{2}}Y\left(2(c_3-c_2),-\mathbf{v}_0;2(c_1-c_3),\mathbf{0};2(c_2-c_1),\mathbf{v}_0|\tau\right)\\ -\frac{1}{\sqrt{6}}Y\left(1+6c_1,\mathbf{v}_0;1+6c_2,-2\mathbf{v}_0;1+6c_3,\mathbf{v}_0|\tau\right)\\ Y\left(0,-\mathbf{v}_1;0,\mathbf{v}_1;0,\mathbf{0}|\tau\right)\\ Y\left(0,\mathbf{v}_2;0,\mathbf{0};0,-\mathbf{v}_2|\tau\right)\\ Y\left(0,\mathbf{0};0,-\mathbf{v}_3;0,\mathbf{v}_3|\tau\right)\\ Y\left(0,\mathbf{0};0,\mathbf{v}_4;0,-\mathbf{v}_4|\tau\right)\\ Y\left(0,-\mathbf{v}_5;0,\mathbf{0};0,\mathbf{v}_5|\tau\right)\\ Y\left(0,\mathbf{v}_6;0,-\mathbf{v}_6;0,\mathbf{0}|\tau\right)\end{pmatrix}, \tag{C.11}$$

where $\mathbf{v}_k\equiv\{1,\rho^{6k},\rho^{5k},\rho^{4k},\rho^{3k},\rho^{2k},\rho^{k}\}$ for $k=0,1,\cdots,6$ and $\mathbf{0}\equiv\{0,0,0,0,0,0,0\}$. From the expressions of q-expansion, we see that the modular form $\widetilde{Y}_{\mathbf{7}}^{(2)}(\tau)$ in the representation $\mathbf{7}$ coincides with $Y_{\mathbf{7}}^{(2)}(\tau)$ of eq. (3.5). Moreover, the two modular octets $Y_{\mathbf{8}a}^{(2)}(\tau)$ and $Y_{\mathbf{8}b}^{(2)}(\tau)$ in eqs. (3.6), (3.7) are linear combinations of $\widetilde{Y}_{\mathbf{8}a}^{(2)}(\tau)$ and $\widetilde{Y}_{\mathbf{8}b}^{(2)}(\tau)$ as follow,

$$Y_{\mathbf{8}a}^{(2)}=\frac{y_1\widetilde{Y}_{\mathbf{8}a}^{(2)}(\tau)+y_2\widetilde{Y}_{\mathbf{8}b}^{(2)}(\tau)}{2\sqrt{2}\pi i},\qquad Y_{\mathbf{8}b}^{(2)}=\frac{z_1\widetilde{Y}_{\mathbf{8}a}^{(2)}(\tau)+z_2\widetilde{Y}_{\mathbf{8}b}^{(2)}(\tau)}{2\sqrt{2}\pi i}, \tag{C.12}$$

with

$$\begin{aligned}&y_1=-\frac{2}{7}(4c_1+c_2-5c_3),\qquad y_2=-\frac{2}{7}(5c_1-4c_2-c_3),\\&z_1=\frac{1}{35}(2c_1-3c_2+c_3),\qquad z_2=\frac{1}{35}(-c_1-2c_2+3c_3)\,.\end{aligned} \tag{C.13}$$

In short, we can construct 23 modular forms of weight 2 and level 7, and they can be decomposed into one septuplet and two octuplets of Γ_7. We can not build the modular multiplet $Y_{\mathbf{3}}^{(2)}(\tau)$ in the representation $\mathbf{3}$ of Γ_7 from the theta function $\theta_3(u,\tau)$.

D Higher weight modular forms and constraints

Through the tensor products of the modular forms $Y^{(2)}_{\mathbf{3}}$, $Y^{(2)}_{\mathbf{7}}$, $Y^{(2)}_{\mathbf{8}a}$ and $Y^{(2)}_{\mathbf{8}b}$, one can find, at weight 4, the following modular multiplets:

$$Y^{(4)}_{\mathbf{1}a}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{1}},\qquad Y^{(4)}_{\mathbf{1}b}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{1}},\qquad Y^{(4)}_{\mathbf{1}c}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{1}},\ Y^{(4)}_{\mathbf{1}d}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{1}}. \tag{D.1}$$

$$\begin{aligned}
&Y^{(4)}_{\mathbf{3}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{3}}, && Y^{(4)}_{\mathbf{3}b}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{3}}, && Y^{(4)}_{\mathbf{3}c}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{3}},\\
&Y^{(4)}_{\mathbf{3}d}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{3}}, && Y^{(4)}_{\mathbf{3}e}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{3}_{\mathbf{A}}}. &&
\end{aligned}\tag{D.2}$$

$$Y^{(4)}_{\bar{\mathbf{3}}a}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\bar{\mathbf{3}}},\qquad Y^{(4)}_{\bar{\mathbf{3}}b}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\bar{\mathbf{3}}},\qquad Y^{(4)}_{\bar{\mathbf{3}}c}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\bar{\mathbf{3}}}. \tag{D.3}$$

$$\begin{aligned}
&Y^{(4)}_{\mathbf{6}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{3}}\right)_{\mathbf{6}}, && Y^{(4)}_{\mathbf{6}b}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{6}}, && Y^{(4)}_{\mathbf{6}c}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{6}},\\
&Y^{(4)}_{\mathbf{6}d}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}}, && Y^{(4)}_{\mathbf{6}e}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{6}f}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{6}_{\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{6}g}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{1}}}, && Y^{(4)}_{\mathbf{6}h}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{6}i}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{6}j}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{6}k}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}}, && Y^{(4)}_{\mathbf{6}l}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}},\\
&Y^{(4)}_{\mathbf{6}m}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}}, && Y^{(4)}_{\mathbf{6}n}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}}, && Y^{(4)}_{\mathbf{6}o}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}},\\
&Y^{(4)}_{\mathbf{6}p}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{6}_{\mathbf{S},\mathbf{2}}}. && &&
\end{aligned}\tag{D.4}$$

$$\begin{aligned}
&Y^{(4)}_{\mathbf{7}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{7}}, && Y^{(4)}_{\mathbf{7}b}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{7}_{\mathbf{S}}}, && Y^{(4)}_{\mathbf{7}c}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}},\\
&Y^{(4)}_{\mathbf{7}d}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{7}}, && Y^{(4)}_{\mathbf{7}e}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{7}_{\mathbf{1}}}, && Y^{(4)}_{\mathbf{7}f}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{7}g}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{7}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{7}h}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{7}i}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{S},\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{7}j}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{7}_{\mathbf{2}}}, && Y^{(4)}_{\mathbf{7}k}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{2}}}, && Y^{(4)}_{\mathbf{7}l}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{S},\mathbf{2}}},\\
&Y^{(4)}_{\mathbf{7}m}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{7}_{\mathbf{A}}}. && &&
\end{aligned}\tag{D.5}$$

$$\begin{aligned}
&Y^{(4)}_{\mathbf{8}a}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{8}}, && Y^{(4)}_{\mathbf{8}b}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{8}c}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{2}}},\\
&Y^{(4)}_{\mathbf{8}d}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{7}}\right)_{\mathbf{8}_{\mathbf{S}}}, && Y^{(4)}_{\mathbf{8}e}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}_{\mathbf{1}}}, && Y^{(4)}_{\mathbf{8}f}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{8}g}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}}, && Y^{(4)}_{\mathbf{8}h}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{1}}}, && Y^{(4)}_{\mathbf{8}i}=\left(Y^{(2)}_{\mathbf{8}b}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{1}}},\\
&Y^{(4)}_{\mathbf{8}j}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}_{\mathbf{2}}}, && Y^{(4)}_{\mathbf{8}k}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{2}}}, && Y^{(4)}_{\mathbf{8}l}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{2}}},\\
&Y^{(4)}_{\mathbf{8}m}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{S},\mathbf{2}}}, && Y^{(4)}_{\mathbf{8}n}=\left(Y^{(2)}_{\mathbf{3}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}}, && Y^{(4)}_{\mathbf{8}o}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}a}\right)_{\mathbf{8}_{\mathbf{3}}},\\
&Y^{(4)}_{\mathbf{8}p}=\left(Y^{(2)}_{\mathbf{7}}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{3}}}, && Y^{(4)}_{\mathbf{8}q}=\left(Y^{(2)}_{\mathbf{8}a}Y^{(2)}_{\mathbf{8}b}\right)_{\mathbf{8}_{\mathbf{A}}}. &&
\end{aligned}\tag{D.6}$$

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Notice that not all of the above modular multiplets are linearly independent. From the q-expansions of $Y_i(\tau)$ given in eq. (3.7), we find the following 297 constraints between the different weight 4 modular multiplets $Y_{\mathbf{r}}^{(4)}$,

$$Y_{\mathbf{1}a}^{(4)} = Y_{\mathbf{1}b}^{(4)} = 300Y_{\mathbf{1}d}^{(4)}, \qquad Y_{\mathbf{1}c}^{(4)} = 0. \tag{D.7}$$

$$Y_{\mathbf{3}a}^{(4)} = -30Y_{\mathbf{3}b}^{(4)} = -\frac{1}{\sqrt{6}}Y_{\mathbf{3}c}^{(4)} = \sqrt{6}Y_{\mathbf{3}d}^{(4)} = -5\sqrt{6}Y_{\mathbf{3}e}^{(4)}. \tag{D.8}$$

$$Y_{\bar{\mathbf{3}}a}^{(4)} = Y_{\bar{\mathbf{3}}b}^{(4)} = Y_{\bar{\mathbf{3}}c}^{(4)} = (0,0,0)^T. \tag{D.9}$$

$$\begin{aligned}
&Y_{\mathbf{6}c}^{(4)} = \frac{1}{2}\left(3\sqrt{2}Y_{\mathbf{6}a}^{(4)} + Y_{\mathbf{6}b}^{(4)}\right), \quad Y_{\mathbf{6}d}^{(4)} = \frac{1}{20}\left(Y_{\mathbf{6}b}^{(4)} - 5\sqrt{2}Y_{\mathbf{6}a}^{(4)}\right), \quad Y_{\mathbf{6}e}^{(4)} = 2\left(3Y_{\mathbf{6}a}^{(4)} + \sqrt{2}Y_{\mathbf{6}b}^{(4)}\right),\\
&Y_{\mathbf{6}f}^{(4)} = 8\sqrt{2}Y_{\mathbf{6}a}^{(4)} - 4Y_{\mathbf{6}b}^{(4)}, \quad Y_{\mathbf{6}g}^{(4)} = \frac{2}{5}\left(Y_{\mathbf{6}b}^{(4)} - 6\sqrt{2}Y_{\mathbf{6}a}^{(4)}\right), \quad Y_{\mathbf{6}h}^{(4)} = 13\sqrt{2}Y_{\mathbf{6}a}^{(4)} - 8Y_{\mathbf{6}b}^{(4)},\\
&Y_{\mathbf{6}i}^{(4)} = \frac{1}{10}\left(2Y_{\mathbf{6}b}^{(4)} - 3\sqrt{2}Y_{\mathbf{6}a}^{(4)}\right), \quad Y_{\mathbf{6}j}^{(4)} = \frac{Y_{\mathbf{6}a}^{(4)}}{10\sqrt{2}}, \quad Y_{\mathbf{6}k}^{(4)} = -2\sqrt{2}Y_{\mathbf{6}a}^{(4)},\\
&Y_{\mathbf{6}l}^{(4)} = 19\sqrt{2}Y_{\mathbf{6}a}^{(4)} - 3Y_{\mathbf{6}b}^{(4)}, \quad Y_{\mathbf{6}m}^{(4)} = \frac{1}{10}\left(Y_{\mathbf{6}b}^{(4)} - \sqrt{2}Y_{\mathbf{6}a}^{(4)}\right), \quad Y_{\mathbf{6}n}^{(4)} = -2\left(7Y_{\mathbf{6}a}^{(4)} + \sqrt{2}Y_{\mathbf{6}b}^{(4)}\right),\\
&Y_{\mathbf{6}o}^{(4)} = \frac{1}{20}\left(14Y_{\mathbf{6}a}^{(4)} + 3\sqrt{2}Y_{\mathbf{6}b}^{(4)}\right), \quad Y_{\mathbf{6}p}^{(4)} = -\frac{Y_{\mathbf{6}b}^{(4)}}{50\sqrt{2}}.
\end{aligned} \tag{D.10}$$

$$\begin{aligned}
&Y_{\mathbf{7}c}^{(4)} = -\frac{1}{10\sqrt{2}}Y_{\mathbf{7}a}^{(4)}, \quad Y_{\mathbf{7}e}^{(4)} = -11Y_{\mathbf{7}a}^{(4)}, \quad Y_{\mathbf{7}d}^{(4)} = -\frac{5}{\sqrt{2}}Y_{\mathbf{7}a}^{(4)},\\
&Y_{\mathbf{7}f}^{(4)} = \frac{1}{2}Y_{\mathbf{7}a}^{(4)}, \quad Y_{\mathbf{7}g}^{(4)} = 5Y_{\mathbf{7}a}^{(4)} + \frac{1}{2}Y_{\mathbf{7}b}^{(4)}, \quad Y_{\mathbf{7}h}^{(4)} = -\frac{3}{10}Y_{\mathbf{7}a}^{(4)},\\
&Y_{\mathbf{7}i}^{(4)} = \frac{1}{600}(14Y_{\mathbf{7}a}^{(4)} + Y_{\mathbf{7}b}^{(4)}), \quad Y_{\mathbf{7}j}^{(4)} = -2Y_{\mathbf{7}a}^{(4)}, \quad Y_{\mathbf{7}k}^{(4)} = -\frac{1}{5}Y_{\mathbf{7}a}^{(4)},\\
&Y_{\mathbf{7}l}^{(4)} = \frac{1}{30\sqrt{3}}(11Y_{\mathbf{7}a}^{(4)} - 2Y_{\mathbf{7}b}^{(4)}), \quad Y_{\mathbf{7}m}^{(4)} = \frac{7}{10}Y_{\mathbf{7}a}^{(4)}.
\end{aligned} \tag{D.11}$$

$$\begin{aligned}
&Y_{\mathbf{8}d}^{(4)} = 2\sqrt{2}Y_{\mathbf{8}a}^{(4)}, \quad Y_{\mathbf{8}e}^{(4)} = \sqrt{\frac{3}{2}}Y_{\mathbf{8}a}^{(4)} + Y_{\mathbf{8}b}^{(4)} + 450Y_{\mathbf{8}c}^{(4)},\\
&Y_{\mathbf{8}f}^{(4)} = \frac{1}{360}\left(-91\sqrt{6}Y_{\mathbf{8}a}^{(4)} - 16Y_{\mathbf{8}b}^{(4)} + 18600Y_{\mathbf{8}c}^{(4)}\right), \quad Y_{\mathbf{8}g}^{(4)} = -\frac{1}{4}Y_{\mathbf{8}a}^{(4)},\\
&Y_{\mathbf{8}h}^{(4)} = \frac{1}{180}\left(35\sqrt{6}Y_{\mathbf{8}a}^{(4)} + 2Y_{\mathbf{8}b}^{(4)} - 8400Y_{\mathbf{8}c}^{(4)}\right), \quad Y_{\mathbf{8}i}^{(4)} = \frac{1}{300}\left(-\sqrt{6}Y_{\mathbf{8}a}^{(4)} - Y_{\mathbf{8}b}^{(4)}\right),\\
&Y_{\mathbf{8}j}^{(4)} = -\frac{15}{4}\left(\sqrt{2}Y_{\mathbf{8}a}^{(4)} - 80\sqrt{3}Y_{\mathbf{8}c}^{(4)}\right), \quad Y_{\mathbf{8}k}^{(4)} = \frac{1}{360}\left(-51\sqrt{2}Y_{\mathbf{8}a}^{(4)} - 8\sqrt{3}(Y_{\mathbf{8}b}^{(4)} - 150Y_{\mathbf{8}c}^{(4)})\right),\\
&Y_{\mathbf{8}l}^{(4)} = \sqrt{6}Y_{\mathbf{8}a}^{(4)} - 300Y_{\mathbf{8}c}^{(4)}, \quad Y_{\mathbf{8}m}^{(4)} = \frac{1}{45}\left(-4\sqrt{6}Y_{\mathbf{8}a}^{(4)} - Y_{\mathbf{8}b}^{(4)} + 150Y_{\mathbf{8}c}^{(4)}\right),\\
&Y_{\mathbf{8}n}^{(4)} = \frac{1}{40}Y_{\mathbf{8}a}^{(4)}, \quad Y_{\mathbf{8}o}^{(4)} = \frac{9}{2\sqrt{2}}Y_{\mathbf{8}a}^{(4)},\\
&Y_{\mathbf{8}p}^{(4)} = -\frac{1}{20\sqrt{2}}Y_{\mathbf{8}a}^{(4)}, \quad Y_{\mathbf{8}q}^{(4)} = \frac{1}{10\sqrt{2}}Y_{\mathbf{8}a}^{(4)}.
\end{aligned} \tag{D.12}$$

These constraints in eqs. (D.7)–(D.12) imply that the linear space of modular forms of weight $k = 4$ and level 7 has dimension 54, as explicitly listed in eqs. (3.8)–(3.16).

There are 82 linearly independent modular forms arising at weight 6 and level $N = 7$ which may be necessary in model construction, we give them in the following,

$$Y^{(6)}_{\mathbf{1}} = \left(Y^{(2)}_{\mathbf{7}} Y^{(4)}_{\mathbf{7}b}\right)_{\mathbf{1}} = Y_{10}Y^{(4)}_{25} + Y_4 Y^{(4)}_{24} + Y^{(4)}_{26}Y_9 + Y^{(4)}_{27}Y_8 + Y^{(4)}_{28}Y_7 + Y^{(4)}_{29}Y_6 + Y^{(4)}_{30}Y_5\,. \tag{D.13}$$

$$Y^{(6)}_{\mathbf{3}a} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{1}a}\right)_{\mathbf{3}} = Y^{(4)}_1 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \tag{D.14}$$

$$Y^{(6)}_{\mathbf{3}b} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{8}a}\right)_{\mathbf{3}} = \begin{pmatrix} \sqrt{3}Y_1Y^{(4)}_{31} + Y_1Y^{(4)}_{32} - \sqrt{6}Y_2Y^{(4)}_{38} - \sqrt{6}Y_3Y^{(4)}_{36} \\ \sqrt{6}Y_1Y^{(4)}_{33} - \sqrt{3}Y_2Y^{(4)}_{31} + Y_2Y^{(4)}_{32} - \sqrt{6}Y_3Y^{(4)}_{37} \\ \sqrt{6}Y_1Y^{(4)}_{35} + \sqrt{6}Y_2Y^{(4)}_{34} - 2Y_3Y^{(4)}_{32} \end{pmatrix}. \tag{D.15}$$

$$Y^{(6)}_{\bar{\mathbf{3}}} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{3}a}\right)_{\bar{\mathbf{3}}} = \begin{pmatrix} Y_2Y^{(4)}_4 - Y_3Y^{(4)}_3 \\ Y_3Y^{(4)}_2 - Y_1Y^{(4)}_4 \\ Y_1Y^{(4)}_3 - Y_2Y^{(4)}_2 \end{pmatrix}. \tag{D.16}$$

$$Y^{(6)}_{\mathbf{6}a} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{3}a}\right)_{\mathbf{6}} = \begin{pmatrix} \sqrt{2}Y_3Y^{(4)}_4 \\ \sqrt{2}Y_1Y^{(4)}_2 \\ Y_1Y^{(4)}_3 + Y_2Y^{(4)}_2 \\ \sqrt{2}Y_2Y^{(4)}_3 \\ Y_1Y^{(4)}_4 + Y_3Y^{(4)}_2 \\ Y_2Y^{(4)}_4 + Y_3Y^{(4)}_3 \end{pmatrix}, \tag{D.17}$$

$$Y^{(6)}_{\mathbf{6}b} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{7}b}\right)_{\mathbf{6}} = \begin{pmatrix} 2Y_1Y^{(4)}_{24} - 2\sqrt{2}Y_2Y^{(4)}_{30} - \sqrt{2}Y_3Y^{(4)}_{28} \\ -\sqrt{2}Y_1Y^{(4)}_{25} + 2Y_2Y^{(4)}_{24} - 2\sqrt{2}Y_3Y^{(4)}_{29} \\ -Y_1Y^{(4)}_{26} + 3Y_2Y^{(4)}_{25} - 2Y_3Y^{(4)}_{30} \\ -2\sqrt{2}Y_1Y^{(4)}_{27} - \sqrt{2}Y_2Y^{(4)}_{26} + 2Y_3Y^{(4)}_{24} \\ 3Y_1Y^{(4)}_{28} - 2Y_2Y^{(4)}_{27} - Y_3Y^{(4)}_{25} \\ -2Y_1Y^{(4)}_{29} - Y_2Y^{(4)}_{28} + 3Y_3Y^{(4)}_{26} \end{pmatrix}. \tag{D.18}$$

$$Y^{(6)}_{\mathbf{7}a} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{6}a}\right)_{\mathbf{7}} = \begin{pmatrix} 2Y_1Y^{(4)}_{10} + 2Y_2Y^{(4)}_9 + 2Y_3Y^{(4)}_7 \\ -2\sqrt{2}Y_2Y^{(4)}_{10} - 2Y_3Y^{(4)}_8 \\ -2Y_1Y^{(4)}_5 - 2\sqrt{2}Y_3Y^{(4)}_9 \\ 3Y_1Y^{(4)}_6 - Y_2Y^{(4)}_5 - \sqrt{2}Y_3Y^{(4)}_{10} \\ -2\sqrt{2}Y_1Y^{(4)}_7 - 2Y_2Y^{(4)}_6 \\ -Y_1Y^{(4)}_8 - \sqrt{2}Y_2Y^{(4)}_7 + 3Y_3Y^{(4)}_5 \\ -\sqrt{2}Y_1Y^{(4)}_9 + 3Y_2Y^{(4)}_8 - Y_3Y^{(4)}_6 \end{pmatrix}, \tag{D.19}$$

$$Y^{(6)}_{\mathbf{7}b} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{6}b}\right)_{\mathbf{7}} = \begin{pmatrix} 2Y_1Y^{(4)}_{16} + 2Y_2Y^{(4)}_{15} + 2Y_3Y^{(4)}_{13} \\ -2\sqrt{2}Y_2Y^{(4)}_{16} - 2Y_3Y^{(4)}_{14} \\ -2Y_1Y^{(4)}_{11} - 2\sqrt{2}Y_3Y^{(4)}_{15} \\ 3Y_1Y^{(4)}_{12} - Y_2Y^{(4)}_{11} - \sqrt{2}Y_3Y^{(4)}_{16} \\ -2\sqrt{2}Y_1Y^{(4)}_{13} - 2Y_2Y^{(4)}_{12} \\ -Y_1Y^{(4)}_{14} - \sqrt{2}Y_2Y^{(4)}_{13} + 3Y_3Y^{(4)}_{11} \\ -\sqrt{2}Y_1Y^{(4)}_{15} + 3Y_2Y^{(4)}_{14} - Y_3Y^{(4)}_{12} \end{pmatrix}, \tag{D.20}$$

$$Y^{(6)}_{\mathbf{7}c} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{7}b}\right)_{\mathbf{7}} = \begin{pmatrix} \sqrt{2}Y_1Y^{(4)}_{30} + \sqrt{2}Y_2Y^{(4)}_{29} + \sqrt{2}Y_3Y^{(4)}_{27} \\ -\sqrt{2}Y_1Y^{(4)}_{24} - 2Y_3Y^{(4)}_{28} \\ -\sqrt{2}Y_2Y^{(4)}_{24} - 2Y_1Y^{(4)}_{25} \\ -Y_1Y^{(4)}_{26} + Y_2Y^{(4)}_{25} + 2Y_3Y^{(4)}_{30} \\ -\sqrt{2}Y_3Y^{(4)}_{24} - 2Y_2Y^{(4)}_{26} \\ Y_1Y^{(4)}_{28} + 2Y_2Y^{(4)}_{27} - Y_3Y^{(4)}_{25} \\ 2Y_1Y^{(4)}_{29} - Y_2Y^{(4)}_{28} + Y_3Y^{(4)}_{26} \end{pmatrix}, \tag{D.21}$$

$$Y^{(6)}_{\mathbf{7}d} = \left(Y^{(2)}_{\mathbf{7}} Y^{(4)}_{\mathbf{1}a}\right)_{\mathbf{7}} = Y^{(4)}_1 \begin{pmatrix} Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \\ Y_{10} \end{pmatrix} = Y^{(4)}_1 Y^{(2)}_{\mathbf{7}}(\tau)\,. \tag{D.22}$$

$$Y^{(6)}_{\mathbf{8}a} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{6}b}\right)_{\mathbf{7}} = \begin{pmatrix} Y_1Y^{(4)}_{16} + Y_2Y^{(4)}_{15} - 2Y_3Y^{(4)}_{13} \\ \sqrt{3}Y_2Y^{(4)}_{15} - \sqrt{3}Y_1Y^{(4)}_{16} \\ \sqrt{2}Y_2Y^{(4)}_{16} - 2Y_3Y^{(4)}_{14} \\ \sqrt{2}Y_3Y^{(4)}_{15} - 2Y_1Y^{(4)}_{11} \\ 2Y_2Y^{(4)}_{11} - \sqrt{2}Y_3Y^{(4)}_{16} \\ 2Y_2Y^{(4)}_{12} - \sqrt{2}Y_1Y^{(4)}_{13} \\ \sqrt{2}Y_2Y^{(4)}_{13} - 2Y_1Y^{(4)}_{14} \\ \sqrt{2}Y_1Y^{(4)}_{15} - 2Y_3Y^{(4)}_{12} \end{pmatrix}, \tag{D.23}$$

$$Y^{(6)}_{\mathbf{8}b} = \left(Y^{(2)}_{\mathbf{3}} Y^{(4)}_{\mathbf{7}b}\right)_{\mathbf{7}} = \begin{pmatrix} 5Y_1Y^{(4)}_{30} - Y_2Y^{(4)}_{29} - 4Y_3Y^{(4)}_{27} \\ -\sqrt{3}Y_1Y^{(4)}_{30} + 3\sqrt{3}Y_2Y^{(4)}_{29} - 2\sqrt{3}Y_3Y^{(4)}_{27} \\ -4Y_1Y^{(4)}_{24} - 3\sqrt{2}Y_2Y^{(4)}_{30} + 2\sqrt{2}Y_3Y^{(4)}_{28} \\ 2\sqrt{2}Y_1Y^{(4)}_{25} - 4Y_2Y^{(4)}_{24} - 3\sqrt{2}Y_3Y^{(4)}_{29} \\ 4\sqrt{2}Y_1Y^{(4)}_{26} + 2\sqrt{2}Y_2Y^{(4)}_{25} + \sqrt{2}Y_3Y^{(4)}_{30} \\ 3\sqrt{2}Y_1Y^{(4)}_{27} - 2\sqrt{2}Y_2Y^{(4)}_{26} + 4Y_3Y^{(4)}_{24} \\ -2\sqrt{2}Y_1Y^{(4)}_{28} - \sqrt{2}Y_2Y^{(4)}_{27} - 4\sqrt{2}Y_3Y^{(4)}_{25} \\ -\sqrt{2}Y_1Y^{(4)}_{29} - 4\sqrt{2}Y_2Y^{(4)}_{28} - 2\sqrt{2}Y_3Y^{(4)}_{26} \end{pmatrix}, \tag{D.24}$$

$$Y^{(6)}_{\mathbf{8}c} = \left(Y^{(2)}_{\mathbf{8}a} Y^{(4)}_{\mathbf{1}a}\right)_{\mathbf{8}} = Y^{(4)}_1 \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{14} \\ Y_{15} \\ Y_{16} \\ Y_{17} \\ Y_{18} \end{pmatrix} = Y^{(4)}_1 Y^{(2)}_{\mathbf{8}a}(\tau)\,, \tag{D.25}$$

$$Y^{(6)}_{\mathbf{8}d} = \left(Y^{(2)}_{\mathbf{8}b} Y^{(4)}_{\mathbf{1}a}\right)_{\mathbf{8}} = Y^{(4)}_1 \begin{pmatrix} Y_{19} \\ Y_{20} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{25} \\ Y_{26} \end{pmatrix} = Y^{(4)}_1 Y^{(2)}_{\mathbf{8}b}(\tau)\,. \tag{D.26}$$

For modular forms of weight 8 ($k = 4$), we find

$$Y^{(8)}_{\mathbf{1}} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{1}a}\right)_{\mathbf{1}} = (Y^{(4)}_1)^2\,. \tag{D.27}$$

$$Y^{(8)}_{\mathbf{3}a} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{3}a}\right)_{\mathbf{3}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_2 \\ Y^{(4)}_3 \\ Y^{(4)}_4 \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{3}a}\,, \tag{D.28}$$

$$Y^{(8)}_{\mathbf{3}b} = \left(Y^{(4)}_{\mathbf{3}a} Y^{(4)}_{\mathbf{8}a}\right)_{\mathbf{3}} = \begin{pmatrix} \sqrt{3}Y^{(4)}_2 Y^{(4)}_{31} + Y^{(4)}_2 Y^{(4)}_{32} - \sqrt{6}Y^{(4)}_3 Y^{(4)}_{38} - \sqrt{6}Y^{(4)}_{36} Y^{(4)}_4 \\ \sqrt{6}Y^{(4)}_2 Y^{(4)}_{33} - \sqrt{3}Y^{(4)}_3 Y^{(4)}_{31} + Y^{(4)}_3 Y^{(4)}_{32} - \sqrt{6}Y^{(4)}_{37} Y^{(4)}_4 \\ \sqrt{6}Y^{(4)}_2 Y^{(4)}_{35} + \sqrt{6}Y^{(4)}_3 Y^{(4)}_{34} - 2Y^{(4)}_{32} Y^{(4)}_4 \end{pmatrix}. \tag{D.29}$$

$$Y^{(8)}_{\bar{\mathbf{3}}} = \left(Y^{(4)}_{\mathbf{3}a} Y^{(4)}_{\mathbf{6}a}\right)_{\bar{\mathbf{3}}} = \begin{pmatrix} \sqrt{2}Y^{(4)}_2 Y^{(4)}_9 + Y^{(4)}_3 Y^{(4)}_8 + Y^{(4)}_4 Y^{(4)}_6 \\ Y^{(4)}_2 Y^{(4)}_8 + \sqrt{2}Y^{(4)}_3 Y^{(4)}_7 + Y^{(4)}_4 Y^{(4)}_5 \\ \sqrt{2}Y^{(4)}_{10} Y^{(4)}_4 + Y^{(4)}_2 Y^{(4)}_6 + Y^{(4)}_3 Y^{(4)}_5 \end{pmatrix}, \tag{D.30}$$

$$Y^{(8)}_{\mathbf{6}a} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{6}a}\right)_{\mathbf{6}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_5 \\ Y^{(4)}_6 \\ Y^{(4)}_7 \\ Y^{(4)}_8 \\ Y^{(4)}_9 \\ Y^{(4)}_{10} \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{6}a}\,, \tag{D.31}$$

$$Y^{(8)}_{\mathbf{6}b} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{6}b}\right)_{\mathbf{6}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_{11} \\ Y^{(4)}_{12} \\ Y^{(4)}_{13} \\ Y^{(4)}_{14} \\ Y^{(4)}_{15} \\ Y^{(4)}_{16} \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{6}b}\,, \tag{D.32}$$

$$Y^{(8)}_{\mathbf{6}c} = \left(Y^{(4)}_{\mathbf{3}a} Y^{(4)}_{\mathbf{3}a}\right)_{\mathbf{6}} = \begin{pmatrix} \sqrt{2}(Y^{(4)}_4)^2 \\ \sqrt{2}(Y^{(4)}_2)^2 \\ 2Y^{(4)}_2 Y^{(4)}_3 \\ \sqrt{2}(Y^{(4)}_3)^2 \\ 2Y^{(4)}_2 Y^{(4)}_4 \\ 2Y^{(4)}_3 Y^{(4)}_4 \end{pmatrix}, \tag{D.33}$$

$$Y^{(8)}_{\mathbf{6}d}=\left(Y^{(4)}_{\mathbf{3}a}Y^{(4)}_{\mathbf{7}a}\right)_{\mathbf{6}}=\begin{pmatrix}2Y^{(4)}_{17}Y^{(4)}_{2}-\sqrt{2}Y^{(4)}_{21}Y^{(4)}_{4}-2\sqrt{2}Y^{(4)}_{23}Y^{(4)}_{3}\\2Y^{(4)}_{17}Y^{(4)}_{3}-\sqrt{2}Y^{(4)}_{18}Y^{(4)}_{2}-2\sqrt{2}Y^{(4)}_{22}Y^{(4)}_{4}\\3Y^{(4)}_{18}Y^{(4)}_{3}-Y^{(4)}_{19}Y^{(4)}_{2}-2Y^{(4)}_{23}Y^{(4)}_{4}\\2Y^{(4)}_{17}Y^{(4)}_{4}-\sqrt{2}Y^{(4)}_{19}Y^{(4)}_{3}-2\sqrt{2}Y^{(4)}_{2}Y^{(4)}_{20}\\-Y^{(4)}_{18}Y^{(4)}_{4}+3Y^{(4)}_{2}Y^{(4)}_{21}-2Y^{(4)}_{20}Y^{(4)}_{3}\\3Y^{(4)}_{19}Y^{(4)}_{4}-2Y^{(4)}_{2}Y^{(4)}_{22}-Y^{(4)}_{21}Y^{(4)}_{3}\end{pmatrix}. \tag{D.34}$$

$$Y^{(8)}_{\mathbf{7}a}=\left(Y^{(4)}_{\mathbf{1}a}Y^{(4)}_{\mathbf{7}a}\right)_{\mathbf{7}}=Y^{(4)}_{1}\begin{pmatrix}Y^{(4)}_{17}\\Y^{(4)}_{18}\\Y^{(4)}_{19}\\Y^{(4)}_{20}\\Y^{(4)}_{21}\\Y^{(4)}_{22}\\Y^{(4)}_{23}\end{pmatrix}=Y^{(4)}_{1}Y^{(4)}_{\mathbf{7}a}, \tag{D.35}$$

$$Y^{(8)}_{\mathbf{7}b}=\left(Y^{(4)}_{\mathbf{1}a}Y^{(4)}_{\mathbf{7}b}\right)_{\mathbf{7}}=Y^{(4)}_{1}\begin{pmatrix}Y^{(4)}_{24}\\Y^{(4)}_{25}\\Y^{(4)}_{26}\\Y^{(4)}_{27}\\Y^{(4)}_{28}\\Y^{(4)}_{29}\\Y^{(4)}_{30}\end{pmatrix}=Y^{(4)}_{1}Y^{(4)}_{\mathbf{7}b}, \tag{D.36}$$

$$Y^{(8)}_{\mathbf{7}c}=\left(Y^{(4)}_{\mathbf{3}a}Y^{(4)}_{\mathbf{6}a}\right)_{\mathbf{7}}=\begin{pmatrix}2Y^{(4)}_{10}Y^{(4)}_{2}+2Y^{(4)}_{3}Y^{(4)}_{9}+2Y^{(4)}_{4}Y^{(4)}_{7}\\-2\sqrt{2}Y^{(4)}_{10}Y^{(4)}_{3}-2Y^{(4)}_{4}Y^{(4)}_{8}\\-2Y^{(4)}_{2}Y^{(4)}_{5}-2\sqrt{2}Y^{(4)}_{4}Y^{(4)}_{9}\\-\sqrt{2}Y^{(4)}_{10}Y^{(4)}_{4}+3Y^{(4)}_{2}Y^{(4)}_{6}-Y^{(4)}_{3}Y^{(4)}_{5}\\-2\sqrt{2}Y^{(4)}_{2}Y^{(4)}_{7}-2Y^{(4)}_{3}Y^{(4)}_{6}\\-Y^{(4)}_{2}Y^{(4)}_{8}-\sqrt{2}Y^{(4)}_{3}Y^{(4)}_{7}+3Y^{(4)}_{4}Y^{(4)}_{5}\\-\sqrt{2}Y^{(4)}_{2}Y^{(4)}_{9}+3Y^{(4)}_{3}Y^{(4)}_{8}-Y^{(4)}_{4}Y^{(4)}_{6}\end{pmatrix}, \tag{D.37}$$

$$Y^{(8)}_{\mathbf{7}d}=\left(Y^{(4)}_{\mathbf{3}a}Y^{(4)}_{\mathbf{6}b}\right)_{\mathbf{7}}=\begin{pmatrix}2Y^{(4)}_{13}Y^{(4)}_{4}+2Y^{(4)}_{15}Y^{(4)}_{3}+2Y^{(4)}_{16}Y^{(4)}_{2}\\-2Y^{(4)}_{14}Y^{(4)}_{4}-2\sqrt{2}Y^{(4)}_{16}Y^{(4)}_{3}\\-2Y^{(4)}_{11}Y^{(4)}_{2}-2\sqrt{2}Y^{(4)}_{15}Y^{(4)}_{4}\\-Y^{(4)}_{11}Y^{(4)}_{3}+3Y^{(4)}_{12}Y^{(4)}_{2}-\sqrt{2}Y^{(4)}_{16}Y^{(4)}_{4}\\-2Y^{(4)}_{12}Y^{(4)}_{3}-2\sqrt{2}Y^{(4)}_{13}Y^{(4)}_{2}\\3Y^{(4)}_{11}Y^{(4)}_{4}-\sqrt{2}Y^{(4)}_{13}Y^{(4)}_{3}-Y^{(4)}_{14}Y^{(4)}_{2}\\-Y^{(4)}_{12}Y^{(4)}_{4}+3Y^{(4)}_{14}Y^{(4)}_{3}-\sqrt{2}Y^{(4)}_{15}Y^{(4)}_{2}\end{pmatrix}. \tag{D.38}$$

$$Y^{(8)}_{\mathbf{8}a} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{8}a}\right)_{\mathbf{8}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_{31} \\ Y^{(4)}_{32} \\ Y^{(4)}_{33} \\ Y^{(4)}_{34} \\ Y^{(4)}_{35} \\ Y^{(4)}_{36} \\ Y^{(4)}_{37} \\ Y^{(4)}_{38} \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{8}a}\,, \tag{D.39}$$

$$Y^{(8)}_{\mathbf{8}b} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{8}b}\right)_{\mathbf{8}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_{39} \\ Y^{(4)}_{40} \\ Y^{(4)}_{41} \\ Y^{(4)}_{42} \\ Y^{(4)}_{43} \\ Y^{(4)}_{44} \\ Y^{(4)}_{45} \\ Y^{(4)}_{46} \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{8}b}\,, \tag{D.40}$$

$$Y^{(8)}_{\mathbf{8}c} = \left(Y^{(4)}_{\mathbf{1}a} Y^{(4)}_{\mathbf{8}c}\right)_{\mathbf{8}} = Y^{(4)}_1 \begin{pmatrix} Y^{(4)}_{47} \\ Y^{(4)}_{48} \\ Y^{(4)}_{49} \\ Y^{(4)}_{50} \\ Y^{(4)}_{51} \\ Y^{(4)}_{52} \\ Y^{(4)}_{53} \\ Y^{(4)}_{54} \end{pmatrix} = Y^{(4)}_1 Y^{(4)}_{\mathbf{8}c}\,, \tag{D.41}$$

$$Y^{(8)}_{\mathbf{8}d} = \left(Y^{(4)}_{\mathbf{3}a} Y^{(4)}_{\mathbf{6}a}\right)_{\mathbf{8}} = \begin{pmatrix} Y^{(4)}_{10} Y^{(4)}_{2} + Y^{(4)}_{3} Y^{(4)}_{9} - 2Y^{(4)}_{4} Y^{(4)}_{7} \\ \sqrt{3} Y^{(4)}_{3} Y^{(4)}_{9} - \sqrt{3} Y^{(4)}_{10} Y^{(4)}_{2} \\ \sqrt{2} Y^{(4)}_{10} Y^{(4)}_{3} - 2Y^{(4)}_{4} Y^{(4)}_{8} \\ \sqrt{2} Y^{(4)}_{4} Y^{(4)}_{9} - 2Y^{(4)}_{2} Y^{(4)}_{5} \\ 2Y^{(4)}_{3} Y^{(4)}_{5} - \sqrt{2} Y^{(4)}_{10} Y^{(4)}_{4} \\ 2Y^{(4)}_{3} Y^{(4)}_{6} - \sqrt{2} Y^{(4)}_{2} Y^{(4)}_{7} \\ \sqrt{2} Y^{(4)}_{3} Y^{(4)}_{7} - 2Y^{(4)}_{2} Y^{(4)}_{8} \\ \sqrt{2} Y^{(4)}_{2} Y^{(4)}_{9} - 2Y^{(4)}_{4} Y^{(4)}_{6} \end{pmatrix}, \tag{D.42}$$

$$Y_{\mathbf{8}e}^{(8)} = \left(Y_{\mathbf{3}a}^{(4)} Y_{\mathbf{6}b}^{(4)}\right)_{\mathbf{8}} = \begin{pmatrix} -2Y_{13}^{(4)}Y_4^{(4)} + Y_{15}^{(4)}Y_3^{(4)} + Y_{16}^{(4)}Y_2^{(4)} \\ \sqrt{3}Y_{15}^{(4)}Y_3^{(4)} - \sqrt{3}Y_{16}^{(4)}Y_2^{(4)} \\ \sqrt{2}Y_{16}^{(4)}Y_3^{(4)} - 2Y_{14}^{(4)}Y_4^{(4)} \\ \sqrt{2}Y_{15}^{(4)}Y_4^{(4)} - 2Y_{11}^{(4)}Y_2^{(4)} \\ 2Y_{11}^{(4)}Y_3^{(4)} - \sqrt{2}Y_{16}^{(4)}Y_4^{(4)} \\ 2Y_{12}^{(4)}Y_3^{(4)} - \sqrt{2}Y_{13}^{(4)}Y_2^{(4)} \\ \sqrt{2}Y_{13}^{(4)}Y_3^{(4)} - 2Y_{14}^{(4)}Y_2^{(4)} \\ \sqrt{2}Y_{15}^{(4)}Y_2^{(4)} - 2Y_{12}^{(4)}Y_4^{(4)} \end{pmatrix}, \tag{D.43}$$

$$Y_{\mathbf{8}f}^{(8)} = \left(Y_{\mathbf{3}a}^{(4)} Y_{\mathbf{7}a}^{(4)}\right)_{\mathbf{8}} = \begin{pmatrix} 5Y_2^{(4)}Y_{23}^{(4)} - 4Y_{20}^{(4)}Y_4^{(4)} - Y_{22}^{(4)}Y_3^{(4)} \\ -\sqrt{3}Y_2^{(4)}Y_{23}^{(4)} - 2\sqrt{3}Y_{20}^{(4)}Y_4^{(4)} + 3\sqrt{3}Y_{22}^{(4)}Y_3^{(4)} \\ -4Y_{17}^{(4)}Y_2^{(4)} + 2\sqrt{2}Y_{21}^{(4)}Y_4^{(4)} - 3\sqrt{2}Y_{23}^{(4)}Y_3^{(4)} \\ -4Y_{17}^{(4)}Y_3^{(4)} + 2\sqrt{2}Y_{18}^{(4)}Y_2^{(4)} - 3\sqrt{2}Y_{22}^{(4)}Y_4^{(4)} \\ 2\sqrt{2}Y_{18}^{(4)}Y_3^{(4)} + 4\sqrt{2}Y_{19}^{(4)}Y_2^{(4)} + \sqrt{2}Y_{23}^{(4)}Y_4^{(4)} \\ 4Y_{17}^{(4)}Y_4^{(4)} - 2\sqrt{2}Y_{19}^{(4)}Y_3^{(4)} + 3\sqrt{2}Y_2^{(4)}Y_{20}^{(4)} \\ -4\sqrt{2}Y_{18}^{(4)}Y_4^{(4)} - 2\sqrt{2}Y_2^{(4)}Y_{21}^{(4)} - \sqrt{2}Y_{20}^{(4)}Y_3^{(4)} \\ -2\sqrt{2}Y_{19}^{(4)}Y_4^{(4)} + \left(-\sqrt{2}\right)Y_2^{(4)}Y_{22}^{(4)} - 4\sqrt{2}Y_{21}^{(4)}Y_3^{(4)} \end{pmatrix}. \tag{D.44}$$

Because of space limitation, we only present modular forms transforming as $\mathbf{3}$ and $\bar{\mathbf{3}}$ under Γ_7 in the following. There are three linearly independent triplet modular forms $Y_{\mathbf{3}a}^{(10)}$, $Y_{\mathbf{3}b}^{(10)}$ and $Y_{\mathbf{3}c}^{(10)}$ of weight 10:

$$\begin{aligned} Y_{\mathbf{3}a}^{(10)} &= \left(Y_{\mathbf{1}a}^{(4)} Y_{\mathbf{3}a}^{(6)}\right)_{\mathbf{3}} = (Y_1^{(4)})^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \\ Y_{\mathbf{3}b}^{(10)} &= \left(Y_{\mathbf{1}a}^{(4)} Y_{\mathbf{3}b}^{(6)}\right)_{\mathbf{3}} = Y_1^{(4)} \begin{pmatrix} Y_1\left(\sqrt{3}Y_{31}^{(4)} + Y_{32}^{(4)}\right) - \sqrt{6}(Y_2Y_{38}^{(4)} + Y_3Y_{36}^{(4)}) \\ \sqrt{6}(Y_1Y_{33}^{(4)} - Y_3Y_{37}^{(4)}) + Y_2\left(Y_{32}^{(4)} - \sqrt{3}Y_{31}^{(4)}\right) \\ \sqrt{6}(Y_1Y_{35}^{(4)} + Y_2Y_{34}^{(4)}) - 2Y_3Y_{32}^{(4)} \end{pmatrix}, \\ Y_{\mathbf{3}c}^{(10)} &= \left(Y_{\mathbf{3}a}^{(4)} Y_{\mathbf{1}}^{(6)}\right)_{\mathbf{3}} = (Y_{10}Y_{25}^{(4)} + Y_4Y_{24}^{(4)} + Y_{26}^{(4)}Y_9 + Y_{27}^{(4)}Y_8 + Y_{28}^{(4)}Y_7 + Y_{29}^{(4)}Y_6 + Y_{30}^{(4)}Y_5) \begin{pmatrix} Y_2^{(4)} \\ Y_3^{(4)} \\ Y_4^{(4)} \end{pmatrix}. \end{aligned} \tag{D.45}$$

We have two linearly independent triplet modular forms $Y_{\bar{\mathbf{3}}a}^{(10)}$ and $Y_{\bar{\mathbf{3}}b}^{(10)}$ of weight 10, which can be chosen as

$$\begin{aligned} Y_{\bar{\mathbf{3}}a}^{(10)} &= \left(Y_{\mathbf{1}a}^{(4)} Y_{\bar{\mathbf{3}}}^{(6)}\right)_{\bar{\mathbf{3}}} = Y_1^{(4)} \begin{pmatrix} Y_2Y_4^{(4)} - Y_3Y_3^{(4)} \\ Y_3Y_2^{(4)} - Y_1Y_4^{(4)} \\ Y_1Y_3^{(4)} - Y_2Y_2^{(4)} \end{pmatrix}, \\ Y_{\bar{\mathbf{3}}b}^{(10)} &= \left(Y_{\mathbf{3}a}^{(4)} Y_{\mathbf{3}b}^{(6)}\right)_{\bar{\mathbf{3}}} \\ &= \begin{pmatrix} Y_3^{(4)}\left(\sqrt{6}Y_1Y_{35}^{(4)} + \sqrt{6}Y_2Y_{34}^{(4)} - 2Y_3Y_{32}^{(4)}\right) + Y_4^{(4)}\left(-\sqrt{6}Y_1Y_{33}^{(4)} + \sqrt{3}Y_2Y_{31}^{(4)} - Y_2Y_{32}^{(4)} + \sqrt{6}Y_3Y_{37}^{(4)}\right) \\ -\sqrt{6}Y_2^{(4)}(Y_1Y_{35}^{(4)} + Y_2Y_{34}^{(4)}) + Y_1Y_4^{(4)}\left(\sqrt{3}Y_{31}^{(4)} + Y_{32}^{(4)}\right) - \sqrt{6}Y_4^{(4)}(Y_2Y_{38}^{(4)} + Y_3Y_{36}^{(4)}) + 2Y_3Y_2^{(4)}Y_{32}^{(4)} \\ Y_2^{(4)}\left(\sqrt{6}Y_1Y_{33}^{(4)} - \sqrt{3}Y_2Y_{31}^{(4)} + Y_2Y_{32}^{(4)} - \sqrt{6}Y_3Y_{37}^{(4)}\right) + Y_3^{(4)}\left(-\sqrt{3}Y_1Y_{31}^{(4)} - Y_1Y_{32}^{(4)} + \sqrt{6}Y_2Y_{38}^{(4)} + \sqrt{6}Y_3Y_{36}^{(4)}\right) \end{pmatrix}. \end{aligned} \tag{D.46}$$

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