

## D-instanton perturbation theory

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**ABSTRACT:** D-instanton world-volume theory has open string zero modes describing collective coordinates of the instanton. The usual perturbative amplitudes in the D-instanton background suffer from infra-red divergences due to the presence of these zero modes, and the usual approach of analytic continuation in momenta does not work since all open string states on a D-instanton carry strictly zero momentum. String field theory is well-suited for tackling these issues. However we find a new subtlety due to the existence of additional zero modes in the ghost sector. This causes a breakdown of the Siegel gauge, but a different gauge fixing consistent with the BV formalism renders the perturbation theory finite and unambiguous. At each order, this produces extra contribution to the amplitude besides what is obtained from integration over the moduli space of Riemann surfaces.

**KEYWORDS:** D-branes, String Field Theory

**ARXIV EPRINT:** [2002.04043](https://arxiv.org/abs/2002.04043)

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## 1 Introduction

String theory is usually formulated using the world-sheet approach. This expresses all perturbative amplitudes in string theory as integrals over the moduli spaces of Riemann surfaces with punctures, with the integrands computed in terms of appropriate correlation functions in the world-sheet conformal field theory of matter and ghost fields. However the integrands are often singular at the boundaries of the moduli spaces, leading to singular integrals. In many cases one can nevertheless define the integral by analytic continuation in the external momenta. However in some cases, involving mass renormalization and vacuum shift, analytic continuation in external momenta is not enough to remove the divergences. In these cases we need to use string field theory to get well defined finite answers for all physical quantities [1].

The problem becomes particularly acute in the presence of D-instantons<sup>1</sup> — D-branes with Dirichlet boundary condition along all non-compact directions including (euclidean) time, since open strings living on such D-branes do not carry any momenta and therefore the divergences cannot be removed by analytic continuation in external momenta. Often one can give physical arguments as to why the divergences cancel [4–6]; however since this requires combining different amplitudes, after cancelling divergences we are left with a finite ambiguity that cannot be fixed. A particular example of this arose in a recent analysis of D-instanton contribution to two dimensional string theory [7]. However, since string field theory is a regular ultra-violet finite quantum field theory with well defined action (up to field redefinition) we do not expect any ambiguity to arise in computation of amplitudes in string field theory. Indeed, one such ambiguity in the two dimensional string

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<sup>1</sup>D-instantons have been recently explored in string field theory in a different context — as classical solution on multiple D3-branes [2, 3]. Our goal here is to study perturbation theory in the presence of D-instantons.

theory was eventually resolved using string field theory, leading to results in agreement with those in the dual matrix model [8].

The divergences in the world-sheet theory in the presence of D-instantons arise from various sources. The first source comprises the collective coordinates of the D-instanton associated with the freedom of translating the D-instanton along the space-time directions transverse to the brane. These collective coordinates correspond to zero modes in string field theory<sup>2</sup> — modes with vanishing quadratic term in the action. Therefore the propagator diverges, leading to divergences in the perturbative amplitude. In the world-sheet description, these show up as logarithmic divergences in the integral over the moduli spaces of Riemann surfaces with punctures. While the conventional world-sheet approach does not give us a systematic procedure for dealing with these divergences, the treatment of these collective modes in string field theory is the same as in ordinary quantum field theory. Instead of treating these modes perturbatively, we leave them unintegrated at the beginning, evaluate the Feynman diagrams using the propagators of the other modes, and after summing over all the Feynman diagrams we integrate over the collective modes. This is expected to recover the energy-momentum conserving delta function which is initially absent in the presence of D-instantons, since space-time translation invariance is broken. This gives an unambiguous procedure for treating the divergence associated with the collective modes. Indeed this treatment of the collective modes was used in [8] to fix a constant in two dimensional string theory that remains ambiguous in the usual world-sheet approach.

The second source of divergence in the perturbative amplitudes, expressed as integrals over moduli spaces of punctured Riemann surfaces, can be traced to open string tachyons if they are present. Normally theories with tachyons are not sensible, unless we can find a new field configuration where tachyons are absent, but D-instantons are different in this respect. The presence of tachyonic open string state on a D-instanton implies that the D-instanton represents a saddle point of the action and therefore the weight factor  $e^S$  in the path integral has a local extremum instead of a local maximum at the solution. Nevertheless such instantons may give sensible contribution to the path integral, as was convincingly demonstrated in the recent analysis in two dimensional string theory [7, 9]. In fact, D-instantons with tachyonic mode may be present even in supersymmetric string theories, e.g. the non-BPS D-instanton in the type IIA string theory. From the point of view of string field theory, since the open string modes do not carry momentum, there is no difficulty in carrying out perturbation theory with tachyons — the propagator of a mode of mass  $m$  is given by  $1/m^2$  irrespective of whether  $m^2$  is positive or negative. However its world-sheet representation, where we represent  $1/m^2$  as  $\int_0^\infty ds e^{-m^2 s}$ , diverges for  $m^2 < 0$ . Therefore, if instead of using the world-sheet representation of the amplitude we use the string field theory representation, there is no divergence in perturbation theory. This has been discussed extensively in [10, 11].

There is a third source of divergence that will be the main focus of this paper. This is due to the presence of additional open string zero modes on the D-instanton that are not associated with the collective coordinates of the D-instanton. For D-instantons in bosonic

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<sup>2</sup>Note that the relevant part of string field theory, describing open strings living on the D-instanton, is a zero dimensional field theory. Therefore we shall use the words mode and field interchangeably.

string theory, these are associated with the pair of states  $|0\rangle$  and  $c_1 c_{-1}|0\rangle$ . These states satisfy the Siegel gauge condition  $b_0|\Psi\rangle = 0$  that is normally used in string field theory, but the associated fields have vanishing kinetic term. Therefore the propagators associated with these modes are infinite and perturbative amplitudes diverge. Furthermore, in this case we cannot remove these divergences by treating them as collective modes. The remedy turns out to be to alter the gauge fixing procedure in the zero mode sector — instead of using the gauge fixed action we use the original gauge invariant action in this sector. Of course this can not be done in an ad hoc fashion, but we show that the Batalin-Vilkovisky (BV) formalism [12–14], that underlies the formulation of open-closed string field theory [15–17], allows us to do this. The net effect of this is that instead of using the states  $|0\rangle$  and  $c_1 c_{-1}|0\rangle$  in the expansion of the string field, we need to use the states  $|0\rangle$  and  $c_0|0\rangle$  in the expansion. This leads to well defined perturbation expansion without any divergent propagator.

The rest of the paper is organized as follows. In section 2 we review the organization of the terms in D-instanton perturbation theory. In particular we discuss why we must include in our analysis certain class of disconnected diagrams but exclude other classes of disconnected diagrams. In section 3 we discuss various types of divergences that arise in perturbation theory in the presence of D-instantons and their remedy. In particular, section 3.1 discusses the divergences due to the collective coordinates and open string tachyons, section 3.2 discusses the divergences due to the zero modes from the ghost sector, and section 3.3 contains a summary of the algorithm needed to tackle all the divergences systematically. In section 4 we demonstrate the need for this new treatment of the ghost zero modes by analyzing a specific amplitude — a disk amplitude with four external collective modes of the open string. We show that in order to get the correct result, we must include the contribution of the out of Siegel gauge mode, associated with the state  $c_0|0\rangle$ , in the computation. In section 5 we discuss similar issues for the ghost sector zero modes for D-instantons in superstring theory.

## 2 Diagrammatics of D-instanton contribution

Let us consider a quantum field theory with instanton solutions. In order to identify the instanton contribution to the Green’s function of a collection of operators, which we shall denote by  $\mathcal{O}$ , we shall divide the path integral over the fields  $\Phi$  into different sectors labelled by their instanton number. For simplicity we shall analyze the contribution up to one instanton sector, but the analysis can be easily generalized to the multi-instanton sector. We denote by  $\Phi_p$  the fluctuations around the vacuum solution and by  $\Phi_I$  the fluctuations around the single instanton solution, and express the correlation function of  $\mathcal{O}$  as<sup>3</sup>

$$\langle \mathcal{O} \rangle = \frac{\int D\Phi_p \exp[S_p] \mathcal{O} + \mathcal{N} e^{-C/g_s} \int D\Phi_I \exp[S_I] \mathcal{O}}{\int D\Phi_p \exp[S_p] + \mathcal{N} e^{-C/g_s} \int D\Phi_I \exp[S_I]}, \quad (2.1)$$

where  $-C/g_s$  is the instanton action and  $\mathcal{N}$  is a normalization constant that gives the ratio of the integration measure in the instanton sector and in the perturbative sector.  $S_p$

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<sup>3</sup>Throughout the paper we shall use the convention that the action  $S$  appears in the integrand of Euclidean path integral as  $e^S$ .

denotes the action of the fluctuating fields  $\Phi_p$  around the vacuum solution and  $S_I$  denotes the action of the fluctuating fields  $\Phi_I$  around the one instanton solution.

If the instanton under consideration represents a D-instanton in string theory, then the various terms in this expansion have clear interpretation.  $\int D\Phi_p \exp[S_p] \mathcal{O}$  gives the amplitudes containing world-sheets that do not have any boundary ending on the D-instanton, but we must allow world-sheets with multiple disconnected components, including vacuum bubbles which do not have any external vertex operator insertion.  $\int D\Phi_I \exp[S_I] \mathcal{O}$  gives the perturbative amplitudes containing world-sheets that may have multiple disconnected components, possibly including vacuum bubbles, but at least one of the world-sheets must have at least one boundary ending on the D-instanton. The factors in the denominator have similar interpretation, except that there is no external vertex operator insertion.

Keeping terms containing at most one power of  $e^{-C/g_s}$ , we can expand (2.1) as

$$\begin{aligned} \langle \mathcal{O} \rangle = & \frac{\int D\Phi_p \exp[S_p] \mathcal{O}}{\int D\Phi_p \exp[S_p]} + \mathcal{N} e^{-C/g_s} \frac{\int D\Phi_I \exp[S_I] \mathcal{O}}{\int D\Phi_p \exp[S_p]} \\ & - \mathcal{N} e^{-C/g_s} \frac{\int D\Phi_p \exp[S_p] \mathcal{O}}{\int D\Phi_p \exp[S_p]} \frac{\int D\Phi_I \exp[S_I]}{\int D\Phi_p \exp[S_p]}. \end{aligned} \tag{2.2}$$

We can now interpret the various terms in string theory as follows.

1. The first term is the perturbative amplitude, possibly containing disconnected world-sheets but there should be no boundary ending on D-instanton. The division by the denominator removes from this all factors containing disconnected bubbles. However, disconnected world-sheets are still allowed as long as each component has at least one vertex operator insertion.
2. The second term represents amplitudes in the instanton background, but the division by the denominator removes all factors containing disconnected bubbles *in the perturbative amplitude*. Note that we do not remove bubbles in the instanton background. For D-instantons this means that we sum over world-sheets for which each connected component has either insertion of an external vertex operator, or a boundary ending on the D-instanton, or both.
3. The third term is a subtraction term containing product of two factors. The first one represents the perturbative amplitude with the bubble diagrams removed. The second term represents vacuum bubble diagrams in the presence of the instanton, but containing no factors with perturbative vacuum bubble. For D-instantons this means that we must remove all diagrams in which all the external state vertex operators end on world-sheets without any boundary ending on the D-instanton, even if they are multiplied by vacuum bubbles containing boundaries that do end on the D-instanton.

Therefore the rules for computing a single D-instanton contribution to a given amplitude is to sum over all world-sheet diagrams, possibly containing disconnected components, but subject to the following conditions:

1. Each of these disconnected components must have either at least one boundary ending on the D-instanton or at least one closed string vertex operator.

2. At least one of the disconnected components must have both, a boundary ending on the D-instanton and a closed string vertex operator insertion.

Each such contribution will be multiplied by a single factor of  $\mathcal{N} e^{-1/g_s}$ , irrespective of the number of disconnected components it has.

### 3 Dealing with divergences

We shall use string field theory to evaluate the D-instanton contribution to the physical amplitudes. As will be explained shortly, this is needed to deal with infrared divergences. The string field theory that is relevant for this problem is the interacting field theory of open and closed strings, with open strings satisfying boundary conditions associated with the D-instanton. The collection of open and closed string fields together correspond to the set of fields  $\Phi_I$  in (2.1), ((2.2), with the open strings describing modes that are localized on the D-instanton and closed strings describing modes that are not localized on the instanton. In contrast, the modes  $\Phi_p$  with action  $S_p$  in (2.1), ((2.2) are described by closed string field theory without any D-instanton background.

In any amplitude, the external states of interest will be closed strings (or in general situation open strings living on D-branes other than D-instantons) — the open strings living on the transient D-instantons do not correspond to asymptotic states. However, a subset of the open string fields represent the collective coordinates of the D-instanton, associated with translation along space-time directions, and we cannot carry out the usual perturbation theory in which these zero modes propagate in the internal state, — they have divergent propagator. Therefore in the path integral over the string fields  $\Phi_I$  in (2.2), we must leave these zero modes unintegrated while integrating over all other open string fields, and carry out integration over these zero modes at the very end. In perturbation theory, this means that we must subtract these zero mode contributions from the internal open string propagators, allow arbitrary number of these zero modes to appear as external states together with the closed string states, sum over all Feynman diagrams and all possible number of insertions of the zero mode fields  $\phi$  in amplitudes with a given set of external closed string states, and at the end integrate over these zero modes  $\phi$  explicitly. On physical grounds, these zero mode integrals are expected to restore the space-time momentum conserving delta functions that are otherwise missing in the amplitudes in the presence of D-instantons. In the following we shall discuss the systematic procedure for doing this analysis in string field theory.<sup>4</sup>

#### 3.1 Tachyons and collective modes

The world-sheet expressions for the amplitudes in string theory often diverge from the region where certain number of vertex operators come together, or, more generally, when a Riemann surface with punctures degenerates. Since the divergences of interest to us will

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<sup>4</sup>The only ambiguity that does not seem to be resolved in the current formulation of string field theory is the overall normalization constant  $\mathcal{N}$  in (2.1), (2.2). This is related to the freedom of adding a constant to the string field theory action around the D-instanton.

arise from integration over the open string fields, we shall focus exclusively on these — divergences associated with closed strings, if present, can be dealt with by following the procedure described in [10]. The origin of these divergences in string field theory can be understood by noting that the world-sheet approach replaces the  $1/L_0$  factor in the Siegel gauge open string propagator by:

$$1/L_0 \rightarrow \int_0^1 dq q^{L_0-1}. \tag{3.1}$$

This is an identity for  $L_0 > 0$  but fails for  $L_0 \leq 0$ . For  $L_0 < 0$  the left hand side is well defined but the right hand side is divergent. The world-sheet description of the amplitude uses the right hand side and is therefore divergent, while string field theory uses the left hand side and gives a finite result. Therefore such divergences in the world-sheet amplitude may be dealt with simply by suitably parametrizing the moduli space of Riemann surfaces near degeneration points by variables induced from string field theory, including  $q$ , and then replacing integrals of the form  $\int_0^1 dq q^{\beta-1}$  by  $1/\beta$  for  $\beta < 0$ .

For  $L_0 = 0$  both sides diverge. This is a reflection of the presence of zero mode(s) in the open string sector. While the world-sheet approach does not provide us with a systematic way of dealing with these divergences, in string field theory typically the zero modes would have definite interpretation and therefore there is an unambiguous procedure for dealing with them. In this subsection we shall describe the procedure for dealing with one set of these zero modes — those associated with the collective coordinates of the D-instanton. We shall denote these zero modes collectively by  $\phi$ . As already mentioned, the solution string field theory offers for dealing with such zero modes is to first carry out the path integral over all string fields other than  $\phi$ , for fixed background  $\phi$ , and then carry out the integration over  $\phi$  explicitly. In the world-sheet computation, this translates to the following algorithm [8]:

1. Removing integration over these zero modes in the path integral corresponds to removing the singular contributions due to these zero modes from the internal open string propagators of the Feynman diagrams. In the world-sheet description, this requires parametrizing the moduli space of Riemann surfaces near degeneration points by variables induced from string field theory, including  $q$ , and then removing the singular contribution to the integral proportional to  $\int_0^1 dq/q$  due to these zero modes.
2. Since we are supposed to carry out the path integral with fixed background  $\phi$ , we have to compute amplitudes with external  $\phi$  states (and closed string states) even though we are ultimately interested in amplitudes with external closed strings only. Near each degeneration point we follow the subtraction scheme mentioned in point 1.
3. After computing the relevant amplitude in background  $\phi$ , we carry out integration over  $\phi$ . This is expected to restore momentum conservation that is broken in the presence of a single D-instanton. For example if  $\xi$  denotes the set of collective coordinates associated with space-time translation and  $p$  denotes the total momentum of



external closed strings in an amplitude, then the amplitude is expected to be proportional to  $e^{ip \cdot \xi}$  so that integration over  $\xi$  gives a factor of  $\delta(p)$ . However, this may not be manifest, since the modes  $\phi$  that arise from string field theory may be related to the collective coordinates  $\xi$  by a field redefinition. In that case, the easiest way to see the momentum conserving delta function arising out of the zero mode integration will be to try to use a specific version of string field theory in which the modes  $\phi$  coming from string field theory coincide with the collective coordinates  $\xi$  without any field redefinition [8]. In such cases one recovers the momentum conserving delta function directly from the integration over the zero modes  $\phi$  arising in string field theory. Alternatively, one could use a generic version of string field theory but find the explicit field redefinition that relates the open string modes  $\phi$  to the collective coordinates  $\xi$  that have the coupling proportional to  $e^{ip \cdot \xi}$  [18]. The Jacobian associated with this field redefinition will have to be taken into account in the analysis. After this one can carry out the integration over the  $\xi$  modes and recover the momentum conserving delta function.

Before concluding this subsection, we shall describe the vertex operator for the zero modes associated with the collective coordinates. Let us for definiteness, consider the zero mode associated with translation along the (euclidean) time coordinate. The unintegrated world-sheet vertex operator associated with the corresponding open string state is given by  $c \partial X$  where  $b, c$  denote the usual world-sheet ghost fields and  $X$  is the world-sheet scalar labelling the time direction. The zero modes associated with translation along other directions can be described in a similar way.

### 3.2 Ghost zero modes and the inadequacy of Siegel gauge

Collective coordinates are not the only open string zero modes in string field theory in the presence of a D-instanton — there are other zero modes arising in the ghost sector that require different treatment. In order to understand this we need to begin with a brief review of the BV formalism [12–14].

In the BV formalism for open-closed string field theory [16, 17], we take a generic open string field  $|\Psi_o\rangle$  or closed string field  $|\Psi_c\rangle$  to be a state in the world-sheet CFT of arbitrary ghost number (subject to the condition  $b_0^- |\Psi_c\rangle = 0 = L_0^- |\Psi_c\rangle$  for closed string fields) and expand it as linear combination of a complete set of basis states.<sup>5</sup> The coefficients of expansion are the dynamical variables of the theory, with the coefficients of the open string states of ghost number  $\leq 1$  and closed string states of ghost number  $\leq 2$  considered as fields, and the coefficients of the open string states of ghost number  $\geq 2$  and closed string states of ghost number  $\geq 3$  considered as anti-fields. Up to signs, the pairing between fields and anti-fields is done via BPZ inner product, with an insertion of  $c_0^-$  in the inner product of closed string states. For example, if  $|\varphi_r\rangle$  denotes a basis of open string states of ghost number  $\leq 1$  and  $|\varphi^r\rangle$  is a basis of open string states of ghost number  $\geq 2$ , satisfying the orthonormality condition  $\langle \varphi^r | \varphi_s \rangle = \delta_s^r$ , and if we expand the open string

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<sup>5</sup>We define  $b_0^\pm = (b_0 \pm \bar{b}_0)$ ,  $L_0^\pm = (L_0 \pm \bar{L}_0)$  and  $c_0^\pm = (c_0 \pm \bar{c}_0)/2$ . Furthermore, we assign ghost number 1 to  $c$ ,  $\bar{c}$  and ghost number  $-1$  to  $b$ ,  $\bar{b}$ .



field as  $\sum_r \{\psi_r|\varphi^r\rangle + \psi^r|\varphi_r\rangle\}$ , then  $\psi_r$  is the anti-field of  $\psi^r$  up to a sign. Similarly if  $|\phi_r\rangle$  denotes a basis of closed string states of ghost number  $\leq 2$  and  $|\phi^r\rangle$  is a basis of closed string states of ghost number  $\geq 3$ , each annihilated by  $b_0^-$  and  $L_0^-$ , and satisfying the orthonormality condition  $\langle\phi^r|c_0^-|\phi_s\rangle = \delta_s^r$ , and if we expand the closed string field as  $\sum_r \{\chi_r|\phi^r\rangle + \chi^r|\phi_r\rangle\}$ , then  $\chi_r$  is the anti-field of  $\chi^r$  up to a sign. It is however possible to define new fields and anti-fields by making a symplectic transformation that preserves the anti-bracket. Therefore if we introduce new orthonormal basis  $|\tilde{\varphi}_r\rangle$  and  $|\tilde{\varphi}^r\rangle$  for open string states and  $|\tilde{\phi}_r\rangle$  and  $|\tilde{\phi}^r\rangle$  for closed string states, with  $\langle\tilde{\varphi}^r|\tilde{\varphi}_s\rangle = \delta_s^r$ ,  $\langle\tilde{\varphi}^r|\tilde{\varphi}^s\rangle = 0$ ,  $\langle\tilde{\varphi}_r|\tilde{\varphi}_s\rangle = 0$  and  $\langle\tilde{\phi}^r|c_0^-|\tilde{\phi}_s\rangle = \delta_s^r$ ,  $\langle\tilde{\phi}^r|c_0^-|\tilde{\phi}^s\rangle = 0$ ,  $\langle\tilde{\phi}_r|c_0^-|\tilde{\phi}_s\rangle = 0$ , and expand the open string field as  $\sum_r \{\tilde{\psi}_r|\tilde{\varphi}^r\rangle + \tilde{\psi}^r|\tilde{\varphi}_r\rangle\}$  and the closed string field as  $\sum_r \{\tilde{\chi}_r|\tilde{\phi}^r\rangle + \tilde{\chi}^r|\tilde{\phi}_r\rangle\}$ , then we can treat  $\tilde{\psi}^r$  and  $\tilde{\chi}^r$  as fields and  $\tilde{\psi}_r$  and  $\tilde{\chi}_r$  as the corresponding anti-fields up to sign.

In the BV formalism, the path integral of string field theory, weighted by the exponential of the action, is to be carried out over a Lagrangian submanifold, which corresponds to setting the anti-fields to zero, possibly after making a symplectic transformation. The result of the path integral can be shown to be (formally) independent of the choice of the Lagrangian submanifold. If we use the original definition of fields and anti-fields and define the Lagrangian submanifold to be the subspace  $\psi_r = 0$ ,  $\chi_r = 0$ , then the remaining open string fields have ghost number  $\leq 1$  and the remaining closed string fields have ghost number  $\leq 2$ . Ghost number conservation then implies that the action depends only on the open string fields of ghost number 1 and closed string fields of ghost number 2, i.e. the classical fields. The integration over the open fields of ghost number  $\leq 0$  and closed string fields of ghost number  $\leq 1$  decouple for physical amplitudes, and effectively corresponds to division by the volume of the gauge group. The resulting path integral can be identified as the conventional path integral over all the fields without any gauge fixing, since all the classical fields — open string fields of ghost number 1 and closed string fields of ghost number 2, are to be integrated over. This is formally the correct path integral, but produces singular perturbation expansion, since the gauge symmetry remains unfixed. In particular the kinetic operator will have zero eigenvalues due to the presence of pure gauge states of the form  $Q_B|s\rangle$ .

On the other hand, if we choose to expand the string fields in the new basis  $|\tilde{\varphi}_r\rangle$ ,  $|\tilde{\varphi}^r\rangle$ ,  $|\tilde{\phi}_r\rangle$  and  $|\tilde{\phi}^r\rangle$ , satisfying

$$c_0|\tilde{\varphi}^r\rangle = 0, \quad b_0|\tilde{\varphi}_r\rangle = 0, \quad c_0^+|\tilde{\phi}^r\rangle = 0, \quad b_0^+|\tilde{\phi}_r\rangle = 0, \quad (3.2)$$

and define the Lagrangian submanifold by setting  $\tilde{\psi}_r$  and  $\tilde{\chi}_r$  to zero, then the remaining open string field  $\sum_r \tilde{\psi}^r|\tilde{\varphi}_r\rangle$  and the closed string field  $\sum_r \tilde{\chi}^r|\tilde{\phi}_r\rangle$  satisfy the Siegel gauge conditions  $b_0|\Psi_o\rangle = 0$ ,  $b_0^+|\Psi_c\rangle = 0$ . The resulting path integral is now carried out over fields of all ghost numbers and corresponds to the usual gauge fixed path integral, leading to well defined perturbation theory in a generic open-closed string field theory. We shall see however that in the presence of D-instantons this procedure leads to singular path integral.

For open string fields living on D-instantons, which do not carry any momentum, special care is needed to deal with the ghost excitations carrying  $L_0 = 0$ . For this let us consider the basis states  $|0\rangle$ ,  $c_0|0\rangle$ ,  $c_1c_{-1}|0\rangle$  and  $c_1c_0c_{-1}|0\rangle$ , and expand the open string

field in this sector as

$$\psi^1 c_0|0\rangle + \psi^2|0\rangle + \psi_1 c_1 c_{-1}|0\rangle + \psi_2 c_1 c_0 c_{-1}|0\rangle. \tag{3.3}$$

In the original formulation,  $\psi^1$  and  $\psi^2$  are fields and  $\psi_1$  and  $\psi_2$  are anti-fields. Therefore the gauge invariant path integral will correspond to setting  $\psi_1$  and  $\psi_2$  to 0. On the other hand the Siegel gauge path integral will correspond to setting  $\psi^1$  and  $\psi^2$  to 0. However in this case the quadratic term in the action, being proportional to  $L_0$ , does not depend on the remaining fields  $\psi^2$  and  $\psi_1$  that multiply the  $L_0 = 0$  states. This makes the path integral over  $\psi^2$  and  $\psi_1$  ill defined in perturbation theory. In particular these will lead to additional logarithmic divergences in the loop amplitudes of the type (3.1) with  $L_0 = 0$  which cannot be regarded as due to the collective modes and therefore cannot be removed by the procedure described in section 3.1. This is already visible e.g. in the annulus amplitude analyzed in [7]. To solve this problem we shall choose the Lagrangian submanifold in this sector to be  $\psi_1 = 0, \psi_2 = 0$ , corresponding to the original definition of fields and anti-fields. In this case the quadratic term in the action, proportional to  $\langle \Psi | Q_B | \Psi \rangle$ , does depend on  $\psi^1$  since  $c_0|0\rangle$  is not BRST invariant, but does not depend on  $\psi^2$  since  $|0\rangle$  is BRST invariant. In fact once we integrate out the modes with  $L_0 > 0$ , for which we can use Siegel gauge condition without any problem, the only field in the expansion of  $|\Psi_o\rangle$  multiplying ghost number  $\neq 1$  state is  $\psi^2$  and as a result the whole effective action becomes independent of  $\psi^2$  due to ghost number conservation of world-sheet correlators. Therefore the integration over  $\psi^2$  factors out of the path integral, and its contribution can be absorbed into the overall normalization factor ( $\mathcal{N}$  in (2.2)), leading to well defined perturbation theory.

To understand this point better, it will be useful to recall the physical significance of  $\psi^2$  integration. Since  $\psi^2$  is the coefficient of a ghost number 0 state  $|0\rangle$  of the open string, it represents a gauge transformation parameter, or equivalently the ghost field corresponding to the gauge transformation parameter. BRST invariance of  $|0\rangle$  shows that the gauge transformation under consideration actually represents a rigid gauge transformation. Indeed, this can be identified with the rigid U(1) gauge transformation under which any open string stretching from the D-instanton to another D-brane picks up a constant phase. Therefore integration over  $\psi^2$  corresponds to division by the volume of this U(1) group. Since this is a constant factor, dropping this integral just changes the overall normalization that can be absorbed in  $\mathcal{N}$ .

However, as in the case of the collective coordinates discussed earlier, the open string gauge transformation parameter (equivalently ghost field)  $\psi^2$  may be related to the rigid U(1) gauge transformation parameter  $\theta$  by a complicated field dependent normalization. This can be detected by comparing the gauge transformation in open string field theory generated by  $\psi^2|0\rangle$  with the U(1) gauge transformation that gives a simple phase  $e^{i\theta}$  for any open string stretching from the D-instanton to another D-brane. If there is such a non-trivial field dependent normalization relating  $\psi^2$  and  $\theta$ , we need to change variable from  $\psi^2$  to  $\theta$ , regarding both as grassmann odd ghost fields, and then drop the integration over  $\theta$ . The Jacobian associated with this change of variables will contribute to the integration measure and therefore to the effective action as in the case of integration over the open string zero mode  $\phi$  discussed earlier.

To summarize, while in the  $L_0 \neq 0$  sector we continue to use the Siegel gauge condition  $b_0|\Psi_o\rangle = 0$ , in the  $L_0 = 0$  sector we use the original definition of fields and anti-fields to define the Lagrangian submanifold, i.e. set the components of the open string field with ghost number  $\geq 2$  to zero. This removes the contribution due to the ghost zero modes from the propagator. Therefore, in the perturbative amplitudes, we can remove the  $q^{-1}$  terms in (3.1) arising due to ghost zero modes, just as we would remove the  $q^{-1}$  terms arising from the zero modes associated with the collective coordinate. However we now have to explicitly include the contribution from the  $\psi^1$  propagator — a contribution that is absent in the usual world-sheet expression for the amplitude.

In order to evaluate the contribution to the amplitude due to the  $\psi^1$  field, we shall need the form of the quadratic term in the action of the field  $\psi^1$ . For later use we shall compare this with the quadratic term in the action for the tachyon field  $\psi^0$  multiplying  $c_1|0\rangle$ . If we expand the open string field  $|\Psi_o\rangle$  of ghost number 1 as

$$|\Psi_o\rangle = \psi^0 c_1|0\rangle + \psi^1 c_0|0\rangle + \dots, \tag{3.4}$$

then the quadratic term in the action is given by,

$$\frac{1}{2}\langle\Psi_o|Q_B|\Psi_o\rangle = \frac{1}{2}(\psi^0)^2 + (\psi^1)^2 + \dots, \tag{3.5}$$

where we have used  $\{Q_B, c_0\} = 2 c_1 c_{-1}$ , the normalization convention

$$\langle 0|c_1 c_0 c_{-1}|0\rangle = 1, \tag{3.6}$$

and the fact that the BPZ conjugation, that takes  $|\Psi_o\rangle$  to  $\langle\Psi_o|$ , is generated by  $z \rightarrow -1/z$ . It follows from (3.5) that the propagator of the tachyon  $\psi^0$  is  $-1$ , which agrees with (3.1). (3.5) also shows that in the same normalization, the propagator for  $\psi^1$  is  $-1/2$ .

### 3.3 Summary of the algorithm

We can summarize the procedure for dealing with the divergences associated with open string degeneration as follows:

1. We compute amplitudes involving external on-shell closed string states and arbitrary number of insertions of the on-shell open string zero modes  $\phi$  associated with space-time translation of the D-instanton. These amplitudes can be expressed as integral over the moduli space of punctured Riemann surfaces.
2. Near any degeneration where a pair of open string punctures are sewed together by a long strip, we change variables so that the integral over the moduli space of Riemann surface is expressed as an integral over the parameters arising from string field theory. One of them corresponds to the sewing parameter  $q$  that comes from Schwinger parameter representation of the propagator as given in (3.1). Others are integration parameters that enter into the definition of the interaction vertex of string field theory. In case of multiple degenerations where the Riemann surface has several long strips, there are multiple sewing parameters  $q_1, q_2, \dots$ , — one for each open string propagator.

3. We expand the integrand as a power series in  $q$ . Using (3.1), an integral of the type  $\int_0^1 dq q^{-1+h}$  is replaced by  $1/h$  both for  $h > 0$  and for  $h < 0$ , as long as  $h \neq 0$ . For multiple degenerations, we do this for each variable  $q_i$ .
4. A term in the integral of the form  $\int_0^1 dq q^{-1}$  is set to zero. This corresponds to dropping the path integral over the Siegel gauge states with  $L_0 = 0$ . These include the zero modes  $\phi$  associated with collective coordinates, as well as the zero modes  $\psi_1, \psi^2$  introduced in (3.3). The justification for dropping the path integral over  $\psi_1$  and  $\psi^2$  has been described in section 3.2. On the other hand, as discussed in section 3.1, the integration over the zero modes corresponding to collective coordinates is supposed to be carried out at the end.
5. We need to compare the open string field theory gauge transformation generated by  $\psi^2|0\rangle$  with the simple U(1) gauge transformation that gives a phase  $e^{i\theta}$  for any open string stretched from the D-instanton to another D-brane. If  $\psi^2$  and  $\theta$  are related by field dependent normalization, we need to change variable from  $\psi^2$  to  $\theta$ , regarding both as grassmann odd ghost fields, and then drop integration over  $\theta$ . The Jacobian associated with this change of variables needs to be taken into account in all subsequent computations.
6. We now need to add the contribution from the intermediate  $\psi^1$  state for each open string propagator. This requires computing the relevant amplitude involving insertion of the states  $c_0|0\rangle$  and multiplying it by the  $\psi^1$  propagator computed from (3.5). Since  $c_0|0\rangle$  is not a primary state, the result will depend on the choice of local coordinate system in which the corresponding vertex operator is inserted. This information comes from string field theory.
7. The range  $0 \leq q \leq 1$  typically will span a subspace of the full moduli space near a degeneration point. We can carry out integration over the rest of the moduli space using the original variables, since there are no divergences coming from this region. This corresponds to contribution from contact term vertices in string field theory.
8. After computing the amplitudes by summing over all Feynman diagrams, we sum over all possible number of insertions of  $\phi$  for a given set of external closed string states. This gives a function of the zero modes  $\phi$ . We then integrate over  $\phi$  to get the D-instanton contribution to the closed string amplitude. On general grounds we expect that there exists appropriate change of variables from  $\phi$  to the collective coordinates  $\xi$  so that integration over  $\phi$  reduces to a form proportional to  $\int d\xi e^{ip \cdot \xi}$ , where  $p$  is the total momentum carried by the external closed strings in an amplitude. This will recover the momentum conserving delta function  $\delta(p)$ .

#### 4 Disk four point function

In this section we shall illustrate the breakdown of the Siegel gauge in the perturbative amplitudes. We shall consider the  $\phi$ - $\phi$ - $\phi$ - $\phi$  four point function on the disk, where, for

definiteness, we shall choose  $\phi$  to be the collective mode associated with the freedom of translating the D-instanton along the (euclidean) time direction. Since this is a tree amplitude, and since  $\psi_1$  is not a classical field, we shall not see the need for dropping  $\psi_1$  in the computation of this amplitude, but we shall see the need for including the contribution from the field  $\psi^1$  separately. Furthermore, since the  $\phi$ - $\phi$ - $\phi$  three point coupling vanishes due to time reversal symmetry, we shall not need to remove the contribution of the  $\phi$  field in the internal propagator.

Amplitudes of this type have been analyzed previously in [19–22] for computing effective potential of massless fields. However these computations used a particular form of string interaction vertex which has an additional  $Z_2$  symmetry known as twist symmetry, and due to the use of twist symmetric three point vertex, they did not encounter tree level breakdown of Siegel gauge for this amplitude. Nevertheless a field closely related to  $\psi^1$  was discussed in [22] (called  $\varphi_2$  there) in the context of heterotic string theory, where it was observed that the coupling of this field to a pair of massless fields vanishes due to a specific symmetry, and therefore this field does not appear as intermediate state in the four point scattering amplitude. The role of twist symmetry in our analysis will be discussed later in this section.

Even though our eventual interest is in computing amplitudes with one or more external closed strings, a disk 4-point function with four  $\phi$ 's could arise as a disconnected part of an amplitude with closed strings, e.g. the product of a disk one-point function of a closed string and disk four point function of open strings. For this reason, it is important to evaluate this amplitude. Our analysis will be independent of whether the other coordinates are compact or non-compact, and they may even be replaced by a  $c = 25$  Liouville theory, representing two dimensional string theory.

Before we proceed with the actual computation, let us discuss what result one should expect. The amplitude under consideration can be interpreted as the contribution to the  $\phi^4$  term in the effective action after integrating out all the open string modes other than those associated with the collective coordinates. Since the effective action should be independent of the collective coordinates, and since the field  $\phi$  is associated with the collective coordinate describing translation of the D-instanton along the time direction, the effective action should not depend on  $\phi$ . One might worry that  $\phi$  may be related to the actual collective coordinate by a field redefinition. However, since the effective action is altogether independent of the collective coordinates, no field redefinition can produce a  $\phi$  dependence of the effective action. Therefore we expect the four point amplitude to vanish. This is what we shall now proceed to verify.

If we denote the world-sheet scalar field corresponding to the time coordinate by  $X$ , then the unintegrated vertex operator for  $\phi$  is  $c\partial X$  and the integrated vertex operator is  $\partial X$ . Then, up to a constant of proportionality, the amplitude is given by:

$$A = \int_0^1 dy \langle c\partial X(0)\partial X(y)c\partial X(1)c\partial X(\infty) \rangle = \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} + 1 \right\}. \quad (4.1)$$

Note that we have included the contribution from only the  $0 \leq y \leq 1$  region since the contribution from the other regions are related to these by permutation of the external

states accompanied by  $SL(2,R)$  transformations, and since all external states are identical, they produce identical contributions. The integral (4.1) diverges from the  $y = 0$  and  $y = 1$  regions. In particular, near  $y = 0$  and  $y = 1$  the integrand in (4.1) has double poles indicating tachyon propagation. This is expected, since the operator product of  $\partial X$  with itself generates an identity operator. There is however no  $\partial X$  in the operator product of  $\partial X$  with itself, therefore we do not need to subtract any collective coordinate contribution from the internal propagator. Our goal will be to show how to extract a finite result from (4.1) following the procedure described in section 3. As we shall see, we also need to include the additional contribution due to the  $\psi^1$  propagator that is not present in the usual perturbative world-sheet amplitudes.

In order to proceed, we need to introduce the three point interaction vertex of three open strings. For external off-shell open string states  $A_1, A_2, A_3$  the vertex takes the form:

$$\langle f_1 \circ A_1(0) f_2 \circ A_2(0) f_3 \circ A_3(0) \rangle, \tag{4.2}$$

where  $f_1, f_2$  and  $f_3$  are three conformal transformations, and  $f \circ A$  is the conformal transform of  $A$  by  $f$ . We shall choose the functions  $f_i$  such that  $f_1(0) = 0, f_2(0) = 1, f_3(0) = \infty$ . We also take the vertex to be cyclically symmetric by requiring that the transformation

$$z \rightarrow \frac{1}{1-z}, \tag{4.3}$$

cyclically permutes  $f_1(w), f_2(w)$  and  $f_3(w)$ . This makes the vertex invariant under cyclic permutation of  $A_1, A_2$  and  $A_3$ . In principle the vertex needs to be fully (anti-)symmetrized under the permutations of  $A_1, A_2$  and  $A_3$ . This can be done by averaging over the permutations of the  $A_i$ 's, but since the vertices we shall use will always have two identical external states, this will be automatic.

For simplicity we shall take the  $f_i$ 's to be  $SL(2,R)$  transformations. The most general  $SL(2,R)$  transformations satisfying the desired properties are labelled by a pair of parameters  $\alpha$  and  $\gamma$ :

$$f_1(w_1) = \frac{2w_1}{2\alpha + w_1(1-\gamma)}, \quad f_2(w_2) = \frac{2\alpha + w_2(1-\gamma)}{2\alpha - w_2(1+\gamma)}, \quad f_3(w_3) = -\frac{2\alpha - w_3(1+\gamma)}{2w_3}. \tag{4.4}$$

We shall take  $\alpha$  to be a large number and ignore terms involving negative powers of  $\alpha$ , although all final results are independent of  $\alpha$ . Denoting by  $z$  the global coordinate on the upper half plane, we can identify  $z$  with  $f_i(w_i)$  near  $w_i = 0$ . Inverting these relations we get:

$$w_1 = \alpha \frac{2z}{2-z+\gamma z}, \quad w_2 = 2\alpha \frac{z-1}{z+1+\gamma(z-1)}, \quad w_3 = 2\alpha \frac{1}{1+\gamma-2z}. \tag{4.5}$$

We shall now consider the  $s, t$  and  $u$ -channel diagrams obtained by gluing a pair of these vertices. Since the external states are all identical, it is sufficient to consider only one of these diagrams — the others give identical contribution. We shall call this the contribution from the amplitude with a propagator — to be distinguished from the contribution from the four point interaction vertex which does not have a propagator. For this we introduce

two upper half planes labelled by  $z, z'$  and local coordinates  $w_i, w'_i$  with  $1 \leq i \leq 3$  on each of these planes and make the identification:

$$w_2 w'_2 = -q. \tag{4.6}$$

Using (4.5) we get:

$$4\alpha^2 \frac{z-1}{z+1+\gamma(z-1)} \frac{z'-1}{z'+1+\gamma(z'-1)} = -q. \tag{4.7}$$

The four external punctures of the four point function are located at  $z = 0, \infty$  and  $z' = 0, \infty$ . In the  $z$  plane they are located at

$$z = \infty, \quad z = 0, \quad z' = \infty \Rightarrow z = \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 + \gamma)^2 q}, \quad z' = 0 \Rightarrow z = \frac{4\alpha^2 + (1 - \gamma)^2 q}{4\alpha^2 + (\gamma^2 - 1)q}. \tag{4.8}$$

We shall now make an  $SL(2, \mathbb{R})$  transformation to bring three of the punctures at  $0, 1$  and  $\infty$ , keeping the fourth puncture between  $0$  and  $1$ . Under  $SL(2, \mathbb{R})$  transformation

$$\hat{z} = z \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 - \gamma)^2 q}, \tag{4.9}$$

the punctures are located at:

$$\begin{aligned} \hat{z} = \infty, & & \hat{z} = 0, \\ \hat{z} = \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 - \gamma)^2 q} & & \hat{z} = 1. \\ = 1 - \frac{q}{\alpha^2} + \frac{(1 + \gamma^2)q^2}{2\alpha^4} + \mathcal{O}\left(\frac{q^3}{\alpha^6}\right), & & \end{aligned} \tag{4.10}$$

On the other hand under  $SL(2, \mathbb{R})$  transformation

$$\tilde{z} = 1 - \frac{1}{z} \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 + \gamma)^2 q}, \tag{4.11}$$

the punctures are located at

$$\begin{aligned} \tilde{z} = 1, & & \tilde{z} = \infty, \\ \tilde{z} = 0, & & \tilde{z} = 1 - \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 + \gamma)^2 q} \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 - \gamma)^2 q} \\ & & = \frac{q}{\alpha^2} - \frac{(1 + \gamma^2)q^2}{2\alpha^4} + \mathcal{O}\left(\frac{q^3}{\alpha^6}\right). \end{aligned} \tag{4.12}$$

Eqs. (4.8), (4.10) and (4.12) give equivalent representations of the puncture locations for the Feynman diagrams with a propagator.

Let us now turn to the amplitude (4.1). For analyzing the singular contribution to (4.1) from near  $y = 1$ , we denote the location of the third puncture in (4.10) by  $y$ . This gives:

$$1 - y = \frac{q}{\alpha^2} - \frac{(1 + \gamma^2)q^2}{2\alpha^4} + \mathcal{O}(q^3/\alpha^6). \tag{4.13}$$



The range  $0 \leq q \leq 1$  corresponds to:

$$1 - \frac{1}{\alpha^2} + \frac{(1 + \gamma^2)}{2\alpha^4} + \mathcal{O}(\alpha^{-6}) \leq y \leq 1. \quad (4.14)$$

We also have

$$(1 - y)^{-2} dy = -\alpha^2 q^{-2} dq + \mathcal{O}(\alpha^{-2}). \quad (4.15)$$

Our strategy will be to change variable from  $y$  to  $q$  in the range (4.14) and interpret the contribution from this region as coming from Feynman diagram with a propagator, with the divergence in the integrand as due to tachyon propagating along the internal propagator.<sup>6</sup> Therefore we write

$$\int_0^1 dy (1 - y)^{-2} = \int_0^{1 - \frac{1}{\alpha^2} + \frac{(1 + \gamma^2)}{2\alpha^4} + \mathcal{O}(\alpha^{-6})} dy (1 - y)^{-2} + \alpha^2 \int_0^1 \frac{dq}{q^2} + \mathcal{O}(\alpha^{-2}). \quad (4.16)$$

Using (3.1), we get the replacement rule:

$$\int_0^1 dq q^{-2} \Rightarrow -1. \quad (4.17)$$

Substituting this into (4.16) we get:

$$\int_0^1 dy (1 - y)^{-2} = -1 + \left\{ \frac{1}{\alpha^2} - \frac{(1 + \gamma^2)}{2\alpha^4} \right\}^{-1} - \alpha^2 + \mathcal{O}(\alpha^{-2}) = \frac{\gamma^2 - 1}{2} + \mathcal{O}(\alpha^{-2}). \quad (4.18)$$

Such change of variable should also be done for the  $y^{-1}$  and 1 terms, but since they are not singular at  $y = 1$ , the change of variable will have no effect on the value of the integral.

Similarly for evaluating the integral  $\int_0^1 dy y^{-2}$ , which is singular near  $y = 0$ , we denote the last puncture in (4.12) by  $y$ . This gives

$$y = 1 - \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 + \gamma)^2 q} \frac{4\alpha^2 + (\gamma^2 - 1)q}{4\alpha^2 + (1 - \gamma)^2 q} = \frac{q}{\alpha^2} - \frac{(1 + \gamma^2)q^2}{2\alpha^4} + \mathcal{O}\left(\frac{q^3}{\alpha^6}\right). \quad (4.19)$$

In this case the range  $0 \leq q \leq 1$  corresponds to

$$0 \leq y \leq \frac{1}{\alpha^2} - \frac{(1 + \gamma^2)}{2\alpha^4} + \mathcal{O}(\alpha^{-6}). \quad (4.20)$$

Also we have

$$dy y^{-2} = \alpha^2 q^{-2} dq + \mathcal{O}(\alpha^{-2}). \quad (4.21)$$

Following the same strategy as before, we write

$$\int_0^1 dy y^{-2} = \alpha^2 \int_0^1 dq q^{-2} + \int_{\frac{1}{\alpha^2} - \frac{(1 + \gamma^2)}{2\alpha^4} + \mathcal{O}(\alpha^{-6})}^1 dy y^{-2} + \mathcal{O}(\alpha^{-2}). \quad (4.22)$$

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<sup>6</sup>Physically, the contribution from the region (4.14) may be regarded as coming from the  $s$ -channel diagram, the contribution from the region (4.20) may be regarded as coming from the  $t$ -channel diagram, and the contribution from the rest of the region of  $y$ -integration may be interpreted as coming from the four point contact interaction. For the particular cyclic ordering we have chosen, there is no  $u$ -channel diagram.

After using the replacement rule (3.1) for the first term, we get,

$$\int_0^1 dy y^{-2} = -\alpha^2 - 1 + \left\{ \frac{1}{\alpha^2} - \frac{(1 + \gamma^2)}{2\alpha^4} \right\}^{-1} + \mathcal{O}(\alpha^{-2}) = \frac{\gamma^2 - 1}{2} + \mathcal{O}(\alpha^{-2}). \quad (4.23)$$

Finally we also have the non-singular integral

$$\int_0^1 dy = 1. \quad (4.24)$$

Adding (4.18), (4.23) and (4.24), taking  $\alpha \rightarrow \infty$  limit, and using (4.1), we get

$$A = \gamma^2. \quad (4.25)$$

This however is not the full story. As argued in section 3.2, we also need to add to this the contribution due to the  $\psi^1$  exchange. From (3.5) we see that this contribution is similar to the tachyon exchange contribution  $A_{\psi^0}$ , except for two differences. First, due to the absence of the 1/2 factor multiplying the  $(\psi^1)^2$  term in (3.5), the  $\psi^1$  propagator is 1/2 of the tachyon propagator and we shall have a factor of 1/2. Second, while  $A_{\psi^0}$  will be proportional to the square of the  $\phi$ - $\phi$ - $\psi^0$  three point coupling  $C_{\phi\phi\psi^0}$ , the  $\psi^1$  exchange contribution  $A_{\psi^1}$  will be proportional to the square of the  $\phi$ - $\phi$ - $\psi^1$  three point coupling  $C_{\phi\phi\psi^1}$ . Therefore we have:

$$A_{\psi^1} = \frac{1}{2} A_{\psi^0} (C_{\phi\phi\psi^1}/C_{\phi\phi\psi^0})^2. \quad (4.26)$$

The total tachyon exchange contribution to the amplitude  $A_{\psi^0}$  is given by the terms in (4.16) and (4.22) from the  $\int_0^1 dq q^{-2}$  part of the integral. Using the replacement rule (3.1), we get:

$$A_{\psi^0} = -2\alpha^2. \quad (4.27)$$

The three point coupling with a pair of on-shell fields  $\phi$ , with vertex operators  $c \partial X$  inserted at 0 and  $\infty$ , and an off-shell field with vertex operator  $V$  inserted at 1, is given by

$$\langle c\partial X(0) f_2 \circ V(1) c\partial X(\infty) \rangle, \quad (4.28)$$

where  $f_2$  has been given in (4.4). For the tachyon  $V = c$  and we have

$$f_2 \circ c(1) = f_2'(1)^{-1} c(f_2(1)) = \alpha c(1). \quad (4.29)$$

Furthermore, SL(2,R) invariance gives,

$$\langle c\partial X(0) c(z) c\partial X(\infty) \rangle = C z, \quad (4.30)$$

where the normalization constant  $C$  depends on the normalization and signature of  $X$ . Using (4.29) and (4.30) we get,

$$C_{\phi\phi\psi^0} = \langle c\partial X(0) \alpha c(1) c\partial X(\infty) \rangle = C \alpha. \quad (4.31)$$

On the other hand, for  $\psi^1$ ,  $V = \partial c$  and we have

$$f_2 \circ \partial c(0) = \partial c(f_2(0)) - \frac{f_2''(0)}{f_2'(0)^2} c(f_2(0)) = \partial c(1) - (1 + \gamma)c(1). \quad (4.32)$$

Using (4.30) we get

$$C_{\phi\phi\psi^1} = \langle c\partial X(0) \{ \partial c(1) - (1 + \gamma)c(1) \} c\partial X(\infty) \rangle = -C\gamma. \quad (4.33)$$

Substituting (4.27), (4.31) and (4.33) into (4.26), we get

$$A_{\psi^1} = \frac{1}{2}(-2\alpha^2) \frac{\gamma^2}{\alpha^2} = -\gamma^2. \quad (4.34)$$

Adding (4.34) to (4.25) we get the net contribution to the  $\phi$ - $\phi$ - $\phi$ - $\phi$  four point function:

$$A_{\phi\phi\phi\phi} = A + A_{\psi^1} = \gamma^2 - \gamma^2 = 0. \quad (4.35)$$

This is consistent with the identification of  $\phi$  with the collective coordinate up to field redefinition.

Note that if we had set  $\gamma = 0$ , then  $C_{\phi\phi\psi^1}$  would have vanished, and as a result there would be no  $\psi^1$  exchange contribution. This is related to the fact that for  $\gamma = 0$  the functions  $f_1$ ,  $f_2$  and  $f_3$  defined in (4.4) are not only cyclically permuted under the  $SL(2, \mathbb{R})$  transformation (4.3), but also has full permutation symmetry, up to a change in the sign of the arguments  $w_i$ . For example the  $z \rightarrow 1 - z$  transformation exchanges  $w_1 \leftrightarrow -w_2$  and sends  $w_3$  to  $-w_3$ . This leads to a ‘twist symmetric’ three point vertex where the twist symmetry is a  $Z_2$  symmetry that assigns quantum number  $(-1)^{h+1}$  to a component field that multiplies a world-sheet state of  $L_0$  eigenvalue  $h$  [23, 24]. Under this symmetry transformation the tachyon  $\psi^0$  is even since it multiplies the state  $c_1|0\rangle$  of  $L_0$  eigenvalue  $-1$ , the zero mode field  $\phi$  is odd and the field  $\psi^1$  is odd. Therefore  $C_{\phi\phi\psi^1}$  vanishes but  $C_{\phi\phi\psi^0}$  does not vanish. For this reason, it is convenient to use a twist symmetric vertex by setting  $\gamma = 0$ , since this will avoid propagating  $\psi^1$  in tree amplitudes. However  $\psi^1$  will still propagate in the loop and its contribution need to be included separately.

## 5 Superstrings

The problem with zero modes of D-instantons associated with world-sheet ghosts is not unique to bosonic string theory. If we denote by  $|-1\rangle$  the NS sector vacuum with picture number  $-1$ , then the analog of the expansion (3.3) for the NS sector string field can be written as<sup>7</sup>

$$\psi^1 \beta_{-1/2} c_0 c_1 |-1\rangle + \psi^2 \beta_{-1/2} c_1 |-1\rangle + \psi_1 \gamma_{-1/2} c_1 |-1\rangle + \psi_2 \gamma_{-1/2} c_0 c_1 |-1\rangle, \quad (5.1)$$

where  $\beta_n$  and  $\gamma_n$  are the modes of the superghost fields  $\beta, \gamma$ . Based on ghost number counting of states, we shall regard  $\psi^1$  and  $\psi^2$  as fields and  $\psi_1$  and  $\psi_2$  as their corresponding

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<sup>7</sup>In the Ramond sector, the full BV formalism requires a doubling of the string fields [17, 25–27], but this can be avoided in the NS sector by identifying the two sets of string fields. For simplicity of notation, this is the approach we are adopting here.

anti-fields. Siegel gauge fixing corresponds to choosing a Lagrangian submanifold that sets  $\psi^1$  and  $\psi_2$  to zero. In the resulting gauge fixed action,  $\psi_1$  and  $\psi^2$  appear as zero modes, causing perturbation theory to diverge. As in the case of bosonic string theory, the remedy is to choose a different Lagrangian submanifold by setting  $\psi_1$  and  $\psi_2$  to zero. In this case the mode  $\psi^2$  decouples from the action by ghost number conservation. On the other hand the mode  $\psi^1$  has a non-zero kinetic term, and its contribution must be included separately in the perturbation theory.

A similar analysis can be carried out in the Ramond sector using the identification of fields and anti-fields described in [17].

## Acknowledgments

I wish to thank Carlo Maccaferri, Xi Yin and Barton Zwiebach for useful discussions. This work was supported in part by the J. C. Bose fellowship of the Department of Science and Technology, India and the Infosys chair professorship.

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