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3d $\mathcal{N} = 2$ minimal SCFTs from wrapped M5-branes

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Abstract: We study CFT data of 3-dimensional superconformal field theories (SCFTs) arising from wrapped two M5-branes on closed hyperbolic 3-manifolds. Via so-called 3d/3d correspondence, central charges of these SCFTs are related to a SL(2) Chern-Simons (CS) invariant on the 3-manifolds. After developing a state-integral model for the invariant, we numerically evaluate the central charges for several closed 3-manifolds with small hyperbolic volume. The computation suggests that the wrapped M5-brane systems give infinitely many discrete SCFTs with small central charges.

Keywords: Chern-Simons Theories, M-Theory, Supersymmetric Gauge Theory

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Contents

1 Introduction and conclusion

Quantum field theory (QFT) has become the dominant language in theoretical physics since the success of quantum electrodynamics. The usage of QFT is not restrict to particle physics but ubiquitous: statistical, condensed matter system and even quantum gravity using holography. In general, QFTs are in the form of

$$\text{QFT} \;:\; (\text{a CFT with flavor symmetry } G)+(\text{deformation})+(\text{gauging } H\subset G)\,.$$

At infrared (IR) limit, a QFT flows to another conformal field theory (CFT). So, the general QFTs can be thought as RG flows between CFTs and thus understanding general CFTs is the first step toward understanding QFTs.

“ Classify consistent CFTs and solve them ”

One rigorous way of defining a CFT is specifying CFT data: spectrum of local operators $\{O_I\}$ and their operator product expansion (OPE) coefficients $\{\lambda_{IJK}\}$. By solving a CFT, we mean determining these CFT data.

In this work, we study 3d $\mathcal{N}=2$ unitary superconformal field theories (SCFTs) *without any flavor symmetry* and with *small central charges*. 3d supersymmetry has not been observed experimentally yet. But there is a concrete proposal for condensed matter system [1, 2] which exhibits an emergent supersymmetry and described by a 3d SCFT called critical Wess-Zumino (cWZ) model. The model is known to be the simplest 3d $\mathcal{N}=2$

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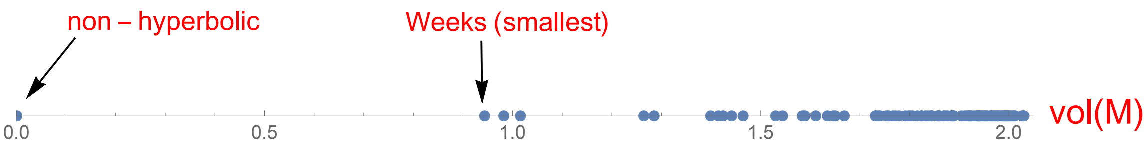

Figure 1. Space of closed 3-manifolds M with $\text{vol}(M) < \text{vol}\big((S^3\backslash \mathbf{5_1^2})_{(5,-1)}\big) \simeq 2.02988$. vol(M) denotes a topological invariant of 3-manifold M called *hyperbolic volume*, the volume measured in the unique hyperbolic metric ($R_{\mu\nu} = -2g_{\mu\nu}$). For each non-zero hyperbolic volume plotted in the graph, there are only finitely many (mostly unique) CH3s. The spectrum is discrete and infinite and has a (non-zero) lower bound 0.9427 which is saturated by the *Weeks* manifold $\big((S^3\backslash \mathbf{5_1^2})_{(5,-1)(5,-2)}\big)$ [4].

SCFT with smallest central charge $c_T/c_T^{free} = \frac{16}{243}\left(16 - \frac{9\sqrt{3}}{\pi}\right) \simeq 0.7268$ [3] where c_T^{free} is the central charge for a free chiral theory. Classifying such simple unitary CFTs is an interesting open question. In two dimensional spacetime, there is a complete classification when $c_T < 1$ called *2d minimal models*. Here, we propose 3d $\mathcal{N} = 2$ *'minimal' SCFTs* based on wrapped M5-brane systems.

An efficient way of constructing 3d $\mathcal{N} = 2$ SCFTs is using wrapped M5-branes system in M-theory:

$$\begin{array}{rl} 11d \text{ space-time:} & \mathbb{R}^{1,2} \times T^*M \times \mathbb{R}^2 \\ & \bigcup \\ N\ M5\text{-branes:} & \mathbb{R}^{1,2} \times M\,. \end{array} \tag{1.1}$$

Here T^*M denotes the cotangent bundle of M. The IR fixed point of the wrapped M5-branes' world-volume theory defines a 3d $\mathcal{N} = 2$ SCFT. It is labelled by an orientable closed hyperbolic 3-manifold (CH3) M and an integer $N \geq 2$. We denote the SCFT as $T_N[M]$.[1] The space of CH3 with small hyperbolic volume is depicted in figure 1. For nomenclature of 3-manifolds, we use a Dehn surgery description (3.4) on S^3 along a White-head link in figure 3. One natural question is

$$\text{“ Solve } T[M] \text{ for closed hyperbolic 3-manifolds } M \text{ ”.} \tag{1.2}$$

As a first step we develop a systematic algorithm for computing the central charge of wrapped M5-brane CFTs. The algorithm can be summarized as:

$$\begin{aligned} &\big(\text{a surgery description (3.4) of } M \text{ and an ideal triangulation (3.5)}\big)\,, \\ &\quad\Longrightarrow \big(\text{State-integral model in eq. (3.9) and (3.11)}\big)\,, \\ &\quad\Longrightarrow \big(\mathcal{Z}\big[T[M] \text{ on } S_b^3\big] \text{ using 3d/3d relation in eq. (2.8)}\big)\,, \\ &\quad\Longrightarrow \big(c_T\big(T[M]\big) \text{ using the relation in eq. (3.2)}\big)\,. \end{aligned}$$

In the procedure, we develop a *state-integral* for SL(2) Chern-Simons theory on *closed hyperbolic 3-manifolds*. We numerically evaluate the central charge for three examples listed in table 1. We have not done careful error analysis but the error seems to be within 2∼3 percents. From the three examples, we see that the central charge is well-

[1] For $N = 2$ case, we skip the subscript “N”.

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	Weeks: $(S^3\backslash \mathbf{5^2_1})_{(5,-1)(5,-2)}$	*Thurston*: $(S^3\backslash \mathbf{5^2_1})_{(5,-1)(1,-2)}$	$(S^3\backslash \mathbf{5^2_1})_{(5,-1)(5,-1)}$
$c_T(T[M])$	0.92	1.01	1.28
vol(M)	0.9427	0.9814	1.2637

Table 1. Central charge of $T[M]$ and hyperbolic volume for three CH3s.

approximated by the hyperbolic volume within a few percent error. We do not have a quantitative understanding for the observation but their positive correlation is somewhat expected. The larger hyperbolic volume means the more topological complexity of the 3-manifold and so the corresponding 3d SCFT is more complicated and has bigger central charge. When the N, number of M5-brane, is very large we can compute the central charge exactly thanks to holography and we can confirm the linear correlation, see eq. (2.7) and (3.2). With this observation on top of *Weeks* manifold having smallest non-zero hyperbolic volume, we expect that the $T[Weeks]$ to be the simplest non-trivial wrapped M5-brane SCFT and there are infinitely many discrete SCFTs with small central charge ($\lesssim 2$). Let us comment a possible caveat of the computation. One basic assumption for the central charge computation is that there is no enhanced abelian symmetries at low energy. If these additional enhanced symmetry appears, we need to consider the mixing between UV $U(1)_R$ with enhanced U(1)s which gives the correct IR superconformal R-charge. The mixing can be determined by the F-maximization principal and we need to compute the central charge and superconformal index taking into account of the mixing.

This work put the first step toward the challenging problem (1.2) and there are several interesting directions worth exploring. We hope to report progresses on these in near future.

- Justify the physical and technical assumption (2.18) used in the central charge computation. We give some circumstantial evidences for them.
- Prove topological invariance of the state-integral model developed in section 3.1. The state-integral model is based on a Dehn surgery representation (3.4) of a 3-manifold. The representation is not unique and we need to show the independence on the choice. We check it perturbatively up to two-loops for several cases.
- In the central charge computation, we assumed no emergent abelian symmetry in the low energy limit. It would be interesting to justify or falsify the assumption.

The paper is organized as follows. In section 2, we introduce wrapped M5-brane SCFTs and their basic properties and a basic dictionary of 3d/3d correspondence. In section 3, a systematic algorithm for computing the central charge is given. It is based on a state-integral for a complex CS theory developed in the section.

2 Wrapped M5-brane SCFT and 3d/3d correspondence

We introduce a 3d SCFT $T[M]$ labelled by a closed 3-manifold M and review basic aspects (holography and 3d/3d correspondence) of the SCFT. For recent studies on the topic, refer to [5–20] (see also recent review [21]).

	$T_N^{\text{full}}[M]$	$T_N[M]$
Global symmetry	$\mathrm{U}(1)_R \times \mathrm{U}(1)_t$	$\mathrm{U}(1)_R^{\text{IR}}$
3d/3d correspondence	"Refined" SL(N) CS theory	SL(N) CS theory
Large N gravity dual	Unknown	$AdS_4 \times M \times S^4$ (for $CH3$)

Table 2. Comparison between $T_N^{\text{full}}[M]$ and $T_N[M]$.

2.1 3d $\mathcal{N} = 2$ SCFT $T_N[M]$

A 3d SCFT $T_N[M]$ is defined as an infrared (IR) fixed point of twisted compactification of 6d A_{N-1} $(2,0)$ theory on a closed 3-manifold M:

$$\begin{aligned} &\text{6d } A_{N-1} \text{ (2,0) theory on } \mathbb{R}^{1,2} \times M \text{ with partial topological twisting along } M \\ &\xrightarrow{\text{IR}} \text{ 3d SCFT } T_N[M]\,. \end{aligned} \tag{2.1}$$

For the partial twisting, we use the usual SO(3) subgroup of SO(5) R-symmetry of the 6d theory. The twisting generically preserves a quarter of supercharges and the resulting 3d theory has $\mathcal{N} = 2$ superconformal symmetry. The metric structure on the 3-manifold is irrelevant in the IR and the 3d SCFT depends only on the topology of the 3-manifold. From M-theoretical perspective, these theories are realized as low-energy world-volume theory of wrapped N M5-branes in (1.1). As pointed out in [16], the 'full' IR CFT has an additional abelian flavor symmetry called $\mathrm{U}(1)_t$ and will be denoted as $T_N^{\text{full}}[M]$. The theory $T_N[M]$ of our interest is obtained as IR fixed point of the $T_N^{\text{full}}[M]$ through a renormalization group (RG) flow triggered by a Higgsing/deformation procedure

$$\left(T_N^{\text{full}}[M] + \text{ Higgsing/deformation}\right) \;\rightsquigarrow\; T_N[M]\,. \tag{2.2}$$

General bottom-up algorithm of constructing theory $T_N[M]$ is introduced in [5] while there are no such a general construction of $T^{\text{full}}[M]$. Not all 3-manifolds M give non-trivial interacting CFTs. Our basic assumptions are

a) For hyperbolic 3-manifold M, the IR fixed theory $T_N[M]$ is non-trivial.

b) For non-hyperbolic 3-manifold M with SO(3) Riemmanian holonomy (for example, $M = S^3$), the corresponding $T_N[M]$ seems to be more or less trivial theories (theories only with topological degree of freedom).[2]

c) M has reduced Riemannina holonomy group (thus non-hyperbolic), i.e, $M = \Sigma \times S^1$ with a Riemann surface Σ. In the case, the resulting 3d SCFT has additional structure, enhanced $\mathcal{N} = 4$ SUSY or additional flavor symmetry.

Simple evidence for a) is

$$\lim_{b\to 0} 2\pi b^2 \mathcal{F}_b(T_N[M]) = \frac{N(N^2-1)}{6}\text{vol}(M)\,. \tag{2.3}$$

[2]The theory $T_N^{\text{full}}[M]$ might not be topological even this case. For example, $T_N^{\text{full}}[S^3/\mathbb{Z}_p]$ is not topological [20, 22, 23].

Here $\mathcal{F}_b$ denotes the free-energy on a squashed 3-sphere S^3_b [24],

$$\begin{aligned}\mathcal{F}_b(\text{a SCFT}) &:= (\text{free-energy of the SCFT on } S^3_b) \\ &:= -\text{Re}\big(\log \mathcal{Z}[\text{the SCFT on } S^3_b]\big)\,. \end{aligned} \tag{2.4}$$

Metrically, the curved background can be realized as

$$S^3_b = \{b^2|z|^2 + b^{-2}|w|^2 = 1 \ : \ (z,w) \in \mathbb{C}^2\}\,, \quad \text{with real } b. \tag{2.5}$$

The geometry has an exact symmetry exchanging $b \leftrightarrow b^{-1}$ and so does the free-energy $\mathcal{F}_b$. The relation in eq. (2.3) can be explained using a 3d/3d relation and perturbative expansion of SL(N) CS theory as we will see in the next section. Since we are interested in a non-trivial 3d $\mathcal{N}=2$ SCFT with small central charge and no extra structures (flavor symmetry or enhacencd SUSY), we concentrate on $N=2$ and the case a).

Holographic dual. Holographic dual to the RG flow (2.1) across dimension was constructed in [25]

$$(\ AdS_7 \times S^4 \text{ solution}\ ) \xrightarrow{\text{Holographic RG}} (\text{ Pernici-Sezgin } AdS_4 \text{ solution in }\ )$$

and M-theory on the AdS_4 solution is proposed as gravity dual of $T_N[M]$. The supergravity solution is

$$AdS_4 \times M \times S^4\,, \tag{2.6}$$

with a warped product metric and the S^4 non-trivially fibred over the M factor. The supergravity solution was found only for closed hyperbolic M. From the holographic computation using supergravity approximation, it has been predicted that [18]

$$\lim_{N\to\infty} \frac{1}{N^3}\mathcal{F}_b(T_N[M]) = \frac{(b+b^{-1})^2}{12\pi}\text{vol}(M)\,. \tag{2.7}$$

2.2 3d/3d relations

3d/3d relation relates the squashed 3-sphere ptn of $T_N[M]$ to ptn of a SL(N) CS theory on M.

$$\begin{aligned}\mathcal{Z}[T_N[M] \text{ on } S^3_b] &= \mathcal{Z}\big[\text{SL}(N)_{k,\sigma} \text{ CS theory on } M\big] \\ &:= \int \frac{[D\mathcal{A}]}{(\text{gauge})} \exp\left(\frac{i(k+\sigma)}{8\pi}\text{CS}[\mathcal{A}] + \frac{i(k-\sigma)}{8\pi}\text{CS}[\bar{\mathcal{A}}]\right), \end{aligned} \tag{2.8}$$

where k and σ are two coupling constants of the complex CS theory. $k \in \mathbb{Z}$ is a quantized CS level and the σ can be either real or purely imaginary. In the 3d/3d relation, they are [15, 17]

$$k = 1 \ \text{ and } \ \sigma = \frac{1-b^2}{1+b^2}\,. \tag{2.9}$$

$\mathcal{A}, \bar{\mathcal{A}}$ denote a pair of SL(N) gauge fields on M and the CS functional is defined as

$$\text{CS}[\mathcal{A}] := \int_M \text{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big)\,. \tag{2.10}$$

Perturbative ptn $\mathcal{Z}^{\overline{\text{hyp}}}_{N;\text{pert}}$ and resurgence. When $b^2 \to 0^+$, the S^3_b-ptn has following asymptotic expansion [7, 8]

$$\mathcal{Z}\big[T_N[M] \text{ on } S^3_b\big] \xrightarrow{b^2 \to 0^+} \sum_\alpha n_\alpha \mathcal{Z}^\alpha_{N;\text{pert}}(M;\hbar)\,. \tag{2.11}$$

Here α labels $\text{SL}(N)$ flat connections on M and n_α are integer coefficients and Z^α_{pert} denotes the formal perturbative expansion around the flat-connection $\mathcal{A}^\alpha$.

$$\mathcal{Z}^\alpha_{N;\text{pert}}(M;\hbar) := \exp\left(\frac{1}{\hbar}S^\alpha_0(M;N) + S^\alpha_1(M;N) + \ldots + \hbar^{n-1}S^\alpha_n(M;N) + \ldots\right). \tag{2.12}$$

Through out the paper, we define

$$\hbar := 2\pi i b^2 \in i\mathbb{R}_+\,. \tag{2.13}$$

S^α_n is the n-loop $\text{SL}(N)$ CS invariant on M. The classical part is

$$S^\alpha_0 = -\frac{1}{2}\text{CS}[\mathcal{A}^\alpha]\,. \tag{2.14}$$

For hyperbolic 3-manifolds, there are two special flat connections, $\mathcal{A}^{\text{hyp}}$ and $\mathcal{A}^{\overline{\text{hyp}}}$, which can be constructed using the unique (complete) hyperbolic structure on M:

$$\mathcal{A}^{\text{hyp}}_N := \rho_N(\omega + ie)\,, \quad \mathcal{A}^{\overline{\text{hyp}}}_N := \rho_N(\omega - ie)\,, \tag{2.15}$$

where e and ω are drei-bein and spin connection for the unique hyperbolic structure respectively and ρ_N is an embedding of $\text{SL}(2)$ into $\text{SL}(N)$ using the N-dimensional representation of $\text{SL}(2) \simeq \text{SU}(2)_{\mathbb{C}}$. Einstein equation with negative cosmology constant become flat connection equation through the above relation. Value of CS functional for these flat connections are related to the hyperbolic volume of 3-manifold:

$$\text{Im}\big(\text{CS}[\mathcal{A}^{\text{hyp}}_N]\big) = -\frac{1}{3}N(N^2-1)\,\text{vol}(M)\,, \quad \text{Im}\big(\text{CS}[\mathcal{A}^{\overline{\text{hyp}}}_N]\big) = \frac{1}{3}N(N^2-1)\,\text{vol}(M)\,. \tag{2.16}$$

These flat connections have most exponentially growing and decaying classical part $e^{\frac{1}{\hbar}S_0}$ when $b \in \mathbb{R}$:

$$\text{Im}\big(\text{CS}[\mathcal{A}^{\text{hyp}}_N]\big) < \text{Im}\big(\text{CS}[\mathcal{A}^{\alpha}_N]\big) < \text{Im}\big(\text{CS}[\mathcal{A}^{\overline{\text{hyp}}}_N]\big)\,,\ \text{for any other flat-connections } \mathcal{A}^\alpha_N. \tag{2.17}$$

From the compatibility with the holographic prediction (2.7) and an argument using a state-integral model,[3] it has been conjectured that [18, 26]

$$n_\alpha \neq 0 \ \text{ only for } \alpha = \overline{\text{hyp}}\,. \tag{2.18}$$

[3]The state-integral model can be interpreted as an integral from localization for a SCFT, which can be identified as $T[M]$ [9], if one choose a proper converging integration contour as a cycle slightly above the real slice. For some knot complements, it is checked that the contour is homologically equivalent to the steepest descendant contour (Lefschetz thimble) associated to the saddle point in (3.21) which corresponds to the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$.

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It implies that the S_b^3-ptn is exponentially decaying at small b which seems to be an universal property of unitary non-topological 3d SCFTs. Actually, the choice (2.18) with $n_{\overline{\text{hyp}}} = \pm 1$ maximizes the free-energy $\mathcal{F}_b$ at small b, see eq. (2.17). We assume that this is the correct choice for the IR SCFT appearing in the 3d/3d relation. The above conjecture can be rephrased in the language of resurgence. For that, first reorganize the perturbative expansion in the following ways:

$$\mathcal{Z}^{\overline{\text{hyp}}}_{N;\text{pert}}(M;\hbar) = \exp\left(\frac{1}{\hbar}S_0^{\overline{\text{hyp}}}(M;N) + S_1^{\overline{\text{hyp}}}(M;N)\right) \times \left(1 + \sum_{n=1}^{\infty} a_n^{\overline{\text{hyp}}}(M;N)(b^2)^n\right),$$

then the conjecture in (2.18) can be stated as:

$$\mathcal{Z}[T_N[M] \text{ on } S_b^3] = \exp\left(\frac{1}{\hbar}S_0^{\overline{\text{hyp}}}(M;N) + S_1^{\overline{\text{hyp}}}(M;N)\right) \times \left(1 + \int_0^\infty d\zeta e^{-\frac{\zeta}{b^2}} B_N^{\overline{\text{hyp}}}(\zeta)\right),$$

$$\text{where } B_N^{\overline{\text{hyp}}}(\zeta) := \sum_{n=1}^{\infty} \frac{a_n^{\overline{\text{hyp}}}(M;N)}{(n-1)!}\zeta^{n-1}\,. \tag{2.19}$$

Here we assume that the series $\{a_n^{\overline{\text{hyp}}}\}_{n=1}^{\infty}$ is Borel summable which is reasonable since the saddle point $\mathcal{A}^{\overline{\text{hyp}}}$ gives the smallest classical contribution and thus other saddle points can not appear as instanton trans-series. On the other hand, it was claimed in [27] that the Borel resummation $\mathcal{Z}_N^{\overline{\text{hyp}}}$ gives the vortex ptn (ptn on $\mathbb{R}^2 \times_q S^1$) instead of S_b^3-ptn. There are two evidences supporting our proposal over their claim: a) At large N and the leading order (N^3) in $1/N$ expansion, the perturbative series $\{S_n^{\overline{\text{hyp}}}(N)\}$ becomes a finite series terminating at two-loops and the answer nicely matches with the holographic prediction (2.7) of S_b^3-ptn [18], b) For $N = 2$ and $M = S^3\backslash\mathbf{4}_1$ (figure-eight knot complement), the Borel resummation is performed explicitly in [27][4]

$$\mathcal{Z}^{\overline{\text{hyp}}}(S^3\backslash\mathbf{4}_1;\hbar = 2\pi i b^2)\big|_{b=1} \simeq 0.37953\,, \tag{2.20}$$

which is a good approximation for the correct S_b^3-ptn of $T_{N=2}[S^3\backslash\mathbf{4}_1]$ computed using a state-integral model. The exact value at $b = 1$ is [28]

$$\mathcal{Z}[T[S^3\backslash\mathbf{4}_1] \text{ on } S_{b=1}^3] = \frac{1}{\sqrt{3}}\left(\exp\left(\frac{\text{vol}(S^3\backslash\mathbf{4}_1)}{2\pi}\right) - \exp\left(-\frac{\text{vol}(S^3\backslash\mathbf{4}_1)}{2\pi}\right)\right) \simeq 0.379568\,. \tag{2.21}$$

Here the hyperbolic volume of $S^3\backslash\mathbf{4}_1$ is

$$\text{vol}(S^3\backslash\mathbf{4}_1) = 2\text{Im}[\text{Li}_2(e^{i\pi/3})] \simeq 2.02988\,. \tag{2.22}$$

3 Central charge of $T[M]$

One basic quantity characterizing a SCFT is central charge c_T which is defined using two point function of stress-energy tensor:

$$T(x)T(0) \sim \frac{c_T}{|x|^{2d}} \times (\text{tensor structure})\,. \tag{3.1}$$

[4]There seems to be a mistake in the sign of classical part in the eq. (6.11) in [27]. After correcting the mistake, $\mathcal{Z}^{\overline{\text{hyp}}}(S^3\backslash\mathbf{4}_1;\hbar = 2\pi i b^2)\big|_{b=1} = e^{-2\times\frac{\text{vol}(S^3\backslash\mathbf{4}_1)}{2\pi}} \times$(eq. (6.23) in [27]).

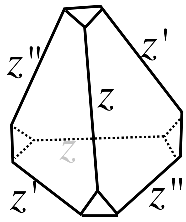

Figure 2. An ideal tetrahedron Δ, tetrahedron with truncated vertices. Hyperbolic structures of Δ are parameterized by edge parameters $(z := e^{Z}, z' := e^{Z'}, z'' = e^{Z''})$ satisfying relations $z' = \frac{1}{1-z}$ and $z'' = 1 - z^{-1}$. These parameters assigned to each pair of boundary edges, as shown in the figure. Geometrically, the logarithm parameters (Z, Z', Z'') measure complex dihedral angles between two faces meeting at the edges. Imaginary parts of these logarithm parameters take values between 0 and π.

For 3d $\mathcal{N} = 2$ SCFTs, the central charge is related to the squashed 3-sphere free energy $\mathcal{F}_b$ (2.4) as follows [29]:

$$c_T = \frac{8}{\pi^2} \frac{\partial^2 \mathcal{F}_b}{\partial b^2}\bigg|_{b=1} . \tag{3.2}$$

We use following normalization

$$c_T(\text{a free chiral theory}) = 1 . \tag{3.3}$$

Combining the 3d/3d correspondence (2.8) and the relation (3.2), we will compute the central charge of $T[M]$.

3.1 A state-integral model for SL(2)$_{k=1}$ CS theory

We review and generalize a state-integral for SL(2) CS theory on hyperbolic 3-manifolds which is believed to be a finite dimensional integral representation of the path integral in the complex CS theory. The generalized state-integral model *is applicable to any closed hyperbolic 3-manifolds* which was not possible for state-integrals [26, 30, 31] in the literature.

Dehn surgery and ideal triangulation. We use a Dehn surgery description of 3-manifold M:

$$M = (S^3\backslash K)_{\{(p_\alpha, q_\alpha)\}_{\alpha=1}^{S\leq|K|}} := \left[(S^3\backslash K) \bigcup_{\alpha=1}^{S} (D^2 \times S^1)_\alpha\right] / \sim , \tag{3.4}$$

and a sufficiently good[5] ideal triangulation of the link complement $S^3\backslash K$:

$$S^3\backslash K = \left(\bigcup_{i=1}^{T} \Delta_i\right) / \sim \ . \tag{3.5}$$

Here K is a link on S^3 of $|K|$ components. A link complement $S^3\backslash K$ is a 3-manifold obtained by removing the tubular neighborhood (topologically $|K|$ copies of solid-tori) of

[5] We assume a positive angle structure of triangulation [17].

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a link K from a 3-sphere S^3. The manifold has $|K|$ torus boundaries and 1-cycles around the link are called 'meridians' and 1-cycles along the link are 'longitudes'. The 3-manifold M in (3.4) is obtained by gluing S solid-tori back to the link complement with following identification:

$$\begin{aligned} & p_\alpha(\alpha\text{-th meridian}) + q_\alpha(\alpha\text{-th longitude}) \\ & \qquad \sim (\text{contractable cycle in } \alpha\text{-th solid-torus})\,. \end{aligned} \tag{3.6}$$

The procedure of gluing solid-torus is called (p_α, q_α)-Dehn filling. (p_α, q_α) is a pair of coprime numbers and the ratio p_α/q_α is called 'slopes'. In short, the 3-manifold is obtained by gluing T ideal tetrahedrons and S solid-tori:

$$T\ :\ \sharp \text{ of ideal tetrahedrons}\,, \quad S\ :\ \sharp \text{ of solid-tori}\,. \tag{3.7}$$

The resulting 3-manifold M has $(|K|-S)$ torus boundaries and when $S = |K|$ it is a closed 3-manifold. *Any closed 3-manifold M can be obtained by a Dehn surgery on S^3* [32, 33].

State-integral model. State-integrals give a finite-integral representation of the CS ptn by properly 'quantizing' the ideal triangulation (3.5) and the Dehn filling (3.6). There are several state-integral models [26, 30, 34], which are believed to be equivalent, based on an ideal triangulation of M. We use the one developed by Dimofte and incorporate Dehn filling into the state-integral model to *cover more general class of 3-manifolds such as closed hyperbolic 3-manifolds*. One systematic way of specifying the gluing rule of an ideal triangulation is using (generalized) Neunmann-Zagier (NZ) datum $(A, B, C, D; f, f'', \nu, \nu_p)$, refer to [35] for the definition, where A, B, C, D are $T \times T$ matrices forming $\mathrm{Sp}(2T, \mathbb{Q})$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2T, \mathbb{Q})\,, \quad \text{with } \det B \neq 0\,, \tag{3.8}$$

and (f, f'', ν, ν_p) are vectors of length T. From these datum, the state-integral (SI) for the link complement is given by [35]

$$\begin{aligned} & \mathcal{Z}_{\mathrm{SI}}(S^3\backslash K; X_1, \ldots, X_{|K|}; \hbar) \\ & = \frac{1}{\sqrt{\det B}} \int \prod_{i=1}^{T} \frac{\Psi_b(Z_i) dZ_i}{\sqrt{2\pi\hbar}} \exp\left[\frac{1}{2\hbar} \vec{Z} \cdot B^{-1} A \vec{Z} + \frac{1}{\hbar}\Big(2u \cdot D B^{-1} u + (2\pi i + \hbar) f \cdot B^{-1} u \right. \\ & \qquad\qquad \left. + \frac{1}{2}(i\pi + \hbar/2)^2 f \cdot B^{-1} \nu - \vec{Z} \cdot B^{-1}\big((i\pi + \hbar/2)\nu + 2u\big)\Big)\right]. \end{aligned} \tag{3.9}$$

Here we define

$$u = \big(X_1, \ldots, X_{|K|}, 0, \ldots, 0\big)\,. \tag{3.10}$$

The quantum dilogarithm function (QDL) Ψ_b is a wave-function on each tetrahedron. See appendix B for the definition and basic properties of the special function. Quantizing the

Dehn fillings in (3.6), we finally have

$$\begin{aligned}&\mathcal{Z}_{\rm SI}\big(M;X_1,\ldots,X_{|K|-S};\hbar\big)\\&=\int\prod_{\alpha=1}^{S}\frac{\Delta_b(X_{\alpha+|K|-s};s_\alpha,q_\alpha)dX_{\alpha+|K|-S}}{(2\pi q_\alpha\hbar)^{1/2}}\exp\left(\frac{1}{\hbar}\sum_{\alpha=1}^{S}\frac{p_\alpha X^2_{\alpha+|K|-s}}{q_\alpha}\right)\mathcal{Z}_{\rm SI}\big(S^3\backslash K;X_1,\ldots,X_{|K|};\hbar\big),\\&\text{with }\Delta_b(X;s,q):=e^{-\frac{i\pi s}{2q}(b^2+b^{-2})}\left(e^{\frac{X}{b^2q}}\sinh\left(\frac{X-i\pi s}{q}\right)-e^{-\frac{X}{b^2q}}\sinh\left(\frac{X+i\pi s}{q}\right)\right).\end{aligned}\tag{3.11}$$

Here s_α is defined to be an integer satisfying $s_\alpha p_\alpha\in q_\alpha\mathbb{Z}-1$. See appendix C for the derivation. The CS wave-function has following naive path-integral interpretation,

$$\begin{aligned}&\mathcal{Z}_{\rm SI}\big(M;X_1,\ldots,X_{|K|-S};\hbar=2\pi i b^2\big)\\&\quad=\int\frac{[d\mathcal{A}]_X}{(\text{gauge})}\exp\left(\frac{i(k+\sigma)}{8\pi}\text{CS}[\mathcal{A}]+\frac{i(k-\sigma)}{8\pi}\text{CS}[\bar{\mathcal{A}}]\right)\Bigg|_{(2.9)},\quad\text{where}\\&[d\mathcal{A}]_X\ :\ \text{Path-integral over SL(2) gauge field on } M \text{ subject to}\\&\qquad\text{boundary conditions fixing } \text{P}e^{\oint_{I-\text{th merdian}}\mathcal{A}}=\begin{pmatrix}e^{X_I}&1\\0&e^{-X_I}\end{pmatrix}.\end{aligned}\tag{3.12}$$

The SL(2) CS wave-function is defined up to a factor [35].

$$\exp\left(\frac{\pi^2}{6\hbar}\alpha+\frac{i\pi}{4}\beta+\frac{\hbar}{24}\gamma\right),\quad\alpha,\beta,\gamma\in\mathbb{Z}.\tag{3.13}$$

The factor is a purely phase factor for real b and irrelevant in free-energy $\mathcal{F}_b$ computation. In the SCFT side of 3d/3d correspondence, (some parts of) the ambiguities comes from local counter-terms in a supergravity on the curved (S^3_b) background [36].

3.1.1 Perturbative expansion

Using the state-integral model above, we can compute the perturbative invariants $\{S^{\rm hyp}_n(M)\}_{n=0}^{\infty}$ (2.12). The state-integral model in (3.9) and (3.11) is of the form:

$$\begin{aligned}&\mathcal{Z}_{\rm SI}\big(M;X_1,\ldots,X_{|K|-S};\hbar\big)\\&\quad=\int\frac{dX_{|K|-S+1}\ldots dX_{|K|}dZ_1\ldots dZ_T}{(2\pi\hbar)^{(T+S)/2}}\exp\Big(\mathcal{W}(Z_1,\ldots,Z_T,X_1,\ldots,X_{|K|};\hbar)\Big).\end{aligned}\tag{3.14}$$

In the limit when $\hbar\to0$, using eq. (B.4)

$$\mathcal{W}(\vec{Z},\vec{X};\hbar)\sim\frac{1}{\hbar}\mathcal{W}_0(\vec{Z},\vec{X})+\mathcal{W}_1(\vec{Z},\vec{X})+\hbar\mathcal{W}_2(\vec{Z},\vec{W})+\ldots.\tag{3.15}$$

Saddle point equations are

- $\dfrac{\partial\mathcal{W}_0}{\partial Z_i}=0\,,\quad\text{for } i=1,\ldots,T$

  $$\Rightarrow A\cdot\vec{Z}+B\cdot\vec{Z}''-i\pi\nu=2u\ \text{ where } Z''_i:=\log(1-e^{-Z_i})\,,$$

- $\dfrac{\partial\mathcal{W}_0}{\partial X_{\alpha+|K|-S}}=0\,,\quad\text{for } \alpha=1,\ldots,S$

  $$\Rightarrow p_\alpha X_{\alpha+|K|-S}+q_\alpha P_{\alpha+|K|-S}=-\text{sign}\left(\text{Re}\left[\frac{X_{\alpha+|K|-S}}{q_\alpha}\right]\right)\pi i\,.\tag{3.16}$$

Here u is defined in (3.10) and we define

$$P_{\alpha+|K|-S} := \left(C \cdot \vec{Z} + D \cdot \vec{Z}'' - i\pi\nu_p\right)_{\alpha+|K|-S}. \tag{3.17}$$

Interpreting the variables Z and Z'' as logarithmic edge parameters of ideal tetrahedrons, these are nothing but gluing equations for the 3-manifold studied in [37]. Solutions to the gluing solution give SL(2) flat connections on M. Refer to [11] for explicit construction of holonomy representation of a flat connection from a solution to the gluing equations. In the map, the solution corresponding to the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$ is characterized by following conditions:

$$\begin{aligned} &0 < \text{Im}[Z_i] < \pi\,, && \text{for all } i = 1, \ldots, T && \text{(hyperbolic)} \\ &X_1 = \ldots = X_{|K|-S} = 0 && && \text{(complete)} \end{aligned} \tag{3.18}$$

Under the first condition, logarithmic edge parameter Z_i determines a hyperbolic structure on Δ_i, see figure 2. The gluing equations are conditions for the hyperbolic structures to be glued smoothly and give a hyperbolic structure on the 3-manifold. For *complete* hyperbolic structure, we additionally need the second conditions requring the meridian holonomies in eq. (3.12) are parabolic. Near each $\mathbb{T}^2$-boundary, the complete hyperbolic metric on M are locally

$$ds^2 = \frac{1}{z^2}\left(dz^2 + ds^2_{\mathbb{T}^2}\right). \tag{3.19}$$

Here z is the (inward) direction transverse to the boundary $\mathbb{T}^2$. Using the metric, one can check that the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$ in (2.15) have parabolic meridian holonomies. For the case when M is hyperbolic and we use an idea triangulation with positive angle structure, there is an unique solution for eq. (3.16) and (3.18) modulo the Weyl-symmetries $(\mathbb{Z}_2)^S$.

$$(\mathbb{Z}_2)^S \;:\; X_{\alpha+|K|-S} \to \pm X_{\alpha+|K|-S} \quad \text{for } \alpha = 1, \ldots, S\,. \tag{3.20}$$

The unique saddle point corresponds to the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$ and we denote

$$\left(X^{\overline{\text{hyp}}}_{\alpha+|K|-S}, Z^{\overline{\text{hyp}}}_i\right) := \text{A solution satisfying eq. (3.16) and (3.18)}\,. \tag{3.21}$$

For non-hyperbolic M, there's no saddle point satisfying these conditions. The formal perturbative expansion of the state-integral around the saddle point defines the perturbative ptn $\mathcal{Z}^{\overline{\text{hyp}}}_{\text{pert}}(M;\hbar)$ (2.12):

$$\begin{aligned} \mathcal{Z}^{\overline{\text{hyp}}}_{\text{pert}}(M;\hbar) &:= 2^S \times \mathcal{Z}^{\overline{\text{hyp}}}_{\text{pert;SI}}\big(M; \vec{X} = \vec{0}; \hbar\big)\,, \\ &:= 2^S \times \left[\text{Perturbative expansion of } \mathcal{Z}_{\text{SI}}\big(M; \vec{X} = \vec{0}; \hbar\big) \text{ around (3.21)}\right]. \end{aligned} \tag{3.22}$$

The overall factor 2^S comes from the fact that there are that many saddle points related by Weyl-symmetries and they all give same perturbative expansion. The state-integral is finite dimensional integration and thus the formal expansion coefficients $\{S^{\overline{\text{hyp}}}_n(M)\}_{n=0}^{\infty}$ are well-defined without any issue of regularization. Refer to [38] for perturbative expansion of the state-integral model in (3.9) using Feynman diagram.

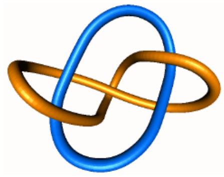

Figure 3. White-head link ($\mathbf{5^2_1}$), the **1**st one among links with **2** components and **5** crossings.

Examples $(S^3\backslash \mathbf{5^2_1})_{(p,q)}$. White-head link ($\mathbf{5^2_1}$) is one of simplest hyperbolic link with two components. The link complement can be decomposed into 4 ideal tetrahedrons (see appendix A):

$$S^3\backslash \mathbf{5^2_1} = \left(\bigcup_{i=1}^4 \Delta_i\right)/\sim \, . \tag{3.23}$$

Using the ideal triangulation, the corresponding state-integral is given by

$$\begin{aligned}\mathcal{Z}_{\rm SI}\big(S^3\backslash \mathbf{5^2_1}; X_1, X_2;\hbar\big) = \frac{1}{\sqrt{2}}\int \prod_{i=1}^4 \frac{\Psi_b(Z_i)dZ_i}{\sqrt{2\pi\hbar}} \exp\bigg[&\frac{2X_1(2Z_1+2Z_4-\hbar-2i\pi)-2Z_3Z_4}{2\hbar} \\ &+\frac{2X_2(-2Z_2-2Z_4+\hbar+2i\pi)+(Z_1+Z_2-Z_3)(Z_1+Z_2-Z_3-\hbar-2i\pi)}{2\hbar}\bigg]\, .\end{aligned} \tag{3.24}$$

Applying the quantum Dehn filling formula (3.11) to the above integral, we obtain the state-integral for $M=(S^3\backslash \mathbf{5^2_1})_{(p,q)}$. For example, when $(p,q)=(5,-1)$

$$\begin{aligned}&\mathcal{Z}_{\rm SI}\big((S^3\backslash \mathbf{5^2_1})_{(5,-1)}; X_1;\hbar\big) \\ &\quad = \int \frac{2\sinh(X_2)\sinh(X_2/b^2)dX_2}{\sqrt{2\pi\hbar}} \exp\left(-\frac{5X_2^2}{\hbar}\right)\mathcal{Z}_{\rm SI}(S^3\backslash \mathbf{5^2_1}; X_1, X_2;\hbar)\, .\end{aligned} \tag{3.25}$$

In the case, the resulting 3-manifold is turned out to be a 3-manifold called 'sister of figure-eight knot-complement'. In `SnapPy`'s census [39], the 3-manifold is denoted as $m003$ and allows an ideal triangulation using two tetrahedrons (see appendix A):

$$m003 = (S^3\backslash \mathbf{5^2_1})_{(5,-1)} = \left(\bigcup_{i=1}^2 \Delta_i\right)/\sim \, . \tag{3.26}$$

From the ideal triangulation, we have an alternative expression for the state-integral model

$$\begin{aligned}\mathcal{Z}_{\rm SI}(m003; X_1;\hbar) = \int \prod_{i=1}^2 \frac{\Psi_b(Z_i)dZ_i}{\sqrt{2\pi\hbar}} \exp\bigg[&\frac{X_1(8Z_1+4Z_2-2\hbar-4i\pi)+8X_1^2}{2\hbar} \\ &+\frac{2Z_1(Z_2-\hbar-2i\pi)+Z_2(Z_2-\hbar-2i\pi)+4Z_1^2}{2\hbar}\bigg]\, .\end{aligned} \tag{3.27}$$

One can check that both expressions, eq. (3.25) and eq. (3.27), give same perturbative invariants $S_n^{\overline{\text{hyp}}}(M)$ modulo (3.13):

$$S_0^{\overline{\text{hyp}}}(m003) = S_0^{\overline{\text{hyp}}}\big((S^3\backslash \mathbf{5_1^2})_{(5,-1)}\big) = \frac{\pi^2}{9} + 2\text{Li}_2\left(\frac{1-\sqrt{3}}{2}\right),$$
$$S_1^{\overline{\text{hyp}}}(m003) = S_1^{\overline{\text{hyp}}}\big((S^3\backslash \mathbf{5_1^2})_{(5,-1)}\big) = -\frac{1}{2}\log(-3-\sqrt{3}i),$$
$$S_2^{\overline{\text{hyp}}}(m003) = S_2^{\overline{\text{hyp}}}\big((S^3\backslash \mathbf{5_1^2})_{(5,-1)}\big) = \frac{1}{864}(9+35\sqrt{3}i).$$

We did similar consistency checks for other examples, $(S^3\backslash \mathbf{5_1^2})_{(3,-2)} = m007, (S^3\backslash \mathbf{5_1^2})_{(5,-2)} = m006$ and $(S^3\backslash \mathbf{5_1^2})_{(2,-3)} = m053$. The matches are delicate and strongly suggests that the state-integral model gives at least the correct perturbative invariants. We leave the general proof showing topological invariance of the perturbative series as future work.

3.2 Numerical evaluation for some $M = (S^3\backslash 5_1^2)_{(5,-1),(p,q)}$

Here we give concrete examples of central charge computation for closed hyperbolic 3-manifolds M. The most technically non-trivial step is finding a converging contour of the state-integral model.

Weeks manifold $= (S^3\backslash 5_1^2)_{(5,-1)(5,-2)} = (m003)_{(5,-2)}$. *Weeks* manifold is the smallest volume hyperbolic 3-manifold. The state-integral is given by[6] (sloppy in the overall factor of the form (3.13))

$$\begin{aligned}
&\mathcal{Z}^{\overline{\text{hyp}}}(\mathit{Weeks}; \hbar = 2\pi i b^2) \\
&= \int_{\Gamma^{\overline{\text{hyp}}}_{\mathit{Weeks}}} \frac{dZ_1 dZ_2 dX}{(2\pi)^3\sqrt{2}}\left(2\cosh\left(\frac{bX}{2}\right)\cosh\left(\frac{X}{2b}\right)\right)\psi_b(Z_1)\psi_b(Z_2) \\
&\qquad \times e^{-\frac{1}{2}(b+b^{-1})(2Z_1+Z_2+2X)-\frac{i}{4\pi}(4Z_1^2+Z_2^2+4Z_2X+3X^2+2Z_1Z_2+8Z_1X)}, \\
&\simeq \int_{\gamma^{\overline{\text{hyp}}}_{\mathit{Weeks}}} \frac{dZ_1 dX}{(2\pi)^2\sqrt{2}}\left(2\cosh\left(\frac{bX}{2}\right)\cosh\left(\frac{X}{2b}\right)\right)\psi_b(Z_1)\psi_b(2X+Z_1) \\
&\qquad \times \exp\left[-(b+b^{-1})(Z_1+X) - \frac{i}{4\pi}(4Z_1^2+3X^2+8Z_1X)\right].
\end{aligned}$$

Using an identity of QDL (B.9), we first integrated out Z_2 along a cycle $\mathbb{E}_{2X+Z_1}$. The contour $\Gamma^{\overline{\text{hyp}}}_{\mathit{Weeks}}$ is a bundle over a 2d cycle $\gamma^{\overline{\text{hyp}}}_{\text{Weeks}} \subset \mathbb{C}^2_{X,Z_1}$ whose fiber is the $\mathbb{E}_{2X+Z_1}$:

$$\begin{array}{ccc}
\mathbb{E}_{2X+Z_1} & \longrightarrow & \Gamma^{\overline{\text{hyp}}}_{\mathit{Weeks}} \\
& & \downarrow \\
& & \gamma^{\overline{\text{hyp}}}_{\mathit{Weeks}}
\end{array} \tag{3.28}$$

[6] We replace the integration variables (Z, X) in the state-integral model by (bZ, bX) to make the symmetry $b \leftrightarrow b^{-1}$ manifest in the integrand.

One particular choice of converging contour $\gamma^{\overline{\text{hyp}}}_{Weeks}$ in the reduced two-dimensional integration is

$$\gamma^{\overline{\text{hyp}}}_{Weeks} := \big\{ \big(Z_1,X\big) = \big(m_1 + iA_{Weeks}(m_1,m_2), m_2 + iB_{Weeks}(m_1,m_2)\big) : (m_1,m_2) \in \mathbb{R}^2 \big\} \subset \mathbb{C}^2,$$

where the continuous functions $A_{Weeks}(m_1,m_2)$ and $B_{Weeks}(m_1,m_2)$ have following asymptotic behavior:

$$\{A_{Weeks}, B_{Weeks}\} = \begin{cases} \{\frac{(b+b^{-1})}{8}, 0\} & \text{if } m_1 \geq \Lambda \quad \text{and} \quad m_2 \geq \Lambda \\ \{\frac{27(b+b^{-1})}{20}, -\frac{11(b+b^{-1})}{20}\} & \text{if } m_1 \geq \Lambda \quad \text{and} \quad m_2 < -\Lambda \\ \{\frac{(b+b^{-1})}{8}, \min(\frac{|m_1|}{2|m_2|}, 1)(b+b^{-1})\} & \text{if } m_1 \leq -\Lambda \quad \text{and} \quad m_2 \geq \Lambda \\ \{2(b+b^{-1}), -\frac{1}{2}(b+b^{-1})\} & \text{if } m_1 \leq -\Lambda \quad \text{and} \quad m_2 \leq -\Lambda \end{cases},$$

with a proper positive number Λ, say 5. For other asymptotic regions, the functions (A_{Weeks}, B_{Weeks}) are given by a linear interpolation of the above. For example,

$$\begin{aligned} &A_{Weeks}(m_1,m_2) = \frac{1}{2\Lambda}(m_1+\Lambda)A_{Weeks}(\Lambda,m_2) + \frac{1}{2\Lambda}(\Lambda-m_1)A_{Weeks}(-\Lambda,m_2)\,, \\ &\text{when } -\Lambda \leq m_1 \leq \Lambda \text{ and } m_2 \geq \Lambda\,. \end{aligned} \tag{3.29}$$

The function can be continuously extended to the remaining finite region $[-\Lambda,\Lambda]^2 \subset \mathbb{R}^2$ without touching poles, see (B.8), in the integrand. Since the integrand is locally holomorphic, small deformations of the contour do not change the final integration. The final result only depends on an homology class of the contour and the extension to the finite region is unique as an element of the homology. Using the contour, we numerically compute

$$\begin{aligned} c_T(T[Weeks]) &= -\frac{8}{\pi^2}\text{Re}\left[\frac{\partial_b^2 \mathcal{Z}^{\overline{\text{hyp}}}\big(Weeks;\hbar=2\pi i b^2\big)}{\mathcal{Z}^{\overline{\text{hyp}}}\big(Weeks;\hbar=2\pi i b^2\big)}\right]_{b=1}, \\ &\simeq 0.93\,. \end{aligned} \tag{3.30}$$

Thurston manifold = $(S^3\backslash 5^2_1)_{(5,-1),(1,-2)}$ = $(m003)_{(1,-2)}$. It is the second smallest hyperbolic closed 3-manifold. After integrating Z_2 using the identity (B.9), the state-integral model reduced to

$$\begin{aligned} &\mathcal{Z}^{\overline{\text{hyp}}}\big(Thurston;\hbar=2\pi i b^2\big) \\ &\quad= \int_{\gamma^{\overline{\text{hyp}}}_{Thurston}} \frac{dZ_1 dX}{(2\pi)^2\sqrt{2}}\left(2\cosh\left(\frac{bX}{2}\right)\cosh\left(\frac{X}{2b}\right)\right)\psi_b(Z_1)\psi_b(2X+Z_1) \\ &\qquad\times\exp\left[-(b+b^{-1})(Z_1+X) - \frac{i}{4\pi}(4Z_1^2+7X^2+8Z_1X)\right]. \end{aligned} \tag{3.31}$$

The converging contour can be constructed in the same way as for $M = Weeks$ case using

$$\{A_{Thurston}, B_{Thurston}\} = \begin{cases} \{\frac{(b+b^{-1})}{8}, 0\} & \text{if } m_1 > \Lambda \quad \text{and} \quad m_2 > \Lambda \\ \{\frac{(b+b^{-1})}{2}, \frac{3(b+b^{-1})}{4}\} & \text{if } m_1 > \Lambda \quad \text{and} \quad m_2 < -\Lambda \\ \{2(b+b^{-1}), -\frac{7}{8}(b+b^{-1})\} & \text{if } m_1 < -\Lambda \quad \text{and} \quad m_2 > \Lambda \\ \{\frac{(b+b^{-1})}{4}, (b+b^{-1})\} & \text{if } m_1 < -\Lambda \quad \text{and} \quad m_2 < -\Lambda \end{cases}.$$

Using the contour, we numerically obtain

$$c_T(T[Thurston]) \simeq 1.01\,. \tag{3.32}$$

(5,-1)-Dehn filling on $m003$, $(S^3\backslash 5^2_1)_{(5,-1),(5,-1)} = (m003)_{(5,-1)}$. The reduced state-integral model for this case is

$$
\begin{aligned}
&\mathcal{Z}^{\overline{\text{hyp}}}(m003_{-5};\hbar = 2\pi i b^2)\\
&= \int_{\gamma^{\overline{\text{hyp}}}_{m003_{-5}}} \frac{dZ_1 dX}{(2\pi)^2}\big(2\sin(bX)\sin(bX/2)\big)\psi_b(Z_1)\psi_b(2X+Z_1)\\
&\quad\times \exp\left[-(b+b^{-1})(Z_1+X) - \frac{i}{4\pi}\big(4Z_1^2 - 2X^2 + 8Z_1X\big)\right].
\end{aligned}
$$

For the contour, we use

$$
\{A_{m003_{-5}}, B_{m003_{-5}}\} = \begin{cases}
\{\frac{(b+b^{-1})}{8}, 0\} & \text{if } m_1 > \Lambda \quad \text{and} \quad m_2 > \Lambda\\
\{2(b+b^{-1}), -\frac{9(b+b^{-1})}{10}\} & \text{if } m_1 > \Lambda \quad \text{and} \quad m_2 < -\Lambda\\
\{\frac{(b+b^{-1})}{8}, (b+b^{-1})\} & \text{if } m_1 < -\Lambda \quad \text{and} \quad m_2 > \Lambda\\
\{2(b+b^{-1}), -\frac{(b+b^{-1})}{2}\} & \text{if } m_1 < -\Lambda \quad \text{and} \quad m_2 < -\Lambda
\end{cases}.
$$

Using the contour, numerically we find

$$
c_T(T[(m003)_{(5,-1)}]) \simeq 1.28\,. \tag{3.33}
$$

Integral Dehn fillings on $m003$, $(S^3\backslash 5^2_1)_{(5,-1),(p,1)} = (m003)_{(p,1)}$ with $p \geq 5$. The reduced state-integral model is

$$
\begin{aligned}
&\mathcal{Z}^{\overline{\text{hyp}}}(m003_p;\hbar = 2\pi i b^2)\\
&= \int_{\gamma^{\overline{\text{hyp}}}_{m003_p}} \frac{dZ_1 dX}{(2\pi)^2}\big(2\sinh(bX)\sin(X/b)\big)\psi_b(Z_1)\psi_b(2X+Z_1)\\
&\quad\times \exp\left[-(b+b^{-1})(Z_1+X) - \frac{i}{4\pi}\big(4Z_1^2 + (8+2p)X^2 + 8Z_1X\big)\right].
\end{aligned}
$$

One particular choice of $\gamma^{\overline{\text{hyp}}}_{m003_p}$ $(p \geq 5)$ is

$$
\gamma^{\overline{\text{hyp}}}_{m003_p} := \left\{\{Z_1, X\} = \left\{m_1 + (b+b^{-1})i, m_2 - \frac{2(b+b^{-1})i}{3\pi}\arctan(m_2)\right\} : m_1, m_2 \in \mathbb{R}\right\} \subset \mathbb{C}^2.
$$

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[7]Nagoya U (Dec 2015), SNU (Dec 2015, Aug 2016), KIAS (Dec 2015), NTU (Jan 2016), IPMU (Feb 2016), PMI (Aug 2016) and Waseda U ("Workshop on Volume Conjecture and Quantum Topology", Sep 2016).

A Ideal triangulation of $S^3\backslash 5^2_1$ and $m003$

Ideal triangulations of 3-manifolds with cusped boundaries are available in a computer software `SnapPy` [39].

Whitehead link complement ($S^3\backslash 5^2_1$). The 3-manifold can be triangulated by 4 ideal tetrahedrons ($T = 4$). Boundary meridian/longitude variables and indepedent internal edges ($\mathcal{C}$) are

$$\begin{aligned}
&2X_1 = -Z_1'' - Z_3'' + Z_4\,, &&P_1 = \frac{Z_1}{2} - Z_3 + \frac{3Z_4}{2} - \frac{Z_1'}{2} - \frac{Z_4'}{2} - Z_1'' - Z_3'' - \frac{Z_2''}{2}\,,\\
&2X_2 = Z_1 - Z_2' - Z_3\,, &&P_2 = Z_1 - \frac{Z_3}{2} - \frac{Z_4}{2} + \frac{Z_3'}{2} - \frac{Z_1'}{2} - \frac{Z_2'}{2} - \frac{Z_4''}{2}\,,\\
&\mathcal{C}_1 = 2Z_1' + Z_1'' + 2Z_2' + Z_2'' + Z_3 + Z_4'' - 2\pi i\,, \\
&\mathcal{C}_2 = Z_1'' + Z_2'' + 2Z_3'' + Z_3 + 2Z_4' + Z_4'' - 2\pi i\,.
\end{aligned} \tag{A.1}$$

Using a linear relation

$$Z_i + Z_i' + Z_i'' = i\pi\,, \tag{A.2}$$

the edge parameter Z_i' can be eliminated. After the elimination, generalized Neumann-Zagier datum $(A, B, C, D; f, f'', \nu, \nu_p)$ are determined by

$$A\cdot\begin{pmatrix} Z_1\\ Z_2\\ Z_3\\ Z_4\end{pmatrix} + B\cdot\begin{pmatrix} Z_1''\\ Z_2''\\ Z_3''\\ Z_4''\end{pmatrix} - i\pi\nu = \begin{pmatrix} 2X_1\\ 2X_2\\ \mathcal{C}_1\\ \mathcal{C}_2\end{pmatrix}, \quad C\cdot\begin{pmatrix} Z_1\\ Z_2\\ Z_3\\ Z_4\end{pmatrix} + D\cdot\begin{pmatrix} Z_1''\\ Z_2''\\ Z_3''\\ Z_4''\end{pmatrix} - i\pi\nu_p = \begin{pmatrix} P_1\\ P_2\\ \Gamma_1\\ \Gamma_2\end{pmatrix}$$
$$A\cdot f + B\cdot f'' = \nu\,, \qquad\qquad C\cdot f + D\cdot f'' = \nu_p\,.$$

Here $\{\Gamma_i\}_{i=1}^2$ are some linear combinations of $\vec{Z}$ and $\vec{Z}''$ chosen to satisfy

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathrm{Sp}(8,\mathbb{Q})\,. \tag{A.3}$$

For example, we can choose

$$\begin{aligned}
\Gamma_1 &= \frac{Z_1}{2} - \frac{Z_3}{4} - \frac{Z_4}{4} + \frac{Z_1''}{4} - \frac{Z_2''}{2} - \frac{Z_3''}{4} - \frac{Z_4''}{4}\,,\\
\Gamma_2 &= \frac{Z_1}{2} + \frac{Z_4}{2} - \frac{5Z_3}{8} + \frac{3Z_1''}{8} + \frac{Z_4''}{8} - \frac{3Z_2''}{8}\,.
\end{aligned}$$

The final expression of the state-integral model is independent on the specific choice of Γ_1 and Γ_2.

Sister of figure-eight knot complement = $(S^3\backslash 5^2_1)_{(5,-1)}$ = ($m003$). The 3-manifold can be triangulated by 2 ideal tetrahedrons ($T = 2$). After eliminating (Z_1', Z_2'), we have

$$\begin{aligned}
&2X = Z_1'' + Z_1 - 2Z_2 - 3Z_2'' + i\pi\,, &&P = -Z_2 - 2Z_2'' + i\pi\,.\\
&\mathcal{C} = Z_1'' + 2Z_1 - Z_2 - 2Z_2''\,, &&\Gamma = Z_1'' + Z_1
\end{aligned}$$

Generalized Neumann-Zagier datum $(A, B, C, D; f, f'', \nu, \nu_p)$ are determined by

$$A \cdot \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + B \cdot \begin{pmatrix} Z_1'' \\ Z_2'' \end{pmatrix} - i\pi\nu = \begin{pmatrix} 2X \\ \mathcal{C} \end{pmatrix}, \quad C \cdot \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + D \cdot \begin{pmatrix} Z_1'' \\ Z_2'' \end{pmatrix} - i\pi\nu_p = \begin{pmatrix} P \\ \Gamma \end{pmatrix}$$
$$A \cdot f + B \cdot f'' = \nu\,, \qquad C \cdot f + D \cdot f'' = \nu_p\,.$$

B Quantum dilogarithm

In this appendix we collect formulas for the noncompact quantum dilogarithm (QDL) function [40]. The function $\Psi_b(Z)$ is defined by

$$\Psi_b(Z) := \begin{cases} \prod_{r=1}^{\infty} \frac{1-q^r e^{-Z}}{1-\tilde{q}^{-r+1} e^{-\tilde{Z}}} & \text{if } |q| < 1 \\ \prod_{r=1}^{\infty} \frac{1-\tilde{q}^r e^{-\tilde{Z}}}{1-q^{-r+1} e^{-Z}} & \text{if } |q| > 1 \end{cases} \tag{B.1}$$

with

$$q := e^{2\pi i b^2}\,, \qquad \tilde{q} := e^{2\pi i b^{-2}}\,, \qquad \tilde{Z} := \frac{1}{b^2} Z\,. \tag{B.2}$$

Integral representation:

$$\log \Psi_b(Z) = \int_{\mathbb{R}+i0^+} \frac{e^{\frac{itZ}{\pi b} + t(b+b^{-1})}}{\sinh(bt)\sinh(b^{-1}t)} \frac{dt}{4t}\,, \qquad \text{for } 0 < \text{Im}[Z] < 2\pi(1+b^2)\,. \tag{B.3}$$

Asymptotic expansion when $\hbar = 2\pi i b^2 \to 0$:

$$\log \Psi_b(Z) \xrightarrow{b^2 \to 0^+} \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \text{Li}_{2-n}(e^{-Z})\,, \qquad \text{for } 0 < \text{Im}[Z] < \pi\,. \tag{B.4}$$

Here B_n is the n-th Bernoulli number with $B_1 = 1/2$. To have $b \leftrightarrow b^{-1}$ symmetry, we define

$$\log \psi_b(x) := \log \Psi_b(bx)\,. \tag{B.5}$$

At $b = 1$, the QDL simplified as

$$\log \psi_{b=1}(x) = \frac{-(2\pi + ix)\log(1 - e^{-x}) + i\text{Li}_2(e^{-x})}{2\pi}\,. \tag{B.6}$$

As $|x| \to \infty$,

$$\begin{aligned} \log \psi_b(x) &\sim \frac{-x^2}{4\pi i} + \frac{1}{2}(b + b^{-1})x \qquad && \text{for Re[x]<0}\,, \\ &\sim 0 && \text{for Re[x]>0}\,. \end{aligned} \tag{B.7}$$

Poles of the $\psi_b(Z)$ are located on

$$\mathbb{Z}_{\leq 0}(2\pi i b) + \mathbb{Z}_{\leq 0}(2\pi i b^{-1})\ . \tag{B.8}$$

Fourier transformation:

$$\frac{e^{\frac{i\pi(b^2+3+b^{-2})}{12}}}{2\pi}\int_{\mathbb{E}_y} dx\psi_b(x)e^{\frac{x^2+2xy-2\pi ix(b+b^{-1})}{4\pi i}} = \psi_b(y) \qquad \text{for Im}(y)>0\,,$$
$$\mathbb{E}_y := \{x+if(x)\ :\ x\in\mathbb{R}\}\subset \text{ where } f \text{ is a function satisfying}$$
$$f\to\begin{cases}-\text{Im}(y)+(b+b^{-1})\pi-\epsilon_1 & \text{if } x\to\infty\\ \epsilon_2 & \text{if } x<\Lambda\end{cases} \text{with small } \epsilon_1,\epsilon_2>0 \text{ and positive } \Lambda. \qquad \text{(B.9)}$$

C Quantum Dehn filling

Classical phase space $P(\partial M)$ and its Lagrangian subvariety $\mathcal{L}(M)$ for the SL(2) CS theory are

$$\begin{aligned}P(\partial M) &= \{\text{SL(2)-flat connections on } \partial M = (\mathbb{T}^2)^{|K|-S}\} = \big(P(\mathbb{T}^2)\big)^{|K|-S}\\ \text{with } P(\mathbb{T}^2) &= (\mathbb{C}^*)^2/\mathbb{Z}_2 = \{(x,p)\in(\mathbb{C}^*)^2 : (x,p)\sim(1/x,1/p)\}\,,\\ \mathcal{L}(M) &= \{\text{SL(2)-flat connections on } M\}\,.\end{aligned} \qquad \text{(C.1)}$$

Here x and p parametrize the SL(2) gauge holonomy around each meridian and longitude respectively:

$$\text{P}e^{\oint_{\text{merdian}}\mathcal{A}} = \begin{pmatrix} x & 1\\ 0 & 1/x\end{pmatrix}\,, \qquad \text{P}e^{\oint_{\text{longitude}}\mathcal{A}} = \begin{pmatrix} p & 1\\ 0 & 1/p\end{pmatrix}\,. \qquad \text{(C.2)}$$

Quantizing them, we have

$$\begin{aligned}P(\partial M) &\rightsquigarrow \mathcal{H}(\partial M) = \big(\mathcal{H}(\mathbb{T}^2)\big)^{|K|-S} && \text{(a Hilbert-space)}\,,\\ \mathcal{L}(M) &\rightsquigarrow \big|\mathcal{Z}(M)\big\rangle\in\mathcal{H}(\partial M) && \text{(a state)}\,.\end{aligned} \qquad \text{(C.3)}$$

Quantization of the phase space $P(\mathbb{T}^2)$ with $k=1$. Phase space $\mathcal{P}(\mathbb{T}^2)$ for $\text{SL}(2)_{k,\sigma}$ CS theory with $k=1$ and $\sigma=\frac{1-b^2}{1+b^2}$ on $\mathbb{R}_t\times\mathbb{T}^2$ is give in (C.1) with following symplectic form ($X:=\log x, P:=\log p$):

$$\Omega = \frac{1}{\pi(1+b^2)}dP\wedge dX + \frac{1}{\pi(1+b^{-2})}d\overline{P}\wedge d\overline{X}\,. \qquad \text{(C.4)}$$

Quantization of the phase space give an infinite dimensional Hilbert-space $\mathcal{H}(\mathbb{T}^2)$ whose position basis are

$$\text{Position bais of } \mathcal{H}(\mathbb{T}^2) = \big\{|X\rangle\ :\ X\in\mathbb{C}\,,\ |X\rangle\sim=|-X\rangle\big\}. \qquad \text{(C.5)}$$

The quantum position/momentum operators acts on the Hilbert-space as

$$\langle X|\hat{x} = \langle X|e^X\,, \quad \langle X|\hat{\tilde{x}} = \langle X|e^{X/b^2}\,, \quad \langle X|\hat{p} = \langle X+i\pi b^2|\,, \quad \langle X|\hat{\tilde{p}} = \langle X+i\pi|\,. \qquad \text{(C.6)}$$

Completeness relation in $\mathcal{H}(\mathbb{T}^2)$ is

$$\frac{1}{4\pi b}\int d\mu|X\rangle\langle X| = \mathbb{I}\,. \qquad \text{(C.7)}$$

Quantization of Dehn filling. For a 3-manifold closed M obtained by gluing two 3-manifolds M_1 and M_2 along a common $\mathbb{T}^2$ boundary with a $\varphi \in \text{SL}(2,\mathbb{Z})$ twist, the SL(2) CS ptn is given by

$$\begin{aligned} &\mathcal{Z}_{\text{SI}}(M = M_1 \cup_\varphi M_2; \hbar) = \langle \mathcal{Z}(M_1) | \hat{\varphi} | \mathcal{Z}(M_2) \rangle \,, \\ &\qquad |\mathcal{Z}(M_i)\rangle \in \mathcal{H}(\mathbb{T}^2)\,, \qquad\qquad i = 1,2\,, \\ &\qquad \hat{\varphi} \;:\; \mathcal{H}(\mathbb{T}^2) \to \mathcal{H}(\mathbb{T}^2)\,. \end{aligned} \tag{C.8}$$

For solid-torus $D_2 \times S^1$, the wave-function is simply given by

$$\langle X | \mathcal{Z}(D_2 \times S^1) \rangle = 4 \sinh(X) \sinh(X/b^2)\,. \tag{C.9}$$

Note that solid-torus can be thought as unknot complement on S^3, $D^2 \times S^1 = S^3 \backslash \mathbf{0}_1$, and we use the canonical polarization where the position (momentum) is an eigenvalue homonomy around the meridian (longitude). The wave-function satisfy a pair of difference equations ($q := e^{2\pi i b^2}, \bar{q} := e^{2\pi i b^{-2}}$):

$$\begin{aligned} &\hat{A}_{K=\mathbf{0}_1}(\hat{x}^2, \hat{p}, q^{1/2}) |\mathcal{Z}(D^2 \times S^1)\rangle = \hat{A}_{K=\mathbf{0}_1}(\hat{\tilde{x}}^2, \hat{\tilde{p}}, \bar{q}^{1/2}) |\mathcal{Z}(D^2 \times S^1)\rangle = 0\,, \\ &\qquad \text{where } \hat{A}_{\mathbf{0}_1}(\hat{x}^2, \hat{p}, q^{1/2}) = \hat{p}^2 + 1 - q^{1/2}\hat{p} - q^{-1/2}\hat{p}\,. \end{aligned} \tag{C.10}$$

Regardless of whether the gauge group is SU(2) or its complexification SL(2), the difference operator $\hat{A}_K$ annihilating the knot-complement wave-function $|Z(S^3 \backslash K)\rangle$ is the same and called 'quantum A-polynomial' of knot K [41]. For a closed 3-manifold $(S^3\backslash K)_{p/q}$ obtained by performing Dehn surgery with a slope p/q on S^3 along a knot K,[8] the CS wave function can be obtained as follows:

$$\begin{aligned} &(S^3 \backslash K)_{p/q} = (D^2 \times S^1) \cup_{\varphi_{p/q}} (S^3 \backslash K)\,, \qquad \varphi_{p/q} := \begin{pmatrix} * & * \\ p & q \end{pmatrix} \in \text{SL}(2,\mathbb{Z})\,, \\ &\mathcal{Z}_{\text{SI}}\big((S^3\backslash K)_{p/q}; \hbar\big) = \langle \mathcal{Z}(D^2 \times S^1) | \hat{\varphi}_{p/q} | \mathcal{Z}(S^3 \backslash K) \rangle\,, \qquad \hat{\varphi}_{p/q} \;:\; \mathcal{H}(\mathbb{T}^2) \to \mathcal{H}(\mathbb{T}^2)\,. \end{aligned} \tag{C.11}$$

Two generators of $\text{SL}(2,\mathbb{Z})$ are

$$\varphi_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varphi_T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{C.12}$$

Quantization of these operators give [17]

$$\begin{aligned} &\hat{\varphi}_S\,,\ \hat{\varphi}_T \;:\; \mathcal{H}(\mathbb{T}^2) \to \mathcal{H}(\mathbb{T}^2)\,, \\ &\qquad \langle X | \hat{\varphi}_S | \psi \rangle = \frac{1}{\sqrt{2\pi} b} \int dY e^{-\frac{XY}{\pi i b^2}} \langle Y | \psi \rangle\,, \\ &\qquad \langle X | \hat{\varphi}_T | \psi \rangle = e^{\frac{1}{2\pi i b^2} X^2} \langle X | \psi \rangle\,, \qquad\qquad \text{for any } |\psi\rangle \in \mathcal{H}(\mathbb{T}^2)\,. \end{aligned} \tag{C.13}$$

[8] We call a link K with one component ($|K| = 1$) a 'knot'.

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For general element $\varphi = \begin{pmatrix} r & s \\ p & q \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$,

$$
\begin{aligned}
\langle X|\hat{\varphi}|\psi\rangle &= \frac{1}{\sqrt{2s}\pi b}\int dY\, e^{\frac{qX^2}{2\pi i b^2 s}+\frac{XY}{\pi i b^2 s}+\frac{rY^2}{2\pi i b^2 s}}\langle Y|\psi\rangle\,, \qquad &\text{for } s\neq 0\,,\\
\langle X|\hat{\varphi}|\psi\rangle &= e^{\frac{pX^2}{2\pi i b^2 r}}\langle X|\psi\rangle\,, &\text{for } s=0\,. \qquad \text{(C.14)}
\end{aligned}
$$

Inserting the completeness relation (C.7), we have

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{SI}}\big((S^3\backslash K)_{p/q};\hbar\big) &= \langle \mathcal{Z}(D^2\times S^1)|\hat{\varphi}|\mathcal{Z}(S^3\backslash K)\rangle\\
&= \frac{1}{4\pi b}\int dX \langle \mathcal{Z}(D^2\times S^1)|X\rangle\langle X|\hat{\varphi}|\mathcal{Z}(S^3\backslash K)\rangle\\
&= \frac{1}{\pi^2 b^2\sqrt{2s}}\int dX dY \sinh(X)\sinh(X/b^2) e^{\frac{qX^2}{2\pi i b^2 s}+\frac{XY}{\pi i b^2 s}+\frac{rY^2}{2\pi i b^2 s}}\langle Y|\mathcal{Z}(S^3\backslash K)\rangle\\
&= \int \frac{\Delta_b(Y;s,q)dY}{(2\pi q\hbar)^{1/2}}\exp\left(\frac{p}{\hbar q}Y^2\right)\mathcal{Z}_{\mathrm{SI}}(S^3\backslash K;Y;\hbar). \qquad \text{(C.15)}
\end{aligned}
$$

Here Δ_b is defined in eq. (3.11). For given (p,q), the s is determined modulo $q\mathbb{Z}$ and the final expression $\mathcal{Z}_{\mathrm{SI}}\big((S^3\backslash K)_{p/q}\big)$ does not depend on the choice of (r,s) modulo the intrinsic ambiguity (3.13). This is compatible with the fact that the resulting 3-manifold does not depends on (r,s) but only on (p,q).

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