

All three-loop four-point correlators of half-BPS operators in planar $\mathcal{N} = 4$ SYM

Dmitry Chicherin,^a James Drummond,^b Paul Heslop^c and Emery Sokatchev^{a,d}

^aLaboratoire d'Annecy-le-Vieux de Physique Théorique (LAPTH),
UMR 5108, Université de Savoie, CNRS,
B.P. 110, F-74941 Annecy-le-Vieux, France

^bSchool of Physics & Astronomy, University of Southampton,
Highfield, Southampton, SO17 1BJ, United Kingdom

^cMathematics Department, Durham University, Science Laboratories,
South Rd, Durham DH1 3LE, United Kingdom

^dPhysics Department, Theory Unit, CERN,
CH -1211, Geneva 23, Switzerland

E-mail: chicherin@lapth.cnrs.fr, j.m.drummond@soton.ac.uk,
paul.heslop@durham.ac.uk, emeri.sokatchev@cern.ch

ABSTRACT: We obtain the planar correlation function of four half-BPS operators of arbitrary weights, up to three loops. Our method exploits only elementary properties of the integrand of the planar correlator, such as its symmetries and singularity structure. This allows us to write down a general ansatz for the integrand. The coefficients in the ansatz are fixed by means of a powerful light-cone OPE relation between correlators with different weights. Our result is formulated in terms of a limited number of functions built from known one-, two- and three-loop conformal integrals. These results are useful for checking recent integrability predictions for the OPE structure constants.

KEYWORDS: AdS-CFT Correspondence, Conformal and W Symmetry, Integrable Field Theories, Scattering Amplitudes

ARXIV EPRINT: [1512.02926](https://arxiv.org/abs/1512.02926)

Contents

1	Introduction	1
2	Generalities and summary of the results	4
2.1	Summary of the results	7
3	Description of the method	9
3.1	General properties of the integrand	9
3.2	The role of planarity	14
3.3	Light-cone OPE relation	16
3.3.1	Origin of the light-cone relation	17
3.3.2	A possible stronger relation	21
3.3.3	Singularities of the integrand	22
3.4	Double short-distance OPE	22
3.5	Integral identities	23
4	OPE analysis of the integrated four-point correlators	24
4.1	Equal weights ($kkkk$)	28
4.2	Weights ($kkk'k'$) and an averaging rule	33
4.3	General consistency checks	34
5	Conclusions	34
A	Uniformity of the correlators	36
B	The superconformal OPE	37

1 Introduction

The correlation functions of gauge invariant operators are the natural object in a conformal field theory like $\mathcal{N} = 4$ super-Yang-Mills. Among them a privileged role is played by the correlators of half-BPS scalar operators. They form short superconformal multiplets whose lowest-weight states are annihilated by half of the Poincaré supercharges. The conformal dimensions and more generally, the two- and three-point correlation functions of the half-BPS operators are protected from quantum corrections, but the four-point functions are not. The OPE spectrum of two half-BPS operators is rich, coupling-dependent and generically contains unprotected (long) supermultiplets. Thus, the four-point correlators of half-BPS operators encode some genuinely dynamical information and hence are interesting objects to study.

Such correlators have attracted a lot of attention in the context of the AdS/CFT correspondence [1–3]. In its simplest form it states that type IIB supergravity on an $\text{AdS}_5 \times S^5$ background is dual to the limit of the gauge theory where the 't Hooft coupling $a = g^2 N_c / (4\pi^2)$ is infinite and the number of colours N_c is large. The compactification of type IIB supergravity on S^5 results in an infinite tower of (generically massive) Kaluza-Klein modes. According to the AdS/CFT conjecture, the half-BPS operators $\mathcal{O}^{(k)}$ of dimension k are dual to the KK modes transforming in the irrep $[0, k, 0]$ of $\text{SU}(4) \sim \text{SO}(6)$.

Among all half-BPS operators the simplest and widely studied one is that of minimal weight $k = 2$. The corresponding supermultiplet $\mathcal{T} = \mathcal{O}^{(2)} + \dots$ is very special, as it contains the conserved R symmetry current, the stress-energy tensor and the Lagrangian of the $\mathcal{N} = 4$ theory. It is dual to the graviton multiplet of the $\text{AdS}_5 \times S^5$ supergravity comprising the *massless* KK modes. In perturbation theory the loop corrections are generated by integrated Lagrangian insertions. The *integrand* of the ℓ -loop correction to the n -point correlator of the stress-tensor multiplet is most naturally obtained from the correlator $\langle \mathcal{T}(1) \dots \mathcal{T}(n + \ell) \rangle$ evaluated at the lowest perturbative (Born) level. This approach was developed and successfully used for calculating the two-loop four-point function $\langle \mathcal{O}^{(2)} \mathcal{O}^{(2)} \mathcal{O}^{(2)} \mathcal{O}^{(2)} \rangle$ in [4]. More recently, by exploiting a hidden permutation symmetry of the correlators $\langle \mathcal{T}(1) \dots \mathcal{T}(4 + \ell) \rangle$ the planar integrand of this four-point function was found up to seven loops [5–7].

Apart from the simplest case of the stress-tensor supermultiplet, the correlators of half-BPS operators of arbitrary weights deserve equal attention. From the AdS point of view, to start bringing out the flavor of the more involved ten-dimensional physics one has to go beyond the massless sector of the theory and consider new examples of supergravity-induced four-point correlators involving BPS operators of higher dimension. The first steps in this direction were made in [8, 9]. In [9] the four-point correlator $\langle \mathcal{O}^{(3)} \mathcal{O}^{(3)} \mathcal{O}^{(3)} \mathcal{O}^{(3)} \rangle$ was obtained at two loops and the matching AdS supergravity amplitude of *massive* KK states was constructed. Later on, the general case of four half-BPS operators of equal weights $\langle \mathcal{O}^{(k)} \mathcal{O}^{(k)} \mathcal{O}^{(k)} \mathcal{O}^{(k)} \rangle$ up to two loops was considered in [10, 11].¹ This study revealed a degeneracy phenomenon: in the large N_c limit only one (at one loop) and two (at two loops) distinct functions of the conformally invariant cross-ratios described the whole variety of $\text{SU}(4)$ channels in these correlators. The degeneracy is lifted at strong coupling (AdS supergravity).

Besides the AdS/CFT duality, another good reason for studying four-point correlators of *unequal* BPS weights comes from the recent advances in integrability. In the paper [15], which generalizes the results of [16, 17] to the non-compact case, the three-point correlators of two half-BPS operators and one unprotected operator in the $\text{SL}(2)$ sector were studied in the one-loop approximation. An integrability based conjecture was made for the values of the corresponding structure constants. Recently, this result was extended to two loops in [18] and to three loops in [19, 20]. In the absence of direct calculations of the relevant three-point functions, use can be made of the OPE of the four-point correlators of half-BPS operators, which produces sum rules for the structure constants. Such tests of the

¹Classes of correlators with different weights have also been studied in AdS supergravity in [12–14].

integrability conjecture are most sensitive if the four-point correlators involve half-BPS operators of *different* weights. Two particular cases, the correlators $\langle \mathcal{O}^{(2)} \mathcal{O}^{(2)} \mathcal{O}^{(k)} \mathcal{O}^{(k)} \rangle$ and $\langle \mathcal{O}^{(2)} \mathcal{O}^{(3)} \mathcal{O}^{(3)} \mathcal{O}^{(4)} \rangle$ have been computed to two loops in [21] and [22], respectively. These results confirm the prediction of [15]. Some preliminary three-loop results of the present paper have already been used by the authors of [20] as a valuable check of their findings.

Finally, another motivation for the study of the whole class of correlators of half-BPS operators is the search for integrability directly at the level of the multipoint correlation functions. The recent advances in integrability give us strong evidence that the spectrum of anomalous dimensions (i.e., the two-point functions) and the OPE structure constants (i.e., the three-point functions) are integrable in planar $\mathcal{N} = 4$ SYM. It is well known that all correlation functions in a conformal theory can eventually be built from these two elementary ingredients. It is therefore reasonable to expect some kind of integrable structure in the higher-point functions as well. The results of the present work give indications in this direction.

In this paper we address the problem of finding the perturbative corrections to the four-point functions of half-BPS operators of arbitrary weights $\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \mathcal{O}^{(k_3)} \mathcal{O}^{(k_4)} \rangle$ up to three loops. We apply and further develop the method proposed in [5, 6]. The idea is not to compute such correlators using standard Feynman rules but rather to predict their *integrands*. As mentioned above, the integrand of the ℓ -loop correction to the four-point function can be viewed as a $(4 + \ell)$ -point correlator with ℓ Lagrangian insertions, calculated at Born level. This is a rational function of the $(4 + \ell)$ space-time points having certain simple properties. They follow from $\mathcal{N} = 4$ superconformal symmetry and also from the known short-distance physical singularities. This allows us to write down the most general ansatz in the form of a polynomial numerator with given conformal weights at each point, and a fixed universal denominator accounting for the expected singularities. We then classify all possible numerators. Their number is drastically reduced if we restrict ourselves only to *planar* configurations.

The next step is to find a way to fix the arbitrary coefficients in the ansatz. Using the light-cone super-OPE of two half-BPS operators, we derive a very simple relation between two correlators with shifted weights at two points, $k_1 k_2 k_3 k_4$ and $k_1 + 1, k_2 + 1, k_3 k_4$, in the limit where these two points become light-like separated. Iterating this relation imposes many consistency conditions on the coefficients in our ansatz, for all possible values of the BPS weights. These conditions allow us to determine all the coefficients at two loops and all but one at three loops. The latter can be fixed by adapting the Euclidean logarithmic singularity criterion on the integrand elaborated in [6].

Our main result is that all possible correlators of four half-BPS operators, in the planar limit and up to three loops, are described by a limited number of conformally invariant functions (9 at two loops and 55 at three loops). This result, which we call *uniformity*, generalises the degeneracy of the one- and two-loop correlators with equal weights observed in [10, 11]. The various functions are made of a small number of one-, two- and three-loop planar conformal integrals, all of which have already appeared in the simplest correlator $\langle \mathcal{O}^{(2)} \mathcal{O}^{(2)} \mathcal{O}^{(2)} \mathcal{O}^{(2)} \rangle$ at three loops [5]. When we convert the integrands that our method produces into conformal integrals, we use a number of identities for the latter [5, 23]. This

reduces the basis of independent integrals in the final result. We would like to emphasise that, unlike the case 2222 where planarity is automatic (absence of non-planar Feynman graphs), this is not so in the general case $k_1 k_2 k_3 k_4$. So, planarity is a key ingredient in our construction. The fact that we are able to unambiguously predict the entire class of planar correlators of half-BPS operators to three loops, using only their elementary properties, can be interpreted as evidence for a new integrable structure.

Having obtained an expression for all the three-loop correlators, we perform an OPE analysis of the results in perturbation theory. We focus on the leading twist contributions to each contributing $\text{su}(4)$ channel present in the joint OPE of $\mathcal{O}^{(k_1)}\mathcal{O}^{(k_2)}$ and $\mathcal{O}^{(k_3)}\mathcal{O}^{(k_4)}$ for many different values of k_1, k_2, k_3, k_4 . We are able to verify predictions from [15, 18–20] for three-point functions of two protected operators and one unprotected one. We also use this approach to formulate many consistency checks on the results obtained from the construction of the Born-level correlators. We are also able to relate the uniformity property of the Born-level correlators to the appearance of wrapping corrections to three-point functions in the approach of [18].

The paper is organised as follows. In section 2 we give the basic definitions and recall some properties of the correlators we discuss. Then we summarise our two- and three-loop results in the form of two tables. The tables list the coefficients in front of the two- and three-loop conformal integrals that form a basis for all the correlators. The finite size of our tables reflects the fact the number of independent functions is limited (uniformity). We only display our results for the *integrals*, not the *integrand*s due to size limitations. Section 3 contains a detailed description of the method we use to predict the integrand. We recall its basic properties and formulate the most general ansatz reflecting these properties. We then explain the role of planarity for drastically restricting the number of possible topologies of the integrands. The examination of the light-cone super-OPE of two half-BPS operators leads us to a powerful relation between pairs of correlators with shifted weights. In this section we also recall the Euclidean log criterion from [6] and the conformal integral identities from [5, 23]. Section 4 is devoted to an independent check of our results via the standard OPE analysis of the integrated correlation function. Section 5 contains our conclusions and possible further developments. Appendix A contains some details of the proof of the uniformity property. Appendix B summarises the necessary information on the superconformal OPE of two half-BPS operators.

2 Generalities and summary of the results

The lowest component of a half-BPS multiplet in $\mathcal{N} = 4$ SYM is a real scalar field of dimension k (with $k \geq 2$) transforming in the irrep $[0, k, 0]$ of the R symmetry group $\text{SO}(6) \sim \text{SU}(4)$. In terms of the elementary fields it can be realised as a single-trace operator

$$\text{tr}(\phi^{\{I_1 \dots \phi^{I_k\}}). \tag{2.1}$$

Here ϕ^I , $I = 1, \dots, 6$ are the $\mathcal{N} = 4$ SYM scalars and $\{, \}$ denotes traceless and weighted symmetrization. A convenient way of handling the $\text{SO}(6)$ indices is to project the opera-

tor (2.1) onto the highest weight state of the irrep $[0, k, 0]$. This can be done with the help of a complex null vector Y^I ($Y^I Y^I = 0$):

$$\mathcal{O}^{(k)}(x, y) = Y^{I_1} \dots Y^{I_k} \text{tr}(\phi^{I_1} \dots \phi^{I_k}). \quad (2.2)$$

We start by summarizing the general properties of the four-point correlator of half-BPS operators in the $\mathcal{N} = 4$ SYM theory,

$$\mathcal{G}_{k_1 k_2 k_3 k_4} = \langle \mathcal{O}^{(k_1)}(x_1, Y_1) \mathcal{O}^{(k_2)}(x_2, Y_2) \mathcal{O}^{(k_3)}(x_3, Y_3) \mathcal{O}^{(k_4)}(x_4, Y_4) \rangle, \quad (2.3)$$

where $k_1 \geq k_2 \geq k_3 \geq k_4 \geq 2$ are the weights of the four half-BPS operators. The allowed combinations of four label are those for which $\sum_{i=1}^4 k_i = 2n$ and $k_1 \leq k_2 + k_3 + k_4$, so that it is possible to connect the four points with free propagators without leaving any scalars unpaired. For our purposes, a further restriction comes from the fact that the so-called ‘extremal’ (with $k_1 = k_2 + k_3 + k_4$) and ‘next-to-extremal’ (with $k_1 = k_2 + k_3 + k_4 - 2$) correlators are protected [24–26], i.e. for them $\mathcal{G}^{\text{loop}}$ does not exist. This amounts to requiring $k_i < \sum_{j \neq i} k_j - 2$.

The correlator (2.3) splits into two parts,

$$\mathcal{G}_{k_1 k_2 k_3 k_4} = \mathcal{G}_{k_1 k_2 k_3 k_4}^0 + \mathcal{G}_{k_1 k_2 k_3 k_4}^{\text{loop}}. \quad (2.4)$$

The first part is a rational function of the space-time coordinates and corresponds to the Born (free) approximation. The second part includes all the loop corrections which involve non-trivial functions originating from Feynman integrals.

The expression for \mathcal{G}^0 is a polynomial in the elementary propagators (Wick contractions) of two free scalars

$$d_{ij} = d_{ji} \equiv 4\pi^2 \langle \phi(x_i, y_i) \phi(x_j, y_j) \rangle = \frac{y_{ij}^2}{x_{ij}^2}, \quad (2.5)$$

where $y_{ij}^2 = Y_i \cdot Y_j$ and $x_{ij}^2 = (x_i - x_j)^2$. Then we can write the general expression

$$\mathcal{G}_{k_1 k_2 k_3 k_4}^0 = \sum_{\{a_{ij}\}} \left(\prod_{1 \leq i < j \leq 4} (d_{ij})^{a_{ij}} \right) C_{\{a_{ij}\}}, \quad (2.6)$$

where $a_{ij} = a_{ji} \geq 0$ (with $i \neq j$) are integers such that $\sum_{j \neq i} a_{ij} = k_i$ for each $i = 1, \dots, 4$. The sum in (2.6) goes over all possible partitions $\{a_{ij}\}$ satisfying the above condition. Each term in the sum has the required conformal and R-symmetry weights at each of the four points. The coefficients $C_{\{a_{ij}\}}$ are numbers obtained by calculating the colour and combinatorial factors of the different free Feynman diagrams. Here is a simple example:

$$\mathcal{G}_{2222}^0 = \frac{N_c^2}{(4\pi^2)^4} (d_{12}d_{23}d_{34}d_{14} + d_{12}d_{24}d_{34}d_{13} + d_{13}d_{23}d_{24}d_{14}), \quad (2.7)$$

where we have displayed only the connected part and the colour factor is given for $N_c \gg 1$.²

²Our colour convention is $\text{tr}(t_a t_b) = \delta_{ab}/2$.

In principle, the interacting (loop) part of the correlator $\mathcal{G}^{\text{loop}}$ has a structure similar to (2.6). The main difference is that the constant coefficients $C_{\{a_{ij}\}}$ are replaced by functions of the two independent conformally invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (2.8)$$

Thus, in general we can write

$$\mathcal{G}_{k_1 k_2 k_3 k_4}^{\text{loop}} = \sum_{\{a_{ij}\}} \prod_{1 \leq i < j \leq 4} (d_{ij})^{a_{ij}} F_{\{a_{ij}\}}(u, v), \quad (2.9)$$

where each function admits a perturbative expansion in the 't Hooft coupling $a = g^2 N_c / (4\pi^2)$,

$$F_{\{a_{ij}\}}(u, v) = \sum_{\ell \geq 1} a^\ell F_{\{a_{ij}\}}^{(\ell)}(u, v). \quad (2.10)$$

$\mathcal{N} = 4$ superconformal symmetry puts additional restrictions on the coefficient functions in (2.9). According to the 'partial non-renormalisation' theorem of ref. [8, 27] (for alternative derivations see also [5, 28]), the interacting part of the correlator takes the *factorised form*

$$\mathcal{G}_{k_1 k_2 k_3 k_4}^{\text{loop}} = C_{k_1 k_2 k_3 k_4} R(1, 2, 3, 4) \times \sum_{\{b_{ij}\}} \left(\prod_{1 \leq i < j \leq 4} (d_{ij})^{b_{ij}} \right) \frac{F_{\{b_{ij}\}}(u, v)}{x_{13}^2 x_{24}^2}, \quad (2.11)$$

where

$$C_{k_1 k_2 k_3 k_4} = \frac{1}{2} \left(\frac{N_c}{2} \right)^{\frac{1}{2} \sum k_i - 2} \frac{k_1 k_2 k_3 k_4}{(4\pi^2)^{\frac{1}{2} \sum k_i}} \quad (2.12)$$

is a normalisation factor (for convenience here quoted only the large N_c limit) and R is a *universal rational prefactor* carrying $SU(4)$ weight 2 and conformal weight 1 at each point. Explicitly,

$$\begin{aligned} R(1, 2, 3, 4) = & d_{12}^2 d_{34}^2 x_{12}^2 x_{34}^2 + d_{13}^2 d_{24}^2 x_{13}^2 x_{24}^2 + d_{14}^2 d_{23}^2 x_{14}^2 x_{23}^2 \\ & + d_{12} d_{23} d_{34} d_{14} (x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) \\ & + d_{12} d_{13} d_{24} d_{34} (x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2) \\ & + d_{13} d_{14} d_{23} d_{24} (x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2), \end{aligned} \quad (2.13)$$

which is fully symmetric in the points 1, 2, 3, 4. The denominator $x_{13}^2 x_{24}^2$ supplies the missing conformal weights, so that the functions $F_{\{b_{ij}\}}(u, v)$ are conformally invariant. The new partitions $\{b_{ij}\}$ in (2.11) satisfy the modified conditions $\sum_{j \neq i} b_{ij} = k_i - 2$ for each $i = 1, \dots, 4$. For the purpose of presentation we organise $\{b_{ij}\}$ into sextuples of integers,

$$\{b_{ij}\} = \{b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\}. \quad (2.14)$$

The simplest example again is

$$\mathcal{G}_{2222}^{\text{loop}} = \frac{2N_c^2}{(4\pi^2)^4} R(1, 2, 3, 4) \times \frac{F(u, v)}{x_{13}^2 x_{24}^2}. \quad (2.15)$$

Here $\{b_{ij}\} = \{0, 0, 0, 0, 0, 0\}$ and the dynamical information is encoded in the single function $F(u, v)$. This is not the only case where the sum on the right-hand side of (2.11) contains only one term. There are several infinite families of such correlators. A straightforward generalization of $\mathcal{G}_{2222}^{\text{loop}}$ is the correlator $\mathcal{G}_{kk22}^{\text{loop}}$ with $k \geq 2$. In this case there is a unique y -structure encoded by the sextuple $\{b_{ij}\} = \{k - 2, 0, 0, 0, 0, 0\}$.

More generally, the correlators with weights $k_1 = a + b + c + 2$, $k_2 = a + 2$, $k_3 = b + 2$, $k_4 = c + 2$ (or equivalently, $k_1 = k_2 + k_3 + k_4 - 4$) are characterised by the unique sextuple $\{b_{ij}\} = \{a, b, c, 0, 0, 0\}$. Such correlators are known as ‘near extremal’ [12] or ‘next-next-to-extremal’ [14]. Another three-parameter family of correlators with a unique y -structure are those containing one (or more) weight-two operator. For example, if $k_4 = 2$ the unique sextuple is $\{b_{ij}\} = \{a, b, 0, c, 0, 0\}$ corresponding to weights $k_1 = a + b + 2$, $k_2 = a + c + 2$, $k_3 = b + c + 2$, $k_4 = 2$.

In addition to the conformal and R-symmetry properties, the correlator may be further restricted by the permutation symmetry of the external points. If two or more of the labels k_i are equal, the operators $\mathcal{O}^{(k_i)}$ are identical and the correlator must be invariant under the permutations of the corresponding points. This symmetry organises the propagator structures $\prod (d_{ij})^{b_{ij}}$ and the coefficient functions $F_{\{b_{ij}\}}$ into equivalence classes.

A further and less obvious symmetry takes place if some $k_i = 2$. In this case $\mathcal{O}^{(2)}$ is the superconformal primary of the energy-momentum supermultiplet, which also contains the Lagrangian of the theory. This results in a rather powerful permutation symmetry between (some of) the external points and the Lagrangian insertion points (see [5] for details).

2.1 Summary of the results

In this subsection we summarise our results for all possible choices of the four labels k_i , up to three loops. We restrict ourselves to the planar limit $N_c \rightarrow \infty$ and planarity of the resulting correlator graphs will be a key input.

The generic expression for the conformally invariant functions $F_{\{b_{ij}\}}(u, v)$ is given in terms of a set of one-, two- and three-loop integrals (with the cross-ratios defined in (2.8)):

$$\begin{aligned} F^{(1)}/x_{13}^2 x_{24}^2 &= g_{1234} \\ F^{(2)}/x_{13}^2 x_{24}^2 &= c_h^1 h_{12;34} + c_h^2 h_{13;24} + c_h^3 h_{14;23} + \frac{1}{2} (c_{gg}^1 x_{12}^2 x_{34}^2 + c_{gg}^2 x_{13}^2 x_{24}^2 + c_{gg}^3 x_{14}^2 x_{23}^2) [g_{1234}]^2 \\ F^{(3)}/x_{13}^2 x_{24}^2 &= c_{gh}^1 x_{12}^2 x_{34}^2 (g \times h)_{12;34} + c_{gh}^2 x_{13}^2 x_{24}^2 (g \times h)_{13;24} + c_{gh}^3 x_{14}^2 x_{23}^2 (g \times h)_{14;23} \\ &\quad + c_L^1 L_{12;34} + c_L^2 L_{13;24} + c_L^3 L_{14;23} + c_E^1 E_{12;34} + c_E^2 E_{13;24} + c_E^3 E_{14;23} \\ &\quad + \frac{1}{2} (c_H^1 + c_H^2 1/v) H_{12;34} + \frac{1}{2} (c_H^3 + c_H^4 u/v) H_{13;24} + \frac{1}{2} (c_H^5 + c_H^6 u) H_{14;23}, \end{aligned} \quad (2.16)$$

where the conformal integrals are defined as follows:

$$\begin{aligned}
 g_{12;34} &= -\frac{1}{4\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \\
 h_{12;34} &= \frac{x_{34}^2}{(4\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)} \\
 E_{12;34} &= \frac{x_{23}^2 x_{24}^2}{(-4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7 x_{16}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{47}^2)} \\
 L_{12;34} &= \frac{x_{34}^4}{(-4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{27}^2 x_{37}^2 x_{47}^2)} \\
 (g \times h)_{12;34} &= \frac{x_{12}^2 x_{34}^4}{(-4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) (x_{16}^2 x_{36}^2 x_{46}^2) (x_{27}^2 x_{37}^2 x_{47}^2) x_{67}^2} \\
 H_{12;34} &= \frac{x_{41}^2 x_{23}^2 x_{34}^2}{(-4\pi^2)^3} \int \frac{d^4 x_5 d^4 x_6 d^4 x_7 x_{57}^2}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2)}. \quad (2.17)
 \end{aligned}$$

The one-loop correlators are always the same independently of the partition $\{b_{ij}\}$, only the normalisation factor (2.12) changes (see [10] for the case of equal weights). Our two- and three-loop results are presented in the form of two tables where the values of the numerical coefficients in front of the various integrals in (2.16) are listed. Some of the two-loop results in table 1 were obtained in the past through direct Feynman graph calculations [4, 9, 11, 14, 21, 22]. The three-loop result for the case \mathcal{G}_{2222} was first obtained in ref. [5] by a method similar to the one used in the present paper. The other results shown in table 2 are new.

We show that the number of functions that encode the quantum corrections of all the correlators at two and three loops is finite. There are 9 independent functions at two loops and 55 functions at three loops. These functions $F_{\{b_{ij}\}}$ have the form (2.16) with the numerical coefficients listed in tables 1, 2. Some of the lines in the tables are double, which means that the two sextuples come with the same function.

We say that a pair of sextuples are two-loop-equivalent, $\{b_{ij}\} \sim \{b'_{ij}\}$, if some of the entries $b_{ij}, b'_{ij} \geq 1$ are different but all the entries $b_{ij}, b'_{ij} = 0$ are the same. The corresponding two-loop functions are equal, $F_{\{b_{ij}\}}^{(2)} = F_{\{b'_{ij}\}}^{(2)}$. Similarly, a pair of sextuples are three-loop-equivalent, $\{b_{ij}\} \sim \{b'_{ij}\}$, if only their entries $b_{ij}, b'_{ij} \geq 2$ can possibly differ but all the entries $b_{ij}, b'_{ij} = 0, 1$ are the same. The corresponding three-loop functions are equal, $F_{\{b_{ij}\}}^{(3)} = F_{\{b'_{ij}\}}^{(3)}$.

To extract a particular correlator $\mathcal{G}_{k_1 k_2 k_3 k_4}^{\text{loop}}$ from the tables, we first need to enumerate all the relevant y -structures encoded by the sextuples $\{b_{ij}\}$ in (2.11), satisfying the conditions $\sum_{j \neq i} b_{ij} = k_i - 2$ for each $i = 1, \dots, 4$. The coefficients of the various integrals making up the functions $F_{\{b_{ij}\}}$ (one representative of each crossing equivalence class) are then listed in the tables.

Let us consider a couple of examples. In the correlator \mathcal{G}_{3322} there is a unique y -structure y_{12}^2 corresponding to $\{1, 0, 0, 0, 0, 0\}$. At two loops we find $F_{\{1, 0, 0, 0, 0, 0\}}^{(2)}$ in the

$\{b_{ij}\}$		c_{gg}^1	c_{gg}^2	c_{gg}^3	c_h^1	c_h^2	c_h^3
$\{0, 0, 0, 0, 0, 0\}$		1	1	1	2	2	2
$\{\beta_1, 0, 0, 0, 0, 0\}$		0	1	1	1	2	2
$\{\beta_1, \beta_2, 0, 0, 0, 0\}$		0	0	1	1	1	2
$\{\beta_1, \beta_2, 0, \beta_3, 0, 0\}$ $\{\beta_1, \beta_2, \beta_3, 0, 0, 0\}$		0	0	0	1	1	1
$\{0, 0, \beta_1, \beta_2, 0, 0\}$		1	1	0	2	2	0
$\{\beta_1, 0, \beta_2, \beta_3, 0, 0\}$		0	1	0	1	2	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, 0, 0\}$		0	0	0	1	1	0
$\{0, \beta_1, \beta_2, \beta_3, \beta_4, 0\}$		1	0	0	2	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, 0\}$		0	0	0	1	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$		0	0	0	0	0	0

Table 1. Numerical coefficients specifying the two-loop functions $F_{\{b_{ij}\}}^{(2)}$, eq. (2.16). All possible sextuples $\{b_{ij}\}$ (up to crossing permutations) are listed. The parameters $\beta_i \geq 1$ in the different lines are independent. The graphs depict the y -structures encoded by the sextuples $\{b_{ij}\}$. A line between points i and j corresponds to $(y_{ij}^2)^{b_{ij}}$ with $b_{ij} \geq 1$.

2nd line of table 1 and at three loops $F_{\{1,0,0,0,0,0\}}^{(3)}$ in the 2nd line of table 2. In the correlator \mathcal{G}_{4444} there are six y -structures which break down into two equivalence classes under crossing symmetry

$y_{14}^4 y_{23}^4$	$\{0, 0, 2, 2, 0, 0\}$	$y_{13}^4 y_{24}^4$	$\{0, 2, 0, 0, 2, 0\}$	$y_{12}^4 y_{34}^4$	$\{2, 0, 0, 0, 0, 2\}$
$y_{13}^2 y_{23}^2 y_{24}^2 y_{14}^2$	$\{0, 1, 1, 1, 1, 0\}$	$y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2$	$\{1, 0, 1, 1, 0, 1\}$	$y_{12}^2 y_{24}^2 y_{34}^2 y_{13}^2$	$\{1, 1, 0, 0, 1, 1\}$

At two loops we find $F_{\{0,0,2,2,0,0\}}^{(2)}$ in the 5th line of table 1 and $F_{\{0,1,1,1,1,0\}}^{(2)}$ in the 8th line. The remaining four functions are obtained by crossing from the previous two. At three loops we find $F_{\{0,0,2,2,0,0\}}^{(3)}$ in the 20th line of table 2 and $F_{\{0,1,1,1,1,0\}}^{(3)}$ in the 28th line.

3 Description of the method

The results listed in the tables have been obtained by using similar ideas to those employed in refs. [5, 6] for constructing (the integrand of) the correlator \mathcal{G}_{2222} . In the case of different BPS weights there appear some important new ingredients. Here we give a brief summary of the method and explain the new key points.

3.1 General properties of the integrand

The loop corrections (2.11) are obtained by the Lagrangian insertion procedure. It amounts to computing the Born-level $(4 + \ell)$ -point correlator with ℓ Lagrangian insertions and then

$\{b_{ij}\}$	$\begin{matrix} 1 & 2 \\ \nearrow & \nwarrow \\ 4 & 3 \end{matrix}$	c_{gh}^1	c_{gh}^2	c_{gh}^3	c_L^1	c_L^2	c_L^3	c_E^1	c_E^2	c_E^3	c_H^1	c_H^2	c_H^3	c_H^4	c_H^5	c_H^6
$\{0, 0, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	2	2	2	6	6	6	4	4	4	2	2	2	2	2	2
$\{1, 0, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	2	2	2	6	6	4	2	2	1	1	2	0	2	0
$\{\beta_1, 0, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	2	2	3	6	6	4	2	2	1	1	2	0	2	0
$\{1, 1, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	-1	2	2	2	6	2	2	1	1	0	1	0	0	0
$\{\beta_1, 1, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	-1	2	3	2	6	2	2	1	1	0	1	0	0	0
$\{\beta_1, \beta_2, 0, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	2	3	3	6	2	2	1	1	0	1	0	0	0
$\{1, 1, 0, 1, 0, 0\}$ $\{1, 1, 1, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	-1	-1	2	2	2	1	1	1	0	0	0	0	0	0
$\{\beta_1, 1, 0, 1, 0, 0\}$ $\{\beta_1, 1, 1, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	-1	-1	3	2	2	1	1	1	0	0	0	0	0	0
$\{\beta_1, \beta_2, 0, 1, 0, 0\}$ $\{\beta_1, \beta_2, 1, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	-1	3	3	2	1	1	1	0	0	0	0	0	0
$\{\beta_1, \beta_2, 0, \beta_3, 0, 0\}$ $\{\beta_1, \beta_2, \beta_3, 0, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	0	3	3	3	1	1	1	0	0	0	0	0	0
$\{0, 0, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	2	2	0	6	6	-2	0	0	4	0	2	0	2	0	0
$\{1, 0, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	2	0	2	6	-2	0	0	2	0	1	0	0	0	0
$\{\beta_1, 0, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	2	0	3	6	-2	0	0	2	0	1	0	0	0	0
$\{1, 1, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	-1	0	2	2	-2	0	0	1	0	0	0	0	0	0
$\{\beta_1, 1, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	-1	0	3	2	-2	0	0	1	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	0	3	3	-2	0	0	1	0	0	0	0	0	0
$\{1, 1, 1, \beta_1, 0, 0\}$ $\{1, 1, \beta_1, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	-1	0	2	2	-1	0	0	1	0	0	0	0	0	0
$\{\beta_1, 1, 1, \beta_2, 0, 0\}$ $\{\beta_1, 1, \beta_2, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	-1	0	3	2	-1	0	0	1	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, \beta_3, 0, 0\}$ $\{\beta_1, \beta_2, \beta_3, 1, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	0	3	3	-1	0	0	1	0	0	0	0	0	0
$\{0, 0, \beta_1, \beta_2, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	2	2	0	6	6	0	0	0	4	0	2	0	2	0	0
$\{1, 0, \beta_1, \beta_2, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	2	0	2	6	0	0	0	2	0	1	0	0	0	0
$\{\beta_1, 0, \beta_2, \beta_3, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	2	0	3	6	0	0	0	2	0	1	0	0	0	0
$\{1, 1, \beta_1, \beta_2, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	-1	0	2	2	0	0	0	1	0	0	0	0	0	0
$\{\beta_1, 1, \beta_2, \beta_3, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	-1	0	3	2	0	0	0	1	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, 0, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	0	3	3	0	0	0	1	0	0	0	0	0	0
$\{1, \beta_1, 0, 0, 1, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	0	2	2	-1	6	0	2	0	1	0	0	0	0	0
$\{\beta_1, \beta_2, 0, 0, 1, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	0	0	2	3	-1	6	0	2	0	1	0	0	0	0	0
$\{0, 1, 1, 1, 1, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	2	0	0	6	-2	-2	0	0	0	0	0	0	0	0	0
$\{1, 1, 1, 1, 1, 0\}$	$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$	-1	0	0	2	-2	-2	0	0	0	0	0	0	0	0	0

$\{b_{ij}\}$		c_{gh}^1	c_{gh}^2	c_{gh}^3	c_L^1	c_L^2	c_L^3	c_E^1	c_E^2	c_E^3	c_H^1	c_H^2	c_H^3	c_H^4	c_H^5	c_H^6
$\{\beta_1, 1, 1, 1, 1, 0\}$		0	0	0	3	-2	-2	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, 1, 1, 1, 0\}$		-1	0	0	2	-1	-2	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, 1, 1, 0\}$		0	0	0	3	-1	-2	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, 1, \beta_2, 1, 0\}$		-1	0	0	2	-1	-1	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, \beta_2, 1, 1, 0\}$		-1	0	0	2	-1	-1	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, \beta_3, 1, 0\}$		0	0	0	3	-1	-1	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, 1, 1, 0\}$		0	0	0	3	-1	-1	0	0	0	0	0	0	0	0	0
$\{0, 1, \beta_1, \beta_2, 1, 0\}$		2	0	0	6	-2	0	0	0	0	0	0	0	0	0	0
$\{1, 1, \beta_1, \beta_2, 1, 0\}$		-1	0	0	2	-2	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, 1, \beta_2, \beta_3, 1, 0\}$		0	0	0	3	-2	0	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, \beta_2, \beta_3, 1, 0\}$		-1	0	0	2	-1	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, 1, 0\}$		0	0	0	3	-1	0	0	0	0	0	0	0	0	0	0
$\{0, \beta_1, \beta_2, \beta_3, \beta_4, 0\}$		2	0	0	6	0	0	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, \beta_2, \beta_3, \beta_4, 0\}$		-1	0	0	2	0	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, 0\}$		0	0	0	3	0	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, 0, 0, 0, 0, 1\}$		0	2	2	-1	6	6	4	0	0	0	0	2	0	2	0
$\{\beta_1, 0, 1, 1, 0, 1\}$		0	2	0	-1	6	-2	0	0	0	0	0	0	0	0	0
$\{\beta_1, 0, \beta_2, \beta_3, 0, 1\}$		0	2	0	-1	6	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, 0, 0, 1, 1\}$		0	0	2	-1	-1	6	0	0	0	0	0	0	0	0	0
$\{1, 1, 1, 1, 1, 1\}$		0	0	0	-2	-2	-2	0	0	0	0	0	0	0	0	0
$\{\beta_1, 1, 1, 1, 1, 1\}$		0	0	0	-1	-2	-2	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, 1, 1, 1\}$		0	0	0	-1	-1	-2	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, 1, \beta_3, 1, 1\}$		0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, 1, 1, 1\}$		0	0	0	-1	-1	-1	0	0	0	0	0	0	0	0	0
$\{1, 1, \beta_1, \beta_2, 1, 1\}$		0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, 1, \beta_2, \beta_3, 1, 1\}$		0	0	0	-1	-2	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, 1, 1\}$		0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
$\{1, \beta_1, \beta_2, \beta_3, \beta_4, 1\}$		0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, 1\}$		0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
$\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2. Numerical coefficients specifying the three-loop functions $F_{\{b_{ij}\}}^{(3)}$, eq. (2.16). All possible sextuples $\{b_{ij}\}$ (up to crossing permutations) are listed. The parameters $\beta_i \geq 2$ in the different lines are independent. A thin line between points i and j corresponds to y_{ij}^2 , i.e. $b_{ij} = 1$, and a thick line to $(y_{ij}^2)^{b_{ij}}$ with $b_{ij} \geq 2$.

integrating over the coordinates of the insertion points,

$$\mathcal{G}_{k_1 k_2 k_3 k_4}^\ell = \int \frac{d^4 x_5 \dots d^4 x_{4+\ell}}{\ell! (-4\pi^2)^\ell} G_{k_1 k_2 k_3 k_4}^\ell \tag{3.1}$$

$$G_{k_1 k_2 k_3 k_4}^\ell = \langle \mathcal{O}^{(k_1)}(1) \mathcal{O}^{(k_2)}(2) \mathcal{O}^{(k_3)}(3) \mathcal{O}^{(k_4)}(4) \mathcal{L}(5) \dots \mathcal{L}(4 + \ell) \rangle_{\text{Born}} . \tag{3.2}$$

Thus the problem is reduced to determining the correlator (3.2).

The cases where one or more $k_i = 2$ are special. The half-BPS scalar operator $\mathcal{O}^{(2)}$ and the Lagrangian \mathcal{L} are members of the same $\mathcal{N} = 4$ supermultiplet, the chiral truncation \mathcal{T} of the stress-tensor supermultiplet,

$$\mathcal{T}(x, y, \rho) = \mathcal{O}^{(2)}(x, y) + \dots + \rho^4 \mathcal{L}(x), \tag{3.3}$$

where $\rho_\alpha^a = \theta_\alpha^a + \theta_\alpha^{a'} y_{a'}^a$ is the SU(4) harmonic projection of the chiral odd variable θ_α^A .³ This projection carries U(1) charge (+1). The operator $\mathcal{O}^{(2)}$ (as well as the whole supermultiplet \mathcal{T}) has charge (+4) in the same units. The Lagrangian \mathcal{L} is chargeless and hence independent of the harmonic variable y (SU(4) singlet). The half-BPS operators $\mathcal{O}^{(k)}(x, y)$ of conformal weight $k > 2$ are the bottom components of other analytic superfields depending on ρ , carrying U(1) charge $2k$. Thus, the integrand of the loop corrections is given by the component $(\rho_1)^0 (\rho_2)^0 (\rho_3)^0 (\rho_4)^0 (\rho_5)^4 \dots (\rho_{4+\ell})^4$ of the super-correlator

$$\langle \mathcal{O}^{(k_1)}(1) \mathcal{O}^{(k_2)}(2) \mathcal{O}^{(k_3)}(3) \mathcal{O}^{(k_4)}(4) \mathcal{T}(5) \dots \mathcal{T}(4 + \ell) \rangle_{\text{Born}}, \tag{3.4}$$

evaluated in the Born approximation. It is invariant under the permutations of the points $(5, \dots, 4 + \ell)$, i.e. it has S_ℓ symmetry. In the special case where p of the $k_i = 2$ this symmetry is enhanced to $S_{p+\ell}$. If some operators have equal weights $k_i = k_j \neq 2$, there is an additional permutation symmetry of those points. The most symmetric case \mathcal{G}_{2222}^ℓ was studied in [5, 6], where the maximal $S_{4+\ell}$ symmetry proved to be extremely helpful in constructing the integrand. In the general case $\mathcal{G}_{k_1 k_2 k_3 k_4}^\ell$ we have less symmetry but are nevertheless able to determine the integrand up to three loops, as explained below.

Superconformal symmetry imposes restrictions on the form of the correlators \mathcal{G}^ℓ . According to the partial non-renormalisation theorem, the loop corrections to any four-point correlator of scalar half-BPS operators are proportional to the rational function $R(1, 2, 3, 4)$ defined in (2.13). It is convenient to turn R into a polynomial multiplying it by the permutation invariant factor $x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2$. The prefactor R has U(1) charge (+4) at each point whereas the correlator $\mathcal{G}_{k_1 k_2 k_3 k_4}$ bears charges $2k_i \geq 4$ at each point. The difference of U(1) charges between \mathcal{G}^ℓ and R can be compensated by a product of propagator factors,

$$G_{k_1 k_2 k_3 k_4}^\ell = C_{k_1 k_2 k_3 k_4} \times \mathcal{I} \times \sum_{\{b_{ij}\}} \left(\prod_{1 \leq i < j \leq 4} (d_{ij})^{b_{ij}} \right) f_{\{b_{ij}\}}^\ell(x_1, \dots, x_{4+\ell}). \tag{3.5}$$

³The complex four-vector y_a^a is part of the SO(6) null vector $Y^I = (1, y, \sqrt{-1 - y^2})$.

where $\mathcal{I} = R x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2$ is a polynomial in both y and x . Explicitly this polynomial is given by

$$\begin{aligned} \mathcal{I} := & x_{14}^4 x_{23}^4 y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2 + x_{12}^2 x_{14}^2 x_{34}^2 x_{23}^2 y_{13}^4 y_{24}^4 + x_{13}^2 x_{14}^2 x_{24}^2 x_{23}^2 y_{12}^4 y_{34}^4 \\ & - x_{12}^2 x_{14}^2 x_{34}^2 x_{23}^2 y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2 - x_{13}^2 x_{14}^2 x_{24}^2 x_{23}^2 y_{12}^2 y_{14}^2 y_{23}^2 y_{34}^2 - x_{13}^2 x_{14}^2 x_{24}^2 x_{23}^2 y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2 \\ & - x_{12}^2 x_{14}^2 x_{34}^2 x_{23}^2 y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2 + x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2 y_{14}^4 y_{23}^4 + x_{12}^4 x_{34}^4 y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2 \\ & - x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2 y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2 + x_{13}^4 x_{24}^4 y_{12}^2 y_{14}^2 y_{23}^2 y_{34}^2 - x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2 y_{12}^2 y_{14}^2 y_{23}^2 y_{34}^2 . \end{aligned} \tag{3.6}$$

This expression accounts for the y -dependence of the integrand of the correlator $\mathcal{G}_{k_1 k_2 k_3 k_4}$. The x -coordinate part is not completely fixed by the superconformal symmetry. It is encoded in the $(4 + \ell)$ -point rational functions $f_{\{b_{ij}\}}^\ell(x)$ having the crossing symmetry S_ℓ (or higher, depending on the weights $k_1 k_2 k_3 k_4$) of (3.4). They can be written in the form

$$f_{\{b_{ij}\}}^\ell = \frac{P_{\{b_{ij}\}}^\ell(x_1, \dots, x_{4+\ell})}{\prod_{1 \leq p < q \leq 4+\ell} x_{pq}^2}, \tag{3.7}$$

where $P_{\{b_{ij}\}}^\ell$ are polynomials of conformal weight $(1 - \ell)$ at each point. To justify the singularity structure of this correlator we need to consider the OPE of the various operators (see section 3.3.3).

All possible numerator terms up to three loops were analysed in [5]. There we had an additional permutation symmetry — not present in the current more general situation — which meant that all terms came with the same coefficient. Here the terms which can appear are the same as there, but the coefficients are different.

At two and three loops then we can write the general ansatz as

$$\begin{aligned} P_{\{b_{ij}\}}^2(x_1, \dots, x_6) &= \sum_{\sigma \in S_6/\text{auto}} a_{\{b_{ij}, \sigma\}}^{(2)} x_{\sigma_1 \sigma_2}^2 x_{\sigma_3 \sigma_4}^2 x_{\sigma_5 \sigma_6}^2 \\ P_{\{b_{ij}\}}^3(x_1, \dots, x_7) &= \sum_{\sigma \in S_7/\text{auto}} a_{\{b_{ij}, \sigma\}}^{(3)} x_{\sigma_1 \sigma_2}^4 x_{\sigma_3 \sigma_4}^2 x_{\sigma_4 \sigma_5}^2 x_{\sigma_5 \sigma_6}^2 x_{\sigma_6 \sigma_7}^2 x_{\sigma_7 \sigma_1}^2, \end{aligned} \tag{3.8}$$

and all that remains is to determine the coefficients $a_{\{b_{ij}, \sigma\}}^{(\ell)}$. Here the sum is over all permutations of $S_{4+\ell}$ which are inequivalent when acting on the monomial. So for example, clearly $x_{12}^2 x_{34}^2 x_{56}^2 = x_{21}^2 x_{34}^2 x_{56}^2$, so in the two-loop case the identity permutation and the permutation (12) give the same monomial and we only sum over one of the two. This is the same as modding out by the automorphism group of the corresponding graph which explains our notation $S_{4+\ell}/\text{auto}$. Furthermore we also explicitly symmetrise over permutations of the integration variables, which further trivially identifies coefficients.

To each term in $f_{\{b_{ij}\}}$ we can draw the corresponding graph

$$\begin{array}{cc}
 \begin{array}{c} \sigma_3 \\ \diagdown \quad \diagup \\ \sigma_2 \quad \sigma_6 \\ \diagup \quad \diagdown \\ \sigma_5 \\ \diagdown \quad \diagup \\ \sigma_4 \end{array} & \begin{array}{c} \sigma_2 \\ \diagdown \quad \diagup \\ \sigma_3 \quad \sigma_7 \\ \diagup \quad \diagdown \\ \sigma_5 \quad \sigma_6 \\ \diagdown \quad \diagup \\ \sigma_4 \end{array} \\
 f_{\{b_{ij}, \sigma\}}^2 & f_{\{b_{ij}, \sigma\}}^3 \cdot
 \end{array} \tag{3.9}$$

An important feature of both these graphs, which we return to in the next subsection, is that they are planar, and they have the property that adding any further edge to either graph makes them non-planar.⁴

We should also note that although at two loops the above structure is the only possibility consistent with conformal weights, at three loops there are three other inequivalent topologies consistent with conformal weights. However these do not contribute to the planar correlation function (since they do not yield planar component correlation functions — a requirement we insist on, as discussed in the following subsection) and so we do not write them out here.

3.2 The role of planarity

A further strong constraint on the polynomial P^ℓ in (3.7) comes from the planar limit. We have classified the possible P^ℓ having the properties discussed in section 3.1 above. The number can be greatly reduced by requiring that the correlator that we want to construct should correspond to the leading colour approximation in the limit $N_c \rightarrow \infty$. If we wished to compute the Born level $(4+\ell)$ -point component correlator (3.4) from standard Feynman diagrams, we would only draw *planar graphs*, i.e. graphs with leading order colour factors. Here we are not using the highly inefficient Feynman diagram technique. Instead, we wish to *predict* the answer based on its elementary properties like symmetries, singularities and now planarity. Our result should arise from the simplification of the sum of many planar Feynman graphs. Following [6], we make the natural assumption that the final expression for any component correlator (i.e. the result of these simplifications) is itself representable as a sum of *planar graphs*. These graphs are formed in the usual way with a propagator $1/x_{ij}^2$ represented by a line between points i and j (we can also represent numerators x_{ij}^2 via dashed lines, but these will not take part in the planarity criterion). We thus assert that every component correlator corresponds to a sum of planar graphs. Equivalently, every term accompanying a given y -structure corresponds to a sum of planar graphs. This turns out to be a strong requirement.

⁴This property is not valid starting from four loops.

To illustrate the power of this we will examine in detail the restriction from planarity on the functions $f_{\{111110\}}^2$ and $f_{\{111111\}}^2$.

Consider the formula for the correlator (3.5). The contribution of the coefficient function $f_{\{111110\}}^2$ to the correlator is

$$\mathcal{I} \times y_{12}^2 y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2 \times \frac{P_{\{111110\}}^2(x_1, \dots, x_6)}{x_{12}^4 x_{13}^4 x_{14}^4 x_{23}^4 x_{24}^4 x_{34}^2 x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2} . \quad (3.10)$$

The powers of 4 in the denominator come from the additional propagator factors $(d_{ij})^{b_{ij}}$. They can never be removed by a numerator term (which in the two-loop case only contain x_{ij}^2 but not x_{ij}^4 , see (3.8)). Let us concentrate on an individual term from the sum in (3.8). In order to obtain a planar contribution all three numerator factors must completely cancel the matching factors in the denominator. This is because each term in $f_{\{b_{ij}\}}^2$ has the topology of an octahedron which is a planar graph, see (3.9). But the addition of any new edges to the octahedron will produce a non-planar graph. In $f_{\{b_{ij}\}}^2$ itself all numerators cancel denominators, and so we conclude that any numerator not cancelling a denominator will automatically yield a non-planar graph.

Graphically, the multiplication by d_{ij} corresponds to attaching further edges $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}$ to the graph in (3.9) for some choice of permutation σ . This is only allowed if all these 5 edges are already existing edges (since as mentioned below (3.9), adding a new edge results in a non-planar graph). It thus becomes apparent that the only possible term in $P_{\{111110\}}^2$ which can yield a planar contribution is

$$P_{\{111110\}}^2 \propto x_{34}^2 x_{15}^2 x_{26}^2 + x_{34}^2 x_{15}^2 x_{26}^2 . \quad (3.11)$$

This is indeed the only non-zero term in our final result given in the penultimate line of table 1. It corresponds to a single orientation of the two-loop ladder integral.

However it is also now clear that a similar analysis in the case $f_{\{111111\}}^2$ — which will have an additional power of x_{34}^2 in the denominator compared to the previous case — will mean there is no numerator that can yield a planar contribution. We conclude that this contribution vanishes, $f_{\{111111\}}^2 = 0$. It is then clear that having all $b_{ij} \geq 1$ does not modify this non-planar topology, therefore planarity alone implies that

$$f_{\{b_{ij}\}}^2 = 0 \quad \text{if all } b_{ij} \geq 1 . \quad (3.12)$$

The analysis at three loops is very similar. However the presence of x_{ij}^4 in the numerator (see (3.8)) means that the effect is slightly delayed and takes place for $b_{ij} \geq 2$ rather than $b_{ij} \geq 1$. We find

$$f_{\{b_{ij}\}}^3 = 0 \quad \text{if all } b_{ij} \geq 2 . \quad (3.13)$$

Note that in the above analysis we have ignored the effect of the polynomial \mathcal{I} on planarity. Indeed terms in \mathcal{I} can cancel denominators and this softens the “non-planarity” of the result. However there are a number of terms in \mathcal{I} with different y factors (3.6) and all terms need to be planar. It turns out that apart from one-loop the presence of this

polynomial does not affect the conclusions. Similarly one should consider the sum over all building block functions rather than focussing on each single coefficient function alone. However, again, doing so does not seem to change the above conclusions. In other words so far we found that it was enough to assume that

$$\left(\prod_{1 \leq i < j \leq 4} (d_{ij})^{b_{ij}} \right) f_{\{b_{ij}\}}^\ell(x_1, \dots, x_{4+\ell}) \tag{3.14}$$

are all given by planar expressions. This is not a necessary consequence of the above planarity requirement. When (3.14) is inserted into the correlator expression (3.5), one multiplies by the polynomial \mathcal{I} — thus removing propagators — and sums over different structures which all mix together. Thus this leaves the possibility that the contributing expressions (3.14) could be non-planar whilst still giving planar component correlators. But in actual fact this never appears to happen in practice.

The next step is to find a way of fixing the coefficients in (3.8), i.e. in the *planar* ansatz for the integrand $G_{k_1 k_2 k_3 k_4}^\ell$ in (3.5). We are going to use two criteria based on the detailed understanding of the OPE. The first amounts to comparing the singular light-like limits of two correlators $\lim_{x_{12}^2 \rightarrow 0} G_{k_1 k_2 k_3 k_4}^\ell$ and $\lim_{x_{12}^2 \rightarrow 0} G_{k_1+1, k_2+1, k_3, k_4}^\ell$. The second criterion, proposed in [6], derives from the requirement that the logarithm of the correlator have simple log divergences in the short-distance limit.

3.3 Light-cone OPE relation

We claim the existence of a powerful relation between different integrands, i.e. Born-level correlators. It is based on the structure of the OPE of two half-BPS operators $\mathcal{O}^{(k_1)}(1)$ and $\mathcal{O}^{(k_2)}(2)$ in the light-cone limit $x_{12}^2 \rightarrow 0$. The key property is that the leading light-cone singularity in each SU(4) channel of the correlator $G_{k_1+1, k_2+1, k_3 k_4}^\ell$ is simply related, in the planar approximation $N_c \rightarrow \infty$, to that of $G_{k_1 k_2 k_3 k_4}^\ell$:

$$\lim_{\substack{x_{12}^2, y_{12} \rightarrow 0 \\ d_{12} \text{ fixed}}} \left[\frac{G_{k_1+1, k_2+1, k_3 k_4}^\ell}{C_{k_1+1, k_2+1, k_3 k_4}} - d_{12} \times \frac{G_{k_1 k_2 k_3 k_4}^\ell}{C_{k_1 k_2 k_3 k_4}} \right] = \alpha d_{12}. \tag{3.15}$$

The limit is taken as follows: $y_{12} = \epsilon \vec{n}$, $x_{12}^2 = \epsilon^2$ with some complex four-vector \vec{n} and $\epsilon \rightarrow 0$. The propagator factor d_{12} in the second term on the left-hand side equalises the conformal and SU(4) weights of the two terms. The claim is that in this limit the two correlators in (3.15) can only differ by terms proportional to d_{12} . The proof is given below in section 3.3.1.

Let us insert the general form of the correlators (3.5), (3.7) in (3.15):

$$\lim_{\substack{x_{12}^2, y_{12} \rightarrow 0 \\ d_{12} \text{ fixed}}} \frac{\mathcal{I}}{\prod_{1 \leq p < q \leq 4+\ell} x_{pq}^2} d_{12} \sum_{\{b_{ij}\}} \left(\prod_{1 \leq i < j \leq 4} (d_{ij})^{b_{ij}} \right) \left[P_{\{b_{ij}\}}^\ell|_{b_{12} \rightarrow b_{12}+1} - P_{\{b_{ij}\}}^\ell \right] = 0, \tag{3.16}$$

where the sextuples $\{b_{ij}\}$ correspond to the labels $k_1 k_2 k_3 k_4$ before the shift. The shift of the label b_{12} accounts for the shifts of k_1 and k_2 (recall that $k_i = \sum_{j \neq i} b_{ij} + 2$).

The reason for the vanishing right-hand side of (3.16) is that in our limit $d_{12}\mathcal{I}/x_{12}^2 \rightarrow (d_{12})^2(x_{13}^2x_{24}^2 - x_{14}^2x_{23}^2)^2y_{13}^2y_{14}^2y_{34}^2$ (see (3.6)), while we expect only $(d_{12})^1$ on the right-hand side of (3.15).⁵ Let us extract the terms with the same y -structure from (3.16). The family of sextuples $\{b^{(k)}\} = \{b_{12}, b_{13} - k, b_{14} + k, b_{23} + k, b_{24} - k, b_{34}\}$ parametrised by an integer k from the interval $[\alpha, \beta]$, $\alpha = -\min(b_{14}, b_{23})$, $\beta = \min(b_{13}, b_{24})$, corresponds to the y -structure $(d_{12}^2)^{b_{12}}(y_{13}^2)^{b_{13}+b_{23}}(y_{14}^2)^{b_{14}+b_{24}}(y_{34}^2)^{b_{34}}$. We deduce the following condition on the polynomials in the ansatz

$$\sum_{k=\alpha}^{\beta} (x_{13}^2x_{24}^2)^{k-\alpha} (x_{14}^2x_{23}^2)^{\alpha+\beta-k} \left[P_{\{b^{(k)}\}}^{\ell} |_{b_{12} \rightarrow b_{12}+1} - P_{\{b^{(k)}\}}^{\ell} \right]_{x_{12}^2=0} = 0. \tag{3.17}$$

In reality, eq. (3.17) in combination with planarity at two and three loops implies the stronger constraint (see appendix A)

$$\left[P_{\{b_{ij}\}}^{\ell} |_{b_{12} \rightarrow b_{12}+1} - P_{\{b_{ij}\}}^{\ell} \right]_{x_{12}^2=0} = 0. \tag{3.18}$$

The constraint (3.18) can be applied to any pair of the four outer points of the correlators. It can be repeated iteratively, shifting the weights at the chosen pair of points by any finite amount. This results in many relations between the coefficients of the numerators $P_{\{b_{ij}\}}^{\ell}$ of correlators of different BPS weights. We use the known correlators $\mathcal{G}_{2222}^2, \mathcal{G}_{2222}^3$ (which have been obtained in [5] by a similar method) as the starting point of the recursion. The planarity requirement of section 3.2, in combination with this light-cone OPE relation, implies that it is sufficient to consider only configurations with $b_{ij} = 0, 1$ (two loops) or $b_{ij} = 0, 1, 2$ (three loops), all cases with higher weights are reduced to these (see appendix A). Then relation (3.18) allows us to fix all the coefficients in our ansatz at two loops and all but one at three loops. To fix the latter we need yet another OPE criterion explained in section 3.4.

3.3.1 Origin of the light-cone relation

Relation (3.15) follows from the light-cone OPE of two half-BPS operators and any third operator $\mathcal{O}_{[a,b,a]}^{\Delta,S}$ of dimension Δ , spin S and in the SU(4) representation with Dynkin labels $[a, b, a]$.⁶ This takes the form

$$\mathcal{O}^{(k_1)}(x_1, y_1)\mathcal{O}^{(k_2)}(x_2, y_2) \sim C_{k_1 k_2 \mathcal{O}_{[a,b,a]}^{\Delta,S}} \frac{(y_{12}^2)^{(k_1+k_2-2a-b)/2}}{(x_{12}^2)^{(k_1+k_2-\Delta+S)/2}} \left([x_{12}]^S [y_{12}]^a \mathcal{O}_{[a,b,a]}^{\Delta,S}(x_2, y_2) + \dots \right). \tag{3.19}$$

Here the dots denote descendant terms, both space-time descendants (x -derivatives of operators) and SU(4) descendants (y -derivatives of operators). Importantly, these descendant terms only appear together with polynomials in x_{12} and y_{12} . We are slightly schematic in our display of indices. The square brackets simply indicate symmetrised tensor products,

⁵The terms with $b_{12} = 0$ in $G_{k_1+1, k_2+1, k_3 k_4}^{\ell}$ are not displayed in (3.16) because they contribute only to the right-hand side in (3.15).

⁶This OPE has been studied in [29, 30]. For a summary see appendix B.

and the indices will be contracted with those of the operator. The range of SU(4) representations on the right-hand side of (3.19) is determined by the tensor product (we assume that $k_1 \geq k_2$)

$$[0, k_1, 0] \otimes [0, k_2, 0] = \bigoplus_{r=0}^{k_2} \bigoplus_{a=0}^{k_2-r} [a, b, a] \quad \text{with } b = k_1 + k_2 - 2a - 2r. \quad (3.20)$$

These are SO(6) tensor representations of rank $2a + b = k_1 + k_2 - 2r$. The OPE $\mathcal{O}^{(k_1)}(1) \times \mathcal{O}^{(k_2)}(2)$ contributes to the correlator (3.2) only those representations which are in the overlap with the tensor product $[0, k_3, 0] \otimes [0, k_4, 0]$.

In $\mathcal{N} = 4$ super-Yang-Mills operators form multiplets. The superconformal primary operators with $r = 0, 1$ are protected (BPS or semishort), the unprotected operators have $r \geq 2$ (see [29, 30]). Further, the protected operators in the product $\mathcal{O}^{(k_1)}(1) \times \mathcal{O}^{(k_2)}(2)$ do not appear in (3.2) because their three-point functions with $\mathcal{O}^{(k_3)}(3) \mathcal{O}^{(k_4)}(4)$ are protected [31] and hence have no loop corrections. Therefore, for our purposes the first sum on the right-hand side of (3.20) starts at $r = 2$ corresponding to long multiplets in the OPE. Each long multiplet contains a number of superdescendant operators, only some of which appear in the OPE of two half BPS scalars. The superdescendants which occur are:

$$\begin{aligned} & \mathcal{O}_{[a,b,a]}^{\Delta,S}, \\ & \mathcal{B}_{[a+1,b,a+1]}^{\Delta+1,S+1}, \mathcal{B}_{[a-1,b+2,a-1]}^{\Delta+1,S+1}, \mathcal{B}_{[a-1,b+2,a-1]}^{\Delta+1,S-1}, \mathcal{A}'_{[a+1,b,a+1]}^{\Delta+1,S-1}, \\ & \mathcal{C}_{[a,b+2,a]}^{\Delta+2,S+2}, \mathcal{C}_{[a,b+2,a]}^{\Delta+2,S}, \mathcal{C}_{[a-2,b+4,a-2]}^{\Delta+2,S}, \mathcal{B}'_{[a+2,b,a+2]}^{\Delta+2,S}, \mathcal{B}_{[a,b+2,a]}^{\Delta+2,S}, \mathcal{B}'_{[a,b+2,a]}^{\Delta+2,S-2}, \\ & \mathcal{D}_{[a-1,b+4,a-1]}^{\Delta+3,S+1}, \mathcal{C}'_{[a+1,b+2,a+1]}^{\Delta+3,S+1}, \mathcal{C}'_{[a+1,b+2,a+1]}^{\Delta+3,S-1}, \mathcal{C}'_{[a-1,b+4,a-1]}^{\Delta+3,S-1}, \\ & \mathcal{D}'_{[a,b+4,a]}^{\Delta+4,S}. \end{aligned} \quad (3.21)$$

The superdescendants are obtained by acting with the supercharges on the primary operator (HWS) $\mathcal{O}_{[a,b,a]}^{\Delta,S}$. The derivation of this from analytic superspace together with more details is in appendix B. Here we will simply consider the first and the last terms. The highest dimension component which occurs is $\mathcal{D}'_{[a,b+4,a]}^{\Delta+4,S} \in Q^4 \bar{Q}^4 \mathcal{O}_{[a,b,a]}^{\Delta,S}$. The super OPE then takes the form

$$\begin{aligned} & \mathcal{O}^{(k_1)}(x_1, y_1) \mathcal{O}^{(k_2)}(x_2, y_2) \\ & \sim \sum_{\Delta, S, [a,b,a]} C_{k_1 k_2 \mathcal{O}_{[a,b,a]}^{\Delta,S}} (d_{12})^{(k_1+k_2-2a-b)/2-2} (x_{12}^2)^{(\Delta-S-2a-b)/2-2} \\ & \quad \times \left[y_{12}^4 [x_{12}]^S [y_{12}]^a \mathcal{O}_{[a,b,a]}^{\Delta,S} + \dots + x_{12}^4 [x_{12}]^S [y_{12}]^a \mathcal{D}'_{[a,b+4,a]}^{\Delta+4,S} + \dots \right], \end{aligned} \quad (3.22)$$

where the dots in the middle denote terms relating to the other superdescendants (listed in (3.21)) and the dots at the end denote conformal and SU(4) descendants. The main point is that each component of the supermultiplet appears with the same OPE coefficient $C_{k_1 k_2 \mathcal{O}_{[a,b,a]}^{\Delta,S}}$.

We now wish to consider the OPE in the limit in which $x_{12}^2 \rightarrow 0$ and $y_{12} \rightarrow 0$ but with the ratio $d_{12} = y_{12}^2/x_{12}^2$ fixed. The leading contribution to the OPE in this limit comes from operators with the minimum value of $\Delta - S - 2a - b$. Well-known superconformal unitarity bounds [32] state that for long representations the superconformal primary satisfies $\Delta - S - 2a - b \geq 2$. Let us thus set $\Delta - S = 2a + b + 2$ in (3.22):

$$\begin{aligned} \mathcal{O}^{(k_1)}(x_1, y_1)\mathcal{O}^{(k_2)}(x_2, y_2) &\sim \sum_{\Delta-S=2a+b+2} C_{k_1 k_2 \mathcal{O}_{[a,b,a]}^{\Delta,S}} (d_{12})^{(k_1+k_2-2a-b)/2-1} [x_{12}]^S \left(y_{12}^2 [y_{12}]^a \mathcal{O}_{[a,b,a]}^{\Delta,S} \right. \\ &\quad \left. + [y_{12}]^{a+1} [x_{12}] \mathcal{B}_{[a+1,b,a+1]}^{\Delta+1,S+1} + y_{12}^2 [y_{12}]^{a-1} [x_{12}] \mathcal{B}_{[a-1,b+2,a-1]}^{\Delta+1,S+1} + [y_{12}]^a [x_{12}]^2 \mathcal{C}_{[a,b+2,a]}^{\Delta+2,S+2} + \dots \right), \end{aligned} \tag{3.23}$$

where we have displayed only the terms of leading twist. In our double limit only one superdescendant survives,

$$\begin{aligned} \lim_{\substack{x_{12}^2, y_{12} \rightarrow 0 \\ d_{12} \text{ fixed}}} \mathcal{O}^{(k_1)}(x_1, y_1)\mathcal{O}^{(k_2)}(x_2, y_2) \\ \sim \sum_{\Delta-S=b+2} C_{k_1 k_2 \mathcal{O}_{[0,b,0]}^{\Delta,S}} (d_{12})^{(k_1+k_2-b)/2-1} [x_{12}]^{S+2} \mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2} + \dots \end{aligned} \tag{3.24}$$

An important property of this type of operators is that they are made entirely from scalars. Indeed, they have twist $(\Delta + 2) - (S + 2) = b + 2$ and SU(4) labels $[0, b + 2, 0]$ implying that the length of the operator (i.e. number of constituent scalars) equals the rank of the SO(6) tensor representation. Trying to replace some of the scalars by fermion bilinears or by gluons either increases the twist or modifies the representation.

The question we want to investigate now is what happens to the OPE structure constant $C_{k_1 k_2 \mathcal{O}_{[0,b,0]}^{\Delta,S}}$ when we increase the BPS weights $k_1 \rightarrow k_1 + 1$, $k_2 \rightarrow k_2 + 1$. We wish to show that⁷

$$C_{k_1+1, k_2+1, \mathcal{O}_{[0,b,0]}^{\Delta,S}} = C_{k_1 k_2 \mathcal{O}_{[0,b,0]}^{\Delta,S}} \times \frac{N_c (k_1 + 1)(k_2 + 1)}{2 k_1 k_2}. \tag{3.25}$$

In other words, the ratio of the two OPE coefficients is *independent of the quantum numbers of the operator* $\mathcal{O}_{[0,b,0]}^{\Delta,S}$. We call this property of the structure constants in the relevant sector of our OPE *universality*.

The structure constant $C_{k_1 k_2 \mathcal{O}_{[0,b,0]}^{\Delta,S}}$ is the same for all the members of the supermultiplet. We find it advantageous to determine it from the three-point function $\langle \mathcal{O}^{(k_1)}(1)\mathcal{O}^{(k_2)}(2)\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2}(3) \rangle$ divided by the two-point function $\langle \mathcal{C}\mathcal{C} \rangle$ (the latter drops out of the ratio (3.25)). The key point in our argument is that the operator $\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2}$ is made from scalars only. Our integrand (3.2) is a Born-level correlator. For the three-point function $\langle \mathcal{O}(1)\mathcal{O}(2)\mathcal{C}(3) \rangle$, where all the operators are made from scalars, the Born approximation coincides with the free theory result. It is obtained by Wick contractions

⁷This result is consistent with the values of the structure constants for the operators in the so-called SL(2) sector considered in [15]. Such operators correspond to the descendants $\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2}$ in (3.21) and they are made only from scalars.

with free scalar propagators. A certain number k of propagators connect points 1 and 2. The remaining $k_1 - k$ scalars at point 1 are connected to point 3 and similarly for point 2. This means that the operator $\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2}$ is made from $k_1 + k_2 - 2k = b + 2$ scalars and S space-time derivatives.⁸ The space-time dependence of the three-point function $\langle \mathcal{O} \mathcal{O} \mathcal{C} \rangle$ is fixed by conformal symmetry, and the structure constants are determined by the colour and combinatorial factors. Let us examine the colour tensor of each operator. For the half-BPS (single-trace) operator $\mathcal{O}^{(k_i)}$ it is the trace $\text{tr}(t_{(a_1 \dots t_{a_{k_i}})})$ of k_i generators of $\text{SU}(N_c)$ (here (\dots) denotes weighted symmetrisation). For $\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2}$ the colour tensor $\gamma_{a_1 \dots a_{k_1+k_2-2k}}^S$ depends on the details of the operator in question.⁹ The combination of the three colour tensors, including the combinatorial factor, has the form

$$\frac{k_1!k_2!}{k!} \text{tr}[t_{(a_1 \dots t_{a_k} t_{b_1 \dots t_{b_{k_1-k}}})}] \text{tr}[t_{(a_1 \dots t_{a_k} t_{c_1 \dots t_{c_{k_2-k}}})}] \gamma_{b_1 \dots b_{k_1-k} c_1 \dots c_{k_2-k}}^S. \quad (3.26)$$

Here the k Wick contractions between points 1 and 2 are realised as contractions of the first k colour tensor indices. In the large N_c limit this becomes¹⁰

$$\left(\frac{N_c^{k-1}}{2^k} k_1 k_2 \right) (k_1 - k)! (k_2 - k)! \text{tr}[t_{(b_1 \dots t_{b_{k_1-k}})} t_{(c_1 \dots t_{c_{k_2-k}})}] \gamma_{b_1 \dots c_{k_2-k}}^S \quad (k \geq 1). \quad (3.27)$$

Now, let us see what happens when we shift $k_1 \rightarrow k_1 + 1$, $k_2 \rightarrow k_2 + 1$. In order to maintain the twist or equivalently the length $k_1 + k_2 - 2k$ fixed, we need to also increase $k \rightarrow k + 1$. Only the first factor in the parentheses in (3.27) changes. Then the ratio of the structure constants is as given by eq. (3.25).

Next, let us insert the OPE limit (3.24) in the correlator (3.2)

$$\lim_{\substack{x_{12}^2, y_{12} \rightarrow 0 \\ d_{12} \text{ fixed}}} \mathcal{G}_{k_1 k_2 k_3 k_4}^\ell = \sum_{\mathcal{O}_{[0,b,0]}^{\Delta,S}; \Delta=S+b+2} C_{k_1 k_2 \mathcal{O}_{[0,b,0]}^{\Delta,S}} d_{12}^{r-1} [x_{12}]^{S+2} \quad (3.28)$$

$$\times \left\langle \left[\mathcal{C}_{[0,b+2,0]}^{\Delta+2,S+2} + \dots \right] \mathcal{O}^{(k_3)}(3) \mathcal{O}^{(k_4)}(4) \mathcal{L}(5) \dots \mathcal{L}(4+\ell) \right\rangle_{\text{Born}},$$

and in its counterpart $\mathcal{G}_{k_1+1, k_2+1, k_3 k_4}^\ell$. The shifted version of the tensor product (3.20) is

$$[0, k_1 + 1, 0] \otimes [0, k_2 + 1, 0] = \bigoplus_{r=0}^{k_2+1} \bigoplus_{a=0}^{k_2+1-r} [a, k_1 + k_2 + 2 - 2a - 2r, a]. \quad (3.29)$$

As before, the long multiplets have $r \geq 2$. Comparing this decomposition with (3.20) for $a = 0$ and $r \geq 2$, we see that the shifted version contains a representation $[0, k_1 + k_2 - 2, 0]$

⁸The number k cannot be zero because otherwise the twist of the superconformal primary $\mathcal{O}_{[0,b,0]}^{\Delta,S}$ will be $\Delta - S = k_1 + k_2$. We have already set $\Delta - S = b + 2 = k_1 + k_2 + 2(1 - r)$, so this would imply $r = 1$ in (3.20). As pointed out earlier, the multiplets with $r = 1$ are protected and do not contribute to the integrand (3.2).

⁹In the case of degeneracy, i.e. existence of several operators \mathcal{C} with the same quantum numbers made from scalars (for an example see [33]), we assume that they have been diagonalised. Then the colour tensor γ^S corresponds to a specific eigenstate.

¹⁰This relation does not hold for $k = 0$ but as we have explained, this case is of no relevance for us.

with $r = 2$. This is an $SO(6)$ tensor of rank $k_1 + k_2 - 2$. The maximal rank for long operators ($r = 2$) in (3.20) is $k_1 + k_2 - 4$. From (3.28) it follows that this extra channel comes with a prefactor d_{12} which account for the right-hand side of (3.15). Note that such a contribution will only be visible if it also appears in the tensor product $[0, k_3, 0] \otimes [0, k_4, 0]$.

Finally, taking into account the universality of the structure constants (3.25) and the normalisation factors (2.12) in (3.15), we see that the contributions of the long supermultiplets common for both correlators coincide in our limit. The only source of difference are the terms with $r = 2$ in the shifted version of (3.28), which are responsible for the right-hand side of (3.15). We have thus proven this important relation.

3.3.2 A possible stronger relation

The examination of our two- and three-loop results in section 2.1 shows that they are compatible with a stronger light-cone relation between two planar correlators:

$$\frac{G_{k_1+1, k_2+1, k_3 k_4}^\ell}{C_{k_1+1, k_2+1, k_3 k_4}} - d_{12} \times \frac{G_{k_1 k_2 k_3 k_4}^\ell}{C_{k_1 k_2 k_3 k_4}} = O(1/x_{12}^2). \quad (3.30)$$

The claim is that in this limit the two correlators in (3.15) can only differ by terms of order $1/x_{12}^2$. Bearing in mind that the correlator $G_{k_1 k_2 k_3 k_4}^\ell$ can have poles in x_{12}^2 up to $1/(x_{12}^2)^{(k_1+k_2-2)/2}$, this involves a remarkable cancellation of much of the correlation functions. Notice that the new relation does not involve any limit of the auxiliary y -variables, i.e. it applies to all the $SU(4)$ channels in the correlators. From (3.30) we deduce the stronger condition on the numerator polynomials in (3.7)

$$P_{\{b_{ij}\}}^\ell|_{b_{12} \rightarrow b_{12}+1} - P_{\{b_{ij}\}}^\ell = O((x_{12}^2)^{b_{12}+1}). \quad (3.31)$$

Here $b_{12} \leq \ell - 2$ in order to match the conformal weights of the left- and right-hand sides. If $b_{12} > \ell - 2$ the right-hand side must vanish.

How could we possibly prove such a relation? The starting point would be the conformal light-cone OPE (3.19) (no need to evoke its supersymmetric version (3.22) anymore). The relation would hold if the universality of the structure constants (3.25) applied to all possible operators, not just the specific $SU(4)$ channels $[0, b, 0]$. To prove this we would be tempted to argue that the structure constants are determined by the free three-point function $\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \mathcal{O}_{[a, b, a]}^{\Delta, S} \rangle_{\text{free}}$. Then the planar colour factor (3.27) would explain the universality, with the exception of the case $k = 0$, i.e. when the operator $\mathcal{O}_{[a, b, a]}^{\Delta, S}$ has maximal length. This would explain the right-hand side in (3.30).

The problem with this argument is that for generic operators $\mathcal{O}_{[a, b, a]}^{\Delta, S}$, whose length does not equal the rank $2a + b$ of the $SO(6)$ representation, we cannot rule out the presence of fermions and gluons in their composition. For such ingredients the three-point function $\langle \mathcal{O} \mathcal{O} \mathcal{O}_{[a, b, a]}^{\Delta, S} \rangle_{\text{Born}}$ is not necessarily free anymore. A simple example is an operator of the type $\text{tr}(F_{\mu\nu}^2)$. It can only talk to the half-BPS scalar operators via interaction vertices, so $\langle \mathcal{O} \mathcal{O} \text{tr}(F^2) \rangle_{\text{Born}} \sim g^2$ and not g^0 as for an operator made of scalars. The colour factors of such three-point functions become more difficult to control. This does not mean that the universality of the structure constants stops working, but at present we cannot make a definitive claim. This issue deserves further study.

We would like to point out that the three constraints — the weaker (3.17), the intermediate (3.18) and the stronger (3.31), are in fact equivalent up to three loops if we assume planarity (see appendix A for the explanation). This is however not true starting from four loops.

3.3.3 Singularities of the integrand

The OPE considerations above allow us to explain the structure of the space-time singularities of our ansatz for the integrand (3.5), (3.7). Consider first the singularities with respect to the four external points, i.e. for $x_{ij}^2 \rightarrow 0$ with $1 \leq i, j \leq 4$. They are determined by the OPE of two half-BPS operators saturating the unitarity bound, eq. (3.23). We see that all the poles in x_{12}^2 appear as propagator factors, accompanied by an extra power of y_{12}^2 if the contribution comes from operators of SO(6) rank $2a + b$. Comparing with (3.5), (3.7) and recalling the definition (2.13), we see exactly the same structure.

Further, the singularities between an external and a Lagrangian insertion points are determined by the OPE

$$\mathcal{O}^{(k)} \times \mathcal{L} \sim \frac{C_{\mathcal{O}\mathcal{O}\mathcal{L}}}{x^4} \mathcal{O}^{(k)} + O\left(\frac{1}{x^2}\right). \tag{3.32}$$

The leading singularity $1/x^4$ does not really appear there because the two-point function of BPS operators is protected and hence $C_{\mathcal{O}\mathcal{O}\mathcal{L}} = 0$. So, this OPE contributes at most a singularity $1/x^2$, as in our ansatz (3.5), (3.7).

Finally, the OPE $\mathcal{L} \times \mathcal{L}$ of two *chiral* Lagrangians has a leading singularity in the form of a contact term, $\delta^4(x)$. Our correlators are always considered for non-coincident points, so we can only see the subleading singularity $1/x^2$ in this OPE.

3.4 Double short-distance OPE

As explained in section 3.3, the powerful recursion relation (3.15) allows us to fix all but one coefficient in our three-loop ansatz (and all at two loops). To fix the single remaining coefficient it is sufficient to consider the simplest correlator (cf. (2.11))

$$\mathcal{G}_{3322}^{\text{loop}} = C_{3322} R d_{12} \sum_{\ell \geq 1} a^\ell F_\ell. \tag{3.33}$$

A new independent restriction on this correlator follows from the Euclidean OPE (coincident points). Let us perform a double OPE in two inequivalent ways,

$$\begin{aligned} \lim_{1 \rightarrow 2, 3 \rightarrow 4} \mathcal{G}_{3322} &= \langle (\mathcal{O}^{(3)}(1) \times \mathcal{O}^{(3)}(2)) (\mathcal{O}^{(2)}(3) \times \mathcal{O}^{(2)}(4)) \rangle \\ \lim_{1 \rightarrow 3, 2 \rightarrow 4} \mathcal{G}_{3322} &= \langle (\mathcal{O}^{(3)}(1) \times \mathcal{O}^{(2)}(3)) (\mathcal{O}^{(3)}(2) \times \mathcal{O}^{(2)}(4)) \rangle. \end{aligned} \tag{3.34}$$

In the first case $x_1 \rightarrow x_2, x_3 \rightarrow x_4$ or $u \rightarrow 0, v \rightarrow 1$ in terms of the conformal cross-ratios (2.8). The criterion derived in [5] is that the function

$$\log \left(1 + 6x_{13}^4 \sum_{\ell \geq 1} a^\ell F_\ell \right) \xrightarrow[u \rightarrow 1]{u \rightarrow 0} \frac{\gamma_K(a)}{2} \log u + O(u^0) \tag{3.35}$$

diverges as a simple logarithm at all orders in a . Here γ_K is the anomalous dimension of the Konishi operator, the leading non-protected operator in the overlap of the OPEs $\mathcal{O}^{(3)} \times \mathcal{O}^{(3)}$ and $\mathcal{O}^{(2)} \times \mathcal{O}^{(2)}$. The numerical coefficient 6 (planar limit) on the left-hand side has been worked out in [5] for the case \mathcal{G}_{2222} by examination of the Born-level OPE and comparison of the free two- and three-point functions. Alternatively, knowing the correlator up to two loops allows us to fix this coefficient by making sure that the log criterion works at two loops.

In the second case $x_1 \rightarrow x_4, x_2 \rightarrow x_3$ the criterion imposes simple logarithmic behaviour on the function

$$\log \left(1 + 4x_{12}^4 \sum_{\ell \geq 1} a^\ell F_\ell \right) \xrightarrow[u \rightarrow 1]{v \rightarrow 0} \frac{\gamma_6(a)}{2} \log v + O(v^0). \tag{3.36}$$

Here γ_6 is the anomalous dimension of the scalar operator of dimension 3 and in the vector representation of $SO(6)$, the leading non-protected operator in the OPE $\mathcal{O}^{(2)} \times \mathcal{O}^{(3)}$. On the left-hand side we used our knowledge of the two-loop correlator from the recursion relation (3.15) to fix the coefficient 4 (notice that it differs from the 6 in (3.35)).

Conditions (3.35) and (3.36) are to be implemented as follows (see [5] for the detailed explanation). We expand the logarithms up to a^3 and obtain restrictions on the linear combinations of two-loop (at level a^2) and three-loop (at level a^3) integrals. The integrals beyond one loop in general diverge stronger than simple logarithms. In order to weaken the divergences, the numerator of the integrand must vanish in the singular regime where an integration point approaches an outer point.¹¹ For example, for the first OPE we choose $x_5 \rightarrow x_1$ or $x_5 \rightarrow x_3$, for the second OPE we choose $x_5 \rightarrow x_1$ or $x_5 \rightarrow x_2$.

We remark that the conditions following from the OPE (3.35) with dominant twist two are not independent from what the light-cone relation (3.15) has already given us. Only the second OPE (3.36) is really useful for our purposes. It fixes the only remaining coefficient and thus fully determines all the correlators up to three loops.

3.5 Integral identities

Once we have fully determined the integrand of a given correlator, we need to turn it into a set of conformal integrals by substituting the polynomials P^ℓ in (3.5) and then the integrand in (3.1). We find the integrals listed in (2.17), appearing in various orientations. We can profit from a number of identities that these integrals satisfy [5, 23],

$$\begin{aligned} h_{12;34} &= h_{34;12} , & h_{12;34} &= h_{21;34} , & h_{12;34} &= h_{12;43} \\ L_{12;34} &= L_{34;12} , & L_{12;34} &= L_{21;34} , & L_{12;34} &= L_{12;43} \\ E_{12;34} &= E_{34;12} , & E_{12;34} &= E_{21;34} , & E_{12;34} &= E_{12;43} \\ H_{12;34} &= H_{34;12} , & H_{12;34} &= H_{21;43} , & H_{21;34} &= 1/v H_{12;34} \\ H_{31;24} &= u/v H_{13;24} , & H_{41;23} &= u H_{14;23} , & & \end{aligned} \tag{3.37}$$

to bring the answer to the form (2.16). The final results are listed in tables 1, 2.

¹¹A similar criterion for the integrand of the four-gluon amplitude was first proposed in [34].

Note that at the level of the *integrand* we distinguish the topology of the three-loop ladder integral $L_{12;34}$ defined in (2.17) from that of the so-called ‘tennis court’ integral,

$$T_{12;34} = \frac{x_{34}^2}{(-4\pi^2)^3} \int \frac{d^4x_5 d^4x_6 d^4x_7 x_{17}^2}{(x_{15}^2 x_{35}^2)(x_{16}^2 x_{46}^2)(x_{37}^2 x_{27}^2 x_{47}^2) x_{56}^2 x_{57}^2 x_{67}^2}. \quad (3.38)$$

The latter does not appear in our result (2.16) for the *integrated* four-point correlation function because of the identity $T_{12;34} = L_{12;34}$ proven in [23].

We remark that the same set of integrals was used in [5] to construct the three-loop correction to the correlator \mathcal{G}_{2222} . The main difference is that in the latter case one has an enhanced permutation symmetry $S_{4+\ell}$ because all the four operators $\mathcal{O}^{(2)}$ belong to the stress-tensor multiplet (3.3). Consequently, the freedom is reduced to a single constant per loop order, up to three loops.

4 OPE analysis of the integrated four-point correlators

Now let us turn to an OPE analysis of the four-point correlation functions. We would like to discuss the constraints that the light-cone OPE places on the functions $F_{\{b_{ij}\}}$ appearing in (2.11). Similar analysis has been performed in [28, 35] and we follow the general discussion therein.¹² We will see that simple consistency conditions in fact require many of the coefficients in the tables 1 and 2 presented before to take precisely the correct values. By performing the OPE analysis we will also be able to present detailed checks of the (derived) tree-level and (conjectured) one-loop formulae for three-point functions of two half-BPS and one long operator presented in [15]. Moreover we will be able to check the recently presented three-loop formulae [19, 20] for the same three-point functions in the case where the long operator has twist two.

In order to have a uniform discussion of the light-cone OPE for the four-point correlators discussed in this paper, we choose to consider the expansion around the limit $x_{12}^2 x_{34}^2 \rightarrow 0$, or equivalently $u \rightarrow 0$ with v fixed. Then, instead of discussing different expansions of a given correlator, we consider our preferred expansion of the various correlators obtained by permuting the ordering of the external operators.

Without loss of generality we can always pick k_4 to be the largest of the weights and order the weights so that $k_1 \leq k_2$. We define the quantity E via

$$E = \frac{1}{2}(k_1 + k_2 + k_3 - k_4). \quad (4.1)$$

We then find it convenient to rewrite the correlation functions as follows (we use the notation $k_{ij} = k_i - k_j$),

$$\mathcal{G}_{k_1 k_2 k_3 k_4} = \mathcal{G}_{k_1 k_2 k_3 k_4}^0 + C_{k_1 k_2 k_3 k_4} d_{14}^a d_{24}^b d_{12}^c d_{34}^{k_3} u^{\frac{1}{2}k_{34}} \mathcal{S}(u, v; \sigma, \tau) \mathcal{H}(u, v; \sigma, \tau). \quad (4.2)$$

In equation (4.2) we have introduced the variables

$$\sigma = \frac{y_{13}^2 y_{24}^2}{y_{12}^2 y_{34}^2}, \quad \tau = \frac{y_{14}^2 y_{23}^2}{y_{12}^2 y_{34}^2} \quad (4.3)$$

¹²For more general and recent approaches see [36, 37].

while the powers on the propagator factors are given by

$$a = k_1 - E, \quad b = k_2 - E, \quad c = E. \quad (4.4)$$

Finally the function \mathcal{S} in (4.2) is a simple polynomial obtained from $R(1, 2, 3, 4)$ and is given by

$$\begin{aligned} \mathcal{S}(u, v; \sigma, \tau) &= R(1, 2, 3, 4) \frac{x_{12}^2 x_{34}^2 x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2 y_{12}^4 y_{34}^4} \\ &= v + \sigma^2 uv + \tau^2 u + \sigma v(v - 1 - u) + \tau(1 - u - v) + \sigma\tau(u - 1 - v). \end{aligned} \quad (4.5)$$

We recall [27, 28] that the presence of the factor \mathcal{S} in (4.2) or $R(1, 2, 3, 4)$ in (2.11) is a reflection of the fact that the quantum loop corrections to the full correlator can only come from intermediate operators in the OPE which belong to long supermultiplets. The free correlator $G_{k_1 k_2 k_3 k_4}^0$ on the other hand receives contributions both from protected operators and long operators,

$$\mathcal{G}_{k_1 k_2 k_3 k_4}^0 = \mathcal{G}_{k_1 k_2 k_3 k_4}^{\text{protected}} + C_{k_1 k_2 k_3 k_4} d_{14}^a d_{24}^b d_{12}^c d_{34}^{k_3} u^{\frac{1}{2}k_{34}} \mathcal{S}(u, v; \sigma, \tau) \mathcal{H}^{(0)}(u, v; \sigma, \tau). \quad (4.6)$$

In order to understand the OPE expansion of the correlator $G_{k_1 k_2 k_3 k_4}$, we first expand the function \mathcal{H} into eigenmodes of the $\mathfrak{su}(4)$ Casimir acting at points 1 and 2 as follows

$$\mathcal{H}(u, v; \sigma, \tau) = \sum_{L \leq m \leq n \leq U} A_{nm}(u, v) Y_{nm}^{(a,b)}(\sigma, \tau). \quad (4.7)$$

The channel with labels n, m corresponds to an exchanged supermultiplet with superconformal primary in the representation with $\mathfrak{su}(4)$ Dynkin labels $[n - m, a + b + 2m, n - m]$.

The bounds on the summation region are given by

$$L = \max(0, E - k_1), \quad U = \min(k_3, E) - 2. \quad (4.8)$$

The functions $Y_{nm}^{(a,b)}$ are given in terms of Jacobi polynomials via

$$Y_{nm}^{(a,b)}(\sigma, \tau) = \frac{P_{n+1}^{(a,b)}(y) P_m^{(a,b)}(\bar{y}) - P_m^{(a,b)}(y) P_{n+1}^{(a,b)}(\bar{y})}{y - \bar{y}}, \quad (4.9)$$

where the variables y and \bar{y} are defined via

$$\sigma = \frac{1}{4}(1 + y)(1 + \bar{y}), \quad \tau = \frac{1}{4}(1 - y)(1 - \bar{y}). \quad (4.10)$$

We recall that the Jacobi polynomials are given by a finite hypergeometric series which can be usefully expressed via Rodrigues's formula as follows,

$$P_n^{(\alpha, \beta)}(z) = \frac{(-1)^n}{2^n n!} (1 - z)^{-\alpha} (1 + z)^{-\beta} \frac{d^n}{dz^n} \left[(1 - z)^\alpha (1 + z)^\beta (1 - z^2)^n \right]. \quad (4.11)$$

The function $\mathcal{H}^{(0)}$ appearing in (4.6) has an expansion directly analogous to (4.7) with coefficients $A_{nm}^{(0)}$ whose precise form will not be important in what follows.

Now the functions A_{nm} appearing in the expansion (4.7) can themselves be expanded in terms of conformal blocks describing the quantum loop corrections to the contributions of conformal primary operators of dimension Δ and spin l .¹³ Specifically we have

$$A_{nm}^{(0)}(u, v) + A_{nm}(u, v) = \sum_{\Delta, l} a_{nm}^{\Delta, l} G_{\Delta}^{(l)}(u, v; k_{21}, k_{43}). \quad (4.12)$$

We are interested in the form of the perturbative quantum corrections, so we will need to expand the above sum over conformal blocks order by order in the Yang-Mills coupling. The order g^0 term will correspond to the contribution of $A_{nm}^{(0)}$, while all higher orders come from the A_{nm} which are themselves directly obtained from the explicit results of the previous sections.

The conformal blocks are given by [38]

$$G_{\Delta}^{(l)}(u, v; \delta, \tilde{\delta}) = \frac{u^{\frac{1}{2}(\Delta-l)}}{x - \bar{x}} \left(x \left(-\frac{1}{2}x \right)^l f_{\Delta+l}^{\delta, \tilde{\delta}}(x) f_{\Delta-l-2}^{\delta, \tilde{\delta}}(\bar{x}) - \bar{x} \left(-\frac{1}{2}\bar{x} \right)^l f_{\Delta+l}^{\delta, \tilde{\delta}}(\bar{x}) f_{\Delta-l-2}^{\delta, \tilde{\delta}}(x) \right), \quad (4.13)$$

with

$$f_{\rho}^{\delta, \tilde{\delta}}(z) = {}_2F_1 \left(\frac{1}{2}(\rho + \delta), \frac{1}{2}(\rho - \tilde{\delta}); \rho; z \right). \quad (4.14)$$

In (4.13) we employ the variables

$$u = x\bar{x}, \quad v = (1-x)(1-\bar{x}). \quad (4.15)$$

For our purposes here it will be sufficient to consider only the leading power in the expansion for small u for each function $A_{nm}(u, v)$, keeping any powers of $\log u$. This corresponds to keeping the leading twist¹⁴ contribution to each distinct $\text{su}(4)$ channel in the expansion of the correlation functions. The limit may be achieved by taking $\bar{x} \rightarrow 0$ with x fixed. In this case we may drop any power suppressed terms from the conformal blocks,

$$G_{\Delta}^{(l)}(u, v; \delta, \tilde{\delta}) = u^{\frac{1}{2}(\Delta-l)} \left(-\frac{1}{2}x \right)^l f_{\Delta+l}^{\delta, \tilde{\delta}}(x) + O(\bar{x}). \quad (4.16)$$

Our task is now to match the explicit expressions for the leading powers in the \bar{x} expansions of A_{nm} , obtained from the limits of the correlation functions G_{k_1, k_2, k_3, k_4} , with the perturbative expansion of the sum over conformal blocks given in (4.12). We write the scaling dimension Δ of a superconformal primary as $\Delta = \Delta_0 + \gamma(\lambda)$, where γ is the anomalous dimension. The free scaling dimension is not a good label for different operators since in the free theory many operators of a given spin may have the same Δ_0 . We therefore label operators of a given spin l which have degenerate free scaling dimensions with an extra index I . We find

$$A_{nm}^{(0)}(u, v) + A_{nm}(u, v) = u^p \sum_{I, l} a_{nm, I, l} u^{\eta_{I, l}} \left(-\frac{1}{2}x \right)^l f_{2p+2l+2\eta_{I, l}}^{k_{21}, k_{43}}(x) + O(u^{p+1}) \quad (4.17)$$

¹³In this section we denote the spin by l and the 't Hooft coupling by λ .

¹⁴The twist T of an operator is the dimension minus the spin $T = \Delta - l$.

where $\eta_{I,l} = \frac{1}{2}\gamma_{I,l}$ and

$$p = \frac{1}{2}(\max(k_{21}, k_{43}) + 2 + 2(n - \max(0, E - k_1))) \quad (4.18)$$

is half the free twist of the leading twist operators in a given $\text{su}(4)$ channel with labels m, n . The coupling dependence in (4.17) is in the quantities $A_{mn}(u, v)$ on the l.h.s. and $a_{nm,I,l}$ and $\gamma_{I,l}$ on the r.h.s. They admit perturbative expansions of the form

$$A_{nm}(u, v) = \sum_{r=1}^{\infty} \lambda^r A_{nm}^{(r)}(u, v), \quad \eta_{I,l}(\lambda) = \sum_{r=1}^{\infty} \lambda^r \eta_{I,l}^{(r)}, \quad a_{nm,I,l}(\lambda) = \sum_{r=0}^{\infty} \lambda^r a_{nm,I,l}^{(r)}. \quad (4.19)$$

The functions $A_{nm}^{(r)}(u, v)$ exhibit logarithmic corrections in their expansions for small u of the form

$$A_{nm}^{(r)}(u, v) = u^p \sum_{s=0}^r (\log u)^s g_{nm,s}^{(r)}(x) + O(u^{p+1}). \quad (4.20)$$

Now we may expand both sides of eq. (4.17) in the coupling. This leads us to expressions of the following general form for the functions $g_{nm,s}^{(r)}$,

$$g_{nm,s}^{(r)}(x) = \sum_{I,l} \left(-\frac{1}{2}x\right)^l [\mathcal{O}_{nm,I,l,s}^{(r)} f_{2t+2l}^{k_{21}, k_{43}}(x)]_{t=p}, \quad (4.21)$$

where $\mathcal{O}_{nm,I,l,s}^{(r)}$ is in general a differential operator (in t) acting on the function f . To simplify the notation a little we suppress the indices n and m (in other words we write $\mathcal{O}_{nm,I,l,s}^{(r)}(x) \equiv \mathcal{O}_{I,l,s}^{(r)}(x)$ and $a_{nm,I,l}^{(r)} \equiv a_{I,l}^{(r)}$). At leading order we simply have a multiplicative operator,

$$\mathcal{O}_{I,l,0}^{(0)} = a_{I,l}^{(0)}. \quad (4.22)$$

At order λ we have

$$\begin{aligned} \mathcal{O}_{I,l,1}^{(1)} &= a_{I,l}^{(0)} \eta_{I,l}^{(1)}, \\ \mathcal{O}_{I,l,0}^{(1)} &= a_{I,l}^{(1)} + a_{I,l}^{(0)} \eta_{I,l}^{(1)} \partial_t. \end{aligned} \quad (4.23)$$

At order λ^2 we find

$$\begin{aligned} \mathcal{O}_{I,l,2}^{(2)} &= a_{I,l}^{(0)} \frac{1}{2} (\eta_{I,l}^{(1)})^2, \\ \mathcal{O}_{I,l,1}^{(2)} &= a_{I,l}^{(1)} \eta_{I,l}^{(1)} + a_{I,l}^{(0)} \eta_{I,l}^{(2)} + a_{I,l}^{(0)} (\eta_{I,l}^{(1)})^2 \partial_t, \\ \mathcal{O}_{I,l,0}^{(2)} &= a_{I,l}^{(2)} + a_{I,l}^{(1)} \eta_{I,l}^{(1)} \partial_t + a_{I,l}^{(0)} \left[\eta_{I,l}^{(2)} \partial_t + \frac{1}{2} (\eta_{I,l}^{(1)})^2 \partial_t^2 \right]. \end{aligned} \quad (4.24)$$

Finally, at order λ^3 we have

$$\begin{aligned} \mathcal{O}_{I,l,3}^{(3)} &= a_{I,l}^{(0)} \frac{1}{6} (\eta_{I,l}^{(1)})^3, \\ \mathcal{O}_{I,l,2}^{(3)} &= a_{I,l}^{(1)} \frac{1}{2} (\eta_{I,l}^{(1)})^2 + a_{I,l}^{(0)} \left[\eta_{I,l}^{(2)} \eta_{I,l}^{(1)} + \frac{1}{2} (\eta_{I,l}^{(1)})^3 \partial_t \right], \\ \mathcal{O}_{I,l,1}^{(3)} &= a_{I,l}^{(2)} \eta_{I,l}^{(1)} + a_{I,l}^{(1)} \left[\eta_{I,l}^{(2)} + (\eta_{I,l}^{(1)})^2 \partial_t \right] + a_{I,l}^{(0)} \left[\eta_{I,l}^{(3)} + 2\eta_{I,l}^{(2)} \eta_{I,l}^{(1)} \partial_t + \frac{1}{2} (\eta_{I,l}^{(1)})^2 \partial_t^2 \right], \\ \mathcal{O}_{I,l,0}^{(3)} &= a_{I,l}^{(3)} + a_{I,l}^{(2)} \eta_{I,l}^{(1)} \partial_t + a_{I,l}^{(1)} \left[\eta_{I,l}^{(2)} \partial_t + \frac{1}{2} (\eta_{I,l}^{(1)})^2 \partial_t^2 \right] + a_{I,l}^{(0)} \left[\eta_{I,l}^{(3)} \partial_t + \eta_{I,l}^{(2)} \eta_{I,l}^{(1)} \partial_t^2 + \frac{1}{6} (\eta_{I,l}^{(1)})^3 \partial_t^3 \right]. \end{aligned} \quad (4.25)$$

These leading-twist OPE expansions are to be compared to the explicit leading-twist results for the perturbative expansion of the four-point correlation functions. As we have seen, these correlation functions are expressed purely in terms of one-, two- and three-loop ladder integrals, as well as, at three-loops, the Easy and Hard integrals.

Let us now compare the expressions (2.4) and (2.11) with the form (4.2) for the four-point correlator. We find

$$\mathcal{H}(u, v; \sigma, \tau) = u^{\frac{1}{2}k_{43}+1}v^{-1} \sum_{\{b_{ij}\}} \sigma^{b_{13}} \tau^{b_{23}} u^{b_{13}+b_{23}} v^{-b_{23}} F_{\{b_{ij}\}}(u, v). \quad (4.26)$$

For fixed k_1, k_2, k_3, k_4 we may regard b_{13} and b_{23} as free variables while the other b_{ij} are related to them via

$$b_{14} = b_{23} + a, \quad b_{24} = b_{13} + b, \quad b_{12} = E - 2 - b_{13} - b_{23}, \quad b_{34} = k_3 - 2 - b_{13} - b_{23}. \quad (4.27)$$

The bounds on b_{13} and b_{23} follow from the fact that all the b_{ij} are non-negative. Defining $n = b_{13} + b_{23}$ we may rewrite (4.26) as

$$\mathcal{H}(u, v; \sigma, \tau) = u^{\frac{1}{2}k_{43}+1}v^{-1} \sum_{L \leq b_{23} \leq n \leq U} \sigma^{n-b_{23}} \tau^{b_{23}} u^n v^{-b_{23}} F_{n, b_{23}}(u, v). \quad (4.28)$$

with

$$L = \max(0, E - k_1), \quad U = \min(k_3, E) - 2. \quad (4.29)$$

We recall that each $F_{\{b_{ij}\}}(u, v)$ is given by a perturbative expansion given in (2.16). We will now consider a few examples of the above expansions on the correlators under consideration.

4.1 Equal weights ($kkkk$)

In the first instance we specialise to the case where all weights are equal, $k_1 = k_2 = k_3 = k_4 = k$, which was studied extensively in [35]. Note that having all k_i equal implies

$$b_{14} = b_{23}, \quad b_{24} = b_{13}, \quad b_{12} = b_{34}, \quad b_{34} = k - 2 - b_{13} - b_{23}. \quad (4.30)$$

In this case the correlator simplifies to

$$\mathcal{G}_{kkkk} = \mathcal{G}_{kkkk}^0 + C_{kkkk} (d_{12})^k (d_{34})^k \mathcal{S}(u, v; \sigma, \tau) \mathcal{H}(u, v; \sigma, \tau). \quad (4.31)$$

The expansion in (4.7) above reduces to an expansion in terms of Legendre polynomials,

$$\mathcal{H}(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq k-2} A_{nm}(u, v) Y_{nm}^{(0,0)}(\sigma, \tau). \quad (4.32)$$

The leading twist of the exchanged operators in the OPE in a given $\text{su}(4)$ channel is $2 + 2n$ (hence $p = 1 + n$). The expansion (4.28) takes the form

$$\mathcal{H}(u, v; \sigma, \tau) = (u/v) \sum_{0 \leq b_{23} \leq n \leq k-2} \sigma^{n-b_{23}} \tau^{b_{23}} u^n v^{-b_{23}} F_{n, b_{23}}(u, v). \quad (4.33)$$

Now we consider the first few values of k .

2222. The case (2222) corresponds to the well-studied case of four stress-tensor multiplets. The one-loop and two-loop results were derived in [4, 39–42] and the OPE analysis was performed in [35]. The form of the correlator in terms of three-loop integrals was obtained in [5] and the asymptotics necessary for the leading twist OPE analysis were derived in [43]. From these results the full two-variable kinematical dependence of the integrals was reconstructed in [44]. From the expansion (4.32) we have only a single $\text{su}(4)$ channel whose leading twist is 2,

$$\mathcal{H}(u, v; \sigma, \tau) = A_{00}(u, v) = \frac{u}{v} F_{\{0,0,0,0,0,0\}}(u, v), \tag{4.34}$$

where the second equality comes from inserting the only allowed values of the $\{b_{ij}\} = \{0, 0, 0, 0, 0, 0\}$ into (4.26).

Expanding the above function for small u and keeping only the leading power in u we have

$$A_{00}^{(r)}(u, v) = u \sum_{s=0}^r (\log u)^s g_{00,s}^{(r)}(x) + O(u^2). \tag{4.35}$$

The explicit forms of the $g_{00,s}^{(r)}(x)$ may then be read off up to three loops from the explicit expression for the function $F_{\{0,0,0,0,0,0\}}$ in terms of the known integrals. The case of leading twist equal to two is special in that there is only a single operator for each spin l . The sum over operators labelled by I in the relations (4.21)–(4.25) may thus be dropped. From the knowledge of the free theory three-point functions for two weight-two protected operators and one twist-two long operator the anomalous dimensions and normalisations may be constructed up to three loops. These have already been explicitly worked out in [35, 43]. Following [43] we reorganise the final expansion in (4.19) to express it as $a_{00,l}(\lambda) = \tilde{a}_{l+2}(\lambda)$, where

$$\begin{aligned} \tilde{a}_l(\lambda) = & 2^{l-1} \left[\frac{\Gamma(l+1 + \tilde{\eta}_l(\lambda))^2}{\Gamma(2l+1 + 2\tilde{\eta}_l(\lambda))} \right]_{\text{rational}} \\ & \times \left[1 + \lambda c_{2,l}^{(1)} + \lambda^2 (c_{4,l}^{(2)} + \zeta_3 c_{1,l}^{(2)}) + \lambda^3 (c_{6,l}^{(3)} + \zeta_3 c_{3,l}^{(3)} + \zeta_5 c_{1,l}^{(3)}) + \dots \right]. \end{aligned} \tag{4.36}$$

The subscript ‘rational’ on the first factor in brackets denotes the fact that one should discard all zeta-value contributions arising from Taylor expanding the Gamma functions in the $\tilde{\eta}_l(\lambda)$.

The quantity $\tilde{a}_l(\lambda)$ in (4.36) corresponds the expression given in [43] for the squared three-point functions of the two protected operators and one long operator in the $[0, 2, 0]$ representation. It is the same as our squared three-point function up to the shift by two in the spin because the intermediate operator in the $[0, 2, 0]$ considered in [43] is a super-descendant of our superconformal primary operator which is in the $[0, 0, 0]$ representation. Similarly the quantity $\tilde{\eta}_l$ is related to our anomalous dimension simply via $\eta_l(\lambda) = \tilde{\eta}_{l+2}(\lambda)$.

Here let us note only the explicit expression of the quantities c and $\tilde{\eta}$ up to two loops,

$$\begin{aligned}
 \tilde{\eta}_l^{(1)} &= 2h_1, \\
 \tilde{\eta}_l^{(2)} &= 2h_{-2,1} - h_{-3} - 2h_{-2}h_1 - 2h_1h_2 - h_3, \\
 c_{2,l}^{(1)} &= -h_2, \\
 c_{1,l}^{(2)} &= 3h_1, \\
 c_{4,l}^{(2)} &= \frac{5}{2}h_{-4} + h_{-2}^2 + 2h_{-3}h_1 + h_{-2}h_2 + h_2^2 + 2h_1h_3 + \frac{5}{2}h_4 - 2h_{-3,1} - h_{-2,2} - 2h_{1,3}.
 \end{aligned}
 \tag{4.37}$$

In the above equations we use the notation h to denote a harmonic sum with argument l . We refer the reader to [43] for the explicit three-loop formulae.

3333. In the case (3333) we have three $\text{su}(4)$ channels,

$$\begin{aligned}
 \mathcal{H}(u, v; \sigma, \tau) &= \sum_{0 \leq m \leq n \leq 1} A_{nm}(u, v) Y_{nm}(\sigma, \tau) \\
 &= \frac{u}{v} \left(F_{\{1,0,0,0,0,1\}} + \sigma u F_{\{0,1,0,0,1,0\}} + \tau \frac{u}{v} F_{\{0,0,1,1,0,0\}} \right).
 \end{aligned}
 \tag{4.38}$$

The relations between the different expansions are given by

$$\begin{aligned}
 A_{00} &= \frac{u}{v} \left(F_{\{1,0,0,0,0,1\}} + \frac{u}{6} \left(F_{\{0,1,0,0,1,0\}} + \frac{1}{v} F_{\{0,0,1,1,0,0\}} \right) \right), \\
 A_{10} &= \frac{1}{6} \frac{u^2}{v} \left(F_{\{0,1,0,0,1,0\}} - \frac{1}{v} F_{\{0,0,1,1,0,0\}} \right), \\
 A_{11} &= \frac{1}{6} \frac{u^2}{v} \left(F_{\{0,1,0,0,1,0\}} + \frac{1}{v} F_{\{0,0,1,1,0,0\}} \right).
 \end{aligned}
 \tag{4.39}$$

The first channel A_{00} has leading twist two. To keep only the leading twist contribution we keep only the first term in (4.39). From the explicit expression for $F_{\{1,0,0,0,0,1\}}$ up to three loops we may then read off the normalisations $a_{00,l}(\lambda)$, corresponding to the squares of the three-point functions for two weight-three protected operators and one twist-two long operator up to three loops. We again write the perturbative expansion in the form (4.36) and we note that the anomalous dimensions η_l are identical to the (2222) case because the exchanged operators are the same.

At one loop the expression for $c_{2,l}^{(1)}$ is identical to the weight 2 case given in (4.37). This follows from the fact that there is only a single one-loop integral, namely g_{1234} . At two loops we find

$$\begin{aligned}
 c_{1,l}^{(2)} &= 0, \\
 c_{4,l}^{(2)} &= \frac{5}{2}h_{-4} + \frac{1}{2}h_{-2}^2 + 3h_{-3}h_1 + h_{-2}h_1^2 + h_{-2}h_2 + h_2^2 + 2h_1h_3 + 2h_4 - h_{-3,1} - h_{1,3} \\
 &\quad - 2h_{-2,1,1} - 2h_{1,-2,1}.
 \end{aligned}
 \tag{4.40}$$

At three loops we have not found an expression valid for arbitrary l analogous to the ones found in [43] for the weight 2 case. We expect that a relatively simple formula in terms

of harmonic sums, similar to the one for the (2222) case, will reproduce the results we have for each spin. However we may still check our results against the expressions found in [19, 20] for the leading spins and we find perfect agreement with the predictions coming from the integrability approach of [18]. In particular we reproduce¹⁵ the table at the top of page 9 of [19]. The parameter η in that table should be set to $\frac{1}{2}$.

Let us now turn to the channel A_{11} which has leading twist four. In this case the supermultiplets of twist four which are exchanged in the OPE contain super descendants which are pure scalar operators in the sl_2 sector studied in [15]. Thus, from this channel we can compare to the predictions made in [15] for the small x expansions of the functions $g_{11,s}^{(r)}(x)$ defined in (4.20). We pull out some simple prefactors to aid comparison:

$$\begin{aligned}
 g_{11,1}^{(1)}(x) &= -\frac{1}{3.2.4 x^4} \left(8x^4 + 16x^5 + \frac{74}{3}x^6 + 34x^7 + \frac{659}{15}x^8 + \dots \right), \\
 g_{11,0}^{(1)}(x) &= -\frac{1}{3.4 x^4} \left(-8x^4 - 14x^5 - \frac{179}{9}x^6 - \frac{155}{6}x^7 - \frac{28663}{900}x^8 + \dots \right), \\
 g_{11,2}^{(2)}(x) &= \frac{1}{3.2^2.4 x^4} \left(24x^4 + 48x^5 + \frac{712}{9}x^6 + \frac{352}{3}x^7 + \frac{72953}{450}x^8 + \dots \right), \\
 g_{11,1}^{(2)}(x) &= \frac{1}{3.2.4 x^4} \left(-64x^4 - 116x^5 - \frac{1619}{9}x^6 - \frac{2287}{9}x^7 - \frac{9094423}{27000}x^8 + \dots \right), \\
 g_{11,3}^{(3)}(x) &= -\frac{1}{3.2^3.4 x^4} \left(\frac{160}{3}x^4 + \frac{320}{3}x^5 + \frac{15688}{81}x^6 + \frac{8488}{27}x^7 + \frac{4732363}{10125}x^8 + \dots \right), \\
 g_{11,2}^{(3)}(x) &= -\frac{1}{3.2^2.4 x^4} \left(-256x^4 - 472x^5 - \frac{65954}{81}x^6 - \frac{34132}{27}x^7 - \frac{122032589}{67500}x^8 + \dots \right).
 \end{aligned} \tag{4.41}$$

The terms in the parentheses reproduce precisely¹⁶ the predicted expansion coming from the conjectured form of the one-loop twist-four structure constants in [15].

Finally we note that we also have explicit data for the channel A_{10} which is also of leading twist four.

4444. In the case (4444) we have six $su(4)$ channels,

$$\begin{aligned}
 \mathcal{H}(u, v; \sigma, \tau) &= \sum_{0 \leq m \leq n \leq 1} A_{nm}(u, v) Y_{nm}(\sigma, \tau) \\
 &= \frac{u}{v} \left(F_{\{2,0,0,0,2\}} + \sigma u F_{\{1,1,0,0,1,1\}} + \tau \frac{u}{v} F_{\{1,0,1,1,0,1\}} \right. \\
 &\quad \left. + \sigma^2 u^2 F_{\{0,2,0,0,2,0\}} + \sigma \tau \frac{u^2}{v} F_{\{0,1,1,1,1,0\}} + \tau^2 \frac{u^2}{v^2} F_{\{0,0,2,2,0,0\}} \right). \tag{4.42}
 \end{aligned}$$

¹⁵We need to rescale their coupling g^2 by a factor of 4, i.e. $g^2|_{\text{there}} = 4\lambda|_{\text{here}}$ and rescale their coefficients by a global factor of 1/4 to match our conventions.

¹⁶Apart from the third term in the final line in (4.41) which appears to be a simple typo in [15].

The distinct $\text{su}(4)$ channels are given by

$$\begin{aligned}
 A_{00} &= \frac{u}{60v^3} (3u^2 F_{\{0,0,2,2,0,0\}} + u^2 v F_{\{0,1,1,1,1,0\}} + 3u^2 v^2 F_{\{0,2,0,0,2,0\}} + 10uv F_{\{1,0,1,1,0,1\}} \\
 &\quad + 10uv^2 F_{\{1,1,0,0,1,1\}} + 60v^2 F_{\{2,0,0,0,2,0\}}), \\
 A_{10} &= -\frac{u^2}{12v^3} (u F_{\{0,0,2,2,0,0\}} - uv^2 F_{\{0,2,0,0,2,0\}} + 2v F_{\{1,0,1,1,0,1\}} - 2v^2 F_{\{1,1,0,0,1,1\}}), \\
 A_{11} &= \frac{u^2}{60v^3} (6u F_{\{0,0,2,2,0,0\}} + uv F_{\{0,1,1,1,1,0\}} + 6uv^2 F_{\{0,2,0,0,2,0\}} + 10v F_{\{1,0,1,1,0,1\}} \\
 &\quad + 10v^2 F_{\{1,1,0,0,1,1\}}), \\
 A_{20} &= \frac{u^3}{60v^3} (2F_{\{0,0,2,2,0,0\}} - v F_{\{0,1,1,1,1,0\}} + 2v^2 F_{\{0,2,0,0,2,0\}}), \\
 A_{21} &= \frac{u^3}{20v^3} (-F_{\{0,0,2,2,0,0\}} + v^2 F_{\{0,2,0,0,2,0\}}), \\
 A_{22} &= \frac{u^3}{60v^3} (F_{\{0,0,2,2,0,0\}} + v F_{\{0,1,1,1,1,0\}} + v^2 F_{\{0,2,0,0,2,0\}}). \tag{4.43}
 \end{aligned}$$

The channel A_{00} again has leading twist two. If we keep only the leading twist contributions, the only term which contributes is the last one on the r.h.s. of the first equation in (4.43). Up to two loops the function $F_{\{2,0,0,0,2\}}$ is identical to $F_{\{1,0,0,0,1\}}$. Thus the results for $a_{0,0,l}(\lambda)$ are identical up to two loops with the (3333) case. At three loops we may again compare with the results of [19, 20]. Again the relevant data is given in the table on page 9 of [19]. This time we must set the variable η in that table to zero. We can see that the variable η in table 9 of [19] is varying exactly in accordance with the coefficient of the single integral $L_{12;34}$, which is finite in the OPE limit $x_{12}^2 \rightarrow 0$.

We may also examine the channel A_{22} where the leading twist is six. Again we can compare to the data presented in [15] in table 4 in appendix C. Expanding A_{22} in the leading twist sector we find perfect agreement with the coefficients for leading spins detailed in that table.

Higher k and wrapping corrections. We have previously noted that there is a uniformity of the functions $F_{\{b_{ij}\}}$ in that there is only one contributing integral, g_{1234} at one loop while functions with any given $b_{ij} \geq 1$ are identified at two loops and those with any given $b_{ij} \geq 2$ are identified at three loops. In particular the leading twist-two channel of the correlator $(kkkk)$ is of the form

$$A_{00}(u, v) = \frac{u}{v} F_{\{k-2,0,0,0,k-2\}} + O(u^2). \tag{4.44}$$

We see that the uniform behaviour of the functions $F_{\{k-2,0,0,0,k-2\}}$ for all k at one loop, $k \geq 3$ at two loops¹⁷ and $k \geq 4$ at three loops implies that the normalisations $a_{00,l}(\lambda)$ will also exhibit such a uniform behaviour. Thus the normalisations $a_{00,l}(\lambda)$ in the cases (5555), (6666) etc. will all be the same as those of (4444) up to three loops. This uniformity in k is precisely what is expected from the nature of possible wrapping contributions to the three-point functions of two protected operators and one twist-two long operator in the approach of [18].

¹⁷This uniformity (called ‘degeneracy’) was first remarked in [10, 11].

4.2 Weights $(kkk'k')$ and an averaging rule

Let us now consider correlation functions with weights $(kkk'k')$ for k and k' not equal. Note that by our assumptions on the ordering of the weights we have $k' > k$. The b_{ij} are related as follows,

$$b_{14} = b_{23}, \quad b_{24} = b_{13}, \quad b_{12} = k - 2 - b_{13} - b_{23}, \quad b_{34} = k' - 2 - b_{13} - b_{23}. \quad (4.45)$$

The correlator simplifies to

$$\mathcal{G}_{kkk'k'} = \mathcal{G}_{kkk'k'}^0 + C_{kkk'k'} (d_{12})^k (d_{34})^{k'} \mathcal{S}(u, v; \sigma, \tau) \mathcal{H}(u, v; \sigma, \tau). \quad (4.46)$$

The expansion in (4.7) above reduces to an expansion in terms of Legendre polynomials just as in the case of equal weights (4.32). The leading twist of the exchanged operators in the OPE in a given $\text{su}(4)$ channel is again $2 + 2n$ (hence $p = 1 + n$).

Since the normalisations $a_{nm,I,l}$ are products of three-point functions we expect them to obey

$$a_{nm,I,l}^{(kkk'k')} = \sqrt{a_{nm,I,l}^{(kkkk)} a_{nm,I,l}^{(k'k'k'k')}}. \quad (4.47)$$

Let us now focus on the case of the twist-two operators (where we can drop the additional index I as there is no operator mixing). We know in this case that, in the free theory and at one loop, the normalisations $a_{00,l}^{(kkkk)}$ are in fact independent of k . We write this explicitly as follows,

$$a_{00,l}^{(kkkk)} = a_{00,l}^{(0)} (1 + \lambda b_l^{(1)} + \lambda^2 b_{l,k}^{(2)} + \lambda^3 b_{l,k}^{(3)} + \dots). \quad (4.48)$$

This means that if we perturbatively expand (4.47) we find

$$a_{00,l}^{(kkk'k')} = a_{00,l}^{(0)} \left(1 + \lambda b_l^{(1)} + \lambda^2 \frac{1}{2} (b_{l,k}^{(2)} + b_{l,k'}^{(2)}) + \lambda^3 \frac{1}{2} (b_{l,k}^{(3)} + b_{l,k'}^{(3)}) \right) + \dots \quad (4.49)$$

In other words, to three loops, we find that the normalisations $a_{00,l}(\lambda)$ for the case $(kkk'k')$ are the average of those for the cases $(kkkk)$ and $(k'k'k'k')$. In particular this means that the leading twist contributions to $A_{00}^{(r)}$ obey

$$A_{00}^{(r),(kkk'k')} = \frac{1}{2} \left(A_{00}^{(r),(kkkk)} + A_{00}^{(r),(k'k'k'k')} \right), \quad r = 1, 2, 3. \quad (4.50)$$

Since the leading twist-two contributions to the A_{00} channels for the correlators $(kkk'k')$ are all given by the functions $(u/v)F_{\{k-2,0,0,0,0,k'-2\}}$ we conclude that in the limit of small u we have

$$F_{\{k-2,0,0,0,0,k'-2\}} = \frac{1}{2} (F_{\{k-2,0,0,0,0,k-2\}} + F_{\{k'-2,0,0,0,0,k'-2\}}) + O(u) \quad (4.51)$$

up to three loops.

Of the integral functions appearing in the expansion (2.16) of correlators up to three loops, the functions with coefficients c_{gg}^1 at two loops and c_{gh}^1 , c_H^4 , and c_H^6 at three loops are all power-suppressed in the OPE limit $x_{12}^2 \rightarrow 0$. The remaining functions can all contribute in the limit to the leading twist expansion of any given $\text{su}(4)$ channel of a

correlation function. Moreover, the remaining functions are all linearly independent in the limit, as can be verified from their explicit expressions [44]. This means that, for those functions which are not power suppressed, one may apply the averaging rule (4.51) directly at the level of the individual integral coefficients.

In fact the only reason we have restricted ourselves to three loops in equation (4.50) is the dependence on k in $a_{00,l}^{(kkkk)}$ at two loops. However, as we have seen from the analysis of the preceding section, this dependence is very mild indeed. In fact $a_{00,l}^{(kkkk)}$ is again independent of k at two loops as long as $k \geq 3$ and even at three loops as long as $k \geq 4$. This means that we can actually extend our average rule beyond three loops in these cases. This may prove a useful tool in higher loop explorations of the correlation functions of $\frac{1}{2}$ -BPS operators.

4.3 General consistency checks

In fact we may use the OPE of the four-point correlation functions to cross-check many of our results from tables 1 and 2. If we allow ourselves to make an ansatz in terms of the known ladder integrals and, at three loops, the Easy and Hard integrals, we find that many of their coefficients are fixed by consistency of the OPE expansion. The reason is that at ℓ loops the operators $\mathcal{O}_{nm,I,l,s}^{(\ell)}$ appearing in (4.21) are explicitly known for $s > 1$ if all the lower loop data $a_{nm,I,l}^{(\ell')}$ and $\eta_{I,l}^{(\ell')}$ are known for $\ell' < \ell$. Moreover if $\eta_{I,l}^{(\ell)}$ is known then one also knows $\mathcal{O}_{nm,I,l,1}^{(\ell)}$. In the case of the exchange of twist-two operators these data may be read off from the lower loop correlators themselves, while the anomalous dimensions to the relevant order are well known. We find that matching such constraints from the form of the OPE fixes many of the constants in the ansatz. As an example, for the correlator (3333), such OPE consistency checks fix all but one coefficient, namely the coefficient c_L^1 in an ansatz for $F_{\{1,0,0,0,1\}}$. This final coefficient can then be determined from the averaging rule we described above in section 4.2, assuming the correlator (2233) is known. The fact that the coefficients obtained in tables 1 and 2 are all consistent with what is essentially an independent check based on the forms of the actual integrated functions further increases our confidence that the values are correct.

5 Conclusions

In this paper we have shown how to construct the four-point correlation functions of half-BPS operators of arbitrary weights, in the planar limit and up to three loops. Our construction uses only elementary properties of the *integrand* of the loop corrections, viewed as a rational correlator at Born level. Knowing its symmetries and singularity structure, we are able to write down a relatively concise ansatz in the planar limit. The unknown coefficients are then determined from a chain of relations between correlators with different weights, following from comparing their light-cone OPEs. Interestingly, we need to consider the set of all such correlators and all the relations between them, in a kind of bootstrap procedure.

We have used the known correlator \mathcal{G}_{2222} as the starting point of the recursion. The three-loop correlator \mathcal{G}_{2222} was found in [5] with the help of another, exceptional property —

the hidden permutation symmetry between external and Lagrangian insertion points for the operators $\mathcal{O}^{(2)}$. We have also used the same symmetry in the present work, if one or more of the four operators are of the type $\mathcal{O}^{(2)}$. So, it may seem that this symmetry is an essential ingredient of the whole construction of integrands. However, we have experimented with a more general ansatz where the symmetry is not taken into account. Using the light-cone OPE consistency conditions from this paper we were able to determine all but two coefficients at two loops and seven at three loops. Then we applied the Minkowski and Euclidean logarithmic divergence criteria from [6] and succeeded in fixing *all the coefficients*, including those in the correlator \mathcal{G}_{2222} . So, the hidden symmetry of [5] may be very helpful but is not indispensable, at least up to three loops.

We would like to emphasise the role of the planar limit in our construction. Not only it greatly reduces the size of our ansatz for the integrand, but most importantly, it is responsible for the universality property of the OPE structure constants discussed in section 3.3. Without the OPE relation (3.15) that follows from this universality we would not be able to go very far in the non-planar case. We interpret this as a sign of some new type of integrability for the correlation functions of half-BPS operators.

To further elucidate the predictability (or integrability) of these correlators we have to see what happens at higher loops. We have some preliminary encouraging results at four loops. We hope that they can be useful for checking the recent integrability predictions for the OPE structure constants [19, 20]. However, we can only provide the answer in terms of four-loop conformal integrals, which will have to be evaluated by some modern techniques [44, 45].

Ultimately, the goal would be to try to construct all correlators of half-BPS operators with an arbitrary number of points and at arbitrary loop levels, using only basic properties of the integrands. In case of success this can shed new and very nontrivial light on the origin of the remarkable properties of scattering amplitudes/Wilson loops. The latter are known to be light-like limits of correlation functions [46, 47]. This duality is most easily seen at the level of their integrands [48–50].

Acknowledgments

E.S. is grateful to Sergio Ferrara, Vasco Gonçalves, Yassen Stanev and Pedro Vieira for helpful discussions. We acknowledge discussions with Burkhard Eden at the early stages of this project. D.C. and E.S. acknowledge partial support by the French National Agency for Research (ANR) under contract StrongInt (BLANC-SIMI-4-2011). The work of D.C. has been supported by the “Investissements d’avenir, Labex ENIGMASS”. P.H. acknowledges support from the STFC Consolidated Grant ST/L000407/1 and the Marie Curie network GATIS (gatis.desy.eu) of the European Union’s Seventh Framework Programme FP7/2007-2013/ under REA Grant Agreement No 317089. J.M.D acknowledges support from ERC grant 648630.

A Uniformity of the correlators

As discussed in section 3.2, the planarity of component terms is a powerful restriction on the correlation functions. The light-cone OPE condition relating different correlators is another powerful condition. In combination they are enough to determine all three-loop correlation functions up to a single unfixed coefficient. However even before performing the detailed analysis leading to this conclusion we can use the two restrictions to deduce a uniform structure for correlation functions of sufficiently high weights.

More concretely, they combine to imply that at two loops

$$f_{\{b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\}}^2 = f_{\{1, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\}}^2 \quad \text{for all } b_{12} \geq 1, b_{13}, \dots, b_{34} \geq 0, \quad (\text{A.1})$$

and similarly at three loops

$$f_{\{b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\}}^3 = f_{\{2, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\}}^3 \quad \text{for all } b_{12} \geq 2, b_{13}, \dots, b_{34} \geq 0. \quad (\text{A.2})$$

The choice of b_{12} here is simply for convenience and similar equations apply for any other b_{ij} . These equations imply that we can restrict our attention to the set $b_{ij} \in \{0, 1\}$ at two loops and $b_{ij} \in \{0, 1, 2\}$ at three loops: all other cases will reduce to these cases. For example the above equation implies that $f_{\{7, 1, 5, 0, 1, 8\}}^3 = f_{\{2, 1, 2, 0, 1, 2\}}^3$.

To see where this comes from we first consider two loops. The light-cone OPE in the form of (3.18) implies that $P_{\{b_{ij}\}}^\ell|_{b_{12} \rightarrow b_{12}+1} - P_{\{b_{ij}\}}^\ell = O(x_{12}^2)$. Since at two loops the numerator has the form (3.8), this means the difference between the two numerators has only two unfixed terms:

$$P_{\{b_{ij}\}}^2|_{b_{12} \rightarrow b_{12}+1} - P_{\{b_{ij}\}}^2 = x_{12}^2 \left(a_1 x_{34}^2 x_{56}^2 + a_2 (x_{35}^2 x_{46}^2 + x_{36}^2 x_{45}^2) \right). \quad (\text{A.3})$$

We can then ask what values of a_1 and a_2 are consistent with planarity. Inserting $P_{\{b_{ij}\}}^2$ into the expression for the corresponding correlation function (3.5) and then replacing it with the right-hand side of (A.3) we see that the constants a_1, a_2 appear as

$$\mathcal{I} \times \prod_{ij \neq 12} d_{ij}^{b_{ij}} \times \frac{x_{12}^2 \left(a_1 x_{34}^2 x_{56}^2 + a_2 (x_{35}^2 x_{46}^2 + x_{36}^2 x_{45}^2) \right)}{(x_{12}^2)^{b_{12}} \prod_{1 \leq p < q \leq 6} x_{pq}^2}. \quad (\text{A.4})$$

If $b_{12} \geq 1$ these terms are non-planar (since the x_{12}^2 in the numerator does not cancel that in the denominator). Thus planarity requires $a_1 = a_2 = 0$ and we deduce (A.1).

The three-loop proof is very similar, the difference being that the presence of the additional power in the numerator at three loops (see (3.8)) delays the universal structure by one level.

Now we briefly explain why (3.18) follows from (3.17) and planarity at two and three loops. We will prove a more general statement from which this one follows. Consider the equation

$$\sum_{k=0}^n (x_{13}^2 x_{24}^2)^k (x_{14}^2 x_{23}^2)^{n-k} Q_k(x) = 0 \quad (\text{A.5})$$

for a set of polynomials $Q_k(x)$ of the form (3.8) at two and three loops, correspondingly, and $n > 0$. We assume that each contribution in (A.5) is planar, i.e. for each k the rational function

$$Q_k / ((x_{13}^2 x_{24}^2)^{n-k} (x_{14}^2 x_{23}^2)^k \prod_{i < j} x_{ij}^2) \tag{A.6}$$

corresponds to a set of planar graphs. Then $Q_k = 0, k = 0, 1, \dots, n$, is the only solution of (A.5).

Indeed, the different terms on the left-hand side of (A.5) can possibly cancel each other only if $Q_k = x_{13}^2 x_{24}^2 f_k(x) + x_{14}^2 x_{23}^2 g_k(x)$ with some polynomials $f_k(x), g_k(x)$. Substituting this in (A.6), we see that all contributions correspond to nonplanar graphs according to the argument around (A.4). Let us mention that this statement relies upon a special property of the two-loop and three-loop planar graphs from (3.9): adding any further edge to either of them makes them non-planar. This property is not valid at four loops and higher.

Finally we show that at two and three loops the strongest criterion (3.31) is a consequence of (3.18) and planarity. At two loops (3.31) is equivalent to (3.18) if $b_{12} = 0$, and if $b_{12} \geq 1$ we can evoke the uniformity property formulated above. At three loops the relationship between the two criteria is more involved. We need to show that $P_{\{b_{ij}\}}^3|_{b_{12}=2} - P_{\{b_{ij}\}}^3|_{b_{12}=1} = O(x_{12}^2)$ and planarity imply that $P_{\{b_{ij}\}}^3|_{b_{12}=2} - P_{\{b_{ij}\}}^3|_{b_{12}=1} = O(x_{12}^4)$. Indeed, let $P_{\{b_{ij}\}}^3|_{b_{12}=2} - P_{\{b_{ij}\}}^3|_{b_{12}=1} = x_{12}^2 f(x)$ with a polynomial $f(x)$ such that $f(x)|_{x_{12}^2=0} \neq 0$. Then following the argument around (A.4) we conclude that

$$(x_{12}^2 f(x)) / \left((x_{12}^2)^{b_{12}} \prod_{i < j} x_{ij}^2 \right) \tag{A.7}$$

with $b_{12} = 1$ necessarily produces nonplanar graphs. The reason is that the x_{12}^2 in the numerator cannot cancel the x_{12}^4 in the denominator.

In conclusion, the three constraints — the weaker (3.17), the intermediate (3.18) and the strongest (3.31), are in fact equivalent up to three loops if we assume planarity. This is however not true starting from four loops.

B The superconformal OPE

In [30] a manifestly superconformal form for the OPE in $\mathcal{N} = 4$ SYM was written down in analytic superspace which has coordinates

$$X^{AA'} = \begin{pmatrix} x^{\alpha\dot{\alpha}} & 0 \\ 0 & y^{aa'} \end{pmatrix}. \tag{B.1}$$

The indices A, A' are superindices carrying representations of $SL(2|2)$ and they split as $A = (\alpha|a)$ and $A' = (\dot{\alpha}|a')$ with $\alpha, \dot{\alpha}$ carrying the left and right spinor representations of the Lorentz group and a, a' two different $SU(2)$ subgroups of $SU(4)$. We have dropped all superspace coordinates which is why we have 0's in the off-diagonal blocks in (B.1) but these can be easily put back in.

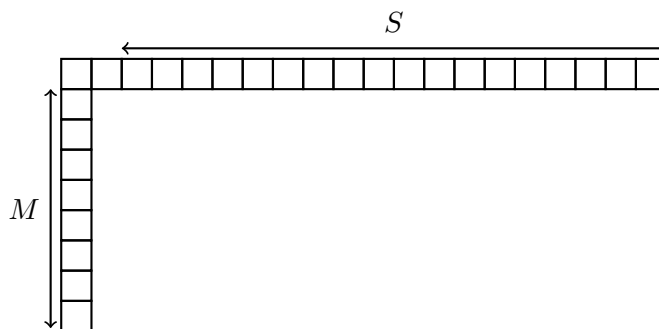
Then the OPE of two half BPS operators takes the form

$$\mathcal{O}_p(X_1)\mathcal{O}_q(X_2) = \sum_{\mathcal{O}} C_{pq\mathcal{O}}(d_{12})^{\frac{1}{2}(p+q-L)}(X_{12})^{\underline{AA'}}\mathcal{O}_{\underline{AA'}}^L(2) + \dots \quad (\text{B.2})$$

Here the sum is over all operators in the theory, $L = \Delta - S$ is the twist of the operator and the underlined index is a multi-index indicating a tensor in a representation determined by the operator. The operators of most interest here are the semi-short operators whose highest weight states have spin S , lie in the $SU(4)$ representation space with Dynkin labels $[MNM]$ and which have twist

$$L = 2M + N + 2 \quad (\text{B.3})$$

For such operators the $SL(2|2)$ representation which the indices $\underline{A}, \underline{A}'$ lie in is given by a hook-shaped Young tableau with top row of length $S + 2$ and left column of height $M + 1$. There are thus $M + S + 2$ boxes in total.



Since these are Young tableaux involving superindices, horizontal boxes correspond to symmetrisation of α, β indices, but anti-symmetrisation of a, b indices, and vice versa for vertical boxes. The HWS is obtained by filling as many of the boxes as possible with a indices. Since the index only takes on 2 values, at most two a indices can be found in the same row (more than two in a row would correspond to antisymmetrising 3 indices and thus vanish). So the HWS has the first two columns filled with a 's and all other columns filled with α 's. Other index choices with fewer a 's and more α s correspond to acting with $Q_{\alpha a}$ on the HWS.

The primed indices follow a similar story (and are in the same representation as the unprimed indices for a nonvanishing OPE of half BPS operators).

Let us consider a simple example, the Konishi operator has $L = 2, S = M = N = 0$ and the corresponding Young tableau is simply the symmetric representation, so $\mathcal{K}_{(AB)(A'B')}$. Writing out all bosonic terms we see that it decomposes into the bosonic terms

$$\begin{aligned} \mathcal{A} & \quad \boxed{a|b} \otimes \boxed{a'|b'} \\ \mathcal{B}_{\beta\dot{\beta}aa'} = Q_{\beta a}\bar{Q}_{\dot{\beta}a'}\mathcal{A} & \quad \boxed{a|\beta} \otimes \boxed{a'|\dot{\beta}} \\ \mathcal{C} = Q_{\alpha\beta}^2\bar{Q}_{\dot{\alpha}\dot{\beta}}^2\mathcal{A} & \quad \boxed{\alpha|\beta} \otimes \boxed{\dot{\alpha}|\dot{\beta}}. \end{aligned} \quad (\text{B.4})$$

We would expect more terms where we make different choices for the two $SL(2|2)$ reps. However these are the only terms we will obtain when switching off the superspace variables since in the OPE we contract with the block diagonal X given in (B.1), thus tying together the primed and unprimed indices.

The general case is similar. Consider any operator with twist $L = 2M + N + 2$, $\mathcal{O}_{AA'}$. There are $S + M + 2$ unprimed superindices and $S + M + 2$ primed superindices, symmetrised according to the above hook-shaped Young tableau. Splitting the superindices into the $SL(2)$ subgroups we obtain the following components

$L-2M-N$ Component operator

$$\begin{aligned}
 2 \quad & \mathcal{A}_{[MNM]}^{\Delta,S} = \mathcal{O}_{bb'\alpha(S)\dot{\alpha}(S)a(M)a'(M)} \\
 0 \quad & \mathcal{B}_{[(M+1)N(M+1)]}^{\Delta+1,S+1} = \mathcal{O}_{\alpha(S+1)\dot{\alpha}(S+1)a(M+1)a'(M+1)} \\
 2 \quad & \mathcal{B}_{[(M-1)(N+2)(M-1)]}^{\Delta+1,S+1} = \mathcal{O}_{bb'\alpha(S+1)\dot{\alpha}(S+1)a(M-1)a'(M-1)} \\
 4 \quad & \mathcal{B}_{[(M-1)(N+2)(M-1)]}^{\Delta+1,S-1} = \mathcal{O}_{\beta\dot{\beta}bb'\alpha(S-1)\dot{\alpha}(S-1)a(M-1)a'(M-1)} \\
 0 \quad & \mathcal{C}_{[M(N+2)M]}^{\Delta+2,S+2} = \mathcal{O}_{\alpha(S+2)\dot{\alpha}(S+2)a(M)a'(M)} \\
 2 \quad & \mathcal{C}_{[M(N+2)M]}^{\Delta+2,S} = \mathcal{O}_{\beta\dot{\beta}\alpha(S)\dot{\alpha}(S)a(M)a'(M)} \\
 4 \quad & \mathcal{C}_{[(M-2)(N+4)(M-2)]}^{\Delta+2,S} = \mathcal{O}_{bb'\beta\dot{\beta}\alpha(S)\dot{\alpha}(S)a(M-2)a'(M-2)} \\
 2 \quad & \mathcal{D}_{[(M-1)(N+4)(M-1)]}^{\Delta+3,S+1} = \mathcal{O}_{\beta\dot{\beta}\alpha(S+1)\dot{\alpha}(S+1)a(M-1)a'(M-1)} \in Q^3\bar{Q}^3\mathcal{A}_{[MNM]}^{\Delta,S} \cdot \quad (\text{B.5})
 \end{aligned}
 \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \in Q^2\bar{Q}^2\mathcal{A}_{[MNM]}^{\Delta,S}$$

These are all the components which occur in the free OPE. Writing out the super OPE (B.2) in components we then get

$$\begin{aligned}
 \mathcal{O}_p(X_1)\mathcal{O}_q(X_2) &= \sum_{\mathcal{O}} C_{pq\mathcal{O}}(d_{12})^{\frac{1}{2}(p+q-L)}(x_{12})^{\alpha(S)\dot{\alpha}(S)}(y_{12})^{a(M-1)a'(M-1)} \\
 &\quad \times \left(y_{12}^2 y_{12}^{a_M a'_M} \mathcal{A}_{\alpha(S)\dot{\alpha}(S)a(M)a'(M)}(2) + \dots \right. \\
 &\quad \left. + x_{12}^2 x_{12}^{\alpha_{S+1}\dot{\alpha}_{S+1}} \mathcal{D}_{\alpha(S+1)\dot{\alpha}(S+1)a(M-1)a'(M-1)}(2) \right). \quad (\text{B.6})
 \end{aligned}$$

However in the interacting theory the semi-short multiplet combines with three others to form a long multiplet. In fact only one of these three multiplets contributes to the OPE. This is another semi-short multiplet with highest weight state $M \rightarrow M + 1, S \rightarrow S - 1, N \rightarrow N, (\mathcal{A}')_{[(M+1),N,(M+1)]}^{\Delta+1,S-1}$. Whilst in the free theory $\mathcal{A}, \mathcal{A}'$ are both superconformal primaries of independent multiplets, in the interacting theory \mathcal{A}' becomes a descendant of $\mathcal{A}, (\mathcal{A}')_{[(M+1),N,(M+1)]}^{\Delta+1,S-1} \in Q\bar{Q}\mathcal{A}_{[M,N,M]}^{\Delta,S}$. In total therefore in the interacting theory we have

the following components contributing to a long supermultiplet

$L-2M-N$	Component operator	
2	$\mathcal{A}_{[MNM]}^{\Delta,S}$	
0	$\mathcal{B}_{[(M+1)N(M+1)]}^{\Delta+1,S+1}$	}
2	$\mathcal{B}_{[(M-1)(N+2)(M-1)]}^{\Delta+1,S+1}$	
4	$\mathcal{B}_{[(M-1)(N+2)(M-1)]}^{\Delta+1,S-1}$	
2	$(\mathcal{A}')_{[(M+1)N(M+1)]}^{\Delta+1,S-1}$	
$\in Q\bar{Q}\mathcal{A}_{[MNM]}^{\Delta,S}$		
0	$\mathcal{C}_{[M(N+2)M]}^{\Delta+2,S+2}$	}
2	$\mathcal{C}_{[M(N+2)M]}^{\Delta+2,S}$	
4	$\mathcal{C}_{[(M-2)(N+4)(M-2)]}^{\Delta+2,S}$	
0	$(\mathcal{B}')_{[(M+2)N(M+2)]}^{\Delta+2,S}$	
2	$(\mathcal{B}')_{[M(N+2)M]}^{\Delta+2,S}$	
4	$(\mathcal{B}')_{[M(N+2)M]}^{\Delta+2,S-2}$	
$\in Q^2\bar{Q}^2\mathcal{A}_{[MNM]}^{\Delta,S}$		
2	$\mathcal{D}_{[(M-1)(N+4)(M-1)]}^{\Delta+3,S+1}$	}
0	$(\mathcal{C}')_{[(M+1)(N+2)(M+1)]}^{\Delta+3,S+1}$	
2	$(\mathcal{C}')_{[(M+1)(N+2)(M+1)]}^{\Delta+3,S-1}$	
4	$(\mathcal{C}')_{[(M-1)(N+4)(M-1)]}^{\Delta+3,S-1}$	
$\in Q^3\bar{Q}^3\mathcal{A}_{[MNM]}^{\Delta,S}$		
2	$(\mathcal{D}')_{[M(N+4)M]}^{\Delta+4,S} \in Q^4\bar{Q}^4\mathcal{A}_{[MNM]}^{\Delta,S}$	(B.7)

Writing out the full super OPE is now equivalent to summing two copies of the semi-short OPE (B.6) with appropriate quantum numbers. In components we then get

$$\begin{aligned}
 \mathcal{O}_p(X_1)\mathcal{O}_q(X_2) &= \sum_{\mathcal{O}} C_{pq}\mathcal{O}(d_{12})^{\frac{1}{2}(p+q-L)}(x_{12})^{\alpha(S)\dot{\alpha}(S)}(y_{12})^{a(M-1)a'(M-1)} \\
 &\times \frac{x_{12}^2}{y_{12}^2} \left(y_{12}^4 y_{12}^{a_M a'_M} \mathcal{A}_{\alpha(S)\dot{\alpha}(S)a(M)a'(M)}(2) + \dots \right. \\
 &\quad \left. + x_{12}^4 x_{12}^{\alpha_S \dot{\alpha}_S} (\mathcal{D}')_{\alpha(S)\dot{\alpha}(S)a(M)a'(M)}(2) \right). \tag{B.8}
 \end{aligned}$$

Note that this long OPE can also be derived directly using superindices rather than by summing two semi-short OPEs as we have done here.

Note also that in the free theory one can write down explicit forms for the single trace operators in question. They have the schematic form

$$\mathcal{O}_{\underline{AA}'} = Tr(\partial_{\underline{AA}'}^{a+s+2} W^L) \tag{B.9}$$

where the derivatives can act on any of the different W s and in general will be linear combinations of such terms.

Finally in the main text we will be interested in considering this OPE in the limit in which both x_{12}^2 and $y_{12} \rightarrow 0$ (but with the ratio fixed). In this limit only the operators from the above list with $L - 2M - N = 0$ survive.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The Large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] B. Eden, C. Schubert and E. Sokatchev, *Three loop four point correlator in $\mathcal{N} = 4$ SYM*, *Phys. Lett. B* **482** (2000) 309 [[hep-th/0003096](#)] [[INSPIRE](#)].
- [5] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Hidden symmetry of four-point correlation functions and amplitudes in $\mathcal{N} = 4$ SYM*, *Nucl. Phys. B* **862** (2012) 193 [[arXiv:1108.3557](#)] [[INSPIRE](#)].
- [6] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in $\mathcal{N} = 4$ SYM*, *Nucl. Phys. B* **862** (2012) 450 [[arXiv:1201.5329](#)] [[INSPIRE](#)].
- [7] R.G. Ambrosio, B. Eden, T. Goddard, P. Heslop and C. Taylor, *Local integrands for the five-point amplitude in planar $\mathcal{N} = 4$ SYM up to five loops*, *JHEP* **01** (2015) 116 [[arXiv:1312.1163](#)] [[INSPIRE](#)].
- [8] P.J. Heslop and P.S. Howe, *Four point functions in $\mathcal{N} = 4$ SYM*, *JHEP* **01** (2003) 043 [[hep-th/0211252](#)] [[INSPIRE](#)].
- [9] G. Arutyunov, F.A. Dolan, H. Osborn and E. Sokatchev, *Correlation functions and massive Kaluza-Klein modes in the AdS/CFT correspondence*, *Nucl. Phys. B* **665** (2003) 273 [[hep-th/0212116](#)] [[INSPIRE](#)].
- [10] G. Arutyunov and E. Sokatchev, *On a large- N degeneracy in $\mathcal{N} = 4$ SYM and the AdS/CFT correspondence*, *Nucl. Phys. B* **663** (2003) 163 [[hep-th/0301058](#)] [[INSPIRE](#)].
- [11] G. Arutyunov, S. Penati, A. Santambrogio and E. Sokatchev, *Four point correlators of BPS operators in $\mathcal{N} = 4$ SYM at order g^4* , *Nucl. Phys. B* **670** (2003) 103 [[hep-th/0305060](#)] [[INSPIRE](#)].
- [12] E. D'Hoker, J. Erdmenger, D.Z. Freedman and M. Pérez-Victoria, *Near extremal correlators and vanishing supergravity couplings in AdS/CFT*, *Nucl. Phys. B* **589** (2000) 3 [[hep-th/0003218](#)] [[INSPIRE](#)].
- [13] L. Berdichevsky and P. Naaijkens, *Four-point functions of different-weight operators in the AdS/CFT correspondence*, *JHEP* **01** (2008) 071 [[arXiv:0709.1365](#)] [[INSPIRE](#)].
- [14] L.I. Uruchurtu, *Next-next-to-extremal Four Point Functions of $\mathcal{N} = 4$ 1/2 BPS Operators in the AdS/CFT Correspondence*, *JHEP* **08** (2011) 133 [[arXiv:1106.0630](#)] [[INSPIRE](#)].
- [15] P. Vieira and T. Wang, *Tailoring Non-Compact Spin Chains*, *JHEP* **10** (2014) 35 [[arXiv:1311.6404](#)] [[INSPIRE](#)].

- [16] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *Tailoring Three-Point Functions and Integrability*, *JHEP* **09** (2011) 028 [[arXiv:1012.2475](#)] [[INSPIRE](#)].
- [17] N. Gromov and P. Vieira, *Quantum Integrability for Three-Point Functions of Maximally Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **111** (2013) 211601 [[arXiv:1202.4103](#)] [[INSPIRE](#)].
- [18] B. Basso, S. Komatsu and P. Vieira, *Structure Constants and Integrable Bootstrap in Planar $\mathcal{N} = 4$ SYM Theory*, [arXiv:1505.06745](#) [[INSPIRE](#)].
- [19] B. Eden and A. Sfondrini, *Three-point functions in $\mathcal{N} = 4$ SYM: the hexagon proposal at three loops*, *JHEP* **02** (2016) 165 [[arXiv:1510.01242](#)] [[INSPIRE](#)].
- [20] B. Basso, V. Goncalves, S. Komatsu and P. Vieira, *Gluing Hexagons at Three Loops*, *Nucl. Phys. B* **907** (2016) 695 [[arXiv:1510.01683](#)] [[INSPIRE](#)].
- [21] M. D'Alessandro and L. Genovese, *A Wide class of four point functions of BPS operators in $\mathcal{N} = 4$ SYM at order g^4* , *Nucl. Phys. B* **732** (2006) 64 [[hep-th/0504061](#)] [[INSPIRE](#)].
- [22] D. Chicherin and E. Sokatchev, *A note on four-point correlators of half-BPS operators in $\mathcal{N} = 4$ SYM*, *JHEP* **11** (2014) 139 [[arXiv:1408.3527](#)] [[INSPIRE](#)].
- [23] J.M. Drummond, J. Henn, V.A. Smirnov and E. Sokatchev, *Magic identities for conformal four-point integrals*, *JHEP* **01** (2007) 064 [[hep-th/0607160](#)] [[INSPIRE](#)].
- [24] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Extremal correlators in the AdS/CFT correspondence*, in *The many faces of the superworld*, M.A. Shifman ed., World Scientific (2000), pp. 332–360 [[hep-th/9908160](#)] [[INSPIRE](#)].
- [25] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Extremal correlators in four-dimensional SCFT*, *Phys. Lett. B* **472** (2000) 323 [[hep-th/9910150](#)] [[INSPIRE](#)].
- [26] B.U. Eden, P.S. Howe, E. Sokatchev and P.C. West, *Extremal and next-to-extremal n point correlators in four-dimensional SCFT*, *Phys. Lett. B* **494** (2000) 141 [[hep-th/0004102](#)] [[INSPIRE](#)].
- [27] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, *Partial nonrenormalization of the stress tensor four point function in $\mathcal{N} = 4$ SYM and AdS/CFT*, *Nucl. Phys. B* **607** (2001) 191 [[hep-th/0009106](#)] [[INSPIRE](#)].
- [28] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, *Nucl. Phys. B* **711** (2005) 409 [[hep-th/0407060](#)] [[INSPIRE](#)].
- [29] B. Eden and E. Sokatchev, *On the OPE of 1/2 BPS short operators in $\mathcal{N} = 4$ SCFT₄*, *Nucl. Phys. B* **618** (2001) 259 [[hep-th/0106249](#)] [[INSPIRE](#)].
- [30] P.J. Heslop and P.S. Howe, *OPEs and three-point correlators of protected operators in $\mathcal{N} = 4$ SYM*, *Nucl. Phys. B* **626** (2002) 265 [[hep-th/0107212](#)] [[INSPIRE](#)].
- [31] P.J. Heslop and P.S. Howe, *Aspects of $\mathcal{N} = 4$ SYM*, *JHEP* **01** (2004) 058 [[hep-th/0307210](#)] [[INSPIRE](#)].
- [32] V.K. Dobrev and V.B. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys. Lett. B* **162** (1985) 127 [[INSPIRE](#)].
- [33] M. Bianchi, B. Eden, G. Rossi and Y.S. Stanev, *On operator mixing in $\mathcal{N} = 4$ SYM*, *Nucl. Phys. B* **646** (2002) 69 [[hep-th/0205321](#)] [[INSPIRE](#)].

- [34] J.L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin and A. Volovich, *The Soft-Collinear Bootstrap: $\mathcal{N} = 4$ Yang-Mills Amplitudes at Six and Seven Loops*, *JHEP* **03** (2012) 032 [[arXiv:1112.6432](#)] [[INSPIRE](#)].
- [35] F.A. Dolan and H. Osborn, *Conformal partial wave expansions for $\mathcal{N} = 4$ chiral four point functions*, *Annals Phys.* **321** (2006) 581 [[hep-th/0412335](#)] [[INSPIRE](#)].
- [36] A. Bissi and T. Lukowski, *Revisiting $\mathcal{N} = 4$ superconformal blocks*, *JHEP* **02** (2016) 115 [[arXiv:1508.02391](#)] [[INSPIRE](#)].
- [37] R. Doobary and P. Heslop, *Superconformal partial waves in Grassmannian field theories*, *JHEP* **12** (2015) 159 [[arXiv:1508.03611](#)] [[INSPIRE](#)].
- [38] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys. B* **599** (2001) 459 [[hep-th/0011040](#)] [[INSPIRE](#)].
- [39] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, *R current correlators in $\mathcal{N} = 4$ super Yang-Mills theory from anti-de Sitter supergravity*, *Nucl. Phys. B* **540** (1999) 247 [[hep-th/9805105](#)] [[INSPIRE](#)].
- [40] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Four point functions in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at two loops*, *Nucl. Phys. B* **557** (1999) 355 [[hep-th/9811172](#)] [[INSPIRE](#)].
- [41] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Simplifications of four point functions in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at two loops*, *Phys. Lett. B* **466** (1999) 20 [[hep-th/9906051](#)] [[INSPIRE](#)].
- [42] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Anomalous dimensions in $\mathcal{N} = 4$ SYM theory at order g^4* , *Nucl. Phys. B* **584** (2000) 216 [[hep-th/0003203](#)] [[INSPIRE](#)].
- [43] B. Eden, *Three-loop universal structure constants in $\mathcal{N} = 4$ SUSY Yang-Mills theory*, [arXiv:1207.3112](#) [[INSPIRE](#)].
- [44] J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington and V.A. Smirnov, *Leading singularities and off-shell conformal integrals*, *JHEP* **08** (2013) 133 [[arXiv:1303.6909](#)] [[INSPIRE](#)].
- [45] S. Caron-Huot and J.M. Henn, *Iterative structure of finite loop integrals*, *JHEP* **06** (2014) 114 [[arXiv:1404.2922](#)] [[INSPIRE](#)].
- [46] L.F. Alday, B. Eden, G.P. Korchemsky, J. Maldacena and E. Sokatchev, *From correlation functions to Wilson loops*, *JHEP* **09** (2011) 123 [[arXiv:1007.3243](#)] [[INSPIRE](#)].
- [47] T. Adamo, M. Bullimore, L. Mason and D. Skinner, *A Proof of the Supersymmetric Correlation Function/Wilson Loop Correspondence*, *JHEP* **08** (2011) 076 [[arXiv:1103.4119](#)] [[INSPIRE](#)].
- [48] B. Eden, G.P. Korchemsky and E. Sokatchev, *From correlation functions to scattering amplitudes*, *JHEP* **12** (2011) 002 [[arXiv:1007.3246](#)] [[INSPIRE](#)].
- [49] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: Part I*, *Nucl. Phys. B* **869** (2013) 329 [[arXiv:1103.3714](#)] [[INSPIRE](#)].
- [50] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: Part II*, *Nucl. Phys. B* **869** (2013) 378 [[arXiv:1103.4353](#)] [[INSPIRE](#)].