Published for SISSA by Springer

Received: July 14, 2015
Accepted: July 31, 2015
Published: August 26, 2015

Implications of $\mathcal{N} = 4$ superconformal symmetry in three spacetime dimensions

Evgeny I. Buchbinder, Sergei M. Kuzenko and Igor B. Samsonov[1]

School of Physics M013, The University of Western Australia,
35 Stirling Highway, Crawley W.A. 6009, Australia

E-mail: evgeny.buchbinder@uwa.edu.au, sergei.kuzenko@uwa.edu.au, igor.samsonov@uwa.edu.au

Abstract: We study implications of $\mathcal{N} = 4$ superconformal symmetry in three dimensions, thus extending our earlier results in [1] devoted to the $\mathcal{N} \leq 3$ cases. We show that the three-point function of the supercurrent in $\mathcal{N} = 4$ superconformal field theories contains two linearly independent forms. However, only one of these structures contributes to the three-point function of the energy-momentum tensor and the other one is present in those $\mathcal{N} = 4$ superconformal theories which are not invariant under the mirror map. We point out that general $\mathcal{N} = 4$ superconformal field theories admit two inequivalent flavour current multiplets and show that the three-point function of each of them is determined by one tensor structure. As an example, we compute the two- and three-point functions of the conserved currents in $\mathcal{N} = 4$ superconformal models of free hypermultiplets. We also derive the *universal* relations between the coefficients appearing in the two- and three-point correlators of the supercurrent and flavour current multiplets in all superconformal theories with $\mathcal{N} \leq 4$ supersymmetry. Our derivation is based on the use of Ward identities in conjunction with superspace reduction techniques.

Keywords: Extended Supersymmetry, Superspaces

ArXiv ePrint: 1507.00221

Dedicated to the memory of Professor Boris M. Zupnik

[1]On leave from Tomsk Polytechnic University, 634050 Tomsk, Russia.

doi:10.1007/JHEP08(2015)125

Contents

1 Introduction

In our recent work [1], the two- and three-point correlation functions of the supercurrent and flavour current multiplets have been computed for three-dimensional (3D) $\mathcal{N}$-extended superconformal field theories with $1 \leq \mathcal{N} \leq 3$. Here we extend the analysis of [1] to the $\mathcal{N} = 4$ case. We also study the reduction of correlation functions in $\mathcal{N}$-extended superconformal field theory to $(\mathcal{N} - 1)$-extended superspace.

In two dimensions, $\mathcal{N} = 3$ supersymmetry automatically implies $\mathcal{N} = 4$ [2, 3] for nonlinear σ-models.[1] What about three dimensions? As far as the supersymmetric nonlinear σ-models are concerned, 3D $\mathcal{N} = 3$ supersymmetry again implies $\mathcal{N} = 4$. Indeed, the proof given in [2, 3] remains valid in three dimensions. Moreover, off-shell $\mathcal{N} = 3$ supersymmetric σ-models can be shown to possess off-shell $\mathcal{N} = 4$ supersymmetry [4]. Analogous results hold for $\mathcal{N} = 3$ super Yang-Mills theories with matter [5]. However, a rather counter-intuitive situation occurs with parity odd Chern-Simons terms. The $\mathcal{N} = 3$ Chern-Simons action[2] exists for any gauge group [6–8]. On the other hand, it is well known that no $\mathcal{N} = 4$ supersymmetric Chern-Simons action can be constructed (for a recent proof, see [9]), although abelian $\mathcal{N} = 4$ BF couplings are abundant [10].

The R-symmetry group is (locally isomorphic to) $\mathsf{SU}(2)$ in $\mathcal{N} = 3$ supersymmetry and $\mathsf{SU}(2)_{\rm L} \times \mathsf{SU}(2)_{\rm R}$ for $\mathcal{N} = 4$. This difference implies that there are two inequivalent $\mathcal{N} = 4$ gauge multiplets [10–12] and two inequivalent $\mathcal{N} = 4$ hypermultiplets [11, 12], as compared with a single vector multiplet and a single hypermultiplet in $\mathcal{N} = 3$ supersymmetry. The

[1] The proof given in [2, 3] is as follows. It is known that $\mathcal{N}$-extended supersymmetry requires the existence of $\mathcal{N} - 1$ anti-commuting complex structures for the σ-model target space. In the $\mathcal{N} = 3$ case, the target space has two such structures, I and J. Their product, $K := I\,J$, is a third complex structure which anti-commutes with I and J, and therefore the σ-model is $\mathcal{N} = 4$ supersymmetric.

[2] The $\mathcal{N} = 3$ Chern-Simons action was constructed for the first time by Zupnik and Hetselius [6] in 3D $\mathcal{N} = 3$ harmonic superspace, and several years later it was re-discovered [7, 8] at the component level.

JHEP08(2015)125

inequivalent $\mathcal{N} = 4$ vector multiplets obey different off-shell constraints and transform in different representations of the R-symmetry group, and similarly for the inequivalent hypermultiplets.[3]

The doubling of gauge and matter multiplets in $\mathcal{N} = 4$ supersymmetry has important implications for the structure of $\mathcal{N} = 4$ superconformal field theory. First of all, there are two inequivalent $\mathcal{N} = 4$ flavour current multiplets whereas there is only one in superconformal models with $\mathcal{N} = 1, 2, 3$ supersymmetry which were considered in [1]. Secondly, we will demonstrate that the three-point function of the $\mathcal{N} = 4$ supercurrent has two independent structures, as compared with a single structure in the $\mathcal{N} = 1, 2, 3$ cases studied in [1].

The zoo of $\mathcal{N} = 4$ superconformal field theories in three dimensions is pretty large. The trivial examples of such theories are provided by models of free hypermultiplets. More interesting are interacting models of hypermultiplets coupled to vector multiplets, with BF couplings for the vector multiplets. The non-abelian $\mathcal{N} = 4$ superconformal field theories include the Gaiotto-Witten models [13] and their generalisations [14]. For all abelian $\mathcal{N} = 4$ superconformal field theories, there exist off-shell realisations. As concerns the non-abelian $\mathcal{N} = 4$ superconformal theories proposed in [13, 14], it is not yet known how to formulate them in $\mathcal{N} = 4$ superspace, which is an interesting open problem. There also exist Chern-Simons-matter theories with $\mathcal{N} = 6$ [15–17] and $\mathcal{N} = 8$ [18–20] superconformal symmetry. The correlation functions of certain conserved currents in these theories can be studied using the $\mathcal{N} = 4$ superfield methods developed in the present paper.

This paper is organised as follows. In section 2 we give a brief review of the superconformal building blocks for the two- and three-point correlation functions in 3D $\mathcal{N}$-extended superspace following the conventions and notation used in [1]. We also elaborate on those properties of the building blocks which are specific to the $\mathcal{N} = 3$ and $\mathcal{N} = 4$ cases. In section 3 we develop a new representation for the correlation functions of the $\mathcal{N} = 3$ flavour current multiplets originally computed in [1]. This representation allows us to easily upgrade the $\mathcal{N} = 3$ flavour current correlators to the $\mathcal{N} = 4$ ones which are derived in section 4. Here we also construct two- and three-point functions of the $\mathcal{N} = 4$ supercurrent and demonstrate that the latter involves two independent tensor structures which distinguish the $\mathcal{N} = 4$ supercurrent correlators from those in the $\mathcal{N} = 1, 2, 3$ cases. In section 5 we consider a particular example of $\mathcal{N} = 4$ superconformal field theories given by the model of free $\mathcal{N} = 4$ hypermultiplets for which we explicitly compute the correlation functions of the supercurrent and the flavour current multiplets. For this model we find important relations between the coefficients in the two- and three-point functions which are interpreted as the manifestations of Ward identities for these correlators. We argue that though these relations between the coefficients are found for the particular model of free hypermultiplets, they hold for generic $\mathcal{N} = 4$ superconformal field theories as well. Section 6 is devoted to the derivation of the Ward identities for the $1 \leq \mathcal{N} \leq 4$ flavour current multiplets. In section 7 we uncover various relations between the coefficients in the two-point and three-point functions both for the supercurrents and flavour current multiplets for all $\mathcal{N} \leq 4$. Finally, in section 8 we discuss the results and some open problems.

[3]The inequivalent vector multiplets and hypermultiplets can be described in terms of superfields that are defined on two different supersymmetric subspaces of the $\mathcal{N} = 4$ harmonic superspace [11, 12].

JHEP08(2015)125

The main body of the paper is accompanied by three technical appendices. Appendix A is devoted to a brief review of 3D off-shell $\mathcal{N}=4$ multiplets. In appendix B we use the 3D $\mathcal{N}=4$ harmonic superspace approach to derive a new representation for the q^+ hypermultiplet propagator which is important in studying the implications of the Ward identity for the correlation functions of the $\mathcal{N}=4$ flavour current multiplets. In appendix C we collect some details of the reduction of the $\mathcal{N}=4$ correlation functions computed in this paper to the $\mathcal{N}=3$ and $\mathcal{N}=2$ superspaces.

2 Superconformal building blocks

This section contains a brief summary of those results in [1] which are necessary for our subsequent analysis. In addition, we elaborate on specific technical features of the $\mathcal{N}=3$ and $\mathcal{N}=4$ cases.

2.1 Superconformal transformations and primary superfields

Consider $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ parametrised by real bosonic (x^a) and fermionic (θ^α_I) coordinates

$$z^A=(x^a,\theta^\alpha_I), \qquad a=0,1,2\,, \qquad \alpha=1,2\,, \qquad I=1,\dots,\mathcal{N}\,.$$

Here the indices 'a' and 'α' are Lorentz and spinor ones, respectively, while 'I' is the R-symmetry index. The 3D $\mathcal{N}$-extended superconformal group $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ cannot be realised to act by smooth transformations on $\mathbb{M}^{3|2\mathcal{N}}$. However, a transitive action of $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ is naturally defined on the so-called compactified Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ in which $\mathbb{M}^{3|2\mathcal{N}}$ is embedded as a dense open domain [4]. In general, only infinitesimal superconformal transformations are well defined on $\mathbb{M}^{3|2\mathcal{N}}$. Such a transformation

$$\delta z^A=\xi z^A \quad\Longleftrightarrow\quad \delta x^a=\xi^a(z)+\mathrm{i}\,\xi^\alpha_I(z)\theta^\beta_I(\gamma^a)_{\alpha\beta}\,, \qquad \delta\theta^\alpha_I=\xi^\alpha_I(z) \tag{2.1}$$

is associated with an even real supervector field on $\mathbb{M}^{3|2\mathcal{N}}$,

$$\xi=\xi^A D_A=\xi^a\partial_a+\xi^\alpha_I D^I_\alpha=-\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta}+\xi^\alpha_I D^I_\alpha\,, \qquad \overline{\xi^A}=\xi^A\,, \tag{2.2}$$

which obeys the equation $[\xi,D^I_\alpha]\propto D^J_\beta$. All solutions of this equation are called the conformal Killing supervector fields of Minkowski superspace. They span a Lie superalgebra (with respect to the standard Lie bracket $[\xi_1,\xi_2]$) that is isomorphic to the superconformal algebra $\mathfrak{osp}(\mathcal{N}|4;\mathbb{R})$.

Explicit expressions for the components $\xi^A=(\xi^a,\xi^\alpha_I)$ of the most general conformal Killing supervector fields are given by eq. (4.4) in [1]. Equivalent results were derived earlier by Park [21] and later in [4]. In the present paper, we will not need these explicit expressions. For our analysis, it suffices to use the relation

$$[\xi,D^I_\alpha]=-(D^I_\alpha\xi^\beta_J)D^J_\beta=\lambda_\alpha{}^\beta(z)D^I_\beta+\Lambda^{IJ}(z)D^J_\alpha-\frac{1}{2}\sigma(z)D^I_\alpha\,. \tag{2.3}$$

JHEP08(2015)125

Here the superfield parameters on the right are expressed in terms of ξ^A as follows:

$$\lambda_{\alpha\beta}(z) = -\frac{1}{\mathcal{N}} D^I_{(\alpha}\xi^I_{\beta)}\,, \quad \Lambda^{IJ}(z) = -2D^{[I}_\alpha \xi^{J]\alpha}\,, \quad \sigma(z) = \frac{1}{\mathcal{N}} D^I_\alpha \xi^\alpha_I = \frac{1}{3}\partial_a \xi^a\,. \tag{2.4}$$

One may think of $\lambda_{\alpha\beta}(z)$, $\Lambda^{IJ}(z)$ and $\sigma(z)$ as the parameters of special local Lorentz, R-symmetry and scale transformations, respectively, due to their action on the covariant derivative given by (2.3). The same interpretation is supported by the explicit expressions for $\lambda_{\alpha\beta}(z)$, $\Lambda^{IJ}(z)$ and $\sigma(z)$ as polynomials in z^A:

$$\lambda^{\alpha\beta}(z) = \lambda^{\alpha\beta} - x^{\gamma(\alpha}b^{\beta)}_\gamma - \frac{\mathrm{i}}{2} b^{\alpha\beta}\theta^\gamma_I \theta_{I\gamma} + 2\mathrm{i}\eta^{(\alpha}_I \theta^{\beta)}_I\,, \tag{2.5a}$$

$$\Lambda_{IJ}(z) = \Lambda_{IJ} + 4\mathrm{i}\eta^\alpha_{[I}\theta_{J]\alpha} + 2\mathrm{i}b_{\alpha\beta}\theta^\alpha_I \theta^\beta_J\,, \tag{2.5b}$$

$$\sigma(z) = \sigma + b_{\alpha\beta}x^{\beta\alpha} + 2\mathrm{i}\theta^\alpha_I \eta_{I\alpha}\,. \tag{2.5c}$$

Here the constant parameters $\lambda_{\alpha\beta}$, Λ^{IJ} and σ correspond to the Lorentz, R-symmetry and scale transformations from $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$, while $b^{\alpha\beta}$ and $\eta_{I\alpha}$ generate the special conformal and S-supersymmetry transformations.

It is the z-dependent parameters (2.4) which appear, along with ξ itself, in the superconformal transformation law[4] of a primary tensor superfield of dimension q

$$\delta\Phi^{\mathcal{I}}_{\mathcal{A}} = -\xi\Phi^{\mathcal{I}}_{\mathcal{A}} - q\,\sigma(z)\Phi^{\mathcal{I}}_{\mathcal{A}} + \lambda^{\alpha\beta}(z)(M_{\alpha\beta})_{\mathcal{A}}{}^{\mathcal{B}}\Phi^{\mathcal{I}}_{\mathcal{B}} + \Lambda_{IJ}(z)(R^{IJ})^{\mathcal{I}}{}_{\mathcal{J}}\Phi^{\mathcal{J}}_{\mathcal{A}}\,. \tag{2.6}$$

Here $\Phi^{\mathcal{I}}_{\mathcal{A}}$ is assumed to transform in some representations of the Lorentz and R-symmetry groups with respect to its indices '$\mathcal{A}$' and '$\mathcal{I}$', respectively. The matrices $M_{\alpha\beta}$ and R^{IJ} in (2.6) are the Lorentz and $\mathsf{SO}(\mathcal{N})$ generators, respectively. It should be mentioned that the R-symmetry subgroup of $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ is $\mathsf{O}(\mathcal{N})$. In what follows, its connected component of the identity, $\mathsf{SO}(\mathcal{N})$, will be referred to as the R-symmetry group.

Consider a correlation function $\langle\Phi_1(z_1)\dots\Phi_n(z_n)\rangle$ of several primary superfields Φ_1, $\dots$, Φ_n (with their indices suppressed) that originate in some superconformal field theory. In terms of this correlation function, the statement of superconformal invariance is

$$\sum_{k=1}^{n}\langle\Phi_1(z_1)\dots\delta\Phi_k(z_k)\dots\Phi_n(z_n)\rangle = 0\,. \tag{2.7}$$

2.2 Two-point functions

In ordinary conformal field theory in d dimensions, a comprehensive discussion of the building blocks for the two- and three-point correlation functions of primary fields was given by Osborn and Petkou [23] who built on the earlier works by Mack [24] and others [25–30]. Their analysis was extended to superconformal field theories formulated in superspace by Osborn and Park [21, 31–33].

In the case of 3D superconformal field theories, the building blocks for the two- and three-point correlation functions were derived first in [21] using the coset construction for

[4] The transformation law (2.6) is a 3D super-extension of the Mack-Salam construction [22].

$\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ and more recently in [1] using the supertwistor approach. All building blocks are composed of the following two-point structures:

$$\boldsymbol{x}_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_{1I}^{(\alpha}\theta_{2I}^{\beta)} - \mathrm{i}\theta_{12I}^{\alpha}\theta_{12I}^{\beta}\,, \tag{2.8a}$$

$$\theta_{12I}^{\alpha} = (\theta_1 - \theta_2)_I^{\alpha}\ . \tag{2.8b}$$

The former transforms homogeneously at z_1 and z_2,

$$\widetilde{\delta}\boldsymbol{x}_{12}^{\alpha\beta} = \left(\frac{1}{2}\delta^{\alpha}{}_{\gamma}\sigma(z_1) - \lambda^{\alpha}{}_{\gamma}(z_1)\right)\boldsymbol{x}_{12}^{\gamma\beta} + \boldsymbol{x}_{12}^{\alpha\gamma}\left(\frac{1}{2}\delta_{\gamma}{}^{\beta}\sigma(z_2) - \lambda_{\gamma}{}^{\beta}(z_2)\right)\,, \tag{2.9a}$$

while the latter involves an inhomogeneous piece in its transformation law,

$$\widetilde{\delta}\theta_{12I}^{\alpha} = \left(\frac{1}{2}\delta^{\alpha}{}_{\beta}\sigma(z_1) - \lambda^{\alpha}{}_{\beta}(z_1)\right)\theta_{12I}^{\beta} + \Lambda_{IJ}(z_2)\theta_{12J}^{\alpha} - \boldsymbol{x}_{12}^{\alpha\beta}\eta_{I\beta}(z_2)\,, \tag{2.9b}$$

with $\eta_{I\alpha}(z) := -\frac{\mathrm{i}}{2}D_{\alpha}^{I}\sigma(z) = \eta_{I\alpha} - b_{\alpha\beta}\theta_I^{\beta}$. Here the variation $\widetilde{\delta}$ is defined to act on an arbitrary n-point function $\mathcal{O}(z_1,\dots,z_n)$ by the rule

$$\widetilde{\delta}\mathcal{O}(z_1,\dots,z_n) = \sum_{i=1}^{n}\xi_{z_i}\mathcal{O}(z_1,\dots,z_n)\ . \tag{2.10}$$

As follows from (2.6), each primary superfield Φ is determined by the following data: (i) its dimension q; (ii) the representation T of the Lorentz group to which Φ belongs; and (iii) the representation D of $\mathsf{SO}(\mathcal{N})$ in which Φ transforms. There are three building blocks, which are descendants of (2.8) and which take care of the above data in the correlation functions of primary superfields.

Firstly, using (2.8a) we define the scalar two-point function

$$\boldsymbol{x}_{12}{}^2 := -\frac{1}{2}\boldsymbol{x}_{12}^{\alpha\beta}\boldsymbol{x}_{12\alpha\beta} \tag{2.11}$$

with the transformation law

$$\widetilde{\delta}\boldsymbol{x}_{12}{}^2 = \Big(\sigma(z_1) + \sigma(z_2)\Big)\boldsymbol{x}_{12}{}^2\ . \tag{2.12}$$

In general, the correlation functions contain multiplicative factors proportional to powers of $\boldsymbol{x}_{12}{}^2$ in such a way as to guarantee the right scaling properties.

Secondly, the Lorentz structure of the primary fields in correlation functions is taken care of by the 2×2 matrix

$$\hat{\underline{\boldsymbol{x}}}_{12} := \frac{\hat{\boldsymbol{x}}_{12}}{\sqrt{-\boldsymbol{x}_{12}{}^2}}\,, \qquad (\varepsilon\hat{\underline{\boldsymbol{x}}}_{12})^2 = \mathbb{1}_2\,, \tag{2.13}$$

where we have used the matrix notation $\hat{\boldsymbol{x}}_{12} = (\boldsymbol{x}_{12}^{\alpha\beta})$ and $\varepsilon = (\varepsilon_{\alpha\beta})$. Its transformation law is

$$\widetilde{\delta}\underline{\boldsymbol{x}}_{12}^{\alpha\beta} = -\lambda^{\alpha}{}_{\gamma}(z_1)\,\underline{\boldsymbol{x}}_{12}^{\gamma\beta} - \underline{\boldsymbol{x}}_{12}^{\alpha\gamma}\,\lambda_{\gamma}{}^{\beta}(z_2)\ . \tag{2.14}$$

Thirdly, the $\mathsf{SO}(\mathcal{N})$ structure of the primary fields in correlation functions is taken care of by the $\mathcal{N}\times\mathcal{N}$ matrix

$$u_{12} = (u_{12}^{IJ}) , \qquad u_{12}^{IJ} = \delta^{IJ} + 2\mathrm{i}\theta_{12}^{\alpha I}(\boldsymbol{x}_{12}^{-1})_{\alpha\beta}\theta_{12}^{\beta J} , \tag{2.15}$$

where

$$(\boldsymbol{x}_{12}^{-1})_{\alpha\beta} = -\frac{\boldsymbol{x}_{12\beta\alpha}}{\boldsymbol{x}_{12}{}^2} \tag{2.16}$$

is the inverse for $(\boldsymbol{x}_{12})^{\alpha\beta}$, that is $(\boldsymbol{x}_{12}^{-1})_{\alpha\beta}(\boldsymbol{x}_{12})^{\beta\gamma} = \delta_\alpha^\gamma$. One may check that the matrix u_{12} is orthogonal and unimodular,

$$u_{12}^{\mathrm{T}} u_{12} = \mathbb{1}_{\mathcal{N}} , \qquad \det u_{12} = 1 . \tag{2.17}$$

It follows from (2.9) that

$$\tilde{\delta} u_{12}^{IJ} = \Lambda^{IK}(z_1) u_{12}^{KJ} - u_{12}^{IK}\Lambda^{KJ}(z_2) . \tag{2.18}$$

The above properties provide the rationale why u_{12}^{IJ} naturally arises in correlation functions of primary superfields with $\mathsf{SO}(\mathcal{N})$ indices.

The two-point correlation function of the primary superfield $\Phi_{\mathcal{A}}^{\mathcal{I}}$ and its conjugate $\bar{\Phi}_{\mathcal{I}}^{\mathcal{A}}$ is fixed by the superconformal symmetry up to a single coefficient c and has the form

$$\langle \Phi_{\mathcal{A}}^{\mathcal{I}}(z_1)\bar{\Phi}_{\mathcal{J}}^{\mathcal{B}}(z_2)\rangle = c\frac{T_{\mathcal{A}}{}^{\mathcal{B}}(\varepsilon\hat{\boldsymbol{x}}_{12})D^{\mathcal{I}}{}_{\mathcal{J}}(u_{12})}{(\boldsymbol{x}_{12}{}^2)^q} \tag{2.19}$$

provided the representations T and D are irreducible. The denominator in (2.19) is fixed by the dimension of Φ.

Before turning to three-point building blocks, it should be pointed out that the two-point structure $\boldsymbol{x}_{12}^{\alpha\beta}$ defined by (2.8a) has the following symmetry property

$$\boldsymbol{x}_{21}^{\alpha\beta} = -\boldsymbol{x}_{12}^{\beta\alpha} . \tag{2.20}$$

It can be decomposed into its symmetric and antisymmetric parts,

$$\boldsymbol{x}_{12}^{\alpha\beta} = x_{12}^{\alpha\beta} + \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\theta_{12}{}^2 , \tag{2.21}$$

where

$$\theta_{12}{}^2 := \theta_{12I}^{\alpha}\theta_{12I\alpha} , \qquad x_{12}^{\alpha\beta} = x_{12}^{\beta\alpha} := (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_{1I}^{(\alpha}\theta_{2I}^{\beta)} . \tag{2.22}$$

As is seen from (2.9a), the two-point structure $x_{12}^{\alpha\beta}$ does not transform homogeneously, unlike $\boldsymbol{x}_{12}^{\alpha\beta}$. However, in practice it is often useful to deal with $x_{12}^{\alpha\beta}$ for concrete calculations.

2.3 Three-point functions

Associated with three superspace points z_1, z_2 and z_3 are the following three-point structures:

$$\boldsymbol{X}_{1\alpha\beta} = -(\boldsymbol{x}_{21}^{-1})_{\alpha\gamma}\boldsymbol{x}_{23}^{\gamma\delta}(\boldsymbol{x}_{13}^{-1})_{\delta\beta} , \tag{2.23a}$$

$$\Theta_{1\alpha}^{I} = (\boldsymbol{x}_{21}^{-1})_{\alpha\beta}\theta_{12}^{I\beta} - (\boldsymbol{x}_{31}^{-1})_{\alpha\beta}\theta_{13}^{I\beta} , \tag{2.23b}$$

$$U_1^{IJ} = u_{12}^{IK} u_{23}^{KL} u_{31}^{LJ} \ . \tag{2.23c}$$

They transform as tensors at the point z_1

$$\begin{aligned}
\tilde{\delta}\boldsymbol{X}_{1\alpha\beta} &= \lambda_\alpha{}^\gamma(z_1)\boldsymbol{X}_{1\gamma\beta} + \boldsymbol{X}_{1\alpha\gamma}\lambda^\gamma{}_\beta(z_1) - \sigma(z_1)\boldsymbol{X}_{1\alpha\beta} \,, && (2.24a)\\
\tilde{\delta}\Theta^I_{1\alpha} &= \left(\lambda_\alpha{}^\beta(z_1) - \frac{1}{2}\delta_\alpha{}^\beta\sigma(z_1)\right)\Theta^I_{1\beta} + \Lambda^{IJ}(z_1)\Theta^J_{1\alpha} \,, && (2.24b)\\
\tilde{\delta}U_1^{IJ} &= \Lambda^{IK}(z_1)U_1^{KJ} - U_1^{IK}\Lambda^{KJ}(z_1) \ . && (2.24c)
\end{aligned}$$

These objects have many properties resembling those of the two-point functions. In particular, the tensor (2.23a) can be decomposed into symmetric and antisymmetric parts similar to (2.21),

$$\boldsymbol{X}_{1\alpha\beta} = X_{1\alpha\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}{\Theta_1}^2 \,, \tag{2.25}$$

where the symmetric spinor $X_{1\alpha\beta} = X_{1\beta\alpha}$ is equivalently represented as a three-vector $X_{1m} = -\frac{1}{2}\gamma_m^{\alpha\beta}X_{1\alpha\beta}$.

Next, the matrix (2.23c) can be expressed in terms of (2.23a) and (2.23b) similarly to (2.15):

$$U_1^{IJ} = \delta^{IJ} + 2\mathrm{i}\Theta^I_{1\alpha}(\boldsymbol{X}_1^{-1})^{\alpha\beta}\Theta^J_{1\beta} = \delta^{IJ} - 2\mathrm{i}\frac{\Theta^I_{1\alpha}\boldsymbol{X}_1^{\beta\alpha}\Theta^J_{1\beta}}{{\boldsymbol{X}_1}^2} \ . \tag{2.26}$$

The matrix $U_1 = (U_1^{IJ})$ is orthogonal and unimodular.

We point out that in (2.23) we have defined the three-point structures which transform as tensors at the point z_1. Performing cyclic permutations of the superspace points z_1, z_2 and z_3 in (2.23) one obtains similar objects which transform as tensors at the superspace points z_2 and z_3. The three-point structures at different superspace points are related to each other as follows

$$\begin{aligned}
\boldsymbol{x}_{13}^{\alpha\alpha'}\boldsymbol{X}_{3\alpha'\beta'}\boldsymbol{x}_{31}^{\beta'\beta} &= -(\boldsymbol{X}_1^{-1})^{\beta\alpha} = \frac{\boldsymbol{X}_1^{\alpha\beta}}{{\boldsymbol{X}_1}^2} \,, && (2.27a)\\
\Theta^I_{1\gamma}\boldsymbol{x}_{13}^{\gamma\delta}\boldsymbol{X}_{3\delta\beta} &= u_{13}^{IJ}\Theta^J_{3\beta} \,, && (2.27b)\\
U_3^{IJ} &= u_{31}^{IK}U_1^{KL}u_{13}^{LJ} \ . && (2.27c)
\end{aligned}$$

Various primary superfields, including the supercurrent, obey certain differential constraints. In order to take into account these constraints in correlation functions, we need rules to evaluate covariant derivatives of the variables (2.23a) and (2.23b) and also those obtained from them by cyclic permutations of the superspace points z_1, z_2 and z_3. Given a function $f(\boldsymbol{X}_3, \Theta_3)$, one can prove the following differential identities:

$$\begin{aligned}
D^I_{(1)\gamma}f(\boldsymbol{X}_3,\Theta_3) &= (\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}u_{13}^{IJ}\mathcal{D}^{J\alpha}_{(3)}f(\boldsymbol{X}_3,\Theta_3) \,, && (2.28a)\\
D^I_{(2)\gamma}f(\boldsymbol{X}_3,\Theta_3) &= \mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\alpha\gamma}u_{23}^{IJ}\mathcal{Q}^{J\alpha}_{(3)}f(\boldsymbol{X}_3,\Theta_3) \,, && (2.28b)
\end{aligned}$$

where we have introduced the operators

$$\mathcal{D}^I_{(3)\alpha} = \frac{\partial}{\partial\Theta^\alpha_{3I}} + \mathrm{i}\gamma^m_{\alpha\beta}\Theta^{I\beta}_3\frac{\partial}{\partial X_3^m} \,, \qquad \mathcal{Q}^I_{(3)\alpha} = \mathrm{i}\frac{\partial}{\partial\Theta^\alpha_{3I}} + \gamma^m_{\alpha\beta}\Theta^{I\beta}_3\frac{\partial}{\partial X_3^m} \ . \tag{2.29}$$

Let Φ, Ψ and Π be primary superfields (with indices suppressed) of dimensions q_1, q_2 and q_3, respectively. The three-point correlation function for these superfields can be found with the use of the ansatz

$$\langle \Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Psi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{B}_1}(\varepsilon\hat{\underline{\boldsymbol{x}}}_{13})T^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{B}_2}(\varepsilon\hat{\underline{\boldsymbol{x}}}_{23})D^{(1)\mathcal{I}_1}{}_{\mathcal{J}_1}(u_{13})D^{(2)\mathcal{I}_2}{}_{\mathcal{J}_2}(u_{23})}{(\boldsymbol{x}_{13}{}^2)^{q_1}(\boldsymbol{x}_{23}{}^2)^{q_2}} \times H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\boldsymbol{X}_3,\Theta_3,U_3)\,, \tag{2.30}$$

where $H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}$ is a tensor constructed in terms of the three-point functions (2.23). The functional form of this tensor is highly constrained by the following conditions:

(i) It should obey the scaling property

$$H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\lambda^2\boldsymbol{X},\lambda\Theta,U) = (\lambda^2)^{q_3-q_2-q_1}H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\boldsymbol{X},\Theta,U) \tag{2.31}$$

in order for the correlation function to have the correct transformation law under the superconformal group.

(ii) When some of the superfields Φ, Ψ and Π obey differential equations such as the conservation conditions of conserved current multiplets, the tensor $H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}$ is constrained by certain differential equations as well. In deriving such equations the identities (2.28) may be useful.

(iii) When two of the superfields Φ, Ψ and Π (or all of them) coincide, the tensor H should obey certain constraints originating from the symmetry under permutations of the superfields, e.g.

$$\langle \Phi^{\mathcal{A}}_{\mathcal{I}}(z_1)\Phi^{\mathcal{B}}_{\mathcal{J}}(z_2)\Pi^{\mathcal{C}}_{\mathcal{K}}(z_3)\rangle = (-1)^{\epsilon(\Phi)}\langle \Phi^{\mathcal{B}}_{\mathcal{J}}(z_2)\Phi^{\mathcal{A}}_{\mathcal{I}}(z_1)\Pi^{\mathcal{C}}_{\mathcal{K}}(z_3)\rangle\,, \tag{2.32}$$

where $\epsilon(\Phi)$ is the Grassmann parity of $\Phi^{\mathcal{A}}_{\mathcal{I}}$.

These constraints fix the functional form of the tensor H (and, hence, the three-point correlation function) up to a few arbitrary constants.

2.4 Specific features of the $\mathcal{N}=3$ case

An important feature of the $\mathcal{N}=3$ case is that the R-symmetry group $\mathsf{SO}(3)$ is related to $\mathsf{SU}(2)$ by the isomorphism $\mathsf{SO}(3)\cong\mathsf{SU}(2)/\mathbb{Z}_2$. This isomorphism makes it possible to convert the $\mathsf{SO}(3)$ index of every isovector Z_I into a pair of isospinor ones,

$$Z_I \;\rightarrow\; Z_i{}^j := \frac{\mathrm{i}}{\sqrt{2}}(\vec{Z}\cdot\vec{\sigma})_i{}^j \equiv Z_I(\tau_I)_i{}^j\,, \qquad Z_i{}^i = 0\,, \tag{2.33}$$

with $\vec{\sigma}$ the Pauli matrices.[5]

The isospinor indices will be raised and lowered using the $\mathsf{SU}(2)$ invariant antisymmetric tensors ε_{ij} and ε^{ij} (normalised as $\varepsilon^{12}=\varepsilon_{21}=1$). The rules for raising and lowering the isospinor indices are

$$\psi^i = \varepsilon^{ij}\psi_j\,, \qquad \psi_i = \varepsilon_{ij}\psi^j\ . \tag{2.34}$$

[5]Our definition of the τ-matrices agrees with the one adopted in [4].

In particular, associated with the matrices $(\tau_I)_i{}^j$, eq. (2.33), are the symmetric matrices $(\tau_I)_{ij} = (\tau_I)_{ji}$ and $(\tau_I)^{ij} = (\tau_I)^{ji}$ which are related to each other by complex conjugation:

$$\overline{(\tau_I)_{ij}} = (\tau_I)^{ij} \ . \tag{2.35}$$

If A_I and B_I are $\mathsf{SO}(3)$ vectors and A_{ij} and B_{ij} are the associated symmetric isotensors, then

$$A_I = A_{ij}(\tau_I)^{ij} \,, \qquad A_I B_I = A_{ij} B^{ij} \,, \tag{2.36}$$

in accordance with the identities

$$(\tau_I)_i{}^k(\tau_J)_k{}^j = -\frac{1}{\sqrt{2}}\varepsilon_{IJK}(\tau_K)_i{}^j - \frac{1}{2}\delta_{IJ}\delta_i{}^j \,, \qquad (\tau_I)_{ij}(\tau_I)_{kl} = \frac{1}{2}(\varepsilon_{ik}\varepsilon_{jl} + \varepsilon_{il}\varepsilon_{jk}) \ . \tag{2.37}$$

Given an antisymmetric second-rank $\mathsf{SO}(3)$ tensor, $\Lambda^{IJ} = -\Lambda^{JI}$, for its counterpart with isospinor indices $\Lambda^{ijkl} = -\Lambda^{klij} = \Lambda^{IJ}(\tau_I)^{ij}(\tau_J)^{kl}$ we have

$$\Lambda^{IJ} = -\Lambda^{JI} \quad \Longrightarrow \quad \Lambda^{IJ}(\tau_I)^{ij}(\tau_J)^{kl} = -\varepsilon^{jl}\Lambda^{ik} - \varepsilon^{ik}\Lambda^{jl} \,, \qquad \Lambda^{ij} = \Lambda^{ji} \ . \tag{2.38}$$

Applying the above conversion to the Grassmann coordinates θ^α_I and the spinor covariant derivatives D^I_α gives

$$\theta^\alpha_{ij} = \theta^\alpha_I(\tau_I)_{ij} \,, \qquad D^{ij}_\alpha = (\tau_I)^{ij} D^I_\alpha \,, \tag{2.39}$$

and similarly for the two- and three-point functions (2.8b) and (2.23b)

$$\theta^{ij}_{12\alpha} = (\tau_I)^{ij}\theta^I_{12\alpha} \,, \qquad \Theta^{ij}_{1\alpha} = (\tau_I)^{ij}\Theta^I_{1\alpha} \ . \tag{2.40}$$

The covariant derivatives D^{ij}_α obey the anti-commutation relation

$$\{D^{ij}_\alpha, D^{kl}_\beta\} = -2\mathrm{i}\varepsilon^{i(k}\varepsilon^{l)j}\partial_{\alpha\beta} \ . \tag{2.41}$$

In terms of the superspace coordinates $z^A = (x^a, \theta^\alpha_{ij})$, the explicit realisation of the covariant derivatives is

$$D^{ij}_\alpha = \frac{\partial}{\partial\theta^\alpha_{ij}} + \mathrm{i}\,\theta^{\beta\,ij}\partial_{\alpha\beta} \ . \tag{2.42}$$

The isomorphism $\mathsf{SO}(3) \cong \mathsf{SU}(2)/\mathbb{Z}_2$ implies that associated with the orthogonal unimodular matrix u^{IJ}_{12} given by (2.15) is a unique, up to sign, unitary and unimodular matrix $\mathbf{u}^{ij}_{12}$ such that

$$(\tau_I)^{ii'}(\tau_J)^{jj'}u^{IJ}_{12} = \frac{1}{2}(\mathbf{u}^{ij}_{12}\mathbf{u}^{i'j'}_{12} + \mathbf{u}^{i'j}_{12}\mathbf{u}^{ij'}_{12}) \ . \tag{2.43}$$

The matrix $\mathbf{u}^{ij}_{12}$ can be chosen as

$$\mathbf{u}^{ij}_{12} = -\varepsilon^{ij} - \mathrm{i}\frac{\theta^{ik}_{12\alpha}\boldsymbol{x}^{\beta\alpha}_{12}\theta^j_{12k\beta}}{\boldsymbol{x}_{12}{}^2} + \frac{1}{8}\varepsilon^{ij}\frac{\theta_{12}{}^4}{\boldsymbol{x}_{12}{}^2} \,, \qquad \theta_{12}{}^4 := (\theta^{\alpha ij}_{12}\theta_{12\alpha ij})^2 \ . \tag{2.44}$$

It is easy to check that (2.44) is indeed unitary and unimodular,

$$\mathbf{u}^\dagger_{12}\mathbf{u}_{12} = \mathbb{1}_2 \,, \qquad \det \mathbf{u}_{12} = 1 \,, \tag{2.45}$$

and obeys the equation (2.43) with u^{IJ}_{12} given by (2.15).

JHEP08(2015)125

The transformation law of the orthogonal matrix u_{12}, eq. (2.18), has the following counterpart in terms of the unitary matrix $\mathbf{u}_{12}$:

$$\tilde{\delta}\mathbf{u}_{12}^{ij} = \Lambda^i_k(z_1)\mathbf{u}_{12}^{kj} + \mathbf{u}_{12}^{ik}\Lambda_k^j(z_2) \ . \tag{2.46}$$

Here the symmetric matrix $\Lambda^{ij}(z)$ with isospinor indices is related to the antisymmetric matrix $\Lambda^{IJ}(z)$ with isovector indices, eq. (2.4), according to the general rule (2.38).

Let us introduce one more 2×2 matrix by the rule

$$n_{12}^{ij} = \frac{\mathbf{u}_{12}^{ij}}{\boldsymbol{x}_{12}} = -\frac{\varepsilon^{ij}}{x_{12}} - \mathrm{i}\frac{\theta^{ik}_{12\alpha}x_{12}^{\alpha\beta}\theta^j_{12k\beta}}{{x_{12}}^3} \ . \tag{2.47}$$

The second expression for n_{12}^{ij} is given in terms of the symmetric part of $\boldsymbol{x}_{12}^{\alpha\beta}$ given by (2.22). It may be shown that the two-point function (2.47) obeys the analyticity condition

$$D^{(ij}_{(1)\alpha}n_{12}^{k)l} = 0 \ . \tag{2.48}$$

This is why n_{12}^{ij} appears in the correlation functions of $\mathcal{N}=3$ flavour current multiplets.

Similarly to (2.43), we can represent (2.26) as

$$(\tau_I)^{ii'}(\tau_J)^{jj'}U_1^{IJ} = \frac{1}{2}(\mathbf{U}_1^{ij}\mathbf{U}_1^{i'j'} + \mathbf{U}_1^{i'j}\mathbf{U}_1^{ij'}) \ , \tag{2.49}$$

where we have introduced the matrix

$$\mathbf{U}_1^{ij} = -\varepsilon^{ij} + \mathrm{i}\Theta^{ik}_{1\alpha}(\boldsymbol{X}_1^{-1})^{\alpha\beta}\Theta^j_{1k\beta} + \frac{1}{8}\varepsilon^{ij}\frac{{\Theta_1}^4}{{\boldsymbol{X}_1}^2} \,, \tag{2.50}$$

which can be expressed as a product of three two-point functions (2.44)

$$\mathbf{U}_1^{ij} = -\mathbf{u}_{12}^{ik}\mathbf{u}_{23\,kl}\mathbf{u}_{31}^{lj} \ . \tag{2.51}$$

As a consequence, the transformation law (2.46) implies

$$\tilde{\delta}\mathbf{U}_1^{ij} = \Lambda^i_k(z_1)\mathbf{U}_1^{kj} + \mathbf{U}_1^{ik}\Lambda_k^j(z_1) \ . \tag{2.52}$$

By analogy with (2.47) we introduce the matrix

$$N_1^{ij} = \frac{\mathbf{U}_1^{ij}}{\boldsymbol{X}_1} = -\frac{\varepsilon^{ij}}{X_1} - \mathrm{i}\frac{\Theta^{ik}_{1\alpha}X_1^{\alpha\beta}\Theta^j_{1k\beta}}{{X_1}^3} \,, \tag{2.53}$$

which obeys the analyticity condition

$$\mathcal{D}^{(ij}_{(1)\alpha}N_1^{k)l} = 0 \,, \tag{2.54}$$

where the derivative $\mathcal{D}^{ij}_\alpha$ is related to (2.29) by the rule (2.39).

Here we have only considered the thee-point functions (2.50) and (2.53) which transform as tensors at z_1. Performing cyclic permutations of the superspace points z_1, z_2 and z_3 leads to similar objects which transform as tensors at z_2 and z_3.

JHEP08(2015)125

2.5 Specific features of the $\mathcal{N}=4$ case

In the case of $\mathcal{N}=4$ supersymmetry, the R-symmetry group $\mathsf{SO}(4)$ possesses the isomorphism $\mathsf{SO}(4) \cong \big(\mathsf{SU}(2)_\mathrm{L} \times \mathsf{SU}(2)_\mathrm{R}\big)/\mathbb{Z}_2$ which can be used to convert each $\mathsf{SO}(4)$ vector index into a pair of $\mathsf{SU}(2)$ ones,[6]

$$Z_I = (\vec{Z}, Z_4) \ \rightarrow \ Z_i{}^{\tilde{k}} := \frac{\mathrm{i}}{\sqrt{2}}(\vec{Z}\cdot\vec{\sigma})_i{}^{\tilde{k}} + \frac{1}{\sqrt{2}} Z_4 \delta_i{}^{\tilde{k}} \equiv Z_I (\tau_I)_i{}^{\tilde{k}} \ . \tag{2.55}$$

The index 'I' is an $\mathsf{SO}(4)$ vector one, while the indices 'i' and '$\tilde{i}$' are, respectively, $\mathsf{SU}(2)_\mathrm{L}$ and $\mathsf{SU}(2)_\mathrm{R}$ spinor indices. Given $\mathsf{SU}(2)_\mathrm{L}$ and $\mathsf{SU}(2)_\mathrm{R}$ spinors ψ_i and $\chi_{\tilde{i}}$, respectively, we will raise and lower their indices by using the antisymmetric tensors $\varepsilon^{ij}, \varepsilon_{ij}$ and $\varepsilon^{\tilde{i}\tilde{j}}, \varepsilon_{\tilde{i}\tilde{j}}$ (normalised by $\varepsilon^{12} = \varepsilon_{21} = \varepsilon^{\tilde{1}\tilde{2}} = \varepsilon_{\tilde{2}\tilde{1}} = 1$) according to the rules:

$$\psi^i = \varepsilon^{ij}\psi_j \ , \quad \psi_i = \varepsilon_{ij}\psi^j \ , \qquad \chi^{\tilde{i}} = \varepsilon^{\tilde{i}\tilde{j}}\chi_{\tilde{j}} \ , \quad \chi_{\tilde{i}} = \varepsilon_{\tilde{i}\tilde{j}}\chi^{\tilde{j}} \ . \tag{2.56}$$

The complex conjugation acts on the τ-matrices as

$$\overline{(\tau_I)_{i\tilde{i}}} = (\tau_I)^{i\tilde{i}} = \varepsilon^{ij}\varepsilon^{\tilde{i}\tilde{j}}(\tau_I)_{j\tilde{j}} \ . \tag{2.57}$$

The τ-matrices have the following properties:

$$(\tau_{(I})_{i\tilde{j}}(\tau_{J)})^{j\tilde{j}} = \frac{1}{2}\delta_{IJ}\delta_i^j \ , \qquad (\tau_{(I})_{j\tilde{i}}(\tau_{J)})^{j\tilde{j}} = \frac{1}{2}\delta_{IJ}\delta_{\tilde{i}}^{\tilde{j}} \ , \qquad (\tau_I)_{i\tilde{i}}(\tau^I)_{j\tilde{j}} = \varepsilon_{ij}\varepsilon_{\tilde{i}\tilde{j}} \ . \tag{2.58}$$

The conversion from $\mathsf{SO}(4)$ to $\mathsf{SU}(2)$ indices works as follows. Associated with an $\mathsf{SO}(4)$ vector A_I is the second-rank isospinor $A_{i\tilde{i}}$ defined by

$$A_{i\tilde{i}} := (\tau_I)_{i\tilde{i}}A^I \quad \longleftrightarrow \quad A_I = (\tau_I)^{i\tilde{i}}A_{i\tilde{i}} \ , \tag{2.59}$$

such that

$$A_I B^I = A_{i\tilde{i}}B^{i\tilde{i}} \ . \tag{2.60}$$

Given an antisymmetric second-rank $\mathsf{SO}(4)$ tensor, $A_{IJ} = -A_{JI}$, its counterpart with isospinor indices, $A_{IJ}(\tau^I)_{i\tilde{i}}(\tau^J)_{j\tilde{j}}$, can be decomposed as

$$A_{IJ} = -A_{JI} \ \longrightarrow \ A_{IJ}(\tau^I)_{i\tilde{i}}(\tau^J)_{j\tilde{j}} = \varepsilon_{ij}A_{\tilde{i}\tilde{j}} + \varepsilon_{\tilde{i}\tilde{j}}A_{ij} \ , \quad A_{ij} = A_{ji} \ , \quad A_{\tilde{i}\tilde{j}} = A_{\tilde{j}\tilde{i}} \ . \tag{2.61a}$$

We also have

$$A_{IJ}(\tau^I)^{i\tilde{i}}(\tau^J)^{j\tilde{j}} = -\varepsilon^{ij}A^{\tilde{i}\tilde{j}} - \varepsilon^{\tilde{i}\tilde{j}}A^{ij} \ . \tag{2.61b}$$

Applying the conversion rule to the Grassmann variables θ^I_α and covariant derivatives D^I_α gives $\theta^{i\tilde{i}}_\alpha = (\tau_I)^{i\tilde{i}}\theta^I_\alpha$ and $D^{i\tilde{i}}_\alpha = (\tau_I)^{i\tilde{i}}D^I_\alpha$, respectively. For the two- and tree-point functions (2.8b) and (2.23b), the same rule gives $\theta^{i\tilde{i}}_{12\alpha} = (\tau_I)^{i\tilde{i}}\theta^I_{12\alpha}$ and $\Theta^{i\tilde{i}}_{1\alpha} = (\tau_I)^{i\tilde{i}}\Theta^I_{1\alpha}$.

The covariant derivatives $D^{i\tilde{k}}_\alpha$ satisfy the anti-commutation relations

$$\{D^{i\tilde{k}}_\alpha, D^{j\tilde{l}}_\beta\} = 2\mathrm{i}\varepsilon^{ij}\varepsilon^{\tilde{k}\tilde{l}}\partial_{\alpha\beta} \ . \tag{2.62}$$

[6] Our definition of the τ-matrices agrees with the one used in [4] and differs from that adopted in [34].

JHEP08(2015)125

In terms of the superspace coordinates $z^A = (x^a, \theta^\alpha_{k\tilde{l}})$, the explicit realisation of the covariant derivatives is

$$D^{k\tilde{l}}_\alpha = \frac{\partial}{\partial\theta^\alpha_{k\tilde{l}}} + \mathrm{i}\,\theta^{\beta k\tilde{l}}\partial_{\alpha\beta} \ . \tag{2.63}$$

Due to the isomorphism $\mathsf{SO}(4) \cong \big(\mathsf{SU}(2)_\mathrm{L} \times \mathsf{SU}(2)_\mathrm{R}\big)/\mathbb{Z}_2$, the orthogonal matrix u^{IJ}_{12} given by (2.15) is equivalent to a pair of $\mathsf{SU}(2)$ matrices $\mathbf{u}^{ij}_{12}$ and $\mathbf{u}^{\tilde{i}\tilde{j}}_{12}$ constrained by

$$u^{IJ}_{12}(\tau_I)^{i\tilde{i}}(\tau_J)^{j\tilde{j}} = \mathbf{u}^{ij}_{12}\mathbf{u}^{\tilde{i}\tilde{j}}_{12} \ . \tag{2.64}$$

The solution to this equation is

$$\begin{aligned}
\mathbf{u}^{ij}_{12} &= -\varepsilon^{ij} - \mathrm{i}\frac{\theta^{\tilde{k}i}_{12\alpha}x^{\alpha\beta}_{12}\theta^j_{12\beta\tilde{k}}}{{x_{12}}^2} - \frac{1}{8}\frac{\varepsilon^{ij}{\theta_{12}}^4}{{x_{12}}^2} + \frac{1}{8}\frac{\varepsilon^{ij}\tilde{\theta}^4_{12}}{{x_{12}}^4} \\
&\quad -\frac{\mathrm{i}}{8}\frac{\theta^{\tilde{k}i}_\alpha x^{\alpha\beta}_{12}\theta^j_{12\beta\tilde{k}}{\theta_{12}}^4}{{x_{12}}^4} + \frac{1}{128}\frac{\varepsilon^{ij}{\theta_{12}}^8}{{x_{12}}^4} \,,
\end{aligned} \tag{2.65a}$$

$$\begin{aligned}
\mathbf{u}^{\tilde{i}\tilde{j}}_{12} &= -\varepsilon^{\tilde{i}\tilde{j}} - \mathrm{i}\frac{\theta^{\tilde{i}k}_{12\alpha}x^{\alpha\beta}_{12}\theta^{\tilde{j}}_{12\beta k}}{{x_{12}}^2} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}{\theta_{12}}^4}{{x_{12}}^2} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}\tilde{\theta}^4_{12}}{{x_{12}}^4} \\
&\quad -\frac{\mathrm{i}}{8}\frac{\theta^{\tilde{i}k}_{12\alpha}x^{\alpha\beta}_{12}\theta^{\tilde{j}}_{12\beta k}{\theta_{12}}^4}{{x_{12}}^4} + \frac{1}{128}\frac{\varepsilon^{\tilde{i}\tilde{j}}{\theta_{12}}^8}{{x_{12}}^4} \,,
\end{aligned} \tag{2.65b}$$

where we have used the notation

$$\theta^4 = (\theta^{I\alpha}\theta^I_\alpha)^2 = (\theta^{\tilde{i}i\alpha}\theta_{\tilde{i}i\alpha})^2 \,, \tag{2.66a}$$

$$\tilde{\theta}^4 = \theta^I_\alpha x^{\alpha\beta}\theta^J_\beta\theta^K_\mu x^{\mu\nu}\theta^L_\nu\varepsilon_{IJKL} = \theta^{\tilde{k}j}_\alpha x^{\alpha\beta}\theta_{i\tilde{k}\beta}\theta^{\tilde{l}i}_\mu x^{\mu\nu}\theta_{j\tilde{l}\nu} - \theta^{\tilde{j}k}_\alpha x^{\alpha\beta}\theta_{k\tilde{i}\beta}\theta^{\tilde{i}l}_\mu x^{\mu\nu}\theta_{l\tilde{j}\nu} \ . \tag{2.66b}$$

Both matrices (2.65) are unitary and unimodular, in particular it holds that

$$\mathbf{u}^{ij}_{12}\mathbf{u}_{12kj} = \delta^i_k \,, \qquad \mathbf{u}^{\tilde{i}\tilde{j}}_{12}\mathbf{u}_{12\tilde{k}\tilde{j}} = \delta^{\tilde{i}}_{\tilde{k}} \ . \tag{2.67}$$

Note also that the expressions (2.65) are defined by the equation (2.64) uniquely, up to an overall sign which we fix as in (2.65) for further convenience.

The transformation law (2.18) implies that the matrices $\mathbf{u}^{ij}_{12}$ and $\mathbf{u}^{\tilde{i}\tilde{j}}_{12}$ defined by (2.64) vary under the infinitesimal superconformal transformations as isospinors at z_1 and z_2,

$$\tilde{\delta}\mathbf{u}^{ij}_{12} = \Lambda^i_k(z_1)\mathbf{u}^{kj}_{12} + \mathbf{u}^{ik}_{12}\Lambda^j_k(z_2) \,, \qquad \tilde{\delta}\mathbf{u}^{\tilde{i}\tilde{j}}_{12} = \Lambda^{\tilde{i}}_{\tilde{k}}(z_1)\mathbf{u}^{\tilde{k}\tilde{j}}_{12} + \mathbf{u}^{\tilde{i}\tilde{k}}_{12}\Lambda^{\tilde{j}}_{\tilde{k}}(z_2) \,, \tag{2.68}$$

where $\Lambda^{ij}(z)$ and $\Lambda^{\tilde{i}\tilde{j}}(z)$ are constructed from $\Lambda^{IJ}(z)$ by the rule (2.61).

Let us define the following matrices:

$$n^{ij}_{12} = \frac{\mathbf{u}^{ij}_{12}}{\boldsymbol{x}_{12}} = -\frac{\varepsilon^{ij}}{x_{12}} - \mathrm{i}\frac{\theta^{\tilde{k}i}_\alpha x^{\alpha\beta}_{12}\theta^j_{12\beta\tilde{k}}}{{x_{12}}^3} + \frac{1}{8}\frac{\varepsilon^{ij}\tilde{\theta}^4_{12}}{{x_{12}}^5} \,, \tag{2.69a}$$

$$n^{\tilde{i}\tilde{j}}_{12} = \frac{\mathbf{u}^{\tilde{i}\tilde{j}}_{12}}{\boldsymbol{x}_{12}} = -\frac{\varepsilon^{\tilde{i}\tilde{j}}}{x_{12}} - \mathrm{i}\frac{\theta^{\tilde{i}k}_{12\alpha}x^{\alpha\beta}_{12}\theta^{\tilde{j}}_{12\beta k}}{{x_{12}}^3} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}\tilde{\theta}^4_{12}}{{x_{12}}^5} \ . \tag{2.69b}$$

Similarly to (2.48), these matrices obey the analyticity conditions

$$D^{\tilde{i}(i}_{(1)\alpha} n^{k)l}_{12} = 0\,, \qquad D^{i(\tilde{i}}_{(1)\alpha} n^{\tilde{k})\tilde{l}}_{12} = 0\,. \tag{2.70}$$

By analogy with the two-point functions (2.64), we introduce three-point matrices with $\mathsf{SU}(2)$ indices

$$U_1^{IJ}(\tau_I)^{i\tilde{i}}(\tau_J)^{j\tilde{j}} = \mathbf{U}_1^{ij}\mathbf{U}_1^{\tilde{i}\tilde{j}}\,, \tag{2.71}$$

which have the following explicit form

$$\begin{aligned}\mathbf{U}_1^{ij} &= -\varepsilon^{ij} - \mathrm{i}\frac{\Theta^{\tilde{k}i}_{1\alpha}X_1^{\alpha\beta}\Theta^{j}_{1\beta\tilde{k}}}{X_1{}^2} - \frac{1}{8}\frac{\varepsilon^{ij}\Theta_1{}^4}{X_1{}^2} + \frac{1}{8}\frac{\varepsilon^{ij}\tilde{\Theta}_1^4}{X_1{}^4}\\ &\quad -\frac{\mathrm{i}}{8}\frac{\Theta^{\tilde{k}i}_{1\alpha}X_1^{\alpha\beta}\Theta^{j}_{1\beta\tilde{k}}\Theta_1{}^4}{X_1{}^4} + \frac{1}{128}\frac{\varepsilon^{ij}\Theta_1{}^8}{X_1{}^4}\,, \end{aligned}\tag{2.72a}$$

$$\begin{aligned}\mathbf{U}_1^{\tilde{i}\tilde{j}} &= -\varepsilon^{\tilde{i}\tilde{j}} - \mathrm{i}\frac{\Theta^{\tilde{i}k}_{1\alpha}X_1^{\alpha\beta}\Theta^{\tilde{j}}_{1\beta k}}{X_1{}^2} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}\Theta_1{}^4}{X_1{}^2} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}\tilde{\Theta}_1^4}{X_1{}^4}\\ &\quad -\frac{\mathrm{i}}{8}\frac{\Theta^{\tilde{i}k}_{1\alpha}X_1^{\alpha\beta}\Theta^{\tilde{j}}_{1\beta k}\Theta_1{}^4}{X_1{}^4} + \frac{1}{128}\frac{\varepsilon^{\tilde{i}\tilde{j}}\Theta_1{}^8}{X_1{}^4}\,. \end{aligned}\tag{2.72b}$$

Here the composites Θ^4 and $\tilde{\Theta}^4$ are defined by the same rules as in (2.66). By construction, the matrices (2.72) transform as tensors at the superspace point z_1

$$\tilde{\delta}\mathbf{U}_1^{ij} = \Lambda^i_k(z_1)\mathbf{U}_1^{kj} + \mathbf{U}_1^{ik}\Lambda^j_k(z_1)\,, \qquad \tilde{\delta}\mathbf{U}_1^{\tilde{i}\tilde{j}} = \Lambda^{\tilde{i}}_{\tilde{k}}(z_1)\mathbf{U}_1^{\tilde{k}\tilde{j}} + \mathbf{U}_1^{\tilde{i}\tilde{k}}\Lambda^{\tilde{j}}_{\tilde{k}}(z_1)\,. \tag{2.73}$$

It is possible to check that the matrices (2.72) can be expressed as products of three matrices of the type (2.65)

$$\mathbf{U}_1^{ij} = -\mathbf{u}_{12}^{ik}\mathbf{u}_{23\,kl}\mathbf{u}_{31}^{lj}\,, \qquad \mathbf{U}_1^{\tilde{i}\tilde{j}} = -\mathbf{u}_{12}^{\tilde{i}\tilde{k}}\mathbf{u}_{23\,\tilde{k}\tilde{l}}\mathbf{u}_{31}^{\tilde{l}\tilde{j}}\,. \tag{2.74}$$

The three-point analogs of (2.69) are

$$N_1^{ij} = \frac{\mathbf{U}_1^{ij}}{\boldsymbol{X}_1} = -\frac{\varepsilon^{ij}}{X_1} - \mathrm{i}\frac{\Theta^{\tilde{k}i}_{1\alpha}X_1^{\alpha\beta}\Theta^{j}_{1\beta\tilde{k}}}{X_1{}^3} + \frac{1}{8}\frac{\varepsilon^{ij}\tilde{\Theta}_1^4}{X_1{}^5}\,, \tag{2.75a}$$

$$N_1^{\tilde{i}\tilde{j}} = \frac{\mathbf{U}_1^{\tilde{i}\tilde{j}}}{\boldsymbol{X}_1} = -\frac{\varepsilon^{\tilde{i}\tilde{j}}}{X_1} - \mathrm{i}\frac{\Theta^{\tilde{i}k}_{1\alpha}X_1^{\alpha\beta}\Theta^{\tilde{j}}_{1\beta k}}{X_1{}^3} - \frac{1}{8}\frac{\varepsilon^{\tilde{i}\tilde{j}}\tilde{\Theta}_1^4}{X_1{}^5}\,. \tag{2.75b}$$

These matrices are analytic with respect to the spinor derivatives (2.29)

$$\mathcal{D}^{\tilde{i}(i}_{(1)\alpha} N_1^{k)l} = 0\,, \qquad \mathcal{D}^{i(\tilde{i}}_{(1)\alpha} N_1^{\tilde{k})\tilde{l}} = 0\,. \tag{2.76}$$

In this section we considered only the three-point functions which transform as tensors at the superspace point z_1. It is straightforward to obtain the analogs of these objects transforming covariantly at z_2 and z_3 by permuting the superspace points.

The two- and three-point superconformal building blocks constructed above are very similar to the $\mathcal{N} = 3$ ones given in the previous subsection. This is not accidental. It turns out that the latter can be found from the former by applying the $\mathcal{N}=4 \to \mathcal{N}=3$ superspace reduction. Indeed, when we switch off one of the four Grassmann variables θ_I at each superspace point, say $\theta_4=0$, the expressions (2.65) prove to coincide with (2.44),

$$\mathbf{u}^{ij}_{12(\mathcal{N}=4)}|_{\theta_4=0} = \mathbf{u}^{\tilde{i}\tilde{j}}_{12(\mathcal{N}=4)}|_{\theta_4=0} = \mathbf{u}^{ij}_{12(\mathcal{N}=3)}\ . \tag{2.77}$$

Here we have attached extra subscripts, ($\mathcal{N}=4$) and ($\mathcal{N}=3$), to the two-point functions to distinguish them. We usually omit these labels if no confusion occurs. For the three-point functions (2.50) and (2.72) we have similar relations

$$\mathbf{U}^{ij}_{1(\mathcal{N}=4)}|_{\theta_4=0} = \mathbf{U}^{\tilde{i}\tilde{j}}_{1(\mathcal{N}=4)}|_{\theta_4=0} = \mathbf{U}^{ij}_{1(\mathcal{N}=3)}\ . \tag{2.78}$$

The superspace reduction rules (2.77) and (2.78) will be important below when we turn to studying the correlators of the $\mathcal{N}=3$ and $\mathcal{N}=4$ flavour current multiplets.

3 Correlation functions for the $\mathcal{N}=3$ flavour current multiplets revisited

Here we obtain a new representation for the correlation functions of the $\mathcal{N}=3$ flavour current multiplets computed in [1]. Such a representation will be more convenient for comparison of the $\mathcal{N}=3$ correlators with $\mathcal{N}=4$ ones.

As discussed in [1], the $\mathcal{N}=3$ flavour current multiplet is described by a primary isovector superfield L^I of dimension 1, which is subject to the conservation equation

$$D^{(I}_\alpha L^{J)} - \frac{1}{3}\delta^{IJ} D^K_\alpha L^K = 0\ . \tag{3.1}$$

Its superconformal transformation law is

$$\delta L^I = -\xi L^I - \sigma(z) L^I + \Lambda^{IJ}(z) L^J\ . \tag{3.2}$$

The dimension of L^I is uniquely fixed by requiring the constraint (3.1) to be invariant under the superconformal transformations.

Consider an $\mathcal{N}=3$ superconformal field theory possessing n flavour current multiplets $L^{I\bar{a}}$, $\bar{a}=1,\dots,n$. Their two- and three-point functions were found in [1] to be

$$\langle L^{I\bar{a}}(z_1) L^{J\bar{b}}(z_2)\rangle = a_{\mathcal{N}=3}\frac{u^{IJ}_{12}\delta^{\bar{a}\bar{b}}}{\boldsymbol{x}_{12}{}^2}\,, \tag{3.3}$$

$$\langle L^{I\bar{a}}(z_1) L^{J\bar{b}}(z_2) L^{K\bar{c}}(z_3)\rangle = b_{\mathcal{N}=3}\, f^{\bar{a}\bar{b}\bar{c}}\frac{u^{II'}_{13}u^{JJ'}_{23}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2} H^{I'J'K}(\boldsymbol{X}_3,\Theta_3)\,, \tag{3.4}$$

where

$$\begin{aligned} H^{IJK}(\boldsymbol{X},\Theta) = \frac{1}{\boldsymbol{X}}\Big[&\varepsilon^{IJK} - U^{LJ}\varepsilon^{LIK} + U^{IL}\varepsilon^{LJK} \\ &-\frac{1}{16}(\delta^{IJ}\varepsilon^{KMN}U^{MN} + \varepsilon^{IMN}U^{MN}U^{KJ} + \varepsilon^{JMN}U^{MN}U^{IK}) \end{aligned}$$

$$+\frac{5}{16}(U^{IJ}\varepsilon^{KMN}U^{MN}+\delta^{IK}\varepsilon^{JMN}U^{MN}+\delta^{JK}\varepsilon^{IMN}U^{MN})\Big] . \quad (3.5)$$

In (3.4), $f^{\bar{a}\bar{b}\bar{c}}$ denotes the completely antisymmetric structure constants of the Lie algebra of the flavour group which is assumed to be simple. The tensor H^{IJK} in (3.5) is expressed in terms of the orthogonal matrix (2.26). The correlation functions (3.3) and (3.4) are fixed by the superconformal symmetry and the conservation condition up to arbitrary coefficients $a_{\mathcal{N}=3}$ and $b_{\mathcal{N}=3}$.

We now switch to the $\mathsf{SU}(2)$ notation, $L^I \to L^{ij} = (\tau_I)^{ij} L^I$, in accordance with the rules introduced in subsection 2.4. Then the conservation equation (3.1) turns into the analyticity condition

$$D_\alpha^{(ij}L^{kl)} = 0\,, \quad (3.6)$$

and the superconformal transformation (3.2) takes the form

$$\delta L^{ij} = -\xi L^{ij} - \sigma(z)L^{ij} + 2\Lambda_k^{(i}(z)L^{j)k}\ . \quad (3.7)$$

Here the symmetric matrix $\Lambda^{ij}(z)$ with isospinor indices is related to the antisymmetric matrix $\Lambda^{IJ}(z)$ with isovector indices, eq. (2.4), according to the general rule (2.38).

Using the relation (2.43) between the two-point building blocks with $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ indices, for (3.3) we immediately get

$$\langle L^{ij\,\bar{a}}(z_1)L^{kl\,\bar{b}}(z_2)\rangle = \frac{a_{\mathcal{N}=3}}{2}\frac{\delta^{\bar{a}\bar{b}}(\mathbf{u}_{12}^{ik}\mathbf{u}_{12}^{jl}+\mathbf{u}_{12}^{jk}\mathbf{u}_{12}^{il})}{\boldsymbol{x}_{12}{}^2}\ . \quad (3.8)$$

Contracting the three-point function (3.4) with three τ-matrices leads to

$$\langle L^{ij\,\bar{a}}(z_1)L^{kl\,\bar{b}}(z_2)L^{mn\,\bar{c}}(z_3)\rangle = b_{\mathcal{N}=3}\, f^{\bar{a}\bar{b}\bar{c}}\frac{\mathbf{u}_{13}^{ii'}\mathbf{u}_{13}^{jj'}\,\mathbf{u}_{23}^{kk'}\,\mathbf{u}_{23}^{ll'}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H_{i'j'\,k'l'}{}^{mn}(\boldsymbol{X}_3,\Theta_3)\,, \quad (3.9)$$

where

$$H^{ij\,kl\,mn} = H^{(ij)(kl)(mn)} = (\tau_I)^{ij}(\tau_J)^{kl}(\tau_K)^{mn}H^{IJK}\ . \quad (3.10)$$

In order to compute the right-hand side of (3.10), it is convenient to rewrite the expression (3.5) in the form

$$H^{IJK}(X,\Theta) = \frac{\varepsilon^{IJK}}{X}+\frac{1}{2}\frac{\delta^{IJ}\varepsilon^{KMN}A^{MN}}{X^3}-\frac{1}{2}\frac{\delta^{IK}\varepsilon^{JMN}A^{MN}}{X^3}-\frac{1}{2}\frac{\delta^{JK}\varepsilon^{IMN}A^{MN}}{X^3}\,, \quad (3.11)$$

where

$$A^{IJ} := \mathrm{i}\Theta^{I\alpha}X_{\alpha\beta}\Theta^{J\beta} = -A^{JI}\ . \quad (3.12)$$

For the first term in the right-hand side of (3.11) we apply the identity

$$\varepsilon^{IJK} = -\sqrt{2}\mathrm{tr}(\tau^I\tau^K\tau^J) = -\sqrt{2}\tau^I_{ij}\tau^J_{kl}\tau^K_{mn}\varepsilon^{mi}\varepsilon^{jk}\varepsilon^{ln}\ . \quad (3.13)$$

The other terms in (3.11) can be rewritten as

$$\delta^{IJ}\varepsilon^{KMN}A^{MN}-\delta^{IK}\varepsilon^{JMN}A^{MN}-\delta^{KJ}\varepsilon^{IMN}A^{MN} = -2\sqrt{2}\tau^I_{ij}\tau^J_{kl}\tau^K_{mn}\varepsilon^{mi}A^{jl}\varepsilon^{kn}\,, \quad (3.14)$$

where

$$A^{mn} = \varepsilon_{kl}\tau_I^{mk}\tau_J^{nl}A^{IJ} = \mathrm{i}\Theta_\alpha^{mk}X^{\alpha\beta}\Theta_{k\beta}^n \; . \tag{3.15}$$

In deriving (3.14), the following identity

$$\tau_K^{mn}\varepsilon^{KMN}A_{MN} = -\sqrt{2}A^{mn} \tag{3.16}$$

may be useful. Now, substituting (3.13) and (3.14) into (3.10) we find

$$H^{ij\,kl\,mn} = -\frac{1}{\sqrt{2}}\left(\frac{\varepsilon^{m(i}\varepsilon^{j)(l}\varepsilon^{k)n}}{X} + \frac{\varepsilon^{m(i}A^{j)(l}\varepsilon^{k)n}}{X^3}\right) - \frac{1}{\sqrt{2}}\left(\frac{\varepsilon^{n(i}\varepsilon^{j)(l}\varepsilon^{k)m}}{X} + \frac{\varepsilon^{n(i}A^{j)(l}\varepsilon^{k)m}}{X^3}\right) . \tag{3.17}$$

Finally, taking into account the explicit expression for A^{jl} given by (3.15), we note that the tensors in the parentheses in (3.17) can be rewritten in terms of the matrix N^{jl} introduced in (2.53),

$$H^{ij\,kl\,mn}(\boldsymbol{X}_3, \Theta_3) = \frac{1}{\sqrt{2}}\left(\varepsilon^{m(i}N_3^{j)(l}\varepsilon^{k)n} + \varepsilon^{n(i}N_3^{j)(l}\varepsilon^{k)m}\right) . \tag{3.18}$$

We arrive at the final expression for the three-point function of the $\mathcal{N} = 3$ flavour current multiplets

$$\langle L^{ij\,\bar{a}}(z_1)L^{kl\,\bar{b}}(z_2)L^{mn\,\bar{c}}(z_3)\rangle = b_{\mathcal{N}=3}\, f^{\bar{a}\bar{b}\bar{c}}\frac{\mathbf{u}_{13}^{ii'}\mathbf{u}_{13}^{jj'}\mathbf{u}_{23}^{kk'}\mathbf{u}_{23}^{ll'}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H_{i'j'\,k'l'}{}^{mn}(\boldsymbol{X}_3, \Theta_3) \, , \tag{3.19a}$$

$$H^{ij\,kl\,mn}(\boldsymbol{X}_3, \Theta_3) = \frac{1}{\sqrt{2}}\left(\frac{\varepsilon^{m(i}\mathbf{U}_3^{j)(l}\varepsilon^{k)n}}{\boldsymbol{X}_3} + \frac{\varepsilon^{n(i}\mathbf{U}_3^{j)(l}\varepsilon^{k)m}}{\boldsymbol{X}_3}\right) . \tag{3.19b}$$

Obviously, this three-point function possesses the correct superconformal properties since it is built out of the covariant two- and three-point objects introduced in subsection 2.4.

After using the identity (2.28a), the conservation law (3.6) implies the following equation on the tensor $H^{ij\,kl\,mn}$:

$$\mathcal{D}_\alpha^{(i'j'}H^{ij)kl\,mn} = 0 \; . \tag{3.20}$$

It is easy to see that (3.18) obeys this equation since the matrix N^{jl} is analytic (2.54).

The three-point correlation function (3.19a) must have the symmetry property

$$\langle L^{ij\,\bar{a}}(z_1)L^{kl\,\bar{b}}(z_2)L^{mn\,\bar{c}}(z_3)\rangle = \langle L^{mn\,\bar{c}}(z_3)L^{kl\,\bar{b}}(z_2)L^{ij\,\bar{a}}(z_1)\rangle \, , \tag{3.21}$$

which implies the following constraint for the tensor $H^{ij\,kl\,mn}$

$$\begin{aligned} H_{mn\,pq}{}^{ij}(-\boldsymbol{X}_1^{\mathrm{T}}, -\Theta_1) = & -\boldsymbol{x}_{13}{}^2\boldsymbol{X}_3{}^2\mathbf{u}_{13mm'}\mathbf{u}_{13nn'}\mathbf{u}_{13pr} \\ & \times\mathbf{U}_3^{rr'}\mathbf{u}_{13qs}\mathbf{U}_3^{ss'}\mathbf{u}_{13}^{ii'}\mathbf{u}_{13}^{jj'}H_{i'j'\,r's'}{}^{m'n'}(\boldsymbol{X}_3, \Theta_3) \; . \end{aligned} \tag{3.22}$$

Using (2.51) one can check that (3.19b) does satisfy this equation.

Finally, we point out that the explicit form of the correlation function (3.19) is analogous to the three-point correlator of flavour current multiplets in 4D $\mathcal{N} = 2$ superconformal theories [35].

4 Correlation functions of conserved $\mathcal{N}=4$ current multiplets

In this section we compute the two- and three-point functions of the $\mathcal{N}=4$ supercurrent and flavour current multiplets.

4.1 Correlation functions of flavour current multiplets

As discussed in [1], there are two inequivalent flavour current multiplets, L_+^{IJ} and L_-^{IJ}, in $\mathcal{N}=4$ superconformal field theories. They are described by primary $\mathsf{SO}(4)$ bivectors, $L_\pm^{IJ}=-L_\pm^{JI}$, subject to the same conservation equation

$$D_\alpha^I L_\pm^{JK} = D_\alpha^{[I} L_\pm^{JK]} - \frac{2}{3} D_\alpha^L L_\pm^{L[J}\delta^{K]I}\,, \tag{4.1}$$

which implies that L_+^{IJ} and L_-^{IJ} have dimension 1. These operators possess the same superconformal transformation law

$$\delta L_\pm^{IJ} = -\xi L_\pm^{IJ} - \sigma(z) L_\pm^{IJ} + 2\Lambda^{K[I}(z) L_\pm^{J]K}\,. \tag{4.2}$$

However, they have different algebraic properties,

$$\frac{1}{2}\varepsilon^{IJKL} L_\pm^{KL} = \pm L_\pm^{IJ}\,, \tag{4.3}$$

and thus L_+^{IJ} and L_-^{IJ} belong to inequivalent representations of $\mathsf{SO}(4)$.

Let us convert the $\mathsf{SO}(4)$ indices of L_+^{IJ} and L_-^{IJ} into $\mathsf{SU}(2)$ ones following the rules described in subsection 2.5, and specifically eq. (2.61). The (anti) self-duality conditions (4.3) imply that

$$(\tau_I)^{i\tilde{i}}(\tau_J)^{j\tilde{j}} L_+^{IJ} = \varepsilon^{\tilde{i}\tilde{j}} L^{ij}\,, \qquad (\tau_I)^{i\tilde{i}}(\tau_J)^{j\tilde{j}} L_-^{IJ} = \varepsilon^{ij} L^{\tilde{i}\tilde{j}}\,. \tag{4.4}$$

Here L^{ij} and $L^{\tilde{i}\tilde{j}}$ are symmetric, $L^{ij}=L^{ji}$ and $L^{\tilde{i}\tilde{j}}=L^{\tilde{j}\tilde{i}}$. Since L_+^{IJ} and L_-^{IJ} have different algebraic properties, the conservation equation (4.1) leads to the two different analyticity conditions:

$$D_\alpha^{\tilde{i}(i} L^{kl)} = 0\,, \tag{4.5a}$$

$$D_\alpha^{i(\tilde{i}} L^{\tilde{k}\tilde{l})} = 0\,, \tag{4.5b}$$

where $D_\alpha^{\tilde{i}i} \equiv D_\alpha^{i\tilde{i}} := (\tau_I)^{i\tilde{i}} D_\alpha^I$. It follows from (4.2) and (4.3) that the superconformal transformation laws of L^{ij} and $L^{\tilde{i}\tilde{j}}$ are

$$\delta L^{ij} = -\xi L^{ij} - \sigma(z) L^{ij} + 2\Lambda_k^{(i}(z) L^{j)k}\,, \tag{4.6a}$$

$$\delta L^{\tilde{i}\tilde{j}} = -\xi L^{\tilde{i}\tilde{j}} - \sigma(z) L^{\tilde{i}\tilde{j}} + 2\Lambda_{\tilde{k}}^{(\tilde{i}}(z) L^{\tilde{j})\tilde{k}}\,, \tag{4.6b}$$

where $\Lambda^{ij}(z)$ and $\Lambda^{\tilde{i}\tilde{j}}(z)$ are constructed from $\Lambda^{IJ}(z)$ by the rule (2.61).

We emphasise that the flavour current multiplets L^{ij} and $L^{\tilde{i}\tilde{j}}$ are completely independent and can be studied independently of each other. Since their properties are very similar, here we will consider in detail only the correlation functions for L^{ij} and comment on the correlators of $L^{\tilde{i}\tilde{j}}$ at the end of this section.

The properties of L^{ij} given by its conservation equation (4.5a) and superconformal transformation (4.6a) are very similar to those of the $\mathcal{N}=3$ flavour current multiplet, eqs. (3.6) and (3.7). This similarity is not accidental since there proves to exist a unique $\mathcal{N}=3$ flavour current multiplet $L^{ij}_{(\mathcal{N}=3)}$ associated with $L^{ij} \equiv L^{ij}_{(\mathcal{N}=4)}$.[7] The former is obtained from the latter through the procedure of $\mathcal{N}=4 \to \mathcal{N}=3$ superspace reduction which has been discussed in the literature in the cases of $\mathcal{N}=4$ Minkowski [5] and anti-de Sitter [36] superspaces (see also [1]). As applied to $L^{ij}_{(\mathcal{N}=4)}$, it works as follows. For the Grassmann coordinates θ^α_I of the $\mathcal{N}=4$ superspace, we make $3+1$ splitting $\theta_I \to (\theta_{\hat I}, \theta_4)$ and then consider the θ_4-independent component of $L^{ij}_{(\mathcal{N}=4)}$. It proves to be the desired $\mathcal{N}=3$ flavour current multiplet,

$$L^{ij}_{(\mathcal{N}=3)} = L^{ij}_{(\mathcal{N}=4)}|_{\theta_4=0} \,. \tag{4.7}$$

In fact, it is possible to define an inverse correspondence, $L^{ij}_{(\mathcal{N}=3)} \to L^{ij}_{(\mathcal{N}=4)}$. Specifically, given an $\mathcal{N}=3$ superfield $L^{ij}_{(\mathcal{N}=3)}$ subject to the constraint (3.6), there exists a unique $\mathcal{N}=4$ superfield $L^{ij}_{(\mathcal{N}=4)}$ obeying the constraint (4.5a) and related to $L^{ij}_{(\mathcal{N}=3)}$ by (4.7). This means that all components in the θ_4-expansion of $L^{ij}_{(\mathcal{N}=4)}$ can be restored from the lowest one given by $L^{ij}_{(\mathcal{N}=3)}$.

The above simple observation appears crucial for finding the correlation functions of the $\mathcal{N}=4$ flavour current multiplets. Indeed, the expressions (3.8) and (3.19) can be considered as the lowest components in the θ_4-expansion of the corresponding correlators of the $\mathcal{N}=4$ flavour current multiplets. Moreover, the full information is encoded in these parts of the correlators since the higher-order corrections in θ_4 can be uniquely restored from these lowest components.

Based on these observations, we propose the following ansatz for the correlation functions of several $\mathcal{N}=4$ flavour current multiplets $L^{ij\,\bar a}$, $\bar a = 1,\dots,n$. The two-point function is

$$\langle L^{ij\,\bar a}(z_1) L^{kl\,\bar b}(z_2)\rangle = \frac{a_{\mathcal{N}=4}}{2}\frac{\delta^{\bar a\bar b}(\mathbf{u}^{ik}_{12}\mathbf{u}^{jl}_{12} + \mathbf{u}^{jk}_{12}\mathbf{u}^{il}_{12})}{\boldsymbol{x}_{12}{}^2}\,, \tag{4.8}$$

while the three-point function reads

$$\langle L^{ij\,\bar a}(z_1) L^{kl\,\bar b}(z_2) L^{mn\,\bar c}(z_3)\rangle = b_{\mathcal{N}=4}\, f^{\bar a\bar b\bar c}\frac{\mathbf{u}^{ii'}_{13}\mathbf{u}^{jj'}_{13}\,\mathbf{u}^{kk'}_{23}\mathbf{u}^{ll'}_{23}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2} H_{i'j'\,k'l'}{}^{mn}(\boldsymbol{X}_3,\Theta_3)\,, \tag{4.9a}$$

$$H^{ij\,kl\,mn}(\boldsymbol{X}_3,\Theta_3) = \frac{1}{\sqrt{2}}\left(\frac{\varepsilon^{m(i}\mathbf{U}_3^{j)(l}\varepsilon^{k)n}}{\boldsymbol{X}_3} + \frac{\varepsilon^{n(i}\mathbf{U}_3^{j)(l}\varepsilon^{k)m}}{\boldsymbol{X}_3}\right)\,. \tag{4.9b}$$

The two- and three-point building blocks $\mathbf{u}^{ij}_{12}$ and $\mathbf{U}^{ij}_3$ used in these expressions were introduced in subsection 2.5. Taking into account the relations (2.77) and (2.78), it is clear that (3.8) and (3.19) are related to (4.8) and (4.9a) via the superspace reduction described above

$$\langle L^{ij\,\bar a}_{(\mathcal{N}=4)}(z_1) L^{kl\,\bar b}_{(\mathcal{N}=4)}(z_2)\rangle|_{\theta_4=0} = \langle L^{ij\,\bar a}_{(\mathcal{N}=3)}(z_1) L^{kl\,\bar b}_{(\mathcal{N}=3)}(z_2)\rangle\,, \tag{4.10a}$$

[7] Here we have attached the labels ($\mathcal{N}=3$) and ($\mathcal{N}=4$) to these superfields to distinguish them. Below, when no confusion is possible, these labels are omitted.

$$\langle L^{ij\,\bar a}_{(\mathcal{N}=4)}(z_1)L^{kl\,\bar b}_{(\mathcal{N}=4)}(z_2)L^{mn\,\bar c}_{(\mathcal{N}=4)}(z_3)\rangle|_{\theta_4=0} = \langle L^{ij\,\bar a}_{(\mathcal{N}=3)}(z_1)L^{kl\,\bar b}_{(\mathcal{N}=3)}(z_2)L^{mn\,\bar c}_{(\mathcal{N}=3)}(z_3)\rangle\ . \tag{4.10b}$$

Let us rewrite the correlation function (4.8) in terms of the two-point matrix (2.69a)

$$\langle L^{ij\,\bar a}(z_1)L^{kl\,\bar b}(z_2)\rangle = \frac{a_{\mathcal{N}=4}}{2}\delta^{\bar a\bar b}(n^{ik}_{12}n^{jl}_{12} + n^{jk}_{12}n^{il}_{12})\ . \tag{4.11}$$

Then, owing to (2.70), it is obvious that (4.11) obeys the conservation condition

$$D^{\tilde i'(i'}_\alpha\langle L^{ij)\bar a}(z_1)L^{kl\,\bar b}(z_2)\rangle = 0 \qquad (z_1\neq z_2)\ . \tag{4.12}$$

In the same manner we express (4.9b) in terms of (2.75a)

$$H^{ij\,kl\,mn}(\boldsymbol{X}_3,\Theta_3) = \frac{1}{\sqrt{2}}\left(\varepsilon^{m(i}N_3^{j)(l}\varepsilon^{k)n} + \varepsilon^{n(i}N_3^{j)(l}\varepsilon^{k)m}\right) \tag{4.13}$$

and observe that

$$\mathcal{D}^{\tilde i'(i'}_\alpha H^{ij)kl\,mn}(\boldsymbol{X},\Theta) = 0\,, \tag{4.14}$$

as a consequence of (2.76). The equations (4.12) and (4.14) prove that the correlation functions of $\mathcal{N}=4$ flavour current multiplets constructed in (4.8) and (4.9) do obey the necessary conservation laws.

As concerns the correlation functions of the flavour current multiplets $L^{\tilde i\tilde j}$, they have the same form as eqs. (4.8) and (4.9) but with the indices $i,j,\ldots$ replaced with $\tilde i,\tilde j\ \ldots$ Note also that all mixed two- and and three-point correlators involving both L^{ij} and $L^{\tilde i\tilde j}$ vanish.

4.2 Correlation functions of the supercurrent

In accordance with [37, 38] (see also [1]), the $\mathcal{N}=4$ supercurrent is described by a primary real scalar J subject to the conservation equation

$$D^{I\alpha}D^K_\alpha J = \frac{1}{4}\delta^{IK}D^{L\alpha}D^L_\alpha J\ . \tag{4.15}$$

Its superconformal transformation law is

$$\delta J = -\xi J - \sigma(z)J\ . \tag{4.16}$$

The constraint (4.15) uniquely fixes the dimension of J to be 1.

Since the supercurrent J is a scalar superfield, its two-point correlation function has a simple form

$$\langle J(z_1)J(z_2)\rangle = \frac{c_{\mathcal{N}=4}}{\boldsymbol{x}_{12}{}^2}\,, \tag{4.17}$$

where $c_{\mathcal{N}=4}$ is a free coefficient. Using (2.21) it is easy to check that (4.17) obeys the conservation law (4.15).

The three-point correlation function of the $\mathcal{N}=4$ supercurrent can be found by making the following ansatz

$$\langle J(z_1)J(z_2)J(z_3)\rangle = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H(\boldsymbol{X}_3,\Theta_3)\,, \tag{4.18}$$

where the function H has the homogeneity property

$$H(\lambda^2 \boldsymbol{X}, \lambda\Theta) = \lambda^{-2} H(\boldsymbol{X}, \Theta)\,, \tag{4.19}$$

for a real positive λ. With the use of (2.28a), one can check that the supercurrent conservation condition (4.15) implies a similar equation for H,

$$\mathcal{D}^{I\alpha}\mathcal{D}^K_\alpha H = \frac{1}{4}\delta^{IK}\mathcal{D}^{L\alpha}\mathcal{D}^L_\alpha H\,, \tag{4.20}$$

where $\mathcal{D}^I_\alpha$ is the generalized spinor covariant derivative (2.29).

The general solution of (4.19) can be represented as a Θ-expansion

$$H(\boldsymbol{X}, \Theta) = \frac{c_1}{\boldsymbol{X}} + c_2\frac{\Theta^2}{\boldsymbol{X}^2} + c_3\frac{\Theta^4}{\boldsymbol{X}^3} + c_4\frac{\Theta^6}{\boldsymbol{X}^4} + c_5\frac{\Theta^8}{\boldsymbol{X}^5} + c_6\frac{\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}}{\boldsymbol{X}^5}\,, \tag{4.21}$$

where c_i are some coefficients. It is useful to rewrite (4.21) in terms of the symmetric part $X_{\alpha\beta}$ of $\boldsymbol{X}_{\alpha\beta}$ given in (2.25). The result is

$$H = \frac{d_1}{X} + d_2\frac{\Theta^2}{X^2} + d_3\frac{\Theta^4}{X^3} + d_4\frac{\Theta^6}{X^4} + d_5\frac{\Theta^8}{X^5} + d_6\frac{\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}X_{\alpha\beta}X_{\gamma\delta}}{X^5}\,, \tag{4.22}$$

where d_i are some coefficients which can be, in principle, expressed in terms of c_i. For the function H in the form (4.22) it is easy to check that it solves (4.20) for

$$d_2 = d_3 = d_4 = d_5 = 0\,, \tag{4.23}$$

and d_1, d_6 are arbitrary real. Thus, if we denote $d_6 = d_{\mathcal{N}=4}$ and $d_1 = \tilde{d}_{\mathcal{N}=4}$, the solution for H is

$$\begin{aligned} H(\boldsymbol{X}, \Theta) &= \frac{\tilde{d}_{\mathcal{N}=4}}{X} + d_{\mathcal{N}=4}\frac{\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}X_{\alpha\beta}X_{\gamma\delta}}{X^5} \\ &= \tilde{d}_{\mathcal{N}=4}\left(\frac{1}{\boldsymbol{X}} + \frac{1}{8}\frac{\Theta^4}{\boldsymbol{X}^3} + \frac{3}{128}\frac{\Theta^8}{\boldsymbol{X}^5}\right) + d_{\mathcal{N}=4}\frac{\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}\boldsymbol{X}_{\alpha\beta}\boldsymbol{X}_{\gamma\delta}}{\boldsymbol{X}^5}\,. \end{aligned} \tag{4.24}$$

Here, in the second line of (4.24), we expressed the function H in terms of $\boldsymbol{X}_{\alpha\beta}$ given in (2.25) and transforming covariantly under the superconformal group.

Using the identities (2.27a) and (2.27b) it is possible to show that for arbitrary $d_{\mathcal{N}=4}$ and $\tilde{d}_{\mathcal{N}=4}$ the expression (4.24) obeys the equation

$$H(-\boldsymbol{X}_1^{\mathrm{T}}, -\Theta_1) = \frac{\boldsymbol{x}_{12}{}^2}{\boldsymbol{x}_{23}{}^2} H(\boldsymbol{X}_3, \Theta_3)\,, \tag{4.25}$$

which must hold as a consequence of the symmetry property

$$\langle J(z_1)J(z_2)J(z_3)\rangle = \langle J(z_3)J(z_2)J(z_1)\rangle\,. \tag{4.26}$$

Thus, the $\mathcal{N} = 4$ supercurrent three-point correlation function (4.18) with H given by (4.24) obeys all the constraints dictated by the superconformal symmetry and the conservation

JHEP08(2015)125

equation with *two* arbitrary real coefficients $d_{\mathcal{N}=4}$ and $\tilde{d}_{\mathcal{N}=4}$. The structure of this three-point correlator has some similarities with that for the 4D $\mathcal{N}=2$ supercurrent computed in [35].

As was demonstrated in [1], the three-point functions of the supercurrent in $\mathcal{N} = 1,2,3$ superconformal theories involve only *one* free parameter. In this regard, our $\mathcal{N}=4$ result given by eqs. (4.18) and (4.24) may look rather puzzling, since every $\mathcal{N}=4$ superconformal field theory is a special $\mathcal{N}=3$ superconformal field theory. The resolution of this puzzle is as follows. We showed in [1] that the $\mathcal{N}=4$ supercurrent consists of two $\mathcal{N}=3$ multiplets, one of which is the $\mathcal{N}=3$ supercurrent and the other multiplet includes conserved currents that are not present in general $\mathcal{N}=3$ superconformal fields theories (the fourth supersymmetry current and the R-symmetry currents associated with the coset space $\mathsf{SO}(4)/\mathsf{SO}(3)$). In subsection C.1, by performing the $\mathcal{N}=4 \to \mathcal{N}=3$ reduction of the $\mathcal{N}=4$ supercurrent correlation function (4.18), we demonstrate that the first term in (4.24) does not contribute to the three-point function of the $\mathcal{N}=3$ supercurrent. Hence, it also does not contribute to the three-point correlation function of the energy-momentum tensor upon further reduction down to the component fields. This means that just like in $\mathcal{N}=1,2,3$ superconformal theories the three-point function of the energy-momentum tensor depends just on a single tensor structure and a single free coefficient $d_{\mathcal{N}=4}$.

4.3 Mixed correlators

For completeness, we also present mixed three-point correlation functions involving both the supercurrent and flavour current multiplets. It is not difficult to see that

$$\langle L^{ij\,\bar{a}}(z_1) J(z_2) J(z_3)\rangle = 0\ . \tag{4.27}$$

However, for the correlator with one supercurrent and two flavour current multiplet insertions we get

$$\langle L^{ij\,\bar{a}}(z_1) J(z_2) L^{kl\,\bar{b}}(z_3)\rangle = \delta^{\bar{a}\bar{b}} \frac{\mathbf{u}_{13}^{ii'}\mathbf{u}_{13}^{jj'}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2} H_{i'j'}{}^{kl}(\boldsymbol{X}_3,\Theta_3)\,, \tag{4.28a}$$

$$H^{ij\,kl}(\boldsymbol{X},\Theta) = c\left(N^{i(k}{}_{\varepsilon}{}^{l)j} + N^{j(k}{}_{\varepsilon}{}^{l)i}\right) = c\frac{\mathbf{U}^{i(k}{}_{\varepsilon}{}^{l)j} + \mathbf{U}^{j(k}{}_{\varepsilon}{}^{l)i}}{\boldsymbol{X}}\,, \tag{4.28b}$$

where c is a constant. The tensor $H^{ij\,kl}$ is expressed in terms of the matrices $\mathbf{U}^{ij}$ and N^{ij} which are given by (2.72a) and (2.75a), respectively. This tensor is found as the general solution of the equations

$$\mathcal{D}_\alpha^{\tilde{m}(m} H^{ij)kl} = 0\,, \tag{4.29a}$$

$$\mathcal{Q}^{I\alpha}\mathcal{Q}^J_\alpha H^{ij\,kl} = \frac{1}{4}\delta^{IJ}\mathcal{Q}^{K\alpha}\mathcal{Q}_{K\alpha} H^{ij\,kl}\,, \tag{4.29b}$$

which are the corollaries of the analyticity of the flavour current multiplet (4.5a) and the supercurrent conservation law (4.15). In deriving the equations (4.29) the identities (2.28) have been used.

The equation (4.29a) immediately follows from the analyticity of the matrix N^{ij}, see (2.76). To check the equation (4.29b) it is convenient to rewrite it in terms of $\mathsf{SU}(2)$ indices

$$\mathcal{Q}^{i(\tilde{i}\alpha}\mathcal{Q}_{\alpha}^{\tilde{j})j}H^{i'j'\,kl}=0\ . \tag{4.30}$$

It is easy to see that this equation is satisfied as a consequence of the following property of the matrix (2.75a)

$$\mathcal{D}^{i(\tilde{i}\alpha}\mathcal{D}_{\alpha}^{\tilde{j})j}N^{kl}=\mathcal{Q}^{i(\tilde{i}\alpha}\mathcal{Q}_{\alpha}^{\tilde{j})j}N^{kl}=0\ . \tag{4.31}$$

Finally, we note that the tensor (4.28b) obeys the constraint

$$H_{ij}{}^{kl}(\boldsymbol{X}_3,\Theta_3)=\frac{\boldsymbol{X}_1}{\boldsymbol{X}_3}\mathbf{u}_{31ii'}\mathbf{u}_{31jj'}\mathbf{u}_{31}^{kk'}\mathbf{u}_{31}^{ll'}H_{k'l'}{}^{i'j'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1)\,, \tag{4.32}$$

which is a corollary of the following symmetry property of the correlation function (4.28a)

$$\langle L^{ij\,\bar{a}}(z_1)J(z_2)L^{kl\,\bar{b}}(z_3)\rangle=\langle L^{kl\,\bar{b}}(z_3)J(z_2)L^{ij\,\bar{a}}(z_1)\rangle\ . \tag{4.33}$$

The equation (4.32) can be easily verified with the use of the relation (2.74) which links together the thee-point and two-point unitary matrices.

5 Free $\mathcal{N}=4$ hypermultiplets

In this section we consider a family of trivial $\mathcal{N}=4$ superconformal field theories — models for free hypermultiplets. In these models, the correlation functions of conserved currents can be computed exactly. Using such results will allow us to derive important relations between the numerical parameters appearing in certain two- and three-point functions in general $\mathcal{N}=4$ superconformal field theories.

5.1 On-shell hypermultiplets

In 3D $\mathcal{N}=4$ supersymmetry, there are two types of free on-shell hypermultiplets, left q^i and right $q^{\tilde{i}}$, that transform as isospinors of the different subgroups $\mathsf{SU}(2)_{\mathrm{L}}$ and $\mathsf{SU}(2)_{\mathrm{R}}$ of the R-symmetry group. They obey the following constraints

$$D_{\alpha}^{\tilde{i}(i}q^{j)}=0\,, \tag{5.1a}$$
$$D_{\alpha}^{i(\tilde{i}}q^{\tilde{j})}=0\,, \tag{5.1b}$$

which are similar to those introduced by Sohnius [39] to describe the $\mathcal{N}=2$ hypermultiplet in four dimensions. These primary superfields possess the superconformal transformation laws

$$\begin{aligned}\delta q^i&=-\xi q^i-\frac{1}{2}\sigma(z)q^i+\Lambda^i_j(z)q^j\ ,\\ \delta q^{\tilde{i}}&=-\xi q^{\tilde{i}}-\frac{1}{2}\sigma(z)q^{\tilde{i}}+\Lambda^{\tilde{i}}_{\tilde{j}}(z)q^{\tilde{j}}\ .\end{aligned} \tag{5.2}$$

The constraints (5.1a) and (5.1b) uniquely fix the dimension of q^i and $q^{\tilde{i}}$ to be $1/2$. Associated with q^i and $q^{\tilde{i}}$ are their conjugates

$$\overline{q^i}=\bar{q}_i=\varepsilon_{ij}\bar{q}^j\ ,\qquad \overline{q^{\tilde{i}}}=\bar{q}_{\tilde{i}}=\varepsilon_{\tilde{i}\tilde{j}}\bar{q}^{\tilde{j}}\ , \tag{5.3}$$

which may be seen to obey the same constraints as q^i and $q^{\tilde{i}}$.

In accordance with the general results in section 2, the two-point correlation functions of the primary superfields q^i and $q^{\tilde{i}}$ with their conjugates are

$$\langle q^i(z_1)\bar{q}^j(z_2)\rangle = \frac{1}{4\pi}\frac{\mathbf{u}_{12}^{ij}}{\boldsymbol{x}_{12}} = \frac{1}{4\pi}n_{12}^{ij}\,, \tag{5.4a}$$

$$\langle q^{\tilde{i}}(z_1)\bar{q}^{\tilde{j}}(z_2)\rangle = \frac{1}{4\pi}\frac{\mathbf{u}_{12}^{\tilde{i}\tilde{j}}}{\boldsymbol{x}_{12}} = \frac{1}{4\pi}n_{12}^{\tilde{i}\tilde{j}}\,, \tag{5.4b}$$

where the matrices $\mathbf{u}_{12}^{ij}$, $\mathbf{u}_{12}^{\tilde{i}\tilde{j}}$ and n_{12}^{ij}, $n_{12}^{\tilde{i}\tilde{j}}$ are defined in (2.65) and (2.69), respectively. On the one hand, the expressions for $\langle q^i(z_1)\bar{q}^j(z_2)\rangle$ and $\langle q^{\tilde{i}}(z_1)\bar{q}^{\tilde{j}}(z_2)\rangle$ in terms of $\mathbf{u}_{12}^{ij}$, $\mathbf{u}_{12}^{\tilde{i}\tilde{j}}$ and $\boldsymbol{x}_{12}$ guarantee that they comply with the requirement of superconformal invariance, eq. (2.7). On the other hand, expressing these correlators in terms of n_{12}^{ij} and $n_{12}^{\tilde{i}\tilde{j}}$ allows one to check easily that these two-point functions obey the analyticity constraints (5.1a) and (5.1b), owing to (2.70).

There exist several off-shell realisations for the hypermultiplet, see appendix A for a review. In any off-shell realisation for the left hypermultiplet, the two-point function (5.4a) may differ only by contact terms from that corresponding to the off-shell formulation. In particular, in the harmonic superspace approach one deals with the q^+-hypermultiplet for which it holds that

$$\langle q^+(z_1,u_1)\,\breve{q}^+(z_2,u_2)\rangle = u^+_{1i}\,\langle q^i(z_1)\,\bar{q}_j(z_2)\rangle\,u_2^{+j} \;+\; \text{contact terms}\,. \tag{5.5}$$

Similar comments apply to the right hypermultiplet correlator (5.4b). For our purposes in this section, it suffices to work with the two-point functions (5.4a) and (5.4b). A careful treatment of the singularities of the two- and three-point functions at coincident points is beyond the scope of this paper.

It should be pointed out that switching off the Grassmann variables in (5.4) leads to to the correctly normalised correlators of free complex scalars,

$$\langle q^i(z_1)\bar{q}_j(z_2)\rangle|_{\theta=0} = \langle\varphi^i(x_1)\bar{\varphi}_j(x_2)\rangle = \frac{1}{4\pi}\delta^i_j\frac{1}{\sqrt{(x_1-x_2)^2}}\,, \tag{5.6a}$$

$$\langle q^{\tilde{i}}(z_1)\bar{q}_{\tilde{j}}(z_2)\rangle|_{\theta=0} = \langle\varphi^{\tilde{i}}(x_1)\bar{\varphi}_{\tilde{j}}(x_2)\rangle = \frac{1}{4\pi}\delta^{\tilde{i}}_{\tilde{j}}\frac{1}{\sqrt{(x_1-x_2)^2}}\,, \tag{5.6b}$$

where $\varphi^i(x) = q^i(z)|_{\theta=0}$, $\varphi^{\tilde{i}}(x) = q^{\tilde{i}}(z)|_{\theta=0}$.

5.2 Two-point correlators

Let us consider a free model of m left hypermultiplets q^i and n right hypermultiplets $q^{\tilde{i}}$. We assume that q^i transforms in an irreducible representation of a simple flavour group $G_{\rm L}$ with generators $\Sigma^{\bar{a}}$. Similarly, $q^{\tilde{i}}$ is assumed to transform in an irreducible representation of another simple flavour group $G_{\rm R}$ with generators $\Sigma^{\tilde{a}}$. Viewing q^i and $q^{\tilde{i}}$ as column vectors and their Hermitian conjugates $\bar{q}_i$ and $\bar{q}_{\tilde{i}}$ as row vectors, the supercurrent J is

$$J = \bar{q}_{\tilde{i}}q^{\tilde{i}} - \bar{q}_i q^i\,, \tag{5.7}$$

and the flavour current multiplets $L^{ij\,\bar{a}}$, $L^{\tilde{i}\tilde{j}\,\tilde{a}}$ are given by

$$L^{\bar{a}}_{ij} = -\mathrm{i}\bar{q}_{(i}\Sigma^{\bar{a}}q_{j)}\,, \qquad L^{\tilde{a}}_{\tilde{i}\tilde{j}} = -\mathrm{i}\bar{q}_{(\tilde{i}}\Sigma^{\tilde{a}}q_{\tilde{j})}\,. \tag{5.8}$$

We assume that the generators of the flavour groups are normalised such that

$$\mathrm{tr}(\Sigma^{\bar{a}}\Sigma^{\bar{b}}) = k_{\mathrm{L}}\delta^{\bar{a}\bar{b}}\,, \qquad \mathrm{tr}(\Sigma^{\tilde{a}}\Sigma^{\tilde{b}}) = k_{\mathrm{R}}\delta^{\tilde{a}\tilde{b}}\,. \tag{5.9}$$

The normalisation constants k_{L} and k_{R} depend on the representations of the flavour groups G_{L} and G_{R} chosen. One can check that, due to the free equations of motion (5.1), the current multiplets (5.7) and (5.8) obey the conservation laws (4.15) and (4.5), respectively.

The notable feature of the supercurrent (5.7) is that J is asymmetric with respect to the left and right hypermultiplets. The supergravity origin of this property will be discussed in section 8.

We compute the two-point correlation functions of the supercurrent and flavour current multiplets for the free hypermultiplets. Since there is no correlation between the superfields q^i and $q^{\tilde{i}}$, the two-point function for the supercurrent is given by

$$\langle J(z_1)J(z_2)\rangle = \langle q^{\tilde{i}}(z_1)\bar{q}_{\tilde{i}}(z_1)q^{\tilde{j}}(z_2)\bar{q}_{\tilde{j}}(z_2)\rangle + \langle q^{i}(z_1)\bar{q}_{i}(z_1)q^{j}(z_2)\bar{q}_{j}(z_2)\rangle\,. \tag{5.10}$$

Performing the Wick contractions and making use of (5.4), we find

$$\langle J(z_1)J(z_2)\rangle = \frac{m}{(4\pi)^2}\frac{\mathbf{u}_{12}^{ij}\mathbf{u}_{12ij}}{\boldsymbol{x}_{12}{}^2} + \frac{n}{(4\pi)^2}\frac{\mathbf{u}_{12}^{\tilde{i}\tilde{j}}\mathbf{u}_{12\tilde{i}\tilde{j}}}{\boldsymbol{x}_{12}{}^2} = \frac{1}{8\pi^2}\frac{m+n}{\boldsymbol{x}_{12}{}^2}\,. \tag{5.11}$$

In a similar way we find two-point correlation functions of flavour current multiplets

$$\langle L^{\bar{a}}_{ij}(z_1)L^{\bar{b}}_{kl}(z_2)\rangle = \frac{k_{\mathrm{L}}}{32\pi^2}\frac{(\mathbf{u}_{12il}\mathbf{u}_{12jk}+\mathbf{u}_{12jl}\mathbf{u}_{12ik})}{\boldsymbol{x}_{12}{}^2}\delta^{\bar{a}\bar{b}}\,, \tag{5.12a}$$

$$\langle L^{\tilde{a}}_{\tilde{i}\tilde{j}}(z_1)L^{\tilde{b}}_{\tilde{k}\tilde{l}}(z_2)\rangle = \frac{k_{\mathrm{R}}}{32\pi^2}\frac{(\mathbf{u}_{12\tilde{i}\tilde{l}}\mathbf{u}_{12\tilde{j}\tilde{k}}+\mathbf{u}_{12\tilde{j}\tilde{l}}\mathbf{u}_{12\tilde{i}\tilde{k}})}{\boldsymbol{x}_{12}{}^2}\delta^{\tilde{a}\tilde{b}}\,. \tag{5.12b}$$

Comparing these correlation functions with (4.8) and (4.17) we find the following values for the coefficients $a_{\mathcal{N}=4}$ and $c_{\mathcal{N}=4}$:

$$a_{\mathcal{N}=4} = \frac{k_{\mathrm{L}}}{16\pi^2}\,, \tag{5.13}$$

$$c_{\mathcal{N}=4} = \frac{m+n}{8\pi^2}\,. \tag{5.14}$$

5.3 Three-point correlators

For the three-point function of the flavour current multiplets $L^{\bar{a}}_{ij}$, which are defined by (5.8), we have

$$\langle L^{\bar{a}}_{ij}(z_1)L^{\bar{b}}_{kl}(z_2)L^{\bar{c}}_{mn}(z_3)\rangle = \mathrm{i}\langle\bar{q}_{(i}(z_1)\Sigma^{\bar{a}}q_{j)}(z_1)\bar{q}_{(k}(z_2)\Sigma^{\bar{b}}q_{l)}(z_2)\bar{q}_{(m}(z_3)\Sigma^{\bar{c}}q_{n)}(z_3)\rangle\,. \tag{5.15}$$

Performing the Wick contractions and using the explicit form of the hypermultiplet two-point function (5.4a) we find

$$\langle L^{\bar{a}}_{ij}(z_1)L^{\bar{b}}_{kl}(z_2)L^{\bar{c}}_{mn}(z_3)\rangle = -\frac{f^{\bar{a}\bar{b}\bar{c}}k_{\mathrm{L}}}{128\pi^3}\frac{\mathbf{u}_{12i(k}\mathbf{u}_{23l)(m}\mathbf{u}_{31n)j}+\mathbf{u}_{12j(k}\mathbf{u}_{23l)(m}\mathbf{u}_{31n)i}}{\boldsymbol{x}_{12}\boldsymbol{x}_{13}\boldsymbol{x}_{23}}\,. \tag{5.16}$$

JHEP08(2015)125

Using the identity $\boldsymbol{x}_{12}{}^2 = \boldsymbol{X}_3{}^2 \boldsymbol{x}_{13}{}^2 \boldsymbol{x}_{23}{}^2$ the denominator in (5.16) can be written as

$$\frac{1}{\boldsymbol{x}_{12}\boldsymbol{x}_{13}\boldsymbol{x}_{23}} = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2\boldsymbol{X}_3} \ . \tag{5.17}$$

With the help of (2.74) the correlation function (5.16) gets exactly the form (4.9) with

$$b_{\mathcal{N}=4} = -\frac{\sqrt{2}}{128\pi^3} k_{\rm L} \ . \tag{5.18}$$

Comparing this coefficient with (5.13) we observe that

$$\frac{b_{\mathcal{N}=4}}{a_{\mathcal{N}=4}} = -\frac{\sqrt{2}}{8\pi} \ . \tag{5.19}$$

Although this relation between the coefficients of the two-point and three-point correlation functions is obtained for the free hypermultiplets, we propose that it is universal for any $\mathcal{N} = 4$ superconformal field theory. Indeed, the relation (5.19) can be considered as a manifestation of a Ward identity relating the two- and three-point correlation functions of the flavour current multiplets. Since both of these correlation functions depend on a single tensor structure the relation between their coefficients can be found by considering a particular theory. The explicit form of the relevant $\mathcal{N} = 4$ Ward identity will be derived in the next section.

The three-point correlation function for the flavour current multiplets $L^{\tilde a}_{\tilde i \tilde j}$ can be analysed in a similar way, with the same relation (5.19) between the coefficients.

Now we turn to computing the three-point correlator for the supercurrent (5.7). In the right-hand side of

$$\begin{aligned}\langle J(z_1)J(z_2)J(z_3)\rangle &= \langle q^{\tilde i}(z_1)\bar q_{\tilde i}(z_1) q^{\tilde j}(z_2)\bar q_{\tilde j}(z_2) q^{\tilde k}(z_3)\bar q_{\tilde k}(z_3)\rangle \\ &\quad - \langle q^{i}(z_1)\bar q_{i}(z_1) q^{j}(z_2)\bar q_{j}(z_2) q^{k}(z_3)\bar q_{k}(z_3)\rangle\end{aligned} \tag{5.20}$$

we perform the Wick contractions and make use of (5.4) to get

$$\begin{aligned}\langle J(z_1)J(z_2)J(z_3)\rangle &= \frac{m}{(4\pi)^3}\frac{\mathbf{u}_{12}{}^{\tilde i}{}_{\tilde j}\mathbf{u}_{23}{}^{\tilde j}{}_{\tilde k}\mathbf{u}_{31}{}^{\tilde k}{}_{\tilde i} + \mathbf{u}_{13}{}^{\tilde i}{}_{\tilde k}\mathbf{u}_{21}{}^{\tilde j}{}_{\tilde i}\mathbf{u}_{32}{}^{\tilde k}{}_{\tilde j}}{\boldsymbol{x}_{13}\boldsymbol{x}_{12}\boldsymbol{x}_{23}} \\ &\quad - \frac{n}{(4\pi)^3}\frac{\mathbf{u}_{12}{}^{i}{}_{j}\mathbf{u}_{23}{}^{j}{}_{k}\mathbf{u}_{31}{}^{k}{}_{i} + \mathbf{u}_{13}{}^{i}{}_{k}\mathbf{u}_{21}{}^{j}{}_{i}\mathbf{u}_{32}{}^{k}{}_{j}}{\boldsymbol{x}_{13}\boldsymbol{x}_{12}\boldsymbol{x}_{23}} \\ &= \frac{2m}{(4\pi)^3}\frac{\mathbf{U}_3{}^{\tilde i}{}_{\tilde i}}{\boldsymbol{x}_{12}\boldsymbol{x}_{23}\boldsymbol{x}_{13}} - \frac{2n}{(4\pi)^3}\frac{\mathbf{U}_3{}^{i}{}_{i}}{\boldsymbol{x}_{12}\boldsymbol{x}_{23}\boldsymbol{x}_{13}} \ .\end{aligned} \tag{5.21}$$

Here, in the last line, we have applied the relations (2.74). Next, using the identity (5.17), we express (5.21) in terms of N^{ij} and $N^{\tilde i \tilde j}$ introduced in (2.75)

$$\langle J(z_1)J(z_2)J(z_3)\rangle = \frac{1}{32\pi^3}\frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}(m\, N_3{}^{\tilde i}{}_{\tilde i} - n\, N_3{}^{i}{}_{i}) \ . \tag{5.22}$$

Taking into account the explicit form of the matrices N^{ij} and $N^{\tilde i \tilde j}$ given in (2.75), we conclude that the correlator has the form (4.24),

$$\langle J(z_1)J(z_2)J(z_3)\rangle = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}\left(\frac{\tilde d_{\mathcal{N}=4}}{X_3} + \frac{d_{\mathcal{N}=4}}{X_3{}^5}\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}X_{\alpha\beta}X_{\gamma\delta}\right) , \tag{5.23a}$$

JHEP08(2015)125

where

$$d_{\mathcal{N}=4} = \frac{m+n}{128\pi^3}\,, \qquad \tilde{d}_{\mathcal{N}=4} = \frac{m-n}{16\pi^3}\,. \tag{5.23b}$$

As discussed at the end of subsection 4.2, it is the $d_{\mathcal{N}=4}$-term in (5.23) which contributes to the three-point function of the energy-momentum tensor upon reduction to the component fields. In accordance with (5.23b), the coefficient $d_{\mathcal{N}=4}$ receives additive contributions from the q^i and $q^{\tilde{i}}$ hypermultiplets. The other coefficient $\tilde{d}_{\mathcal{N}=4}$ is non-zero when $m \neq n$. It is known that the mirror map[8] $\mathfrak{M}$ [12, 40] turns every left hypermultiplet q^i into a right one, $q^{\tilde{i}}$, and vice versa, see appendix A. Invariance under the mirror map implies that the theory has the same number of the q^i and $q^{\tilde{i}}$ hypermultiplets. Thus, we conclude that the non-vanishing value of $\tilde{d}_{\mathcal{N}=4}$ indicates that the superconformal theory under consideration is not invariant under the mirror map.

The ratio of the coefficient $d_{\mathcal{N}=4}$ in the three-point function (5.23) with $c_{\mathcal{N}=4}$, which determines the two-point correlator (5.14), is

$$\frac{d_{\mathcal{N}=4}}{c_{\mathcal{N}=4}} = \frac{1}{16\pi}\,. \tag{5.24}$$

Although we have found this relation for the special model of free $\mathcal{N}=4$ hypermultiplets, we expect that (5.24) is universal for all $\mathcal{N}=4$ superconformal models, as a consequence of a Ward identity. Indeed, there is a Ward identity relating the two- and three-point functions of the energy-momentum tensor (see, e.g., [23]). Since in $\mathcal{N}=4$ superconformal field theories each of them is determined by a single tensor structure, the relation between their coefficients can be found by considering a particular theory. Of course, it is possible to derive a superfield Ward identity expressing the $\mathcal{N}=4$ superconformal symmetry. Since its main application is to give another derivation of (5.24), we will not indulge in this technical issue in the present paper.

6 Ward identities for flavour current multiplets

The Ward identities play an important role in quantum field theory as they relate different Green functions. In this section we derive Ward identities for flavour current multiplets in $\mathcal{N}$-extended superconformal field theories, with $1 \leq \mathcal{N} \leq 4$. Such Ward identities relate the two- and three-point correlation functions of the flavour current multiplets and, in principle, allow one to relate the parameters in these correlators. The common feature of the four supersymmetry types $1 \leq \mathcal{N} \leq 4$ is that for each of these cases the Yang-Mills multiplet possesses an unconstrained prepotential formulation.

To derive the Ward identities we will use a standard field theoretic construction that can be described as follows. Consider a superconformal field theory that possesses a flavour current multiplet L (with all indices suppressed). We gauge the flavour symmetry by coupling the theory to a background vector multiplet described by an unconstrained prepotential V which will be the source for L. An n-point function for L is obtained by computing

[8]The mirror map is not directly related to mirror symmetry [41].

n functional derivatives of the generating functional $Z[V]$ with respect to V and then switching V off,

$$\mathrm{i}^n \langle L(1)\dots L(n)\rangle = \frac{\delta^n Z[V]}{\delta V(1)\dots\delta V(n)}\bigg|_{V=0} , \qquad (6.1)$$

where the operator insertions on the left are taken at distinct points. The Ward identities follow from the condition of the gauge invariance of $Z[V]$.

6.1 $\mathcal{N} = 1$ superconformal theories

In $\mathcal{N} = 1$ superconformal field theory, the flavour current multiplet is described by a primary real spinor superfield $L^{\bar{a}}_\alpha$ of dimension 3/2 subject to the conservation condition

$$D^\alpha L^{\bar{a}}_\alpha = 0 , \qquad (6.2)$$

with $\bar{a}$ being the flavour index (see [1] for more details). We now gauge the flavour symmetry by coupling the theory to a background vector multiplet described by a spinor prepotential $V^{\bar{a}}_\alpha$, which is real but otherwise unconstrained (see [42] for the details). The gauge transformation law of $V^{\bar{a}}_\alpha$ is

$$\delta_\lambda V^{\bar{a}}_\alpha = D_\alpha \lambda^{\bar{a}} - f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}}_\alpha \lambda^{\bar{c}} , \qquad (6.3)$$

with the superfield gauge parameters $\lambda^{\bar{a}}$ being real but otherwise unconstrained. The gauge prepotential $V^{\bar{a}}_\alpha$ is the source for the flavour current multiplet in the sense that

$$\mathrm{i}\langle L^{\bar{a}}_\alpha(z)\rangle_V = \frac{\delta Z[V]}{\delta V^{\alpha\bar{a}}(z)} , \qquad (6.4)$$

where $Z[V]$ is the generating functional. As usual, $\langle\dots\rangle_V$ denotes a correlation function in the presence of the background field. The gauge invariance of $Z[V]$ implies that

$$\int \mathrm{d}^{3|2}z\, (D_\alpha \lambda^{\bar{a}} - f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}}_\alpha \lambda^{\bar{c}}) \frac{\delta Z[V]}{\delta V^{\bar{a}}_\alpha(z)} = 0 . \qquad (6.5)$$

Since the gauge parameters $\lambda^{\bar{a}}$ are arbitrary superfields, we conclude that

$$\left(D^\alpha \frac{\delta}{\delta V^{\alpha\bar{a}}} - f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}\,\alpha} \frac{\delta}{\delta V^{\alpha\bar{c}}} \right) Z[V] = 0 . \qquad (6.6)$$

Varying this identity twice and switching off the source $V^{\bar{a}}_\alpha$, we end up with the Ward identity for $\mathcal{N} = 1$ flavour current multiplets

$$\begin{aligned} D^\alpha \langle L^{\bar{a}}_\alpha(z) L^{\bar{b}}_\beta(z_1) L^{\bar{c}}_\gamma(z_2)\rangle + \mathrm{i} f^{\bar{a}\bar{b}\bar{d}} \delta^{3|2}(z-z_1) \langle L^{\bar{d}}_\beta(z_1) L^{\bar{c}}_\gamma(z_2)\rangle& \\ + \mathrm{i} f^{\bar{a}\bar{c}\bar{d}} \delta^{3|2}(z-z_2) \langle L^{\bar{b}}_\beta(z_1) L^{\bar{d}}_\gamma(z_2)\rangle = 0 .& \end{aligned} \qquad (6.7)$$

Here $\delta^{3|2}(z-z')$ is the $\mathcal{N} = 1$ superspace delta-function.

6.2 $\mathcal{N} = 2$ superconformal theories

The $\mathcal{N} = 2$ flavour current multiplet is described by a primary real scalar superfield $L^{\bar{a}}$ of dimension 1 subject to the conservation equation

$$\left(D^{\alpha(I} D^{J)}_{\alpha} - \frac{1}{2} \delta^{IJ} D^{\alpha K} D^{K}_{\alpha} \right) L^{\bar{a}} = 0 \,, \tag{6.8}$$

see [1] for more details.

In this subsection it is useful to deal with complex Grassmann coordinates θ^α and $\bar{\theta}^\alpha$ for $\mathcal{N} = 2$ superspace that are related to the real ones, θ^α_I, as follows:

$$\theta^\alpha = \frac{1}{\sqrt{2}} (\theta^\alpha_1 + \mathrm{i} \theta^\alpha_2) \,, \qquad \bar{\theta}^\alpha = \frac{1}{\sqrt{2}} (\theta^\alpha_1 - \mathrm{i} \theta^\alpha_2) \;. \tag{6.9}$$

The corresponding spinor covariant derivatives are

$$D_\alpha = \frac{1}{\sqrt{2}} (D^1_\alpha - \mathrm{i} D^2_\alpha) \,, \qquad \bar{D}_\alpha = -\frac{1}{\sqrt{2}} (D^1_\alpha + \mathrm{i} D^2_\alpha) \;. \tag{6.10}$$

In this basis, the conservation equations (6.8) turn into the conditions

$$D^2 L^{\bar{a}} = 0 \,, \qquad \bar{D}^2 L^{\bar{a}} = 0 \,, \tag{6.11}$$

which mean that $L^{\bar{a}}$ is a real linear superfield.

We gauge the flavour symmetry by coupling the theory to a background vector multiplet described by a prepotential $V^{\bar{a}}$, which is real but otherwise unconstrained [43, 44]. The gauge transformation of the prepotential is

$$\delta_\lambda V^{\bar{a}} = \frac{\mathrm{i}}{2} (\bar{\lambda}^{\bar{a}} - \lambda^{\bar{a}}) + \frac{1}{2} f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}} (\lambda^{\bar{c}} + \bar{\lambda}^{\bar{c}}) + \dots \,, \tag{6.12}$$

where the gauge parameter $\lambda^{\bar{a}}$ is an arbitrary chiral scalar superfield. The ellipsis in (6.12) stands for those terms which are at least quadratic in $V^{\bar{a}}$, and therefore are irrelevant for the Ward identity relating the two- and three-point correlation functions. Below, we will systematically neglect the $\mathcal{O}(V^2)$-terms in the gauge transformation of $V^{\bar{a}}$.

The gauge prepotential $V^{\bar{a}}$ is the source for the flavour current multiplet $L^{\bar{a}}$ which is obtained from the generating functional $Z[V]$ by

$$\mathrm{i} \langle L^{\bar{a}}(z) \rangle_V = \frac{\delta Z[V]}{\delta V^{\bar{a}}(z)} \;. \tag{6.13}$$

The gauge invariance of the generating functional is expressed as

$$\int \mathrm{d}^{3|4} z \left(\frac{\mathrm{i}}{2} (\bar{\lambda}^{\bar{a}} - \lambda^{\bar{a}}) + \frac{1}{2} f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}} (\lambda^{\bar{c}} + \bar{\lambda}^{\bar{c}}) + \dots \right) \frac{\delta Z[V]}{\delta V^{\bar{a}}(z)} = 0 \;. \tag{6.14}$$

Since the gauge parameters $\lambda^{\bar{a}}$ are arbitrary chiral superfields, we end up with the following identity for the generating functional Z:

$$\bar{D}^2 \left(\frac{\delta}{\delta V^{\bar{a}}(z)} - \mathrm{i} f^{\bar{a}\bar{b}\bar{c}} V^{\bar{b}}(z) \frac{\delta}{\delta V^{\bar{c}}(z)} + \dots \right) Z[V] = 0 \tag{6.15}$$

JHEP08(2015)125

and its conjugate. Varying this equation twice and switching off the gauge superfield $V^{\bar{a}}$, we obtain the Ward identity relating the two- and three-point correlation functions of $\mathcal{N}=2$ flavour current multiplets

$$\begin{aligned}\bar{D}^2\langle L^{\bar{a}}(z)L^{\bar{e}}(z_1)L^{\bar{d}}(z_2)\rangle - 4f^{\bar{a}\bar{e}\bar{c}}\delta_+(z,z_1)\langle L^{\bar{c}}(z_1)L^{\bar{d}}(z_2)\rangle \\ -4f^{\bar{a}\bar{d}\bar{c}}\delta_+(z,z_2)\langle L^{\bar{e}}(z_1)L^{\bar{c}}(z_2)\rangle = 0\ .\end{aligned} \tag{6.16}$$

Here $\delta_+(z,z')$ is the chiral delta-function; it is expressed in terms of the full superspace delta-function $\delta^{3|4}(z-z')$ in the standard way

$$\delta_+(z,z') = -\frac{1}{4}\bar{D}^2\delta^{3|4}(z-z')\ . \tag{6.17}$$

6.3 $\mathcal{N}=3$ superconformal theories

It is known that the conventional 3D $\mathcal{N}=3$ Minkowski superspace $\mathbb{M}^{3|6}$ is not suitable to realise off-shell $\mathcal{N}=3$ supersymmetric theories. The adequate superspace setting for them [6] is $\mathbb{M}^{3|6}\times\mathbb{C}P^1$, which is an extension of $\mathbb{M}^{3|6}$ by the compact coset space $\mathsf{SU}(2)/\mathsf{U}(1)$ associated with the R-symmetry group.[9] The most general $\mathcal{N}=3$ supersymmetric gauge theories in three dimensions can be described using either the harmonic superspace techniques [6, 12] or the projective ones [34]. These formulations are 3D analogues of the 4D $\mathcal{N}=2$ harmonic [45, 46] and projective [47–49] superspace approaches (see also [50] for a review of the projective superspace formalism). The 3D $\mathcal{N}=3$ projective superspace setting has been used to construct the most general off-shell $\mathcal{N}=3$ superconformal σ-models [4] and supergravity-matter couplings [34]. The 3D $\mathcal{N}=3$ harmonic superspace has been shown to be efficient for studying the quantum aspects of $\mathcal{N}=3$ superconformal theories [51]. It also provides an elegant description of the ABJM theory [52]. In this subsection we will use the harmonic superspace to derive Ward identities for $\mathcal{N}=3$ flavour current multiplets.

We will use $\mathsf{SU}(2)$ harmonic variables u_i^+ and u_i^- constrained by

$$u^{+i}u_j^- - u^{-i}u_j^+ = \delta_j^i\,, \qquad \overline{u^{+i}} = u_i^-\ . \tag{6.18}$$

Associated with these variables there are the following vector fields

$$\partial^{++} = u_i^+\frac{\partial}{\partial u_i^-}\,, \quad \partial^{--} = u_i^-\frac{\partial}{\partial u_i^+}\,, \quad \partial^0 = u_i^+\frac{\partial}{\partial u_i^+} - u_i^-\frac{\partial}{\partial u_i^-}\,, \tag{6.19}$$

which form the $\mathsf{SU}(2)$ algebra

$$[\partial^0,\partial^{++}] = 2\partial^{++}\,, \quad [\partial^0,\partial^{--}] = -2\partial^{--}\,, \quad [\partial^{++},\partial^{--}] = \partial^0\ . \tag{6.20}$$

Using these harmonic variables allows one to introduce a new basis for the Grassmann variables θ_α^{ij} and the spinor covariant derivatives D_α^{ij}:

$$\theta_\alpha^{ij} \quad\longrightarrow\quad (\theta_\alpha^{++},\theta_\alpha^{--},\theta_\alpha^0) = (u_i^+u_j^+\theta_\alpha^{ij},\ u_i^-u_j^-\theta_\alpha^{ij},\ u_i^+u_j^-\theta_\alpha^{ij})\,, \tag{6.21a}$$

[9]For every positive integer $\mathcal{N}$, the 3D $\mathcal{N}$-extended superconformal group $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ is a transformation group of the compactified Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ in which Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ is embedded as a dense open domain [4]. In the $\mathcal{N}=3$ case, $\mathsf{OSp}(3|4;\mathbb{R})$ is also defined to act transitively on $\overline{\mathbb{M}}^{3|6}\times\mathbb{C}P^1$, as shown in [4].

$$D_\alpha^{ij} \quad \longrightarrow \quad (D_\alpha^{++}, D_\alpha^{--}, D_\alpha^0) = (u_i^+ u_j^+ D_\alpha^{ij},\, u_i^- u_j^- D_\alpha^{ij},\, u_i^+ u_j^- D_\alpha^{ij}) \ . \tag{6.21b}$$

As discussed in section 3, the $\mathcal{N}=3$ flavour current multiplet is described by a primary superfield $L^{ij} = L^{ji}$ subject to the conservation law (3.6). Associated with this superfield are the following harmonic projections:

$$L^{++} = u_i^+ u_j^+ L^{ij} \ , \quad L^{--} = u_i^- u_j^- L^{ij} \ , \quad L^0 = u_i^+ u_j^- L^{ij} \ . \tag{6.22}$$

It is sufficient to study only one of these projections, say L^{++}, since the others can be obtained by acting on L^{++} with ∂^{--}. By construction, L^{++} is annihilated by ∂^{++},

$$\partial^{++} L^{++} = 0 \ . \tag{6.23}$$

It is important that the equation (3.6) has the following corollary

$$D_\alpha^{++} L^{++} = 0 \, , \tag{6.24}$$

which is usually referred to as the analyticity condition.

The main feature of harmonic superspace is that it allows one to introduce new off-shell multiplets that are annihilated by D_α^{++}. Such superfields are defined on a supersymmetric subspace of $\mathbb{M}^{3|6} \times \mathbb{C}P^1$ known as the analytic subspace. It is parametrised by coordinates

$$\zeta = (x_A^a, \theta_\alpha^{++}, \theta_\alpha^0, u_i^\pm) \, , \tag{6.25}$$

where

$$x_A^a = x^a + \mathrm{i}\gamma^a_{\alpha\beta}\theta^{++\alpha}\theta^{--\beta} \ . \tag{6.26}$$

In the analytic coordinate basis for $\mathbb{M}^{3|6} \times \mathbb{C}P^1$ consisting of the variables ζ and θ_α^{--}, the spinor covariant derivative D_α^{++} becomes short,

$$D_\alpha^{++} = \frac{\partial}{\partial \theta^{--\alpha}} \, , \tag{6.27}$$

while the harmonic derivative ∂^{++} acquires additional terms

$$\mathscr{D}^{++} = \partial^{++} + 2\mathrm{i}\gamma^a_{\alpha\beta}\theta^{++\alpha}\theta^{0\alpha}\frac{\partial}{\partial x_A^a} + \theta^{++\alpha}\frac{\partial}{\partial\theta^{0\alpha}} + 2\theta^{0\alpha}\frac{\partial}{\partial\theta^{--\alpha}} \ . \tag{6.28}$$

Therefore, in the analytic basis, the equation (6.24) tells us that $L^{++} = L^{++}(\zeta)$ while (6.23) becomes a non-trivial constraint

$$\mathscr{D}^{++} L^{++} = 0 \ . \tag{6.29}$$

We are prepared to derive Ward identities in a superconformal field theory possessing flavour current multiplets $L^{++\bar{a}}$. For this we gauge the flavour symmetry by coupling the theory to a background vector multiplet described by a prepotential $V^{++\bar{a}}$ which is an analytic real superfield. Its gauge transformation reads

$$\delta_\lambda V^{++\bar{a}} = \mathscr{D}^{++}\lambda^{\bar{a}} - f^{\bar{a}\bar{b}\bar{c}} V^{++\bar{b}}\lambda^{\bar{c}} \, , \tag{6.30}$$

where the gauge parameters $\lambda^{\bar a}$ are unconstrained analytic superfields. The gauge prepotential $V^{++\bar a}$ is the source for the flavour current multiplet $L^{++\bar a}$ which is obtained from the generating functional $Z[V]$ by

$$\mathrm{i}\langle L^{++\bar a}(\zeta)\rangle_V = \frac{\delta Z[V]}{\delta V^{++\bar a}(\zeta)} \ . \tag{6.31}$$

The gauge invariance of $Z[V]$ implies the equation

$$\int \mathrm{d}\zeta^{(-4)}(\mathscr{D}^{++}\lambda^{\bar a} - f^{\bar a\bar b\bar c}V^{++\bar b}\lambda^{\bar c})\frac{\delta Z[V]}{\delta V^{++\bar a}(\zeta)} = 0\,, \tag{6.32}$$

where $\mathrm{d}\zeta^{(-4)}$ is the invariant measure on the analytic subspace (6.25). Since the gauge parameters $\lambda^{\bar a}$ in (6.32) are arbitrary, we conclude that

$$\left(\mathscr{D}^{++}\frac{\delta}{\delta V^{++\bar a}(\zeta)} - f^{\bar a\bar b\bar c}V^{++\bar b}(\zeta)\frac{\delta}{\delta V^{++\bar c}(\zeta)}\right) Z[V] = 0\ . \tag{6.33}$$

Finally, varying this relation twice and switching off the gauge superfield V^{++} we end up with the Ward identity for the correlation functions of flavour current multiplets $L^{++\bar a}$

$$\begin{aligned}\mathscr{D}^{++}_{(\zeta)}\langle L^{++\bar a}(\zeta)L^{++\bar b}(\zeta_1)L^{++\bar c}(\zeta_2)\rangle + \mathrm{i}f^{\bar a\bar b\bar d}\delta_A^{(4,0)}(\zeta,\zeta_1)\langle L^{++\bar d}(\zeta_1)L^{++\bar c}(\zeta_2)\rangle&\\ +\mathrm{i}f^{\bar a\bar c\bar d}\delta_A^{(4,0)}(\zeta,\zeta_2)\langle L^{++\bar b}(\zeta_1)L^{++\bar d}(\zeta_2)\rangle &= 0\,,\end{aligned} \tag{6.34}$$

where $\delta_A^{(4,0)}(\zeta,\zeta')$ is the delta-function in the analytic subspace.

6.4 $\mathcal{N} = 4$ superconformal theories

The R-symmetry group of the $\mathcal{N} = 4$ super-Poincaré algebra is $\mathsf{SU}(2)_\mathrm{L} \times \mathsf{SU}(2)_\mathrm{R}$. It is the superspace (A.1) which is adequate to formulate general off-shell $\mathcal{N} = 4$ supersymmetric theories. Hence, one can introduce harmonic variables for either of the $\mathsf{SU}(2)$ subgroups, or for both of them. For studying Ward identities involving correlation functions of the left flavour current multiplets L^{ij} it is sufficient to introduce harmonic variables for the subgroup $\mathsf{SU}(2)_\mathrm{L}$ which acts on the indices i, j. We will use the same harmonic variables $u_i^\pm$ constrained by (6.18) and the corresponding harmonic derivatives (6.19). Now we project the $\mathcal{N} = 4$ Grassmann variables $\theta_\alpha^{i\tilde i}$ and spinor covariant derivatives $D_\alpha^{i\tilde i}$ as

$$\theta_\alpha^{i\tilde i} \longrightarrow (\theta_\alpha^{\tilde i+}, \theta_\alpha^{\tilde i-}) = (u_i^+\theta_\alpha^{i\tilde i},\, u_i^-\theta_\alpha^{i\tilde i})\,, \tag{6.35}$$

$$D_\alpha^{i\tilde i} \longrightarrow (D_\alpha^{\tilde i+}, D_\alpha^{\tilde i-}) = (u_i^+D_\alpha^{i\tilde i},\, u_i^-D_\alpha^{i\tilde i})\ . \tag{6.36}$$

The flavour current multiplet L^{ij} has the same harmonic projections as in (6.22). The equation (6.23) remains unchanged in the $\mathcal{N} = 4$ case while the analyticity constraint (6.24) turns into

$$D_\alpha^{\tilde i+}L^{++} = 0\ . \tag{6.37}$$

This equation follows from (4.5a) by contracting the indices i, k, l with the u^+-harmonics.

Let us consider the analytic subspace of the $\mathcal{N} = 4$ harmonic superspace parametrised by the variables

$$\zeta = (x_A^a, \theta_\alpha^{\tilde i+}, u_i^\pm)\,, \tag{6.38}$$

where

$$x_A^a = x^a + \mathrm{i}\gamma^a_{\alpha\beta}\theta^{\tilde{i}+\alpha}\theta^{-\beta}_{\tilde{i}} \ . \tag{6.39}$$

In the coordinate system $(\zeta, \theta^{\tilde{i}-}_\alpha)$, the spinor covariant derivative $D^{\tilde{i}+}_\alpha$ becomes short,

$$D^{\tilde{i}+}_\alpha = \frac{\partial}{\partial\theta^{-\alpha}_{\tilde{i}}} \,, \tag{6.40}$$

while the covariant harmonic derivatives ∂^{++} and ∂^{--} acquire additional terms

$$\mathscr{D}^{++} = \partial^{++} + \mathrm{i}\gamma^a_{\alpha\beta}\theta^{\tilde{i}+\alpha}\theta^{+\beta}_{\tilde{i}}\frac{\partial}{\partial x^a_A} + \theta^{\tilde{i}+}_\alpha\frac{\partial}{\partial\theta^{\tilde{i}-}_\alpha} \,, \tag{6.41a}$$

$$\mathscr{D}^{--} = \partial^{--} + \mathrm{i}\gamma^a_{\alpha\beta}\theta^{\tilde{i}-\alpha}\theta^{-\beta}_{\tilde{i}}\frac{\partial}{\partial x^a_A} + \theta^{\tilde{i}-}_\alpha\frac{\partial}{\partial\theta^{\tilde{i}+}_\alpha} \ . \tag{6.41b}$$

The crucial feature of using the analytic coordinates in the $\mathcal{N}=4$ harmonic superspace is that the equation (6.37) is automatically solved by the analytic superfield $L^{++} = L^{++}(\zeta)$ while (6.23) turns into a non-trivial constraint

$$\mathscr{D}^{++}L^{++} = 0 \ . \tag{6.42}$$

Once the analytic subspace (6.38) in the $\mathcal{N}=4$ superspace is introduced, the further derivation of the Ward identity for L^{++} goes exactly the same way as in section 6.3 and the equations (6.30)–(6.33) remain unchanged. Thus, we end up with the Ward identity for L^{++} exactly in the form (6.34)

$$\begin{aligned}\mathscr{D}^{++}_{(\zeta)}\langle L^{++\bar{a}}(\zeta)L^{++\bar{b}}(\zeta_1)L^{++\bar{c}}(\zeta_2)\rangle + \mathrm{i}f^{\bar{a}\bar{b}\bar{d}}\delta^{(4,0)}_A(\zeta,\zeta_1)\langle L^{++\bar{d}}(\zeta_1)L^{++\bar{c}}(\zeta_2)\rangle& \\ +\mathrm{i}f^{\bar{a}\bar{c}\bar{d}}\delta^{(4,0)}_A(\zeta,\zeta_2)\langle L^{++\bar{b}}(\zeta_1)L^{++\bar{d}}(\zeta_2)\rangle = 0 \ .&\end{aligned} \tag{6.43}$$

In a similar way one can find the Ward identity for the right flavour current multiplet $L^{\tilde{i}\tilde{j}}$ by introducing the harmonic variables for the subgroup $\mathsf{SU}(2)_{\mathrm{R}}$ of the R-symmetry group.

It is instructive to check that the Ward identity (6.43) is satisfied for the free hypermultiplets.

Consider the action for a single hypermultiplet

$$S = \int \mathrm{d}\zeta^{(-4)}\breve{q}^+\mathscr{D}^{++}q^+ \,, \tag{6.44}$$

where q^+ is constrained by

$$D^{\tilde{i}+}_\alpha q^+ = 0 \,, \tag{6.45}$$

and the same constraints hold for its smile-conjugate $\breve{q}^+$. The superfield q^+ contains infinitely many auxiliary component fields at the component level. These auxiliary fields vanish on the equation of motion

$$\mathscr{D}^{++}q^+ = 0 \,, \tag{6.46}$$

which implies that the hypermultiplet superfields take the form

$$q^+(z,u) = u_i^+ q^i(z)\,, \qquad \breve{q}^+(z,u) = u^{+i}\bar{q}_i(z)\ . \tag{6.47}$$

The two-point function $\langle q^+(\zeta_1)\breve{q}^+(\zeta_2)\rangle$ corresponding to the action (6.44) is

$$\langle q^+(\zeta_1)\breve{q}^+(\zeta_2)\rangle = -\mathrm{i}G^{(+,+)}(\zeta_1,\zeta_2)\ . \tag{6.48}$$

Here we have introduced the Green function $G^{(+,+)}(\zeta_1,\zeta_2)$ as a solution of the equation

$$\mathscr{D}^{++}G^{(+,+)}(\zeta_1,\zeta_2) = -\delta_A^{(3,1)}(\zeta_1,\zeta_2)\ . \tag{6.49}$$

It can be represented explicitly in the following form (see appendix B for the details)

$$G^{(+,+)}(\zeta_1,\zeta_2) = -\frac{\mathrm{i}}{4\pi}\frac{(u_1^+u_2^+)}{\sqrt{\hat{x}^a_{12}\hat{x}_{12a}}}\,, \tag{6.50}$$

where

$$\hat{x}^a_{12} = x^a_{A\,1} - x^a_{A\,2} - \frac{\mathrm{i}}{(u_1^+u_2^+)}[(u_1^-u_1^+)\theta_1^{+\tilde{i}}\gamma^a\theta^+_{1\tilde{i}} - (u_1^+u_2^-)\theta_2^{+\tilde{i}}\gamma^a\theta^+_{2\tilde{i}} + 2\theta_1^{+\tilde{i}}\gamma^a\theta^+_{2\tilde{i}}] \tag{6.51}$$

is a manifestly analytic coordinate difference that is invariant under Q-supersymmetry transformations. We point out that the two-point function (6.48) is related to (5.4a) according to eq. (5.5).

Let us consider a free superconformal theory describing a column vector $\boldsymbol{q}^+$ of several q^+ hypermultiplets and their smile conjugates $\breve{\boldsymbol{q}}^+$ viewed as a row vector. The corresponding action, which is the sum of n free actions (6.44) is invariant under rigid flavour transformations

$$\delta\boldsymbol{q}^+ = \mathrm{i}\lambda^{\bar{a}}\Sigma^{\bar{a}}\boldsymbol{q}^+\,, \qquad \delta\breve{\boldsymbol{q}}^+ = -\mathrm{i}\lambda^{\bar{a}}\breve{\boldsymbol{q}}^+\Sigma^{\bar{a}}\,, \tag{6.52}$$

with constant real parameters $\lambda^{\bar{a}}$ and Hermitian generators $\Sigma^{\bar{a}}$ of the flavour group. We gauge this symmetry by coupling the hypermultiplets to an analytic gauge prepotential $V^{++} = V^{++\bar{a}}(\zeta)\Sigma^{\bar{a}}$ taking its values in the Lie algebra of the flavour group,

$$S = \int \mathrm{d}\zeta^{(-4)}\breve{\boldsymbol{q}}^+\mathscr{D}^{++}\boldsymbol{q}^+ \quad\longrightarrow\quad \int \mathrm{d}\zeta^{(-4)}\breve{\boldsymbol{q}}^+\Big(\mathscr{D}^{++} + \mathrm{i}V^{++}\Big)\boldsymbol{q}^+\ . \tag{6.53}$$

From the action obtained we read off the flavour current multiplet

$$L^{++\bar{a}}(\zeta) = \mathrm{i}\breve{\boldsymbol{q}}^+\Sigma^{\bar{a}}\boldsymbol{q}^+\ . \tag{6.54}$$

By construction, $L^{++\bar{a}}$ respects the analyticity constraint (6.37). It also obeys the condition (6.42) on the mass shell. Thus, our new representation (6.54) for the flavour current multiplet is equivalent to (5.8) we used before.

Let us use the new representation (6.54) to compute the two- and three-point functions of the flavour current multiplets. Performing the Wick contractions gives

$$\langle L^{++\bar{a}}(\zeta_1)L^{++\bar{b}}(\zeta_2)\rangle = k_{\mathrm{L}}\delta^{\bar{a}\bar{b}}\langle q^+(\zeta_1)\bar{q}^+(\zeta_2)\rangle\langle q^+(\zeta_1)\bar{q}^+(\zeta_2)\rangle\,, \tag{6.55}$$

$$\langle L^{++\bar{a}}(\zeta_1)L^{++\bar{b}}(\zeta_2)L^{++\bar{c}}(\zeta_3)\rangle = -k_{\rm L} f^{\bar{a}\bar{b}\bar{c}} \langle q^+(\zeta_1)\bar{q}^+(\zeta_3)\rangle \langle q^+(\zeta_1)\bar{q}^+(\zeta_2)\rangle \langle q^+(\zeta_2)\bar{q}^+(\zeta_3)\rangle , \tag{6.56}$$

where the propagator is given by (6.48). Now we use the explicit form of the hypermultiplet Green's function (6.50) and obtain a new representation for the correlators of the flavour current multiplets

$$\langle L^{++\bar{a}}(\zeta_1)L^{++\bar{b}}(\zeta_2)\rangle = a_{\mathcal{N}=4}\delta^{\bar{a}\bar{b}}\frac{(u_1^+u_2^+)^2}{\hat{x}_{12}{}^2} , \tag{6.57}$$

$$\langle L^{++\bar{a}}(\zeta_1)L^{++\bar{b}}(\zeta_2)L^{++\bar{c}}(\zeta_3)\rangle = \sqrt{2}b_{\mathcal{N}=4}f^{\bar{a}\bar{b}\bar{c}}\frac{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}{\sqrt{\hat{x}_{12}{}^2\hat{x}_{23}{}^2\hat{x}_{31}{}^2}} . \tag{6.58}$$

These expressions are manifestly analytic in all arguments. They are equivalent to (4.8) and (4.9) modulo contact terms which vanish for non-coincident superspace points. We stress that for generic values of $a_{\mathcal{N}=4}$ and $b_{\mathcal{N}=4}$ the form of the correlation functions (6.57) and (6.58) is universal for any $\mathcal{N}=4$ superconformal theory although they were derived for free hypermultiplets. The values of the coefficients $a_{\mathcal{N}=4}$ and $b_{\mathcal{N}=4}$ for the case of free hypermultiplets are given by (5.13) and (5.18), respectively.

Recall that the hypermultiplet Green's function (6.50) obeys the equation (6.49). Using this equation we compute the derivative of the expression (6.58)

$$\begin{aligned}&\mathscr{D}^{++}_{(1)}\langle L^{++\bar{a}}(\zeta_1)L^{++\bar{b}}(\zeta_2)L^{++c}(\zeta_3)\rangle = 4\sqrt{2}{\rm i}\pi b_{\mathcal{N}=4}f^{\bar{a}\bar{b}\bar{c}}[\delta_A^{(4,0)}(\zeta_1,\zeta_2)-\delta_A^{(4,0)}(\zeta_1,\zeta_3)]\frac{(u_2^+u_3^+)}{\hat{x}_{12}{}^2}\\ &= 4\sqrt{2}{\rm i}\pi\frac{b_{\mathcal{N}=4}}{a_{\mathcal{N}=4}}f^{\bar{a}\bar{b}\bar{d}}[\delta_A^{(4,0)}(\zeta_1,\zeta_2)-\delta_A^{(4,0)}(\zeta_1,\zeta_3)]\langle L^{++\bar{c}}(\zeta_2)L^{++\bar{d}}(\zeta_3)\rangle .\end{aligned} \tag{6.59}$$

Hence, the correlation functions (6.57) and (6.58) obey the Ward identity (6.43) if the coefficients $a_{\mathcal{N}=4}$ and $b_{\mathcal{N}=4}$ are related to each other by the equation (5.19) which was found previously for the case of free hypermultiplets. Here we have demonstrated that it holds for every $\mathcal{N}=4$ superconformal field theory.

7 Relations between correlation functions in superconformal field theories with $1 \leq \mathcal{N} \leq 4$

The study of correlation functions performed in the present paper is the continuation of our earlier work [1]. In [1] and in sections 3 and 4 of the present paper, we derived explicit expressions for the two- and three-point correlation functions of the supercurrent and flavour current multiplets in three-dimensional $\mathcal{N}$-extended superconformal field theories with $1 \leq \mathcal{N} \leq 4$. As was discussed above, the coefficients of the two- and three-point function are not independent but are related by the Ward identities. The aim of this section is to derive the relations between these coefficients for $1 \leq \mathcal{N} \leq 4$. Our derivation will be based on the following two observations.

- If both the two- and the three-point functions are fixed up to overall coefficients and are related to each other by the Ward identities, we can find a universal relation between the coefficients by considering a particular theory. We have already used this observation to obtain the relations (5.19) and (5.24) which are valid in any $\mathcal{N}=4$ superconformal field theory.[10]

- Every $\mathcal{N}=4$ superconformal theory can also be viewed as a special $\hat{\mathcal{N}}$-extended superconformal theory, with $\hat{\mathcal{N}} \leq 3$. Since the relevant two- and three-point functions in theories with $\mathcal{N}=1,2,3$ supersymmetries are fixed up to overall coefficients[11] [1] we can find similar universal relations between the coefficients using eqs. (5.19) and (5.24).

To perform explicit calculations we use the fact that the correlation functions of conserved currents for different $\mathcal{N}$ are related to each other by the superspace reduction. Indeed, as explained in [1], the supercurrents in $1 \leq \mathcal{N} \leq 3$ superconformal theories can be derived from the supercurrent J in the $\mathcal{N}=4$ theory by applying covariant spinor derivatives and switching off some of the Grassmann coordinates. The flavour current multiplets in $1 \leq \mathcal{N} \leq 3$ theories can also be derived from the $\mathcal{N}=4$ flavour current multiplets by applying the rules of the superspace reduction discussed in [1].

7.1 Superspace reduction of the supercurrent correlation functions

7.1.1 $\mathcal{N}=3$ supercurrent

Let us start with the $\mathcal{N}=4$ supercurrent J whose correlation functions are given by (4.17), (4.18) and (4.24). The $\mathcal{N}=3$ supercurrent J_α is related to J as follows

$$J_\alpha = \mathrm{i} D^4_\alpha J| \,, \tag{7.1}$$

where the bar-projection means that we set $\theta^\alpha_4 = 0$. Hence, for the correlation functions of J_α we have

$$\begin{aligned} \langle J_\alpha(z_1) J_\beta(z_2)\rangle &= -D^4_{(1)\alpha} D^4_{(2)\beta} \langle J(z_1) J(z_2)\rangle| \,, & (7.2)\\ \langle J_\alpha(z_1) J_\beta(z_2) J_\gamma(z_3)\rangle &= -\mathrm{i} D^4_{(1)\alpha} D^4_{(2)\beta} D^4_{(3)\gamma} \langle J(z_1) J(z_2) J(z_3)\rangle| \,. & (7.3)\end{aligned}$$

Computation of the required derivatives of (4.17) and (4.18) is a straightforward but tedious task. The details of this procedure are given in subsection C.1. Here we present the results:

$$\begin{aligned} \langle J_\alpha(z_1) J_\beta(z_2)\rangle &= \mathrm{i} c_{\mathcal{N}=3} \frac{\boldsymbol{x}_{12\alpha\beta}}{\boldsymbol{x}_{12}{}^4} \,, & (7.4\text{a})\\ \langle J_\alpha(z_1) J_\beta(z_2) J_\gamma(z_3)\rangle &= d_{\mathcal{N}=3} \frac{\boldsymbol{x}_{13\alpha\alpha'} \boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4 \boldsymbol{x}_{23}{}^4} H^{\alpha'\beta'}{}_\gamma(\boldsymbol{X}_3, \Theta_3) \,, & (7.4\text{b})\end{aligned}$$

[10] The coefficient $\tilde{d}_{\mathcal{N}=4}$ does not contribute to the three-point function of the energy-momentum tensor and, hence, does not appear in the Ward identities. Thus, it is not related to the coefficients $c_{\mathcal{N}=4}$ and $d_{\mathcal{N}=4}$ in a universal manner.

[11] In the case of the three-point function of flavour current multiplets in $\mathcal{N}=2$ superconformal field theories, there is a second structure proportional to the totally symmetric tensor of the flavour symmetry group [1]. However, this structure does not contribute to the three-point functions of conserved currents. Hence, it does not contribute to the Ward identities and can be ignored for our discussion.

$$H^{\alpha\beta}{}_{\gamma}(\boldsymbol{X},\Theta) = \frac{1}{\boldsymbol{X}^5}\Big[(\delta^\beta_\gamma \boldsymbol{X}^{\alpha\rho} + \delta^\alpha_\gamma \boldsymbol{X}^{\rho\beta})\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\rho\varepsilon_{IJK}$$
$$+\boldsymbol{X}^{\beta\alpha}\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\gamma\varepsilon_{IJK} + 2\boldsymbol{X}^{\alpha\mu}\boldsymbol{X}^{\nu\beta}\Theta^I_\mu\Theta^J_\nu\Theta^K_\gamma\varepsilon_{IJK}\Big], \quad (7.4c)$$

where

$$c_{\mathcal{N}=3} = 2c_{\mathcal{N}=4}\,, \quad (7.5a)$$
$$d_{\mathcal{N}=3} = 4d_{\mathcal{N}=4}\,. \quad (7.5b)$$

Eqs. (7.4) are precisely the expressions for the correlation functions of the $\mathcal{N}=3$ supercurrent obtained in [1]. Using eq. (5.24) we obtain the following relation between the coefficients

$$\frac{d_{\mathcal{N}=3}}{c_{\mathcal{N}=3}} = \frac{1}{8\pi}\,. \quad (7.6)$$

We expect that this relation is valid in any $\mathcal{N}=3$ superconformal theory.

7.1.2 $\mathcal{N}=2$ supercurrent

The $\mathcal{N}=2$ supercurrent $J_{\alpha\beta}$ is related to the $\mathcal{N}=3$ supercurrent J_α as follows

$$J_{\alpha\beta} = D^3_\alpha J_\beta|\,, \quad (7.7)$$

where the bar-projection means that $\theta^\alpha_3 = 0$. Hence, the correlation functions of $\mathcal{N}=2$ supercurrents can be found from (7.4) by the rules

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)\rangle = -D^3_{(1)\alpha}D^3_{(2)\beta}\langle J_{\alpha'}(z_1)J_{\beta'}(z_2)\rangle|\,, \quad (7.8)$$
$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = -D^3_{(1)\alpha}D^3_{(2)\beta}D^3_{(3)\gamma}\langle J_{\alpha'}(z_1)J_{\beta'}(z_2)J_{\gamma'}(z_3)\rangle|\,. \quad (7.9)$$

The details of computations of these derivatives are given in subsection C.2. The resulting expressions are:

$$\langle J_{\alpha\beta}(z_1)J^{\alpha'\beta'}(z_2)\rangle = c_{\mathcal{N}=2}\frac{\boldsymbol{x}_{12\alpha}{}^{(\alpha'}\boldsymbol{x}_{12\beta}{}^{\beta')}}{\boldsymbol{x}_{12}{}^6}\,, \quad (7.10)$$
$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = d_{\mathcal{N}=2}\frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^6}H^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(\boldsymbol{X}_3,\Theta_3)\,, \quad (7.11)$$

where

$$H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(\boldsymbol{X},\Theta) = \frac{2\mathrm{i}}{\boldsymbol{X}^3}\left[\varepsilon^{\alpha(\beta}\varepsilon^{\beta')\alpha'}\Theta^\gamma_I\Theta^{\gamma'}_J + \varepsilon^{\alpha(\gamma}\varepsilon^{\gamma')\alpha'}\Theta^\beta_I\Theta^{\beta'}_J + \varepsilon^{\beta(\gamma}\varepsilon^{\gamma')\beta'}\Theta^\alpha_I\Theta^{\alpha'}_J\right]\varepsilon^{IJ}$$
$$+\frac{\mathrm{i}}{\boldsymbol{X}^5}\left[3\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\gamma\gamma'}\Theta^\beta_I\Theta^{\beta'}_J + 3\boldsymbol{X}^{\beta\beta'}\boldsymbol{X}^{\gamma\gamma'}\Theta^\alpha_I\Theta^{\alpha'}_J - 5\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\beta\beta'}\Theta^\gamma_I\Theta^{\gamma'}_J\right]\varepsilon^{IJ}$$
$$+\frac{\mathrm{i}}{\boldsymbol{X}^5}\left[5\varepsilon^{\alpha(\gamma}\varepsilon^{\gamma')\alpha'}\boldsymbol{X}^{\beta\beta'} + 5\varepsilon^{\beta(\gamma}\varepsilon^{\gamma')\beta'}\boldsymbol{X}^{\alpha\alpha'} - 3\varepsilon^{\alpha(\beta}\varepsilon^{\beta')\alpha'}\boldsymbol{X}^{\gamma\gamma'}\right]\boldsymbol{X}^{\delta\delta'}\Theta^I_\delta\Theta^J_{\delta'}\varepsilon_{IJ}$$
$$+\frac{5}{2}\frac{\mathrm{i}}{\boldsymbol{X}^7}\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\beta\beta'}\boldsymbol{X}^{\gamma\gamma'}\boldsymbol{X}^{\delta\delta'}\Theta^I_\delta\Theta^J_{\delta'}\varepsilon_{IJ}\,. \quad (7.12)$$

The coefficients $c_{\mathcal{N}=2}$ and $d_{\mathcal{N}=2}$ in (7.10) and (7.11) are related to $c_{\mathcal{N}=3}$ and $d_{\mathcal{N}=3}$ as

$$c_{\mathcal{N}=2} = -4c_{\mathcal{N}=3}\,, \qquad d_{\mathcal{N}=2} = -6d_{\mathcal{N}=3}\,. \quad (7.13)$$

Eqs. (7.10), (7.11), (7.12) represent precisely the correlation functions of the $\mathcal{N}=2$ supercurrent obtained in [1].

Taking into account (7.6), we find the ratio of the coefficients (7.13):

$$\frac{d_{\mathcal{N}=2}}{c_{\mathcal{N}=2}} = \frac{3}{16\pi}\,, \tag{7.14}$$

which we expect to be valid in any $\mathcal{N}=2$ superconformal theory.

7.1.3 $\mathcal{N}=1$ supercurrent

Consider now the reduction of the $\mathcal{N}=2$ supercurrent $J_{\alpha\beta}$ to the $\mathcal{N}=1$ supercurrent

$$J_{\alpha\beta\gamma} = \mathrm{i}D^2_\alpha J_{\beta\gamma}|\,, \tag{7.15}$$

where the bar-projection means that $\theta^\alpha_2 = 0$. The corresponding relation for the supercurrent correlation functions reads

$$\langle J_{\alpha\alpha'\alpha''}(z_1)J_{\beta\beta'\beta''}(z_2)\rangle = -D^2_{(1)\alpha}D^2_{(2)\beta}\langle J_{\alpha'\alpha''}(z_1)J_{\beta'\beta''}(z_2)\rangle|\,, \tag{7.16}$$

$$\langle J_{\alpha\alpha'\alpha''}(z_1)J_{\beta\beta'\beta''}(z_2)J_{\gamma\gamma'\gamma''}(z_3)\rangle = -\mathrm{i}D^2_{(1)\alpha}D^2_{(2)\beta}D^2_{(3)\gamma}\langle J_{\alpha'\alpha''}(z_1)J_{\beta'\beta''}(z_2)J_{\gamma'\gamma''}(z_3)\rangle|. \tag{7.17}$$

The computations of these expressions were performed in [1]. Here we give only the result:

$$\langle J_{\alpha\beta\gamma}(z_1)J^{\alpha'\beta'\gamma'}(z_2)\rangle = \mathrm{i}c_{\mathcal{N}=1}\frac{\boldsymbol{x}_{12\alpha}{}^{(\alpha'}\boldsymbol{x}_{12\beta}{}^{\beta'}\boldsymbol{x}_{12\gamma}{}^{\gamma')}}{\boldsymbol{x}_{12}{}^8}\,, \tag{7.18}$$

$$\begin{aligned}\langle J_{\alpha\alpha'\alpha''}(z_1)J_{\beta\beta'\beta''}(z_2)J_{\gamma\gamma'\gamma''}(z_3)\rangle &= \mathrm{i}d_{\mathcal{N}=1}\frac{\boldsymbol{x}_{13\alpha}{}^{\rho}\boldsymbol{x}_{13\alpha'}{}^{\rho'}\boldsymbol{x}_{13\alpha''}{}^{\rho''}\boldsymbol{x}_{23\beta}{}^{\sigma}\boldsymbol{x}_{23\beta'}{}^{\sigma'}\boldsymbol{x}_{23\beta''}{}^{\sigma'}}{\boldsymbol{x}_{13}{}^8\boldsymbol{x}_{23}{}^8}\\ &\quad\times H_{\rho\rho'\rho''\,\sigma\sigma'\sigma''\,\gamma\gamma'\gamma''}(\boldsymbol{X}_3,\Theta_3)\,,\end{aligned} \tag{7.19}$$

where the tensor $H^{\alpha\alpha'\alpha''\,\beta\beta'\beta''\,\gamma\gamma'\gamma''} = (\gamma_m)^{\alpha'\alpha''}(\gamma_n)^{\beta'\beta''}(\gamma_k)^{\gamma'\gamma''}H^{\alpha m\,\beta n\,\gamma k}$ has the following complicated but explicit form:

$$\begin{aligned}
H^{\alpha m\,\beta n\,\gamma k}(X,\Theta) &= (\gamma_p)^{\alpha\beta}\Big[\Theta^\gamma C^{(mnp),k} + \frac{1}{2}(\gamma_r)^\gamma{}_\delta\Theta^\delta\varepsilon^{krq}\eta_{qq'}C^{(mnp),q'} + (\gamma_r)^\gamma{}_\delta\Theta^\delta D^{(mnp),(kr)}\Big]\,,\\
C^{mnp,k} &= \frac{1}{X^3}(\eta^{mn}\eta^{kp} + \eta^{mk}\eta^{np} + \eta^{nk}\eta^{mp})\\
&\quad + \frac{3}{X^5}(X^mX^k\eta^{np} + X^nX^k\eta^{mp} + X^pX^k\eta^{mn})\\
&\quad - \frac{5}{X^5}(X^mX^n\eta^{pk} + X^nX^p\eta^{mk} + X^mX^p\eta^{nk}) - \frac{5}{X^7}X^mX^nX^pX^k\,,\\
D^{(mnp),(kr)} &= \varepsilon^{mks}\eta_{ss'}T^{(np),r,s'} + \varepsilon^{nks}\eta_{ss'}T^{(mp),r,s'} + \varepsilon^{pks}\eta_{ss'}T^{(mn),r,s'}\\
&\quad + \varepsilon^{mrs}\eta_{ss'}T^{(np),k,s'} + \varepsilon^{nrs}\eta_{ss'}T^{(mp),k,s'} + \varepsilon^{prs}\eta_{ss'}T^{(mn),k,s'}\,,\\
T^{(np),r,s} &= \frac{1}{2}\left[\frac{\eta^{nr}X^pX^s + \eta^{pr}X^nX^s - \eta^{np}X^rX^s}{X^5} + \frac{3X^nX^pX^rX^s}{X^7}\right]\,.
\end{aligned} \tag{7.20}$$

It is important that the coefficients $c_{\mathcal{N}=1}$ and $d_{\mathcal{N}=1}$ in (7.18) and (7.19) are expressed in terms of $c_{\mathcal{N}=2}$ and $d_{\mathcal{N}=2}$ as

$$c_{\mathcal{N}=1} = 6c_{\mathcal{N}=2}\,, \qquad d_{\mathcal{N}=1} = -5d_{\mathcal{N}=2}\,. \tag{7.21}$$

From (7.14) we find the ratio of these coefficients:

$$\frac{d_{\mathcal{N}=1}}{c_{\mathcal{N}=1}} = -\frac{5}{32\pi}\,. \tag{7.22}$$

7.2 Correlation functions of flavour current multiplets

We now turn to deriving relations between the coefficients in the two- and three-point correlators of flavour current multiplets.

7.2.1 $\mathcal{N}=3$ flavour current multiplets

The two- and three-point correlation functions of $\mathcal{N}=4$ flavour current multiplets are found in the form (4.8) and (4.9). They contain free coefficients $a_{\mathcal{N}=4}$ and $b_{\mathcal{N}=4}$ which are related to each other by (5.19). Owing to the identities (4.10), the same relation must hold for the coefficients among two-point and three-point functions in $\mathcal{N}=3$ superconformal theories

$$\frac{b_{\mathcal{N}=3}}{a_{\mathcal{N}=3}} = -\frac{\sqrt{2}}{8\pi} \,. \tag{7.23}$$

7.2.2 $\mathcal{N}=2$ flavour current multiplets

Let us consider the reduction of the $\mathcal{N}=3$ $\langle LLL\rangle$ correlator to $\mathcal{N}=2$ superspace. Recall that the $\mathcal{N}=2$ flavour current multiplet is described by a primary real dimension-1 superfield L subject to the constraint

$$\Big(D^{\alpha I}D^J_\alpha - \frac{1}{2}\delta^{IJ}D^{K\alpha}D^K_\alpha\Big)L = 0\,, \tag{7.24}$$

which defines the $\mathcal{N}=2$ linear multiplet. Such a superfield can be obtained by bar-projecting one of the three components of the $\mathcal{N}=3$ flavour current multiplet L^I,

$$L = L^3| \,, \tag{7.25}$$

where the bar-projection assumes that $\theta^\alpha_3 = 0$. Hence, the correlation functions of the $\mathcal{N}=2$ flavour current multiplets can be obtained by evaluating the bar-projections

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = \langle L^{3\,\bar a}(z_1)L^{3\,\bar b}(z_2)\rangle| \,, \tag{7.26a}$$

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle = \langle L^{3\,\bar a}(z_1)L^{3\,\bar b}(z_2)L^{3\,\bar c}(z_3)\rangle| \,. \tag{7.26b}$$

Now, given the explicit form of the correlation functions of $\mathcal{N}=3$ flavour current multiplets, eqs. (3.3) and (3.4), we derive

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = a_{\mathcal{N}=3}\frac{\delta^{\bar a\bar b}}{\boldsymbol{x}_{12}{}^2}\,, \tag{7.27a}$$

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle = -\frac{1}{2}b_{\mathcal{N}=3}\frac{f^{\bar a\bar b\bar c}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}\frac{\mathrm{i}\Theta_3^{\hat I\alpha}X_{3\alpha\beta}\Theta_3^{\hat J\beta}\varepsilon_{\hat I\hat J}}{X_3{}^3}\,, \tag{7.27b}$$

where $\hat I$, $\hat J$ are the $\mathsf{SO}(2)$ indices. Recall that the $\mathcal{N}=2$ flavour current correlation functions were found in [1] in the form

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = a_{\mathcal{N}=2}\frac{\delta^{\bar a\bar b}}{\boldsymbol{x}_{12}{}^2}\,, \tag{7.28a}$$

$$\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)L^{\bar{c}}(z_3)\rangle = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}\left[f^{\bar{a}\bar{b}\bar{c}}b_{\mathcal{N}=2}\frac{\mathrm{i}\varepsilon_{\hat{I}\hat{J}}\Theta_3^{\hat{I}\alpha}X_{3\alpha\beta}\Theta_3^{\hat{J}\beta}}{X_3{}^3}+d^{\bar{a}\bar{b}\bar{c}}\frac{\tilde{b}_{\mathcal{N}=2}}{X_3}\right] . \quad (7.28b)$$

Comparing these expressions with (7.27) we conclude that

$$a_{\mathcal{N}=2} = a_{\mathcal{N}=3}\,, \qquad b_{\mathcal{N}=2} = -\frac{1}{2}b_{\mathcal{N}=3}\,, \qquad \tilde{b}_{\mathcal{N}=2} = 0\,. \qquad (7.29)$$

As a consequence of (7.23) we find the ratio of coefficients $b_{\mathcal{N}=2}$ and $a_{\mathcal{N}=2}$

$$\frac{b_{\mathcal{N}=2}}{a_{\mathcal{N}=2}} = \frac{\sqrt{2}}{16\pi}\,. \qquad (7.30)$$

Let us point out that $\tilde{b}_{\mathcal{N}=2}$ is found to be zero because the last term in (7.28b) cannot be lifted to $\mathcal{N}=3$ supersymmetry, but in generic $\mathcal{N}=2$ supersymmetric theories it is not necessarily zero. It can be shown that this term does not contribute to the three-point function of conserved currents [1] and, hence, is irrelevant for our present discussion.

7.2.3 $\mathcal{N}=1$ flavour current multiplets

Finally, let us discuss the reduction of the $\mathcal{N}=2$ flavour current multiplets correlation functions down to $\mathcal{N}=1$. The $\mathcal{N}=1$ flavour current is described by a primary dimension-3/2 superfield L_α obeying the conservation law $D^\alpha L_\alpha = 0$. It can be obtained from the $\mathcal{N}=2$ flavour current multiplet L by the rule

$$L_\alpha = \mathrm{i}D^2_\alpha L|\,, \qquad (7.31)$$

where the bar-projection assumes that $\theta_2^\alpha = 0$. The corresponding relations among the correlation functions of $\mathcal{N}=2$ and $\mathcal{N}=1$ flavour current multiplets were found in [1]

$$\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)\rangle = -D^2_{(1)\alpha}D^2_{(2)\beta}\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)\rangle| = 2\mathrm{i}a_{\mathcal{N}=2}\delta^{\bar{a}\bar{b}}\frac{\boldsymbol{x}_{12\alpha\beta}}{\boldsymbol{x}_{12}{}^4}\,, \qquad (7.32)$$

$$\begin{aligned}\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)L^{\bar{c}}_\gamma(z_3)\rangle &= -\mathrm{i}D^2_{(1)\alpha}D^2_{(2)\beta}D^2_{(3)\gamma}\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)L^{\bar{c}}(z_3)\rangle| \\ &= 2\mathrm{i}b_{\mathcal{N}=2}\frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}\frac{X_3^{\alpha\beta}\Theta_3^\gamma-\varepsilon^{\alpha\gamma}X_3^{\beta\rho}\Theta_{3\rho}-\varepsilon^{\beta\gamma}X_3^{\alpha\rho}\Theta_{3\rho}}{X_3{}^3}\,. \quad (7.33)\end{aligned}$$

The same expressions for these correlation functions were found in [1] by using the superconformal invariance and conservation conditions. The free coefficients of these correlation functions are related to the ones in (7.32) and (7.33)

$$a_{\mathcal{N}=1} = 2a_{\mathcal{N}=2}\,, \qquad b_{\mathcal{N}=1} = 2b_{\mathcal{N}=2}\,. \qquad (7.34)$$

Hence, these coefficients have the same ratio as in (7.30)

$$\frac{b_{\mathcal{N}=1}}{a_{\mathcal{N}=1}} = \frac{\sqrt{2}}{16\pi}\,. \qquad (7.35)$$

JHEP08(2015)125

8 Concluding comments

In this paper, we have studied some implications of $\mathcal{N} = 4$ superconformal symmetry in three dimensions. A rather unexpected result of our analysis is that the three-point function of the supercurrent in $\mathcal{N} = 4$ superconformal field theories is allowed to possess *two* independent tensor structures, which is a consequence of the superconformal symmetry and the conservation equation. It may look surprising since any $\mathcal{N} = 4$ superconformal field theory can also be thought of as a special case of one with $\mathcal{N} < 4$ and, as we showed in [1], similar three-point functions in superconformal field theories with $\mathcal{N} < 4$ contain only one tensor structure. An apparent disagreement has a simple resolution. From the viewpoint of $\mathcal{N} = 3$ (or even less extended) supersymmetry, the $\mathcal{N} = 4$ supercurrent consists of two $\mathcal{N} = 3$ multiplets, one of which is the $\mathcal{N} = 3$ supercurrent and the other contains additional currents, like the R-symmetry currents. Such $\mathcal{N} \to \mathcal{N} - 1$ decompositions can be found in the introduction of [1]. As we explained in section 4 (see also subsection C.1), only one tensor structure in the three-point function of the $\mathcal{N} = 4$ supercurrent contributes to the three-point function of the $\mathcal{N} = 3$ (and, hence, $\mathcal{N} < 3$) supercurrent. Thus, just like in general $\mathcal{N} \leq 3$ superconformal theories, the three-point correlator of the energy-momentum tensor in $\mathcal{N} = 4$ superconformal theories is determined by a single tensor structure. As concerned the second tensor structure, in section 5 we pointed out that it is present in those $\mathcal{N} = 4$ superconformal field theories which are not invariant under the mirror map (see also below). In the case of free $\mathcal{N} = 4$ hypermultiplet models, it is proportional to the difference between the number of left and right supermultiplets with respect to the R-symmetry group $\mathsf{SU}(2)_{\rm L} \times \mathsf{SU}(2)_{\rm R}$.

Another important result of the paper consists in the relations between the coefficients of the two- and three-point correlation functions of the supercurrent and flavour current multiplets in *all* $1 \leq \mathcal{N} \leq 4$ superconformal theories. These relations are derived in section 7 and the analysis is based on two observations. First, if both the two- and three-point functions of either the supercurrent or the flavour current multiplets are fixed up to a single coefficient and are related to each other by the Ward identities, we can derive the *universal* ratio of the coefficients by simply considering any specific theory. Second, as already mentioned, any $\mathcal{N}$-extended supersymmetric theory is a special case of a $(\mathcal{N}-1)$-extended theory. In particular, any $\mathcal{N} = 4$ superconformal theory can be considered as an $\mathcal{N} = 1$, $\mathcal{N} = 2$ or $\mathcal{N} = 3$ superconformal theory. As a result, we can derive all *universal* relations between the coefficients of the two- and three-point functions by considering one relatively simple specific example, namely, the $\mathcal{N} = 4$ superconformal theory of free hypermultiplets.

The hypermultiplet supercurrent (5.7) is asymmetric with respect to the left and right hypermultiplets. More generally, given an $\mathcal{N} = 4$ superconformal theory that is invariant with respect to the mirror map, its supercurrent must change sign under the mirror map $\mathfrak{M}$. A simple illustrating example is provided by the model describing an equal number of left and right hypermultiplets. The corresponding supercurrent

$$J = \bar{q}_{\tilde{i}} q^{\tilde{i}} - \bar{q}_i q^i \tag{8.1}$$

is odd under the mirror map $q^{\tilde{i}} \longleftrightarrow q^i$. This property has its origin in $\mathcal{N} = 4$ conformal supergravity. To explain this important point, we have to recall three results from supergravity. Firstly, as shown in [34], the $\mathcal{N}=4$ super-Cotton tensor $X(z)$ changes its sign under the mirror map.[12] Secondly, the off-shell action $S_{\rm CSG}$ for $\mathcal{N}=4$ conformal supergravity [53] proves to be invariant under the mirror map, and its variation can be represented as

$$\delta S_{\rm CSG} \propto \int {\rm d}^{3|8}z\, E\, \delta H X\,, \qquad E = {\rm Ber}(E_M{}^A)\,. \tag{8.2}$$

Here ${\rm d}^{3|8}z\, E$ is the integration measure of $\mathcal{N}=4$ curved superspace, and $H(z)$ denotes the conformal supergravity prepotential (see also [37, 38]). Therefore, the prepotential H changes its sign under the mirror map. Thirdly, given a system of matter multiplets coupled to conformal supergravity, an infinitesimal disturbance of H changes the matter action $S_{\rm matter}$ as follows

$$\delta S_{\rm matter} = \int {\rm d}^{3|8}z\, E\, \delta H J\,, \tag{8.3}$$

where $J(z)$ is the matter supercurrent. If $S_{\rm matter}$ is invariant with respect to $\mathfrak{M}$, then J is indeed odd under the mirror map.

As shown in subsection 4.2, the most general expression for the three-point function of the $\mathcal{N}=4$ supercurrent is

$$\langle J(z_1) J(z_2) J(z_3)\rangle = \frac{1}{\boldsymbol{x}_{13}{}^2 \boldsymbol{x}_{23}{}^2} \left(\frac{\tilde{d}_{\mathcal{N}=4}}{X_3} + \frac{d_{\mathcal{N}=4}}{X_3{}^5} \varepsilon_{IJKL} \Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta} X_{\alpha\beta}X_{\gamma\delta} \right)\,. \tag{8.4}$$

The parameter $\tilde{d}_{\mathcal{N}=4}$ must vanish, $\tilde{d}_{\mathcal{N}=4}=0$, in every theory invariant under the mirror map. The second term in (8.4) is odd under the mirror map due to the property

$$\mathfrak{M}:\ \varepsilon_{IJKL} \Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta} X_{\alpha\beta}X_{\gamma\delta} \longrightarrow -\varepsilon_{IJKL} \Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta} X_{\alpha\beta}X_{\gamma\delta}\,. \tag{8.5}$$

All other building blocks in (8.4) are invariant under $\mathfrak{M}$.

It would be interesting to extend the results of the present paper to the cases of superconformal theories with $\mathcal{N}>4$. The $\mathcal{N}>4$ supercurrent is described by a primary superfield J^{IJKL} of dimension 1 subject to the conservation law [37, 38]

$$D^I_\alpha J^{JKLP} = D^{[I}_\alpha J^{JKLP]} - \frac{4}{\mathcal{N}-3} D^Q_\alpha J^{Q[JKL}\delta^{P]I}\,, \quad I=1,\dots,\mathcal{N}\,. \tag{8.6}$$

The construction of the correlation functions involving the supercurrent J^{IJKL} has its own complications due to a large number of R-symmetry indices. We postpone this problem for later study.

Acknowledgments

We are grateful to Joseph Novak for comments on the manuscript. This work is supported in part by the Australian Research Council, project DP140103925. The work of E.I.B. is also supported by the ARC Future Fellowship FT120100466.

[12]The algebra of covariant derivatives for $\mathcal{N}=4$ conformal supergravity is known to be invariant under the mirror map [34]. The super-Cotton tensor is obtained from the completely antisymmetric curvature tensor X^{IJKL}, which is invariant under the mirror map, by the rule $X^{IJKL}=\varepsilon^{IJKL}X$. Since the Levi-Civita tensor ε^{IJKL} changes its sign under the mirror map, eq. (A.28), the same is true of the super-Cotton tensor.

A Comments on off-shell hypermultiplets

The superfield constraints (5.1a) and (5.1b) are on-shell. There exist off-shell models for left and right hypermultiplets such that their equations of motion are equivalent to the constraints (5.1a) and (5.1b). Before discussing such off-shell hypermultiplets, some general comments are in order. Off-shell descriptions exist for many 3D $\mathcal{N}=4$ supersymmetric field theories such as general $\mathcal{N}=4$ nonlinear σ-models [4]. However, it turns out that the conventional $\mathcal{N}=4$ Minkowski superspace $\mathbb{M}^{3|8}$ is not suitable to realise the most interesting off-shell couplings. An adequate superspace setting for them is an extension of $\mathbb{M}^{3|8}$ by auxiliary bosonic dimensions parametrising a compact manifold, in the spirit of the superspace [56] $\mathbb{M}^{4|8}\times\mathbb{C}P^1$ which is at the heart of the 4D $\mathcal{N}=2$ harmonic [45, 46] and projective [47–49] superspace approaches.[13] All known off-shell $\mathcal{N}=4$ supersymmetric field theories in three dimensions can be realised in the following superspace[14] [11, 12]

$$\mathbb{M}^{3|8}\times\mathbb{C}P^1_{\rm L}\times\mathbb{C}P^1_{\rm R}=\mathbb{M}^{3|8}\times[\mathsf{SU}(2)/\mathsf{U}(1)]_{\rm L}\times[\mathsf{SU}(2)/\mathsf{U}(1)]_{\rm R}\,, \tag{A.1}$$

which may be called harmonic or projective depending on the type of $\mathcal{N}=4$ off-shell multiplets one is interested in. All such multiplets are functions over either $\mathbb{C}P^1_{\rm L}$ or $\mathbb{C}P^1_{\rm R}$. For definiteness, let us consider left multiplets associated with $\mathbb{C}P^1_{\rm L}$. Our presentation below is similar to [50].

Let $v_{\rm L}\equiv(v^i)\in\mathbb{C}^2\setminus\{0\}$ be homogeneous coordinates for $\mathbb{C}P^1_{\rm L}$, and $v_{\rm L}{}^\dagger=(\overline{v^i}):=(\bar v_i)$ be their conjugates (in what follows, the subscripts 'L' and 'R' will always be omitted if no confusion may occur). Any superfield living in $\mathbb{M}^{3|8}\times\mathbb{C}P^1$ may be identified with a function $\phi(z,v,\overline{v})$ that only scales under arbitrary re-scalings of v:

$$\phi(z,c\,v,\bar c\,\overline{v})=c^{n_+}\,\bar c^{n_-}\,\phi(z,v,\overline{v})\,,\qquad c\in\mathbb{C}^*\equiv\mathbb{C}\setminus\{0\} \tag{A.2}$$

for some parameters $n_\pm$ such that n_+-n_- is an integer. Since $v^\dagger v=\bar v_i v^i\neq 0$, we can always choose $n_-=0$ by redefining $\phi(z,v,\bar v)\to\phi(z,v,\bar v)/(v^\dagger v)^{n_-}$. Any superfield with the homogeneity property

$$\phi^{(n)}(z,c\,v,\bar c\,\overline{v})=c^n\phi^{(n)}(z,v,\overline{v})\,,\qquad c\in\mathbb{C}^* \tag{A.3}$$

is said to have weight n. Let us introduce fermionic operators

$$\mathfrak{D}^{\tilde i}_\alpha\equiv D^{\tilde i(1)}_\alpha:=v_iD^{i\tilde i}_\alpha\,, \tag{A.4}$$

where $v_i:=\varepsilon_{ij}\,v^j$. In accordance with (2.62), these operators strictly anticommute with each other,

$$\{\mathfrak{D}^{\tilde i}_\alpha,\mathfrak{D}^{\tilde j}_\beta\}=0\,, \tag{A.5}$$

[13] The relationship between the 4D $\mathcal{N}=2$ harmonic and projective superspace formulations is spelled out in [54].

[14] For every positive integer $\mathcal{N}$, the 3D $\mathcal{N}$-extended superconformal group $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ is a transformation group of the so-called compactified Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ in which $\mathbb{M}^{3|2\mathcal{N}}$ is embedded as a dense open domain [4]. In the $\mathcal{N}=4$ case, $\mathsf{OSp}(4|4;\mathbb{R})$ is also defined to act transitively on $\overline{\mathbb{M}}^{3|8}\times\mathbb{C}P^1_{\rm L}\times\mathbb{C}P^1_{\rm R}$, as shown in [4].

which allows us to introduce left *isochiral multiplets* (following the terminology of [55]) constrained by

$$\mathfrak{D}^{\tilde{i}}_{\alpha}\phi^{(n)}(z,v,\bar{v}) = 0 \ . \tag{A.6}$$

These constraints are consistent with the homogeneity condition (A.3).

Given an isochiral superfield $\phi^{(n)}(z,v^i,\bar{v}_j)$, its complex conjugate

$$\bar{\phi}^{(n)}(z,\bar{v}_i,v^j) := \overline{\phi^{(n)}(z,v^i,\bar{v}_j)} \tag{A.7}$$

is no longer isochiral. However, by analogy with the 4D $\mathcal{N}=2$ case [45, 56] one can define a modified conjugation that maps every isochiral superfield $\phi^{(n)}(z,v,\bar{v})$ into an isochiral one $\breve{\phi}^{(n)}(z,v,\bar{v})$ of the same weight defined as follows:

$$\phi^{(n)}(v^i,\bar{v}_j) \longrightarrow \bar{\phi}^{(n)}(\bar{v}_i,v^j) \longrightarrow \bar{\phi}^{(n)}\Big(\bar{v}_i \to -v_i, v^j \to \bar{v}^j\Big) =: \breve{\phi}^{(n)}(v^i,\bar{v}_j) \ . \tag{A.8}$$

The weight-n isochiral superfield $\breve{\phi}^{(n)}(z,v,\bar{v})$ is said to be the smile-conjugate of $\phi^{(n)}(z,v,\bar{v})$. One can check that

$$\breve{\breve{\phi}}^{(n)}(z,v,\bar{v}) = (-1)^n \phi^{(n)}(z,v,\bar{v}) \ . \tag{A.9}$$

Therefore, if the weight n is even, real isochiral superfields can be defined, $\breve{\phi}^{(2m)} = \phi^{(2m)}$.

Within the 3D $\mathcal{N}=4$ projective superspace approach [4], off-shell multiplets are described in terms of weight-n isochiral superfields $Q^{(n)}(z,v)$,

$$\mathfrak{D}^{\tilde{i}}_{\alpha}Q^{(n)} = 0\,, \qquad Q^{(n)}(z,c\,v) = c^n\, Q^{(n)}(z,v)\,, \quad c \in \mathbb{C}^*\,, \tag{A.10}$$

which are *holomorphic* over an *open domain* of $\mathbb{C}P^1_{\rm L}$,

$$\frac{\partial}{\partial \bar{v}_i} Q^{(n)} = 0 \ . \tag{A.11}$$

Such isochiral superfields are called left *projective multiplets* of weight n. The action principle in projective superspace involves a contour integral, and not an integral over $\mathbb{C}P^1$. This is why there is no need for projective multiplets to be smooth over $\mathbb{C}P^1$. This approach is useful to construct the most general $\mathcal{N}=4$ supersymmetric σ-models, both in Minkowski superspace [4] and in supergravity [34]. The structure of superconformal projective multiplets is well understood [4].

Somewhat different isochiral superfields are used in the framework of the 3D $\mathcal{N}=4$ harmonic superspace approach [11, 12]. The equivalence $v^i \sim c\,v^i$, which is intrinsic to $\mathbb{C}P^1$, allows one to switch to the description in terms of normalised isotwistors:

$$u^{+i} := \frac{v^i}{\sqrt{v^\dagger v}}\,, \qquad u^-_i := \frac{\bar{v}_i}{\sqrt{v^\dagger v}} = \overline{u^{+i}} \quad \Longrightarrow \quad \big({u_i}^-, {u_i}^+\big) \in \mathsf{SU}(2) \ . \tag{A.12}$$

The variables $u^\pm_i$ are called *harmonics*. They are defined modulo the equivalence relation $u^\pm_i \sim {\rm e}^{\pm {\rm i}\alpha} u^\pm_i$, with $\alpha \in \mathbb{R}$. It is clear that the harmonics parametrize the coset space

$\mathsf{SU}(2)/\mathsf{U}(1) \cong S^2$. Given an isochiral superfield $\phi^{(n)}(z, v, \bar{v})$ we can associate with it the following superfield

$$\varphi^{(n)}(z, u^+, u^-) := \phi^{(n)}\left(z, \frac{v}{\sqrt{v^\dagger v}}, \frac{\bar{v}}{\sqrt{v^\dagger v}}\right) = \frac{1}{(\sqrt{v^\dagger v})^n}\phi^{(n)}(z, v, \bar{v}) \tag{A.13}$$

obeying the homogeneity condition

$$\varphi^{(n)}(z, \mathrm{e}^{\mathrm{i}\alpha}\, u^+, \mathrm{e}^{-\mathrm{i}\alpha}\, u^-) = \mathrm{e}^{\mathrm{i}n\alpha}\, \varphi^{(n)}(z, u^+, u^-) \ . \tag{A.14}$$

This property tells us that $\varphi^{(n)}(z, u^\pm)$ has $\mathsf{U}(1)$ charge n. Thus the weight of $\phi^{(n)}(z, v, \bar{v})$ is replaced with the $\mathsf{U}(1)$ charge of $\varphi^{(n)}(z, u^\pm)$. It is obvious that we have the one-to-one correspondence $\phi^{(n)}(z, v, \bar{v}) \longleftrightarrow \varphi^{(n)}(z, u^\pm)$. The fermionic operators (A.4) turn into

$$D^{\tilde{i}+}_\alpha := \frac{1}{\sqrt{v^\dagger v}}\mathfrak{D}^{\tilde{i}}_\alpha = u^+_i D^{i\tilde{i}}_\alpha \, , \tag{A.15}$$

and therefore the isochirality condition (A.6) takes the form

$$D^{\tilde{i}+}_\alpha \varphi^{(n)}(z, u^\pm) = 0 \ . \tag{A.16}$$

In harmonic superspace, every isochiral superfield $\varphi^{(n)}(z, u^\pm)$ is required to be a *smooth* charge-n function over $\mathsf{SU}(2)$ or, equivalently, a *smooth* tensor field over the two-sphere S^2. Such a superfield is called left *analytic*. It can be represented, say for $n \geq 0$, by a convergent Fourier series

$$\varphi^{(n)}(z, u^\pm) = \sum_{p=0}^{\infty} \varphi^{(i_1 \ldots i_{n+p} j_1 \ldots j_p)}(z)\, u^+_{i_1} \ldots u^+_{i_{n+p}} u^-_{j_1} \ldots u^-_{j_p} \, , \tag{A.17}$$

in which the coefficients $\varphi^{i_1 \ldots i_{n+2p}}(z) = \varphi^{(i_1 \ldots i_{n+2p})}(z)$ are ordinary $\mathcal{N} = 4$ superfields obeying first-order differential constraints that follow from (A.16). The beauty of this approach is that the power of harmonic analysis can be used.

We are now prepared to discuss off-shell hypermultiplets. In harmonic superspace, the most suitable off-shell description of a single hypermultiplet makes use of an analytic superfield $q^+(z, u^\pm) \equiv q^{(1)}(z, u^\pm)$ and its smile-conjugate $\breve{q}^+(z, u^\pm)$. The free hypermultiplet equation of motion, which corresponds to the action (6.44), is

$$\partial^{++} q^+ = 0 \quad \Longrightarrow \quad q^+(z, u^\pm) = q^i(z) u^+_i \, , \tag{A.18}$$

where $q^i(z)$ obeys the constraint (5.1a).

In projective superspace, the most suitable off-shell description of a single hypermultiplet makes use of an arctic multiplet $\Upsilon^{(1)}(z, v)$ and its its smile-conjugate $\breve{\Upsilon}^{(1)}(z, v)$. By definition, $\Upsilon^{(1)}(z, v)$ is a weight-1 projective multiplet which is holomorphic over the so-called north chart $\mathbb{C}$ of $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Here the point $\infty \in \mathbb{C}P^1$ is identified with the "north pole" $v^i_{\text{north}} \sim (0, 1)$. In the north chart, it is useful to introduce a complex (*inhomogeneous*) coordinate ζ defined by

$$v^i = v^1\, (1, \zeta) \, , \qquad \zeta := \frac{v^2}{v^1} \, , \qquad\quad i = 1, 2 \ . \tag{A.19}$$

The arctic multiplet $\Upsilon^{(1)}(z,v)$ looks like

$$\Upsilon^{(1)}(z,v) = v^1 \sum_{k=0}^{\infty} \Upsilon_k(z)\zeta^k \,, \tag{A.20}$$

and its smile-conjugate *antarctic* multiplet $\breve{\Upsilon}^{(1)}(z,v)$, is

$$\breve{\Upsilon}^{(1)}(z,v) = v^2 \sum_{k=0}^{\infty} \bar{\Upsilon}_k(z) \frac{(-1)^k}{\zeta^k} \,. \tag{A.21}$$

The dynamics of the free polar hypermultiplet is described by the action

$$S = \frac{1}{2\pi}\oint_\gamma v_i \mathrm{d}v^i \int \mathrm{d}^3x\, D^{(-4)}\mathcal{L}^{(2)}(z,v)\Big|_{\theta=0}\,, \qquad \mathcal{L}^{(2)} = \breve{\Upsilon}^{(1)}\Upsilon^{(1)}\,, \tag{A.22}$$

where we have defined

$$D^{(-4)} := \frac{1}{48}D^{(-2)\tilde{i}\tilde{j}}D^{(-2)}_{\tilde{i}\tilde{j}}\,, \quad D^{(-2)}_{\tilde{i}\tilde{j}} := D^{(-1)\gamma}_{\tilde{i}}D^{(-1)}_{\tilde{j}\gamma}\,, \quad D^{(-1)\tilde{i}}_{\alpha} := \frac{1}{(v,u)}u_i D^{i\tilde{i}}_{\alpha}\,. \tag{A.23}$$

The fourth-order operator $D^{(-4)}$ in (A.22) involves a constant isotwistor u_i constrained only by the condition $(v,u) := v^i u_i \neq 0$ which must hold along the closed integration contour γ. The action (A.22) proves to be independent of u_i. It can be shown that the equation of motion, which follows from the action (A.22), is

$$\Upsilon^{(1)}(z,v) = v^1\Big(\Upsilon_0(z) + \Upsilon_1(z)\zeta\Big) \equiv q^i(z)v_i\,, \tag{A.24}$$

where $q^i(z)$ obeys the constraint (5.1a). Thus, the q^+ hypermultiplet and the polar hypermultiplet provide two different off-shell realisations for the hypermultiplet. Both actions (6.44) and (A.22) are superconformal.

There is a family of isochiral multiplets that are holomorphic over $\mathbb{C}P^1$, and therefore they are suitable for both the harmonic and projective superspace settings. These are the so-called $\mathcal{O}(n)$ multiplets, where $n = 1, 2, \ldots$,

$$H^{(n)}(z,v) = H^{i_1\ldots i_n}(z)v_{i_1}\ldots v_{i_n}\,, \qquad D^{\tilde{j}(j}_{\alpha}H^{i_1\ldots i_n)} = 0\,. \tag{A.25}$$

Such a multiplet is (i) on-shell for $n = 1$ and describes a free hypermultiplet; and (ii) off-shell for $n > 1$. When n is even, one can define real multiplets with respect to the smile-conjugation. The flavour current multiplet L^{ij} is described by a real $\mathcal{O}(2)$ multiplet $L^{(2)}$. It may be shown that real $\mathcal{O}(2n)$ multiplets with $n > 1$ can be used to describe neutral hypermultiplets. However, the corresponding free hypermultiplet actions are not superconformal.

The mirror map [12, 40] is defined as

$$\mathfrak{M}:\ \mathsf{SU}(2)_{\rm L} \longleftrightarrow \mathsf{SU}(2)_{\rm R}\,. \tag{A.26}$$

It changes the tensor types of superfields as $\mathrm{D}^{(p/2)}_{\rm L}\otimes\mathrm{D}^{(q/2)}_{\rm R} \to \mathrm{D}^{(q/2)}_{\rm L}\otimes\mathrm{D}^{(p/2)}_{\rm R}$, where $\mathrm{D}^{(p/2)}$ denotes the spin-$p/2$ representation of $\mathsf{SU}(2)$. The mirror map interchanges the on-shell left q^i and right $q^{\tilde{i}}$ hypermultiplets,

$$\mathfrak{M}\cdot q^i = q^{\tilde{i}}\,, \qquad \mathfrak{M}\cdot q^{\tilde{i}} = q^i\,. \tag{A.27}$$

It also interchanges the left L^{ij} and right $L^{\tilde{i}\tilde{j}}$ flavour current multiplets. Since the latter multiplets are (anti) self-dual, eq. (4.3), the mirror map must act on the Levi-Civita tensor ε^{IJKL} as

$$\mathfrak{M} \cdot \varepsilon^{IJKL} = -\varepsilon^{IJKL} \,. \tag{A.28}$$

B $\mathcal{N} = 4$ hypermultiplet propagator

The free equation of motion for q^+ hypermultiplet is $\mathscr{D}^{++}q^+ = 0$, where $\mathscr{D}^{++}$ is defined in (6.41a). By definition, the Green function of the free hypermultiplet $G^{(+,+)}(\zeta_1, \zeta_2)$ obeys the equation

$$\mathscr{D}^{++}G^{(+,+)}(\zeta_1, \zeta_2) = -\delta_A^{(3,1)}(\zeta_1, \zeta_2) \,, \tag{B.1}$$

where $\delta_A^{(3,1)}(\zeta_1, \zeta_2)$ is the analytic delta functions. The solution to this equation is very similar to the four-dimensional q-hypermultiplet Green's function [45, 57]

$$G^{(+,+)}(\zeta_1, \zeta_2) = \frac{1}{\Box}(D_1^+)^4(D_2^+)^4 \frac{\delta^3(x_1 - x_2)\delta^8(\theta_1 - \theta_2)}{(u_1^+ u_2^+)^3} \,, \tag{B.2}$$

where $(D^+)^4 = \frac{1}{16}(D^{+\tilde{1}\alpha}D_\alpha^{+\tilde{1}})(D^{+\tilde{2}\beta}D_\beta^{+\tilde{2}})$. To check that (B.2) obeys (B.1) one has to take into account that $\mathscr{D}^{++}$ commutes with $D_\alpha^{+\tilde{i}}$ and hits only the harmonic distribution in (B.2) producing the harmonic delta-function (see [46] for a review of properties of harmonic distributions)

$$\partial^{++}\frac{1}{(u_1^+ u_2^+)^3} = \frac{1}{2}(\partial^{--})^2\delta^{(3,-3)}(u_1, u_2) \,. \tag{B.3}$$

This harmonic delta function is part of the analytic delta function

$$\delta_A^{(3,1)}(\zeta_1, \zeta_2) = (D_2^+)^4\delta^3(x_1 - x_2)\delta^8(\theta_1 - \theta_2)\delta^{(3,-3)}(u_1, u_2) \,. \tag{B.4}$$

As a result we have

$$\mathscr{D}_1^{++}G^{(+,+)}(\zeta_1, \zeta_2) = \frac{1}{2}\frac{1}{\Box}(D_1^+)^4(\mathscr{D}_1^{--})^2\delta_A^{(3,1)}(\zeta_1, \zeta_2) = -\delta_A^{(3,1)}(\zeta_1, \zeta_2) \,. \tag{B.5}$$

Here we applied the identity

$$(D^+)^4(\mathscr{D}^{--})^2\phi_A = -2\Box\phi_A \,, \tag{B.6}$$

which holds for arbitrary analytic superfield ϕ_A.

We point out that the operator $1/\Box$ in (B.2) acts only on the bosonic delta-function $\delta^3(x_1 - x_2)$ and gives the scalar field Green's function $G(x_1, x_2)$ which we represent as the integral over the proper time s

$$\frac{1}{\Box}\delta^3(x_1 - x_2) = -G(x_1, x_2) = -\mathrm{i}\int_0^\infty \mathrm{d}s \, U(x_1, x_2|s) \,, \tag{B.7}$$

where $U(x_1, x_2|s)$ is the heat kernel of the three-dimensional d'Alembert operator

$$U(x_1, x_2|s) = \frac{\mathrm{i}}{(4\pi\mathrm{i}s)^{3/2}} e^{\mathrm{i}\frac{(x_1 - x_2)^2}{4s}} \,. \tag{B.8}$$

The integration over the proper time in (B.7) can be done explicitly, and the result has slightly different forms for the point inside and outside the lightcone

$$G(x_1,x_2)=\begin{cases}\frac{1}{4\pi}\frac{1}{\sqrt{-(x_1-x_2)^2}} & (x_1-x_2)^2<0\\ \frac{\mathrm{i}}{4\pi}\frac{1}{\sqrt{(x_1-x_2)^2}} & (x_1-x_2)^2>0\ .\end{cases} \tag{B.9}$$

These two cases can be unified in a single formula such that

$$\frac{1}{\Box}\delta^3(x_1-x_2)=-\frac{\mathrm{i}}{4\pi}\frac{1}{\sqrt{(x_1-x_2)^2}} \tag{B.10}$$

is valid for $(x_1-x_2)^2\neq 0$. Then, we rewrite (B.2) as

$$G^{(+,+)}(\zeta_1,\zeta_2)=-\frac{\mathrm{i}}{4\pi}(D_1^+)^4(D_2^+)^4\left(\frac{1}{\sqrt{(x_1-x_2)^2}}\frac{\delta^8(\theta_1-\theta_2)}{(u_1^+u_2^+)^3}\right)\ . \tag{B.11}$$

It is important to realize that the supersymmetrized coordinate difference (6.51) at coincident Grassmann coordinates is simply

$$\hat{x}^a_{12}|_{\theta_1=\theta_2}=(x_1-x_2)^a\ . \tag{B.12}$$

Thus, in (B.11) we can apply the identity

$$\frac{\delta^8(\theta_1-\theta_2)}{\sqrt{(x_1-x_2)^2}}=\frac{\delta^8(\theta_1-\theta_2)}{\sqrt{\hat{x}_{12}{}^2}} \tag{B.13}$$

and use the analyticity of (6.51) in both superspace arguments to represent (B.11) as follows

$$G^{(+,+)}(\zeta_1,\zeta_2)=-\frac{\mathrm{i}}{4\pi}\frac{1}{\sqrt{\hat{x}_{12}{}^2}}(D_1^+)^4(D_2^+)^4\frac{\delta^8(\theta_1-\theta_2)}{(u_1^+u_2^+)^3}\ . \tag{B.14}$$

Finally, we employ the identity

$$(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)=(u_1^+u_2^+)^4 \tag{B.15}$$

to get the following final expression for the hypermultiplet Green's function

$$G^{(+,+)}(\zeta_1,\zeta_2)=-\frac{\mathrm{i}}{4\pi}\frac{(u_1^+u_2^+)}{\sqrt{\hat{x}_{12}{}^2}}\ . \tag{B.16}$$

This representation of the hypermultiplet Green's function was used in section 6.4 in studying Ward identities of $\mathcal{N}=4$ flavour current multiplets. Note that a similar representation of the four-dimensional hypermultiplet propagator was found in [58] (see also [46]).

C Superspace reduction of correlation functions

The procedure of superspace reduction of supercurrent correlation functions is straightforward, but quite tedious. It was applied in [1] to find the relations among three-point correlation functions of the $\mathcal{N}=2$ and $\mathcal{N}=1$ supercurrents. Here we will follow the same procedure to perform the $\mathcal{N}=4\to\mathcal{N}=3\to\mathcal{N}=2$ reductions of the supercurrent correlators.

JHEP08(2015)125

C.1 $\mathcal{N}=4 \to \mathcal{N}=3$ reduction of the correlation functions for the supercurrent

The $\mathcal{N}=4$ supercurrent is described by the primary scalar superfield J of dimension 1. When reduced to the $\mathcal{N}=3$ superspace, it has two independent $\mathcal{N}=3$ superfield components: a scalar S and a spinor J_α [1]

$$S = J| \,, \tag{C.1a}$$

$$J_\alpha = \mathrm{i} D^4_\alpha J| \,, \tag{C.1b}$$

where the bar-projection means $\theta_{4\alpha}=0$. The $\mathcal{N}=4$ supercurrent conservation condition (4.15) turns to the following constraints for the $\mathcal{N}=3$ superfields S and J_α

$$\left(D^{\hat{I}\alpha} D^{\hat{J}}_\alpha - \frac{1}{3}\delta^{\hat{I}\hat{J}} D^{\hat{K}\alpha} D^{\hat{K}}_\alpha \right) S = 0 \,, \tag{C.2a}$$

$$D^{\hat{I}\alpha} J_\alpha = 0 \,. \tag{C.2b}$$

Here $\hat{I},\hat{J},\hat{K}=1,2,3$ are the indices of $\mathsf{SO}(3)$ group.

The superfield J_α is the $\mathcal{N}=3$ supercurrent. In components, it contains the energy-momentum tensor, conserved currents of $\mathcal{N}=3$ supersymmetry and conserved currents of the $\mathsf{SO}(3)$ subgroup of the $\mathsf{SO}(4)$ R-symmetry of $\mathcal{N}=4$ theory. The $\mathcal{N}=3$ scalar contains among its components the current of the fourth supersymmetry and the currents of the remaining $\mathsf{SO}(4)/\mathsf{SO}(3)$ R-symmetry. Therefore, when we consider an $\mathcal{N}=4$ superconformal theory in the $\mathcal{N}=3$ superspace, the conserved quantities are described by the following four types of three-point correlation functions

$$\langle SSS \rangle \,, \quad \langle SSJ_\alpha \rangle \,, \quad \langle SJ_\alpha J_\beta \rangle \,, \quad \langle J_\alpha J_\beta J_\gamma \rangle \,. \tag{C.3}$$

In this appendix we derive these correlators from the three-point function of the $\mathcal{N}=4$ supercurrent which was obtained in section 4.2 in the form

$$\langle J(z_1) J(z_2) J(z_3) \rangle = \frac{1}{\boldsymbol{x}_{13}{}^2 \boldsymbol{x}_{23}{}^2} H(X_3, \Theta_3) \,, \tag{C.4a}$$

$$H(X_3,\Theta_3) = \frac{\tilde{d}_{\mathcal{N}=4}}{X_3} + d_{\mathcal{N}=4} \frac{\varepsilon_{IJKL} \Theta_3^{I\alpha} \Theta_3^{J\beta} \Theta_3^{K\gamma} \Theta_3^{L\delta} X_{3\alpha\beta} X_{3\gamma\delta}}{X_3{}^5} \,. \tag{C.4b}$$

The distinguishing feature of this correlation function as compared to the ones in the $\mathcal{N}=1,2,3$ superconformal theories is that it has two completely different terms with two independent parameters $\tilde{d}_{\mathcal{N}=4}$ and $d_{\mathcal{N}=4}$. As we will show further, the two terms in (C.4b) contribute to different correlators (C.3).

C.1.1 Correlator $\langle SSS \rangle$

Since the superfield S is just the lowest component of J, see (C.1a), its three-point correlator appears simply by switching off the Grassmann coordinate $\theta_{4\alpha}$ at each superspace point

$$\langle S(z_1) S(z_2) S(z_3) \rangle = \langle J(z_1) J(z_2) J(z_3) \rangle | = \frac{1}{\boldsymbol{x}_{13}{}^2 \boldsymbol{x}_{23}{}^2} \frac{\tilde{d}_{\mathcal{N}=4}}{X_3} \,. \tag{C.5}$$

Note that the last term in (C.4b) vanishes in this reduction and only the first term with the coefficient $\tilde{d}_{\mathcal{N}=4}$ survives.

C.1.2 Correlator $\langle J_\alpha S S\rangle$

To compute this correlation function we have to hit (C.4) by one spinor covariant derivative

$$\langle J_\alpha(z_1)S(z_2)S(z_3)\rangle = \mathrm{i}D^4_{(1)\alpha}\langle J(z_1)J(z_2)J(z_3)\rangle| = \mathrm{i}D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H(X_3,\Theta_3)|\ . \tag{C.6}$$

Note that the spinor covariant derivative acts on the two-point function (2.8a) by the rule

$$D^I_{(1)\alpha}\boldsymbol{x}^{\mu\nu}_{12} = -2\mathrm{i}\delta^\mu_\alpha\theta^{I\nu}_{12}\ . \tag{C.7}$$

Hence, all terms in which the derivative $D^4_{(1)\alpha}$ hits the bosonic two-point and three-point structures vanish under the bar-projection and only the last term in (C.4b) contributes

$$\begin{aligned}\langle J_\alpha(z_1)S(z_2)S(z_3)\rangle &= \mathrm{i}d_{\mathcal{N}=4}\frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}D^4_{(1)\alpha}\frac{\varepsilon_{IJKL}\Theta_3^{I\mu}\Theta_3^{J\nu}\Theta_3^{K\rho}\Theta_3^{L\sigma}X_{3\mu\nu}X_{3\rho\sigma}}{X_3{}^5}\Big| \\ &= -\mathrm{i}d_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\beta}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\mathcal{D}^{4\beta}_{(3)}\frac{\varepsilon_{IJKL}\Theta_3^{I\mu}\Theta_3^{J\nu}\Theta_3^{K\rho}\Theta_3^{L\sigma}X_{3\mu\nu}X_{3\rho\sigma}}{X_3{}^5}\Big| \\ &= -4\mathrm{i}d_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\beta}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\frac{X_3^{\mu\nu}X_3^{\rho\sigma}\Theta^{\hat I}_{3\mu}\Theta^{\hat J}_{3\nu}\Theta^{\hat K}_{3\gamma}\varepsilon_{\hat I\hat J\hat K}}{X_3{}^5}\ .\end{aligned} \tag{C.8}$$

Here, in the second line, we applied the identity (2.28a). Note that, in contrast to (C.5), this correlation function depends on the coefficient $d_{\mathcal{N}=4}$ rather than $\tilde d_{\mathcal{N}=4}$.

C.1.3 Correlator $\langle J_\alpha J_\beta S\rangle$

To compute this correlation function we have to hit (C.4) by two spinor covariant derivatives

$$\langle J_\alpha(z_1)S(z_2)J_\beta(z_3)\rangle = D^4_{(3)\beta}D^4_{(1)\alpha}\langle J(z_1)J(z_2)J(z_3)\rangle| = D^4_{(3)\beta}D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H(X_3,\Theta_3)|\ . \tag{C.9}$$

As is seen from (C.7), when two covariant spinor derivatives hit the correlation function (C.4), only the following two terms survive under the bar-projection

$$\langle J_\alpha(z_1)S(z_2)J_\beta(z_3)\rangle = A+B\,, \tag{C.10a}$$

$$A = \frac{1}{\boldsymbol{x}_{23}{}^2}\left(D^4_{(3)\beta}D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}\right)H(X_3,\Theta_3)|\,, \tag{C.10b}$$

$$B = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}D^4_{(3)\beta}D^4_{(1)\alpha}H(X_3,\Theta_3)|\ . \tag{C.10c}$$

In the part A we easily compute the derivatives owing to (C.7)

$$D^4_{(3)\beta}D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}\Big| = 2\mathrm{i}\frac{\boldsymbol{x}_{13\alpha\beta}}{\boldsymbol{x}_{13}{}^4}\ . \tag{C.11}$$

Thus, for (C.10b) we have

$$A = 2\mathrm{i}\frac{\boldsymbol{x}_{13\alpha\beta}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\frac{\tilde d_{\mathcal{N}=4}}{X_3}\ . \tag{C.12}$$

In the part B given by (C.10c) two derivatives hit the function H. For one of them we apply the identity (2.28a) to represent it in the form

$$D^4_{(3)\beta}D^4_{(1)\alpha}H| = D^4_{(3)\beta}\boldsymbol{x}^{-1}_{13\gamma\alpha}u^{4I}_{13}\mathcal{D}^{I\gamma}H| = \boldsymbol{x}^{-1}_{13\gamma\alpha}[D^4_{(3)\beta}\mathcal{D}^{4\gamma}H| + (D^4_{(3)\beta}u^{4\hat I}_{13})\mathcal{D}^{\hat I\gamma}H|]\ . \tag{C.13}$$

The two terms in the right-hand side of (C.13) give the following two contributions to the correlation function

$$B_1 = \frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4} D^4_{(3)\beta}\mathcal{D}^4_{(3)\gamma}\frac{\tilde{d}_{\mathcal{N}=4}}{X_3}\Big| \,, \tag{C.14a}$$

$$B_2 = \frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4} (D^4_{(3)\gamma}u^{4\hat{I}}_{13})\mathcal{D}^{\hat{I}}_{(3)\gamma}\frac{\tilde{d}_{\mathcal{N}=4}}{X_3}\Big| \,. \tag{C.14b}$$

Here we have taken into account that the last term in (C.4b) does not contribute to (C.13).

In the right-hand side of (C.14a) we use the explicit form (2.29) of the derivative $\mathcal{D}^4_\gamma$ to represent this expression as

$$\begin{aligned} B_1 &= \mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}(D^4_{(3)\beta}\Theta^{4\delta}_{(3)})\partial_{\gamma\delta}\frac{1}{X_3}\Big| \\ &= -\mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}(\boldsymbol{x}^{-1}_{13}{}^{\delta}{}_{\beta} - \boldsymbol{x}^{-1}_{23}{}^{\delta}{}_{\beta})\frac{X_{3\gamma\delta}}{X_3{}^3} \\ &= -\mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}\left(-\boldsymbol{X}_{3\beta\delta} + \mathrm{i}\varepsilon_{\beta\delta}\frac{\theta_{13}{}^2}{\boldsymbol{x}_{13}{}^2} + 2\mathrm{i}\boldsymbol{x}^{-1}_{13\beta\mu}\theta^{\hat{I}\mu}_{13}\theta^{\hat{I}\nu}_{32}\boldsymbol{x}^{-1}_{32\nu\delta}\right)\frac{X_3^{\gamma\delta}}{X_3{}^3} \,. \end{aligned} \tag{C.15}$$

Here, in the last line, we applied the identity

$$\boldsymbol{x}^{-1}_{13\alpha\beta} - \boldsymbol{x}^{-1}_{23\alpha\beta} = -\boldsymbol{X}_{3\beta\alpha} + \mathrm{i}\varepsilon_{\beta\alpha}\frac{\theta_{13}{}^2}{\boldsymbol{x}_{13}{}^2} + 2\mathrm{i}\boldsymbol{x}^{-1}_{13\beta\mu}\theta^{\hat{I}\mu}_{13}\theta^{\hat{I}\nu}_{32}\boldsymbol{x}^{-1}_{32\nu\alpha} \,. \tag{C.16}$$

In this identity, only the first term in the right-hand side is given by the three-point structure while the other two terms are non-covariant in the sense that they are represented by the combination of two-point superconformal structures and cannot be expressed solely in terms of three-point ones. These non-covariant terms should cancel against the contributions form (C.14b). Indeed, using the definition (2.15) we compute the derivatives in (C.14b)

$$\begin{aligned} B_2 &= 2\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{13}{}^2}\boldsymbol{x}^{-1}_{13\beta\rho}\theta^{\hat{I}\rho}_{13}\Theta^{\hat{I}\delta}_3\partial_{\gamma\delta}\frac{1}{X_3} \\ &= \tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha}{}^{\gamma}\theta_{13}{}^2}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^2}\frac{X_{3\gamma\beta}}{X_3{}^3} + 2\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\boldsymbol{x}^{-1}_{13\beta\rho}\boldsymbol{x}^{-1}_{\delta\sigma}\theta^{\hat{I}\rho}_{13}\theta^{\hat{I}\sigma}_{23}\frac{X_3^{\gamma\delta}}{X_3{}^3} \,. \end{aligned} \tag{C.17}$$

Thus, in the sum of (C.15) and (C.17) only one term remains which we represent in the following form using (2.25)

$$B_1 + B_2 = \mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\frac{\boldsymbol{X}_{3\beta\delta}X_3^{\gamma\delta}}{X_3{}^3} = \mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}\left(-\frac{\delta^\gamma_\beta}{X_3} - \frac{\mathrm{i}}{2}\frac{X^\gamma_{3\beta}\Theta_3{}^2}{X_3{}^3}\right) \,. \tag{C.18}$$

Finally, we put together the contributions (C.12) and (C.18) and get the resulting expression for the correlation function (C.9) in the form

$$\langle J_\alpha(z_1)S(z_2)J_\beta(z_3)\rangle = \mathrm{i}\tilde{d}_{\mathcal{N}=4}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^2}H^\gamma_\beta(X_3,\Theta_3)\,, \tag{C.19a}$$

$$H^\gamma_\beta(X,\Theta) = \mathrm{i}\frac{\delta^\gamma_\beta}{X} + \frac{1}{2}\frac{X^\gamma_\beta\Theta^2}{X^3} \,. \tag{C.19b}$$

One can verify that the tensor (C.19b) obeys the equations

$$\mathcal{D}^{\hat{I}}_{\alpha} H^{\alpha}_{\beta} = 0\,, \qquad \left(\mathcal{D}^{\hat{I}\alpha}\mathcal{D}^{\hat{J}}_{\alpha} - \frac{1}{3}\delta^{\hat{I}\hat{J}}\mathcal{D}^{\hat{K}\alpha}\mathcal{D}^{\hat{K}}_{\alpha}\right) H^{\gamma}_{\beta} = 0\,, \tag{C.20}$$

which are the corollaries of (C.2).

It is interesting to note that the correlation function (C.19a) depends only on the parameter $\tilde{d}_{\mathcal{N}=4}$ similar to (C.5).

C.1.4 Correlator $\langle J_\alpha J_\beta J_\gamma \rangle$

To compute the correlation function with three $\mathcal{N}=3$ supercurrents J_α we have to hit (C.4) by three spinor covariant derivatives

$$\langle J_\alpha(z_1) J_\beta(z_2) J_\gamma(z_3)\rangle = -\mathrm{i} D^4_{(1)\alpha} D^4_{(2)\beta} D^4_{(3)\gamma} \langle J(z_1) J(z_2) J(z_3)\rangle|\,. \tag{C.21}$$

First of all, we point out that the first term in (C.4b) does not contribute to (C.21). Indeed, due to the identity (C.7), when three derivatives hit this term we always get the contribution which vanishes under the bar-projection

$$-\,\mathrm{i}\tilde{d}_{\mathcal{N}=4} D^4_{(1)\alpha} D^4_{(2)\beta} D^4_{(3)\gamma} \frac{1}{\boldsymbol{x}_{13}{}^2 \boldsymbol{x}_{23}{}^2 X_3}\bigg| = 0\,. \tag{C.22}$$

Hence, we need to consider only the second term in (C.4b).

Taking into account (C.22) we represent (C.21) as a sum of two contributions

$$\langle J_\alpha(z_1) J_\beta(z_2) J_\gamma(z_3)\rangle = A + B\,, \tag{C.23a}$$

$$A = \frac{\mathrm{i}}{\boldsymbol{x}_{13}{}^2}\left(D^4_{(3)\gamma} D^4_{(2)\beta}\frac{1}{\boldsymbol{x}_{23}{}^2}\right) D^4_{(1)\alpha}\tilde{H}(X_3,\Theta_3) - \frac{\mathrm{i}}{\boldsymbol{x}_{23}{}^2}\left(D^4_{(3)\gamma} D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}\right) D^4_{(2)\beta}\tilde{H}(X_3,\Theta_3)\bigg|\,, \tag{C.23b}$$

$$B = \frac{\mathrm{i}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2} D^4_{(3)\gamma} D^4_{(2)\beta} D^4_{(1)\alpha}\tilde{H}(X_3,\Theta_3)\bigg|\,, \tag{C.23c}$$

where $\tilde{H}$ is the second term in (C.4b)

$$\tilde{H}(X,\Theta) = d_{\mathcal{N}=4}\frac{\varepsilon_{IJKL}\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}X_{\alpha\beta}X_{\gamma\delta}}{X^5} = \frac{4d_{\mathcal{N}=4}}{X^5}(\Theta^{\hat{I}}_{\alpha}\Theta^{\hat{J}}_{\mu}\Theta^{\hat{K}}_{\nu}\varepsilon_{\hat{I}\hat{J}\hat{K}})\Theta^4_{\delta}X^{\alpha\mu}X^{\nu\delta}\,. \tag{C.24}$$

In the right-hand side of (C.23b) we apply the following relations

$$D^4_{(3)\gamma} D^4_{(2)\beta}\frac{1}{\boldsymbol{x}_{23}{}^2}\bigg| = 2\mathrm{i}\frac{\boldsymbol{x}_{23\beta\gamma}}{\boldsymbol{x}_{23}{}^4}\,, \qquad D^4_{(3)\gamma} D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}\bigg| = 2\mathrm{i}\frac{\boldsymbol{x}_{13\alpha\gamma}}{\boldsymbol{x}_{13}{}^4}\,. \tag{C.25}$$

Next, using the identities (2.28) we have

$$D^4_{(1)\alpha}\tilde{H}(X_3,\Theta_3)| = -\frac{\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^2}\mathcal{D}^{4\rho}_{(3)}\tilde{H}(X_3,\Theta_3)|\,, \quad D^4_{(2)\beta}\tilde{H}(X_3,\Theta_3)| = \frac{\boldsymbol{x}_{23\beta\rho}}{\boldsymbol{x}_{23}{}^2}\mathcal{D}^{4\rho}_{(3)}\tilde{H}(X_3,\Theta_3)|\,, \tag{C.26}$$

where the derivative of (C.24) reads

$$\mathcal{D}^{4\rho}\tilde{H}(X,\Theta)| = 4d_{\mathcal{N}=4}\frac{X^{\rho\beta}X^{\gamma\delta}\varepsilon_{\hat{I}\hat{J}\hat{K}}\Theta^{\hat{I}}_{\beta}\Theta^{\hat{J}}_{\gamma}\Theta^{\hat{K}}_{\delta}}{X^5}\,. \tag{C.27}$$

Substituting now (C.25)–(C.27) into (C.23b) we get the corresponding contribution to the correlation function

$$A = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H_{(A)}^{\alpha'\beta'}{}_{\gamma}\,, \tag{C.28a}$$

$$H_{(A)}^{\alpha'\beta'}{}_{\gamma} = 8d_{\mathcal{N}=4}\frac{\varepsilon_{\hat{I}\hat{J}\hat{K}}\Theta^{\hat{I}}_{\gamma'}\Theta^{\hat{J}}_{\mu}\Theta^{\hat{K}}_{\nu}X^{\mu\nu}(X^{\beta'\gamma'}\delta^{\alpha'}_{\gamma}+X^{\alpha'\gamma'}\delta^{\beta'}_{\gamma})}{X^5}\,. \tag{C.28b}$$

Using (2.28) we get the following representation for the part (C.23c)

$$B = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H_{(B)}^{\alpha'\beta'}{}_{\gamma}(X_3,\Theta_3)\,, \tag{C.29}$$

where

$$H^{\alpha\beta}_{(B)\gamma} = \Big(-2\Theta^{\hat{I}}_{\gamma}\frac{\partial}{\partial\Theta^{\hat{I}}_{\beta}}\frac{\partial}{\partial\Theta^{4}_{\alpha}}+X_{\mu\gamma}\frac{\partial}{\partial X_{\beta\mu}}\frac{\partial}{\partial\Theta^{4}_{\alpha}}+X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\mu}}\frac{\partial}{\partial\Theta^{4}_{\beta}}-X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\beta}}\frac{\partial}{\partial\Theta^{4}_{\mu}}\Big)\tilde{H}|\,. \tag{C.30}$$

Computing the derivatives of the function (C.24) which are necessary for (C.30) gives

$$\begin{aligned}
&-2\Theta^{\hat{I}}_{\gamma}\frac{\partial}{\partial\Theta^{\hat{I}}_{\beta}}\frac{\partial}{\partial\Theta^{4}_{\alpha}}\tilde{H}| \\
&\quad=\frac{16d_{\mathcal{N}=4}}{X^5}X^{\alpha\mu}X^{\beta\nu}\Theta^{\hat{I}}_{\gamma}\Theta^{\hat{J}}_{\mu}\Theta^{\hat{K}}_{\nu}\varepsilon_{\hat{I}\hat{J}\hat{K}}+\frac{8d_{\mathcal{N}=4}}{X^5}X^{\alpha\beta}X^{\mu\nu}\Theta^{\hat{I}}_{\gamma}\Theta^{\hat{J}}_{\mu}\Theta^{\hat{K}}_{\nu}\varepsilon_{\hat{I}\hat{J}\hat{K}}\,, \qquad\text{(C.31)}\\
&\Big(X_{\mu\gamma}\frac{\partial}{\partial X_{\beta\mu}}\frac{\partial}{\partial\Theta^{4}_{\alpha}}+X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\mu}}\frac{\partial}{\partial\Theta^{4}_{\beta}}-X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\beta}}\frac{\partial}{\partial\Theta^{4}_{\mu}}\Big)\tilde{H}| \\
&\quad=-\frac{4d_{\mathcal{N}=4}}{X^5}X^{\alpha\beta}X^{\mu\nu}\Theta^{\hat{I}}_{\gamma}\Theta^{\hat{J}}_{\alpha}\Theta^{\hat{K}}_{\beta}\varepsilon_{\hat{I}\hat{J}\hat{K}}-\frac{8d_{\mathcal{N}=4}}{X^5}X^{\alpha\mu}X^{\beta\nu}\Theta^{\hat{I}}_{\gamma}\Theta^{\hat{J}}_{\alpha}\Theta^{\hat{K}}_{\beta}\varepsilon_{\hat{I}\hat{J}\hat{K}}\\
&\qquad-\frac{4d_{\mathcal{N}=4}}{X^5}X^{\mu\nu}(X^{\alpha\rho}\delta^{\beta}_{\gamma}+X^{\beta\rho}\delta^{\alpha}_{\gamma})\Theta^{\hat{I}}_{\rho}\Theta^{\hat{J}}_{\mu}\Theta^{\hat{K}}_{\nu}\varepsilon_{\hat{I}\hat{J}\hat{K}}\,. \qquad\text{(C.32)}
\end{aligned}$$

Finally, we collect the results of computations (C.28b), (C.31) and (C.32) in a single expression

$$\langle J_{\alpha}(z_1)J_{\beta}(z_2)J_{\gamma}(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H^{\alpha'\beta'}{}_{\gamma}(X_3,\Theta_3)\,, \tag{C.33a}$$

$$\begin{aligned}
H^{\alpha\beta}{}_{\gamma}(X,\Theta) = \frac{4d_{\mathcal{N}=4}}{X^5}\Big[&(\delta^{\beta}_{\gamma}X^{\alpha\rho}+\delta^{\alpha}_{\gamma}X^{\rho\beta})X^{\mu\nu}\Theta^{\hat{I}}_{\mu}\Theta^{\hat{J}}_{\nu}\Theta^{\hat{K}}_{\rho}\varepsilon_{\hat{I}\hat{J}\hat{K}}\\
&+X^{\beta\alpha}X^{\mu\nu}\Theta^{\hat{I}}_{\mu}\Theta^{\hat{J}}_{\nu}\Theta^{\hat{K}}_{\gamma}\varepsilon_{\hat{I}\hat{J}\hat{K}}+2X^{\alpha\mu}X^{\nu\beta}\Theta^{\hat{I}}_{\mu}\Theta^{\hat{J}}_{\nu}\Theta^{\hat{K}}_{\gamma}\varepsilon_{\hat{I}\hat{J}\hat{K}}\Big]\,.
\end{aligned} \tag{C.33b}$$

This expression for $H^{\alpha\beta}{}_{\gamma}$ coincides with (7.4c) upon the replacement $X_{\alpha\beta}\to\boldsymbol{X}_{\alpha\beta}$. Although $X_{\alpha\beta}$ and $\boldsymbol{X}_{\alpha\beta}$ differ in a Θ-dependent term, see (2.25), one can check that these additional terms do not contribute to (C.33b). To match the expressions (7.4b) and (C.33) one has to make also the identification of their parameters (7.5b).

C.1.5 Reduction of the two-point function

The superspace reduction of the two-point function of $\mathcal{N}=4$ supercurrents (4.17) to the $\mathcal{N}=3$ superspace is much simpler than the same procedure for the three-point functions. Indeed, the correlator of the superfield S immediately follow from (4.17)

$$\langle S(z_1)S(z_2)\rangle = \langle J(z_1)J(z_2)\rangle| = \frac{c_{\mathcal{N}=4}}{\boldsymbol{x}_{12}{}^2}\,, \tag{C.34}$$

while for the correlation function of the $\mathcal{N}=3$ supercurrent J_α we have

$$\langle J_\alpha(z_1)J_\beta(z_2)\rangle = D^4_{(2)\beta}D^4_{(1)\alpha}\langle J(z_1)J(z_2)\rangle| = c_{\mathcal{N}=4}D^4_{(2)\beta}D^4_{(1)\alpha}\frac{1}{\boldsymbol{x}_{12}{}^2}\Big|\,. \tag{C.35}$$

Applying the identity (C.11) we find

$$\langle J_\alpha(z_1)J_\beta(z_2)\rangle = 2\mathrm{i}c_{\mathcal{N}=4}\frac{\boldsymbol{x}_{12\alpha\beta}}{\boldsymbol{x}_{12}{}^4}\,. \tag{C.36}$$

Comparing this two-point function with (7.4a) allows us to get the relation (7.5a) among the coefficients $c_{\mathcal{N}=3}$ and $c_{\mathcal{N}=4}$.

C.2 $\mathcal{N}=3 \to \mathcal{N}=2$ reduction of the supercurrent correlation function

Recall that the $\mathcal{N}=3$ supercurrent J_α contains the following two independent $\mathcal{N}=2$ supermultiplets [1]:

$$R_\alpha := J_\alpha|\,, \qquad D^{\hat I\alpha}R_\alpha = 0\,; \tag{C.37a}$$

$$J_{\alpha\beta} := D^3_{(\alpha}J_{\beta)}|\,, \qquad D^{\hat I\alpha}J_{\alpha\beta} = 0\,, \qquad \hat I = 1,2\,. \tag{C.37b}$$

Here $J_{\alpha\beta}$ is the $\mathcal{N}=2$ supercurrent, while R_α contains the third supersymmetry current and two R-symmetry currents corresponding to $\mathsf{SO}(3)/\mathsf{SO}(2)$. In this appendix we consider all two- and three-point correlation functions of R_α and $J_{\alpha\beta}$ which follow from the corresponding correlators of $\mathcal{N}=3$ supercurrent J_α.

C.2.1 Two-point correlators

Consider the two-point correlation function of the $\mathcal{N}=3$ supercurrent (7.4a). Obviously, the two-point correlator of the superfield R_α has the same form in $\mathcal{N}=2$ superspace

$$\langle R_\alpha(z_1)R_\beta(z_2)\rangle = \langle J_\alpha(z_1)J_\beta(z_2)\rangle| = \mathrm{i}c_{\mathcal{N}=3}\frac{\boldsymbol{x}_{12\alpha\beta}}{\boldsymbol{x}_{12}{}^4}\,, \tag{C.38}$$

where the bar-projection assumes $\theta_{3\alpha}=0$.

To find the two-point function of $\mathcal{N}=2$ supercurrent we need to hit (7.4a) by two spinor covariant derivatives

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)\rangle = -D^3_{(1)\alpha}D^3_{(2)\beta}\langle J_{\alpha'}(z_1)J_{\beta'}(z_2)\rangle| = \mathrm{i}c_{\mathcal{N}=3}D^3_{(2)\beta}D^3_{(1)\alpha}\frac{\boldsymbol{x}_{12\alpha'\beta'}}{\boldsymbol{x}_{12}{}^4}\Big|\,. \tag{C.39}$$

It is straightforward to compute these derivatives using the definition of the two-point structure (2.8a)

$$\langle J_{\alpha\beta}(z_1)J^{\alpha'\beta'}(z_2)\rangle = -4c_{\mathcal{N}=3}\frac{\boldsymbol{x}_{12\alpha}{}^{(\alpha'}\boldsymbol{x}_{12\beta}{}^{\beta')}}{\boldsymbol{x}_{12}{}^6}\,. \tag{C.40}$$

Comparing this expression with (7.10) we find the relation among the coefficients $c_{\mathcal{N}=2}$ and $c_{\mathcal{N}=3}$ given in (7.13).

JHEP08(2015)125

C.2.2 Three-point correlators involving R_α

There are three correlation functions involving R_α:

$$\langle R_\alpha(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle \ , \quad \langle J_{\alpha\delta}(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle \ , \quad \langle J_{\alpha\delta}(z_1)J_{\beta\rho}(z_2)R_\gamma(z_3)\rangle \ .$$

It is easy to see that

$$\langle R_\alpha(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle = \langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle| = 0 \ . \tag{C.41}$$

Indeed, the tensor $H_{\alpha\beta\gamma}$ which defines the $\mathcal{N}=3$ supercurrent correlator (7.4c) vanishes under the bar-projection owing to $\Theta^I_\alpha\Theta^J_\beta\Theta^K_\gamma\varepsilon_{IJK}| = 0$. Similarly, it is possible to show that

$$\langle J_{\alpha\delta}(z_1)J_{\beta\rho}(z_2)R_\gamma(z_3)\rangle = -D^3_{(1)\delta}D^3_{(2)\rho}\langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle| = 0 \ . \tag{C.42}$$

Thus, we need to consider only $\langle J_{\alpha\delta}(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle$ which is non-trivial

$$\begin{aligned}\langle J_{\alpha\delta}(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle &= D^3_{(1)\delta}\langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle| \\ &= -d_{\mathcal{N}=3}\frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{13\delta\delta'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^4}\mathcal{D}^{3\delta'}H^{\alpha'\beta'}{}_\gamma| \ . \end{aligned} \tag{C.43}$$

Here we have applied the identity (2.28a) and the representation (7.4b) for the $\mathcal{N}=3$ supercurrent three-point function. It is easy to evaluate the derivative of the tensor (7.4c) since under the bar projection only those terms survive in which $\mathcal{D}^{3\delta}$ acts on the generalized Grassmann variable Θ^I_α but not on $\boldsymbol{X}_{\mu\nu}$. As a result, we get the following representation for the correlator (C.43)

$$\langle J_{\alpha\delta}(z_1)R_\beta(z_2)R_\gamma(z_3)\rangle = d_{\mathcal{N}=3}\frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{13\delta\delta'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^4}H^{\delta'\alpha'\beta'}{}_\gamma(X_3,\Theta_3) \ , \tag{C.44a}$$

$$\begin{aligned}H^{\delta\alpha\beta}{}_\gamma(X,\Theta) = \frac{1}{X^5}\Big[&\delta^\beta_\gamma X^{\alpha\delta}X^{\mu\nu}(\Theta\Theta)_{\mu\nu} + 2\delta^\beta_\gamma X^{\alpha\mu}X^{\delta\nu}(\Theta\Theta)_{\mu\nu} \\ &+2X^{\alpha\delta}X^{\nu\beta}(\Theta\Theta)_{\nu\gamma} + 2\delta^{(\alpha}_\gamma X^{\delta)\beta}X^{\mu\nu}(\Theta\Theta)_{\mu\nu} \\ &+4\delta^{(\delta}_\gamma X^{\alpha)\mu}X^{\nu\beta}(\Theta\Theta)_{\mu\nu} + 4X^{\beta(\alpha}X^{\delta)\nu}(\Theta\Theta)_{\nu\gamma}\Big] \ , \end{aligned} \tag{C.44b}$$

where we use the notation $(\Theta\Theta)_{\mu\nu} = \Theta^I_\mu\Theta^J_\nu\varepsilon_{IJ}$.

C.2.3 Three-point correlator of the $\mathcal{N}=2$ supercurrent

Consider now the three-point function of the $\mathcal{N}=2$ supercurrent

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = -D^3_{(1)\alpha}D^3_{(2)\beta}D^3_{(3)\gamma}\langle J_{\alpha'}(z_1)J_{\beta'}(z_2)J_{\gamma'}(z_3)\rangle| \ . \tag{C.45}$$

The $\mathcal{N}=3$ supercurrent three-point correlation function is found in the form (C.33) involving the tensor $H^{\alpha\beta}{}_\gamma$. For the following calculations it will be convenient to use the form of this tensor with the pair of spinor indices $\alpha\beta$ converted into a vector one m

$$H^m_\gamma = -\frac{1}{2}\gamma^m_{\alpha\beta}H^{\alpha\beta}{}_\gamma = -\mathrm{i}\frac{6}{X^3}\Theta^3_\gamma(\Theta\Theta)^m + \mathrm{i}\frac{18}{X^5}X^mX^p\Theta^3_\gamma(\Theta\Theta)_p$$

$$
\begin{aligned}
&-\mathrm{i}\frac{2}{X^3}\varepsilon^{pmq}(\gamma_q)^\mu_\gamma\Theta^3_\mu(\Theta\Theta)_p+\mathrm{i}\frac{8}{X^5}X^mX_r\varepsilon^{prq}(\gamma_q)^\mu_\gamma\Theta^3_\mu(\Theta\Theta)_p\\
&+\mathrm{i}\frac{4}{X^5}\varepsilon^{mrq}(\gamma_q)^\mu_\gamma X_rX^p\Theta^3_\mu(\Theta\Theta)_p+\mathrm{i}\frac{2}{X^5}X_rX^q\varepsilon^{mrq}(\gamma_q)^\mu_\gamma\Theta^3_\mu(\Theta\Theta)_p\,, \qquad \text{(C.46)}
\end{aligned}
$$

where we employ the short-notation $(\Theta\Theta)_m$ for

$$
(\Theta\Theta)_m=-\frac{\mathrm{i}}{2}(\gamma_m)^{\alpha\beta}\Theta^I_\alpha\Theta^J_\beta\varepsilon_{IJ}\,. \qquad \text{(C.47)}
$$

We substitute (C.33a) into (C.45) and represent it as a sum of two parts with specific distribution of covariant spinor derivatives on the factors

$$
D^3_{(3)\gamma}D^3_{(2)\beta}D^3_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\alpha''}\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H^{\alpha''\beta''}{}_{\gamma'}(X_3,\Theta_3)|=A+B\,, \qquad \text{(C.48a)}
$$

$$
\begin{aligned}
A=&\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}\left(D^3_{(3)\gamma}D^3_{(2)\beta}\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}\right)D^3_{(1)\alpha}H^{\alpha''\beta''}{}_{\gamma'}\\
&-\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}\left(D^3_{(3)\gamma}D^3_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}\right)D^3_{(2)\beta}H^{\alpha''\beta''}{}_{\gamma'}|\,, \qquad \text{(C.48b)}\\
B=&\frac{\boldsymbol{x}_{13\alpha'\alpha''}\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}D^3_{(3)\gamma}D^3_{(2)\beta}D^3_{(1)\alpha}H^{\alpha''\beta''}{}_{\gamma'}|\,. \qquad \text{(C.48c)}
\end{aligned}
$$

One can check that in (C.48a) the terms in which the covariant spinor derivatives are distributed in other ways vanish under the bar-projection. Now consider the computations of contribution (C.48b) and (C.48c) separately.

In the part A given by (C.48b) we need the following relations

$$
\begin{aligned}
D^3_{(3)\gamma}D^3_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}|&=\frac{2\mathrm{i}}{\boldsymbol{x}_{13}{}^6}(\boldsymbol{x}_{13\alpha\alpha''}\boldsymbol{x}_{13\alpha'\gamma}+\boldsymbol{x}_{13\alpha\gamma}\boldsymbol{x}_{13\alpha'\alpha''})\,,\\
D^3_{(3)\gamma}D^3_{(2)\beta}\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}|&=\frac{2\mathrm{i}}{\boldsymbol{x}_{23}{}^6}(\boldsymbol{x}_{23\beta\beta''}\boldsymbol{x}_{23\beta'\gamma}+\boldsymbol{x}_{23\beta\gamma}\boldsymbol{x}_{23\beta'\beta''})\,, \qquad \text{(C.49)}
\end{aligned}
$$

which follow from the definition (2.8a). With the use of identities (2.28) we have

$$
\begin{aligned}
D^3_{(1)\alpha}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|&=-\frac{\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^2}\mathcal{D}^{3\rho}_{(3)}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|\,,\\
D^3_{(2)\beta}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|&=\frac{\boldsymbol{x}_{23\beta\rho}}{\boldsymbol{x}_{23}{}^2}\mathcal{D}^{3\rho}_{(3)}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|\,. \qquad \text{(C.50)}
\end{aligned}
$$

Taking into account (C.49) and (C.50) we represent the part A in the form

$$
A=\frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^6}H^{\rho\rho'\,\sigma\sigma'}_{(A)}{}_{\gamma\gamma'}(X_3,\Theta_3)\,, \qquad \text{(C.51)}
$$

where

$$
H^{\rho\rho'\,\sigma\sigma'}_{(A)}{}_{\gamma\gamma'}=-4\mathrm{i}(\delta^\sigma_\gamma\mathcal{D}^{3\rho}H^{\rho'\sigma'}{}_{\gamma'}+\delta^{\sigma'}_\gamma\mathcal{D}^{3\rho}H^{\rho'\sigma}{}_{\gamma'}+\delta^\rho_\gamma\mathcal{D}^{3\sigma}H^{\rho'\sigma'}{}_{\gamma'}+\delta^{\rho'}_\gamma\mathcal{D}^{3\sigma}H^{\rho\sigma'}{}_{\gamma'})|\,. \qquad \text{(C.52)}
$$

In the expression (C.48c) we use the identities (2.28) to represent it in the form

$$
\begin{aligned}
D^3_{(3)\gamma}D^3_{(2)\beta}D^3_{(1)\alpha}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|&=\mathrm{i}\boldsymbol{x}^{-1}_{13\rho\alpha}\boldsymbol{x}^{-1}_{23\sigma\beta}D^3_{(3)\gamma}u^{3J}_{13}u^{3K}_{23}\mathcal{Q}^{K\sigma}\mathcal{D}^{J\rho}H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|\\
&=\mathrm{i}(\boldsymbol{x}^{-1}_{13})^\rho{}_\alpha(\boldsymbol{x}^{-1}_{23})^\sigma{}_\beta D^3_{(3)\gamma}[\mathcal{Q}^3_\sigma\mathcal{D}^3_\rho+u^{31}_{13}\mathcal{Q}^3_\sigma\mathcal{D}^1_\rho+u^{32}_{13}\mathcal{Q}^3_\sigma\mathcal{D}^2_\rho
\end{aligned}
$$

JHEP08(2015)125

$$+u^{31}_{23}\mathcal{Q}^1_\sigma\mathcal{D}^3_\rho+u^{32}_{23}\mathcal{Q}^2_\sigma\mathcal{D}^3_\rho]H^{\alpha'\beta'\gamma}(X_3,\Theta_3)|\,. \tag{C.53}$$

Below we consider various terms in this expression separately.

For the first term in the square brackets in (C.53) we use explicit forms of generalised spinor covariant derivative and the supercharge (2.29) to rewrite it as

$$\begin{aligned}&D^3_{(3)\gamma}\mathcal{Q}^3_\sigma\mathcal{D}^3_\rho H^{\alpha'\beta'\gamma}|\\&=[(\boldsymbol{x}^{-1}_{13})^\mu{}_\gamma-(\boldsymbol{x}^{-1}_{23})^\mu{}_\gamma]\left(\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{3\rho}}+\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{3\rho}}-\frac{\partial}{\partial X^{\rho\sigma}}\frac{\partial}{\partial\Theta^{3\mu}}\right)H^{\alpha'\beta'\gamma}|\,.\end{aligned} \tag{C.54}$$

Here we used the fact that in the bar-projection only those terms survive in which the derivative $D^3_{(3)\gamma}$ acts on $\Theta^{3\mu}$ and produces the factor $[(\boldsymbol{x}^{-1}_{13})^\mu{}_\gamma-(\boldsymbol{x}^{-1}_{23})^\mu{}_\gamma]$. As is pointed out in [1], this factor cannot be expressed solely in terms of $\boldsymbol{X}_{3\alpha\beta}$ and $\Theta^I_{3\alpha}$, but it involves the two-point structures as well

$$(\boldsymbol{x}^{-1}_{13})_{\alpha\beta}-(\boldsymbol{x}^{-1}_{23})_{\alpha\beta}=-\boldsymbol{X}_{3\alpha\beta}+\mathrm{i}\frac{\varepsilon_{\alpha\beta}}{\boldsymbol{x}_{23}{}^2}\theta^2_{23}+2\mathrm{i}(\boldsymbol{x}^{-1}_{13})_{\alpha\mu}\theta^\mu_{13}\theta^\nu_{32}(\boldsymbol{x}^{-1}_{32})_{\nu\beta}\,. \tag{C.55}$$

The last two terms here are non-covariant in the sense that they are expressed in terms of two-point superconformal invariants rather than the three-point ones. Then taking into account (C.55) we rewrite (C.54) as

$$\begin{aligned}D^3_{(3)\gamma}\mathcal{Q}^3_\sigma\mathcal{D}^3_\rho H^{\alpha'\beta'\gamma}|=&-\boldsymbol{X}_3{}^\mu{}_\gamma\left(\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{3\rho}}+\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{3\rho}}-\frac{\partial}{\partial X^{\rho\sigma}}\frac{\partial}{\partial\Theta^{3\mu}}\right)H^{\alpha'\beta'\gamma}|\\&+\text{non-covariant terms},\end{aligned} \tag{C.56}$$

where the 'non-covariant terms' are those which correspond to the last two terms in (C.55). Here we do not write down these terms explicitly as they cancel against the contributions coming from the remaining terms in the square brackets in (C.53)[15]

$$\begin{aligned}&D^3_{(3)\gamma}[u^{31}_{13}\mathcal{Q}^3_\sigma\mathcal{D}^1_\rho+u^{32}_{13}\mathcal{Q}^3_\sigma\mathcal{D}^2_\rho+u^{31}_{23}\mathcal{Q}^1_\sigma\mathcal{D}^3_\rho+u^{32}_{23}\mathcal{Q}^2_\sigma\mathcal{D}^3_\rho]H^{\alpha'\beta'\gamma}|\\&\quad=2\left(\Theta^1_\gamma\frac{\partial}{\partial\Theta^{1\sigma}}\frac{\partial}{\partial\Theta^{3\rho}}+\Theta^2_\gamma\frac{\partial}{\partial\Theta^{2\sigma}}\frac{\partial}{\partial\Theta^{3\rho}}\right)H^{\alpha'\beta'\gamma}|-\text{non-covariant terms}.\end{aligned} \tag{C.57}$$

Thus, when we take the sum of (C.56) and (C.57) these 'non-covariant terms' cancel and we get the contribution to the $\mathcal{N}=2$ supercurrent correlation functions in the form

$$B=\frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^6}H_{(B)}^{\rho\rho'\sigma\sigma'}{}_{\gamma\gamma'}(X_3,\Theta_3)\,, \tag{C.58}$$

where

$$\begin{aligned}H_{(B)}^{\rho\rho'\sigma\sigma'}{}_{\gamma\gamma'}(X,\Theta)=\mathrm{i}\Bigg(&2\Theta^1_\gamma\frac{\partial}{\partial\Theta^1_\rho}\frac{\partial}{\partial\Theta^3_\sigma}+2\Theta^2_\gamma\frac{\partial}{\partial\Theta^2_\rho}\frac{\partial}{\partial\Theta^3_\sigma}-X_{\mu\gamma}\frac{\partial}{\partial X_{\sigma\mu}}\frac{\partial}{\partial\Theta^3_\rho}\\&-X_{\mu\gamma}\frac{\partial}{\partial X_{\rho\mu}}\frac{\partial}{\partial\Theta^3_\sigma}+X_{\mu\gamma}\frac{\partial}{\partial X_{\rho\sigma}}\frac{\partial}{\partial\Theta^3_\mu}\Bigg)H^{\rho'\sigma'}{}_{\gamma'}(X,\Theta)|\,.\end{aligned} \tag{C.59}$$

[15] This cancellation has been explicitly demonstrated in appendix C.1 of [1] for the case of superspace reduction of the $\mathcal{N}=2$ supercurrent correlation function down to $\mathcal{N}=1$. In the present case the cancellation of the non-covariant terms can be checked in the same way.

Summarizing (C.51) and (C.58) we find the $\mathcal{N}=2$ supercurrent three-point correlation function

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6\boldsymbol{x}_{23}{}^6} H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}(X_3,\Theta_3)\,, \tag{C.60}$$

where the tensor $H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}$ is expressed in terms of derivatives of the tensor $H^{\alpha\beta}{}_{\gamma}$ in the $\mathcal{N}=3$ theory given by (C.33b)

$$\begin{aligned}
H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} &= H_{(A)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} + H_{(B_1)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} + H_{(B_2)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} + H_{(B_3)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}\,, && \text{(C.61a)}\\
H_{(A)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} &= 2\mathrm{i}\bigg(\delta_\gamma^\sigma\frac{\partial}{\partial\Theta^3_\rho}H^{\rho'\sigma'}{}_{\gamma'} + \delta_\gamma^{\sigma'}\frac{\partial}{\partial\Theta^3_\rho}H^{\rho'\sigma}{}_{\gamma'}\\
&\qquad + \delta_\gamma^{\rho}\frac{\partial}{\partial\Theta^3_\sigma}H^{\rho'\sigma'}{}_{\gamma'} + \delta_\gamma^{\rho'}\frac{\partial}{\partial\Theta^3_\sigma}H^{\rho\sigma'}{}_{\gamma'}\bigg)\Big|\,, && \text{(C.61b)}\\
H_{(B_1)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} &= 2\mathrm{i}\left(\Theta^1_\gamma\frac{\partial}{\partial\Theta^1_\rho}\frac{\partial}{\partial\Theta^3_\sigma} + \Theta^2_\gamma\frac{\partial}{\partial\Theta^2_\rho}\frac{\partial}{\partial\Theta^3_\sigma}\right)H^{\rho'\sigma'}{}_{\gamma'}\Big|\,, && \text{(C.61c)}\\
H_{(B_2)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} &= -\mathrm{i}\left(X_{\mu\gamma}\frac{\partial}{\partial X_{\sigma\mu}}\frac{\partial}{\partial\Theta^3_\rho} + X_{\mu\gamma}\frac{\partial}{\partial X_{\rho\mu}}\frac{\partial}{\partial\Theta^3_\sigma}\right)H^{\rho'\sigma'}{}_{\gamma'}\Big|\,, && \text{(C.61d)}\\
H_{(B_3)}^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} &= \mathrm{i}X_{\mu\gamma}\frac{\partial}{\partial X_{\rho\sigma}}\frac{\partial}{\partial\Theta^3_\mu}H^{\rho'\sigma'}{}_{\gamma'}\Big|\,. && \text{(C.61e)}
\end{aligned}$$

As a result, the problem is reduced to computing the derivatives of the tensor (C.33b).

Let us convert the spinor indices of $H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}$ into the vector ones

$$H^{mnk} = -\frac{1}{8}\gamma^m_{\rho\rho'}\gamma^n_{\sigma\sigma'}(\gamma^k)^{\gamma\gamma'}H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}\,. \tag{C.62}$$

It is known that the tensor H^{mnk} which defines the three-point correlation function of $\mathcal{N}=2$ supercurrent can be represented in the form [1]

$$H^{mnk} = (\Theta\Theta)_p C^{mnp,k}\,, \tag{C.63}$$

where $(\Theta\Theta)_p$ is given in (C.47) and $C^{mnp,k}$ is symmetric and traceless in the indices mnp,

$$C^{mnp,k} = C^{(mnp),k}\,, \qquad \eta_{mn}C^{mnp,k} = 0\,. \tag{C.64}$$

Hence, the tensor $H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}$ has the following symmetry property

$$H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'} = H^{(\rho\rho'\sigma\sigma')}{}_{\gamma\gamma'}\,. \tag{C.65}$$

As a consequence, the relation (C.62) can equivalently be rewritten as

$$H^{mnk} = -\frac{1}{8}\gamma^m_{\rho\sigma}\gamma^n_{\rho'\sigma'}(\gamma^k)^{\gamma\gamma'}H^{\rho\rho'\,\sigma\sigma'}{}_{\gamma\gamma'}\,, \tag{C.66}$$

and it appears to be more convenient to use the tensor (C.46) for further computations. Indeed, the expression (C.61b) can be rewritten as a derivative of (C.46)

$$H^{mnp}_{(A)} = 2\mathrm{i}(\gamma^n)_{\rho\sigma}(\gamma^k)^{\gamma\sigma}\frac{\partial}{\partial\Theta^3_\rho}H^m_\gamma = C^{mnp,k}_{(A)}(\Theta\Theta)_p\,,$$

$$
\begin{aligned}
C^{mnp,k}_{(A)} &= -24\left(\frac{1}{X^3}\eta^{kn}\eta^{mp} - \frac{3}{X^5}\eta^{nk}X^mX^p\right) \\
&\quad -\frac{24}{X^5}(X^mX^k\eta^{np} + X^kX^p\eta^{mn}) + \frac{24}{X^5}(\eta^{kp}X^mX^n + \eta^{km}X^nX^p)\ . \qquad \text{(C.67)}
\end{aligned}
$$

In contrast to (C.64), the tensor $C^{mnp,k}_{(A)}$ is not symmetric and traceless in the indices mnp. However, when all contributions (C.61) are taken into account, the resulting tensor $C^{mnp,k}$ should obey the symmetry (C.64). Hence, it is sufficient for the following to consider only the symmetric part of (C.67) in the indices mnp

$$
\begin{aligned}
C^{(mnp),k}_{(A)} &= -\frac{8}{X^3}(\eta^{nk}\eta^{mp} + \eta^{mk}\eta^{np} + \eta^{kp}\eta^{mn}) \\
&\quad +\frac{40}{X^5}(\eta^{kn}X^mX^p + \eta^{km}X^nX^p + \eta^{kp}X^mX^n) \\
&\quad -\frac{16}{X^5}(X^kX^m\eta^{nr} + X^kX^n\eta^{mp} + X^kX^p\eta^{mn})\ . \qquad \text{(C.68)}
\end{aligned}
$$

In the same way using the tensor (C.46) we compute the expression (C.61c)–(C.61d),

$$
H^{mnk}_{(B_1)} = \frac{\mathrm{i}}{2}\gamma^n_{\rho\sigma}(\gamma^k)^{\gamma\gamma'}\left(\Theta^1_\gamma\frac{\partial}{\partial\Theta^1_\rho}\frac{\partial}{\partial\Theta^3_\sigma} + \Theta^2_\gamma\frac{\partial}{\partial\Theta^2_\rho}\frac{\partial}{\partial\Theta^3_\sigma}\right)H^m_{\gamma'} = C^{mnp,k}_{(B_1)}(\Theta\Theta)_p\,, \qquad \text{(C.69a)}
$$

$$
\begin{aligned}
C^{mnp,k}_{(B_1)} &= -\frac{12}{X^3}\eta^{mn}\eta^{kp} + \frac{36}{X^5}X^mX^n\eta^{kp} \\
&\quad -\frac{12}{X^5}(X^kX^m\eta^{np} + X^kX^n\eta^{mp}) + \frac{12}{X^5}(\eta^{km}X^nX^p + \eta^{kn}X^mX^p)\,, \qquad \text{(C.69b)}
\end{aligned}
$$

$$
H^{mnk}_{(B_2)} = -\frac{\mathrm{i}}{2}\gamma^n_{\rho\sigma}(\gamma^k)^{\gamma\gamma'}X_{3\mu\gamma}\frac{\partial}{\partial X_{3\sigma\mu}}\frac{\partial}{\partial\Theta^3_\rho}H^m_{\gamma'} = C^{mnp,k}_{(B_2)}\,, \qquad \text{(C.70a)}
$$

$$
\begin{aligned}
C^{mnp,k}_{(B_2)} &= \frac{6}{X^3}\eta^{nk}\eta^{mp} + \frac{12}{X^3}(\eta^{mk}\eta^{np} + \eta^{mn}\eta^{kp}) \\
&\quad -\frac{42}{X^5}\eta^{nk}X^mX^p - \frac{78}{X^5}\eta^{km}X^nX^p - \frac{30}{X^5}\eta^{kp}X^mX^n \\
&\quad +\frac{30}{X^5}\eta^{mn}X^kX^p + \frac{6}{X^5}X^mX^k\eta^{np} + \frac{24}{X^5}X^nX^k\eta^{mp}\,, \qquad \text{(C.70b)}
\end{aligned}
$$

$$
H^{mnk}_{(B_3)} = -\frac{\mathrm{i}}{2}(\gamma^k)^{\gamma\gamma'}X_{3\mu\gamma}\frac{\partial}{\partial X_{3n}}\frac{\partial}{\partial\Theta^3_\mu}H^m_{\gamma'} = C^{mnp,k}_{(B_3)}(\Theta\Theta)_p\,, \qquad \text{(C.71a)}
$$

$$
\begin{aligned}
C^{mnp,k}_{(B_3)} &= -\frac{12}{X^3}\eta^{mk}\eta^{np} + \frac{12}{X^5}\eta^{nk}X^mX^p + \frac{48}{X^5}\eta^{mk}X^nX^p \\
&\quad -\frac{18}{X^5}(\eta^{mp}X^kX^n + \eta^{mn}X^kX^p) - \frac{6}{X^5}\eta^{np}X^kX^m + \frac{30}{X^7}X^mX^nX^kX^p\ . \qquad \text{(C.71b)}
\end{aligned}
$$

We need only symmetric parts of the tensors (C.69b), (C.70b) and (C.71b) in the indices mnp:

$$
C^{(mnp)k}_{(B_1)} = -\frac{4}{X^3}(\eta^{mn}\eta^{kp} + \eta^{kn}\eta^{mp} + \eta^{km}\eta^{np})
$$

$$
\begin{aligned}
&+\frac{20}{X^5}(\eta^{kp}X^mX^n+\eta^{km}X^nX^p+\eta^{kn}X^mX^p)\\
&-\frac{8}{X^5}(X^kX^m\eta^{np}+X^kX^n\eta^{mp}+X^kX^p\eta^{mn})\,, \qquad \text{(C.72a)}\\
C^{(mnp)k}_{(B_2)} &= \frac{10}{X^3}(\eta^{km}\eta^{np}+\eta^{kn}\eta^{mp}+\eta^{kp}\eta^{mn})\\
&-\frac{50}{X^5}(\eta^{kn}X^mX^p+\eta^{km}X^nX^p+\eta^{kp}X^mX^p)\\
&+\frac{20}{X^5}(X^kX^p\eta^{mn}+X^kX^n\eta^{mp}+X^kX^m\eta^{np})\,, \qquad \text{(C.72b)}\\
C^{(mnp)k}_{(B_3)} &= -\frac{4}{X^3}(\eta^{km}\eta^{np}+\eta^{kn}\eta^{mp}+\eta^{kp}\eta^{mn})\\
&+\frac{20}{X^5}(\eta^{kn}X^mX^p+\eta^{km}X^nX^p+\eta^{kp}X^mX^p)\\
&-\frac{14}{X^5}(X^kX^p\eta^{mn}+X^kX^n\eta^{mp}+X^kX^m\eta^{np})+\frac{30}{X^7}X^mX^nX^kX^p\,. \qquad \text{(C.72c)}
\end{aligned}
$$

The sum of (C.67) and (C.72) is

$$
\begin{aligned}
C^{mnp,k} &= C^{(mnp)k}_A+C^{(mnp)k}_{B_1}+C^{(mnp)k}_{B_2}+C^{(mnp)k}_{B_3}\\
&= -6d_{\mathcal{N}=3}\bigg[\frac{1}{X^3}(\eta^{km}\eta^{np}+\eta^{kn}\eta^{mp}+\eta^{kp}\eta^{mn})\\
&-\frac{5}{X^5}(\eta^{kp}X^mX^n+\eta^{kn}X^mX^p)\\
&-\frac{5}{X^5}(\eta^{kp}X^mX^n+\eta^{kn}X^mX^p+\eta^{km}X^nX^p)\\
&+\frac{3}{X^5}(X^kX^m\eta^{np}+X^kX^n\eta^{mp}+X^kX^p\eta^{mn})-\frac{5}{X^7}X^mX^nX^kX^p\bigg]\,. \qquad \text{(C.73)}
\end{aligned}
$$

Substituting this tensor back into (C.63) we find the $\mathcal{N}=2$ supercurrent correlation function in the form (7.11) where the parameter $d_{\mathcal{N}=2}$ is related to $d_{\mathcal{N}=3}$ as

$$
d_{\mathcal{N}=2}=-6d_{\mathcal{N}=3}\,. \qquad \text{(C.74)}
$$

References

[1] E.I. Buchbinder, S.M. Kuzenko and I.B. Samsonov, *Superconformal field theory in three dimensions: correlation functions of conserved currents*, *JHEP* **06** (2015) 138 [arXiv:1503.04961] [INSPIRE].

[2] L. Álvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model*, *Commun. Math. Phys.* **80** (1981) 443 [INSPIRE].

[3] J.A. Bagger, *Supersymmetric sigma models*, lectures given at the 1984 *NATO Advanced Study Institute on Supersymmetry*, Bonn Germany August 1984, reprinted in *Supergravities in diverse dimensions*, vol. 1, A. Salam and E. Sezgin eds., North-Holland/World Scientific, (1989), pg. 569.

[4] S.M. Kuzenko, J.-H. Park, G. Tartaglino-Mazzucchelli and R. Unge, *Off-shell superconformal nonlinear σ-models in three dimensions*, *JHEP* **01** (2011) 146 [arXiv:1011.5727] [INSPIRE].

[5] B.M. Zupnik, *Three-dimensional $\mathcal{N}=4$ supersymmetry in harmonic $\mathcal{N}=3$ superspace*, *Theor. Math. Phys.* **165** (2010) 1315 [*Teor. Mat. Fiz.* **165** (2010) 97] [arXiv:1005.4750] [INSPIRE].

[6] B.M. Zupnik and D.V. Khetselius, *Three-dimensional extended supersymmetry in the harmonic superspace* (in Russian), *Sov. J. Nucl. Phys.* **47** (1988) 730 [*Yad. Fiz.* **47** (1988) 1147] [INSPIRE].

[7] H.-C. Kao and K.-M. Lee, *Selfdual Chern-Simons systems with an $\mathcal{N}=3$ extended supersymmetry*, *Phys. Rev.* **D 46** (1992) 4691 [hep-th/9205115] [INSPIRE].

[8] H.-C. Kao, *Selfdual Yang-Mills Chern-Simons Higgs systems with an $\mathcal{N}=3$ extended supersymmetry*, *Phys. Rev.* **D 50** (1994) 2881 [INSPIRE].

[9] S.M. Kuzenko and J. Novak, *Supergravity-matter actions in three dimensions and Chern-Simons terms*, *JHEP* **05** (2014) 093 [arXiv:1401.2307] [INSPIRE].

[10] R. Brooks and S.J. Gates, Jr., *Extended supersymmetry and super-BF gauge theories*, *Nucl. Phys.* **B 432** (1994) 205 [hep-th/9407147] [INSPIRE].

[11] B.M. Zupnik, *Harmonic superspaces for three-dimensional theories*, in *Supersymmetries and quantum symmetries*, J. Wess and E. Ivanov eds., Springer, Berlin Germany (1999), pg. 116 [hep-th/9804167] [INSPIRE].

[12] B. Zupnik, *Harmonic superpotentials and symmetries in gauge theories with eight supercharges*, *Nucl. Phys.* **B 554** (1999) 365 [*Erratum ibid.* **B 644** (2002) 405] [hep-th/9902038] [INSPIRE].

[13] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings, and the θ-angle in $\mathcal{N}=4$ super Yang-Mills theory*, *JHEP* **06** (2010) 097 [arXiv:0804.2907] [INSPIRE].

[14] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *$\mathcal{N}=4$ superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [arXiv:0805.3662] [INSPIRE].

[15] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$\mathcal{N}=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [INSPIRE].

[16] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *$\mathcal{N}=5,6$ superconformal Chern-Simons theories and M2-branes on orbifolds*, *JHEP* **09** (2008) 002 [arXiv:0806.4977] [INSPIRE].

[17] O. Aharony, O. Bergman and D.L. Jafferis, *Fractional M2-branes*, *JHEP* **11** (2008) 043 [arXiv:0807.4924] [INSPIRE].

[18] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108] [INSPIRE].

[19] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955] [INSPIRE].

[20] A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys.* **B 811** (2009) 66 [arXiv:0709.1260] [INSPIRE].

[21] J.-H. Park, *Superconformal symmetry in three-dimensions*, *J. Math. Phys.* **41** (2000) 7129 [hep-th/9910199] [INSPIRE].

[22] G. Mack and A. Salam, *Finite component field representations of the conformal group*, *Annals Phys.* **53** (1969) 174 [INSPIRE].

[23] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [hep-th/9307010] [INSPIRE].

[24] G. Mack, *Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory*, *Commun. Math. Phys.* **53** (1977) 155 [INSPIRE].

[25] S. Ferrara, A.F. Grillo and R. Gatto, *Manifestly conformal-covariant expansion on the light cone*, *Phys. Rev.* **D 5** (1972) 3102 [INSPIRE].

[26] S. Ferrara, A.F. Grillo and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161 [INSPIRE].

[27] K. Koller, *The significance of conformal inversion in quantum field theory*, *Commun. Math. Phys.* **40** (1974) 15 [INSPIRE].

[28] I.T. Todorov, M.C. Mintchev and V.P. Petkova, *Conformal invariance in quantum field theory*, Scuola Normale Superiore, Pisa Italy (1978) [INSPIRE].

[29] E.S. Fradkin and M.Y. Palchik, *Recent developments in conformal invariant quantum field theory*, *Phys. Rept.* **44** (1978) 249 [INSPIRE].

[30] Y.S. Stanev, *Stress-energy tensor and* U(1) *current operator product expansions in conformal QFT*, *Bulg. J. Phys.* **15** (1988) 93 [INSPIRE].

[31] H. Osborn, $\mathcal{N} = 1$ *superconformal symmetry in four-dimensional quantum field theory*, *Annals Phys.* **272** (1999) 243 [hep-th/9808041] [INSPIRE].

[32] J.-H. Park, *Superconformal symmetry in six-dimensions and its reduction to four-dimensions*, *Nucl. Phys.* **B 539** (1999) 599 [hep-th/9807186] [INSPIRE].

[33] J.-H. Park, *Superconformal symmetry and correlation functions*, *Nucl. Phys.* **B 559** (1999) 455 [hep-th/9903230] [INSPIRE].

[34] S.M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, *Off-shell supergravity-matter couplings in three dimensions*, *JHEP* **03** (2011) 120 [arXiv:1101.4013] [INSPIRE].

[35] S.M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in* $\mathcal{N} = 2$ *superconformal theory*, *Class. Quant. Grav.* **17** (2000) 665 [hep-th/9907107] [INSPIRE].

[36] D. Butter, S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *Nonlinear* σ*-models with AdS supersymmetry in three dimensions*, *JHEP* **02** (2013) 121 [arXiv:1210.5906] [INSPIRE].

[37] D. Butter, S.M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, *Conformal supergravity in three dimensions: new off-shell formulation*, *JHEP* **09** (2013) 072 [arXiv:1305.3132] [INSPIRE].

[38] S.M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, $\mathcal{N} = 6$ *superconformal gravity in three dimensions from superspace*, *JHEP* **01** (2014) 121 [arXiv:1308.5552] [INSPIRE].

[39] M.F. Sohnius, *Supersymmetry and central charges*, *Nucl. Phys.* **B 138** (1978) 109 [INSPIRE].

[40] B.M. Zupnik, *Three-dimensional* $\mathcal{N} = 4$ *superconformal superfield theories*, *Theor. Math. Phys.* **162** (2010) 74 [arXiv:0905.1179] [INSPIRE].

[41] K.A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys. Lett.* **B 387** (1996) 513 [hep-th/9607207] [INSPIRE].

JHEP08(2015)125

[42] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200] [INSPIRE].

[43] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyper-Kähler metrics and supersymmetry*, *Commun. Math. Phys.* **108** (1987) 535 [INSPIRE].

[44] B.M. Zupnik and D.G. Pak, *Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities*, *Theor. Math. Phys.* **77** (1988) 1070 [*Teor. Mat. Fiz.* **77** (1988) 97] [INSPIRE].

[45] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained* $\mathcal{N} = 2$ *matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. Quant. Grav.* **1** (1984) 469 [*Erratum ibid.* **2** (1985) 127] [INSPIRE].

[46] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge U.K. (2001).

[47] A. Karlhede, U. Lindström and M. Roček, *Selfinteracting tensor multiplets in* $\mathcal{N} = 2$ *superspace*, *Phys. Lett.* **B 147** (1984) 297 [INSPIRE].

[48] U. Lindström and M. Roček, *New hyper-Kähler metrics and new supermultiplets*, *Commun. Math. Phys.* **115** (1988) 21 [INSPIRE].

[49] U. Lindström and M. Roček, $\mathcal{N} = 2$ *super Yang-Mills theory in projective superspace*, *Commun. Math. Phys.* **128** (1990) 191 [INSPIRE].

[50] S.M. Kuzenko, *Lectures on nonlinear* σ*-models in projective superspace*, *J. Phys.* **A 43** (2010) 443001 [arXiv:1004.0880] [INSPIRE].

[51] I.L. Buchbinder, E.A. Ivanov, O. Lechtenfeld, N.G. Pletnev, I.B. Samsonov and B.M. Zupnik, *Quantum* $\mathcal{N} = 3$, $D = 3$ *Chern-Simons matter theories in harmonic superspace*, *JHEP* **10** (2009) 075 [arXiv:0909.2970] [INSPIRE].

[52] I.L. Buchbinder, E.A. Ivanov, O. Lechtenfeld, N.G. Pletnev, I.B. Samsonov and B.M. Zupnik, *ABJM models in* $\mathcal{N} = 3$ *harmonic superspace*, *JHEP* **03** (2009) 096 [arXiv:0811.4774] [INSPIRE].

[53] D. Butter, S.M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, *Conformal supergravity in three dimensions: off-shell actions*, *JHEP* **10** (2013) 073 [arXiv:1306.1205] [INSPIRE].

[54] S.M. Kuzenko, *Projective superspace as a double punctured harmonic superspace*, *Int. J. Mod. Phys.* **A 14** (1999) 1737 [hep-th/9806147] [INSPIRE].

[55] A.A. Roslyi and A.S. Schwarz, *Supersymmetry in a space with auxiliary dimensions*, *Commun. Math. Phys.* **105** (1986) 645 [INSPIRE].

[56] A.A. Rosly, *Super Yang-Mills constraints as integrability conditions* (in Russian), in *Proceedings of the International Seminar on Group Theoretical Methods in Physics*, Zvenigorod U.S.S.R. (1982), vol. 1, M.A. Markov ed., Nauka, Moscow U.S.S.R. (1983), pg. 263, english translation in *Group Theoretical Methods in Physics*, vol. 3, M.A. Markov, V.I. Man'ko and A.E. Shabad eds., Harwood Academic Publishers, London U.K. (1987), pg. 587.

[57] A. Galperin, E.A. Ivanov, V. Ogievetsky and E. Sokatchev, *Harmonic supergraphs. Green functions*, *Class. Quant. Grav.* **2** (1985) 601 [INSPIRE].

[58] E. Sokatchev, 3*-point functions in* $\mathcal{N} = 4$ *Yang-Mills in harmonic superspace*, in *Supersymmetries and quantum symmetries*, J. Wess and E.A. Ivanov eds., *Lect. Notes Phys.* **524** (1999) 106 [INSPIRE].

JHEP08(2015)125