Supersymmetric gauge theories on five-manifolds

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ABSTRACT: We construct rigid supersymmetric gauge theories on Riemannian five-manifolds. We follow a holographic approach, realizing the manifold as the conformal boundary of a six-dimensional bulk supergravity solution. This leads to a systematic classification of five-dimensional supersymmetric backgrounds with gravity duals. We show that the background metric is furnished with a conformal Killing vector, which generates a transversely holomorphic foliation with a transverse Hermitian structure. Moreover, we prove that any such metric defines a supersymmetric background. Finally, we construct supersymmetric Lagrangians for gauge theories coupled to arbitrary matter on such backgrounds.

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Contents

1 Introduction

There has recently been considerable work on defining and studying supersymmetric gauge theories on curved backgrounds. The main reason for this interest is that these quantum field theories possess classes of observables that may be computed exactly using localization methods. Such non-perturbative results allow for quantitative tests of various conjectured dualities, and have also led to the discovery of new dualities. A primary example is the AdS/CFT correspondence, where exact strong-coupling field theory calculations may be compared to semi-classical gravity.

In this paper we focus on rigid supersymmetry in $d = 5$ dimensions, which is currently not as well-developed as its lower-dimensional cousins. Supersymmetric gauge theories were constructed and studied on the round S^5 in [1–4]. The product background $S^1 \times S^4$ studied in [5, 6] leads to the superconformal index. As in lower dimensions, the first constructions of non-conformally flat backgrounds were produced via various ad hoc methods. These include the squashed S^5 geometries of [7, 8], and the product backgrounds $S^3 \times \Sigma_2$ [9, 10] and $S^2 \times M_3$ [11–13]. In the latter two cases the spheres are round, while supersymmetry on the Riemann surface Σ_2 or three-manifold M_3 is achieved via a topological twist utilizing the $\mathrm{SU}(2)_R$ symmetry of the theory. These constructions have been used to successfully test AGT-type correspondences.

A systematic method for constructing rigid supersymmetric field theories on curved backgrounds, in any dimension d, was initiated in [14]. Here one first couples the field

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theory to off-shell supergravity, and then takes a decoupling limit in which the gravity multiplet becomes a non-dynamical background field. This approach was applied to five-dimensional Poincaré supergravity [15–17] in the series of papers [18–20].[1] Supersymmetry of the background requires a certain generalized Killing spinor equation to hold, whose related geometry was investigated in [18], together with an algebraic "dilatino" equation which was studied in [19]. The latter reference recasts these conditions into local geometric constraints on the five-manifold M_5. As in lower dimensions, one finds that the background is parametrized by various arbitrary functions/tensors. In particular (M_5, g) is equipped with a Killing vector field $\xi = \partial_\psi$, with dual one-form $S^2(\mathrm{d}\psi + \rho)$ and transverse four-dimensional metric $g^{(4)}$, where locally the function $S = \|\xi\|$ and tensors ρ and $g^{(4)}$ are ξ-invariant but otherwise freely specifiable.[2] The authors of [19] furthermore show that *locally* all deformations of the background fields lead to Q-exact deformations of the action, where Q is the supercharge. Despite this generality, these backgrounds apparently don't include the conformally flat $S^1 \times S^4$ geometry mentioned above [19]. We shall comment further on these issues later.

In [1] a twisted version of $\mathcal{N} = 1$ super-Yang-Mills theory is defined on *contact* five-manifolds (M_5, η). Here η is a contact one-form, meaning that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form. On a Sasaki-Einstein five-manifold [22] one can construct $\mathcal{N} = 1$ super-Yang-Mills coupled to matter [23]. This is essentially because the two Killing spinors on a Sasaki-Einstein manifold satisfy the same Killing spinor equations as those on the round sphere. For the special class of toric $(\mathrm{U}(1)^3$-invariant) Sasaki-Einstein manifolds of [24] the localized perturbative partition function has been computed in [25–27], with the last reference also giving a conjectured formula for the full partition function. The authors of [28] furthermore show that one can define a twisted version of $\mathcal{N} = 2$ super-Yang-Mills theory on any *K-contact* five-manifold. We also note that K-contact geometry arises as a special case in [18].

In the present paper we instead take a holographic approach, similar to [29] in lower dimensions, to construct rigid supersymmetry in five dimensions. Here M_5 is realized as the conformal boundary of a six-dimensional bulk solution of Romans $F(4)$ gauged supergravity [30]. Some of the groundwork for this was laid in [31, 32], where supergravity duals of the squashed five-sphere backgrounds of [7, 8] were constructed (see also [33, 34] for holographic duals to the supersymmetric Rényi entropy in five dimensions). We begin with a general supersymmetric asymptotically locally AdS solution to the Romans theory, and extract the conditions this imposes on the five-dimensional conformal boundary. Although the resulting spinor equations are quite complicated, we will show they are completely equivalent to a very simple geometric structure. We find that M_5 is equipped with a conformal Killing vector $\xi = \partial_\psi$ which generates a *transversely holomorphic foliation*. This is compatible with an almost contact form $\eta = \mathrm{d}\psi + \rho$, where up to global constraints that we describe the norm $S = \|\xi\|$ and ρ are arbitrary, and the transverse metric $g^{(4)}$ is *Hermitian*. The only other remaining freedom is an arbitrary function α (such that $S\alpha$ is ξ-invariant), which together with the metric determines all the remaining background data.

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[1]See [21] for the construction of supersymmetric Lorentzian backgrounds within the superspace formulation of five-dimensional conformal supergravity.

[2]There are also additional freely specifiable fields, which determine the rest of the background.

This structure is similar to the rigid limit of Poincaré supergravity described above, but with the addition of an integrable transverse complex structure and Hermitian metric. In fact it is a natural hybrid of the "real" three-dimensional rigid supersymmetric geometry studied in [35, 36] and the four-dimensional supersymmetric geometry of [37, 38] (where the four-manifold is complex with a compatible Hermitian metric).

The outline of the rest of the paper is as follows. In section 2 we summarize the form of supersymmetric asymptotically locally AdS solutions to Romans supergravity, in particular extracting the Killing spinor equations on the conformal boundary M_5. These are then used as a starting point for a purely five-dimensional analysis in section 3. We show that the spinor equations are completely equivalent to a simple geometric structure on M_5, and present a number of subclasses and examples, including many of the examples referred to above. In section 4 we construct $\mathcal{N} = 1$ supersymmetric gauge theories formed of vector and hypermultiplets on this background geometry. Our conclusions are presented in section 5.

2 Rigid supersymmetry from holography

The bosonic fields of the six-dimensional Romans supergravity theory [30] consist of the metric, a scalar field X, a two-form potential B, together with an $\mathrm{SO}(3)_R \sim \mathrm{SU}(2)_R$ R-symmetry gauge field A^i with field strength $F^i = \mathrm{d}A^i - \frac{1}{2}\varepsilon_{ijk}A^j \wedge A^k$, where $i = 1, 2, 3$. Here we are working in a gauge in which the Stueckelberg one-form is zero, and we set the gauge coupling constant to 1. The Euclidean signature equations of motion for this theory may be found in [32], although we will not require their explicit form here.

A solution is supersymmetric provided there exists a non-trivial $\mathrm{SU}(2)_R$ doublet of Dirac spinors ϵ_I, $I = 1, 2$, satisfying the following Killing spinor and dilatino equations

$$D_M\epsilon_I = \frac{\mathrm{i}}{4\sqrt{2}}(X + \tfrac{1}{3}X^{-3})\Gamma_M\Gamma_7\epsilon_I - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{NP}(\Gamma_M{}^{NP} - 6\delta_M{}^N\Gamma^P)\epsilon_I \tag{2.1}$$
$$-\frac{1}{48}X^2H_{NPQ}\Gamma^{NPQ}\Gamma_M\Gamma_7\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F_{NP}^i(\Gamma_M{}^{NP} - 6\delta_M{}^N\Gamma^P)\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,,$$

$$0 = -\mathrm{i}X^{-1}\partial_M X\Gamma^M\epsilon_I + \frac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2H_{MNP}\Gamma^{MNP}\Gamma_7\epsilon_I$$
$$-\frac{1}{8\sqrt{2}}X^{-1}F_{MN}\Gamma^{MN}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F_{MN}^i\Gamma^{MN}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,. \tag{2.2}$$

Here Γ_M are taken to be Hermitian and generate the Clifford algebra $\mathrm{Cliff}(6, 0)$ in an orthonormal frame, $M = 0, \ldots, 5$. We have defined the chirality operator $\Gamma_7 = \mathrm{i}\Gamma_{012345}$, which satisfies $(\Gamma_7)^2 = 1$. The covariant derivative acting on the spinor is $D_M\epsilon_I = \hat{\nabla}_M\epsilon_I + \frac{\mathrm{i}}{2}A_M^i(\sigma_i)_I{}^J\epsilon_J$, where $\hat{\nabla}_M = \partial_M + \frac{1}{4}\Omega_M{}^{NP}\Gamma_{NP}$ denotes the Levi-Civita spin connection while σ_i, $i = 1, 2, 3$, are the Pauli matrices.

Given a supersymmetric asymptotically locally AdS solution we may introduce a radial coordinate r, so that the conformal boundary is at $r = \infty$ and the metric admits an expansion of the form

$$\mathrm{d}s^2 = \frac{9}{2}\frac{\mathrm{d}r^2}{r^2} + r^2\left[g_{\mu\nu} + \frac{1}{r^2}g_{\mu\nu}^{(2)} + \cdots\right]\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{2.3}$$

Here x^μ, $\mu = 1, \ldots, 5$, are coordinates on the conformal boundary, which has metric $g = (g_{\mu\nu})$. Notice that the particular form of the metric in (2.3) is not reparametrization invariant under $r \to \Lambda r$, where $\Lambda = \Lambda(x^\mu)$. However, the correction terms under such a transformation are subleading in the $1/r$ expansion. This will play an important role in the next section.

For simplicity we shall mainly consider Abelian solutions in which $A^1 = A^2 = 0$, and $A^3 \equiv A$, with field strength $F \equiv \mathrm{d}A$. Similarly to the metric (2.3) we then write the following general expansions for the remaining bosonic fields

$$
\begin{aligned}
X &= 1 + \frac{1}{r^2} X_2 + \cdots , \\
B &= rb - \frac{1}{r^2} \mathrm{d}r \wedge A^{(0)} + \cdots \\
A &= a + \cdots ,
\end{aligned}
\tag{2.4}
$$

where we define $f \equiv \mathrm{d}a$. Some of the terms a priori present in these expansions are set to zero by the equations of motion; for example, the $O(1/r)$ term in the expansion of X [32]. The Killing spinors similarly admit an expansion of the form

$$
\epsilon_I = \sqrt{r} \begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}} \begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + O(r^{-3/2}) .
\tag{2.5}
$$

Here we have used the orthonormal frame

$$
E^0 = \frac{3}{\sqrt{2}} \frac{\mathrm{d}r}{r} , \qquad E^\mu = r e^\mu + \cdots
\tag{2.6}
$$

for the metric (2.3). Furthermore, the spin connection expands as

$$
\Omega_\mu{}^{0\nu} = -\frac{\sqrt{2}}{3} \delta_\mu{}^\nu + \frac{1}{r^2} \omega_\mu{}^\nu + \cdots .
\tag{2.7}
$$

Also as in [32] we consider a "real" class of solutions for which ϵ_I satisfies the symplectic Majorana condition $\varepsilon_I{}^J \epsilon_J = \mathcal{C}_6 \epsilon_I^* \equiv \epsilon_I^c$, where $\mathcal{C}_6$ denotes the charge conjugation matrix, satisfying $\Gamma_M^T = \mathcal{C}_6^{-1} \Gamma_M \mathcal{C}_6$. The bosonic fields are all taken to be real, with the exception of the B-field which is purely imaginary. With these reality properties, one can show that the Killing spinor equation (2.1) and dilatino equation (2.2) for ϵ_2 are simply the charge conjugates of the corresponding equations for ϵ_1. In this way we effectively reduce to a single Killing spinor $\epsilon \equiv \epsilon_1$, with $\mathrm{SU}(2)_R$ doublet (ϵ, ϵ^c). We then note the following large r expansions of bilinears:[3]

$$
\begin{aligned}
\epsilon^\dagger \Gamma_7 \epsilon &= 4\alpha S + \cdots , \\
\mathrm{i}\epsilon^\dagger \Gamma_7 \Gamma_{(1)} \epsilon &= 2S r K_2 - 3\sqrt{2} \mathrm{d}r + \cdots .
\end{aligned}
\tag{2.8}
$$

Here we have defined $\Gamma_{(1)} \equiv \Gamma_M E^M$ and

$$
S \equiv \chi^\dagger \chi .
\tag{2.9}
$$

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[3]Here we take the spinors to be Grassmann even.

We also note that the bilinear $\epsilon^\dagger \Gamma_{(1)}\epsilon$ is a Killing one-form in the bulk [32]. This will hence restrict to a conformal Killing vector on the boundary at $r = \infty$.

Substituting the expansions (2.5) into the bulk Killing spinor equation (2.1), at the first two orders we obtain

$$\left(\nabla_\mu + \frac{\mathrm{i}}{2}a_\mu\right)\chi = -\frac{\sqrt{2}}{3}\mathrm{i}\gamma_\mu\varphi - \frac{\mathrm{i}}{12\sqrt{2}}b_{\nu\sigma}\gamma_\mu{}^{\nu\sigma}\chi + \frac{\mathrm{i}}{3\sqrt{2}}b_{\mu\nu}\gamma^\nu\chi\,, \tag{2.10}$$

$$\left(\nabla_\mu + \frac{\mathrm{i}}{2}a_\mu\right)\varphi = -\frac{\mathrm{i}}{6\sqrt{2}}b_{\mu\nu}\gamma^\nu\varphi + \frac{1}{16\sqrt{2}}f_{\nu\sigma}\gamma_\mu{}^{\nu\sigma}\chi - \frac{3}{8\sqrt{2}}f_{\mu\nu}\gamma^\nu\chi \tag{2.11}$$

$$+ \frac{1}{48}(\mathrm{d}b)_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\gamma_\mu\chi - \frac{1}{36}A^{(0)}_\nu\gamma_\mu{}^\nu\chi + \frac{1}{12}A^{(0)}_\mu\chi + \frac{\mathrm{i}}{2}\omega_\mu{}^\nu\gamma_\nu\chi\,.$$

Here γ_μ generate the Clifford algebra $\mathrm{Cliff}(5,0)$ in an orthonormal frame, while ∇ denotes the Levi-Civita spin connection for the boundary metric g. Similarly, the bulk dilatino equation (2.2) implies

$$-\frac{1}{6\sqrt{2}}b_{\mu\nu}\gamma^{\mu\nu}\varphi - \frac{\sqrt{2}}{3}X_2\chi + \frac{\mathrm{i}}{8\sqrt{2}}f_{\mu\nu}\gamma^{\mu\nu}\chi + \frac{\mathrm{i}}{24}(\mathrm{d}b)_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\chi - \frac{\mathrm{i}}{18}A^{(0)}_\mu\gamma^\mu\chi = 0\,. \tag{2.12}$$

As explained in [32], equation (2.10) may be rewritten in the form of a charged conformal Killing spinor equation, with additional b-field couplings. Setting $b = 0$ one obtains the standard charged conformal Killing spinor equation, whose solutions (twistor spinors) have been studied in the holographic context for three-manifolds and four-manifolds in [29, 38–41]. On the other hand, previous work on rigid supersymmetry in five dimensions [18–20] has used Killing spinor equations of a different form, without the coupling to φ in (2.10). We may make closer contact with this work by noting that supersymmetry in the bulk also implies the algebraic relation

$$\varphi = -\alpha\chi - \frac{\mathrm{i}}{2}(K_2)_\nu\gamma^\nu\chi\,. \tag{2.13}$$

This follows from the bilinear expansions (2.8).

In the remainder of the paper we shall take equations (2.10), (2.11), (2.12), and (2.13) as our starting point for a purely five-dimensional analysis.

3 Background geometry

In this section we begin with a Riemannian five-manifold (M_5, g), on which we'd like to define rigid supersymmetric gauge theories. The gauge/gravity correspondence implies this should be possible, provided the spinor equations derived in the previous section hold.

Let us summarize the background data. In addition to the real metric g, we have two generalized Killing spinors χ, φ. Globally these are spin^c spinors, being sections of the spin bundle of M_5 tensored with $L^{-1/2}$, $\chi, \varphi \in \Gamma[\mathrm{Spin}(M_5) \otimes L^{-1/2}]$, where L is the complex line bundle for which the real gauge field a is a connection. This Abelian gauge field is a background field for $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$, with (χ, χ^c), (φ, φ^c) forming $\mathrm{SU}(2)_R$ doublets, where $\chi^c \equiv \mathcal{C}\chi^*$ with $\mathcal{C}$ the five-dimensional charge conjugation matrix. The spinors χ, φ then satisfy the coupled Killing spinor equations (2.10), (2.11), where the background b-field is

taken to be a purely imaginary two-form, $A^{(0)}$ is a purely imaginary one-form, while $\omega_{\mu\nu} = g_{\nu\sigma}\omega_\mu{}^\sigma$ is real and symmetric. Furthermore, χ and φ are related algebraically by (2.13), which introduces the additional background fields α and K_2, which are respectively a real function and real one-form. Finally we have the dilatino equation (2.12), which introduces the real background function X_2.

In the remainder of this section we shall analyse the geometric constraints that these equations impose on (M_5, g). Although the background data and equations (2.10)–(2.13) appear a priori complicated, in fact we shall see that the geometry they are equivalent to is very simple.

3.1 Differential constraints

In the analysis that follows it is convenient to assume that the spin^c spinor χ is nowhere zero. More generally χ could vanish along some locus $Z \subset M_5$, and the local geometry we shall derive below is valid on $M_5 \backslash Z$. If Z is non-empty one would need to impose suitable boundary conditions, although we shall not consider this further in this paper. A nowhere zero spin^c spinor equips (M_5, g) with a local SU(2) structure. Specifically, we may define the bilinears

$$S \equiv \chi^\dagger \chi \,, \qquad\qquad K_1 \equiv \frac{1}{S}\chi^\dagger \gamma_{(1)}\chi \,,$$
$$J \equiv -\frac{\mathrm{i}}{S}\chi^\dagger \gamma_{(2)}\chi \,, \qquad \Omega \equiv -\frac{1}{S}(\chi^c)^\dagger \gamma_{(2)}\chi \,. \tag{3.1}$$

Here we have introduced the notation $\gamma_{(n)} \equiv \frac{1}{n!}\gamma_{\mu_1\cdots\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$, where x^μ, $\mu = 1,\ldots,5$, are local coordinates on M_5. Since χ is nowhere zero the scalar function S is strictly positive, and it makes sense to normalize the bilinears as in (3.1). We note that K_1 is a real unit length one-form, while J is a real two-form with square length $\|J\|^2 = 2$. Here the square norm of a p-form ϕ is defined via $\|\phi\|^2 \mathrm{vol}_5 = \phi \wedge *\phi$, where $*$ denotes the Hodge duality operator on (M_5, g) and vol_5 denotes the Riemannian volume form. The complex bilinear Ω is globally a two-form valued in the line bundle L^{-1}.

That χ, or equivalently the bilinears (3.1), defines a local SU(2) structure follows from some simple group theory. The spin group is $\mathrm{Spin}(5) \cong \mathrm{Sp}(2) \subset \mathrm{U}(4)$, with the latter acting in the fundamental representation on the spinor space $\mathbb{C}^4$. The stabilizer of a non-zero spinor is then $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. When M_5 is spin and L is trivial, so that $\chi \in \Gamma[\mathrm{Spin}(M_5)]$, this defines a global SU(2) structure. However, more generally we require only that M_5 is spin^c, and in this case the *global* stabilizer group is enlarged to U(2): the additional U(1) factor rotates the spinor by a phase, which may be undone by a U(1) gauge transformation. To see this in more detail we introduce a local orthonormal frame e^a, $a = 1,\ldots,5$, so that

$$K_1 = e^5 \,, \qquad J = e^1 \wedge e^2 + e^3 \wedge e^4 \,, \qquad \Omega = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \,, \tag{3.2}$$

where the metric is $g = \sum_{a=1}^5 (e^a)^2$. The $\mathrm{U}(2) = \mathrm{SU}(2) \times_{\mathbb{Z}_2} \mathrm{U}(1)$ structure group acts in the obvious way on the $\mathbb{C}^2$ spanned by $e^1 + \mathrm{i}e^2, e^3 + \mathrm{i}e^4$. This leaves K_1, J and the metric g invariant, but rotates Ω by the determinant of the U(2) transformation. In order for this to be undone by a gauge transformation, this identifies the line bundle as $L = \Lambda^{2,0}$. The latter is the space of Hodge type $(2,0)$-forms for the four-dimensional vector bundle spanned by

e^1, e^2, e^3, e^4, and with almost complex structure I for which $e^1 + \mathrm{i}e^2$ and $e^3 + \mathrm{i}e^4$ are $(1,0)$-forms. Thus our rigid supersymmetric geometry will in general be equipped with a global U(2) structure on M_5 (or more precisely on $M_5 \setminus Z$).

The one-form $SK_1 = \chi^\dagger \gamma_{(1)} \chi$ arises simply from the restriction of the bulk Killing one-form $\epsilon^\dagger \Gamma_{(1)} \epsilon$ to the conformal boundary, and thus defines a conformal Killing one-form on (M_5, g). This is easily confirmed from the Killing spinor equation (2.10) for χ, which implies

$$\nabla_{(\mu}(SK_1)_{\nu)} = \mathcal{L}_\xi(\log S)g_{\mu\nu}\,, \tag{3.3}$$

where we have introduced the dual vector field ξ, defined by $g(\xi, \cdot) = SK_1$, and $\mathcal{L}$ denotes the Lie derivative.

One finds that the spinor equations (2.10)–(2.13) imply the following differential constraints:

$$\mathrm{d}S = -\frac{\sqrt{2}}{3}\left(SK_2 + \mathrm{i}i_\xi b\right)\,, \qquad \mathrm{d}(S\alpha) = -\frac{1}{2\sqrt{2}}i_\xi \mathrm{d}a\,, \tag{3.4}$$

$$\mathrm{d}(SK_1) = \frac{2\sqrt{2}}{3}\left[2\alpha SJ + SK_1 \wedge K_2 + \mathrm{i}Sb - \frac{\mathrm{i}}{2}i_\xi(*b)\right]\,, \tag{3.5}$$

$$\mathrm{d}(SK_2) = \mathrm{i}i_\xi \mathrm{d}b - \mathrm{i}\mathcal{L}_\xi(\log S)b\,, \tag{3.6}$$

$$\mathrm{d}(SJ) = -\sqrt{2}K_2 \wedge (SJ)\,, \tag{3.7}$$

$$\mathrm{d}(S\Omega) = -\mathrm{i}\left(a - 2\sqrt{2}\alpha K_1 - \mathrm{i}\sqrt{2}K_2\right) \wedge (S\Omega)\,. \tag{3.8}$$

Here $(i_V\phi)_{a_1\cdots a_{p-1}} = V^b\phi_{ba_1\cdots a_{p-1}}$ defines the interior contraction of a vector V into a p-form ϕ. Notice that the background data X_2, $A^{(0)}$ and $\omega_{\mu\nu}$ in (2.10)–(2.13) does not enter equations (3.4)–(3.8): they simply drop out (one only needs to use the reality properties we specified, together with the fact that $\omega_{\mu\nu} = \omega_{\nu\mu}$ is symmetric).

It is straightforward to verify that (3.4)–(3.8) are invariant under the Weyl transformations

$$\begin{aligned}
\alpha &\to \Lambda^{-1}\alpha\,, & a &\to a\,, & K_2 &\to K_2 - \frac{3}{\sqrt{2}}\mathrm{d}\log\Lambda\,, \\
S &\to \Lambda S\,, & K_1 &\to \Lambda K_1\,, & b &\to \Lambda b\,, \\
g &\to \Lambda^2 g\,, & J &\to \Lambda^2 J\,, & \Omega &\to \Lambda^2 \Omega\,.
\end{aligned} \tag{3.9}$$

This symmetry is of course inherited from invariance under the change of radial variable $r \to \Lambda r$ in the bulk. If S is nowhere zero notice that one might use this symmetry to set $S \equiv 1$.

Using equations (3.4)–(3.8) one can show that the conformal Killing vector ξ preserves all of the background geometric structure, provided one rescales the fields by appropriate powers of S according to their Weyl weights in (3.9). For instance, contracting ξ into the second equation in (3.4) shows that $\mathcal{L}_\xi(S\alpha) = 0$. On the other hand, taking the exterior derivative of the same equation one finds $\mathcal{L}_\xi \mathrm{d}a = 0$. One can hence locally choose a gauge in which a is invariant under $\mathcal{L}_\xi$, so that the second equation in (3.4) is solved by

$$S\alpha = \frac{1}{2\sqrt{2}}i_\xi a\,. \tag{3.10}$$

In a similar way, one can show that also $S^{-1}b$ and $S^{-2}J$ are invariant under $\mathcal{L}_\xi$, while $S^{-2}\Omega$ is invariant under $\mathcal{L}_\xi$ in the gauge choice for which (3.10) holds. Notice that the first equation in (3.4) implies that $i_\xi K_2 = -\frac{3}{\sqrt{2}}\mathcal{L}_\xi(\log S)$.

Without loss of generality it is convenient to henceforth impose $\mathcal{L}_\xi S = 0$.[4] In terms of the bulk expansion in section 2 this means choosing the radial coordinate r to be independent of the bulk Killing vector. This is a natural choice, which in turn implies that $\mathcal{L}_\xi S = 0$ and SK_1 is Killing, and we shall make this convenient (partial) conformal gauge choice in the following. We may then introduce a local coordinate ψ so that

$$\xi = \partial_\psi \,. \tag{3.11}$$

The condition $\mathcal{L}_\xi S = 0$ is then equivalent to S being independent of ψ.

3.2 Geometric structure

The Killing vector ξ has norm S, and the dual one-form K_1 may be written locally as

$$K_1 = S(\mathrm{d}\psi + \rho) \;\equiv\; S\eta \,, \tag{3.12}$$

where $i_\xi\rho = 0$. Notice that η has Weyl weight zero and norm $1/S$. The local frame e_1, e_2, e_3, e_4 provide a basis for $D = \ker\eta$, and D inherits an almost complex structure from J. One then defines an endomorphism Φ of the tangent bundle of M_5 by

$$\Phi\mid_D = I\,, \qquad \Phi\mid_\xi = 0\,, \tag{3.13}$$

where I is the almost complex structure. One easily verifies that $\Phi^2 = -1 + \xi \otimes \eta$, which is a defining relation of an *almost contact structure*. Moreover, the five-dimensional metric takes the form

$$\mathrm{d}s^2_{M_5} = S^2\eta^2 + \mathrm{d}s^2_4 \,, \tag{3.14}$$

where $\mathrm{d}s^2_4$ is Hermitian with respect to I. Although ξ is Killing, this structure is in general *not* a K-contact structure, which is a stronger condition. In particular the latter requires [42] that $\mathrm{d}\eta$ is the fundamental $(1,1)$-form J associated to the transverse almost complex structure (which in general is not the case here), which in turn implies that η is a contact form, i.e. that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form (which in general is also not the case here). Notice that since ξ is nowhere zero, its orbits define a foliation of M_5.

Let us now turn to the differential constraints (3.4)–(3.8). The two equations (3.4) allow us to write

$$b = \mathrm{i}S\eta \wedge \left(K_2 + \frac{3}{\sqrt{2}}\mathrm{d}\log S\right) + b_\perp\,, \qquad a \;=\; 2\sqrt{2}S\alpha\eta + a_\perp\,, \tag{3.15}$$

where $b_\perp$ and $a_\perp$ are *basic* forms for the foliation defined by ξ; that is, they are invariant under, and have zero interior contraction with, ξ. Recall that in writing the gauge field in the form in (3.15) we have made a (partial) gauge choice, as in (3.10). This leaves a

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[4]An exception being the $S^1 \times S^4$ geometry discussed in section 3.3.

residual gauge freedom $a_\perp \to a_\perp + \mathrm{d}\lambda$, where λ is a basic (ξ-invariant) function. The equation (3.6) is simply equivalent to b being invariant under ξ.

The differential constraint (3.5) reduces to

$$\mathrm{d}\rho = \frac{\sqrt{2}}{3S}\left(-\mathrm{i}*_4 b_\perp + 2\mathrm{i}b_\perp + 4\alpha J\right)\ . \tag{3.16}$$

Here $*_4$ is the Hodge dual with respect to the transverse four-dimensional metric $\mathrm{d}s_4^2$, with volume form $e^1 \wedge e^2 \wedge e^3 \wedge e^4$. It is then convenient to introduce

$$b_\perp = b^+ + b^-\ , \tag{3.17}$$

decomposing into the transversely self-dual and anti-self-dual parts. Equation (3.16) is then equivalent to

$$b^+ = \mathrm{i}\left(4\alpha J - \frac{3}{\sqrt{2}}S\mathrm{d}\rho^+\right)\,, \qquad b^- = -\frac{\mathrm{i}}{\sqrt{2}}S\mathrm{d}\rho^-\ . \tag{3.18}$$

The constraint (3.7) simply identifies

$$\theta \equiv J \lrcorner \mathrm{d}J = -\sqrt{2}K_2 - \mathrm{d}\log S\,, \tag{3.19}$$

with the *Lee form* θ of the transverse four-dimensional Hermitian structure. That is, every four-dimensional Hermitian structure with fundamental two-form J satisfies $\mathrm{d}J = \theta \wedge J$. Finally, the differential constraint (3.8) now reads

$$\mathrm{d}\Omega = (\theta - \mathrm{i}a_\perp) \wedge \Omega\ . \tag{3.20}$$

This implies that the almost complex structure I is *integrable*, thus defining a *transversely holomorphic foliation* of M_5. We may introduce local coordinates ψ, z_1, z_2 adapted to the foliation, where the transition functions between the z_1, z_2 coordinates are holomorphic.

Notice that we may rewrite (3.20) as

$$\mathrm{d}\Omega = -\mathrm{i}a_{\mathrm{Chern}} \wedge \Omega\,, \tag{3.21}$$

where we have defined

$$a_{\mathrm{Chern}} \equiv a_\perp - I(\theta)\,, \tag{3.22}$$

and $I(\theta) \equiv -i_{\theta^\#}J$, where $\theta^\#$ is the vector field dual to θ. To obtain an explicit expression for the Chern connection a_{Chern}, we begin by noting that $\Omega \wedge \bar{\Omega} = 2J \wedge J$. Using local coordinates z^α, $\alpha = 1, 2$, for the transverse space we may write

$$\Omega = f\,\mathrm{d}z^1 \wedge \mathrm{d}z^2\,, \qquad J = \frac{\mathrm{i}}{2}g^{(4)}_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}}\,, \tag{3.23}$$

which implies that $|f| = \sqrt{\det g^{(4)}}$. Notice that globally f is a section of L^{-1}, where $L \cong \Lambda^{2,0} \equiv \mathcal{K}$ is the canonical bundle. Writing $f = |f|\mathrm{e}^{\mathrm{i}\phi}$ we then have

$$\mathrm{d}\Omega = \mathrm{d}\log f \wedge \Omega = \mathrm{i}\left(\frac{1}{2}\mathrm{d}^c \log \det g^{(4)} + \mathrm{d}\phi\right) \wedge \Omega\,, \tag{3.24}$$

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where $\mathrm{d}^c \equiv I \circ \mathrm{d}$. We thus recognize (up to gauge)

$$a_{\mathrm{Chern}} = -\frac{1}{2}\mathrm{d}^c \log \det g^{(4)} \tag{3.25}$$

as the Chern connection on the canonical bundle.

The geometric content of the differential constraints (3.4)–(3.8) may hence be summarized as follows. M_5 is equipped with a transversely Hermitian structure, so that the metric takes the form

$$\mathrm{d}s_{M_5}^2 = S^2(\mathrm{d}\psi + \rho)^2 + \mathrm{d}s_4^2 \ . \tag{3.26}$$

Here the Killing vector is $\xi = \partial_\psi$, which generates a transversely holomorphic foliation. The almost contact form is $\eta = (\mathrm{d}\psi + \rho)$, and $\mathrm{d}s_4^2$ is a transverse Hermitian metric. One is also free to specify the functions α and S. Given this data, the remaining background fields a and b that enter (3.4)–(3.8) are determined via

$$\begin{aligned}
a &= 2\sqrt{2}S\alpha\eta + a_{\mathrm{Chern}} + I(\theta)\,, \\
b &= -\frac{\mathrm{i}}{\sqrt{2}}S\eta \wedge (\theta - 2\,\mathrm{d}\log S) + 4\mathrm{i}\alpha J - \frac{\mathrm{i}}{\sqrt{2}}S(3\mathrm{d}\rho^+ + \mathrm{d}\rho^-) \ .
\end{aligned} \tag{3.27}$$

In particular the choice of a transverse Hermitian metric $g^{(4)}$ fixes the two-form J, and hence the Lee form θ, while the Hodge type $(2,0)$-form Ω and Chern connection a_{Chern} in (3.25) are also determined up to gauge. Notice that the terms $S\alpha\eta$ and $I(\theta)$ entering the formula for a in (3.27) are both global one-forms on M_5, implying that globally a is a connection on $L = \Lambda^{2,0}$.

We shall furthermore show in section 3.4 that *any* choice of transversely Hermitian structure on M_5 of the above form gives a supersymmetric background. In particular the remaining background fields X_2, $A^{(0)}$, and $\omega_{\mu\nu}$ appearing in the spinor equations (2.10)–(2.13) are also determined by the above geometric data.

3.3 Examples

In this section we shall present some explicit examples of the above construction. These include all explicit examples appearing in the literature (within the Abelian truncation on which we are mostly focusing), including examples with six-dimensional gravity duals, plus large families of new solutions.

General families. We begin by noting some special families of backgrounds:

- Setting $\rho = 0$ and $S \equiv 1$ gives a product metric $M_5 = \mathbb{R} \times M_4$ or $M_5 = S^1 \times M_4$, where M_4 is any Hermitian four-manifold. Notice this four-manifold geometry is the same as the rigid supersymmetric geometry one finds in four dimensions [29, 37]. The first reference here follows a similar holographic approach to the present paper, while the second takes a rigid limit of "new minimal" supergravity in four dimensions.

- If $\mathrm{d}\theta = 0$ then the transverse Hermitian metric is locally conformally Kähler.

    - If furthermore $\theta = 0$ then the transverse four-metric is Kähler.

- If $\theta = 0$ and $\mathrm{d}\rho$ is a positive constant multiple of J then the five-metric is locally conformally Sasakian. Supersymmetric gauge theories on Sasaki-Einstein manifolds, for which furthermore $S \equiv 1$ and g is a positively curved Einstein metric, were defined in [23], and further studied in [25–28].

- We may take any circle bundle over a product of Riemann surfaces $S^1 \hookrightarrow \Sigma_1 \times \Sigma_2$. The Hermitian metric may be taken to be simply a product of two metrics on the Riemann surfaces, while ρ is the connection one-form for the fibration. One can generalize this further by allowing S^1 orbibundles over a product of orbifold Riemann surfaces.

  - If we only fibre over Σ_1, this leads to direct product $M_3 \times \Sigma_2$ solutions, where M_3 is a Seifert fibred three-manifold. Notice this three-manifold geometry is the same as the rigid supersymmetric geometry in three dimensions [35]. Maximally supersymmetric Yang-Mills theory has been studied on similar backgrounds in [9–13], including the direct products $S^3 \times \Sigma_2$ and $M_3 \times S^2$. Here the spheres are equipped with round metrics and the associated canonical spinors, while the spinors on Σ_2 and M_3 are constructed by topologically twisting with the $\mathrm{SU}(2)_R$ symmetry.

- Finally, if $\mathrm{d}\rho$ has Hodge type $(1,1)$ the transversely holomorphic foliation admits a complexification [43], i.e. adding a radial direction to ξ we naturally have a complex six-manifold M_6, with a transversely holomorphic foliation. Notice that Sasakian geometry and the direct product $S^1 \times M_4$ are special cases. When the orbits of ξ all close, M_5 fibres over a Hermitian four-orbifold M_4, and the associated U(1) orbibundle is the unit circle in a Hermitian holomorphic line orbibundle over M_4. The corresponding complex M_6 is then simply the total space of the associated $\mathbb{C}^*$ bundle over M_4.

Squashed Sasaki-Einstein. We have already noted that a Sasakian five-manifold is a particular case of a supersymmetric background. Recall that Sasakian metrics take the form

$$\mathrm{d}s_5^2 = \eta^2 + \mathrm{d}s_4^2 \,, \tag{3.28}$$

where η defines a contact structure on M_5, with Reeb Killing vector field ξ, and $\mathrm{d}s_4^2$ is a transverse Kähler metric. Moreover $\mathrm{d}\eta = \mathrm{d}\rho = 2J$. If the transverse Kähler metric $g^{(4)}$ is Einstein, then the metric (3.28) is said to be a *squashed* Sasaki-Einstein metric.[5] For a given choice of transverse Kähler-Einstein metric, we obtain a two-parameter family of backgrounds, parametrized by the constants c_1, c_2:

$$S \equiv 1 \,, \qquad \alpha = c_1 \,, \qquad K_2 = -\frac{1}{\sqrt{2}}\theta \equiv 0 \,,$$
$$a = c_2 \eta \,, \qquad b = \mathrm{i}(4c_1 - 3\sqrt{2})J \,. \tag{3.29}$$

The Kähler-Einstein metric $g^{(4)}$ satisfies the Einstein equation $\mathrm{Ric}^{(4)} = 2(2\sqrt{2}c_1 - c_2)g^{(4)}$. Notice that we have presented the solution (3.29) in a different gauge choice to (3.10).

---

[5]In the mathematical literature [42] these are called η-Sasaki-Einstein metrics.

We may impose the latter gauge choice by simply transforming $a \to a + (2\sqrt{2}c_1 - c_2)\mathrm{d}\psi$, although the form of a in (3.29) makes it clear that we may take a to be a global one-form on M_5 for this particular class of solutions.

When g_4 is taken to be the standard metric on $\mathbb{CP}^2$, the above geometry is a squashed five-sphere. This corresponds to the conformal boundary of the 1/4 BPS bulk Romans supergravity solutions constructed in [32].

Black hole boundary. In this section we consider the conformal boundary of the 1/2 BPS topological black hole solutions constructed in [33]. We begin with the following product metric on $S^1 \times \mathbb{H}^4$, where $\mathbb{H}^4$ is hyperbolic four-space:

$$\mathrm{d}s_5^2 = \mathrm{d}\tau^2 + \frac{1}{q^2+1}\mathrm{d}q^2 + q^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi_1^2 + \cos^2\vartheta\mathrm{d}\varphi_2^2)\,. \tag{3.30}$$

Here τ is a periodic coordinate on S^1, q is a radial coordinate with $q \in [0, \infty)$, $\vartheta \in [0, \frac{\pi}{2}]$ while φ_1, φ_2 have period 2π. The metric in brackets is simply the round metric on a unit radius S^3. For this solution b vanishes identically, while a is gauge-equivalent to zero. The Killing spinors for this background [33] in general depend on four integration constants (being 1/2 BPS), but for simplicity here we present only the "toric" solution discussed in [33]. The remaining fields are then

$$S = \sqrt{q^2+1}\,, \qquad \alpha = -\frac{3}{2\sqrt{2}\sqrt{q^2+1}}\,,$$
$$K_2 = -\frac{3}{\sqrt{2}}\frac{q}{q^2+1}\mathrm{d}q = -\frac{3}{2\sqrt{2}}\mathrm{d}\log(q^2+1)\,, \tag{3.31}$$

while in a gauge[6] in which $a = 0$ the U(2) structure is given by

$$K_1 = \frac{1}{\sqrt{q^2+1}}\left[\mathrm{d}\tau + q^2(\cos^2\vartheta\mathrm{d}\varphi_2 - \sin^2\vartheta\mathrm{d}\varphi_1)\right]\,,$$
$$J = \frac{q^2}{2}\sin 2\vartheta\,\mathrm{d}\vartheta \wedge (\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2) + \frac{q}{(q^2+1)}\mathrm{d}q \wedge \left[\mathrm{d}\tau + \sin^2\vartheta\mathrm{d}\varphi_1 - \cos^2\vartheta\mathrm{d}\varphi_2\right]\,,$$
$$\Omega = -\frac{q\,\mathrm{e}^{\mathrm{i}(\varphi_1 - \tau - \varphi_2)}}{2\sqrt{q^2+1}}\left[\sin 2\vartheta\,(q\,\mathrm{d}\tau - \mathrm{i}\mathrm{d}q) \wedge (\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2) + q\sin 2\vartheta\,\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \right.$$
$$\left. + 2\mathrm{i}\,q\,\mathrm{d}\vartheta \wedge (\mathrm{d}\tau + \sin^2\vartheta\mathrm{d}\varphi_1 - \cos^2\vartheta\mathrm{d}\varphi_2) - 2\,\mathrm{d}q \wedge \mathrm{d}\vartheta\right]\,. \tag{3.32}$$

The supersymmetric Killing vector is

$$\xi = g(SK_1,\,\cdot\,) = \partial_\tau + \partial_{\varphi_2} - \partial_{\varphi_1}\,. \tag{3.33}$$

Furthermore, notice that rescaling J by $1/(q^2+1)$ leads to a closed two-form, hence showing that the Hermitian metric transverse to ξ is conformal to a Kähler metric. Moreover, one can also check that the almost contact form $\eta = K_1/S$ is a contact form in this case, i.e. that $\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta$ is a volume form.

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[6] This is different to the gauge choice (3.10), where instead $a = -3\mathrm{d}\tau$ for this solution.

Conformally flat $S^1 \times S^4$. In this section we consider the conformally flat metric on $S^1 \times S^4$, which we may write as

$$\mathrm{d}s_5^2 = \mathrm{d}\tau^2 + \mathrm{d}s_{S^4}^2 \,, \tag{3.34}$$

where

$$\mathrm{d}s_{S^4}^2 = \mathrm{d}\beta^2 + \sin^2\beta(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi_1^2 + \cos^2\vartheta \mathrm{d}\varphi_2^2) \,. \tag{3.35}$$

Here τ is a periodic coordinate on S^1, while the metric in brackets in (3.35) is simply the round metric on a unit radius S^3, as in the previous black hole boundary example. The polar coordinate $\beta \in [0, \pi]$. The metric (3.34) of course arises as the conformal boundary of Euclidean AdS in global coordinates, and as such the background fields $a = 0 = b$. There are many Killing spinors in this case, and here we simply choose one so as to present simple expressions for the remaining background data. We find

$$S = \mathrm{e}^{-\tau} \,, \qquad \alpha = 0 \,, \qquad K_2 = \frac{3}{\sqrt{2}} \mathrm{d}\tau \,. \tag{3.36}$$

The U(2) structure is given by

$$\begin{aligned}
K_1 &= \sin\beta \, \mathrm{d}\beta - \cos\beta \, \mathrm{d}\tau \,, \\
J &= \sin^2\beta \sin(\varphi_1 + \varphi_2) \bigg\{ \cot(\varphi_1 + \varphi_2) \, (\mathrm{d}\vartheta \wedge \mathrm{d}\tau - \cot\beta \, \mathrm{d}\beta \wedge \mathrm{d}\vartheta) - \sin^2\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi_1 \\
&\quad - \cos^2\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi_2 + \sin\vartheta \cos\vartheta \Big[(\cot\beta \, \mathrm{d}\beta + \mathrm{d}\tau) \wedge (\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2) \\
&\quad - \cot(\varphi_1 + \varphi_2) \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \Big] \bigg\} \,, \\
\Omega &= \mathrm{i} \sin^2\beta \sin(\varphi_1 + \varphi_2) \Big[\cot\beta \, \mathrm{d}\beta \wedge \mathrm{d}\vartheta - \mathrm{d}\vartheta \wedge \mathrm{d}\tau + \sin\vartheta \cos\vartheta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \Big] \\
&\quad + \sin^2\beta \sin\vartheta \Big[\sin\vartheta + \mathrm{i} \cos\vartheta \cos(\varphi_1 + \varphi_2) \Big] \Big(\cot\beta \, \mathrm{d}\beta \wedge \mathrm{d}\varphi_1 - \cot\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi_2 \\
&\quad + \mathrm{d}\tau \wedge \mathrm{d}\varphi_1 \Big) + \sin^2\beta \cos\vartheta \Big[\cos\vartheta - \mathrm{i} \sin\vartheta \cos(\varphi_1 + \varphi_2) \Big] \Big(\cot\beta \, \mathrm{d}\beta \wedge \mathrm{d}\varphi_2 \\
&\quad + \tan\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi_1 + \mathrm{d}\tau \wedge \mathrm{d}\varphi_2 \Big) \,. \tag{3.37}
\end{aligned}$$

Notice that in this example we obtain a *conformal* Killing vector from the Killing spinor bilinear, but not a Killing vector. As described at the end of section 3.1, we may always make a Weyl transformation of the background to obtain a Killing vector. In the case at hand this corresponds to the Weyl factor $\Lambda = \mathrm{e}^\tau$, and the corresponding Weyl-transformed metric is then (locally) flat, with the Weyl-transformed J and Ω both closed and hence defining a transverse hyperKähler structure. Nevertheless, the fact that the metric (3.34) leads to a conformal Killing vector explains why this background is missing from the rigid supersymmetric geometry in [18, 19]: in the latter references the corresponding bilinear is necessarily a Killing vector. This also suggests that the conjecture made in [19] is likely to be correct: that is, to obtain the $S^1 \times S^4$ background from a rigid limit of supergravity, one should begin with *conformal* supergravity in five dimensions, rather than Poincaré supergravity.[7]

---

[7]See [44] for a related discussion on this point.

Squashed S^5. We consider the squashed five-sphere metric

$$
\mathrm{d}s_5^2 = \frac{1}{s^2}(\mathrm{d}\tau + C)^2 + \mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)
$$
$$
+\frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi)^2 , \tag{3.38}
$$

where $s \in (0,1]$ is the squashing parameter and

$$
C \equiv -\frac{1}{2}\sin^2\sigma(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi) . \tag{3.39}
$$

The coordinates $\sigma, \beta, \vartheta, \varphi$ are coordinates on the base $\mathbb{CP}^2$, with β having period 4π, φ having period 2π, while $\sigma \in [0, \frac{\pi}{2}]$, $\vartheta \in [0, \pi]$, and $\frac{1}{2}\mathrm{d}C$ is the Kähler two-form on $\mathbb{CP}^2$. For the "toric" family discussed in [31, 32] we find

$$
S = \frac{\cos^2\sigma}{b_2} + \frac{\sin^2\sigma}{b_1} , \tag{3.40}
$$

where

$$
b_1 = 1 + \sqrt{1-s^2} , \qquad b_2 = 1 - \sqrt{1-s^2} . \tag{3.41}
$$

The other background fields are, in an appropriate gauge (i.e. not that in (3.10)),

$$
\alpha = \frac{b_1(b_1+b_2)(b_1 - 7b_2 + (b_1-b_2)\cos 2\sigma)}{4\sqrt{2}(b_1\cos^2\sigma + b_2\sin^2\sigma)} ,
$$
$$
a = \frac{b_1 - b_2}{2b_2}(\mathrm{d}\tau + C) ,
$$
$$
b = -\frac{\mathrm{i}(b_1-b_2)}{2\sqrt{2}b_1b_2(b_1+b_2)}\mathrm{d}C ,
$$
$$
K_2 = \frac{\sqrt{2}\,(b_1-b_2)\,\sin 2\sigma}{b_1\cos^2\sigma + b_2\sin^2\sigma}\,\mathrm{d}\sigma = -\sqrt{2}\,\mathrm{d}\log(b_1\cos^2\sigma + b_2\sin^2\sigma) . \tag{3.42}
$$

The U(2) structure is

$$
K_1 = \frac{1}{4b_1b_2(b_1+b_2)\left(b_1\cos^2\sigma + b_2\sin^2\sigma\right)}\left[(b_1+b_2)(b_1 - b_2 + (b_1+b_2)\cos 2\sigma)\mathrm{d}\tau\right.
$$
$$
\left. -\frac{1}{2}\sin^2\sigma\left((b_1-b_2)^2\cos 2\sigma + b_1^2 - 4b_1b_2 - b_2^2\right)(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi)\right] ,
$$
$$
J = \frac{\sin\sigma}{8\,b_1b_2\,(b_1+b_2)^2\,\left(b_1\cos^2\sigma + b_2\sin^2\sigma\right)}\left[4\cos\sigma\left(2(b_1+b_2)\,\mathrm{d}\sigma\wedge\mathrm{d}\tau\right.\right.
$$
$$
\left.\left. -b_1\mathrm{d}\sigma\wedge(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi)\right) + 2\sin\vartheta\sin\sigma\,(b_1\cos^2\sigma + b_2\sin^2\sigma)\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\right] ,
$$
$$
\Omega = \frac{\sin\sigma\,\mathrm{e}^{\mathrm{i}(\tau-\beta)}}{8b_1b_2(b_1+b_2)^2\left(b_1\cos^2\sigma + b_2\sin^2\sigma\right)}\left[-\sin 2\sigma\left(\mathrm{i}\sin\vartheta\,(\,b_1\mathrm{d}\varphi\wedge\mathrm{d}\beta\right.\right.
$$
$$
\left.+2(b_1+b_2)\,\mathrm{d}\tau\wedge\mathrm{d}\varphi - 2(b_1+b_2)\,\mathrm{d}\vartheta\wedge\mathrm{d}\tau + b_1\,\mathrm{d}\vartheta\wedge(\mathrm{d}\beta + \cos\vartheta\mathrm{d}\varphi)\right)
$$
$$
\left. -4\,(b_1\cos^2\sigma + b_2\sin^2\sigma)\,(\sin\vartheta\,\mathrm{d}\sigma\wedge\mathrm{d}\varphi + \mathrm{i}\,\mathrm{d}\vartheta\wedge\mathrm{d}\sigma)\right] . \tag{3.43}
$$

The supersymmetric Killing vector is

$$\xi = b_1 \partial_\tau + 2(b_1 + b_2)\partial_\beta \ . \tag{3.44}$$

One also computes

$$\begin{aligned}
\eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta &= \frac{b_1^3 b_2^3 \, (b_1 + b_2)^2}{2 \left(b_1 \cos^2 \sigma + b_2 \sin^2 \sigma \right)^5} \left((b_1 - b_2)^2 \cos 2\sigma + b_1^2 - 4b_1 b_2 - b_2^2 \right) \\
&\quad \times \left((b_1^2 - b_2^2) \cos 2\sigma + b_1^2 - 6b_1 b_2 + b_2^2 \right) \mathrm{vol}_5 \ ,
\end{aligned} \tag{3.45}$$

where vol_5 denotes the Riemannian volume form and $\eta = K_1/S$ is the almost contact form. The right hand side of (3.45) can have non-trivial zeros, and we thus see that in general η does not define a contact structure. These backgrounds arise as the conformal boundary of the 3/4 BPS solutions of Romans supergravity constructed in [31, 32].

3.4 From geometry to supersymmetry

In this section we will show that *any* choice of transversely Hermitian structure on M_5 defines a supersymmetric background. The background $U(1)_R$ gauge field a and the b-field are given in terms of the geometry by (3.27). It then remains to show that the geometry also determines the fields X_2, $A^{(0)}$ and $\omega_{\mu\nu}$, in such a way that the original spinor equations (2.10)–(2.13) are satisfied.

We first examine the Killing spinor equation (2.10) for χ. In order to proceed it is convenient to choose a set of projection conditions (see for example [45])

$$\gamma_{12}\chi = \gamma_{34}\chi = \mathrm{i}\chi \,, \qquad \gamma_5 \chi = \chi \ . \tag{3.46}$$

These allow one to substitute for the fields b and K_2 in terms of the geometry, via (3.27) and (3.19), into the right hand side of equation (2.10). In doing this calculation it is also convenient to write $\Omega = J_2 + \mathrm{i} J_1$, $J = J_3$ so that

$$J_1 = e_{14} + e_{23} \,, \qquad J_2 = e_{13} - e_{24} \,, \qquad J_3 = e_{12} + e_{34} \ . \tag{3.47}$$

Notice that J_i, $i = 1, 2, 3$ span the transverse self-dual two-forms, and hence may be used as a basis thereof. One can furthermore make use of various identities that easily follow from (3.46), such as $\mathrm{i}\gamma_m \chi = J_{mn}\gamma^n \chi$, where $m, n = 1, \ldots, 4$, and $(\beta^-)_{mn}\gamma^{mn}\chi = 0$ for any transverse anti-self-dual two-form β^-.

In this way it is straightforward to show that the $\mu = 5$ (the ψ direction) component of (2.10) simply imposes $\partial_\psi \chi = 0$.[8] Thus χ is independent of ψ. Taking instead $\mu = m$, $m = 1, 2, 3, 4$, one finds (2.10) is equivalent to

$$\nabla_m^{(4)}\chi = \frac{1}{4}\theta^n \gamma_{mn}\chi - \frac{\mathrm{i}}{2}(a_\perp)_m \chi + \frac{1}{2}(\partial_m \log S)\chi \,, \tag{3.48}$$

where $\nabla^{(4)}$ denotes the Levi-Civita spin connection for the transverse four-dimensional metric. Recall that the latter metric is Hermitian. It is then more natural to express

---

[8]Without loss of generality we take the four-dimensional frame $e^1, \ldots, e^4$ to be independent of the Killing vector $\xi = \partial_\psi$.

equation (3.48) in terms of an appropriate Hermitian connection, which preserves both the metric and the two-form J. The Chern connection is such a connection, defined by

$$\nabla_m^{\text{Chern}}\chi = \partial_m\chi + \frac{1}{4}(\omega_m^{\text{Chern}})_{pq}\gamma^{pq}\chi\,,$$

$$\text{where} \qquad (\omega_m^{\text{Chern}})_{pq} \equiv (\omega_m^{(4)})_{pq} + \frac{1}{2}J_m{}^n(\mathrm{d}J)_{npq}\,. \tag{3.49}$$

This coincides with the Levi-Civita connection if and only if $\mathrm{d}J = 0$ (equivalently $\theta = 0$), so that the metric is Kähler.

Next, let us notice that under the Weyl transformation (3.9) we have $\chi \to \Lambda^{1/2}\chi$, so that it is also natural to introduce

$$\tilde{\chi} \equiv S^{-1/2}\chi\,, \tag{3.50}$$

so that $\tilde{\chi}$ is Weyl invariant. In this notation (3.48) becomes

$$\nabla_m^{\text{Chern}}\tilde{\chi} + \frac{\mathrm{i}}{2}a_{\text{Chern}}\tilde{\chi} = 0\,, \tag{3.51}$$

where recall that $a_{\text{Chern}} = a_\perp - I(\theta)$ is the Chern connection for the canonical bundle $\mathcal{K} \equiv \Lambda^{2,0}$, given explicitly by (3.25). It is then a standard fact, and is straightforward to show, that any Hermitian space admits a canonical solution $\tilde{\chi}$ to (3.51). Specifically, any Hermitian space admits a canonical spin^c structure, with twisted spin bundles $\text{Spin}^c = \text{Spin} \otimes \mathcal{K}^{-1/2}$. In four dimensions this is isomorphic to

$$\text{Spin}^c \cong \left(\Lambda^{0,0} \oplus \Lambda^{0,2}\right) \oplus \Lambda^{0,1}\,, \tag{3.52}$$

where $\Lambda^{p,q}$ denotes the bundle of forms of Hodge type (p, q). In the case at hand, these are defined transversely to the foliation generated by the Killing vector ξ. Under (3.52) the Killing spinor $\tilde{\chi} = S^{-1/2}\chi$ is a section of the trivial line bundle $\Lambda^{0,0}$. Moreover, the Chern connection restricted to this summand is flat, with the induced connection $-\frac{1}{2}a_{\text{Chern}}$ on the twist factor $\mathcal{K}^{-1/2}$ effectively cancelling that coming from the spin bundle. Concretely, in terms of local complex coordinates z^α, $\alpha = 1, 2$, we have $(\omega^{\text{Chern}})_\alpha{}^\beta = (\partial g^{(4)})_{\alpha\bar{\gamma}}(g^{(4)})^{\bar{\gamma}\beta}$, and using the projection conditions (3.46) one can show this precisely cancels the contribution from (3.25). The spin^c spinor $\tilde{\chi}$ is simply a constant length section of this flat line bundle. Put simply, the rescaled Killing spinor $\tilde{\chi} = S^{-1/2}\chi$ is constant.

Next we turn to the dilatino equation (2.12). Substituting for φ in terms of χ, using (2.13), after a somewhat lengthy computation one finds the dilatino equation holds provided

$$A^{(0)} = -\frac{9}{4} * \left(\mathrm{d} * b - \frac{\mathrm{i}\sqrt{2}}{3}b \wedge b\right)\,, \tag{3.53}$$

and

$$X_2 = -4\alpha^2 - \frac{1}{4}\langle K_2, K_2\rangle - \frac{\mathrm{i}}{6\sqrt{2}}S\langle\eta, A^{(0)}\rangle - \frac{3}{16}\langle \mathrm{d}a_\perp, J\rangle - \frac{3}{4\sqrt{2}}\langle K_2, \mathrm{d}\log S\rangle\,. \tag{3.54}$$

Here we have introduced the notation $\phi_1 \wedge *\phi_2 = \frac{1}{p!}\langle\phi_1, \phi_2\rangle\,\mathrm{vol}_5$ for the inner product between two p-forms ϕ_1, ϕ_2. Notice that the expression (3.53) for the imaginary one-form

$A^{(0)}$ coincides with that in [32], which was derived by solving the bulk equations of motion near the conformal boundary, in terms of the boundary data. Notice that under the Weyl scaling (3.9) we have

$$A^{(0)} \to \frac{1}{\Lambda}\left(A^{(0)} + \frac{9}{2}i_{\mathrm{d}\log\Lambda^{\#}}b\right), \qquad X_2 \to \frac{1}{\Lambda^2}X_2 \,. \tag{3.55}$$

The fact that X_2 has Weyl weight -2 is clearly consistent with the bulk expansion (2.4), but the "anomalous" transformation of $A^{(0)}$ in (3.55) naively appears to contradict (2.4), for which $A^{(0)}$ has Weyl weight -1. However, this is where the comment above equation (2.4) is relevant: the reparametrization $r \to \Lambda r$ does *not* preserve the *subleading* terms in the metric (2.3). It is therefore not a strict symmetry of the system we have defined. However, the *leading order* terms in the expansions (2.3), (2.4) *are* invariant. This explains why the differential constraints (3.4)–(3.8) have the Weyl symmetry (3.9), while the higher order term $A^{(0)}$ arising in the expansion of the B-field does not. One could restore the full Weyl symmetry by adding a cross term $9\frac{\mathrm{d}r}{r}\mathscr{C}_\mu \mathrm{d}x^\mu$ into the metric (2.3), so that

$$\mathscr{C} \to \mathscr{C} - \mathrm{d}\log\Lambda \,, \tag{3.56}$$

under $r \to \Lambda r$ preserves the form of the metric. Then $\mathscr{C}$ is a new background field on M_5, and one finds

$$A^{(0)} = -\frac{9}{4}*\left[(\mathrm{d} + 2\mathscr{C}\wedge)*b - \frac{\mathrm{i}\sqrt{2}}{3}b\wedge b\right] \,. \tag{3.57}$$

This now has Weyl weight -1, as expected, and the anomalous variation in (3.55) arises simply because we have made the gauge choice $\mathscr{C} = 0$ in our original expansion. In general notice that a field of Weyl weight w will couple to a Weyl covariant derivative $\mathrm{D}_\mu \equiv \partial_\mu + w\,\mathscr{C}_\mu$, and $w = 2$ for $*b$.

It remains to show that the background geometry implies the φ Killing spinor equation (2.11). At this point notice that everything is fixed uniquely in terms of the free functions α and S, and the transversely Hermitian structure on M_5, apart from the higher order spin connection term $\omega_{\mu\nu}$ which appears in (2.11). After a lengthy computation, in our orthonormal frame one finds the expression

$$\omega_{55} = -6\sqrt{2}\alpha^2 - \frac{1}{3\sqrt{2}}\langle K_2, K_2\rangle - \sqrt{2}X_2 - \frac{1}{2\sqrt{2}}\langle \mathrm{d}a_\perp, J\rangle - \langle K_2, \mathrm{d}\log S\rangle\,,$$

$$\omega_{5m} = \left[-\frac{\mathrm{i}}{3\sqrt{2}}i_{K_2^{\#}}b_\perp + i_{\mathrm{d}\log S^{\#}}\left(2\alpha J + \frac{1}{\sqrt{2}}S\mathrm{d}\rho^-\right)\right]_m = \omega_{m5}\,,$$

$$\omega_{mn} = \frac{\sqrt{2}}{3}(K_2)_m(K_2)_n - \nabla^{(4)}{}_{(m}(K_2)_{n)} - \left(\frac{4}{3}S\alpha\,\mathrm{d}\rho^- + \frac{1}{\sqrt{2}}\mathrm{d}a_\perp^-\right)_{mp}J^p{}_n$$

$$+\left(2\sqrt{2}\alpha^2 + \frac{\sqrt{2}}{3}X_2 - \frac{1}{3\sqrt{2}}\langle K_2, K_2\rangle + \frac{1}{4\sqrt{2}}\langle \mathrm{d}\alpha_\perp, J\rangle\right)\delta_{mn}\,. \tag{3.58}$$

This is manifestly real and symmetric, apart from the last term in the penultimate line. However, it is straightforward to show that $(\beta^-)_{mp}J^p{}_n$ is symmetric for any transverse

anti-self-dual two-form β^-. Thus (2.11) is satisfied provided $\omega_{\mu\nu}$ is given by (3.58). We conclude this subsection by noting the following formula

$$\omega_\mu{}^\mu \;=\; 2\sqrt{2}\alpha^2 + \frac{\sqrt{2}}{3}X_2 - \langle K_2, \frac{1}{\sqrt{2}}K_2 + \mathrm{d}\log S\rangle + \frac{1}{2\sqrt{2}}\langle \mathrm{d}a_\perp, J\rangle - \nabla_m^{(4)}K_2^m \;. \qquad (3.59)$$

This trace will appear in the supersymmetric Lagrangians constructed in section 4.

3.5 Summary

A supersymmetric asymptotically locally AdS solution to six-dimensional Romans supergravity leads to the coupled spinor equations (2.10)–(2.13) on the conformal boundary M_5. These are a rather complicated looking set of equations for the spin^c spinors χ, φ, depending on the large number of background fields $g, X_2, a, A^{(0)}, b$ and $\omega_{\mu\nu}$ on M_5, with φ and χ related to each other by the further background fields α and K_2 via (2.13). However, we have shown these equations are completely equivalent to a very simple geometric structure:

(i) The five-manifold M_5 is equipped with a transversely holomorphic foliation, with the one-dimensional leaves generated by the (conformal) Killing vector field $\xi = \partial_\psi$. This structure is a natural odd-dimensional cousin of a complex manifold, and means we may cover M_5 locally with coordinates ψ, z_1, z_2, where the transition functions between the z_1, z_2 coordinates are holomorphic (more formally we have an open cover $\{U_i\}$ and submersions $f_i : U_i \to \mathbb{C}^2$ with one-dimensional fibres, such that on overlaps $U_i \cap U_j$ we have $f_j = g_{ji} \circ f_i$ where g_{ji} are biholomorphisms of open sets in $\mathbb{C}^2$).

(ii) This foliation is compatible with an almost contact form $\eta = \mathrm{d}\psi + \rho$. Choose a particular $\rho = \rho_0$, which notice is defined only locally in the foliation patches, gluing together to give the global η. Then for fixed foliation any other choice of ρ is related to this by $\rho = \rho_0 + \nu$, where ν is a *global basic one-form*. That is, ν is a global one-form on M_5 satisfying $\mathcal{L}_\xi\nu = 0 = i_\xi\nu$.

(iii) One can choose an *arbitrary* transverse Hermitian metric $\mathrm{d}s_4^2$, invariant under ξ and compatible with the foliation.

(iv) Finally, one is free to choose the ξ-invariant real functions α and S (with S nowhere zero).

An interesting special case is when all the leaves of the foliation are closed, so that ξ generates a U(1) action on M_5 and ψ is a periodic coordinate. In this case M_5 fibres over a complex Hermitian orbifold $M_4 = M_5/\mathrm{U}(1)$, where η is a global angular form for the U(1) orbibundle. Different choices of ν in (ii) above are then simply different connections on this bundle, with (iii) giving different Hermitian metrics on M_4.

We have shown that any choice of the data (i)–(iv) determines a supersymmetric background, solving the spinor equations (2.10)–(2.13), and conversely any such solution determines a choice of the above geometric data. Furthermore, solving (2.10)–(2.13) is equivalent to finding a supersymmetric asymptotically locally AdS solution to Romans supergravity, to the first few orders in an expansion around the conformal boundary M_5.

Of course whether or not this extends to a complete non-singular supergravity solution, as some of the explicit examples in section 3.3 do, is another matter.

4 Supersymmetric gauge theories

In this section we construct $\mathcal{N} = 1$ supersymmetric gauge theories formed of vector and hypermultiplets on the background geometry described in section 3.

We adopt the same notation as [2], in particular using ξ and η to denote five-dimensional Killing spinors. The γ_μ are 4×4 Hermitian matrices which form a basis of Cliff$(5,0)$ in an orthonormal frame. A complete set of 4×4 matrices is given by $(1_4, \gamma_\mu, \gamma_{\mu\nu})$ and we choose $\gamma_{\mu\nu\rho\sigma\tau} = -\epsilon_{\mu\nu\rho\sigma\tau}$ with $\epsilon_{12345} = +1$. The five-dimensional charge conjugation matrix, $\mathcal{C} = (\mathcal{C}_{\alpha\beta})$, is unitary and anti-symmetric in the spinor indices $\alpha, \beta = 1, 2, 3, 4$ of Spin$(5) \cong$ Sp(2). The matrices $\mathcal{C}\gamma_\mu$ are anti-symmetric in spinor indices whereas $\mathcal{C}\gamma_{\mu\nu}$ are symmetric. Spinor bilinears are denoted $(\eta\gamma^{\mu_1\cdots\mu_n}\xi) = \eta^\alpha(\mathcal{C}\gamma^{\mu_1\cdots\mu_n})_{\alpha\beta}\xi^\beta$. Finally, the Fierz identity for Grassmann odd spinors in five dimensions is

$$\gamma^A \eta^\alpha (\xi\gamma^B\lambda) = -\frac{1}{4}(\eta\xi)\gamma^A\gamma^B\lambda^\alpha - \frac{1}{4}(\eta\gamma_\mu\xi)\gamma^A\gamma^\mu\gamma^B\lambda^\alpha + \frac{1}{8}(\eta\gamma_{\mu\nu}\xi)\gamma^A\gamma^{\mu\nu}\gamma^B\lambda^\alpha, \quad (4.1)$$

where γ^A, γ^B denote arbitrary elements of Cliff$(5,0)$.

4.1 Supersymmetry algebra

An off-shell $\mathcal{N} = 1$ vector multiplet in five dimensions consists of a gauge field $\mathcal{A}_\mu$, a real scalar σ, a gaugino λ_I, and a triplet of auxiliary scalars D_{IJ}, all transforming in the adjoint representation of the gauge group G. Here $I, J = 1, 2$ are SU$(2)_R$ symmetry indices. The gaugino is a symplectic-Majorana spinor which satisfies $(\lambda_I^\alpha)^* = \varepsilon^{IJ}\mathcal{C}_{\alpha\beta}\lambda_J^\beta$ whilst the auxiliary scalars satisfy $(D_{IJ})^\dagger = \varepsilon^{IK}\varepsilon^{JL}D_{KL}$, where recall that ε^{IJ} is the Levi-Civita symbol.

We introduce the following covariant derivatives:

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - \mathrm{i}[\mathcal{A}_\mu, \mathcal{A}_\nu], \\
D_\mu\sigma &= \partial_\mu\sigma - \mathrm{i}[\mathcal{A}_\mu, \sigma], \\
D_\mu\lambda_I &= \nabla_\mu\lambda_I - \mathrm{i}[\mathcal{A}_\mu, \lambda_I], \\
D_\mu D_{IJ} &= \partial_\mu D_{IJ} - \mathrm{i}[\mathcal{A}_\mu, D_{IJ}],
\end{aligned} \quad (4.2)$$

where ∇ is the Levi-Civita spin connection. In general we may consider turning on an SU$(2)_R$ background gauge field a_μ^i, $i = 1, 2, 3$, or equivalently we may introduce

$$V_{\mu IJ} \equiv -\frac{\mathrm{i}}{2}a_\mu^i(\sigma_i)_{IJ}, \quad (4.3)$$

where σ_i, $i = 1, 2, 3$, denote the Pauli matrices. In section 2 recall that for simplicity we restricted to an Abelian background gauge field, with $a^1 = a^2 = 0$, $a^3 = a$, but in this section we will relax this assumption. There is also a background two-form b-field and we choose to introduce the gauge field $\mathscr{C}_\mu$ associated with restoring Weyl invariance — see

the earlier discussion around equation (3.56). With this background gauge field active we modify the covariant derivatives to

$$\mathrm{D}_\mu \sigma = D_\mu \sigma - \mathscr{C}_\mu \sigma \,,$$

$$\mathrm{D}_\mu \lambda_I = D_\mu \lambda_I - \frac{3}{2}\mathscr{C}_\mu \lambda_I - V_{\mu I}{}^J \lambda_J \,,$$

$$\mathrm{D}_\mu D_{IJ} = D_\mu D_{IJ} - 2\mathscr{C}_\mu D_{IJ} - 2V_{\mu(I}{}^K D_{J)K} \,, \tag{4.4}$$

so that they are covariant with respect to both Weyl and R-symmetry transformations. These correspond to Weyl weights $w = (-1, 0, -\frac{3}{2}, -2)$ for the gauge multiplet $(\sigma, \mathcal{A}_\mu, \lambda_I, D_{IJ})$.

Given this background data we consider the following (conformal) supersymmetry variations:

$$\delta_\xi \sigma = \mathrm{i}\varepsilon^{IJ}\xi_I \lambda_J \,,$$

$$\delta_\xi \mathcal{A}_\mu = \mathrm{i}\varepsilon^{IJ}\xi_I \gamma_\mu \lambda_J \,,$$

$$\delta_\xi \lambda_I = -\frac{1}{2}\gamma^{\mu\nu}\xi_I \mathcal{F}_{\mu\nu} + \gamma^\mu \xi_I \mathrm{D}_\mu \sigma - D_{IJ}\xi^J + \frac{\mathrm{i}}{3\sqrt{2}}\gamma^{\mu\nu}\xi_I b_{\mu\nu}\sigma - \frac{2\sqrt{2}\mathrm{i}}{3}\tilde{\xi}_I \sigma \,,$$

$$\delta_\xi D_{IJ} = -2\mathrm{i}\xi_{(I}\gamma^\mu \mathrm{D}_\mu \lambda_{J)} + 2[\sigma, \xi_{(I}\lambda_{J)}] + \frac{2\sqrt{2}}{3}\tilde{\xi}_{(I}\lambda_{J)} - \frac{1}{6\sqrt{2}}\xi_{(I}\gamma^{\mu\nu}\lambda_{J)}b_{\mu\nu} \,. \tag{4.5}$$

This has Grassmann odd supersymmetry parameters ξ_I, $\tilde{\xi}_I$. We find that these transformations close onto

$$[\delta_\xi, \delta_\eta]\sigma = -\mathrm{i}v^\nu \mathrm{D}_\nu \sigma - \frac{\sqrt{2}\mathrm{i}}{3}\varrho\sigma \,, \tag{4.6}$$

$$[\delta_\xi, \delta_\eta]\mathcal{A}_\mu = -\mathrm{i}v^\nu \mathcal{F}_{\nu\mu} + D_\mu \Upsilon \,,$$

$$[\delta_\xi, \delta_\eta]\lambda_I = -\mathrm{i}v^\nu \mathrm{D}_\nu \lambda_I + \mathrm{i}[\Upsilon, \lambda_I] - \frac{\sqrt{2}\mathrm{i}}{3}\left[\frac{3}{2}\varrho\lambda_I + R_I{}^J \lambda_J - \frac{1}{4}\Theta^{\alpha\beta}\gamma_{\alpha\beta}\lambda_I\right] \,,$$

$$[\delta_\xi, \delta_\eta]D_{IJ} = -\mathrm{i}v^\nu \mathrm{D}_\nu D_{IJ} + \mathrm{i}[\Upsilon, D_{IJ}] - \frac{\sqrt{2}\mathrm{i}}{3}\left[2\varrho D_{IJ} + R_I{}^K D_{JK} + R_J{}^K D_{IK}\right] \,,$$

where we have defined

$$v^\mu = 2\varepsilon^{IJ}\xi_I \gamma^\mu \eta_J \,,$$

$$\Upsilon = -2\mathrm{i}\varepsilon^{IJ}\xi_I \eta_J \sigma \,,$$

$$\varrho = -2\mathrm{i}\varepsilon^{IJ}(\xi_I \tilde{\eta}_J - \eta_I \tilde{\xi}_J) \,,$$

$$R_{IJ} = -3\mathrm{i}(\xi_I \tilde{\eta}_J + \xi_J \tilde{\eta}_I - \eta_I \tilde{\xi}_J - \eta_J \tilde{\xi}_I) \,,$$

$$\Theta^{\alpha\beta} = -2\mathrm{i}\varepsilon^{IJ}(\tilde{\xi}_I \gamma^{\alpha\beta}\eta_J - \tilde{\eta}_I \gamma^{\alpha\beta}\xi_J) - 2\mathrm{i}\varepsilon^{IJ}(\xi_I \eta_J)b^{\alpha\beta} + \frac{\mathrm{i}}{4}\varepsilon^{\mu\nu\rho\alpha\beta}b_{\mu\nu}v_\rho \,, \tag{4.7}$$

and $R_I{}^J = \varepsilon^{JK}R_{IK}$, provided that the spinors $(\xi, \tilde{\xi})$ and $(\eta, \tilde{\eta})$ satisfy the SU(2)$_R$-covariantization of the (χ, φ) spinor equations (2.10)–(2.12). More precisely

$$\mathrm{D}_\mu \xi_I = -\frac{\sqrt{2}\mathrm{i}}{3}\gamma_\mu \tilde{\xi}_I - \frac{\mathrm{i}}{12\sqrt{2}}b_{\nu\rho}\gamma_\mu{}^{\nu\rho}\xi_I + \frac{\mathrm{i}}{3\sqrt{2}}b_{\mu\nu}\gamma^\nu \xi_I \,, \tag{4.8}$$

$$\mathrm{D}_\mu\tilde\xi_I = -\frac{\mathrm{i}}{6\sqrt2}b_{\mu\nu}\gamma^\nu\tilde\xi_I - \frac{1}{16}\mathrm{D}^\nu b^{\rho\sigma}\gamma_{\mu\nu\rho\sigma}\xi_I + \frac{1}{16}\mathrm{D}_\mu b_{\nu\rho}\gamma^{\nu\rho}\xi_I - \frac{1}{8}\mathrm{D}_\nu b_{\mu\rho}\gamma^{\nu\rho}\xi_I$$
$$+ \frac{\mathrm{i}}{8\sqrt2}V_{\nu\rho I}{}^J\gamma_\mu{}^{\nu\rho}\xi_J - \frac{3\mathrm{i}}{4\sqrt2}V_{\mu\nu I}{}^J\gamma^\nu\xi_J - \frac{1}{36}A^{(0)}_\nu\gamma_\mu{}^\nu\xi_I + \frac{1}{12}A^{(0)}_\mu\xi_I + \frac{\mathrm{i}}{2}\omega_{\mu\nu}\gamma^\nu\xi_I\,,$$
$$0 = -\frac{1}{6\sqrt2}b_{\mu\nu}\gamma^{\mu\nu}\tilde\xi_I - \frac{\sqrt2}{3}X_2\xi_I + \frac{\mathrm{i}}{8}\mathrm{D}_\mu b_{\nu\rho}\gamma^{\mu\nu\rho}\xi_I - \frac{\mathrm{i}}{18}A^{(0)}_\mu\gamma^\mu\xi_I - \frac{1}{4\sqrt2}V_{\mu\nu I}{}^J\gamma^{\mu\nu}\xi_J\,,$$

with $V_{\mu\nu}{}^{IJ} \equiv 2\partial_{[\mu}V_{\nu]}{}^{IJ} - 2V_{[\mu}{}^{K(I}V_{\nu]K}{}^{J)}$. Recall that b has Weyl weight $w = 1$, while the spinors have weight $w = \pm1/2$.

It is crucial for the closure of the algebra that $\omega_{\mu\nu} = \omega_{\nu\mu}$, which is the same condition used in deriving the differential constraints (3.4)–(3.8). Also as for that computation the closure of the supersymmetry algebra is insensitive to the explicit form of $\omega_{\mu\nu}$, $A^{(0)}$ or X_2. Let us also notice that the supersymmetry variations (4.5) reduce to those of the round S^5 in [2] (in particular $b \equiv 0$ for the round S^5, and $\tilde\xi_I^{\text{here}} = \frac{3}{\sqrt2}\mathrm{i}\tilde\xi_I^{\text{there}}$).

We now consider the on-shell hypermultiplet which consists of two complex scalars q_I and a spinor ψ, all transforming in an arbitrary representation of the gauge group. A system of r hypermultiplets is described by q_I^A, ψ^A with $A = 1,\ldots,2r$. The fields satisfy the reality conditions $(q_I^A)^* = \Omega_{AB}\varepsilon^{IJ}q_J^B$ and $(\psi^{A\alpha})^* = \Omega_{AB}\mathcal{C}_{\alpha\beta}\psi^{B\beta}$ with Ω_{AB} being the invariant tensor of $\mathrm{Sp}(r)$. The supersymmetry variations for the system of r hypermultiplets coupled to the vector multiplet are

$$\delta_\xi q_I^A = -2\mathrm{i}\xi_I\psi^A\,,$$
$$\delta_\xi\psi^A = \varepsilon^{IJ}\gamma^\mu\xi_I\mathrm{D}_\mu q_J^A + \mathrm{i}\varepsilon^{IJ}\xi_I\sigma q_J^A - \sqrt2\mathrm{i}\varepsilon^{IJ}\tilde\xi_I q_J^A\,. \qquad (4.9)$$

The commutator of two supersymmetry transformations leads to

$$[\delta_\xi,\delta_\eta]q_I^A = -\mathrm{i}v^\mu\mathrm{D}_\mu q_I^A + \mathrm{i}\Upsilon q_I^A - \frac{\sqrt2\mathrm{i}}{3}\left[\frac{3}{2}\varrho q_I^A + R_I{}^J q_J^A\right],$$
$$[\delta_\xi,\delta_\eta]\psi^A = -\mathrm{i}v^\mu\mathrm{D}_\mu\psi^A + \mathrm{i}\Upsilon\psi^A - \frac{\sqrt2\mathrm{i}}{3}\left[2\varrho\psi^A - \frac{1}{4}\Theta^{\alpha\beta}\gamma_{\alpha\beta}\psi^A\right]$$
$$+ \frac{1}{2}v_\rho\Gamma^\rho\left(\mathrm{i}\gamma^\mu\mathrm{D}_\mu\psi^A + \sigma\psi^A + \varepsilon^{IJ}\lambda_I q_J^A - \frac{1}{4\sqrt2}\gamma^{\mu\nu}\psi^A b_{\mu\nu}\right)$$
$$- \varepsilon^{KL}(\xi_K\eta_L)\left(\mathrm{i}\gamma^\mu\mathrm{D}_\mu\psi^A + \sigma\psi^A + \varepsilon^{IJ}\lambda_I q_J^A - \frac{1}{4\sqrt2}\gamma^{\mu\nu}\psi^A b_{\mu\nu}\right), \qquad (4.10)$$

where

$$\mathrm{D}_\mu q_I^A = \partial_\mu q_I^A - \mathrm{i}\mathcal{A}_\mu q_I^A - \frac{3}{2}\mathscr{C}_\mu q_I^A - V_{\mu I}{}^J q_J^A\,,$$
$$\mathrm{D}_\mu\psi^A = \nabla_\mu\psi^A - \mathrm{i}\mathcal{A}_\mu\psi^A - 2\mathscr{C}_\mu\psi^A\,. \qquad (4.11)$$

Closure of the algebra occurs only on-shell and this identifies the fermionic equation of motion as

$$E_\psi \equiv \mathrm{i}\gamma^\mu\mathrm{D}_\mu\psi^A + \sigma\psi^A + \varepsilon^{IJ}\lambda_I q_J^A - \frac{1}{4\sqrt2}\gamma^{\mu\nu}\psi^A b_{\mu\nu} = 0\,. \qquad (4.12)$$

Acting on E_ψ with the supersymmetry transformations gives the bosonic equation of motion:

$$\varepsilon^{IJ}\left(D^\mu D_\mu q_J^A + \sigma^2 q_J^A - \frac{1}{3} X_2 q_J^A + \frac{1}{\sqrt{2}}\omega_\mu{}^\mu q_J^A - 2(\psi^A \lambda_J)\right) + i D^{IJ} q_J^A = 0 \;. \qquad (4.13)$$

4.2 Lagrangians

The action for a vector multiplet in five dimensions is determined by the prepotential $\mathscr{F}(\mathcal{V})$, which is a real and gauge invariant function of the vector superfield $\mathcal{V}$. Gauge invariance limits the prepotential to being at most cubic in $\mathcal{V}$ [46] and classically it takes the form

$$\mathscr{F}(\mathcal{V}) = \mathrm{Tr}\left[\frac{1}{2\mathsf{g}^2}\mathcal{V}^2 + \frac{k}{6}\mathcal{V}^3\right]. \qquad (4.14)$$

Here g is the dimensionful gauge coupling constant and k is a real constant which is subject to a quantization condition dependent on the gauge group [47]. Writing the components of the vector superfield as $\mathcal{V}^a T_a = (\sigma^a T_a, \mathcal{A}_\mu^a T_a, \lambda_I^a T_a, D_{IJ}^a T_a)$ where T_a are generators of the gauge group in the adjoint representation we find the cubic prepotential term in our curved backgrounds to be

$$\begin{aligned}
\mathcal{L}_{\text{cubic}} = {}&\mathrm{d}_{abc}\left[\frac{1}{24}\epsilon^{\mu\nu\rho\sigma\tau}\mathcal{A}_\mu^a \mathcal{F}_{\nu\rho}^b \mathcal{F}_{\sigma\tau}^c + \frac{i}{8}\varepsilon^{IJ}(\lambda_I^a \gamma^{\mu\nu}\lambda_J^b)\mathcal{F}_{\mu\nu}^c + \frac{i}{4}D^{a,IJ}(\lambda_I^b \lambda_J^c)\right] \\
&+ \mathrm{d}_{abc}\sigma^a\left[\frac{1}{4}\mathcal{F}_{\mu\nu}^b \mathcal{F}^{c,\mu\nu} - \frac{1}{2}D_\mu\sigma^b D^\mu\sigma^c - \frac{1}{4}D_{IJ}^b D^{c,IJ}\right. \\
&\qquad\qquad\left. - \frac{i}{2\sqrt{2}}\sigma^b \mathcal{F}_{\mu\nu}^c b^{\mu\nu} + \frac{1}{3}\sigma^b\sigma^c\left(\frac{\sqrt{2}}{3}\omega_\mu{}^\mu + \frac{2}{3}X_2 - \frac{5}{18}b_{\mu\nu}b^{\mu\nu}\right)\right. \qquad (4.15) \\
&\qquad\qquad\left. + \frac{i}{2}\varepsilon^{IJ}(\lambda_I^b \gamma^\mu D_\mu\lambda_J^c) - \frac{1}{2}\varepsilon^{IJ}\lambda_I^b[\lambda_J,\sigma]^c + \frac{1}{8\sqrt{2}}\varepsilon^{IJ}(\lambda_I^b \gamma^{\mu\nu}\lambda_J^c)b_{\mu\nu}\right].
\end{aligned}$$

Here $\mathrm{d}_{abc} \propto \frac{k}{\pi^2}\mathrm{Tr}\left(T_{(a}T_b T_{c)}\right)$ is a symmetric invariant tensor of the gauge group. It vanishes[9] for all simple gauge groups except U(1) or SU(N) with $N \geq 3$. The Lagrangian $\mathcal{L}_{\text{cubic}}$ is invariant under the superconformal transformations (4.5) provided the supersymmetry parameters satisfy (4.8), and in addition $A^{(0)}$ is given by

$$A^{(0)} = -\frac{9}{4} * \left((\mathrm{d} + 2\mathscr{C}\wedge) * b - \frac{i\sqrt{2}}{3}b \wedge b\right), \qquad (4.16)$$

which matches precisely the expression (3.57) in section 3.

The quadratic term in the prepotential includes Yang-Mills kinetic terms and is not conformally invariant. We therefore expect to break conformality by using the relation

$$\tilde{\xi}_I = -\alpha_I{}^J \xi_J - \frac{i}{2}(K_2)_\mu \gamma^\mu \xi_I\,, \qquad (4.17)$$

---

[9]For example, taking the gauge group to be $G = \mathrm{SO}(N)$ so that the Lie algebra generators satisfy $T_a^t = -T_a$ then $\mathrm{Tr}\left(T_{(a}T_b T_{c)}\right) = \mathrm{Tr}\left(T_{(a}T_b T_{c)}\right)^t = -\mathrm{Tr}\left(T_{(a}T_b T_{c)}\right)$.

which is the $SU(2)_R$-covariantization of (2.13). The Lagrangian describing the quadratic piece can be found from $\mathcal{L}_{\text{cubic}}$ by identifying one of the vector superfields with a constant supersymmetry preserving Abelian vector multiplet [7]. That is

$$\mathcal{L}_{\text{YM}} = \frac{1}{2g^2} \mathcal{V}^{(1)} \text{Tr}[\mathcal{V}^2] \,, \tag{4.18}$$

where $\mathcal{V}^{(1)} = (\sigma^{(1)}, \mathcal{A}_\mu^{(1)}, \lambda_I^{(1)}, D_{IJ}^{(1)})$. We choose $\sigma^{(1)} = 1$ and $\lambda_I^{(1)} = 0$. Then $\mathcal{V}^{(1)}$ is supersymmetry preserving if the fermion variation

$$
\begin{aligned}
\delta_\xi \lambda_I^{(1)} &= -\frac{1}{2}\gamma^{\mu\nu}\xi_I \mathcal{F}_{\mu\nu}^{(1)} - D_{IJ}^{(1)}\xi^J + \frac{i}{3\sqrt{2}}\gamma^{\mu\nu}\xi_I b_{\mu\nu} - \frac{2\sqrt{2}i}{3}\alpha_{IJ}\xi^J - \frac{\sqrt{2}}{3}(K_2)_\mu\gamma^\mu\xi_I \,, \\
&= 0 \,,
\end{aligned}
\tag{4.19}
$$

holds for non-trivial spinor parameters ξ_I and some choice of $D_{IJ}^{(1)}$, $\mathcal{A}_\mu^{(1)}$ such that $\mathcal{F}^{(1)} = d\mathcal{A}^{(1)}$. Here we have substituted for $\tilde{\xi}_I$ using (4.17). To progress, note that there are two natural one-forms in our geometry namely K_1 and K_2. If we concentrate on K_1 which, with $S = 1$, satisfies (3.5)

$$dK_1 = \frac{2\sqrt{2}}{3}\left[2\alpha J + K_1 \wedge K_2 + ib - \frac{i}{2}i_\xi(*b)\right] \,, \tag{4.20}$$

then upon $SU(2)_R$-covariantizing and multiplying by $-\frac{1}{2}\gamma^{\mu\nu}\xi_I$ we find

$$0 = -\frac{1}{2}\gamma^{\mu\nu}\xi_I\left((dK_1)_{\mu\nu} - \frac{i\sqrt{2}}{3}b_{\mu\nu}\right) - \frac{\sqrt{2}}{3}\gamma^\mu\xi_I(K_2)_\mu - \frac{8\sqrt{2}i}{3}\alpha_{IJ}\xi^J \,. \tag{4.21}$$

To derive the previous equation we have used the projection conditions satisfied by the background geometry: $(K_1)_\mu\gamma^\mu\chi = \chi$ and $J_{\mu\nu}\gamma^{\mu\nu}\chi = 4i\chi$, along with $(K_1)^\mu(K_2)_\mu = 0 = (K_1)^\mu\mathscr{C}_\mu$ and $-i(K_1)^\mu b_{\mu\nu} = (K_2)_\nu + \frac{3}{\sqrt{2}}\mathscr{C}_\nu$. Comparing this to (4.19) gives the constant vector multiplet as

$$\mathcal{V}^{(1)} = (\sigma^{(1)}, \mathcal{A}_\mu^{(1)}, \lambda_I^{(1)}, D_{IJ}^{(1)}) = (1, (K_1)_\mu, 0, 2\sqrt{2}i\alpha_{IJ}) \,, \tag{4.22}$$

and the corresponding Yang-Mills Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{YM}} = \frac{1}{g^2}\text{Tr}\Bigg[&\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}D_\mu\sigma D^\mu\sigma - \frac{1}{4}D_{IJ}D^{IJ} + \frac{i}{2}\epsilon^{IJ}(\lambda_I\gamma^\mu D_\mu\lambda_J) - \frac{1}{2}\varepsilon^{IJ}\lambda_I[\lambda_J,\sigma] \\
&+ \frac{1}{8}\epsilon^{\mu\nu\rho\tau}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}(K_1)_\tau - \frac{i}{\sqrt{2}}\sigma\mathcal{F}_{\mu\nu}b^{\mu\nu} + \frac{1}{2}\sigma\mathcal{F}^{\mu\nu}(dK_1)_{\mu\nu} \\
&- 2\sqrt{2}i\sigma D^{IJ}\alpha_{IJ} + \sigma^2\left(\frac{\sqrt{2}}{3}\omega_\mu{}^\mu + \frac{2}{3}X_2 - \frac{5}{18}b_{\mu\nu}b^{\mu\nu} - \frac{i}{2\sqrt{2}}(dK_1)_{\mu\nu}b^{\mu\nu}\right) \\
&+ \frac{i}{8}\varepsilon^{IJ}(\lambda_I\gamma^{\mu\nu}\lambda_J)(dK_1)_{\mu\nu} + \frac{1}{8\sqrt{2}}\varepsilon^{IJ}(\lambda_I\gamma^{\mu\nu}\lambda_J)b_{\mu\nu} - \frac{1}{\sqrt{2}}(\lambda_I\lambda_J)\alpha^{IJ}\Bigg] \,.
\end{aligned}
\tag{4.23}
$$

The second candidate one-form is K_2 but taking $\mathcal{F}^{(1)} = dK_2$ does not lead to (4.19).

The superconformal Lagrangian for the vector coupled hypermultiplets exists irrespective of the gauge group and is straightforward to construct: we simply integrate the equations of motion (4.12) and (4.13) found from closing the superalgebra to find

$$\mathcal{L}_{\mathrm{hm}} = \Omega_{AB} \bigg[-\frac{1}{2}\varepsilon^{IJ}\mathrm{D}^\mu q_I^A \mathrm{D}_\mu q_J^B + \frac{1}{2}\varepsilon^{IJ} q_I^A \sigma^2 q_J^B + \frac{\mathrm{i}}{2} q_I^A D^{IJ} q_J^B$$
$$- 2\varepsilon^{IJ} q_I^A (\psi^B \lambda_J) + \varepsilon^{IJ} q_I^A q_J^B \left(\frac{1}{2\sqrt{2}}\omega_\mu{}^\mu - \frac{1}{6}X_2 \right)$$
$$+ \mathrm{i}(\psi^A \gamma^\mu \mathrm{D}_\mu \psi^B) + \psi^A \sigma \psi^B - \frac{1}{4\sqrt{2}}(\psi^A \gamma^{\mu\nu} \psi^B) b_{\mu\nu} \bigg] . \qquad (4.24)$$

5 Discussion

In this paper we have constructed rigid supersymmetric gauge theories with matter on a general class of five-manifold backgrounds. By construction these are the most general backgrounds that arise as conformal boundaries of six-dimensional Romans supergravity solutions. We find that (M_5, g) is equipped with a conformal Killing vector which generates a transversely holomorphic foliation. In particular the transverse metric $g^{(4)}$ is an arbitrary Hermitian metric with respect to the transverse complex structure. This is a natural hybrid/generalization of the rigid supersymmetric geometries in three and four dimensions constructed in [35, 37, 38], and includes many previous constructions as special cases.

It is interesting to compare the geometry we find to the rigid limit of Poincaré supergravity [18, 19] and the twisting of [28]. In the former case the backgrounds naively appear to be more general, as there is no almost complex structure singled out, nor integrability condition. However, they don't include the $S^1 \times S^4$ geometry relevant for the supersymmetric index, which as we showed in section 3.3 *is* included in our backgrounds. In fact the singling out of the almost complex structure associated to $J = J_3$, where recall that $\Omega = J_2 + \mathrm{i}J_1$, in our geometry is almost certainly related to the fact that in section 3 we focused on the case where we turn on only an Abelian $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$. This was motivated in part for simplicity, and in part because the known solutions to Romans supergravity discussed previously also have this property. Nevertheless, the supersymmetry variations and Lagrangians we constructed in section 4 are valid for an arbitrary background $\mathrm{SU}(2)_R$ gauge field, and it should be relatively straightforward to analyse the geometric constraints in this more general case. Indeed, this is certainly necessary, and presumably sufficient, to reproduce the partially topologically twisted backgrounds $S^2 \times M_3$ of [11–13], since the $\mathrm{SU}(2)$ spin connection of M_3 is twisted by $\mathrm{SU}(2)_R$. On the other hand recall that the twisting in [28] requires that M_5 be a K-contact manifold. This shares many features with our geometry, with one important difference: for a K-contact manifold the transverse two-form J is closed, so the corresponding foliation is *transversely symplectic*; however, our case is in some sense precisely the opposite, namely transversely holomorphic. These intersect precisely for Sasakian manifolds. It is interesting that these various approaches generally seem to lead to different supersymmetric geometries, with varying degrees of overlap.

Given the geometry we find and the results of [48], it is natural to conjecture that the partition function and other BPS observables depend only on the transversely holomorphic

foliation, i.e. for fixed such foliation they are independent of the choice of the remaining background data (functions S, α, the one-form ν defined in section 3.5, and the transverse Hermitian metric $g^{(4)}$). It will be interesting to verify that this is indeed the case, and to compute these quantities using localization methods. Notice that *locally* a transversely holomorphic foliation always looks like $\mathbb{R} \times \mathbb{C}^2$, which perhaps also explains why in [19] the authors found that *locally* all deformations of their backgrounds were Q-exact. Finally, our construction allows one to address holographic duals of these questions, which we plan to return to in future work.

Acknowledgments

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