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Factorization of the 3d superconformal index

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Abstract: We prove that 3d superconformal index for general $\mathcal{N} = 2$ $\mathrm{U}(N)$ gauge group with fundamentals and anti-fundmentals with/without Chern-Simons terms is factorized into vortex and anti-vortex partition function. We show that for simple cases, 3d vortex partition function coincides with a suitable topological open string partition function. We provide much more elegant derivation at the index level for $\mathcal{N} = 2$ Seiberg-like dualities of unitary gauge groups with fundamantal matters and $\mathcal{N} = 4$ mirror symmetry

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1 Introduction

Recently, there has been renewed interest in nonperturbative dualities between three dimensional theories such as mirror symmetry and Seiberg-like dualities. This is explained in part by the availability of sophisticated tools such as the partition function on S^3 and the superconformal index. Using these tools, one can give impressive evidences for various 3d dualities. Some of works in this area are [1]–[19].

It turns out that the partition function has another interesting property, i.e., it is factorized into vortex and anti-vortex partition function [20]. Schematically

$$Z(z, \bar{z}) = Z_{\text{vortex}}(z) Z_{\text{antivortex}}(\bar{z}) = |Z_{\text{vortex}}(z)|^2 \tag{1.1}$$

where z traces the vortex number while $\bar{z}$ traces the anti-vortex number. This is reminiscent of the conformal blocks of the 2-dimensional conformal field theories. The above factorization was shown to hold for abelian gauge theories. Thus it is more desirable to show that this factorization holds for the general nonabelian cases. And it would be an interesting question if the similar holds for 3d superconformal index. In fact, it is recently shown that similar factorization holds for 2-dimensional $\mathcal{N}=2$ supersymmetric partition function in terms of vortex and anti-vortex partition function [21, 22]. Since 3d index is the partition function defined on $S^1 \times S^2$, the two sphere partition function is recovered from the 3d index by taking the radius of S^1 to be small. Thus we expect that the factorization should hold for 3d superconformal index as well.

The purpose of this paper is to show explicitly that such factorization indeed occurs for 3d superconformal index. More explicitly we show that for U(N) gauge theories with N_f fundamental and $\tilde{N}_f$ fundamentals, the index is factorized into vortex and anti-vortex partition function on $R^2 \times S^1$ whenever $\max(N_f, \tilde{N}_f) \geq N$. This is the condition of the existence of the vortex solutions of the underlying field theories. We show the fatorization by explicit residue evaluation of the associated matrix integral of the index, similar to 2d case.

The factorized form of the index has a number of merits and we just explore a few of them in this paper, relegating the full explorations elsewhere. The first one is that we have the explicit expressions of the index after the matrix integral. Obviously since we have the explicit expressions for the index, it would be much more convenient to explore the various dualities. Previously the index is expanded in power series of the conformal dimension of the gauge invariant BPS operators. In this way, one can check various dualities by working out the index of the both sides to some orders in operator dimensions. Though it certainly gives impressive evidences, in this way the full analytic proof cannot be achieved. We will show that explicit factorized formulae of the index reveal much more transparent structures of the dualities. We will see this by working out the index of the dual pairs of Aharony duality with unitary gauge group. The proof of the equality of the index is reduced to proving the nontrivial identity. And we prove the identity for simple cases. Thus we provide the first step toward the analytic proof for general cases.

Furthermore in 2d case, the vortex partition function has the direct connection to the topological open string amplitude. We expect that similar holds for 3d vortex partition function since 2d vortex partition function is so called the homological limit of 3d vortex partition function. We show that vortex partition function is the same as topological open string partition function for simple cases but certainly has the obvious generalizations for much more numerous examples, This is also resonant with the recent proposal by Iqbal and Vafa [24] that the integrand of the 3d superconformal index is given by the square of the topolgical open string amplitude. It would be interesting to explore the precise relation between the 3d vortex partition function and the open topological string.

The content of the paper is as follows. In section 2, we summarize the basic structures of the superconformal index. We carefully study the U(1) gauge theory with a fundamental chiral multiplet with Chern-Simons (CS) level $-1/2$ following [23], find subtleties such as the relative phase of the different monopole sector, in the usual index computation, which will be useful for later computation. In section 3, we firstly work out the factorization for U(1) gauge theory without CS terms, which is technically simpler. Then we summarize the factorization of the general cases, deferring the full proof to the appendix. We also work out the explicit examples of the factorization and show that the associated vortex partition function admits topological open string interpretation. Furthermore we show that in some of the examples vortex partition function can be understood as 3d defect of the 5d field theory. In section 4, we apply the factorized index to understand the $\mathcal{N}=2$ Seiberg-like dualities for unitary gauge group, known as Aharony duality. Factorized index reveals much more clearly such duality should hold at the index level. We briefly touch upon the $\mathcal{N}=4$ Seiberg-like dualities and mirror symmetry and postpone the further explorations elsewhere.

As this work is close to end, we receive the related paper by [25]. As far as we understand , they do not give the general formulae for the factorized index as we do.

2 3d superconformal index

2.1 Summary of the 3d superconformal index

Let us discuss the superconformal index for $\mathcal{N}=2$ $d=3$ superconformal field theories (SCFT). The bosonic subgroup of the 3d $\mathcal{N}=2$ superconformal group is $\mathrm{SO}(2,3)\times\mathrm{SO}(2)$. There are three Cartan elements denoted by ϵ, j_3 and R which come from three factors $\mathrm{SO}(2)_\epsilon\times\mathrm{SO}(3)_{j_3}\times\mathrm{SO}(2)_R$ in the bosonic subgroup, respectively. The superconformal index for an $\mathcal{N}=2$ $d=3$ SCFT is defined as follows [26]:

$$I(x,t)=\mathrm{Tr}(-1)^F\exp(-\beta'\{Q,S\})x^{\epsilon+j_3}\prod_a t_a^{F_a} \tag{2.1}$$

where Q is a supercharge with quantum numbers $\epsilon=\frac{1}{2}, j_3=-\frac{1}{2}$ and $R=1$, and $S=Q^\dagger$. The trace is taken over the Hilbert space in the SCFT on $\mathbb{R}\times S^2$ (or equivalently over the space of local gauge-invariant operators on $\mathbb{R}^3$). The operators S and Q satisfy the following anti-commutation relation:

$$\{Q,S\}=\epsilon-R-j_3:=\Delta. \tag{2.2}$$

As usual, only BPS states satisfying the bound $\Delta=0$ contribute to the index, and therefore the index is independent of the parameter β'. If we have additional conserved charges f_a commuting with the chosen supercharges (Q,S), we can turn on the associated chemical potentials t_a, and then the index counts the number of BPS states weighted by their quantum numbers.

The superconformal index is exactly calculable using the localization technique [27, 28]. It can be written in the following form:

$$I(x,t)=\sum_{m\in\mathbb{Z}^N/S^N}\int da\,\frac{1}{|\mathcal{W}_m|}e^{-S^{(0)}_{CS}(a,m)}e^{ib_0(a,m)}\prod_a t_a^{q_{0a}(m)}x^{\epsilon_0(m)}\exp\left[\sum_{n=1}^\infty\frac{1}{n}f_{\mathrm{tot}}(e^{ina},t^n,x^n)\right]. \tag{2.3}$$

Here we use the notation $a\equiv\vec{a}=\{a_1,a_2,\cdots,a_N\}$, $m\equiv\vec{m}=\{m_1,m_2,\cdots,m_N\}$ where N is the rank of the gauge group and $da\equiv\prod_{j=1}^N da_j$. The origin of this formula is as follows. To compute the trace over the Hilbert space on $S^2\times\mathbb{R}$, we use path-integral on

$S^2 \times S^1$ with suitable boundary conditions on the fields. The path-integral is evaluated using localization, which means that we have to sum or integrate over all BPS saddle points. The saddle points are spherically symmetric configurations on $S^2 \times S^1$ which are labeled by magnetic fluxes on S^2 and holonomy along S^1. The magnetic fluxes are denoted by $\{m_j\}$ and take values in the cocharacter lattice of G (i.e. in $\mathrm{Hom}(\mathrm{U}(1), T)$, where T is the maximal torus of G), while the eigenvalues of the holonomy are denoted $\{a_j\}$ and take values in T. $S^{(0)}_{CS}(a, m)$ is the classical action for the (monopole+holonomy) configuration on $S^2 \times S^1$, $\epsilon_0(m)$ is the Casimir energy of the vacuum state on S^2 with magnetic flux m, $q_{0a}(m)$ is the f_a-charge of the vacuum state, and $b_0(a, m)$ represents the contribution coming from the electric charge of the vacuum state. The last factor comes from taking the trace over a Fock space built on a particular vacuum state. $|\mathcal{W}_m|$ is the order of the Weyl group of the part of G which is left unbroken by the magnetic fluxes m . These ingredients in the formula for the index are given by the following explicit expressions:

$$
\begin{aligned}
S^{(0)}_{CS}(a, m) &= i \sum_{\rho \in R_F} \kappa \rho(m) \rho(a), \\
b_0(a, m) &= -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)| \rho(a), \\
q_{0a}(m) &= -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)| f_a(\Phi), \\
\epsilon_0(m) &= \frac{1}{2} \sum_{\Phi} (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\alpha(m)|,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
f_{\text{tot}}(x, t, e^{ia}) &= f_{\text{vector}}(x, e^{ia}) + f_{\text{chiral}}(x, t, e^{ia}), \\
f_{\text{vector}}(x, e^{ia}) &= -\sum_{\alpha \in G} e^{i\alpha(a)} x^{|\alpha(m)|}, \\
f_{\text{chiral}}(x, t, e^{ia}) &= \sum_{\Phi} \sum_{\rho \in R_\Phi} \left[e^{i\rho(a)} \prod_a t_a^{f_a} \frac{x^{|\rho(m)| + \Delta_\Phi}}{1 - x^2} - e^{-i\rho(a)} \prod_a t_a^{-f_a} \frac{x^{|\rho(m)| + 2 - \Delta_\Phi}}{1 - x^2} \right]
\end{aligned} \tag{2.5}
$$

where $\sum_{\rho \in R_F}$, $\sum_\Phi$, $\sum_{\rho \in R_\Phi}$ and $\sum_{\alpha \in G}$ represent summations over all fundamental weights of G, all chiral multiplets, all weights of the representation R_Φ, and all roots of G, respectively.

We will find it convenient to rewrite the integrand in (2.3) as a product of contributions from the different multiplets. First, note that the single particle index f enters via the so-called Plethystic exponential:

$$
\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(x^n, t^n, z^n = e^{ina}, m) \right) \tag{2.6}
$$

while we define $z_j = e^{ia_j}$. It will be convenient to rewrite this using the q-Pochhammer symbol, defined for n finite or infinite:

$$
(a; q)_n = \prod_{k=0}^{n-1} \left(1 - a q^k \right). \tag{2.7}
$$

Specifically, consider a single chiral field Φ, whose single particle index is given by:[1]

$$\sum_{\rho\in R_\Phi}\left(e^{i\rho(a)}{t_a}^{f_a(\Phi)}\frac{x^{|\rho(m)|+\Delta_\Phi}}{1-x^2}-e^{-i\rho(a)}{t_a}^{-f_a(\Phi)}\frac{x^{|\rho(m)|+2-\Delta_\Phi}}{1-x^2}\right). \tag{2.8}$$

Then we can write the Plethystic exponential of this as follows:

$$\prod_{\rho\in R_\Phi}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(e^{in\rho(a)}{t_a}^{nf_a(\Phi)}\frac{x^{n|\rho(m)|+n\Delta_\Phi}}{1-x^{2n}}-e^{-in\rho(a)}{t_a}^{-nf_a(\Phi)}\frac{x^{n|\rho(m)|+2n-n\Delta_\Phi}}{1-x^{2n}}\right)\right]. \tag{2.9}$$

By rewriting the denominator as a geometric series and interchanging the order of summations, one finds that this becomes:

$$\prod_{\rho\in R_\Phi}\frac{(e^{-i\rho(a)}{t_a}^{-f_a(\Phi)}x^{|\rho(m)|+2-\Delta_\Phi};x^2)_\infty}{(e^{i\rho(a)}{t_a}^{f_a(\Phi)}x^{|\rho(m)|+\Delta_\Phi};x^2)_\infty}. \tag{2.10}$$

The full index will involve a product of such factors over all the chiral fields in the theory, as well as the contribution from the gauge multiplet. It is given by:

$$\begin{aligned}&I(x,t)\\&=\sum_{m\in\mathbb{Z}^N/S_N}\oint\left(\prod_{j=1}^{N}\frac{dz_j}{2\pi i z_j}\right)\frac{1}{|\mathcal{W}_m|}e^{-S_{CS}(a,m)}Z_{\text{gauge}}(x,z,m)\prod_{\Phi}Z_\Phi(x,t,z,m)\end{aligned} \tag{2.11}$$

where

$$Z_{\text{gauge}}(x,z,m)=\prod_{\alpha\in ad(G)}x^{-|\alpha(m)|/2}\left(1-e^{i\alpha(a)}x^{|\alpha(m)|}\right) \tag{2.12}$$

$$\begin{aligned}&\prod_{\Phi}Z_\Phi\left(x,t,\tilde{t},\tau,z,m\right)\\&=\prod_{\rho\in R_\Phi}\left(x^{(1-\Delta_\Phi)}e^{-i\rho(a+\pi)}\prod_a t_a^{-f_a(\Phi)}\right)^{|\rho(m)|/2}\frac{\left(e^{-i\rho(a)}\prod t_a^{-f_a(\Phi)}x^{|\rho(m)|+2-\Delta_\Phi};x^2\right)_\infty}{\left(e^{i\rho(a)}\prod t_a^{f_a(\Phi)}x^{|\rho(m)|+\Delta_\Phi};x^2\right)_\infty}\end{aligned} \tag{2.13}$$

Note that by shifting $t_a\to t_a x^{c_a}$, one can change the value of the R-charge Δ_Φ. Hence Δ_Φ remains the free parameter for generic cases.

We are mainly interested in this ordinary index and work out the factorization. However two important generalizations are worthy of mention, which will be useful in comparison with the result of [23] in the following subsection. The first one is the notion of the generalized index. When we turn on the chemical potential t_a, this can be regarded as turning on a Wilson line for a fixed background gauge field. The generalized index is obtained when we turn on the nontrivial magnetic flux n_a for the corresponding background gauge field. Only the contribution to the chiral multiplets are changed and this is given by

[1] Note that a in $\rho(a)$ and the subscript a in t_a or f_a denote the different objects.

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the replacement $\rho(m) \to \rho(m) + \sum_a f_a(\Phi) n_a$

$$\begin{aligned} Z_\Phi(x,t,z,m) = \prod_{\rho\in R_\Phi} \left(x^{(1-\Delta_\Phi)} e^{-i\rho(a)} \prod_a t_a{}^{-f_a(\Phi)} \right)^{|\rho(m)/2+\sum_a f_a(\Phi)n_a/2|} \\ \times \frac{(e^{-i\rho(a)} t_a{}^{-f_a(\Phi)} x^{|\rho(m)+\sum_a f_a(\Phi)n_a|+2-\Delta_\Phi}; x^2)_\infty}{(e^{i\rho(a)} t_a{}^{f_a(\Phi)} x^{|\rho(m)+\sum_a f_a(\Phi)n_a|+\Delta_\Phi}; x^2)_\infty}. \end{aligned} \tag{2.14}$$

Here n_a should take integer value as does m_j.

For every $\mathrm{U}(N)$ gauge group, we can define another abelian symmetry $\mathrm{U}(1)_T$ whose conserved current is $*F$ of overall $\mathrm{U}(1)$ factor. To couple this topological current to background gauge field we introduce BF term $\int A_{BG} \wedge \mathrm{tr} dA + \cdots$ and terms needed for supersymmetric completion. This introduces to the index

$$z^n w^{\sum_j m_j} \tag{2.15}$$

where n is the new discrete parameter representing the topological charge of $\mathrm{U}(1)_T$ while w is the chemical potential for $\mathrm{U}(1)_T$.

2.2 Comparision to DGG

In the paper by Dimofte, Gaiotto and Gukov [23] (DGG), the simplest mirror pair of $\mathcal{N}=2$ theory was considered and along with it revealed some subtleties in the index computation. The claim is that the theory of one free chiral multiplet with global $\mathrm{U}(1)$ symmetry at CS level $\frac{1}{2}$ is mirror to $\mathrm{U}(1)$ gauge theory at CS level $-\frac{1}{2}$, coupled to a single fundamental chiral multiplet.[2] According to DGG, for the free chiral theory the index is given by

$$\mathcal{I}_\Delta(m;q,\zeta) = \left(-q^{\frac{1}{2}}\right)^{\frac{1}{2}(m+|m|)} \zeta^{-\frac{1}{2}(m+|m|)} \prod_{r=0}^{\infty} \frac{1-q^{r+\frac{1}{2}|m|+1}\zeta^{-1}}{1-q^{r+\frac{1}{2}|m|}\zeta}. \tag{2.16}$$

Note that we use the zero R-charge for the free chiral but value of R-charge can be altered by shifting $\zeta \to \zeta x^\alpha$ for a suitable α. The index of $\mathrm{U}(1)$ theory is [23]

$$\mathcal{I}_{\mathrm{U}(1)}\left(m';q,\zeta'\right) = \sum_{m\in\mathbb{Z}} \oint \frac{d\zeta}{2\pi i\zeta} \zeta'^m \zeta^{m'} \left(-q^{\frac{1}{2}}\right)^{-\frac{1}{2}(m-|m|)} \zeta^{\frac{1}{2}(m-|m|)} \prod_{r=0}^{\infty} \frac{1-q^{r+\frac{1}{2}|m|+1}\zeta^{-1}}{1-q^{r+\frac{1}{2}|m|}\zeta}. \tag{2.17}$$

It is proved that $\mathcal{I}_\Delta(m;q,\zeta) = \mathcal{I}_{\mathrm{U}(1)}(m;q,\zeta)$.

In order to compare it to our index, let us slightly change the variables as follows:

$$\mathcal{I}_{\mathrm{U}(1)}\left(m';x^2,w\right) = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m z^{m'} (-x)^{-\frac{1}{2}(m-|m|)} z^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1-z^{-1}x^{|m|+2+2k}}{1-zx^{|m|+2k}}. \tag{2.18}$$

Note that $\mathrm{U}(1)$ gauge theory has topological $\mathrm{U}(1)$ global symmetry whose current is given by $*F$ and w corresponds to its chemical potential. Under the mirror map, the global symmetry of chiral theory is mapped to the topological symmetry. Hence ζ is mapped to

[2] We use a convention of the opposite sign for the CS level to DGG.

w. The expression appearing at DGG is slightly different from the standard expression one obtains following the prescription specified at the previous subsection or at [28]. For U(1) with CS level $-1/2$, the index is given by[3]

$$I(x,w,m') = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi iz} w^m z^{m'} x^{|m|/2} (-z)^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1-z^{-1}x^{|m|+2+2k}}{1-zx^{|m|+2k}}. \tag{2.19}$$

The term $x^{|m|/2}$ comes from the zero point energy contribution. At first DGG expression appears to change the zero point energy for positive and negative flux sector. However the computation in [28] shows that the one-loop determinant is symmetric under $m \to -m$ hence the zero point energy should be symmetric under $m \to -m$, which comes from the suitable regularization of one-loop determinant. The resolution is that if we assign different R-charge in the free theory by $\zeta \to \zeta q^\alpha$ we modify the U(1) theory by $w \to wx^{2\alpha}$. Using this freedom, if one shifts $w \to wx^{-1/2}$ one obtains DGG eq. (2.18) from the standard computation eq. (2.19). On the other hand, in the U(1) theory there's no freedom to change the assigned R-charge of the charged chiral field and we assign zero R-charge for the scalar of the chiral multiplet. One might worry that this R-charge can violate the unitarity of the SCFT. However, the chiral field itself is not a gauge invariant operator. Furthermore all of the gauge invariant operators of the theory are captured by the index of the free chiral theory due to the mirror symmetry. Thus the assigned zero R-charge does not lead to any inconsistency. Furthermore one can show that the standard index of the U(1) theory eq. (2.19) is equal to the free chiral theory with the canonical R-charge $1/2$, i.e.,

$$\mathcal{I}_\Delta(m;q,\zeta) = \left(q^{\frac{1}{2}}\right)^{\frac{1}{4}(m+|m|)} (-\zeta)^{-\frac{1}{2}(m+|m|)} \prod_{r=0}^{\infty} \frac{1-q^{r+\frac{1}{2}|m|+\frac{3}{4}}\zeta^{-1}}{1-q^{r+\frac{1}{2}|m|+\frac{1}{4}}\zeta} \tag{2.20}$$

with $m=m'$, $q=x^2$ and $\zeta = w$. Thus if we use the standard index computation we have the duality between U(1) theory with CS level $-1/2$ with one charged chiral with zero R-charge and the free chiral with CS level $1/2$ with the standard R-charge assignment.

On the other hand DGG assigns subtle relative phase factor $(-1)^{\frac{1}{2}(m+|m|)}$ between positive and negative flux sector. This phase factor cannot be derived from the usual index computation since it concerns on the relative phase of the different flux sector. In DGG, this relative phase factor have been checked extensively so we include this phase in later computations. It turns out that this phase is crucial for the factorization of the indices.

For reference, for U(1) theory with CS level κ with N_f fundamental chiral and $\tilde{N}_f$ anti-fundamental chiral , the flavor symmetry is $\mathrm{U}(1)_A \times \mathrm{SU}(N_f) \times \mathrm{SU}(\tilde{N}_f)$. There is also the topological symmetry $\mathrm{U}(1)_T$. The index we will use is as follows:

$$\begin{aligned} I(x,t,\tilde{t},\tau,w) = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi iz} w^m x^{\frac{1}{2}(N_f+\tilde{N}_f)|m|} (-z)^{-\kappa m - \frac{1}{2}(N_f-\tilde{N}_f)|m|} \tau^{-\frac{1}{2}(N_f+\tilde{N}_f)|m|} \\ \times \frac{\prod_{a=1}^{N_f}\left(z^{-1}t_a^{-1}\tau^{-1}x^{|m|+2};x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(zt_a\tau x^{|m|};x^2\right)_\infty} \frac{\prod_{a=1}^{\tilde{N}_f}\left(z\tilde{t}_a^{-1}\tau^{-1}x^{|m|+2};x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(z^{-1}\tilde{t}_a\tau x^{|m|};x^2\right)_\infty} \end{aligned} \tag{2.21}$$

[3]The factor $(-1)^{\frac{1}{2}(m-|m|)}$ will be explained in the next paragraph.

where $w, \tau, t_a, \tilde{t}_a$ are the fugacities for $\mathrm{U}(1)_T$, $\mathrm{U}(1)_A$, Cartans of $\mathrm{SU}(N_f), \mathrm{SU}(\tilde{N}_f)$ respectively. Note that we include the additional phase $(-1)^{-\kappa m-\frac{1}{2}(N_f-\tilde{N}_f)|m|}$ to the original index. Similar factor will be included for non-abelian cases as well.

3 Factorization

3.1 U(1) theory without CS terms

We first consider the factorization for the abelian case without Chern-Simons terms. Similar but slightly more complicated derivation works for $\mathrm{U}(N)$ theories with fundamentals and anti-fundamentals in the presence of CS terms. The general derivation is relegated to the appendix. The superconformal index for a $\mathrm{U}(1)$ gauge theory is given by

$$I(x,t,w) = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m \prod_{\Phi} Z_\Phi(x,t,z,m). \tag{3.1}$$

If we considers N_f fundamental and $\tilde{N}_f$ antifundamental chiral multiplets, the matter contribution $\prod_\Phi Z_\Phi$ is given by

$$\begin{aligned} &\prod_{\Phi} Z_\Phi\left(x,t,\tilde{t},\tau,z,m\right) \\ &= x^{(1-\Delta_\Phi)(N_f+\tilde{N}_f)|m|/2}(-z)^{-(N_f-\tilde{N}_f)|m|/2}\tau^{-(N_f+\tilde{N}_f)|m|/2} \\ &\quad\times \frac{\prod_{a=1}^{N_f}\left(z^{-1}t_a^{-1}\tau^{-1}x^{|m|+2-\Delta_\Phi};x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(zt_a\tau x^{|m|+\Delta_\Phi};x^2\right)_\infty} \frac{\prod_{a=1}^{\tilde{N}_f}\left(z\tilde{t}_a^{-1}\tau^{-1}x^{|m|+2-\Delta_\Phi};x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(z^{-1}\tilde{t}_a\tau x^{|m|+\Delta_\Phi};x^2\right)_\infty} \end{aligned} \tag{3.2}$$

where $\{t_a\}$ and $\{\tilde{t}_a\}$ correspond to fugacities for the $\mathrm{SU}(N_f)\times\mathrm{SU}(\tilde{N}_f)$ flavor symmetry; τ is a fugacity for $\mathrm{U}(1)_A$ as in the previous section. $(a;q)_n$ is the q-Pochhammer symbol defined by

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right). \tag{3.3}$$

Note that $N_f+\tilde{N}_f$ should be an even integer due to the quantization of the effective CS level, which leads to the parity anomaly free condition. Shortly we will see that this condition is necessary for sensible factorization. More generally odd integer values of $N_f+\tilde{N}_f$ are allowed in the presence of the level $\kappa \in (2\mathbb{Z}+1)/2$ CS terms.

We define

$$A_\infty(m) \equiv \frac{\prod_{a=1}^{N_f}\left(z^{-1}t_a^{-1}\tau^{-1}x^{|m|+2-\Delta_\Phi};x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(zt_a\tau x^{|m|+\Delta_\Phi};x^2\right)_\infty} \frac{\prod_{a=1}^{\tilde{N}_f}\left(z\tilde{t}_a^{-1}\tau^{-1}x^{|m|+2-\Delta_\Phi};x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(z^{-1}\tilde{t}_a\tau x^{|m|+\Delta_\Phi};x^2\right)_\infty} \tag{3.4}$$

and $A_n(m)$ is the above equation with ∞ replaced by n in each of the q-Pochhammer symbol. The basic idea is that in the evaluation of the index we replace $A_\infty(m)$ by $A_n(m)$. After the evaluation, we take the limit $n\to\infty$. Note that $(a;x^2)_\infty$ is convergent if the series

$\sum_k ax^{2k}$ is convergent. Thus if we choose x real with $0 < x < 1$, $(a;x^2)_\infty$ is convergent. This means that for a given small number ϵ, one can find $n(\epsilon)$,

$$|(a;x^2)_\infty - (a;x^2)_n| < \epsilon \tag{3.5}$$

$$1 < |\frac{(a;x^2)_n}{(a;x^2)_\infty}| < 1+\epsilon \tag{3.6}$$

with $-\frac{\pi}{2} < \arg(a) < \frac{\pi}{2}$. The second equation implies

$$1 < |\frac{1}{(1-ax^{2n})(1-ax^{2n+2})\cdots}| < 1+\epsilon \tag{3.7}$$

which also implies

$$|\frac{1}{(1-ax^{2n+\alpha})(1-ax^{2n+\alpha+2})\cdots}| < 1+\epsilon. \tag{3.8}$$

for any positive α. When $\frac{\pi}{2} < \arg(a) < \frac{3\pi}{2}$, we have instead

$$1 < |(1+(-a)x^{2n})(1+(-a)x^{2n+2})\cdots| < 1+\epsilon \tag{3.9}$$

$$1 < |(1+(-a)x^{2n+\alpha})(1+(-a)x^{2n+\alpha+2})\cdots| < 1+\epsilon \tag{3.10}$$

Using these, one can show that for a given ϵ one can find n such that

$$|A_\infty(m) - A_n(m)| < \epsilon \tag{3.11}$$

independent of m.

Thus if we consider $I(x,t,w)$ and $I(x,t,w)_n$ with $A_\infty(m)$ being replaced by $A_n(m)$,

$$\begin{aligned} &|I(x,t,w) - I(x,t,w)_n| \\ &< \left|\sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m x^{(1-\Delta_\Phi)(N_f+\tilde{N}_f)|m|/2} (-z)^{-(N_f-\tilde{N}_f)|m|/2} \tau^{-(N_f+\tilde{N}_f)|m|/2} \left[A_\infty(m) - A_n(m)\right]\right| \\ &< \left|\sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m x^{(1-\Delta_\Phi)(N_f+\tilde{N}_f)|m|/2} (-z)^{-(N_f-\tilde{N}_f)|m|/2} \tau^{-(N_f+\tilde{N}_f)|m|/2} \epsilon\right| \end{aligned} \tag{3.12}$$

If we require τ, z, w have the unit norm and $0 < x < 1$, the last line is $O(\epsilon)$ since $1-\Delta_\Phi > 0$ for the fundamental scalar fields satisfying the unitary bound. Thus the index is well approximated by $I(x,t,w)_n$ and we simply take $n \to \infty$ after the evaluation of the integral. Then we turn to the residue integration for the evaluation of $I(x,t,w)_n$.

From now on we set $\Delta_\Phi = 0$, which can be restored by deforming $\tau \to \tau x^{\Delta_\Phi}$. Thus, we will require $0 < x < 1$ as well as $|t_a\tau|, |\tilde{t}_a\tau| < 1$.[4]

For finite n, with $N_f > \tilde{N}_f$ $A_n(m)$ has simple poles at $z = \frac{1}{t_a\tau x^{|m|+2l}}$ and $z = \tilde{t}_a\tau x^{|m|+2l}$ with $a = 1, \cdots, N_f$, $l = 0, \cdots, n-1$. In addition, $A_n(m)$ has the pole of order $n(N_f - \tilde{N}_f)$ at $z = 0$ while at $z = \infty$ it has the zero of order $n(N_f - \tilde{N}_f)$. This implies that the expression (3.2) has the pole of order $(n + \frac{|m|}{2})(N_f - \tilde{N}_f)$ at $z = 0$ and the zero of order

[4]The result to be derived below will be extended to other parameter regime by the usual analytic continuation.

$(n+\frac{|m|}{2})(N_f-\tilde{N}_f)$ at $z=\infty$ Thus it is natural to evaluate the integration with unit circle contour by deforming the contour to include the poles located outside the unit circle. This picks up the poles from the fundamental chiral multiplets. Since the contour around the infinite circle gives the vanishing result, we simply have to evaluate the residues of outside poles. As $n\to\infty$, this corresponds to summing over the residues at $z=\frac{1}{t_a\tau x^{|m|+2l}}$ with $a=1,\cdots,N_f$, $l=0,\cdots,\infty$.

Summing the residues the index is given by

$$
\begin{aligned}
&I^{N_f>\tilde{N}_f}(x,t,\tilde{t},\tau,w)\\
&=\sum_{m\in\mathbb{Z}}\sum_{b=1}^{N_f}\sum_{l=0}^{\infty}(-1)^{-\delta|m|}w^m t_b^{\delta|m|}\tau^{-\tilde{N}_f|m|}x^{\mathfrak{N}|m|+\delta(|m|^2+2|m|l)}\\
&\quad\times\frac{\prod_{a=1}^{N_f}\left(t_bt_a^{-1}x^{2|m|+2l+2};x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(t_b^{-1}t_ax^{-2l};x^2\right)'_\infty}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_b^{-1}\tilde{t}_a^{-1}\tau^{-2}x^{-2l+2};x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2x^{2|m|+2l};x^2\right)_\infty}.
\end{aligned}
\tag{3.13}
$$

where the prime $'$ for the q-Pochhammer symbol means that the zero factor of the q-Pochhammer symbol which arises when a equals to b is dropped. We have defined $\mathfrak{N}$ and δ such that

$$
N_f=\mathfrak{N}+\delta,\qquad \tilde{N}_f=\mathfrak{N}-\delta. \tag{3.14}
$$

Carefully reorganizing the expression in (3.13) as explained in the appendix in detail, we have the factorized form of the index. Here we only introduce the main idea. The expression in (3.13) turns out to be symmetric under $|m|+l\leftrightarrow l$. This fact lead us to define $n\equiv l+\frac{|m|}{2}+\frac{m}{2}$ and $\bar{n}\equiv l+\frac{|m|}{2}-\frac{m}{2}$. Then the sum over $\{|m|+l,l\}$, $\sum_{m\in\mathbb{Z}}\sum_{l=0}^{\infty}$ is replaced by the sum over $\{n,\bar{n}\}$, $\sum_{n=0}^{\infty}\sum_{\bar{n}=0}^{\infty}$. Then the index is written as[5]

$$
\begin{aligned}
&I^{N_f>\tilde{N}_f}(x,t,\tilde{t},\tau,w)\\
&=\sum_{b=1}^{N_f}\left[\frac{\prod_{a=1(\neq b)}^{N_f}\left(t_bt_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_\infty}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_b^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\prod_{a=1(\neq b)}^{N_f}\left(t_b^{-1}t_a;x^2\right)_\infty}\right]\\
&\quad\times\left[\sum_{n=0}^{\infty}(-1)^{-\delta n}w^n t_b^{\delta n}\tau^{-\tilde{N}_fn}x^{\mathfrak{N}n+\delta n^2}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_n}{\prod_{a=1}^{N_f}\left(t_bt_a^{-1}x^2;x^2\right)_n}\right]\\
&\quad\times\left[\sum_{\bar{n}=0}^{\infty}(-1)^{-\delta\bar{n}}w^{-\bar{n}} t_b^{\delta\bar{n}}\tau^{-\tilde{N}_f\bar{n}}x^{\mathfrak{N}\bar{n}+\delta\bar{n}^2}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_{\bar{n}}}{\prod_{a=1}^{N_f}\left(t_bt_a^{-1}x^2;x^2\right)_{\bar{n}}}\right],
\end{aligned}
\tag{3.15}
$$

or more concisely

$$
I^{N_f>\tilde{N}_f}(x,t,\tilde{t},\tau,w)=\sum_{b=1}^{N_f}Z^b_{\text{pert}}\left(x,t,\tilde{t},\tau\right)Z^b_{\text{vortex}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right)Z^b_{\text{anti}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right) \tag{3.16}
$$

[5] Eq. (3.15) was also obtained in [7] for $N_f=\tilde{N}_f=2$ in terms of the basic hypergeometric series ${}_2\phi_1$. They also obtained the expression with $w=1$ for general $N_f=\tilde{N}_f$.

where $\mathfrak{w} = (-1)^{-\delta}(-w)$. Note that the condition that $N_f + \tilde{N}_f$ is an even integer implies that $(-1)^{-\delta}$ is a well-defined sign factor; i.e. it is always real valued. The first component Z^b_{pert}, which we call the perturbative part, is given by

$$Z^b_{\text{pert}}\left(x,t,\tilde{t},\tau\right) = \frac{\prod_{a=1(\neq b)}^{N_f}\left(t_b t_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_\infty}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_b^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\prod_{a=1(\neq b)}^{N_f}\left(t_b^{-1}t_a;x^2\right)_\infty}. \tag{3.17}$$

If we think of analytic continuation of q-Pochhammer symbol, Z^b_{pert} is also factorized as follows:

$$Z^b_{\text{pert}}\left(x,t,\tilde{t},\tau\right) = Z^b_{\text{1-loop}}\left(x,t,\tilde{t},\tau\right) Z^b_{\text{1-loop}}\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1}\right) \tag{3.18}$$

where the 1-loop contribution $Z^b_{\text{1-loop}}$ is given by

$$Z^b_{\text{1-loop}} = \frac{\prod_{a=1(\neq b)}^{N_f}\left(t_b t_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_\infty}. \tag{3.19}$$

In addition, the second and the third components, which we call the vortex partition function and the antivortex partition function respectively, are given by

$$\begin{aligned} Z^b_{\text{vortex}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right) &= \sum_{n=0}^{\infty}\mathfrak{w}^n\mathfrak{I}^b_n\left(x,t,\tilde{t},\tau\right), \\ Z^b_{\text{anti}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right) &= \sum_{n=0}^{\infty}\mathfrak{w}^{-n}\mathfrak{I}^b_n\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1}\right) \end{aligned} \tag{3.20}$$

where

$$\begin{gathered} \mathfrak{I}^b_n\left(x,t,\tilde{t},\tau\right) = \prod_{k=1}^{n}\frac{\prod_{a=1}^{\tilde{N}_f}2\sinh\frac{-i\tilde{M}_a-iM_b-2i\mu+2\gamma(k-1)}{2}}{2\sinh\gamma(k-1-n)\prod_{a=1(\neq b)}^{N_f}2\sinh\frac{iM_a-iM_b+2\gamma k}{2}}, \\ t=e^{iM}, \qquad \tilde{t}=e^{i\tilde{M}}, \qquad \tau=e^{i\mu}, \qquad x=e^{-\gamma}. \end{gathered} \tag{3.21}$$

They correspond to the $\mathcal{N}=2$ vortex partition function on $\mathbb{R}^2\times S^1$. The vortex partition function on $\mathbb{R}^2\times S^1$ was considered in [20, 29]. In [29], vortex quantum mechanics was considered and the 1-loop contribution, which is the vortex zero sector, was not worked out. On the other hand, the 1-loop contribution was checked in the other example [20] by matching to the 2d result. We will see that our vortex partition function as well as the 1-loop contribution matches the known result.

The index for $N_f < \tilde{N}_f$ is simply obtained by interchanging $t_a \leftrightarrow \tilde{t}_a$ as well as $N_f \leftrightarrow \tilde{N}_f$.

For $N_f = \tilde{N}_f$, $A_n(m)$ has the finite value at $z=0$ and $z=\infty$ which is

$$\prod_{a=1}^{N_f}\prod_{k=0}^{n-1}t_a^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2, \tag{3.22}$$

which goes to zero as $n\to\infty$ assuming $|t_a^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2|<1$, which is compatible with the original ranges of parameters. Thus we have the simple poles of $z=0$ and $z=\infty$ in the

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index formula (3.1). In the limit $n \to \infty$ the residues at $z = 0$, $z = \infty$ are zero. Thus we can simply sum over either the poles outside of the unit circle or poles inside of the unit circle. The previous results for $N_f \neq \tilde{N}_f$ still holds for $N_f = \tilde{N}_f$. We will sometimes omit the superscript $N_f \geq \tilde{N}_f$ of $I^{N_f \geq \tilde{N}_f}(x,t,\tilde{t},\tau,w)$ when we consider $N_f \geq \tilde{N}_f$ cases.

One can also check the result is reduced to the known result of 2d partition function of $\mathcal{N} = 2$ theories in the small radius limit of S^1 [21, 22]. The same is true of non-abelian cases, which we will summarize in the next subsection. In the evaluation of the non-abelian index, one can follow the similar limit procedure to the above.

3.2 Factorization: summary of U(N) cases

Now we summarize the factorized index formula for non-abelian cases in the presence of Chern-Simons terms. The superconformal index in the presence of nonzero CS term is written as

$$\begin{aligned} &I(x,t,w) \\ &= \sum_{m\in\mathbb{Z}^N/S_N} \oint \left(\prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j}\right) \frac{1}{|\mathcal{W}_m|} w^{\sum_j m_j} e^{-S_{CS}(a,m)} Z_{\text{gauge}}(x,z,m) \prod_{\Phi} Z_\Phi(x,t,z,m) \end{aligned} \tag{3.23}$$

where $S_{CS}^{(0)}(a,m) = i\sum_{\rho\in R_F} \kappa\rho(m)\rho(a)$. The CS term with level κ induces the classical action term in the path integral. It leads to the pole at $z_i = 0$ or $z_i = \infty$ according to the sign of κm. As shown in the appendix, one can show that the residues at these poles are zero. The contour integral over the holonomy variables of the gauge group can be written as

$$\begin{aligned} &I\left(x,t,\tilde{t},\tau,w\right) \\ &= \sum_{\sigma\in S_{N_f}/(S_N\times S_{N_f-N})} Z_{\text{pert}}\left(x,\sigma(t),\tilde{t},\tau\right) Z_{\text{vortex}}\left(x,\sigma(t),\tilde{t},\tau,\mathfrak{w}\right) Z_{\text{anti}}\left(x,\sigma(t),\tilde{t},\tau,\mathfrak{w}\right) \end{aligned} \tag{3.24}$$

where σ is an element of the quotient group $S_{N_f}/(S_N \times S_{N_f-N})$ where S_{N_f} is the symmetric group of degree N_f; S_N and S_{N_f-N} are its subgroups whose elements are the permutations of the first N elements and the last $N_f - N$ elements respectively. The fugacity for vorticity $\mathfrak{w}$ is related to the chemical potential w for U$(1)_T$ by $\mathfrak{w} = (-1)^{-\kappa-\delta}(-w)$. Note that the quantization condition of the effective CS level: $\kappa + (N_f + \tilde{N}_f)/2 \in \mathbb{Z}$ implies that $\mathfrak{w} = (-1)^{-\kappa-\delta}(-w)$ has a well-defined sign; i.e., $(-1)^{-\kappa-\delta}$ is always real valued. The perturbative part is given by

$$Z_{\text{pert}}\left(x,t,\tilde{t},\tau\right) = Z_{1-\text{loop}}\left(x,t,\tilde{t},\tau\right) Z_{1-\text{loop}}\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1}\right) \tag{3.25}$$

where

$$Z_{1-\text{loop}}\left(x,t,\tilde{t},\tau\right) = \left(\prod_{i<j}^{N} 2\sinh\frac{1}{2}\left(iM_i - iM_j\right)\right)\left(\prod_{j=1}^{N} \frac{\prod_{a=1(\neq j)}^{N_f}\left(t_j t_a^{-1} x^2; x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_j \tilde{t}_a \tau^2; x^2\right)_\infty}\right) \tag{3.26}$$

while the vortex and antivortex partition function are given by

$$
\begin{aligned}
Z_{\text{vortex}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right) &= \sum_{n=\vec{0}}^{\vec{\infty}} \mathfrak{w}^{\sum_j n_j} \mathfrak{I}_{(n_j)}\left(x,t,\tilde{t},\tau\right), \\
Z_{\text{anti}}\left(x,t,\tilde{t},\tau,\mathfrak{w}\right) &= \sum_{n=\vec{0}}^{\vec{\infty}} \mathfrak{w}^{-\sum_j n_j} \mathfrak{I}_{(n_j)}\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1}\right)
\end{aligned} \tag{3.27}
$$

where

$$
\begin{aligned}
&\mathfrak{I}_{(n_j)}\left(x,t,\tilde{t},\tau\right) \\
&= e^{-S_0} \prod_{j=1}^{N} \prod_{k=1}^{n_j} \frac{\prod_{a=1}^{\tilde{N}_f} 2\sinh \frac{-i\tilde{M}_a - iM_j - 2i\mu + 2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N} 2\sinh \frac{iM_i - iM_j + 2\gamma(k-1-n_i)}{2}\right)\left(\prod_{a=N+1}^{N_f} 2\sinh \frac{iM_a - iM_j + 2\gamma k}{2}\right)}
\end{aligned} \tag{3.28}
$$

and $e^{-S_0} = e^{\kappa \sum_j (iM_j n_j + i\mu n_j - \gamma n_j^2)}$, which appears due to the nonzero CS term.

3.3 Factorization of mirror of one free chiral

Let us consider a U(1) theory with a single chiral multiplet in the fundamental representation. If one also turns on the level $-\frac{1}{2}$ CS interaction and the fixed background magnetic flux m' corresponding to the topological global symmetry $\mathrm{U}(1)_T$, the index is given by

$$
I(x,w',m') = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w'^m z^{m'} x^{\frac{1}{2}|m|} (-z)^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1} x^{|m|+2-\Delta_\phi+2k}}{1 - z x^{|m|+\Delta_\phi+2k}}. \tag{3.29}
$$

This can be written as

$$
I(x,w,m') = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m z^{m'} (-x)^{-\frac{1}{2}(m-|m|)} z^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1} x^{|m|+2+2k}}{1 - z x^{|m|+2k}} \tag{3.30}
$$

where we redefined $w' x^{\frac{1}{2}} \to w$. A factor $x^{\frac{1}{2}}$ is additionally absorbed to w for later convenience.

One may consider its mirror description, a single free chiral theory with the level $\frac{1}{2}$ CS interaction. As introduced in the previous section the index for the mirror description is given by [23]

$$
\mathcal{I}_\Delta(m;q,\zeta) = \left(-q^{\frac{1}{2}}\right)^{\frac{1}{2}(m+|m|)} \zeta^{-\frac{1}{2}(m+|m|)} \prod_{r=0}^{\infty} \frac{1 - q^{r+\frac{1}{2}|m|+1} \zeta^{-1}}{1 - q^{r+\frac{1}{2}|m|} \zeta} \tag{3.31}
$$

where m and ζ are magnetic flux and the Wilson line of the fixed background U(1) vector field coupling to the conserved current of the U(1) flavor symmetry. Again the parameters are identified with ours as follows:

$$
m = m', \qquad q = x^2, \qquad \zeta = w.
$$

It was argued in [23] that the index (3.30) agrees with (3.31).

Here we revisit the index agreement using the factorized form of the index. The factorized form of (3.30) is given as follows:[6]

$$
\begin{aligned}
&I(x,w,m') \qquad\qquad (3.32)\\
&=\sum_{n=0}^{\infty}\left[w^n x^{-m'n-\frac{n(n+1)}{2}}\left(\prod_{k=1}^{n}2\sinh\gamma k\right)^{-1}\right]\times\sum_{\bar{n}=0}^{\infty}\left[(-w)^{-\bar{n}}x^{-m'\bar{n}+\frac{\bar{n}(\bar{n}+1)}{2}}\left(\prod_{k=1}^{\bar{n}}2\sinh\gamma k\right)^{-1}\right].
\end{aligned}
$$

As before the first summation corresponds to the vortex partition function while the second summation corresponds to the antivortex partition function. One may check that the vortex partition function can be written as a Plethystic exponential:

$$
\begin{aligned}
Z_{\text{vortex}}(x,w,m') &\equiv \sum_{n=0}^{\infty}\left[w^n x^{-m'n-\frac{n(n+1)}{2}}\left(\prod_{k=1}^{n}2\sinh\gamma k\right)^{-1}\right]\\
&= \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{w^n x^{-m'n}}{1-x^{2n}}\right].
\end{aligned} \tag{3.33}
$$

Likewise, the antivortex partition function also has the Plethystic exponential form:

$$
\begin{aligned}
Z_{\text{anti}}(x,w,m') &\equiv \sum_{\bar{n}=0}^{\infty}\left[(-w)^{-\bar{n}}x^{-m'\bar{n}+\frac{\bar{n}(\bar{n}+1)}{2}}\left(\prod_{k=1}^{\bar{n}}2\sinh\gamma k\right)^{-1}\right]\\
&= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\frac{w^{-n}x^{n(-m'+2)}}{1-x^{2n}}\right].
\end{aligned} \tag{3.34}
$$

On the other hand, it was pointed out in [23] that the free chiral index (3.31) has a more concise form as follows:

$$
\mathcal{I}_{\Delta}(m;q,\zeta) = \prod_{r=0}^{\infty}\frac{1-q^{r-\frac{1}{2}m+1}\zeta^{-1}}{1-q^{r-\frac{1}{2}m}\zeta}. \tag{3.35}
$$

One can see that the denominator, which comes from the scalar, is exactly the vortex partition function while the numerator, which comes from the fermion, is the antivortex partition function:

$$
\begin{aligned}
Z_{\text{vortex}}(q^{\frac{1}{2}},\zeta,m) &= \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{\zeta^n q^{-\frac{1}{2}mn}}{1-q^n}\right] = \prod_{r=0}^{\infty}\frac{1}{1-q^{r-\frac{1}{2}m}\zeta},\\
Z_{\text{anti}}(q^{\frac{1}{2}},\zeta,m) &= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\zeta^{-n}q^{(-\frac{1}{2}m+1)n}}{1-q^n}\right] = \prod_{r=1}^{\infty}1-q^{r-\frac{1}{2}m+1}\zeta^{-1}.
\end{aligned} \tag{3.36}
$$

[6]Compared with the general formula, the power in x has $x^{-\frac{n(n+1)}{2}}$ while the general formula appearing at the appendix has $x^{-\frac{n^2}{2}}$. The reason is that (3.30) matches with the free theory with zero R-charge for the free chiral while the standard factorized formula matches with the free chiral with canonical R-charge. Two expressions are related by the shift $w \to wx^{-\frac{1}{2}}$.

3.4 Relation to topological open string amplitude

The form of the vortex partition function has the close relation to the topological open string amplitude. As the first example we consider the vortex partition function for U(1) gauge theory with Chern-Simons level $-1/2$ with a single chiral multiplet. As already explained at the previous subsection, the vortex partition function is given by

$$\begin{aligned}&\sum_{n=0}^{\infty} w^n x^{-\frac{n(n+1)}{2}} \prod_{k=1}^{n} (2\sinh\gamma k)^{-1} \\ &= \sum_{n=0}^{\infty} \frac{w^n}{(1-e^{-2\gamma})(1-e^{-4\gamma})\cdots(1-e^{-2n\gamma})}\end{aligned} \tag{3.37}$$

with $x = e^{-\gamma}$. Now consider the topological open string for a Lagrangian brane in C^3 as explained in [24, 30]:[7]

$$\begin{aligned}Z_{\text{brane}}(z,t,q) &= \sum s_{\mu^t}(z) C_{00\mu}(t,q) \\ &= \sum_{n=0}^{\infty} \frac{t^{\frac{n}{2}} z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{i=0}^{\infty} \frac{1}{1-q^i t^{\frac{1}{2}} z}.\end{aligned} \tag{3.38}$$

This coincides with the vortex partition function if we identify $z = w, t = 1, q = e^{-2\gamma}$. To compare with the index of the free chiral field with the canonical R-charge we need the shift $z \to z\sqrt{q}$. Then

$$|Z_{\text{brane}}|^2 = \frac{Z_{\text{brane}}(z,q)}{Z_{\text{brane}}(\bar{z},q)} = \frac{\prod_{n=1}^{\infty}(1-zq^{n-\frac{1}{2}})}{\prod_{n=1}^{\infty}(1-\bar{z}q^{n-\frac{1}{2}})} \tag{3.39}$$

which coincides with the free chiral index as explained in [24]. To compare with the free chiral index of arbitrary R-charge or its mirror dual, one simply change the open string modulus $z \to zq^{\alpha}$ for a suitable α. Note that it's crucial to have Chern-Simons term to match the vortex partition function with the topological open string amplitude.

For this simple example, we generalize the matching between the homological vortex partition function of two dimensions and the topological open string partition function to the full 3d K-theoretic vortex partition function.[8] In [30], many more examples of the matching between the 2d vortex partition function and the topological open string were found. We expect that this surely lifts to the matching between the 3d vortex partition function and the topological open string. Furthermore in the homological version, 2d vortex theory is realized as the surface operator of 4d gauge theories. We expect that this lifts to the 3d defect operator in 5d superconformal field theories. We will work out a simple example in the next subsection.

As a next example, we can consider U(1) gauge theory with one fundamental and one antifundamental chiral theory. As will be shown in the subsection 4.1, the superconformal

[7]We use the refined vertex formalism to write down the topological string partition function. For the notation, please refer to [31]. s_μ in the formula denotes the Schur function.

[8]This relation is parallel to 4d Nekrasov partition function and its 5d version.

index of the theory is given by

$$I^{N=N_f=1} = Z_{\rm pert}^{N=N_f=1} \times Z_{\rm vortex}^{N=N_f=1} \times Z_{\rm anti}^{N=N_f=1} \tag{3.40}$$

where

$$\begin{aligned}
Z_{\rm pert}^{N=N_f=1} &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{\tau^{2n} - \tau^{-2n} x^{2n}}{1 - x^{2n}}\right], \\
Z_{\rm vortex}^{N=N_f=1} &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{(\tau^{-n} - \tau^n) x^n}{1 - x^{2n}}\right], \\
Z_{\rm anti}^{N=N_f=1} &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} w^{-n} \frac{(\tau^{-n} - \tau^n) x^n}{1 - x^{2n}}\right].
\end{aligned} \tag{3.41}$$

Here N denotes the rank of the gauge group while $N_f = \tilde{N}_f$ denotes the number of fundamental and antifundamental multiplets. Note that the vortex and antivortex parts as well as perturbative part are given by the free chiral indices. Hence this U(1) theory can again be written in terms of topological open string amplitude.

If one considers the more general U(1) non-chiral theory with $N_f = \tilde{N}_f = N$, the index can be written as

$$I\left(x, t, \tilde{t}, \tau, w\right) = \sum_{b=1}^{N} \left(Z_{1-\rm loop}^b Z_{\rm vortex}^b\right) \times \left(Z_{1-\rm loop, anti}^b Z_{\rm anti}^b\right) \tag{3.42}$$

where

$$\begin{aligned}
Z_{1-\rm loop}^b\left(x, t, \tilde{t}, \tau\right) &= \prod_{k=1}^{\infty} \frac{\prod_{a=1(\neq b)}^{N} 1 - t_b t_a^{-1} x^{2k}}{\prod_{a=1}^{N} 1 - t_b \tilde{t}_a \tau^2 x^{2(k-1)}}, \\
Z_{1-\rm loop, anti}^b\left(x, t, \tilde{t}, \tau\right) &\equiv Z_{1-\rm loop}^b\left(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1}\right) = \prod_{k=1}^{\infty} \frac{\prod_{a=1}^{N} 1 - t_b^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{2k}}{\prod_{a=1(\neq b)}^{N} 1 - t_b^{-1} t_a x^{2(k-1)}}, \\
Z_{\rm vortex}^b\left(x, t, \tilde{t}, \tau, -w\right) &= \sum_{n=0}^{\infty} \left[w \tau^{-N} x^N\right]^n \prod_{k=1}^{n} \frac{1 - t_b \tilde{t}_b \tau^2 x^{2(k-1)}}{1 - x^{2k}} \prod_{a=1(\neq b)}^{N} \frac{1 - t_b \tilde{t}_a \tau^2 x^{2(k-1)}}{1 - t_b t_a^{-1} x^{2k}}, \\
Z_{\rm anti}^b\left(x, t, \tilde{t}, \tau, -w\right) &= \sum_{\bar{n}=0}^{\infty} \left[w^{-1} \tau^{-N} x^N\right]^{\bar{n}} \prod_{k=1}^{\bar{n}} \frac{1 - t_b \tilde{t}_b \tau^2 x^{2(k-1)}}{1 - x^{2k}} \prod_{a=1}^{N} \frac{1 - t_b \tilde{t}_a \tau^2 x^{2(k-1)}}{1 - t_b t_a^{-1} x^{2k}}.
\end{aligned} \tag{3.43}$$

The vortex partition function here is the same as that of [20]. In fact, with identifications

$$\begin{aligned}
t_b t_a^{-1} &= e^{-2\pi b D_{ab}}, \\
t_b \tilde{t}_a \tau^2 &= e^{-2\pi b C_{ab}}, \\
x^2 &= q, \\
w \tau^{-N} x^N &= z,
\end{aligned} \tag{3.44}$$

one can see that

$$Z_{\rm vortex}^b = Z_V^{(b)}, \qquad Z_{1-\rm loop}^b = Z_{1-\rm loop}^{(b)} \tag{3.45}$$

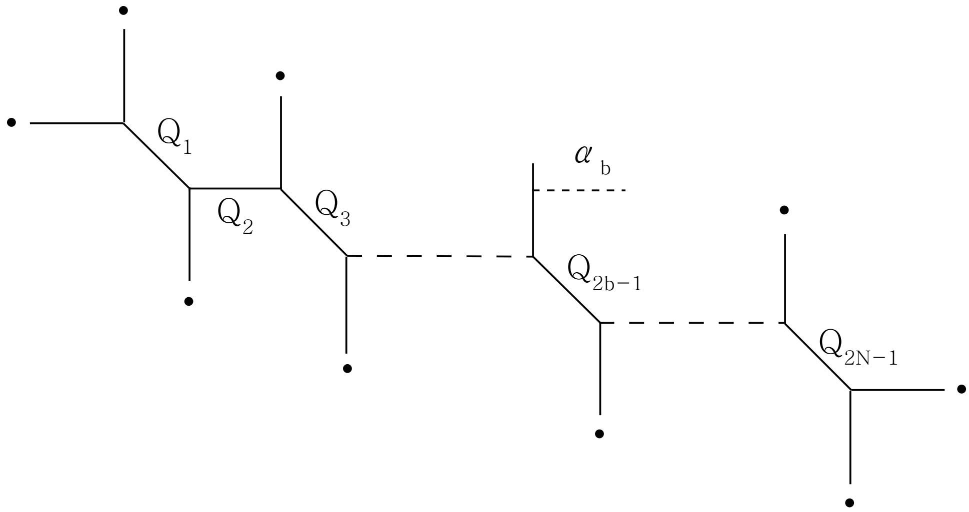

Figure 1. A strip geometry.

where $Z_V^{(b)}, Z_{1-\text{loop}}^{(b)}$ are the components of the partition function for the non-chiral theory given in [20].

As examined in [20] one can also check that the vortex partition function Z^b_{vortex} is exactly the same as the open topological string partition function on the Lagrangian brane placed at the b-th gauge leg of the toric diagram in figure 1, i.e., $\alpha_b \in \{\mathbf{1}^n | n = 0, 1\cdots\}$. The corresponding topological partition function is given by [20, 32]

$$Z^b_{\text{top}} = \sum_n A_n^{(b)} z^n, \tag{3.46}$$

$$A_n^{(b)} \equiv \frac{\mathcal{K}_{\substack{\bullet\cdots 1^n \cdots\bullet \\ \bullet\cdots\bullet}}}{\mathcal{K}_{\substack{\bullet\cdots\bullet \\ \bullet\cdots\bullet}}} = \frac{1}{\prod_{k=1}^n (1-q^k)} \frac{\prod_{a\geq b}\prod_{k=1}^n (1-Q_{\alpha_b\beta_a} q^{k-1}) \prod_{a<b}\prod_{k=1}^n (1-Q_{\beta_a\alpha_b} q^{-(k-1)})}{\prod_{a>b}\prod_{k=1}^n (1-Q_{\alpha_b\alpha_a} q^{k-1}) \prod_{a<b}\prod_{k=1}^n (1-Q_{\alpha_a\alpha_b} q^{-(k-1)})}, \tag{3.47}$$

where the Kähler parameters are defined by

$$\begin{aligned} Q_{\alpha_a\alpha_{a'}} &= \prod_{k=a}^{a'-1} Q_{2k-1}Q_{2k}, \\ Q_{\alpha_a\beta_{a'}} &= Q_{\alpha_a\alpha_{a'}} Q_{2a'-1}, \\ Q_{\beta_a\alpha_{a'}} &= Q_{\alpha_a\alpha_{a'}} Q_{2a-1}^{-1}. \end{aligned} \tag{3.48}$$

For a fixed b, if we identify the parameters as follows:

$$z\prod_{a<b} Q_{2a-1}^{-1} = w\tau^{-N}x^N, \tag{3.49}$$

$$\begin{aligned} Q_{\alpha_b\beta_a} &= t_b\tilde{t}_a\tau^2, & a \geq b \\ {Q_{\beta_a\alpha_b}}^{-1} &= t_b\tilde{t}_a\tau^2, & a < b \\ Q_{\alpha_b\alpha_a} &= t_b t_a^{-1}x^2, & a > b \\ {Q_{\alpha_a\alpha_b}}^{-1} &= t_b t_a^{-1}x^2, & a < b \end{aligned} \tag{3.50}$$

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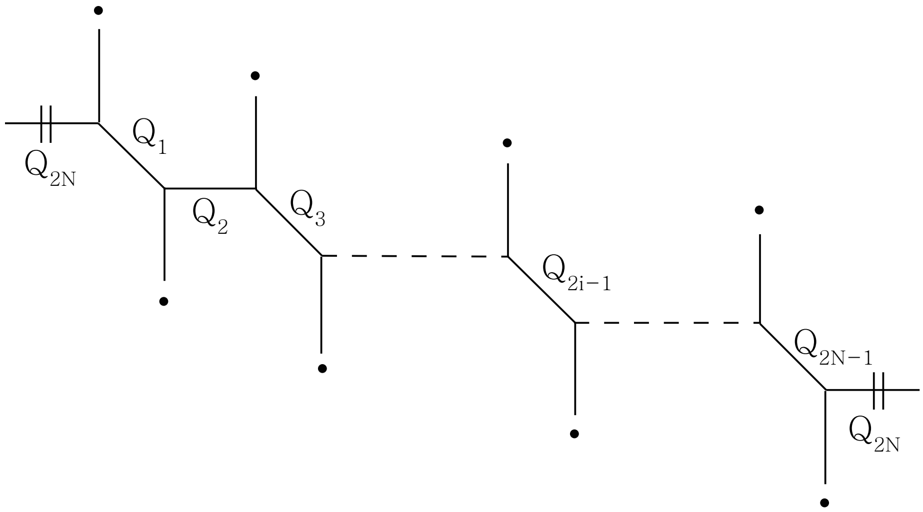

Figure 2. A necklace $\mathrm{U}(1)^N$ quiver theory.

one immediately sees that the topological partition function $Z^b_{\rm top}$ is the same as the vortex partition function for abelian theories, $Z^b_{\rm vortex}$.

3.5 The partition function for $\mathbf{U(1)^N}$ quiver theories and closed topological string

More interestingly the closed string geometry on the strip geometry considered in [20], for which α_b is now the trivial representation, has the close relation to the 5d partition function on $S^1\times S^4$. In this case the 5d gauge theory defined on the strip geometry is $\mathrm{U}(1)^N$ quiver theory.

If we consider first the closed string amplitude on the strip geometry it can be worked out using the refined topological vertex. In [33] a similar geometry given in figure. 2 was examined where the leftmost leg and the rightmost leg are identified. If we disconnect that leg we again obtain the strip geometry we are interested in. The closed string amplitude for figure. 2 is given by

$$
\begin{aligned}
&Z^{\rm inst}_L\left(\tilde{Q};q,t\right)\\
&=\sum_{\{\lambda_{2\alpha}\}}\prod_{\alpha=1}^{N}\frac{\tilde{Q}_{2\alpha}^{|\lambda_{2\alpha}|}\prod_{s\in\lambda_{2\alpha-2}}\left(1-\tilde{Q}_{2\alpha-1}q^{\ell_{2\alpha-2}(s)}t^{a_{2\alpha}(s)+1}\right)\prod_{s\in\lambda_{2\alpha}}\left(1-\tilde{Q}_{2\alpha-1}q^{-\ell_{2\alpha}(s)-1}t^{-a_{2\alpha-2}(s)}\right)}{\prod_{s\in\lambda_{2\alpha}}\left(1-q^{\ell_{2\alpha}(s)}t^{a_{2\alpha}(s)+1}\right)\left(1-q^{-\ell_{2\alpha}(s)-1}t^{-a_{2\alpha}(s)}\right)}
\end{aligned}
\tag{3.51}
$$

where $\tilde{Q}_\alpha=Q_\alpha\left(\frac{q}{t}\right)^{(-1)^{\alpha+1}\frac{1}{2}}$.[9] As in the other examples of geometric engineering, this closed topological string amplitude leads to the Nekrasov instanton partition function. In 5-dimensions full instanton partition function was not worked out for theories with

[9]If we consider the 2d partition $\lambda=\{\lambda_1\geq\lambda_2\cdots\}$, this can be represented by a Young diagram. We draw the λ_1 boxes on the leftmost column and λ_2 boxes on the next-leftmost column and so on. For an element $s=(i,j)\in\lambda$, $a(s)$ denotes the boxes on the right and $l(s)$ denotes the boxes on top, i.e., $a(i,j)=\lambda^t_j-i, l(i,j)=\lambda_i-j$. For more details, refer to [31].

bifundamental fields. However Nekrasov partition function of such quiver in four-dimension was worked out in [34]. One can see that this can be obtained from the closed string amplitude. The unrefined version of the amplitude is obtained by setting $t = q$:

$$\begin{aligned}
&Z_L^{\text{inst}}\left(\tilde{Q}_\alpha; q, q\right) \\
&= \sum_{\{\lambda_{2\alpha}\}} \prod_{\alpha=1}^{N} \tilde{Q}_{2\alpha}^{|\lambda_{2\alpha}|} \prod_{s\in\lambda_{2\alpha}} \frac{\left(1-\tilde{Q}_{2\alpha+1} q^{\ell_{2\alpha}(s)+a_{2\alpha+2}(s)+1}\right)\left(1-\tilde{Q}_{2\alpha-1} q^{-\ell_{2\alpha}(s)-a_{2\alpha-2}(s)-1}\right)}{\left(1-q^{\ell_{2\alpha}(s)+a_{2\alpha}(s)+1}\right)\left(1-q^{-\ell_{2\alpha}(s)-a_{2\alpha}(s)-1}\right)} \\
&= \sum_{\{\lambda_{2\alpha}\}} \prod_{\alpha=1}^{N} \left(\tilde{Q}_{2\alpha}\tilde{Q}_{2\alpha+1}^{1/2}\tilde{Q}_{2\alpha-1}^{1/2}\right)^{|\lambda_{2\alpha}|} q^{\sum_{s\in\lambda_{2\alpha}}[a_{2\alpha+2}(s)-a_{2\alpha-2}(s)]/2} \\
&\quad \times \prod_{s\in\lambda_{2\alpha}} \frac{\sinh\frac{\beta}{2}[\hbar h_{2\alpha,2\alpha+2}(s)+M_{2\alpha+1}]\sinh\frac{\beta}{2}[\hbar h_{2\alpha,2\alpha-2}(s)-M_{2\alpha-1}]}{\sinh^2\frac{\beta}{2}\hbar h_{2\alpha,2\alpha}(s)}
\end{aligned} \tag{3.52}$$

where we define $\tilde{Q}_\alpha \equiv e^{-\beta M_\alpha}$, $q \equiv e^{-\beta\hbar}$. M_α and $\hbar$ are the parameters relevant in four-dimensions. $h_{\alpha,\beta}(s)$ is the hook length defined by $h_{\alpha,\beta}(s) = \ell_\alpha(s) + a_\beta(s) + 1$. In order to obtain the four-dimensional partition function, one would take $\beta \to 0$:

$$\begin{aligned}
&Z_L^{\text{inst}}\left(\tilde{Q}_\alpha; q, q\right)\Big|_{\beta\to 0} \\
&= \sum_{\{\lambda_{2\alpha}\}} \prod_{\alpha=1}^{N} \prod_{s\in\lambda_{2\alpha}} \frac{[\hbar h_{2\alpha,2\alpha+2}(s)+M_{2\alpha+1}][\hbar h_{2\alpha,2\alpha-2}(s)-M_{2\alpha-1}]}{[\hbar h_{2\alpha,2\alpha}(s)]^2},
\end{aligned} \tag{3.53}$$

which is the same as the partition function for quiver theories given in [34] with identifications $M_{2\alpha+1} = a_{2\alpha} - a_{2\alpha+2} + m$ where m denotes the mass of the bifundamentals.[10] Thus it is quite reasonable that the above topolgical string amplitude gives the 5d Nekrasov partition function for $\mathrm{U}(1)^N$ quiver theories. One can cut the leftmost leg and the the rightmost leg, which are identified, by taking $Q_{2N} \to 0$. This gives rise to the closed string amplitude for the strip geometry we originally considered. The amplitude is given by

$$\begin{aligned}
&Z_L^{\text{inst}}\left(\tilde{Q}; q, t\right)\Big|_{Q_{2N}\to 0} \\
&= \sum_{\{\lambda_{2\alpha}\}} \prod_{\alpha=1}^{N-1} \tilde{Q}_{2\alpha}^{|\lambda_{2\alpha}|} \prod_{s\in\lambda_2}\left(1-\tilde{Q}_1 q^{-\ell_2(s)-1} t^{-a_\emptyset(s)}\right) \prod_{s\in\lambda_{2N-2}}\left(1-\tilde{Q}_{2N-1} q^{\ell_{2N-2}(s)} t^{a_\emptyset(s)+1}\right) \\
&\quad \times \frac{\prod_{\alpha=2}^{N-1}\prod_{s\in\lambda_{2\alpha-2}}\left(1-\tilde{Q}_{2\alpha-1} q^{\ell_{2\alpha-2}(s)} t^{a_{2\alpha}(s)+1}\right)\prod_{s\in\lambda_{2\alpha}}\left(1-\tilde{Q}_{2\alpha-1} q^{-\ell_{2\alpha}(s)-1} t^{-a_{2\alpha-2}(s)}\right)}{\prod_{\alpha=1}^{N-1}\prod_{s\in\lambda_{2\alpha}}\left(1-q^{\ell_{2\alpha}(s)} t^{a_{2\alpha}(s)+1}\right)\left(1-q^{-\ell_{2\alpha}(s)-1} t^{-a_{2\alpha}(s)}\right)}
\end{aligned} \tag{3.54}$$

where $a_\emptyset(s = (i, j)) = -i$.

One might wonder since abelian theory is trivial in 5d so that its nonperturbative part is also trivial. However abelian theories can have small instantons and it's quite subtle how to include them. For example if we consider 5d U(1) $\mathcal{N} = 2^*$ theory and if we define its

[10] The hook length $h_{\alpha,\beta}(s)$ is denoted by $\ell_{\alpha\beta}(s)$ in [34].

nonperturbative completion to give the Nekrasov partition function, 5d partition function of U(1) $\mathcal{N} = 2^*$ theory on S^5 gives the index of single M5 brane in 6d [35–37].

The general structure of the 5d index worked out at [38] has the structure

$$\int da PE(f_{\mathrm{mat}}(x, y, e^{ia}, t) + f_{\mathrm{vec}}(x, y, e^{ia}))|I_{\mathrm{inst}}(x, y, e^{ia}, t, q)|^2 \tag{3.55}$$

where da is the Haar measure for the gauge group, PE denotes Plethystic exponential, which gives the one-loop determinant and I_{inst} is Nekrasov instanton partition function. x, y is the chemical potential for Cartans of Lorentz symmetry $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \subset \mathrm{SO}(5)$, $x = e^{-\gamma_1}, y = e^{-\gamma_2}$ and t is the usual chemical potential for the flavor symmetry. Here $\mathrm{SU}(2)_1$ is also twisted with $\mathrm{SU}(2)_R$ R-symmetry. Finally q is introduced to track the instanton number. Thus for $\mathrm{U}(1)^N$ quiver 5d partition function has the same form where I_{inst} is now identified with closed string amplitude. This is consistent with the recent proposal by [24].

In addition, the perturbative part is also factorized and the whole index can be written as

$$I = \int da|I_{\mathrm{pert}}(x, y, e^{ia}, t)I_{\mathrm{inst}}(x, y, e^{ia}, t, q)|^2. \tag{3.56}$$

Now let's check if perturbative part matches. In the refined vertex formalism, the preturbative part is automatically built in. In our case, it is given by [33]

$$Z_{\mathrm{pert}} = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\sum_\alpha \tilde{Q}^n_{2\alpha-1}}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}\right). \tag{3.57}$$

This should match the one-loop determinant of the bifundamental fields. The general expression for the one-loop determinant for the matter fields are given by

$$f_{\mathrm{mat}}(x, y, a) = \frac{x}{(1 - xy)(1 - x/y)}\sum_w (e^{-i\vec{w}\cdot\vec{\alpha}} + e^{i\vec{w}\cdot\vec{\alpha}}) \tag{3.58}$$

where w is the weight of the representation. We suppress the chemical potential of the flavor symmetry. We can see that it has the explicit factorized structure and we can just look for $e^{-i\vec{w}\cdot\vec{\alpha}}$ part to compare with the topological string expression. For bifundamentals, we have

$$f_{\mathrm{pert}} = \frac{x}{(1 - xy)(1 - x/y)}(e^{-i(\alpha_1-\alpha_2)} + e^{-i(\alpha_2-\alpha_3)} + \cdots + e^{-i(\alpha_N-\alpha_1)}). \tag{3.59}$$

This coincides with the corresponding topological string expression if we identify

$$x = \sqrt{tq}, \qquad y = \sqrt{\frac{q}{t}}, \qquad \tilde{Q}_{2k-1} = e^{-i(\alpha_k-\alpha_{k-1})}. \tag{3.60}$$

Furthermore since the open string amplitude was obtained by introducing the Lagrangian brane in the strip geometry, and this leads to the 3d index of the nontrivial SCFT, it is natural to expect that introducing Lagrangian brane corresponds to introducing the surface operator in 5d $\mathrm{U}(1)^N$ quiver theory. This is the T-dual of the Hanany-Witten set up for the surface operator in 4-dimension so we expect this is the 3d defect of the 5d theory.

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This lead to an interesting lesson that apparently trivial 5d theory[11] can have nontrivial defect operator, which corresponds to nontrivial 3d SCFT. Furthermore we saw that the partition function of 5d theory with the defect operator matches the closed+open string amplitude since the the vertex partition function appearing at [20], is normalized by the closed string partition function. The vortex partition function has the structure

$$Z^{b}_{\text{vortex}} = \sum_{n} z^{n} \frac{K(1^{n})}{K(0)} \tag{3.61}$$

where $K(1^n)$ is the string partition function with the insertion of the brane with the representation (1^n) while $K(0)$ denotes the string partition function with the trivial representation, i.e., the closed string partition function.

4 $\mathcal{N} = 2$ Seiberg-like dualities

4.1 Simple cases

In this section we consider Seiberg-like (or Aharony duality) for three dimensional $\mathrm{U}(N)$ gauge theories with $\mathcal{N} = 2$ supersymmetries proposed in [39]. The duality relates two gauge theories which we call the "original" theory and the "dual" theory. Two dual theories have different gauge groups and matter contents but they flow to the same theory in the infrared.

The original theory is a $\mathrm{U}(N)$ gauge theory which consists of N_f fundamental chiral multiplets Q_a and N_f anti-fundamental chiral multiplets $\tilde{Q}^a$ as well as $\mathrm{U}(N)$ vector multiplets. This theory has no superpotential. On the other hand, the dual theory is a $\mathrm{U}(N_f - N)$ gauge theory with N_f pairs of fundamental q^a and anti-fundamental $\tilde{q}_a$ chiral multiplets. In addition, the dual theory contains gauge singlet chiral multiplets, ${M_a}^b$ and $V_\pm$, and they couple to the charged matters through the superpotential, $W = q^a {M_a}^b \tilde{q}_b + V_+ \tilde{V}_- + V_- \tilde{V}_+$. Here $\tilde{V}_\pm$ are chiral superfields corresponding to monopole operators which parametrize the Coulomb branch of the dual theory. The global symmetry of both theories is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_A \times \mathrm{U}(1)_T$ where $\mathrm{SU}(N_f)^2$ is the flavor symmetry acting on the fundamental and anti-fundamental matters, $\mathrm{U}(1)_A$ is an axial symmetry rotating fundamental and anti-fundamental matters by the same phase and $\mathrm{U}(1)_T$ is a topological symmetry whose current is given by $*\mathrm{Tr}F$.

Under the duality, mesonic operators $Q_a \tilde{Q}^b$ and monopole operators with topological charges ± 1 of the original theory are mapped to singlet fields ${M_a}^b$ and $V_\pm$ of the dual theory, respectively. This duality map together with the superpotential W determines global charge assignment of chiral fields of the dual theory.

The superconformal indices for several dual pairs have been computed. The indices are expanded by conformal dimensions of BPS operators and show agreement between BPS spectra of two dual theories at some leading orders. Here we present factorized expressions of superconformal indices for simple cases that shows 3d Seiberg-like dualities in a clearer way.

[11] One way to see the 5d index computation of the this theory is to regard it as a twisted partition function on $S^1 \times S^4$.

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Let us first consider the U(1) gauge theory with $N_f = 1$ flavor which would give the simplest duality model. The proposed dual theory is the U(0) theory, i.e. non-gauge theory, with chiral multiplets M and $V_\pm$ with a superpotential $W = -V_+V_-M$. After vortex-antivortex factorization, the superconformal index of the original theory is given by

$$I^{N=N_f=1} = Z_{\text{pert}}^{N=N_f=1} \times Z_{\text{vortex}}^{N=N_f=1} \times Z_{anti}^{N=N_f=1} \,. \tag{4.1}$$

Firstly the perturbative part is written as

$$Z_{\text{pert}}^{N=N_f=1} = \prod_{l=0}^{\infty} \frac{1-\tau^{-2}x^{2l+2}}{1-\tau^2 x^{2l}} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{\tau^{2n} - \tau^{-2n}x^{2n}}{1-x^{2n}}\right]. \tag{4.2}$$

where we rewrite the expression as a Plethystic exponential. The vortex index is the sum over all vortex number n's. After some calculation it can be written as a Plethystic exponential form:

$$Z_{\text{vortex}}^{N=N_f=1} = \sum_{n=0}^{\infty} (-w)^n \prod_{k=1}^{n} \frac{\tau^{-1}x^{-(k-1)} - \tau x^{k-1}}{x^{-(k-1-n)} - x^{k-1-n}} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{(\tau^{-n} - \tau^n)x^n}{1-x^{2n}}\right] \tag{4.3}$$

where we used the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{4.4}$$

The anti-vortex index is easily obtained from the vortex index by replacing w to w^{-1}. Therefore, it turns out that the superconformal index of $N = N_f = 1$ theory can be rewritten as a simple Plethystic exponential form

$$\begin{aligned} I^{N=N_f=1} &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(x^n, \tau^n, w^n)\right], \\ f(x,\tau,w) &= \frac{\tau^2 x^{2\Delta_Q} - \tau^{-2}x^{2-2\Delta_Q}}{1-x^2} + \frac{\tau^{-1}x^{1-\Delta_Q} - \tau x^{1+\Delta_Q}}{1-x^2}(w + w^{-1}), \end{aligned} \tag{4.5}$$

where we restored R-charge Δ_Q of the chiral boson Q of the original theory. Amazingly this form of the index is exactly the same as the superconformal index of the dual theory. The function f is identical to the single letter index in the dual theory. As the chiral field M of the dual theory is identified with the meson operator $Q\tilde{Q}$ of the original theory, its R-charge and $\text{U}(1)_A$ charge are $2\Delta_Q$ and $+2$ respectively, and therefore the letter index of M is given by the first term of f. The second term of f comes from the letter contribution of dual chiral multiplets $V_\pm$ which is mapped to monopole operators with $\text{U}(1)_T$ charges ± 1. In general, zero point energies and $\text{U}(1)_A$ charges of monopole operators with GNO charge $(\pm 1, 0, \cdots, 0)$ for $N = N_f$ theories are

$$\begin{aligned} \epsilon_0 &= N_f(1-\Delta_Q) - (N-1) = 1 - N\Delta_Q \,, \\ b_{\text{U}(1)_A} &= -N_f = -N \end{aligned} \tag{4.6}$$

from (2.4). One can then see that the single letter index of $V_\pm$ for $N = N_f = 1$ case agrees with the second term of f.

This theory is known to be mirror-dual to the XYZ theory [40]. The chiral fields $M, V_\pm$ in the dual theory correspond to X, Y, Z fields of the superpotential $W = -MV_+V_-$, so that they should have R-charges $\Delta_M = \Delta_V = \frac{2}{3}$. As shown in [41], the R-charge of the original chiral field is determined to be $\Delta_Q = \frac{1}{3}$ in IR, and therefore one can see from (4.5) that the dual chiral fields have the correct R-charges in the IR fixed point.

More generally, one can express the superconformal indice for U(N) gauge theories with $N_f = N$ fundamental and anti-fundamental matters in duality manifest forms using the factorization. The dual theory is a U(0) theory with chiral multiplet ${M_a}^b$ and $V_\pm$. For $N_f = N$, the dual theory is known to have a superpotential of the form $W = -V_+V_-\det(M)$ [40]. The vortex index reduces to Plethystic exponential forms

$$\begin{aligned} Z^{N=N_f}_{\text{vortex}} =& \sum_{\vec{n}=0}^{\infty}(-w)^n \prod_{i,j}^{N}\prod_{k=1}^{n_i} \frac{t_i^{-1/2}\tilde{t}_j^{-1/2}\tau^{-1}x^{-(k-1)} - t_i^{1/2}\tilde{t}_j^{1/2}\tau x^{k-1}}{t_i^{-1/2}t_j^{1/2}x^{-(k-1-n_i)} - t_i^{1/2}t_j^{-1/2}x^{k-1-n_i}} \\ =& \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}w^n \frac{(\tau^{-Nn}-\tau^{Nn})x^n}{1-x^{2n}}\right] . \end{aligned} \tag{4.7}$$

We explicitly checked the last identity for some low values of n and N. Together with the antivortex partition function, this can be interpreted as the multi-particle index for singlet chiral fields $V_\pm$ of the dual theory. All of the t dependence are cancelled out, which is expected since $V_\pm$ are the flavor singlets. Restoring R-charge by shifting $\tau \to \tau x^{\Delta_Q}$, one can check the chiral field V_+ has correct R-charge, $1 - N\Delta_Q$, and U$(1)_A$ charge, $-N$. Then the superconformal index after combining the perturbative part can also be rewritten as duality manifest form

$$\begin{aligned} I^{N=N_f} =& Z^{N=N_f}_{\text{pert}} \times Z^{N=N_f}_{\text{vortex}} \times Z^{N=N_f}_{\text{anti}} \\ =& \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f^{N=N_f}(x^n, t^n, \tilde{t}^n, \tau^n, w^n)\right] , \\ Z^{N=N_f}_{\text{pert}} =& \prod_{i,j}^{N}\prod_{l=0}^{\infty}\frac{1 - t_i^{-1}\tilde{t}_j^{-1}\tau^{-2}x^{2l+2-2\Delta_Q}}{1 - t_i\tilde{t}_j\tau^2 x^{2l+2\Delta_Q}} , \\ f^{N=N_f} =& \sum_{i,j}^{N}\frac{t_i\tilde{t}_j\tau^2x^{2\Delta_Q} - t_i^{-1}\tilde{t}_j^{-1}\tau^{-2}x^{2-2\Delta_Q}}{1-x^2} + \frac{\tau^{-N}x^{1-N\Delta_Q} - \tau^N x^{1+N\Delta_Q}}{1-x^2}(w + w^{-1}) . \end{aligned} \tag{4.8}$$

This precisely agrees with superconformal index of the dual theory with $N \times N$ chiral fields ${M_i}^j$ and two chiral fields $V_\pm$. When $N > 1$, the dual theories flow to free theories. One can check it first for $N = 2$, where the Z-extrimization of [41] determines R-charge of the original chiral fields as $\Delta_Q = \frac{1}{4}$. Then R-charges of the dual chiral fields are fixed to be $\Delta_M = \Delta_V = \frac{1}{2}$ and so the dual theory is obviously free. For $N > 2$, it seems to be impossible to have free dual theory by adjusting the original R-charge Δ_Q. However, as we see from the index formula (4.8), the index of the IR conformal theory is written as that of non-interacting free fields and therefore IR degrees of freedom can carry new U(1) charges

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for accidental symmetry which emerges only at the IR fixed point. The UV R-symmetry then mixes with this extra U(1) symmetry so that the dual chiral fields $M, V_\pm$ become free fields in IR.

Note that the proof of the duality is reduced to that of the identity eq. (4.7) and we provide the proof for $N_f = N = 1$ case. Though we do not give the analytic proof for all cases, the situation is much more improved. Previously at the index level we simply compared the index expressions order by order since we did not have the explicit expressions to all orders in x. The similar pattern emerges for more general cases.

Now we consider further generalization to $N_f > N$ theories. Unlike the previous cases which are mostly free theories, dual theories are now interacting gauge theories. Let us first consider U(1) gauge theory with $N_f = 2$ pairs of fundamental and anti-fundamental matters. The dual theory is also U(1) gauge theory with $N_f = 2$ flavors, but has additional 2×2 chiral fields $M_a{}^b$ and two chiral fields $V_\pm$. The superconformal index of the original theory is

$$I^{(1,2)}{=}\sum_{\sigma\in S_2} Z^{(1,2)}_{\text{pert}}(x,\sigma(t),\tilde{t},\tau)\times Z^{(1,2)}_{\text{vortex}}(x,\sigma(t),\tilde{t},\tau,w)\times Z^{(1,2)}_{\text{anti}}(x,\sigma(t),\tilde{t},\tau,w^{-1}),$$
$$Z^{(1,2)}_{\text{pert}}{=}\prod_{l=0}^{\infty}\left[\frac{1-t_1t_2^{-1}x^{2l+2}}{1-t_1^{-1}t_2x^{2l}}\prod_{a=1}^{2}\frac{1-t_1^{-1}\tilde{t}_a^{-1}\tau^{-2}x^{2l+2-2\Delta_Q}}{1-t_1\tilde{t}_a\tau^{2}x^{2l+2\Delta_Q}}\right],$$
$$Z^{(1,2)}_{\mathit{vortex}}{=}\sum_{n=0}^{\infty}(-w)^n\prod_{k=1}^{n}\frac{\prod_{a=1}^{2}(t_1^{-1/2}\tilde{t}_a^{-1/2}\tau^{-1}x^{-(k-1)-\Delta_Q}-t_1^{1/2}\tilde{t}_a^{1/2}\tau x^{k-1+\Delta_Q})}{(x^{-(k-1-n)}-x^{k-1-n})(t_1^{-1/2}t_2^{1/2}x^{-k}-t_1^{1/2}t_2^{-1/2}x^{k})} \tag{4.9}$$

and $Z^{(1,2)}_{\text{anti}} = Z^{(1,2)}_{\text{vortex}}(w\to w^{-1})$ where $I^{(N,N_f)}$ denotes the index of the original theory with $U(N)$ gauge group and N_f flavors, and $\sigma(t)$ runs over permutations of $\{t_1,t_2\}$. In fact this index also has the duality manifest expression. The perturbative part $Z^{(1,2)}_{\text{pert}}$ with exchange of t_a's can be rewritten as

$$\begin{aligned}Z^{(1,2)}_{\text{pert}}(t_1{\leftrightarrow}t_2)&{=}\prod_{l=0}^{\infty}\left[\frac{1{-}t_1^{-1}t_2x^{2l+2}}{1{-}t_1t_2^{-1}x^{2l}}\prod_{a=1}^{2}\frac{1{-}t_1\tilde{t}_a\tau^2x^{2l+2\Delta_Q}}{1{-}t_1^{-1}\tilde{t}_a^{-1}\tau^{-2}x^{2l+2(1-\Delta_Q)}}\cdot\prod_{a,b}^{2}\frac{1{-}t_a^{-1}\tilde{t}_b^{-1}\tau^{-2}x^{2l+2-2\Delta_Q}}{1-t_a\tilde{t}_b\tau^2x^{2l+2\Delta_Q}}\right]\\&{=}\tilde{Z}^{(1,2)}_{\mathit{pert}}\times\prod_{l=0}^{\infty}\prod_{a,b}^{2}\frac{1{-}t_a^{-1}\tilde{t}_b^{-1}\tau^{-2}x^{2l+2-2\Delta_Q}}{1-t_a\tilde{t}_b\tau^2x^{2l+2\Delta_Q}}\end{aligned} \tag{4.10}$$

where $\tilde{Z}^{(1,2)}_{\text{pert}} \equiv Z^{(1,2)}_{\mathit{pert}}\big(t{\to}t^{-1},\tilde{t}{\to}\tilde{t}^{-1},\tau{\to}\tau^{-1},\Delta_Q{\to}1{-}\Delta_Q\big)$. We shall identify $\tilde{Z}^{(1,2)}_{\text{pert}}$ to the perturbative part of charged chiral fields $q^a,\tilde{q}_a$ in the dual theory. Also the second infinity product term in the second line of (4.10) will be identified with the index contribution of the meson field $M_a{}^b$ of the dual theory. Similarly, we define the dual vortex index as $\tilde{Z}^{(1,2)}_{\text{vortex}} \equiv Z^{(1,2)}_{\mathit{vortex}}\big(t{\to}t^{-1},\tilde{t}{\to}\tilde{t}^{-1},\tau{\to}\tau^{-1},\Delta_Q{\to}1{-}\Delta_Q\big)$ and find that [7]

$$Z^{(1,2)}_{\text{vortex}}(t_1,t_2)=\tilde{Z}^{(1,2)}_{\text{vortex}}(t_2,t_1)\times\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}w^n\frac{\tau^{-2n}x^{2n(1-\Delta_Q)}-\tau^{2n}x^{2n\Delta_Q}}{1-x^{2n}}\right] \tag{4.11}$$

using one of Heien's ${}_2\phi_1$-transformation formulae

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right] \tag{4.12}$$

where the basic hypergeometric series ${}_{r+1}\phi_r$ is given by

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1 \; a_2 \; \cdots \; a_{r+1} \\ b_1 \; b_2 \; \cdots \; b_r \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n}{(b_1;q)_n \cdots (b_r,q)_n \, (q;q)_n} z^n. \tag{4.13} $$

The Plethystic exponential term on the right hand side of (4.11) corresponds to the index contribution from chiral fields $V_\pm$ of the dual theory. Finally, collecting all the result, the original index becomes

$$ I^{(1,2)} = \tilde{I}^{(1,2)} \times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f^{(1,2)}(x^n, t^n, \tilde{t}^n, \tau^n, w^n)\right] \tag{4.14} $$

where

$$ \begin{aligned} \tilde{I}^{(1,2)} &= \sum_{\sigma \in S_2} \tilde{Z}^{(1,2)}_{\text{pert}}(\sigma(t)) \times \tilde{Z}^{(1,2)}_{\text{vortex}}(\sigma(t)) \times \tilde{Z}^{(1,2)}_{\text{anti}}(\sigma(t)) \,, \\ f^{(1,2)} &= \sum_{a,b}^{2} \frac{t_a \tilde{t}_b \tau^2 x^{2\Delta_Q} - t_a^{-1} \tilde{t}_b^{-1} \tau^{-2} x^{2-2\Delta_Q}}{1-x^2} + \frac{\tau^{-2} x^{2(1-\Delta_Q)} - \tau^2 x^{2\Delta_Q}}{1-x^2} (w + w^{-1}) \end{aligned} \tag{4.15} $$

This is exactly the same as the superconformal index of the dual theory, which is a U(1) gauge theory with charged chiral multiplets $q^a, \tilde{q}_a$, singlet chiral multiplets ${M_a}^b$ and $V_\pm$. The superpotential W implies that R-charges for $q, \tilde{q}$ are $1-\Delta_Q$ and other charges are opposite to $Q, \tilde{Q}$ of the original theory. Thus the index $\tilde{I}^{(1,2)}$ encodes the contributions from the chiral multiplets $q, \tilde{q}$. One can also check that the single letter index $f^{(1,2)}$ represents the letter indices for ${M_a}^b$ and $V_\pm$ with correct R-charge and global charges.

4.2 General cases

One can generalize the $N = 1, N_f = 2$ example in the previous subsection to general N, N_f in the same way. The index contribution of the singlet matters ${M_a}^b, V_\pm$ is straightforward, and the contribution of $q, \tilde{q}$ is obtained by replacing $N \to N_f - N$, $t, \tilde{t} \to t^{-1}, \tilde{t}^{-1}$, $\tau \to \tau^{-1} x$ in the original index. We again take $\Delta_\Phi = 0$ for simplicity. Thus, the superconformal index for the dual theory is given by

$$ \begin{aligned} & I(x, t, \tilde{t}, \tau, w) \\ &= \left(\prod_{a,b=1}^{N_f} \frac{(t_a^{-1} \tilde{t}_b^{-1} \tau^{-2} x^2; x^2)_\infty}{(t_a \tilde{t}_b \tau^2; x^2)_\infty}\right) \frac{(w^{-1} \tau^{N_f} x^{1-N_f+N}; x^2)_\infty}{(w \tau^{-N_f} x^{N_f-N+1}; x^2)_\infty} \frac{(w \tau^{N_f} x^{1-N_f+N}; x^2)_\infty}{(w^{-1} \tau^{-N_f} x^{N_f-N+1}; x^2)_\infty} \\ & \quad \times \sum_{\sigma \in S_{N_f}/(S_{N_f-N} \times S_N)} \tilde{Z}_{\text{pert}}\left(x, \sigma(t), \tilde{t}, \tau\right) \tilde{Z}_{\text{vortex}}\left(x, \sigma(t), \tilde{t}, \tau, -w\right) \tilde{Z}_{\text{anti}}\left(x, \sigma(t), \tilde{t}, \tau, -w\right) \end{aligned} \tag{4.16} $$

where

$$ \begin{aligned} \tilde{Z}_{\text{pert}}\left(x, t, \tilde{t}, \tau\right) &= Z^{N \to N_f - N}_{\text{pert}}\left(x, t^{-1}, \tilde{t}^{-1}, \tau^{-1} x\right), \\ \tilde{Z}_{\text{vortex}}\left(x, t, \tilde{t}, \tau, -w\right) &= Z^{N \to N_f - N}_{\text{vortex}}\left(x, t^{-1}, \tilde{t}^{-1}, \tau^{-1} x, -w\right), \\ \tilde{Z}_{\text{anti}}\left(x, t, \tilde{t}, \tau, -w\right) &= Z^{N \to N_f - N}_{\text{anti}}\left(x, t^{-1}, \tilde{t}^{-1}, \tau^{-1} x, -w\right). \end{aligned} \tag{4.17} $$

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Note that σ is now an element of $S_{N_f}/(S_{N_f-N}\times S_N)$, not in $S_{N_f}/(S_N\times S_{N_f-N})$. With a little algebra one can show that the following identity holds:

$$\begin{aligned}
&\prod_{b\in\{b_j\}}\frac{\prod_{a=1(\neq b)}^{N_f}\left(t_b t_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_\infty}\frac{\prod_{a=1}^{N_f}\left(t_b^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\prod_{a=1(\neq b)}^{N_f}\left(t_b^{-1}t_a;x^2\right)_\infty}\\
&=\left(\frac{\prod_{a,b\in\{b_j\}^c(a\neq b)}1-t_b t_a^{-1}}{\prod_{a,b\in\{b_j\}(a\neq b)}1-t_b^{-1}t_a}\right)\left(\prod_{a,b=1}^{N_f}\frac{\left(t_a^{-1}\tilde{t}_b^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\left(t_a\tilde{t}_b\tau^2;x^2\right)_\infty}\right)\\
&\times\left(\prod_{b\in\{b_j\}^c}\frac{\prod_{a=1(\neq b)}^{N_f}\left(t_b^{-1}t_a x^2;x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(t_b^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}\frac{\prod_{a=1}^{N_f}\left(t_b\tilde{t}_a\tau^2;x^2\right)_\infty}{\prod_{a=1(\neq b)}^{N_f}\left(t_b t_a^{-1};x^2\right)_\infty}\right)
\end{aligned}\tag{4.18}$$

for an arbitrary subset $\{b_j\}\subset\{1,\cdots,N_f\}$. $\{b_j\}^c$ is given by $\{b_j\}^c=\{1,\cdots,N_f\}-\{b_j\}$. It suggests that we can write the perturbative part of the original index as follows:

$$\begin{aligned}
Z_{\text{pert}}\left(x,\sigma(t),\tilde{t},\tau\right)&=\tilde{Z}_{\text{pert}}\left(x,\sigma^c(t),\tilde{t},\tau\right)\times\left(\prod_{a,b=1}^{N_f}\frac{\left(t_a^{-1}\tilde{t}_b^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\left(t_a\tilde{t}_b\tau^2;x^2\right)_\infty}\right)\\
&=\tilde{Z}_{\text{pert}}\left(x,\sigma^c(t),\tilde{t},\tau\right)\times\exp\left[\sum_{n=1}^\infty\frac{1}{n}f_M\left(x^n,t^n,\tilde{t}^n,\tau^n\right)\right]
\end{aligned}\tag{4.19}$$

where f_M is exactly the letter index for the Mesons ${M_a}^b$:

$$f_M(x,t,\tilde{t},\tau)=\sum_{a,b=1}^{N_f}\frac{t_a\tilde{t}_b\tau^2-t_a^{-1}\tilde{t}_b^{-1}\tau^{-2}x^2}{1-x^2}.\tag{4.20}$$

For a given $\sigma\in S_{N_f}/S_N\times S_{N_f-N}$, we have defined its complementary permutation $\sigma^c\in S_{N_f}/S_{N_f-N}\times S_N$ as follows:

$$\sigma^c=\begin{pmatrix}1&\cdots&N_f-N&N_f-N+1&\cdots&N_f\\ \sigma(N+1)&\cdots&\sigma(N_f)&\sigma(1)&\cdots&\sigma(N)\end{pmatrix};\tag{4.21}$$

i.e., the operation ^{c} swaps the first N and the last N_f-N of the given permutation. Now every term of each remaining part has a nonzero power of w except 1. In addition, Z_{vortex} of the original theory only has positive powers of w while Z_{anti} has negative powers of w. Therefore, we conjecture the following identities:

$$\begin{aligned}
Z_{\text{vortex}}\left(x,\sigma(t),\tilde{t},\tau,-w\right)&=\tilde{Z}_{\text{vortex}}\left(x,\sigma^c(t),\tilde{t},\tau,-w\right)\times\frac{\left(w\tau^{N_f}x^{1-N_f+N};x^2\right)_\infty}{\left(w\tau^{-N_f}x^{N_f-N+1};x^2\right)_\infty}\\
&=\tilde{Z}_{\text{vortex}}\left(x,\sigma^c(t),\tilde{t},\tau,-w\right)\times\exp\left[\sum_{n=1}^\infty\frac{1}{n}f_+\left(x^n,t^n,\tilde{t}^n,\tau^n,w^n\right)\right],
\end{aligned}\tag{4.22}$$

$$Z_{\text{anti}}\left(x,\sigma(t),\tilde{t},\tau,-w\right)=\tilde{Z}_{\text{anti}}\left(x,\sigma^c(t),\tilde{t},\tau,-w\right)\times\frac{\left(w^{-1}\tau^{N_f}x^{1-N_f+N};x^2\right)_\infty}{\left(w^{-1}\tau^{-N_f}x^{N_f-N+1};x^2\right)_\infty}$$
$$=\tilde{Z}_{\text{anti}}\left(x,\sigma^c(t),\tilde{t},\tau,-w\right)\times\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f_-\left(x^n,t^n,\tilde{t}^n,\tau^n,w^n\right)\right],\tag{4.23}$$

$$f_\pm\left(x,t,\tilde{t},\tau,w\right)=w^{\pm}\frac{\tau^{-N_f}x^{N_f-N+1}-\tau^{N_f}x^{1-N_f+N}}{1-x^2},\tag{4.24}$$

which are generalizations of the identities for special N,N_f in the previous subsection. We also check validity of these formulae by extensive numerical computation. Note that $f_++f_-=f_{V_+}+f_{V_-}$ where $f_{V_\pm}$ are the letter indices for the singlets $V_\pm$:

$$f_{V_\pm}\left(x,t,\tilde{t},\tau,w\right)=\frac{w^{\pm}\tau^{-N_f}x^{N_f-N+1}-w^{\mp}\tau^{N_f}x^{1-N_f+N}}{1-x^2}.\tag{4.25}$$

More specifically f_+ contains the contribution from the scalar of V_+ and the fermion of V_- while f_- contains the contribution from the fermion of V_+ and the scalar of V_-. We have seen that for every component the contribution of a certain choice of σ_{orig} for the original theory is exactly the same as the contribution of its complementary permutation $\sigma_{\text{dual}}={\sigma_{\text{orig}}}^c$. Summing over all possible σ we have exactly the same index for the original and dual theories, for any N and $N_f\geq N$. We also have learned that the perturbative contribution of Q_a and $\tilde{Q}^a$ maps to that of $q^a,\tilde{q}_a$ and the contribution of ${M_a}^b$ while the vortex and antivortex contributions of $Q_a,\tilde{Q}^a$ map to those of $q^a,\tilde{q}_a$ and the contributions of $V_\pm$. This is indeed the expected result and we confirm this by the explicit evaluation of the dual-pair indices.

4.3 $\mathcal{N}=4$ Seiberg-like duality and mirror symmetry

$\mathcal{N}=4$ Seiberg-like dualities were proposed in [4, 29] based on brane configuration of Type IIB string theory. Under the duality, a U(N) gauge theory with N_f fundamental hypermultiplets is conjectured to be dual to another U(N_f-N) theory with N_f hypers in the infrared.

In this section we consider the simplest example of $\mathcal{N}=4$ Seiberg-like duality. At low energy, the $\mathcal{N}=4$ U(1) gauge theory with one fundamental hypermultiplet and the free theory of one hypermultiplet flow to the same theory. This is also the simplest example of the mirror symmetry. The free hypermultiplet is so called the twisted hypermultiplet in the context of mirror symmetry. As two theories are simple enough, we can easily compare two superconformal indices of them and check this duality conjecture. In the case at hand, the U(1) gauge multiplet of the original theory couples to one fundamental and anti-fundamental chiral matters while the adjoint chiral matter is decoupled from the U(1) vector multiplet. So there is a similarity between this U(1) theory and $\mathcal{N}=2$ U(1) gauge theory with $N_f=\tilde{N}_f=1$ chiral matters up to the decoupled adjoint chiral. In fact, once we assign the correct global charges to $\mathcal{N}=2$ fields, it is easy to write the superconformal index of $\mathcal{N}=4$ U(1) gauge theory using $\mathcal{N}=2$ result. Then the superconformal index for

U(1) theory with $N_f = 1$ fundamental hypermultiplet after factorization becomes

$$\begin{aligned} I^{\mathcal{N}=4}_{N=N_f=1} &= I^{\mathcal{N}=2}_{N=N_f=1}(\Delta_Q = \frac{1}{2}, \tau = y^{1/2}) \times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n(y^{-n} - y^n)}{1 - x^{2n}}\right] \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{y^{-n/2}x^{n/2} - y^{n/2}x^{3n/2}}{1 - x^{2n}}(w^n + w^{-n})\right] \end{aligned} \tag{4.26}$$

Here $I^{\mathcal{N}=2}_{N=N_f=1}$ is the index of (4.5) for $\mathcal{N} = 2$ theory, and we set R-charge of bosonic fields to be $\Delta_Q = \frac{1}{2}$ and introduced the chemical potential y for the off-diagonal $\mathrm{U}(1)_A$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SO}(4)$ R-symmetry. The exponential term on the right hand side of the first line comes from the adjoint chiral multiplet. The final expression is written as the Plethystic exponential of one free hypermultiplet that agrees with the duality proposal. In the dual theory w is interpreted as the U(1) flavor chemical potential. Note that this also perfectly matches with mirror symmetry. Under the mirror symmetry the monopole operator of the U(1) theory is mapped to the twisted free hypermultiplet. Note that w at eq. (4.26) is the vortex number, which is nothing but the monopole charge. In the mirror side this is mapped to the charge of the flavor symmetry of the free hypermultiplet. The detailed exploration of the mirror symmetry and $\mathcal{N} = 4$ Seiberg-like duality in terms of the factorization will appear elsewhere.

5 Concluding remarks

There would be manifold generalizations one can pursue related to the current work. The first one is the direct proof of the factorization using the localization. For the 2d partition function, it is explicitly worked out in [22]. Certainly it is more desirable to consider more general gauge groups and general matters, which will have applications for Seiberg-like dualities for classical groups with two index matters, which was explored in [42].

For simple cases, we already saw the vortex partition function coincides with the corresponding topological open string amplitude. Such pattern will hold for more general cases and it would be desirable to work out explicitly. In [30], the 2d vortex arises as the surface operator of the 4d supersymmetric gauge theory and we expect that this will be lifted to the 3d defect to the 5d SCFT. The 3d SCFT realized as the IR limit of the 3d gauge theory flows to the 2d CFT upon the dimensional reduction. Thus many of the properties of 2d CFTs such as conformal blocks and $tt*$ equations will be lifted to the corresponding 3d CFTs, which is interesting to explore. In the same spirit, the relation between 3d mirror symmetry and 2d mirror symmetry would be worked out in similar way. 2d mirror symmetry in the nonabelian gauge group setup is explored recently [44, 45] and it would be interesting to find its relation to 3d mirror symmetry.

Finally the proof of the duality such as Seiberg-like duality, mirror symmetry will be greatly simplified with the factorized form of the index and it is worth attempting analytic proof of the index equality for dual pairs.

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A Factorization: nonabelian cases

In this section we will derive the factorized expression of the superconformal index for a $\mathrm{U}(N)$ gauge theory in the presence of Chern-Simons terms. Firstly the superconformal index is given by

$$\begin{aligned} & I(x,t,w) \\ &= \sum_{m\in\mathbb{Z}^N/S_N} \oint \left(\prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j}\right) \frac{1}{|\mathcal{W}_m|} w^{\sum_j m_j} e^{-S_{CS}(a,m)} Z_{\text{gauge}}(x,z,m) \prod_{\Phi} Z_\Phi(x,t,z,m) \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} e^{-S_{CS}(a,m)} &= e^{-i\mathrm{Tr}_{CS}(a+\pi)m} \\ &= \prod_{j=1}^{N} (-z_j)^{-\kappa m_j}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} Z_{\text{gauge}}(x,z,m) &= \prod_{\alpha\in ad(G)} x^{-|\alpha(m)|/2} \left(1 - e^{i\alpha(a)} x^{|\alpha(m)|}\right) \\ &= \prod_{\substack{i,j=1 \\ (i\neq j)}}^{N} x^{-|m_i-m_j|/2} \left(1 - z_i z_j^{-1} x^{|m_i-m_j|}\right), \end{aligned} \tag{A.3}$$

$$\begin{aligned} & \prod_{\Phi} Z_\Phi\left(x,t,\tilde{t},\tau,z,m\right) \\ &= \prod_{\rho\in R_\Phi} \left(x^{(1-\Delta_\Phi)} e^{-i\rho(a+\pi)} \prod_a t_a^{-f_a(\Phi)}\right)^{|\rho(m)|/2} \frac{\left(e^{-i\rho(a)} \prod t_a^{-f_a(\Phi)} x^{|\rho(m)|+2-\Delta_\Phi}; x^2\right)_\infty}{\left(e^{i\rho(a)} \prod t_a^{f_a(\Phi)} x^{|\rho(m)|+\Delta_\Phi}; x^2\right)_\infty} \\ &= \prod_{j=1}^{N} x^{(1-\Delta_\Phi)(N_f+\tilde{N}_f)|m_j|/2} (-z_j)^{-(N_f-\tilde{N}_f)|m_j|/2} \tau^{-(N_f+\tilde{N}_f)|m_j|/2} \\ &\quad \times \frac{\prod_{a=1}^{N_f}\left(z_j^{-1} t_a^{-1} \tau^{-1} x^{|m_j|+2-\Delta_\Phi}; x^2\right)_\infty}{\prod_{a=1}^{N_f}\left(z_j t_a \tau x^{|m_j|+\Delta_\Phi}; x^2\right)_\infty} \frac{\prod_{a=1}^{\tilde{N}_f}\left(z_j \tilde{t}_a^{-1} \tau^{-1} x^{|m_j|+2-\Delta_\Phi}; x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(z_j^{-1} \tilde{t}_a \tau x^{|m_j|+\Delta_\Phi}; x^2\right)_\infty}. \end{aligned} \tag{A.4}$$

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$(a;q)_n$ is the q-Pochhammer symbol defined by

$$(a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right). \tag{A.5}$$

We have included a phase shift $a_j \to a_j + \pi$ for nonzero magnetic flux vacua. The resulting phase is the nontrivial phase discussed in the subsection 2.2. $\{t_a\}$ and $\{\tilde{t}_a\}$ correspond to fugacities for the $\mathrm{SU}(N_f) \times \mathrm{SU}(\tilde{N}_f)$ flavor symmtery of N_f fundamental and $\tilde{N}_f$ antifundamental chiral multiplets while τ is a fugacity for $\mathrm{U}(1)_A$. Note that $\kappa + (N_f + \tilde{N}_f)/2$ should be an integer due to the quantization of the effective CS level, which leads to the parity anomaly free condition. We will have the sensible factorization only for parity anomaly-free theories. In addition, we will set $\Delta_\Phi = 0$, which can be restored by deforming $\tau \to \tau x^{\Delta_\Phi}$. We follow the similar limit procedure to the abelian case for the evaluation of the integral so we require $0 < x < 1$ as well as $|t_a \tau|, |\tilde{t}_a \tau| < 1$. After the residue computation, we extend the result to other parameter regime by analytic continuation. For those ranges of parameters, poles from the fundamental chiral multiplet contribution lie outside the integration contour $|z| = 1$ while poles from the antifundamental chiral multiplet contribution lie inside the integration contour. In addition, the integrand may also have poles or zeroes at the origin and at infinity. This can be handled in a similar way to the abelian case and one can choose the contour so that there would be no contribution from the origin or at infinity. When N_f and $\tilde{N}_f$ are the same, poles of finite order may exist both at the origin and at infinity depending on the sign of $N \pm \kappa m_j$.

Firstly we deal with the $N_f > \tilde{N}_f$ case. As in the abelian case, we can take poles from outside of the unit cirle, which are those from the fundamental chiral multiplet contribution: $z_j = t_{b_j}^{-1} \tau^{-1} x^{-|m_j| - 2l_j}$ for $b_j = 1, \cdots, N_f$ and $l_j = 0, 1, \cdots$. Summing the residues the index is given by

$$\begin{aligned}
&I^{N_f > \tilde{N}_f}(x, t, \tilde{t}, \tau, w) \\
&= \sum_{m \in \mathbb{Z}^N / S_N} \sum_{b=\vec{1}}^{\vec{N}_f} \sum_{l=\vec{0}}^{\vec{\infty}} \frac{1}{|\mathcal{W}_m|} \\
&\quad \times \left[\prod_{j=1}^{N} (-1)^{-\kappa m_j - \delta |m_j|} w^{m_j} t_{b_j}^{\kappa m_j + \delta |m_j|} \tau^{\kappa m_j - \tilde{N}_f |m_j|} \right. \\
&\qquad \left. \times x^{\kappa m_j (|m_j| + 2l_j) - \sum_{i \neq j} |m_i - m_j|/2 + \mathfrak{N} |m_j| + \delta (|m_j|^2 + 2|m_j| l_j)} \right] \\
&\quad \times \left[\prod_{\substack{i,j=1 \\ (i \neq j)}}^{N} 1 - t_{b_i}^{-1} t_{b_j} x^{|m_i - m_j| - |m_i| + |m_j| - 2l_i + 2l_j} \right] \\
&\quad \times \left[\frac{\prod_{a=1}^{N_f} \left(t_{b_j} t_a^{-1} x^{2|m_j| + 2l_j + 2}; x^2\right)_\infty}{\prod_{a=1}^{N_f} \left(t_{b_j}^{-1} t_a x^{-2l_j}; x^2\right)'_\infty} \frac{\prod_{a=1}^{\tilde{N}_f} \left(t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{-2l_j + 2}; x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f} \left(t_{b_j} \tilde{t}_a \tau^2 x^{2|m_j| + 2l_j}; x^2\right)_\infty} \right]
\end{aligned} \tag{A.6}$$

where the prime $'$ for the q-Pochhammer symbol means that the zero factor of the q-Pochhammer symbol which arises when a equals to b_j is dropped. We have defined $\mathfrak{N}$ and δ such that $N_f = \mathfrak{N}+\delta$ and $\tilde{N}_f = \mathfrak{N}-\delta$. $\sum_{b=\vec{1}}^{\tilde{N}_f}$ and $\sum_{l=\vec{0}}^{\vec{\infty}}$ are short expressions for $\sum_{b_1=1}^{N_f}\cdots\sum_{b_N=1}^{N_f}$ and $\sum_{l_1=0}^{\infty}\cdots\sum_{l_N=0}^{\infty}$. Now let us focus on each factor. They can be rewritten in the following ways:

$$\prod_{\substack{i,j=1\\(i\neq j)}}^{N} 1-t_{b_i}^{-1}t_{b_j}x^{|m_i-m_j|-|m_i|+|m_j|-2l_i+2l_j}$$
$$=\prod_{i<j}^{N}(-4x^{m_i-m_j})\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma\,(n_i-n_j)}{2}\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma\,(\bar{n}_i-\bar{n}_j)}{2}, \tag{A.7}$$

$$\left(t_{b_j}t_a^{-1}x^{2|m_j|+2l_j+2};x^2\right)_\infty=\frac{\left(t_{b_j}t_a^{-1}x^2;x^2\right)_\infty}{\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{|m_j|+l_j}}, \tag{A.8}$$

$$\left(t_{b_j}^{-1}t_ax^{-2l_j};x^2\right)'_\infty=(-1)^{l_j}t_{b_j}^{-l_j}t_a^{l_j}x^{-l_j^2-l_j}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{l_j}\left(t_{b_j}^{-1}t_a;x^2\right)'_\infty, \tag{A.9}$$

$$\left(t_{b_j}^{-1}\tilde{t}_a^{-1}\tau^{-2}x^{-2l_j+2};x^2\right)_\infty=(-1)^{l_j}t_{b_j}^{-l_j}\tilde{t}_a^{-l_j}\tau^{-2l_j}x^{-l_j^2+l_j}\left(t_{b_j}\tilde{t}_a\tau^2;x^2\right)_{l_j}\left(t_{b_j}^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty, \tag{A.10}$$

$$\left(t_{b_j}\tilde{t}_a\tau^2x^{2|m_j|+2l_j};x^2\right)_\infty=\frac{\left(t_{b_j}\tilde{t}_a\tau^2;x^2\right)_\infty}{\left(t_{b_j}\tilde{t}_a\tau^2;x^2\right)_{|m_j|+l_j}} \tag{A.11}$$

where $t=e^{iM}$, $\tilde{t}=e^{i\tilde{M}}$, $\tau=e^{i\mu}$ and $x=e^{-\gamma}$. We also defined $n_j\equiv l_j+\frac{|m_j|}{2}+\frac{m_j}{2}$ and $\bar{n}_j\equiv l_j+\frac{|m_j|}{2}-\frac{m_j}{2}$. We have assumed $m_1\geq\cdots\geq m_N$ without any loss of generality. Gathering those results one can rewrite the whole index as follows:

$$\begin{aligned}
&I^{N_f>\tilde{N}_f}(x,t,\tilde{t},\tau,w)\\
&=\sum_{m\in\mathbb{Z}^N/S_N}\sum_{b=\vec{1}}^{\tilde{N}_f}\sum_{l=\vec{0}}^{\vec{\infty}}\frac{(-1)^{N(N-1)/2}}{|\mathcal{W}_m|}\\
&\quad\times\Bigg[\prod_{j=1}^{N}(-1)^{-\kappa(n_j-\bar{n}_j)-\delta(n_j+\bar{n}_j)}w^{n_j-\bar{n}_j}t_{b_j}^{\kappa(n_j-\bar{n}_j)+\delta(n_j+\bar{n}_j)}\tau^{\kappa(n_j-\bar{n}_j)-\tilde{N}_f(n_j+\bar{n}_j)}\\
&\qquad\qquad\times x^{\kappa(n_j-\bar{n}_j)(n_j+\bar{n}_j)+\mathfrak{N}(n_j+\bar{n}_j)+\delta[(|m_j|+l_j)^2+l_j^2]}\Bigg]\\
&\quad\times\left[\prod_{i<j}4\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma\,(n_i-n_j)}{2}\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma\,(\bar{n}_i-\bar{n}_j)}{2}\right]\\
&\quad\times\Bigg[\prod_{j=1}^{N}\frac{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_{b_j}\tilde{t}_a\tau^2;x^2\right)_\infty}\frac{\prod_{a=1}^{\tilde{N}_f}\left(t_{b_j}^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j}^{-1}t_a;x^2\right)_\infty}
\end{aligned} \tag{A.12}$$

$$\times\frac{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}\tilde t_a\tau^2;x^2\right)_{|m_j|+l_j}}{\prod_{a=1}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{|m_j|+l_j}}\frac{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}\tilde t_a\tau^2;x^2\right)_{l_j}}{\prod_{a=1}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{l_j}}\Bigg].$$

again with n and $\bar n$ given by $n_j=l_j+\frac{|m_j|}{2}+\frac{m_j}{2}$, $\bar n_j=l_j+\frac{|m_j|}{2}-\frac{m_j}{2}$. We can replace the summation $\sum_{m\in\mathbb{Z}^N/S_N}$ by $\sum_{m\in\mathbb{Z}^N}\frac{|\mathcal{W}_m|}{N!}$. Note that $\{|m_j|+l_j,l_j\}=\{n_j,\bar n_j\}$ for all values of m_j and l_j. Since the expression is symmetric under $|m_j|+l_j\leftrightarrow l_j$, we just replace $\{|m_j|+l_j,l_j\}$ by $\{n_j,\bar n_j\}$ in the expression and rearrange the summations as $\sum_{m\in\mathbb{Z}^N}\sum_{l=\vec 0}^{\vec\infty}=\sum_{n=\vec 0}^{\vec\infty}\sum_{\bar n=\vec 0}^{\vec\infty}$. The index is then written in the factorized form as follows:

$$\begin{aligned}
&I^{N_f>\tilde N_f}(x,t,\tilde t,\tau,w)\\
&=\frac{(-1)^{N(N-1)/2}}{N!}\sum_{b=\vec 1}^{\vec N_f}\Bigg\{\left[\prod_{j=1}^{N}\frac{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_\infty}{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}\tilde t_a\tau^2;x^2\right)_\infty}\frac{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}^{-1}\tilde t_a^{-1}\tau^{-2}x^2;x^2\right)_\infty}{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j}^{-1}t_a;x^2\right)_\infty}\right]\\
&\quad\times\left[\sum_{n=\vec 0}^{\vec\infty}\left(\prod_{j=1}^{N}(-1)^{-\kappa n_j-\delta n_j}w^{n_j}t_{b_j}^{\kappa n_j+\delta n_j}\tau^{\kappa n_j-\tilde N_f n_j}x^{\kappa n_j^2+\mathfrak{N}n_j+\delta n_j^2}\right)\right.\\
&\qquad\left.\times\left(\prod_{i<j}2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}\right)\left(\prod_{j=1}^{N}\frac{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}\tilde t_a\tau^2;x^2\right)_{n_j}}{\prod_{a=1}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{n_j}}\right)\right]\\
&\quad\times\left[\sum_{\bar n=\vec 0}^{\vec\infty}\left(\prod_{j=1}^{N}(-1)^{\kappa\bar n_j-\delta\bar n_j}w^{-\bar n_j}t_{b_j}^{-\kappa\bar n_j+\delta\bar n_j}\tau^{-\kappa\bar n_j-\tilde N_f\bar n_j}x^{-\kappa\bar n_j^2+\mathfrak{N}\bar n_j+\delta\bar n_j^2}\right)\right.\\
&\qquad\left.\times\left(\prod_{i<j}2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(\bar n_i-\bar n_j)}{2}\right)\left(\prod_{j=1}^{N}\frac{\prod_{a=1}^{\tilde N_f}\left(t_{b_j}\tilde t_a\tau^2;x^2\right)_{\bar n_j}}{\prod_{a=1}^{N_f}\left(t_{b_j}t_a^{-1}x^2;x^2\right)_{\bar n_j}}\right)\right].
\end{aligned}\tag{A.13}$$

If $b_i=b_j$ for $i\neq j$, the index vanishes because it has the antisymmetric contribution of n_i and n_j. Together with the flavor symmetry it implies that one can arrange b_j in ascending order, $b_1<\cdots<b_N$. And the summation $\frac{1}{N!}\sum_{b_1,\cdots,b_N=1}^{N_f}$ is accordingly replaced by $\sum_{1\le b_1<\cdots<b_N\le N_f}$. Using

$$\begin{aligned}
&\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}\right)\left(\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde N_f}2\sinh\frac{-iM_{b_j}-i\tilde M_a-2i\mu+2\gamma(k-1)}{2}}{\prod_{a=1}^{N_f}2\sinh\frac{-iM_{b_j}+iM_a+2\gamma k}{2}}\right)\\
&=(-1)^{\sum_j n_j}\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}}{2}\right)\\
&\quad\times\left(\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde N_f}2\sinh\frac{-i\tilde M_a-iM_{b_j}-2i\mu+2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}+2\gamma(k-1-n_i)}{2}\right)\left(\prod_{a\in\{b_j\}^c}2\sinh\frac{iM_a-iM_{b_j}+2\gamma k}{2}\right)}\right)
\end{aligned}\tag{A.14}$$

where $\{b_j\}^c = \{1, \cdots, N_f\} - \{b_j\}$, the whole index is finally written in the following concise form:

$$I^{N_f > \tilde{N}_f}(x, t, \tilde{t}, \tau, w) = \sum_{\substack{1 \le b_1 < \cdots \\ < b_N \le N_f}} Z^{\{b_j\}}_{\text{pert}}\left(x, t, \tilde{t}, \tau\right) Z^{\{b_j\}}_{\text{vortex}}\left(x, t, \tilde{t}, \tau, \mathfrak{w}\right) Z^{\{b_j\}}_{\text{anti}}\left(x, t, \tilde{t}, \tau, \mathfrak{w}\right) \tag{A.15}$$

where $\mathfrak{w} = (-1)^{-\kappa-\delta}(-w)$. Also recall $t = e^{iM}$, $\tilde{t} = e^{i\tilde{M}}$, $\tau = e^{i\mu}$, $x = e^{-\gamma}$. Note that the parity anomaly free condition $\kappa + (N_f + \tilde{N}_f)/2$ being an integer implies that $(-1)^{-\kappa-\delta}$ is a well-defined sign factor; i.e., it is always real valued. The first component $Z^{\{b_j\}}_{\text{pert}}$, which we call the perturbative part, is given by

$$\begin{aligned} &Z^{\{b_j\}}_{\text{pert}}\left(x, t, \tilde{t}, \tau\right) \\ &= \left(\prod_{\substack{i,j=1\\(i\neq j)}}^{N} 2\sinh\frac{1}{2}\left(iM_{b_i} - iM_{b_j}\right)\right)\left(\prod_{j=1}^{N} \frac{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j} t_a^{-1} x^2; x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_{b_j}\tilde{t}_a \tau^2; x^2\right)_\infty} \frac{\prod_{a=1}^{\tilde{N}_f}\left(t_{b_j}^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2; x^2\right)_\infty}{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j}^{-1} t_a; x^2\right)_\infty}\right). \end{aligned} \tag{A.16}$$

If we think of analytic continuation of q-Pochhammer symbol, $Z^{\{b_j\}}_{\text{pert}}$ is also factorized as follows:

$$Z^{\{b_j\}}_{\text{pert}}\left(x, t, \tilde{t}, \tau\right) = Z^{\{b_j\}}_{1-\text{loop}}\left(x, t, \tilde{t}, \tau\right) Z^{\{b_j\}}_{1-\text{loop}}\left(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1}\right) \tag{A.17}$$

where

$$Z^{\{b_j\}}_{1-\text{loop}}\left(x, t, \tilde{t}, \tau\right) = \left(\prod_{i<j}^{N} 2\sinh\frac{1}{2}\left(iM_{b_i} - iM_{b_j}\right)\right)\left(\prod_{j=1}^{N} \frac{\prod_{a=1(\neq b_j)}^{N_f}\left(t_{b_j} t_a^{-1} x^2; x^2\right)_\infty}{\prod_{a=1}^{\tilde{N}_f}\left(t_{b_j}\tilde{t}_a \tau^2; x^2\right)_\infty}\right). \tag{A.18}$$

In addition, the second and the third component of eq. (A.15), which we call the vortex partition function and the antivortex partition function respectively, are given by

$$\begin{aligned} Z^{\{b_j\}}_{\text{vortex}}\left(x, t, \tilde{t}, \tau, \mathfrak{w}\right) &= \sum_{n=\vec{0}}^{\vec{\infty}} \mathfrak{w}^{\sum_j n_j} \mathfrak{J}^{\{b_j\}}_{(n_j)}\left(x, t, \tilde{t}, \tau\right), \\ Z^{\{b_j\}}_{\text{anti}}\left(x, t, \tilde{t}, \tau, \mathfrak{w}\right) &= \sum_{n=\vec{0}}^{\vec{\infty}} \mathfrak{w}^{-\sum_j n_j} \mathfrak{J}^{\{b_j\}}_{(n_j)}\left(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1}\right) \end{aligned} \tag{A.19}$$

where

$$\begin{aligned} &\mathfrak{J}^{\{b_j\}}_{(n_j)}\left(x, t, \tilde{t}, \tau\right) \\ &= e^{-S_0} \prod_{j=1}^{N}\prod_{k=1}^{n_j} \frac{\prod_{a=1}^{\tilde{N}_f} 2\sinh\frac{-i\tilde{M}_a - iM_{b_j} - 2i\mu + 2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N} 2\sinh\frac{iM_{b_i} - iM_{b_j} + 2\gamma(k-1-n_i)}{2}\right)\left(\prod_{a\in\{b_j\}^c} 2\sinh\frac{iM_a - iM_{b_j} + 2\gamma k}{2}\right)} \end{aligned} \tag{A.20}$$

where $e^{-S_0} = e^{\kappa\sum_j(iM_{b_j}n_j + i\mu n_j - \gamma n_j^2)}$. They correspond to the $\mathcal{N} = 2$ vortex partition functions on $\mathbb{R}^2 \times S^1$. We will see at the end of the appendix that the vortex partition function obtained in this way is perfectly consistent with the $\mathcal{N} = 4, 3$ results obtained in [29].

The index for $N_f < \tilde{N}_f$ is simply obtained by interchanging $t_a \leftrightarrow \tilde{t}_a$ as well as $N_f \leftrightarrow \tilde{N}_f$, and $\kappa \to -\kappa$. For $N_f = \tilde{N}_f$ on the other hand, the integrand may have a pole at the origin if $N + \kappa m_j > 0$ and have a pole at infinity if $N - \kappa m_j > 0$. Therefore we should take them into account. For $N + \kappa m_j > 0$ there is a pole at the origin, whose residue is given by

$$\begin{aligned}
&\mathrm{Res}(\dots, 0) \\
&= \left(\prod_{j=1}^{N} x^{-\sum_{i\neq j}|m_i-m_j|/2+N_f|m_j|}\tau^{-N_f|m_j|}\right)\left(\prod_{j=1}^{N}\lim_{z_j\to 0}\frac{1}{(N+\kappa m_j-1)!}\frac{\partial^{N+\kappa m_j-1}}{\partial z_j^{N+\kappa m_j-1}}\right) \\
&\left[\left(\prod_{\substack{i,j=1\\(i\neq j)}}^{N} z_j - z_i x^{|m_i-m_j|}\right)\left(\prod_{j=1}^{N}\prod_{a=1}^{N_f}\prod_{k=0}^{\infty}\frac{z_j - t_a^{-1}\tau^{-1}x^{|m_j|+2+2k}}{1-z_j t_a \tau x^{|m_j|+2k}}\frac{1-z_j\tilde{t}_a^{-1}\tau^{-1}x^{|m_j|+2+2k}}{z_j - \tilde{t}_a \tau x^{|m_j|+2k}}\right)\right]
\end{aligned} \tag{A.21}$$

Let us recall the $N = 1, \kappa = 0$ case discussed in the main text. In that case one has a vanishing infinite product:

$$\sim \lim_{n\to\infty}\prod_{k=0}^{n} t_a^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2 = 0 \tag{A.22}$$

assuming $|t_a^{-1}\tilde{t}_a^{-1}\tau^{-2}x^2| < 1$. Here n denotes the subscript of q-Pochhammer symbol $(a : q)_n$ appearing in the index expression. We start from finite n in the index, then take $n \to \infty$ limit as we do for the abelian case. For general N and κ, there are $N + \kappa m_j - 1$ differentiations with respect to z_j. When each of them acts, an additional factor arises. Nevertheless, we still have a vanishing infinite product because there are only the finite number of such additional factors, which are not singular. Therefore, the residue still vanishes. In the same manner if there is a pole at infinity, its residue also vanishes. In conclusion, although there might be poles at the origin and at infinity for $N_f = \tilde{N}_f$, the previous results for $N_f \neq \tilde{N}_f$ still holds for $N_f = \tilde{N}_f$. We will sometimes omit the superscript $N_f \geq \tilde{N}_f$ of $I^{N_f \geq \tilde{N}_f}\left(x, t, \tilde{t}, \tau, w\right)$ when we consider $N_f \geq \tilde{N}_f$ cases.

One might write the factorized index in a slightly different way using permutations of the fugacities $\{t_a\}$ for the $\mathrm{SU}(N_f)$ flavor symmetry as follows:

$$\begin{aligned}
&I\left(x, t, \tilde{t}, \tau, w\right) \\
&= \sum_{\sigma\in S_{N_f}/(S_N\times S_{N_f-N})} Z_{\text{pert}}\left(x, \sigma(t), \tilde{t}, \tau\right) Z_{\text{vortex}}\left(x, \sigma(t), \tilde{t}, \tau, \mathfrak{w}\right) Z_{\text{anti}}\left(x, \sigma(t), \tilde{t}, \tau, \mathfrak{w}\right)
\end{aligned} \tag{A.23}$$

where we have defined $Z_{\text{pert/vortex/anti}} \equiv Z^{\{b_j\}}_{\text{pert/vortex/anti}}$ with $b_j = 1, \cdots N$. σ is an element of the quotient group $S_{N_f}/(S_N \times S_{N_f-N})$ where S_{N_f} is the symmetric group of degree N_f; S_N and S_{N_f-N} are its subgroups whose elements are the permutations of the first N elements and the last $N_f - N$ elements respectively. $\mathfrak{w}$ is again given by $\mathfrak{w} = (-1)^{-\kappa-\delta}(-w)$ as before. Now one can compare Z_{vortex}, or more precisely $\mathfrak{I}_{(n_j)} \equiv \mathfrak{I}^{\{1,\cdots,N\}}_{(n_j)}$ in $Z_{\text{vortex}} = \sum_n \mathfrak{w}^{\sum_j n_j}\mathfrak{I}_{(n_j)}$, with the vortex partition function on $\mathbb{R}^2 \times S^1$, $I_{(k_1,k_2,\cdots,k_N)}$, obtained in [29].

Recall that $\mathfrak{I}_{(n_j)}$ is given by

$$\begin{aligned}&\mathfrak{I}_{(n_j)}\left(x,t,\tilde{t},\tau\right)\\&=e^{-S_0}\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde{N}_f}2\sinh\frac{-i\tilde{M}_a-iM_j-2i\mu+2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N}2\sinh\frac{iM_i-iM_j+2\gamma(k-1-n_i)}{2}\right)\left(\prod_{a=N+1}^{N_f}2\sinh\frac{iM_a-iM_j+2\gamma k}{2}\right)}\end{aligned}\tag{A.24}$$

where $e^{-S_0}=e^{\kappa\sum_j(iM_jn_j+i\mu n_j-\gamma n_j^2)}$. We compare our $\mathcal{N}=2$ result with the $\mathcal{N}=4,3$ results in [29] and observe the perfect consistency. Although the results in [29] also have a contribution of an adjoint matter, the contribution can be easily separated from the whole partition function. Therefore, we can compare both results. In order to compare the results we should redefine our γ to $2i\gamma$ and κ to $-\kappa$. With the following identifications of mass parameters:

$$\begin{aligned}iM_j+i\mu&=\mu_j+2i\gamma, &\qquad j&=0,\cdots,N,\\iM_a+i\mu&=\mu_a-2i\gamma, & a&=N+1,\cdots,N_f,\\i\tilde{M}_b+i\mu&=-\mu_b+2i\gamma, & b&=1,\cdots,\tilde{N}_f=N_f,\end{aligned}\tag{A.25}$$

where our parameters are on the left hand side while the parameters in [29] are on the right hand side, we found

$$\begin{aligned}\prod_{i=1}^{N}-\frac{1}{2\sinh\frac{iM_i-iM_j+2\gamma(k-1-n_i)}{2}}&=z_vz_{\text{fund}}^N,\\\prod_{a=N+1}^{N_f}-\frac{1}{2\sinh\frac{iM_a-iM_j+2\gamma k}{2}}&=z_{\text{fund}}^{N_f-N},\\\prod_{a=1}^{\tilde{N}_f}-2\sinh\frac{-i\tilde{M}_a-iM_j-2i\mu+2\gamma(k-1)}{2}&=z_{\text{anti}}^{N_f}.\end{aligned}\tag{A.26}$$

This strongly indicates that our result gives the correct $\mathcal{N}=2$ vortex partition function on $\mathbb{R}^2\times S^1$.

B Detailed calculations

In this section we give a derivation of (A.14) in the previous section:

$$\begin{aligned}&\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}\right)\left(\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde{N}_f}2\sinh\frac{-iM_{b_j}-i\tilde{M}_a-2i\mu+2\gamma(k-1)}{2}}{\prod_{a=1}^{N_f}2\sinh\frac{-iM_{b_j}+iM_a+2\gamma k}{2}}\right)\\&=(-1)^{\sum_j n_j}\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}}{2}\right)\\&\quad\times\left(\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde{N}_f}2\sinh\frac{-i\tilde{M}_a-iM_{b_j}-2i\mu+2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}+2\gamma(k-1-n_i)}{2}\right)\left(\prod_{a\in\{b_j\}^c}2\sinh\frac{iM_a-iM_{b_j}+2\gamma k}{2}\right)}\right).\end{aligned}$$

In order to derive the above identity it is convenient to write the first factor on the left hand side in terms of q-Pochhammer symbol. It is given by

$$\prod_{i<j}^{N} 2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}=\prod_{i<j}^{N}\left(t_{b_i}^{1/2}t_{b_j}^{-1/2}x^{n_i-n_j}\right)\frac{\left(t_{b_i}^{-1}t_{b_j}x^{-2(n_i-n_j)};x^2\right)_\infty}{\left(t_{b_i}^{-1}t_{b_j}x^{-2(n_i-n_j)+2};x^2\right)_\infty}. \tag{B.1}$$

Then each q-pochhammer symbol in the numerator and in the denominator can be written as follows:

$$\left(t_{b_i}^{-1}t_{b_j}x^{-2(n_i-n_j)};x^2\right)_\infty=\left(t_{b_i}^{-1}t_{b_j};x^2\right)_\infty\times\frac{\left(\prod_{k=0}^{n_i-1}-t_{b_i}^{-1}t_{b_j}x^{-2-2k}\right)\left(t_{b_i}t_{b_j}^{-1}x^2;x^2\right)_{n_i}}{\left(t_{b_i}^{-1}t_{b_j}x^{-2n_i};x^2\right)_{n_j}}, \tag{B.2}$$

$$\left(t_{b_i}^{-1}t_{b_j}x^{-2(n_i-n_j)+2};x^2\right)_\infty=\left(t_{b_i}^{-1}t_{b_j}x^2;x^2\right)_\infty\times\frac{\left(\prod_{k=0}^{n_i-1}-t_{b_i}^{-1}t_{b_j}x^{2n_j-2k}\right)\left(t_{b_i}t_{b_j}^{-1}x^{-2n_j};x^2\right)_{n_i}}{\left(t_{b_i}^{-1}t_{b_j}x^2;x^2\right)_{n_j}}. \tag{B.3}$$

Combining them with the factor $t_{b_i}^{1/2}t_{b_j}^{-1/2}x^{n_i-n_j}$, we have the following expression:

$$\begin{aligned}&\prod_{i<j}^{N} 2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}\\&=\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}}{2}\times\frac{\left(x^{-n_i-n_j-2n_in_j}\right)\left(t_{b_i}t_{b_j}^{-1}x^2;x^2\right)_{n_i}\left(t_{b_i}^{-1}t_{b_j}x^2;x^2\right)_{n_j}}{\left(t_{b_i}t_{b_j}^{-1}x^{-2n_j};x^2\right)_{n_i}\left(t_{b_i}^{-1}t_{b_j}x^{-2n_i};x^2\right)_{n_j}}\\&=\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}}{2}\right)\left(\prod_{\substack{i,j=1\\(i\neq j)}}^{N}\prod_{k=1}^{n_j}\frac{2\sinh\frac{iM_{b_i}-iM_{b_j}+2\gamma k}{2}}{2\sinh\frac{iM_{b_i}-iM_{b_j}+2\gamma(k-1-n_i)}{2}}\right)\end{aligned} \tag{B.4}$$

where we used the fact that the second factor after the first equality sign is symmetric under $i\leftrightarrow j$. Then using the following equality

$$\begin{aligned}&\prod_{a=1}^{N_f}2\sinh\frac{-iM_{b_j}+iM_a+2\gamma k}{2}\\&=\left(\prod_{i=1}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}+2\gamma k}{2}\right)\left(\prod_{a\in\{b_j\}^c}2\sinh\frac{iM_a-iM_{b_j}+2\gamma k}{2}\right),\end{aligned} \tag{B.5}$$

we finally obtain the identity (A.14):

$$\left(\prod_{i<j}^{N}2\sinh\frac{iM_{b_i}-iM_{b_j}-2\gamma(n_i-n_j)}{2}\right)\left(\prod_{j=1}^{N}\prod_{k=1}^{n_j}\frac{\prod_{a=1}^{\tilde N_f}2\sinh\frac{-iM_{b_j}-i\tilde M_a-2i\mu+2\gamma(k-1)}{2}}{\prod_{a=1}^{N_f}2\sinh\frac{-iM_{b_j}+iM_a+2\gamma k}{2}}\right)$$

$$= (-1)^{\sum_j n_j} \left(\prod_{i<j}^{N} 2\sinh \frac{iM_{b_i} - iM_{b_j}}{2} \right)$$

$$\times \left(\prod_{j=1}^{N} \prod_{k=1}^{n_j} \frac{\prod_{a=1}^{\tilde{N}_f} 2\sinh \frac{-i\tilde{M}_a - iM_{b_j} - 2i\mu + 2\gamma(k-1)}{2}}{\left(\prod_{i=1}^{N} 2\sinh \frac{iM_{b_i} - iM_{b_j} + 2\gamma(k-1-n_i)}{2} \right) \left(\prod_{a\in\{b_j\}^c} 2\sinh \frac{iM_a - iM_{b_j} + 2\gamma k}{2} \right)} \right).$$

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