

Lifshitz to AdS flow with interpolating p -brane solutions

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ABSTRACT: In continuation with our studies of Lifshitz like Dp -brane solutions, we propose a class of 1/4 BPS supersymmetric interpolating solutions which interpolate between IR Lifshitz solutions and UV AdS solutions smoothly. We demonstrate properties of these classical solutions near the two fixed points. These interpolating solutions are then used to calculate the entanglement entropies of strip-like subsystems. With these bulk solutions the entropy functional also gets modified. We also make a curious observation about the electric-magnetic duality and the thermal entropy of the Hodge-dual Lifshitz Dp brane systems.

KEYWORDS: AdS-CFT Correspondence, p -branes, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

Recently a significant amount of work is being carried out [1]–[33], on the construction of the string duals of some strongly coupled quantum systems near the critical fixed points, exhibiting Lifshitz type scaling symmetries [4]

$$t \rightarrow \lambda^a t, \quad x^i \rightarrow \lambda x^i. \tag{1.1}$$

Namely the time and space coordinates in the CFT do scale *asymmetrically*. Some of these systems exhibit a non-fermi liquid or strange metallic behaviour at ultra low temperatures, see for details [21, 22]. There are also issues related to the entanglement entropy of the quantum subsystems [21, 34, 35]. The entanglement entropy of the subsystems can be defined geometrically as the area of a minimal surface within the bulk, with specific boundary conditions [34, 35]. Recently, a class of Lifshitz and Schrödinger type spacetimes have been constructed in type II string theory and M theory, exhibiting a fixed amount of supersymmetry [9, 13]. The Lifshitz like solutions have also been shown to arise in [14, 16, 23, 24] and from intersecting D-branes in [26, 27]. Our main focus in this work is a class of 1/4 BPS Lifshitz Dp solutions [9, 13], which can be generically obtained as vanishing horizon double limits of the boosted black p-branes vacua [13]. The supersymmetry makes these Lifshitz solutions more interesting, because we can find more definitive predictions about the boundary nonrelativistic CFT. Let us note that some of these Lifshitz IR solutions have problems in the UV region. In the paper [11], an specific resolution of the UV problem was attempted for D3-brane Lifshitz solutions. Particularly, the solutions were modified

such that they remain well behaved classical geometries even in the UV region. Here in this article we extend that particular approach to all Lifshitz Dp solutions given in [13]. We write down a new class of 1/4 supersymmetric solutions which can interpolate between (IR) Lifshitz solutions and (UV) AdS solutions smoothly. We demonstrate various properties of these classical solutions in the IR and UV asymptotic regions. These interpolating bulk solutions are then used to calculate the entanglement entropy of strip-like subsystems of the boundary CFT. In general, the entropy functional gets modified. We also make a curious observation about the effect of electric-magnetic duality on the thermal entropy of the electric/magnetic (Hodge) dual Lifshitz solutions. For example, the entropy of the near extremal Lifshitz Dp -brane goes as

$$S_{(p)} \sim T^{\frac{1}{\tilde{p}}} \tag{1.2}$$

where \tilde{p} is nothing but the number of spatial world-volume directions of the corresponding (magnetic) dual Lifshitz $D\tilde{p}$ -brane. The same relation holds good for the pair of non-extremal Lifshitz M2 and M5-branes.

The paper is planned in the following way. In the section 2 we review the basic properties of the maximally supersymmetric $\text{AdS} \times S$ vacua in type II string theory. In section 3 we study the 1/4 BPS Lifshitz Dp -brane vacua and obtain expressions for their thermal quantities at finite temperature. We do show how various thermodynamical quantities behave when vanishing temperature limit is taken. We explore the effect of electric-magnetic duality on the thermal entropy. In section 4 we write down new interpolating solutions which are well behaved in the UV. We obtain the entanglement entropy using these smooth interpolating solutions. The entanglement entropy expression matches with the recent works [32, 33] for the strip like subsystems. The conclusions are given in the section 5.

2 Dp -branes and relativistic CFTs

The maximally supersymmetric near horizon Dp -brane solutions are given by [20]

$$ds_{\text{AdS}}^2 = R_p^2 r^{\frac{p-3}{2}} \left[r^{5-p} [(dx^-)^2 - dx^+ dx^- + d\vec{x}_{(p-1)}^2] + \frac{dr^2}{r^2} + d\Omega_{(8-p)}^2 \right],$$

$$e^\phi = (2\pi)^{2-p} g_{\text{YM}}^2 R_p^{3-p} r^{\frac{(7-p)(p-3)}{4}} \tag{2.1}$$

along with a suitable $(p+2)$ -form field strength

$$F_{p+2} = (7-p) R_p^{2p-2} r^{6-p} dr \wedge dx^+ \wedge dx^- \wedge [dx_{(p-1)}] \tag{2.2}$$

for the electric type Dp -branes ($p < 3$) and a $(8-p)$ form

$$F_{8-p} = (7-p) R_p^4 \omega_{8-p} \tag{2.3}$$

for the magnetic type (hodge dual) Dp -branes ($p > 3$). Specially for D3-brane case we have $F_5 = 4(1 + \star)\omega_5$, which is self-dual 5-form field strength. We have introduced x^+, x^- as lightcone coordinates along the world volume of the branes, and $\vec{x}_{(p-1)}$ represents other $(p-1)$ spatial directions parallel to the Dp -brane, and as usual r is the radial (holographic) coordinate. The interpretation of various parameters can be found in [20] and also given in [13].

A proposal. One should note that, in these conformally $\text{AdS}_{p+2} \times S^{8-p}$ solutions, we have taken a slightly modified AdS metric elements: $r^{5-p}[(dx^-)^2 - dx^+ dx^- + d\vec{x}_{(p-1)}^2] + \frac{dr^2}{r^2}$. Namely we have introduced a constant g_{--} component. Doing this is actually harmless as it still remains an AdS geometry. The constant g_{--} term can be reabsorbed by a coordinate shifts like $x^+ \rightarrow x^+ + x^-$, if the need arises. However, certain global symmetries of the metric, such as the lightcone boost $x^- \rightarrow \lambda x^-$, $x^+ \rightarrow \frac{1}{\lambda} x^+$, are spontaneously broken in this new modified frame. The inclusion of g_{--} component in these solutions is useful in the following way. We shall be considering (nonrelativistic) Lifshitz-like solutions, having nontrivial g_{--} deformations. In these solutions, we shall take x^- to be mostly a compact direction. Note that, when x^- is compactified, in order to trust our classical string metric, it makes sense to keep g_{--} finite instead of taking it to be vanishing, also see comments in ref. [5]. Of course, in the ‘shifted’ (x^+, x^-) frame, as in (2.1), the DLCQ of boundary CFT will have a slightly changed energy-mass relationship; see the discussion in the appendix of [13]. For the case of D3-branes, such UV geometry was specifically proposed as a resolution of ‘UV problem’ of the $a = 3$ Lifshitz solutions of [13]. We must emphasize that all our Lifshitz solutions, as given below in section 3, are good only in some intermediate range of z , while they do have problem in extreme UV region, i.e. as we get near to the AdS boundary. We propose that, all those Lifshitz solutions ($p < 5$) can be modified so as to include the spacetime in eq. (2.1) as the asymptotic metric in UV.

Let us redefine the radial coordinate

$$r^{p-5} = z^2 \quad \text{for } p \neq 5 \tag{2.4}$$

Similar, redefinition of the radial coordinate can be done for D5-brane separately, if required. With z as holographic coordinate and some scaling of the brane coordinates the above solutions can be brought to the form

$$ds^2 = R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ \frac{(dx^-)^2}{z^2} + \frac{-dx^+ dx^- + d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right\} + d\Omega_{(8-p)}^2 \right]$$

$$e^\phi = (2\pi)^{2-p} g_{\text{YM}}^2 R_p^{3-p} z^{\frac{(7-p)(p-3)}{2(p-5)}} \tag{2.5}$$

along with the $(p + 2)$ -form flux. One can find that under the dilatations the coordinates would rescale as

$$z \rightarrow \xi z, \quad x^\pm \rightarrow \xi x^\pm, \quad \vec{x} \rightarrow \xi \vec{x} \tag{2.6}$$

while the dilaton and the string metric in (2.5) conformally rescale as

$$g_{MN} \rightarrow \xi^{\frac{p-3}{p-5}} g_{MN}, \quad e^\phi \rightarrow \xi^{\frac{(7-p)(p-3)}{2(p-5)}} e^\phi \tag{2.7}$$

Note this latter conformal rescaling is the standard Weyl rescaling behaviour, of non-conformal Dp-branes AdS solutions [20], giving rise to the RG flow in the boundary CFT. From eq. (2.6) the dynamical exponent of time is $a \equiv a_{\text{rel}} = 1$, so that the boundary theories are $(p + 1)$ -dimensional ‘relativistic’ CFT $_{(p+1)}$ with sixteen supercharges. Note, once x^- is taken to be a coordinate on a circle, the boundary CFT becomes a DLCQ theory

and is a p -dimensional theory. While the compactification of the bulk solution (2.5) along x^- and S^{8-p} , results in $(p+1)$ -dimensional (Einstein) metric given as

$$\begin{aligned}
 ds_{p+1}^2 &\sim z^{\left(\frac{p-5}{p-1} + \frac{p-3}{p-5}\right)} \left[-\frac{(dx^+)^2}{z^2} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right] \\
 &= z^{\frac{2(p^2-7p+14)}{(p-1)(p-5)}} \left[-\frac{(dx^+)^2}{z^2} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right] \equiv z^{\frac{2\theta}{d}} ds_{\text{AdS}_{p+1}}^2. \quad (2.8)
 \end{aligned}$$

From where we can read the hyperscaling parameter, as it is known now, to be

$$\theta = \frac{p^2 - 7p + 14}{p - 5} \equiv \theta_{\text{rel}}. \quad (2.9)$$

Note that, $d \equiv p - 1$ gives the total number of spatial directions of the boundary CFT_p . Let us mention here that there is also a running $(p+1)$ -dimensional dilaton field

$$e^{-2\phi_{(p+1)}} \sim z^{\frac{p-5}{2}} \quad (2.10)$$

as well as other form fields arising out of reduction of $(p+2)$ -form field strength. These solutions are extremal solutions.

2.1 The thermal entropy of the relativistic theory

In order to know the thermal behaviour of the boundary CFT, one includes black holes in the bulk anti-de Sitter geometry. In our coordinates the near extremal Dp solutions are

$$\begin{aligned}
 ds^2 &= R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ -\frac{(f-1)(dx^+)^2}{4z^2} + \frac{-dx^- dx^+ + (dx^-)^2 + d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{fz^2} \right\} + d\Omega_{(8-p)}^2 \right] \\
 &= R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ -\frac{f(dx^+)^2}{4z^2} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{fz^2} \right\} + \frac{1}{z^2} \left(dx^- - \frac{1}{2} dx^+ \right)^2 + d\Omega_{(8-p)}^2 \right] \quad (2.11)
 \end{aligned}$$

where function

$$f = 1 - \left(\frac{z}{z_0} \right)^{\frac{2p-14}{p-5}}$$

vanishes at $z = z_0$ ($z_0 > 0$) as it is the location of the horizon. As usual with black hole Dp -branes, the dilaton and other flux form fields remain unchanged. Corresponding thermal CFTs have a definite temperature behaviour. For example, the entropy density, s , of the relativistic theories [20]

$$s \equiv \frac{S}{V_d} \sim 2\pi r^- T^{\frac{p-9}{p-5}}, \quad T \sim z_0^{-1} \quad (2.12)$$

where V_d is the volume of the d -dimensional spatial ensemble box. There is also a chemical potential $\mu \sim \frac{1}{2r^-}$, which is trivial, as the corresponding charge density is vanishing. This is simply an artefact of our coordinate choice (shifted lightcone frame). This could be undone by a gauge choice, but we do not worry about it here. Thus the system is still a

Dp-brane	d	a_{rel}	θ_{rel}	$s \sim T^\alpha$
1	0	1	-2	T^2
2	1	1	$-\frac{4}{3}$	$T^{\frac{7}{3}}$
3	2	1	-1	T^3
4	3	1	-2	T^5

Table 1. Dynamical scaling exponents and θ parameter arising out of the relativistic Dp brane solutions.

canonical ensemble with a fixed number of particles. Using the expression for θ_{rel} given above, entropy is also expressible as

$$s \sim T^{p-1-\theta_{\text{rel}}} \equiv T^{\frac{d-\theta_{\text{rel}}}{a_{\text{rel}}}} \tag{2.13}$$

Note that the dynamical exponent of time coordinate in relativistic solutions is simply unity. Thus literally there is a hyperscaling violation as ($\theta_{\text{rel}} \neq 0$) in these relativistic systems too, due to the nontrivial conformal factors in the metrics. This is an all familiar terrain so far. We prepare a table of the corresponding CFT data in the table 1. The exponent α of the T in the entropy expression increases with the increase in the dimensionality of the relativistic ensemble.

3 $\frac{1}{4}$ -BPS Lifshitz Dp-branes

The Lifshitz like Dp solutions with eight supersymmetries are given by [9, 13]

$$\begin{aligned}
 ds_{\text{lif}}^2 &= R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ \frac{\beta^2}{z^{4/(p-5)}} (dx^-)^2 + \frac{-dx^+ dx^- + d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right\} + d\Omega_{(8-p)}^2 \right], \\
 e^\phi &= (2\pi)^{2-p} (g_{\text{YM}})^2 R_p^{3-p} z^{\frac{(7-p)(p-3)}{2(p-5)}}
 \end{aligned} \tag{3.1}$$

with the $(p+2)$ -form flux, given above (for $p \neq 5$). Here β is arbitrary scale parameter and can be absorbed by scaling the lightcone coordinates. These solutions can simply be obtained by employing ‘vanishing horizon double limits’ of the boosted black Dp-branes solutions [9, 13]. These could also be described as conformally AdS spacetimes with plane wave, having momentum along x^- . In these Lifshitz like solutions the light cone coordinates do scale *asymmetrically* under the dilatations

$$z \rightarrow \xi z, \quad x^- \rightarrow \xi^{2-a} x^-, \quad x^+ \rightarrow \xi^a x^+, \quad \vec{x} \rightarrow \xi \vec{x} \tag{3.2}$$

with the dynamical exponent of time $a = a_{\text{lif}} = \frac{2p-12}{p-5}$. At the same time the dilaton field and the metric in eq. (3.1) conformally rescale as in eq. (2.7). These Lifshitz solutions (3.1), on explicit compactifications along x^- and S^{8-p} , generically give rise to $(p+1)$ -dimensional noncompact Lifshitz metrics (in Einstein frame)

$$ds_{\text{lif}_{p+1}}^2 \sim z^{\frac{2(p^2-6p+7)}{(p-1)(p-5)}} \left(-\frac{(dx^+)^2}{\beta^2 z^{2a_{\text{lif}}}} + \frac{d\vec{x}_{p-1}^2 + dz^2}{z^2} \right), \tag{3.3}$$

with

$$a_{\text{lif}} = \frac{2p - 12}{p - 5} \quad (3.4)$$

Thus the hyperscaling parameter

$$\theta_{\text{lif}} = \frac{p^2 - 6p + 7}{p - 5}, \quad (3.5)$$

for all $0 < p \leq 6$ but $p \neq 5$. Note that θ is never vanishing in these Lifshitz solutions (3.3) or in the relativistic solutions (2.8). In fact, we generally find that

$$\theta_{\text{lif}} > \theta_{\text{rel}} \quad (3.6)$$

for all p cases. The physically interesting cases are with $p = 2, 3, 4$, and they all satisfy $a \geq \frac{\theta}{d} + 1$. The corresponding boundary nonrelativistic CFTs do have spatial dimensions $d = 1, 2, 3$ respectively. These systems could hopefully be realized in nature.

3.1 The thermal entropy of a Lifshitz system

The thermal behavior of the entropy at the Lifshitz fixed points could be studied if we consider black holes in the Lifshitz solutions (3.1). It is described by the following type of black hole solutions [13]

$$ds_{\text{Lif}}^2 = R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ -\frac{f(dx^+)^2}{4z^2 g} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{fz^2} \right\} + \frac{g}{z^2} \left(dx^- - \frac{1+f}{4g} dx^+ \right)^2 + d\Omega_{(8-p)}^2 \right] \quad (3.7)$$

where functions

$$f = 1 - \left(\frac{z}{z_0} \right)^{\frac{2p-14}{p-5}}$$

while $g(z) \equiv \frac{1}{4} \left(\frac{z}{z_{\text{IR}}} \right)^{\frac{2(p-7)}{p-5}}$, where $z_{\text{IR}} > 0$ is some an intermediate IR scale. Also $z_0 > z_{\text{IR}}$ is the black hole horizon. Note that, the dilaton and the $(p+2)$ -form field strengths remain same as in the relativistic solutions.

The thermal entropy of the system (not the entanglement entropy) is obtained by estimating the area of the black hole horizon, It can be summarised by the same type of expression as in the relativistic case, namely

$$s \sim (2\pi r^-) T^{\frac{d-\theta_{\text{lif}}}{a_{\text{lif}}}}, \quad T \sim z_0^{-a_{\text{lif}}}. \quad (3.8)$$

While the chemical potential and the charge density is given by

$$\mu_N \sim \frac{1}{r_-} \left(\frac{z_{\text{IR}}}{z_0} \right)^{\frac{2p-14}{p-5}}, \quad \rho \sim r_-^2 z_{\text{IR}}^{\frac{2p-14}{5-p}} \quad (3.9)$$

where $d \equiv p - 1$ is the number of spatial dimensions of the CFT. Note, the thermal behaviour of the system, particularly in very low temperature limit $T \rightarrow 0$ (as $r_h \equiv$

Dp-brane	d	a_{lif}	θ_{lif}	s
D1	0	$\frac{5}{2}$	$\frac{-1}{2}$	$T^{\frac{1}{5}}$
D2	1	$\frac{8}{3}$	$\frac{1}{3}$	$T^{\frac{1}{4}}$
D3	2	3	1	$T^{\frac{1}{3}}$
D4	3	4	1	$T^{\frac{1}{2}}$
D5	4	1	3	T^1

Table 2. Dynamical scaling exponents of the Lifshitz solutions.

$1/z_0 \rightarrow 0$) can be determined at a fixed charge density ($z_{\text{IR}} = \text{fixed}$) when the chemical potential is taken as $\mu_N \rightarrow 0$ in a specific manner. From (3.8) and (3.9) it is

$$s \sim T^{\frac{d-\theta_{\text{lif}}}{a_{\text{lif}}}}, \quad T \sim r_h^{a_{\text{lif}}} \sim 0, \quad \mu_N \sim r_h^{\frac{2p-14}{p-5}} \sim 0, \quad \rho = \text{fixed} \quad (3.10)$$

Especially for $p = 3$ case we have

$$s \sim T^{\frac{1}{3}}, \quad T \sim r_h^3, \quad \mu_N \sim r_h^4, \quad \rho = \text{fixed} \quad (3.11)$$

which matches with the result [9]. Since horizon size vanishes in this limit this is an extremal limit.

It is useful to note from the table 2 that the dynamical exponents of time, a_{lif} , for these Lifshitz geometries are all positive definite and generally $a_{\text{lif}} > a_{\text{rel}}$. But also a very interesting observation follows. For a given Lifshitz Dp-brane type (electric or magnetic) the exponent of T in entropy expression (3.8) is universally fixed by the unique fraction $\frac{1}{\tilde{p}}$, see the table 2, where \tilde{p} is the number of spatial directions of the corresponding *electric/magnetic* dual $D\tilde{p}$ -brane. This distinct Hodge-dual behavior of the thermal entropy of the Lifshitz system at low temperatures is remarkably present for all the Dp solutions. Therefore the entropy of the thermal Lifshitz system given in (3.8) can also be written as a simple expression

$$s_{(p)} \sim T^{\frac{1}{\tilde{p}}}. \quad (3.12)$$

Thus for example if $p = 1$, we would take $\tilde{p} = 5$, for $p = 2$, we should take $\tilde{p} = 4$, for $p = 3$ (self-dual), we should take $\tilde{p} = 3$, and for $p = 4$, we should take $\tilde{p} = 2$, and so on. We get the empirical identity

$$\tilde{p} = 6 - p = \frac{a_{\text{lif}}}{d - \theta_{\text{lif}}} \quad (3.13)$$

which is indeed true.

The same behaviour as (3.12) is also seen in the case of M-theory Lifshitz type solutions in the next section.

3.2 Lifshitz solutions in eleven dimensions

There do exist Lifshitz solution in M-theory as well, obtainable from ‘vanishing horizon double limits’ of corresponding ‘boosted black M2-branes’ [9],

$$ds_{\text{lifM2}}^2 = r^2 \left(-dx^+ dx^- + \frac{\beta^2}{4r^3} (dx^-)^2 + dy^2 \right) + \frac{1}{4} \frac{dr^2}{r^2} + d\Omega_7^2, \quad (3.14)$$

Mp-brane	d	a_{lif}	θ_{lif}	$s \sim T^\alpha$
M2	1	$\frac{5}{2}$	$\frac{1}{2}$	$T^{\frac{1}{5}}$
M5	4	4	2	$T^{\frac{1}{2}}$

Table 3. Dynamical scaling exponents of the Lifshitz M2 and M5 solutions.

with 4-form field strength F_4 being an ‘electric type flux’. We should call them electrically charged Lifshitz membrane solutions. These solutions have dynamical exponent of time as $a_{\text{lif}} = \frac{5}{2}$. Note that x^- should be taken to be compact and one could take it to be the 11-th circle of M-theory. The boundary theory would be a 1 + 1 dimensional CFT. The value of θ can be determined by going to the Einstein frame in noncompact directions spanned by the coordinates (x^+, y, r) , and it is given in the table 3.

Similarly double limits of boosted black M5-branes give us following ‘magnetically charged’ Lifshitz M5 solution

$$ds_{\text{lif}_{\text{M5}}}^2 = r^2 \left(-dx^+ dx^- + \frac{\beta^2}{4r^6} (dx^-)^2 + dy_1^2 + \dots + dy_4^2 \right) + 4 \frac{dr^2}{r^2} + d\Omega_4^2 \quad (3.15)$$

where the F_4 flux is taken along S^4 . These M5-brane Lifshitz vacua have dynamical exponent $a_{\text{lif}} = 4$ and the boundary theory is (1 + 4)-dimensional CFT.

Of course, the two Lifshitz vacua (3.14) and (3.15) in M-theory ought to be rightfully seen as electric-magnetic (Hodge) dual of each other. These 1/4 BPS (extremal) solutions describe boundary theories at respective Lifshitz fixed points. Making these solutions slightly off-extremal, that is including black holes in the IR region of the solutions, we could study the behaviour of their thermal CFTs. The respective thermal entropies are summarised in the table 3.

As discussed above that M2 and M5 Lifshitz vacua are electric-magnetic dual of each other in the same sense as ordinary relativistic M2-brane is Hodge-dual to M5-brane and vice versa. As a curious observation we find that for M-theory Lifshitz solutions the expressions of the entropy are

$$\begin{aligned} s_{\text{lif}_{\text{M2}}} &\sim T^{\frac{1}{5}} && \text{for M2} \\ s_{\text{lif}_{\text{M5}}} &\sim T^{\frac{1}{2}} && \text{for M5} \end{aligned} \quad (3.16)$$

If we pair them up, the expressions could then be summarised simply by an expression

$$s_{\text{lif}_{\text{M}(p)}} \sim T^{\frac{1}{\tilde{p}}} \quad (3.17)$$

where \tilde{p} is to be taken as the integer number counting the spatial world-volume directions of dual $\text{M}\tilde{p}$ -brane. For example, for M2-brane, $p = 2$, $\tilde{p} = 5$ and vice versa. Not only this, we can see this effect of Hodge-duality in the case of $\text{D}p$ -brane Lifshitz vacua too.

On the other hand at the (relativistic) UV fixed point, the thermal entropy of the CFTs goes as

$$s_{(p)} \sim T^{\alpha_p} \quad (3.18)$$

where $\alpha_p \geq 2$ and is usually a growing number as p increases, generally for all p -branes in ten or eleven dimensions.

4 Interpolating solutions

In this section we first take up the issue of bad UV behaviour of our Lifshitz geometries of the last sections. Then we propose a remedy so as to regularise these solutions in order to include proper AdS metric in the UV region.

4.1 Problem with Lifshitz solutions in the UV region

As we noted, once lightcone coordinate x^- is compactified, i.e. $x^- \sim x^- + 2\pi r^-$, the Lifshitz geometries (3.1) do provide a valid holographic description of a p -dimensional nonrelativistic CFT, but only in a finite z (energy) range. These solutions cannot be trusted in the far UV region. For example, let us take the D3 case, the string metric in this case cannot be trusted near the boundary (UV region) because the physical size of x^- circle

$$\frac{R_{\text{phys}}^-}{l_s} = \frac{R_3}{l_s} \beta r^- z \tag{4.1}$$

becomes sub-stringy when $z \rightarrow 0$. This is true for all other Lifshitz like solutions given in (3.1), (3.14) and (3.15). Thus this UV problem exists whenever x^- is a circle! There are a few possible ways to tackle this problem however.

1. Of course, standard thing we could do is to include higher derivative (world-sheet) corrections to the IR Lifshitz solutions when the size of x^- starts becoming sub-stringy.
2. Alternatively, as suggested in ref. [5] for the Schrödinger type solutions, it will be appropriate to go over to a T-dual type II string picture where the T-dualised x^- circle will have a finite size.
3. The third possibility could be that, it is quite plausible, to regularize the Lifshitz solutions so as to include appropriate boundary (UV) configuration, such as discussed in [11] for the D3 case.

In general, we naively expect a boundary Lifshitz theory to flow towards becoming a relativistic theory at high energies. Hence, we can think of attaching suitable boundary configuration, like the conformally AdS geometry such as in eq. (2.5), to the Lifshitz solutions (3.1). This we can do for all p cases.¹ Such D3 brane solutions with regularized UV behaviour become [11]

$$\begin{aligned} ds_{D3}^2 &= R_3^2 \left[\left(\frac{1}{z^2} + \beta^2 z^2 \right) (dx^-)^2 + \frac{-dx^+ dx^- + d\vec{x}_{(2)}^2}{z^2} + \frac{dz^2}{z^2} + d\Omega_{(5)}^2 \right] \\ e^\phi &= (2\pi)^{-1} g_{\text{YM}}^2, \quad F_{(5)} = 4R_3^4 (1 + \star) \omega_5 \end{aligned} \tag{4.2}$$

It should be noted that the solutions (4.2) no longer have the asymmetric scaling properties possessed by the purely Lifshitz solutions (3.1). These interpolating solutions (4.2) behave like solitons which interpolate between the IR Lifshitz and the UV relativistic fixed points.

¹This can be achieved by making a shift $x^+ \rightarrow x^+ - x^-$ in the above Lifshitz solutions.

Namely, in the deep IR region ($z \sim \infty$) it flows towards a Lifshitz fixed point described by $a = 3, \theta = 1$. The thermal entropy of the 2 + 1 boundary CFT at IR fixed point behaves as

$$s \sim v_2(2\pi r^-)T^{\frac{1}{3}}, \tag{4.3}$$

see the table 2. This is an entirely expected behaviour. For example, this behaviour automatically emerges when vanishing horizon double limits are employed on the thermal quantities in thermal CFT [9, 13]. While in the deep UV region, as $z \sim 0$, the solution (4.2) tends to become a conformally AdS configuration with $a = 1, \theta = -1$. The thermal entropy of the 2 + 1-dimensional CFT at the UV fixed point behaves as

$$S \sim v_2(2\pi r^-)T^3 \tag{4.4}$$

which is an expected behaviour of a relativistic 3D CFT, see the table 1. (Note that we have r^- in the above expressions because the coordinate x^- is compact having radius r^- .) Thus we have an interpolating soliton solution of type II string theory which takes us from a Lifshitz solution in IR to a relativistic solution in UV. That is, the Lifshitz theory at the IR fixed point also has a needed UV completion in terms of relativistic fixed point. This appears to be true at least in the supersymmetric examples considered here, although it may not be entirely true when there is no supersymmetry in the system.²

4.2 Interpolating Dp solutions

It is worth while to write down the interpolating solutions for all p -branes, which behave like a Lifshitz solution eq. (3.1) in the extreme IR and as a relativistic solution eq. (2.1) in the far UV region. The interpolating soliton solutions can be written as (for $p \neq 5$)

$$\begin{aligned} ds_{\text{Int}}^2 &= R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ \left(\frac{1}{z^2} + \frac{\beta^2}{z^{4/(p-5)}} \right) (dx^-)^2 + \frac{-dx^+ dx^- + d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right\} + d\Omega_{(8-p)}^2 \right] \\ &= R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ \frac{K}{z^2} (dx^-)^2 + \frac{-dx^+ dx^- + d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right\} + d\Omega_{(8-p)}^2 \right] \\ e^\phi &= (2\pi)^{2-p} g_{\text{YM}}^2 R_p^{3-p} z^{\frac{(7-p)(p-3)}{2p-10}} \end{aligned} \tag{4.5}$$

with the $(p + 2)$ -form flux. Where the new function

$$K(z) = 1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}} \right)^{\frac{2p-14}{p-5}} \tag{4.6}$$

is also a harmonic function and plays the role of the interpolating function. The parameter $z_{\text{IR}} > 0$ is an intermediate IR scale and can be related to β . It is being called interpolating solution because the metric (4.5) smoothly connects Lifshitz and AdS regions, even when x^- is compact. It is much like a ‘wormhole’ geometry, the size of x^- circle stays finite. In the asymptotic UV region ($z \ll z_{\text{IR}}$) where $K \approx 1$, it starts behaving relativistically, while for

²The flows from Lifshitz solutions have been studied earlier in suitable phenomenological settings by [36–38]. We thank the anonymous referee for the information.

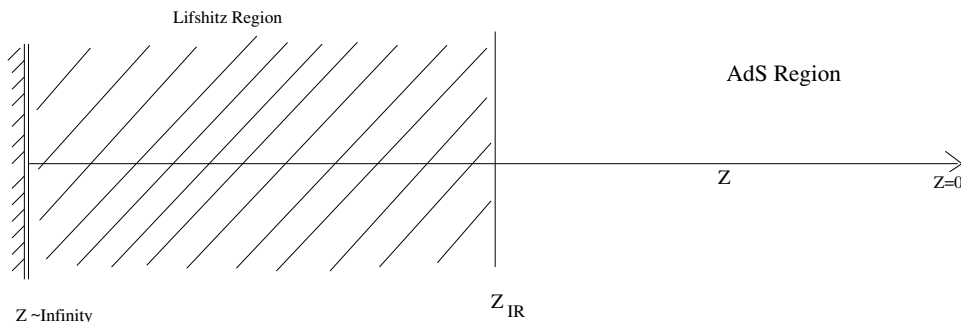


Figure 1. In zero temperature solutions the Lifshitz window (the shaded region) starts at $z \sim \infty$ ($r \sim 0$) and ends at z_{IR} .

$z \gg z_{\text{IR}}$ where $K \approx \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}}$ it behaves like a Lifshitz spacetime. Note that, since these solutions are interpolating solitonic configurations any scaling symmetry of the metric (4.5) is explicitly broken. The *scaling* or dilatation symmetry of the metric becomes explicit in extreme IR or UV regions only. This interpolating geometry is depicted schematically in the figure 1.

The explicit compactification of the metric (4.5) gives a $(p+1)$ dimensional spacetime

$$ds_{p+1}^2 = L^2 z^{\frac{2(p^2-7p+14)}{(p-1)(p-5)}} K^{\frac{1}{p-1}} \left[-\frac{(dx^+)^2}{4z^2 K} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right] \quad (4.7)$$

where K is given above in (4.6). There is a running $(p+1)$ -dimensional dilaton field

$$\begin{aligned} e^{-2\phi_{(p+1)}} &\sim z^{\frac{p-5}{2}} \sqrt{K} \\ A_{(1)} &= -\frac{1}{2K} dx^+ \end{aligned} \quad (4.8)$$

where L^2 is a specific size factor which follows from compactification.

It is also plausible to include black holes in these interpolating solutions (4.5). This can be done systematically by employing the boost, see [13], and only changes occur in the spacetime metric

$$ds_{\text{Lif}}^2 = R_p^2 z^{\frac{p-3}{p-5}} \left[\left\{ -\frac{f(dx^+)^2}{4z^2 K} + \frac{d\vec{x}_{(p-1)}^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{fz^2} \right\} + \frac{K}{z^2} (dx^- - A)^2 + d\Omega_{(8-p)}^2 \right] \quad (4.9)$$

where 1-form

$$A \equiv \frac{(1+f) + \lambda^{-2}(1-f)}{4K} dx^+$$

and the harmonic functions

$$\begin{aligned} f(z) &= 1 - \left(\frac{z}{z_0}\right)^{\frac{2p-14}{p-5}} \\ K(z) &= 1 + \frac{\lambda^2 - 1}{4\lambda} \left(\frac{z}{z_0}\right)^{\frac{2p-14}{p-5}} \equiv 1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2p-14}{p-5}} \end{aligned} \quad (4.10)$$

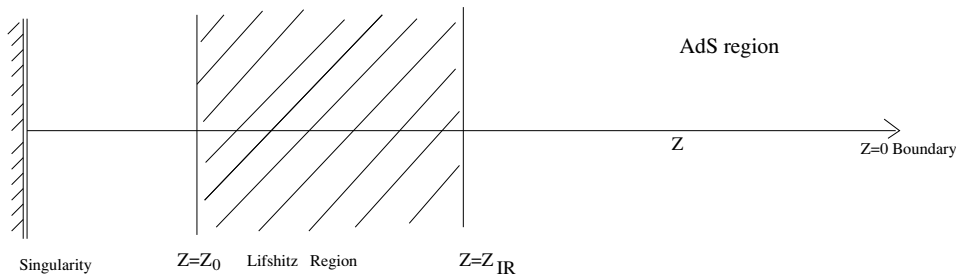


Figure 2. The Lifshitz window appears as the shaded region. It starts at z_0 and ends at z_{IR} .

The dilaton and other form fields remain unchanged. The $z = z_0$ is the location of the black hole horizon. Note that λ is the boost parameter in the above. In the absence of boost, $\lambda = 1$, then $K = 1$. Since the Lifshitz region for many physical applications would be some intermediate (IR) region, it would be worth while to take $z_0 > z_{\text{IR}} > 0$, and this is always guaranteed from (4.10). In this way, the black hole singularity is capped by its horizon. We call the intermediate region $z_0 \geq z \geq z_{\text{IR}}$ as the Lifshitz window region where parameter z_{IR} provides the effective width of the window beyond the horizon. While in the deep UV region, $z \ll z_{\text{IR}}$ the solutions become asymptotically conformally AdS, see the figure 2. Note that the size of Lifshitz window depends on the boost, it can be widened if we take λ sufficiently large. Specially if $\lambda = 1$ the Lifshitz region altogether disappears and we get ordinary AdS black hole solutions. The Lifshitz BH solutions (4.9) with an intermediate Lifshitz region should present a good IR description (at finite temperature) of a boundary Lifshitz theory. The black hole horizon provides an effective IR (thermal) cut-off scale in the dual CFT.

4.3 Entanglement entropy

In order to find the entanglement entropy of the CFT, we shall use the interpolating zero temperature solutions like (4.5) or (4.7). According to Ryu-Takayanagi proposal [34, 35], if we pick up a subsystem A (with its boundary ∂A), the subsystem has an entanglement of its states with its parent system. Then the entanglement entropy of the subsystem A can be given geometrically in terms of the area of an extremal surface $X_{(p-1)}$ (space like $(p-1)$ -dimensional surface) ending on to the boundary ∂A . Thus we have

$$S_{\text{Ent}}(A) = \frac{1}{4G_{p+1}} [\text{Area}]_X \tag{4.11}$$

The extremal surface X extends well inside the bulk geometry. We pick up the subsystem A to be a rectangular strip along $x_1(z), x^-(y)$ at any fixed time. Note that, x^+ is identified with boundary time coordinate and it does not depend upon y . Also as per our study we have to take x^- being a compact coordinate. The range of the coordinates is $-l/2 \leq x_1 \leq l/2$ and the regulated size of other coordinates is $0 \leq x^i \leq l^i$. (For noncompact x^- the subsystem A must be thought off as a strip of finite width l stretched along spatial direction x^- .) For our calculations we shall consider the $(p+1)$ -dimensional Einstein metric

as in (4.7). Then

$$S_{\text{Ent}} = \frac{1}{4G_{p+1}} \int \sqrt{g_X} \tag{4.12}$$

where g_X is the induced metric on the $(p - 1)$ -dimensional extremal surface X . Note after the compactification along x^- the strip becomes just an interval along x_1 . Using the compactified metric (4.7), we find that

$$S_{\text{Ent}} = \frac{V_{p-2}L^d}{2G_{p+1}} \int_{z_*}^{z_\infty} dz z^{\frac{9-p}{p-5}} \sqrt{K} \sqrt{\frac{4}{(5-p)^2} + (x'_1)^2} \tag{4.13}$$

where $z_\infty \approx 0$ is the UV cut-off and z_* is the turning point. V_{p-2} is the size of the ensemble box stretched along rest of the spatial directions, x_2, \dots, x_{p-1} . K is as given in (4.6). The extremal surface satisfies the first order equation

$$\frac{dx_1}{dz} = \frac{2}{5-p} \frac{C z^{\frac{p-9}{p-5}}}{\sqrt{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}} - C^2 z^{\frac{2(p-9)}{p-5}}}} \tag{4.14}$$

where C is the integration constant. The turning point arises where $x'_1|_{z_*} = \infty$. While near the boundary point $x'_1|_{z_\infty} \sim 0$. Finding solutions of first order differential equation (4.14) is much like solving a classical orbit in the central force problem with given boundary (initial) conditions. The term $C^2 z^{\frac{2(p-9)}{p-5}}$ plays the role of a repulsive centrifugal type force, while the term $-\frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}}$ behaves like an attractive central force. Thus Lifshitz deformation in the IR region is of attractive nature while the repulsive forces mainly come from the curvature of AdS spacetime. This gives finally the entropy formula

$$S_{\text{Ent}} = \frac{V_{p-2}L^d}{2G_{p+1}} \int_{z_*}^{z_\infty} dz z^{\frac{9-p}{p-5}} \frac{2}{(5-p)} \frac{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}}}{\sqrt{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}} - C^2 z^{\frac{2(p-9)}{p-5}}}} \tag{4.15}$$

This expression matches with other calculations in the literature [32, 33]. If we set $1/z_{\text{IR}}$ to be zero, the expression (4.15) reduces to the entanglement entropy in the relativistic CFT system. It can be seen that the turning point of the extremal surface in the purely AdS case appears at the value $z = z_c \equiv C^{\frac{5-p}{p-9}}$. Thus we always have $z_* > z_c$ for the Lifshitz system. Thus the area of the extremal surface is larger in the Lifshitz case. Hence the entanglement entropy of the Lifshitz system is generally larger compared to the relativistic (AdS) case. That is

$$S_{\text{Ent}}^{\text{Lifshitz}} > S_{\text{Ent}}^{\text{AdS}}. \tag{4.16}$$

At the finite temperature, looking at eqs. (4.9) and (4.10), we find that

$$S_{\text{Ent}} = \frac{V_{p-2}L^d}{2G_{p+1}} \int_{z_*}^{z_\infty} dz z^{\frac{9-p}{p-5}} \sqrt{K} \sqrt{\frac{4}{(5-p)^2 f} + (x'_1)^2} \tag{4.17}$$

where $f(z)$ is given earlier in (4.10). We always have $z_0 > z_{\text{IR}}$. The extremal surface satisfies the first order equation

$$\frac{dx_1}{dz} = \frac{2}{5-p} \frac{1}{\sqrt{f}} \frac{Cz^{\frac{p-9}{p-5}}}{\sqrt{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}} - C^2 z^{\frac{2(p-9)}{p-5}}}} \quad (4.18)$$

This gives finally the entanglement entropy formula (at finite temperature)

$$S_{\text{Ent}} = \frac{V_{p-2} L^d}{2G_{p+1}} \int_{z_*}^{z_\infty} dz z^{\frac{9-p}{p-5}} \frac{2}{(5-p)} \frac{1}{\sqrt{f}} \frac{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}}}{\sqrt{1 + \frac{1}{4} \left(\frac{z}{z_{\text{IR}}}\right)^{\frac{2(p-7)}{p-5}} - C^2 z^{\frac{2(p-9)}{p-5}}}}. \quad (4.19)$$

5 Conclusion

We have presented quarter BPS Lifshitz Dp -brane vacua and obtained explicit expressions for their thermal quantities at finite temperature. We studied how various quantities behave if the low temperature limit is taken, at fixed charge density. We also studied how Lifshitz Dp -brane systems are mapped under electric-magnetic duality. For example the entropy of the near extremal Lifshitz Dp -brane goes as

$$S_{(p)} \sim T^{\frac{1}{\tilde{p}}} \quad (5.1)$$

where \tilde{p} is an integer giving us the number of spatial world-volume directions of the magnetic dual Lifshitz $D\tilde{p}$ -brane. Thus

$$\tilde{p} = 6 - p = \frac{a_{\text{lif}}}{d - \theta_{\text{if}}}. \quad (5.2)$$

Surprisingly, the same behaviour persists also for the extremal Lifshitz M2 and M5-brane vacua, which are electric-magnetic duals of each other in M-theory. Thus the Lifshitz systems though being inherently nonrelativistic do encode deep quantum relationships such as electric-magnetic duality. Any measurement of these Lifshitz thermal exponents, say $s \sim T^{\frac{1}{4}}$, $s \sim T^{\frac{1}{3}}$ or $s \sim T^{\frac{1}{2}}$ in condensed matter systems with 1, 2 or 3 spatial dimensions, respectively, could be taken as a signature test of electric-magnetic (Hodge) duality in nonrelativistic string systems. It would also be useful to further understand the basic reason behind it.

We have written down the interpolating solutions as well. These class of solutions are well behaved and can be trusted for the classical analysis in the UV region also. The entanglement entropy is calculated by using these interpolating solutions and its expression matches with the recent works of [32, 33]. We also find that the entanglement entropy of the Lifshitz system is generally larger compared to the relativistic (AdS) case.

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