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On the integrability of planar $\mathcal{N} = 2$ superconformal gauge theories

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Abstract: We study the integrability properties of planar $\mathcal{N} = 2$ superconformal field theories in four dimensions. We show that the spin chain associated to the planar dilation operator of $\mathcal{N} = 2$ superconformal QCD fails to be integrable at two loops. In our analysis we focus on a closed SU(2|1) sector, whose two-loop spin chain we fix by symmetry arguments (up to a few undetermined coefficients). It turns out that the Yang-Baxter equation for magnon scattering is not satisfied in this sector. On the other hand, we suggest that the closed SU(2, 1|2) sector, which exists in any $\mathcal{N} = 2$ superconformal gauge theory, may be integrable to all loops. We summarize the known results in the literature that are consistent with this conjecture.

Keywords: Supersymmetric gauge theory, 1/N Expansion, Integrable Field Theories

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Contents

1 Introduction and summary

There is by now overwhelming evidence that planar $\mathcal{N} = 4$ super-Yang Mills theory is a completely integrable model (see [1] for a comprehensive review). To which extent integrability survives in less symmetric (and more realistic) gauge theories is an important question, both because integrability is a very useful computational tool, and because exploring a larger set of examples should shed light on its conceptual origin, which is still mysterious. In fact, the first instances of integrability in a four-dimensional gauge theory were found in QCD itself [2–7]. However, with hindsight, the integrability properties of large N_c QCD discovered so far can be understood as being "inherited" from the maximally supersymmetric theory. For example, a large sector of QCD composite operators has identical one-loop renormalization as the analogous sector in $\mathcal{N} = 4$ SYM.[1] At higher loops, the analysis of the QCD dilation operator is complicated by the breaking of conformal invariance and by the (non-universal) dependence on the regulator. A parallel story holds for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Yang-Mills theories in the usual 't Hooft limit (large N_c, fixed N_f), see [9–13] and references therein. This motivates us to explore integrability in the cleaner theoretical laboratory of theories that remain exactly conformal at the quantum level. The main question one would like to answer is whether integrability in less symmetric conformal gauge theories is always an "accidental" remnant of the $\mathcal{N} = 4$ integrability (and under which conditions do such accidents occur), or whether genuinely new structures are also possible.

A large class of four-dimensional conformal theories are the $\mathcal{N} = 2$ supersymmetric theories with vanishing one-loop beta function. A well-known non-renormalization theorem

[1]The maximal one-loop integrable sector in QCD is the SU(2, 2) sector described in [8]. It contains the SL(2, $\mathbb{R}$) sector of maximal helicity "quasipartonic" lightcone operators. In this latter sector, the planar dilation operator has been shown to coincide with that of $\mathcal{N} = 4$ SYM also at two loops [9, 10], up to overall factors that capture the non-vanishing beta function and the non-universal regulator dependence.

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guarantees that the beta function remains zero in the full quantum theory. Perhaps the simplest example (beyond $\mathcal{N} = 4$ SYM itself) is $\mathcal{N} = 2$ superconformal QCD (SCQCD), the theory with gauge group $SU(N_c)$ and $2N_c$ fundamental hypermultiplets. Integrability is *at best* expected in the planar Veneziano limit of large N_c *and* large $N_f \equiv 2N_c$, with fixed 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$.

The dilation operator of planar SCQCD defines, as usual, the Hamiltonian of a spin chain.[2] We review its symmetry structure in section 2. Closed chains correspond to flavor singlet gauge-invariant operators of the schematic form [24, 25] $\mathrm{Tr}\left(\varphi^{k_1}\mathcal{M}^{k_2}\varphi^{k_3}\mathcal{M}^{k_4}\ldots\right)$. Here φ denotes any of the color-adjoint elementary "letters", for example $\varphi = (\mathcal{D}^n\lambda)^a{}_b$, where $\mathcal{D}$ is a gauge-covariant derivative, λ a gaugino field, and $a, b = 1, \ldots N_c$ color indices. The symbol $\mathcal{M}$ stands for any of the gauge-adjoint composite dimers obtained by the flavor contraction of a fundamental and a antifundamental letter, for example $\mathcal{M}^a{}_b = Q^{ai}\bar{Q}_{bi}$, where Q is the squark field and $i = 1, \ldots N_f$ a flavor index. One can also consider open chains with open flavor indices at the endpoints.

The one-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD was evaluated in the sector of composite operators made of elementary scalar fields in [25], and for the full theory in [26]. The question of its integrability is still not completely settled. Despite some early intriguing hints [25], the spectrum of anomalous dimensions does not exhibit [27] the systematic pairing of opposite-parity eigenvalues that is one of the hallmarks of integrability [15, 17, 18, 28]. It is often easy to *disprove* integrability by setting up a position-space Bethe ansatz and showing that the n-body magnon S-matrix does *not* factorize. In our case, this is not straightforward because the S-matrix of external *dimeric* magnons ($\mathcal{M}$'s moving on the chain) is hard to calculate. On the other hand, the S-matrix of the *elementary* (single-letter) magnons is unaffected at one loop by the presence of the dimers, and trivially coincides with a restriction of the $\mathcal{N} = 4$ S-matrix — an instance of "accidental" one-loop integrability inherited from $\mathcal{N} = 4$ SYM.

As it turns out, it is easier to test integrability at two loops. In section 3 we consider a simple closed $SU(2|1)$ sector, and fix its two-loop Hamiltonian using symmetry, up to a few undetermined parameters. This sector is particularly interesting because it is structurally different from any subsector of $\mathcal{N} = 4$ SYM, as the dimers play a crucial role. The asymptotic excitations on the $SU(2|1)$ chain are gauginos λ_α, where α is an $SU(2)$ Lorentz index. In section 4 we evaluate their two-body scattering matrix and find that it fails to satisfy the Yang-Baxter equation, which conclusively shows that the Hamiltonian of $\mathcal{N} = 2$ SCQCD is *not* completely integrable at higher loops. This would have required a novel integrability structure (not present in $\mathcal{N} = 4$ SYM), which fails to materialize.

There is however still hope for all-loop integrability in other closed subsectors. As we have mentioned, one can identify sectors for which the one-loop dilation operator is identical to a restriction of the $\mathcal{N} = 4$ dilation operator. The largest such sector that remains closed to all orders is the $SU(2, 1|2)$ sector, which consists entirely of letters belonging to the $\mathcal{N} = 2$ vector multiplet, and it is thus a universal sector present in all $\mathcal{N} = 2$ superconformal gauge

[2] See e.g. [14–20] and the reviews [21, 22] for a very partial list of references on the evaluation of the dilation operator in $\mathcal{N} = 4$ SYM. See also [23] for a review of the dilation operator in deformations of $\mathcal{N} = 4$ SYM.

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theories with a Lagrangian description. Of course, in any given theory, all the other fields (such as the fundamental hypermultiplets of SCQCD) do affect the renormalization of the $SU(2,1|2)$ sector, so at sufficiently high order the dilation operator *will* differ from the one of $\mathcal{N}=4$ SYM. Nevertheless, consideration of the symmetry structure of the magnon S-matrix and of the holographic sigma model (when available) lead us to conjecture in section 5 that the $SU(2,1|2)$ sector may remain integrable to all orders. The simplest scenario is that, in any given theory, the dilation operator in this sector coincides with the one in $\mathcal{N}=4$ SYM, up to a model-dependent redefinition of the 't Hooft coupling [29] — a mild but still non-trivial deformation. Analogous (though less compelling) speculations apply to the universal $SU(2,1|1)$ sector that is present in any $\mathcal{N}=1$ superconformal gauge theory, and even to the purely bosonic $SU(2,1)$ sector of QCD, near the Banks-Zaks fixed point at the upper edge of the conformal window.

2 Preliminaries: symmetry structure of the $\mathcal{N}=2$ SCQCD spin chain

The field content of $\mathcal{N}=2$ superconformal QCD comprises an $\mathcal{N}=2$ vector multiplet $\{\phi, \lambda_\alpha{}^{\mathcal{I}}, \mathcal{F}_{\alpha\beta}\}$ and its conjugate, in the adjoint representation of the $SU(N_c)$ gauge group, and $N_f=2N_c$ hypermultiplets $\{Q^{\mathcal{I}}, \psi_\alpha, \bar{\tilde{\psi}}_{\dot\alpha}\,; \bar{Q}_{\mathcal{I}}, \tilde{\psi}_\alpha, \bar{\psi}_{\dot\alpha}\}$, in the (anti)fundamental representation of $SU(N_c)$. Here $\alpha=\pm$ and $\dot\alpha=\dot\pm$ are Lorentz indices, and $\mathcal{I}=\pm$ an $SU(2)_R$ R-symmetry index. We have suppressed color and flavor indices.

States of the spin chain are constructed by stringing together color-adjoint single letters from the vector multiplet, and color-adjoint two-letter "dimers" from the hypermultiplets, e.g. $\psi_i\bar{Q}^i$, where $i=1,\dots N_f$ is a contracted flavor index. Furthermore, each letter can be acted upon by an arbitrary number of covariant derivatives.

The $\mathcal{N}=2$ superconformal group is $SU(2_\alpha, 2_{\dot\alpha}|2_{\mathcal{I}})$, where the subscripts serve to emphasize the Lorentz and R-symmetry subgroups: $SU(2_\alpha)\times SU(2_{\dot\alpha})\times SU(2_{\mathcal{I}})\times U(1)_R \subset SU(2_\alpha, 2_{\dot\alpha}|2_{\mathcal{I}})$. The spin chain vacuum is the chiral state $\mathrm{Tr}\,\phi^k$. It breaks the superconformal group to the subgroup $PSU(2_{\dot\alpha}|2_{\mathcal{I}})\times SU(2_\alpha)\ltimes\mathbb{R}$, where $\mathbb{R}$ is a central generator that gets identified with the spin chain Hamiltonian. In accordance with Goldstone's theorem, broken symmetry generators are manifested as gapless excitations of the spin chain called magnons. Table 1 shows the symmetry generators of the $\mathcal{N}=2$ superconformal algebra. The diagonal boxed generators correspond to the symmetry preserved by the vacuum while the off-diagonal ones are broken and correspond to Goldstone magnons, which transform in the bifundamental representation of $PSU(2_{\dot\alpha}|2_{\mathcal{I}})\times SU(2_\alpha)$.

A priori, the two-body magnon S-matrix when decomposed according to $SU(2_{\dot\alpha}|2_{\mathcal{I}})\times SU(2_\alpha)$ quantum numbers will take the form

$$S_{SU(2_{\dot\alpha},2_\alpha|2_{\mathcal{I}})} = S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})}\times S^{\mathbf{1}}_{SU(2_\alpha)} + S'_{SU(2_{\dot\alpha}|2_{\mathcal{I}})}\times S^{\mathbf{3}}_{SU(2_\alpha)}\,, \tag{2.1}$$

where the superscripts **1** and **3** denote the singlet and triplet $SU(2_\alpha)$ representations. Remarkably, the product of two fundamental $SU(2|2)$ representations consists of a single irreducible representation, which implies that the $SU(2|2)$ two-body S-matrix is completely fixed by symmetry, up to an overall phase [30]. Thus, the total two-body S-matrix of our

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	$SU(2_{\dot\beta})$	$SU(2_{\mathcal{J}})$	$SU(2_\beta)$
$SU(2_{\dot\alpha})$	$\dot{\mathcal{L}}_{\dot\alpha}{}^{\dot\beta}$	$\bar{\mathcal{Q}}_{\mathcal{J}\dot\alpha}$	$\mathcal{D}^{\dagger}_{\beta\dot\alpha}$
$SU(2_{\mathcal{I}})$	$\bar{\mathcal{S}}^{\mathcal{I}\dot\beta}$	$\mathcal{R}_{\mathcal{J}}{}^{\mathcal{I}}$	$\lambda^{\dagger\mathcal{I}}_{\beta}$
$SU(2_\alpha)$	$\mathcal{D}^{\alpha\dot\beta}$	$\lambda_{\mathcal{J}}{}^{\alpha}$	$\mathcal{L}_{\beta}{}^{\alpha}$

Table 1. The $\mathcal{N}=2$ superconformal generators. The boxed generators are preserved by the choice of the spin chain vacuum while the unboxed ones are broken and correspond to Goldstone excitations. The broken generators are identified with the corresponding magnon: the upper-right column contains magnon creation operators while the lower-left row contains magnon annihilation operators.

model factorizes as

$$S_{SU(2_\alpha,2_{\dot\alpha}|2_{\mathcal{I}})} = S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})} \times S_{SU(2_\alpha)}\,. \tag{2.2}$$

The $S_{SU(2_{\dot\alpha}|2_{\mathcal{I}})}$ factor is the two-body S-matrix of the magnons in the $SU(2_\alpha)$ highest weight state, namely $\{\,\lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}\,\}$, while $S_{SU(2_\alpha)}$ is the two-body S-matrix of the magnons in the $SU(2_{\dot\alpha}|2_{\mathcal{I}})$ highest weight state, namely $\{\lambda_\alpha{}^+\}$.

The symmetry analysis also helps us organize the calculation of the dilation generator. We can identify two "orthogonal" all-order closed subsectors, associated with either factor of the two-body S-matrix. Exciting an arbitrary number of $SU(2_\alpha)$ highest weight magnons $\{\,\lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}\}$ above the spin chain vacuum $\mathrm{Tr}\,\phi^k$, and demanding closure of the dilation operator, we obtain a subsector with enhanced $SU(2,1|2)$ symmetry, spanned by the following letters:

$$SU(2,1|2)\text{ sector:} \qquad (\mathcal{D}_{+\dot\alpha})^n\{\,\phi, \lambda_+{}^{\mathcal{I}}, \mathcal{F}_{++}\,\}\,. \tag{2.3}$$

Here the covariant derivatives are understood to be totally symmetrized at each site, so for example $(\mathcal{D}_{+\dot\alpha})^n\phi$ is shorthand for $\mathcal{D}_{+\{\dot\alpha_1}\mathcal{D}_{+\dot\alpha_2}\ldots\mathcal{D}_{+\dot\alpha_n\}}\phi$. The introduction of the self-dual field strength $\mathcal{F}_{++} = [\mathcal{D}_{+\dot+}, \mathcal{D}_{+\dot-}]$ is necessary to achieve closure of the dilation operator because of the transition $\epsilon_{\mathcal{I}\mathcal{J}}\lambda_+{}^{\mathcal{I}}\lambda_+{}^{\mathcal{J}} \leftrightarrow \phi\mathcal{F}_{++}$.

Similarly, considering the $SU(2_{\dot\alpha}|2_{\mathcal{I}})$ highest weight magnons $\{\lambda_\alpha{}^+\}$, and demanding closure we obtain a sector with $SU(2|1)$ symmetry:

$$SU(2|1)\text{ sector:} \qquad \{\,\phi, \lambda_\alpha{}^+, \mathcal{M}^{++}\,\}\,, \tag{2.4}$$

where we have introduced the notation $\mathcal{M}^{\mathcal{I}\mathcal{J}} \equiv Q^{\mathcal{I}}_i\bar{Q}^{i\mathcal{J}}$. Inclusion of the $\mathcal{M}^{++}$ dimer is forced at two loops by the transition $\epsilon^{\alpha\beta}\lambda_\alpha{}^+\lambda_\beta{}^+ \leftrightarrow \phi\mathcal{M}^{++}$.

In the rest of the paper we will consider separately these two subsectors. The $SU(2,1|2)$ sector exists in any $\mathcal{N}=2$ gauge theory, including $\mathcal{N}=4$ SYM, while the $SU(2|1)$ sector is special to $\mathcal{N}=2$ SCQCD and has the potential to reveal a new integrability structure.

3 The two-loop Hamiltonian in the SU(2|1) sector

In this section we will use symmetry arguments to fix the two-loop Hamiltonian of the $SU(2|1)$ sectors, up to a few arbitrary coefficients. With this result at hand, we will

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proceed in the following section to calculate the two-body scattering of magnons and test integrability of the sector. To avoid cluttering we will suppress the "+" $SU(2)_R$ index and write the letters as

$$\{ \phi, \lambda_\alpha, \mathcal{M} \} . \tag{3.1}$$

At one loop the sector decomposes into $\{\phi, \lambda_\alpha\}$ and $\{\phi, \mathcal{M}\}$. Each of these subsectors is separately integrable: the first one, because it is identical to the corresponding sector in $\mathcal{N} = 4$ SYM. The second one, because its Hamiltonian turns out to be trivial [25] — the dimer $\mathcal{M}$ does not move on the ϕ chain so each string of ϕ's and $\mathcal{M}$'s is already an exact eigenstate. The SU(2|1) sector becomes interesting at two loops, where interaction with $\mathcal{M}$ affects the scattering of the asymptotic λ_α magnons.

To avoid an explicit Feynman diagram calculation we will use the approach of [17], where the symmetry algebra was used to restrict the form of the spin chain Hamiltonian in the SU(2|3) subsector of $\mathcal{N} = 4$ SYM. In that case, the two-loop Hamiltonian turned out to be completely fixed by symmetry.

Parity. It will be useful to define a "parity" operation on the states of the chain. As explained in [27], $\mathcal{N} = 2$ SCQCD admits a parity transformation that commutes with the Hamiltonian at all loops. The transformations relevant for the fields in the SU(2|1) subsector are

$$\phi^a{}_b \leftrightarrow -\phi^b{}_a \,, \quad \lambda^a{}_b \leftrightarrow -\lambda^b{}_a \,, \quad \mathcal{M}^a{}_b \leftrightarrow -\mathcal{M}^b{}_a \,. \tag{3.2}$$

This is just transposition of adjoint indices with an extra minus sign. The action on a single trace state is then (using a ket notation for the states of the chain):

$$P|A_1 \dots A_L\rangle = (-1)^{L+f(f+1)/2}|A_L \dots A_1\rangle \,, \tag{3.3}$$

where f is the number of fermionic fields and L is the length of the state considering $\mathcal{M}$ as a *single-site* object.

3.1 Symmetry analysis

The states of the sector furnish a representation of the SU(2|1) algebra. In the interacting theory, the symmetry generators can be written as a perturbation series in the coupling constant [17, 28],

$$\mathcal{J}(g) = \sum_{k=0}^{\infty} g^k \mathcal{J}_k \,. \tag{3.4}$$

As usual when working with spin chains we will focus in the *local* action of the generators, the *complete* action being a sum of local terms. Following [17] we will represent the action of a generator by the symbol

$$\mathcal{J}_k \sim \left\{ {a_1 \dots a_n \atop b_1 \dots b_m} \right\} . \tag{3.5}$$

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This replaces the string of fields $a_1 \dots a_n$ by $b_1 \dots b_m$ and gives zero otherwise. To obtain the total action we apply this transformation at each site of the closed chain. For example,

$$\{{}^{AB}_{CD}\}|ABEABF\rangle = |CDEABF\rangle + 0 + 0 + |ABECDF\rangle + 0 + 0\,. \tag{3.6}$$

Of course, we will pick up an extra minus sign each time a fermionic generator ($\mathcal{Q}$ or $\mathcal{S}$) hops a fermionic field. An interaction with $n+m$ entries will be said to have $n+m$ legs. Because corrections to the generators have their origin in planar perturbation theory, the number of legs is restricted by the order of the coupling constant we are considering. The counting is easier if we forget for a moment our definition of $\mathcal{M}$ and consider Q as fundamental field of our sector. The number of legs is then restricted by,

$$n + m = k + 2\,, \tag{3.7}$$

where k is the order of the coupling.[3] Now, if a Q field sits at the far right in the upper or lower row of (3.5), we know that the next field to its right will be a $\bar{Q}$, in order to have a flavor singlet. An analogous analysis holds for a $\bar{Q}$ sitting in the far left. This means that after writing the $\mathcal{J}$ generators using the Q and $\bar{Q}$ fields, we can replace all the Q's($\bar{Q}$'s) in the far right(left) with an $\mathcal{M}$ symbol, in addition to the explicit $Q\bar{Q} = \mathcal{M}$ replacement.

The SU(2|1) algebra. To obtain the SU(2|1) algebra we start from the full SU(2, 2|2) generators:[4]

$$\{\,\mathcal{L}_\alpha{}^\beta, \dot{\mathcal{L}}_{\dot\alpha}{}^{\dot\beta}, \mathcal{R}_\mathcal{I}{}^\mathcal{J}, \mathcal{P}_{\alpha\dot\beta}, \mathcal{K}^{\alpha\dot\beta}, D, r, \mathcal{Q}_\alpha{}^\mathcal{I}, \mathcal{S}_\mathcal{I}{}^\alpha, \bar{\mathcal{Q}}_{\dot\alpha\mathcal{I}}, \bar{\mathcal{S}}^{\dot\alpha\mathcal{I}}\,\}\,, \tag{3.8}$$

where $\mathcal{L}$ and $\dot{\mathcal{L}}$ are the Lorentz generators, $\mathcal{R}$ and r correspond to $\mathrm{SU}(2)_R$ and the U(1) r-charge, D is the dilation operator and $\mathcal{Q}$ and $\mathcal{S}$ are the supercharges. We now define

$$\mathcal{Q}_\alpha \equiv \mathcal{Q}_\alpha{}^+\,, \tag{3.9}$$
$$\mathcal{S}^\alpha \equiv \mathcal{S}_+{}^\alpha\,, \tag{3.10}$$
$$\mathcal{U} \equiv \mathcal{R}_+{}^+ + \tfrac{1}{2}\left(D_0 - r\right)\,, \tag{3.11}$$
$$\delta\mathcal{H} \equiv \delta D\,. \tag{3.12}$$

We have split the interacting dilation generator as

$$D = D_0 + \delta D\,, \tag{3.13}$$

where D_0 measures the classical conformal dimension and δD its quantum corrections.[5] The SU(2|1) generators are then:

$$\mathcal{J} = \{\mathcal{L}_\alpha{}^\beta, \mathcal{U}, \delta\mathcal{H}, \mathcal{Q}_\alpha, \mathcal{S}^\alpha\}\,. \tag{3.14}$$

[3] As in [17], we use gauge invariance of cyclic states to increase the legs of the generators to its maximum value, i.e. $k+2$ at order k in the coupling.

[4] We follow the conventions of [26].

[5] To be consistent with (3.12) we also define $\mathcal{H}_0 \equiv D_0$, although $\mathcal{H}_0$ is not an SU(2|1) generator.

As in [17], we enhanced the algebra by the extra central U(1) generator δH. The commutation relations are easy to obtain from the original SU(2, 2|2) commutators. Generators carrying SU(2) Lorentz indices transform canonically according to:

$$[\mathcal{L}_\alpha{}^\beta, \mathcal{J}_\gamma] = \delta^\beta_\gamma \mathcal{J}_\alpha - \tfrac{1}{2}\delta^\beta_\alpha \mathcal{J}_\gamma\,, \qquad [\mathcal{L}_\alpha{}^\beta, \mathcal{J}^\gamma] = -\delta^\gamma_\alpha \mathcal{J}_\beta + \tfrac{1}{2}\delta^\beta_\alpha \mathcal{J}^\gamma\,. \tag{3.15}$$

The only non-zero anti-commutator is:

$$\{\mathcal{S}^\beta, \mathcal{Q}_\alpha\} = \mathcal{L}_\alpha{}^\beta + \delta^\beta_\alpha(\mathcal{U} + \tfrac{1}{2}\delta\mathcal{H}) \tag{3.16}$$

and the non-zero $\mathcal{U}$-charges are:

$$[\mathcal{U}, \mathcal{Q}_\alpha] = -\tfrac{1}{2}\mathcal{Q}_\alpha\,, \qquad [\mathcal{U}, \mathcal{S}^\alpha] = \tfrac{1}{2}\mathcal{S}^\alpha\,. \tag{3.17}$$

Also,

$$[\mathcal{J}, \delta\mathcal{H}] = 0\,, \tag{3.18}$$

confirming that $\delta\mathcal{H}$ is indeed a central element.

Note that $\mathcal{U}$ is defined in terms of generators that do not receive quantum corrections and therefore it will not be modified in the interacting theory. The same applies to $\mathcal{L}_\alpha{}^\beta$ if we choose a regularization scheme consistent with Lorentz symmetry. In general, different regularization schemes can differ in which generators will be quantum deformed, but the physical outcome (in this case, the eigenvalues of the dilation operator) must of course be the same in all schemes. Our algebraic analysis takes the simplest form in a scheme where the Lorentz generators maintain the tree level form. An example of such a scheme is dimensional regularization, where Lorentz invariance is manifest at each step.

3.2 The interacting generators

The tree-level representation of the SU(2|1) algebra reads

$$\begin{aligned} \mathcal{U} &= \left\{ {\phi \atop \phi} \right\} + \tfrac{1}{2}\left\{ {\alpha \atop \alpha} \right\}, \\ \mathcal{L}_\alpha{}^\beta &= \left\{ {\alpha \atop \beta} \right\} - \tfrac{1}{2}\delta^\alpha_\beta\left\{ {\gamma \atop \gamma} \right\}, \\ (\mathcal{Q}_\alpha)_0 &= e^{i\beta_1}\left\{ {\phi \atop \alpha} \right\}, \\ (\mathcal{S}^\alpha)_0 &= e^{-i\beta_1}\left\{ {\alpha \atop \phi} \right\}, \end{aligned} \tag{3.19}$$

where the subscript "0" indicates that we are working at tree level. The idea is to consider perturbative deformations of these generators and restrict their form using the SU(2|1) algebra. In principle, there should be fluctuations in the length, but because we consider the dimeric impurity $\mathcal{M}$ as a single-site object, the length always stays constant. For $\mathcal{H}_2$ we have:

$$\begin{aligned} \mathcal{H}_2 = {} & c_0\left\{ {\phi\phi \atop \phi\phi} \right\} + c_1\left\{ {\phi\mathcal{M} \atop \phi\mathcal{M}} \right\} + c_2\left\{ {\mathcal{M}\phi \atop \mathcal{M}\phi} \right\} + c_3\left\{ {\mathcal{M} \atop \mathcal{M}} \right\} + c_4\left\{ {\phi\alpha \atop \phi\alpha} \right\} + c_5\left\{ {\alpha\phi \atop \alpha\phi} \right\} \\ & + c_6\left\{ {\phi\alpha \atop \alpha\phi} \right\} + c_7\left\{ {\alpha\phi \atop \phi\alpha} \right\} + c_8\left\{ {\alpha\mathcal{M} \atop \alpha\mathcal{M}} \right\} + c_9\left\{ {\mathcal{M}\alpha \atop \mathcal{M}\alpha} \right\} + c_{10}\left\{ {\alpha\beta \atop \alpha\beta} \right\} + c_{11}\left\{ {\alpha\beta \atop \beta\alpha} \right\}. \end{aligned} \tag{3.20}$$

Imposing invariance under parity we obtain:

$$c_1 = c_2\,, \qquad c_4 = c_5\,, \qquad c_6 = c_7\,, \qquad c_8 = c_9\,. \tag{3.21}$$

In addition, protection of $\phi\phi$ implies $c_0 = 0$.[6] This still leaves seven independent coefficients. Imposing that the algebra commutation relations are satisfied perturbatively eliminates six of them, leaving us with one undetermined parameter, $c_1 \equiv \alpha_1^2$, which is associated with a rescaling of the coupling and cannot be fixed by algebraic means. The procedure is now completely algorithmic and it was described in detail in [17]. For each perturbative correction we consider the most general ansatz consistent with conservation of classical energy, r-charge and equation (3.7). Consistency of the algebra commutations relations significantly reduces the number of independent parameters. As extra input we use the fact that in the SU(1|1) subsector spanned by $\{\,\phi, \lambda_+\,\}$ the two-loop Hamiltonian of $\mathcal{N} = 2$ SCQCD should be identical to the corresponding Hamiltonian in $\mathcal{N} = 4$ SYM [31]. We present our results in tables 2 and 3. At first sight, there seems to be a high number of independent coefficients, however most of them are unphysical. The two coefficients $\{\,\alpha_1, \alpha_3\,\}$ can be reabsorbed by a redefinition of the coupling,[7]

$$g \to \alpha_1 g + \alpha_3 g^3\,. \tag{3.22}$$

The six coefficients $\{\,\beta_1, \beta_2, \delta_1, \delta_2, \delta_3, \delta_4\,\}$ correspond to similarity transformations and never show up in physical quantities like anomalous dimensions or S-matrix elements. We are then left with $\{\,\eta,\, \chi\,\}$ which do show up in physical quantities and therefore cannot be ignored. However, the S-matrix elements that we will study in the next section happen to be independent of $\{\,\eta,\, \chi\,\}$.

4 The magnon S-matrix in the SU(2|1) sector

We now proceed to calculate the magnon two-body S-matrix in the SU(2|1) sector, and to check whether it satisfies the Yang-Baxter equation. Let us start by defining the momentum eigenstate of a single excitation,

$$|\lambda_\alpha(p)\rangle = \sum_k e^{ipk}|\alpha_k\rangle\,, \tag{4.1}$$

where k labels the position of the particle,

$$|\alpha_k\rangle = |\ldots\phi\overset{k}{\downarrow}\!\!\lambda_\alpha\phi\ldots\rangle\,. \tag{4.2}$$

Its dispersion relation is easily obtained by acting with the Hamiltonian:

$$\mathcal{H}|\lambda_\alpha(p)\rangle = g^2\alpha_1^2\left[(2 - e^{ip} - e^{-ip}) + g^2\alpha_1^2(-3 + 2(e^{ip} + e^{-ip}) - \frac{1}{2}(e^{2ip} + e^{-2ip}))\right]|\lambda_\alpha(p)\rangle\,, \tag{4.3}$$

[6]In [17] this condition was obtained using the algebra constraints, in our case we have to give it as extra input.

[7]Note of course that $\alpha_1 \neq 0$, otherwise the whole one-loop Hamiltonian $\mathcal{H}_2$ would vanish. The actual value of α_1 could be fixed by comparison with the explicit perturbative calculation [26]: $\mathcal{H}_{here} = D_{there}$, and $\alpha_1^2 = 2$.

$$
\begin{aligned}
\mathcal{H}_0 =& \left\{{\phi \atop \phi}\right\} + 2\left\{{\mathcal{M} \atop \mathcal{M}}\right\} + \tfrac{3}{2}\left\{{\alpha \atop \alpha}\right\}, \\
\mathcal{H}_2 =& \,\alpha_1^2\Big(\left\{{\phi\mathcal{M} \atop \phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi \atop \mathcal{M}\phi}\right\}\Big) + 2\alpha_1^2\left\{{\mathcal{M} \atop \mathcal{M}}\right\} + \alpha_1^2\Big(\left\{{\phi\alpha \atop \phi\alpha}\right\} + \left\{{\alpha\phi \atop \alpha\phi}\right\}\Big) - \alpha_1^2\Big(\left\{{\phi\alpha \atop \alpha\phi}\right\} + \left\{{\alpha\phi \atop \phi\alpha}\right\}\Big) \\
& + \alpha_1^2\Big(\left\{{\alpha\mathcal{M} \atop \alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha \atop \mathcal{M}\alpha}\right\}\Big) + \alpha_1^2\left\{{\alpha\beta \atop \alpha\beta}\right\} + \alpha_1^2\left\{{\alpha\beta \atop \beta\alpha}\right\}, \\
\mathcal{H}_3 =& -\alpha_1^3\, e^{i\beta_2}\,\varepsilon_{\alpha\beta}\Big(\left\{{\alpha\beta \atop \phi\mathcal{M}}\right\} + \left\{{\alpha\beta \atop \mathcal{M}\phi}\right\}\Big) - \alpha_1^3\, e^{-i\beta_2}\,\varepsilon^{\alpha\beta}\Big(\left\{{\phi\mathcal{M} \atop \alpha\beta}\right\} + \left\{{\mathcal{M}\phi \atop \alpha\beta}\right\}\Big), \\
\mathcal{H}_4 =& \,(-\tfrac{3}{2}\alpha_1^4 + 2\alpha_1\alpha_3)\Big(\left\{{\phi\phi\alpha \atop \phi\phi\alpha}\right\} + \left\{{\alpha\phi\phi \atop \alpha\phi\phi}\right\}\Big) + (\alpha_1^4 - \alpha_1\alpha_3)\Big(\left\{{\phi\phi\alpha \atop \phi\alpha\phi}\right\} + \left\{{\alpha\phi\phi \atop \phi\alpha\phi}\right\}\Big) \\
& - \tfrac{1}{2}\alpha_1^2\Big(\left\{{\phi\phi\alpha \atop \alpha\phi\phi}\right\} + \left\{{\alpha\phi\phi \atop \phi\phi\alpha}\right\}\Big) + (\alpha_1^4 - \alpha_1\alpha_3)\Big(\left\{{\phi\alpha\phi \atop \alpha\phi\phi}\right\} + \left\{{\phi\alpha\phi \atop \phi\phi\alpha}\right\}\Big) \\
& + (-\tfrac{5}{4}\alpha_1^2 + \alpha_1\alpha_3 - \eta + \chi)\Big(\left\{{\phi\phi\mathcal{M} \atop \phi\phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi\phi \atop \mathcal{M}\phi\phi}\right\}\Big) \\
& + (-\tfrac{31}{4}\alpha_1^2 + 7\alpha_1\alpha_3 + \chi)\Big(\left\{{\phi\mathcal{M} \atop \phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi \atop \mathcal{M}\phi}\right\}\Big) + (\alpha_1^4 - 2\alpha_1\alpha_3 + \eta)\Big(\left\{{\phi\mathcal{M} \atop \mathcal{M}\phi}\right\} + \left\{{\mathcal{M}\phi \atop \phi\mathcal{M}}\right\}\Big) \\
& + (\tfrac{19}{2}\alpha_1^4 - 10\alpha_1\alpha_3 + 2\eta - 2\chi)\left\{{\mathcal{M}\phi\mathcal{M} \atop \mathcal{M}\phi\mathcal{M}}\right\} + 2\eta\left\{{\mathcal{M}\mathcal{M} \atop \mathcal{M}\mathcal{M}}\right\} \\
& + (-2\alpha_1^4 + 2\alpha_1\alpha_3 - \eta + \chi + i\alpha_1^2(\delta_1 + \delta_2))\Big(\left\{{\alpha\phi\mathcal{M} \atop \phi\alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\phi\alpha \atop \mathcal{M}\alpha\phi}\right\}\Big) \\
& + (-2\alpha_1^4 + 2\alpha_1\alpha_3 - \eta + \chi - i\alpha_1^2(\delta_1 + \delta_2))\Big(\left\{{\phi\alpha\mathcal{M} \atop \alpha\phi\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha\phi \atop \mathcal{M}\phi\alpha}\right\}\Big) \\
& + (-\tfrac{13}{4}\alpha_1^4 + 3\alpha_1\alpha_3 - \eta + \chi)\Big(\left\{{\phi\alpha\mathcal{M} \atop \phi\alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha\phi \atop \mathcal{M}\alpha\phi}\right\}\Big) \\
& + (-2\alpha_1^4 + 2\alpha_1\alpha_3 + \eta)\Big(\left\{{\alpha\mathcal{M} \atop \alpha\mathcal{M}}\right\} + \left\{{\mathcal{M}\alpha \atop \mathcal{M}\alpha}\right\}\Big) + (2\alpha_1^4 - 2\alpha_1\alpha_3 + \eta)\Big(\left\{{\alpha\mathcal{M} \atop \mathcal{M}\alpha}\right\} + \left\{{\mathcal{M}\alpha \atop \alpha\mathcal{M}}\right\}\Big) \\
& + (-\tfrac{1}{4}\alpha_1^4 + \alpha_1\alpha_3)\Big(\left\{{\phi\alpha\beta \atop \phi\alpha\beta}\right\} + \left\{{\beta\alpha\phi \atop \beta\alpha\phi}\right\}\Big) + (-\tfrac{7}{4}\alpha_1^4 + \alpha_1\alpha_3)\Big(\left\{{\phi\alpha\beta \atop \phi\beta\alpha}\right\} + \left\{{\beta\alpha\phi \atop \alpha\beta\phi}\right\}\Big) \\
& + (\alpha_1^4 - \alpha_1\alpha_3 - i\alpha_1^2\delta_1)\Big(\left\{{\phi\alpha\beta \atop \alpha\phi\beta}\right\} + \left\{{\beta\alpha\phi \atop \beta\phi\alpha}\right\}\Big) + (\alpha_1^4 - \alpha_1\alpha_3 + i\alpha_1^2\delta_1)\Big(\left\{{\alpha\phi\beta \atop \phi\alpha\beta}\right\} + \left\{{\beta\phi\alpha \atop \beta\alpha\phi}\right\}\Big) \\
& + (\tfrac{1}{4}\alpha_1^4 + i\alpha_1^2\delta_3)\Big(\left\{{\phi\alpha\beta \atop \beta\phi\alpha}\right\} + \left\{{\beta\alpha\phi \atop \alpha\phi\beta}\right\}\Big) + (\tfrac{1}{4}\alpha_1^4 - i\alpha_1^2\delta_3)\Big(\left\{{\beta\phi\alpha \atop \phi\alpha\beta}\right\} + \left\{{\alpha\phi\beta \atop \beta\alpha\phi}\right\}\Big) \\
& + (-\tfrac{7}{2}\alpha_1^4 + 4\alpha_1\alpha_3)\left\{{\alpha\phi\beta \atop \alpha\phi\beta}\right\} + \tfrac{1}{2}\alpha_1^2\left\{{\alpha\phi\beta \atop \beta\phi\alpha}\right\} \\
& + (-\tfrac{7}{2}\alpha_1^4 + 4\alpha_1\alpha_3 - \eta + \chi)\Big(\left\{{\mathcal{M}\alpha\beta \atop \mathcal{M}\alpha\beta}\right\} + \left\{{\beta\alpha\mathcal{M} \atop \beta\alpha\mathcal{M}}\right\}\Big) \\
& + (\tfrac{3}{2}\alpha_1^4 - 2\alpha_1\alpha_3 + \eta - \chi)\Big(\left\{{\mathcal{M}\alpha\beta \atop \mathcal{M}\beta\alpha}\right\} + \left\{{\beta\alpha\mathcal{M} \atop \alpha\beta\mathcal{M}}\right\}\Big) \\
& + (-\tfrac{9}{4}\alpha_1^4 + 3\alpha_1\alpha_3)\Big(\left\{{\alpha\beta\gamma \atop \alpha\gamma\beta}\right\} + \left\{{\gamma\beta\alpha \atop \beta\gamma\alpha}\right\}\Big) + (\tfrac{1}{2}\alpha_1^4 - 2\alpha_1\alpha_3)\Big(\left\{{\alpha\beta\gamma \atop \beta\gamma\alpha}\right\} + \left\{{\gamma\beta\alpha \atop \alpha\gamma\beta}\right\}\Big) \\
& + (-\tfrac{1}{2}\alpha_1^4 + 2\alpha_1\alpha_3)\left\{{\alpha\beta\gamma \atop \gamma\beta\alpha}\right\}.
\end{aligned}
$$

Table 2. The Hamiltonian up to order g^4.

hence,

$$E^{\lambda}(p) = 4(g^2\alpha_1^2 - 2g^4\alpha_1^4)\sin^2\frac{p}{2} + 2g^4\alpha_1^4\sin^2 p + O(g^6)\,. \tag{4.4}$$

To extract the S-matrix we will use the familiar perturbative asymptotic Bethe ansatz, see e.g. [32]. For the $SU(2_\alpha)$ singlet two-body state we define:

$$
\begin{aligned}
|\lambda_{[\alpha}\lambda_{\beta]}\rangle =& \sum_{k<l-1} \Psi_{\mathbf{1}}(k,l)|\ldots\phi\underset{\downarrow}{\overset{k}{\lambda}}_{[\alpha}\phi\ldots\phi\underset{\downarrow}{\overset{l}{\lambda}}_{\beta]}\phi\ldots\rangle \\
& + \sum_k \Psi_n(k)|\ldots\phi\underset{\downarrow}{\overset{k}{\lambda}}_{[\alpha}\underset{\downarrow}{\overset{k+1}{\lambda}}_{\beta]}\phi\ldots\rangle + \sum_k \Psi_{\mathcal{M}}(k)|\ldots\phi\underset{\downarrow}{\overset{k}{\mathcal{M}}}\phi\ldots\rangle\,,
\end{aligned} \tag{4.5}
$$

valid up to order g^2. The Ψ's correspond Schrödinger wave functions and k and l label the positions of the particles in the ϕ vacuum. At this order in perturbation theory a transition

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$$
\begin{aligned}
(\mathcal{Q}_\alpha)_0 &= e^{i\beta_1}\left\{\begin{smallmatrix}\phi\\ \alpha\end{smallmatrix}\right\},\\
(\mathcal{Q}_\alpha)_1 &= \alpha_1\, e^{i(\beta_1+\beta_2)}\varepsilon_{\alpha\beta}\left\{\begin{smallmatrix}\beta\\ \mathcal{M}\end{smallmatrix}\right\},\\
(\mathcal{Q}_\alpha)_2 &= ie^{i\beta_1}(\delta_1+\delta_2+\delta_4)\Big(\left\{\begin{smallmatrix}\phi\phi\\ \phi\alpha\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}\phi\phi\\ \alpha\phi\end{smallmatrix}\right\}\Big)+e^{i\beta_1}\big(\tfrac14\alpha_1^2+i\delta_4\big)\Big(\left\{\begin{smallmatrix}\phi\mathcal{M}\\ \alpha\mathcal{M}\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}\mathcal{M}\phi\\ \mathcal{M}\alpha\end{smallmatrix}\right\}\Big)\\
&\quad + e^{i\beta_1}\big(\tfrac14\alpha_1^2+i\delta_3\big)\Big(\left\{\begin{smallmatrix}\phi\beta\\ \beta\alpha\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}\beta\phi\\ \alpha\beta\end{smallmatrix}\right\}\Big)+ie^{i\beta_1}(\delta_2+\delta_4)\Big(\left\{\begin{smallmatrix}\phi\beta\\ \alpha\beta\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}\beta\phi\\ \beta\alpha\end{smallmatrix}\right\}\Big),\\
(\mathcal{S}^\alpha)_0 &= e^{-i\beta_1}\left\{\begin{smallmatrix}\alpha\\ \phi\end{smallmatrix}\right\},\\
(\mathcal{S}^\alpha)_1 &= \alpha_1\, e^{-i(\beta_1+\beta_2)}\varepsilon^{\alpha\beta}\left\{\begin{smallmatrix}\mathcal{M}\\ \beta\end{smallmatrix}\right\},\\
(\mathcal{S}^\alpha)_2 &= -ie^{-i\beta_1}(\delta_1+\delta_2+\delta_4)\Big(\left\{\begin{smallmatrix}\phi\alpha\\ \phi\phi\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}\alpha\phi\\ \phi\phi\end{smallmatrix}\right\}\Big)+e^{-i\beta_1}\big(\tfrac14\alpha_1^2-i\delta_4\big)\Big(\left\{\begin{smallmatrix}\alpha\mathcal{M}\\ \phi\mathcal{M}\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}\mathcal{M}\alpha\\ \mathcal{M}\phi\end{smallmatrix}\right\}\Big)\\
&\quad + e^{-i\beta_1}\big(\tfrac14\alpha_1^2-i\delta_3\big)\Big(\left\{\begin{smallmatrix}\beta\alpha\\ \phi\beta\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}\alpha\beta\\ \beta\phi\end{smallmatrix}\right\}\Big)-ie^{-i\beta_1}(\delta_2+\delta_4)\Big(\left\{\begin{smallmatrix}\alpha\beta\\ \phi\beta\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}\beta\alpha\\ \beta\phi\end{smallmatrix}\right\}\Big).
\end{aligned}
$$

Table 3. Fermionic SU(2|1) generators up to order g^2.

$\lambda_{[\alpha}\lambda_{\beta]} \to \mathcal{M}$ is possible and this is taken into account by the last term in (4.5). In order to solve the scattering problem we consider the following ansatz:

$$
\begin{aligned}
\Psi_{\mathbf{1}}(k,l) &= e^{i(p_1k+p_2l)} + S_{\mathbf{1}}(p_2,p_1)e^{i(p_1l+p_2k)}\,,\\
\Psi_n(k) &= S_n(p_2,p_1)e^{i(p_1+p_2)k}\,,\\
\Psi_{\mathcal{M}}(k) &= S_{\mathcal{M}}(p_2,p_1)e^{i(p_1+p_2)k}\,.
\end{aligned} \tag{4.6}
$$

Here $S_{\mathbf{1}}(p_2,p_1)$, $S_n(p_2,p_1)$ and $S_{\mathcal{M}}(p_2,p_1)$ are functions of g and represent the different scattering amplitudes. Imposing the Schrödinger equation

$$
\mathcal{H}|\lambda_{[\alpha}\lambda_{\beta]}\rangle = E(p_1,p_2)|\lambda_{[\alpha}\lambda_{\beta]}\rangle\,, \tag{4.7}
$$

for the separate cases $l > k+2$, $l = k+2$ and $l = k+1$ we can solve for the scattering amplitudes to order g^2. The interesting term is $S_{\mathbf{1}}(p_2,p_1)$, which governs the asymptotic magnon scattering,

$$
\begin{aligned}
S_{\mathbf{1}}(p_2,p_1) &= -\frac{1-2e^{ip_2}+e^{i(p_1+p_2)}}{1-2e^{ip_1}+e^{i(p_1+p_2)}} \\
&\quad\times\left(1+2ig^2\alpha_1^2\frac{(\cos p_1-2\cos(p_1-p_2)+\cos p_2)\sin\frac{p_1}{2}\sin\frac{p_2}{2}(\sin p_1-\sin p_2)}{\cos(\frac{p_1-p_2}{2})(3-2\cos p_1-2\cos p_2+\cos(p_1+p_2))}+O(g^4)\right).
\end{aligned} \tag{4.8}
$$

In the triplet sector the ansatz is simpler since $\lambda_{\{\alpha}\lambda_{\beta\}}$ does not mix with $\mathcal{M}$,

$$
|\lambda_{\{\alpha}\lambda_{\beta\}}\rangle = \sum_{k<l-1}\Psi_{\mathbf{3}}(k,l)|\ldots\phi\overset{k}{\lambda}_{\{\alpha}\phi\ldots\phi\overset{l}{\lambda}_{\beta\}}\phi\ldots\rangle + \sum_k \Psi_{\mathbf{3}n}(k)|\ldots\phi\overset{k}{\lambda}_{\{\alpha}\overset{k+1}{\lambda}_{\beta\}}\phi\ldots\rangle\,, \tag{4.9}
$$

where

$$
\begin{aligned}
\Psi_{\mathbf{3}}(k,l) &= e^{i(p_1k+p_2l)} + S_{\mathbf{3}}(p_2,p_1)e^{i(p_1l+p_2k)}\,,\\
\Psi_{\mathbf{3}n}(k) &= S_{\mathbf{3}n}(p_2,p_1)e^{i(p_1+p_2)k}\,.
\end{aligned} \tag{4.10}
$$

We find

$$
S_{\mathbf{3}}(p_2,p_1) = -1 - ig^2\alpha_1^2(\sin p_1 - \sin(p_1-p_2) - \sin p_2) + O(g^4)\,. \tag{4.11}
$$

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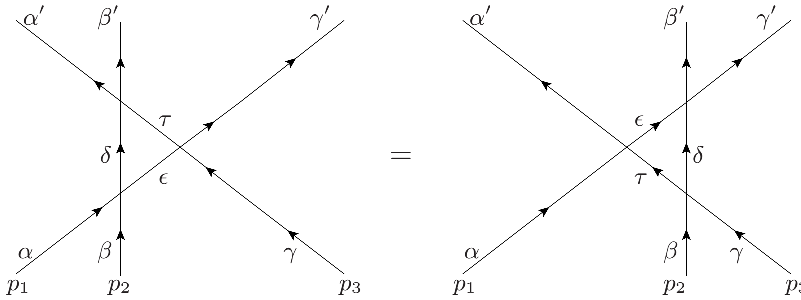

Figure 1. Diagrammatic representation of the Yang-Baxter equation.

Checking the Yang-Baxter equation. We are finally ready to check the Yang-Baxter equation for the two-body magnon S-matrix. The equation reads (see figure 1 for the index flow)

$$S_{\alpha\beta}^{\delta\epsilon}(p_1,p_2)S_{\epsilon\gamma}^{\tau\gamma'}(p_1,p_3)S_{\delta\tau}^{\alpha'\beta'}(p_2,p_3) = S_{\epsilon\delta}^{\beta'\gamma'}(p_1,p_2)S_{\alpha\tau}^{\alpha'\epsilon}(p_1,p_3)S_{\beta\gamma}^{\tau\delta}(p_2,p_3)\,. \tag{4.12}$$

Defining:

$$A(p_1,p_2) = S_{\mathbf{3}}(p_1,p_2)\,, \tag{4.13}$$

$$B(p_1,p_2) = \frac{1}{2}(S_{\mathbf{1}}(p_1,p_2) - S_{\mathbf{3}}(p_1,p_2))\,, \tag{4.14}$$

we can rewrite the S-matrix in terms of the identity operator $\mathbb{I}$ and the trace operator $\mathbb{K}$,

$$S(p_1,p_2) = A(p_1,p_2)\mathbb{I} + B(p_1,p_2)\mathbb{K}\,. \tag{4.15}$$

As explained e.g. in [25], the Yang-Baxter equation is equivalent to the single constraint

$$\begin{aligned} 0 \overset{?}{=} & 2B(p_1,p_2)A(p_1,p_3)B(p_2,p_3) + A(p_1,p_2)A(p_1,p_3)B(p_2,p_3) + B(p_1,p_2)A(p_1,p_3)A(p_2,p_3) \\ & + B(p_1,p_2)B(p_1,p_3)B(p_2,p_3) - A(p_1,p_2)B(p_1,p_3)A(p_2,p_3)\,. \end{aligned} \tag{4.16}$$

A necessary condition for factorization of many-body scattering is the vanishing of the right-hand side. However, working at order g^2 we obtain

$$64i\alpha_1^2 e^{i(p_1+p_2+p_3)}\frac{\sin\left(\frac{p_1}{2}\right)^2\sin\left(\frac{p_2}{2}\right)^2\sin\left(\frac{p_3}{2}\right)^2\tan\left(\frac{p_1-p_2}{2}\right)\tan\left(\frac{p_1-p_3}{2}\right)\tan\left(\frac{p_2-p_3}{2}\right)}{(1+e^{i(p_1+p_2)}-2e^{ip_2})(1+e^{i(p_1+p_3)}-2e^{ip_3})(1+e^{i(p_2+p_3)}-2e^{ip_3})}\,, \tag{4.17}$$

which is certainly non-zero.[8] Failure of the Yang-Baxter equation conclusively shows that the SU(2|1) sector is not integrable at two loops.

[8] The only solution is the trivial solution $\alpha_1 \equiv 0$, which sets to zero the whole interacting Hamiltonian, see table 2.

5 The universal SU(2, 1|2) sector

The SU(2, 1|2) sector (2.3) consists entirely of letters that belong to the $\mathcal{N}=2$ vector multiplet, and it is then present in any $\mathcal{N}=2$ gauge theory. Diagrammatic arguments [31] show that the planar dilation operator in this sector is the same up to *two loops* in any $\mathcal{N}=2$ superconformal theory, as it coincides to that order with a restriction of the $\mathcal{N}=4$ SYM dilation operator. The model dependence kicks in at three loops.[9]

Choosing the usual chiral vacuum $\operatorname{Tr}\phi^k$, the Goldstone magnons $\{\lambda_+{}^{\mathcal{I}}, \mathcal{D}_{+\dot{\alpha}}\}$ transform in the fundamental representation of $SU(2_{\dot{\alpha}}|2_{\mathcal{I}})$. Their two-body S-matrix $S_{SU(2_{\dot{\alpha}}|2_{\mathcal{I}})}$ is uniquely determined up to an overall phase by the SU(2|2) symmetry [30], and thus, just as is the case in $\mathcal{N}=4$ SYM, it automatically satisfies the Yang-Baxter equation. This is a first hint to suspect that this sector may be generically integrable, at least in the sense of the asymptotic Bethe ansatz on the infinite chain.[10] Of course, factorization of the n-body S-matrix into two-body S-matrices is a stronger condition than Yang-Baxter, and an explicit test at three loops will be required. A three-loop diagrammatic analysis is in progress [29]. The strongest conjecture [29] suggested by this perturbative study is that the SU(2, 1|2) Hamiltonian of any $\mathcal{N}=2$ superconformal gauge theory can be mapped to that of $\mathcal{N}=4$ SYM by a redefinition of the 't Hooft coupling, $g^2 \to f(g^2) = g^2 + O(g^6)$. This would be a trivial operation from the viewpoint of the integrable structure. Indeed recall that it is still somewhat of a mystery why the dispersion relation of the $\mathcal{N}=4$ SYM magnons takes the exact form

$$\Delta - |r| = \sqrt{1 + 8g^2 \sin^2 \frac{p}{2}}\,, \tag{5.1}$$

while integrability alone would be compatible with the replacement $g^2 \to f(g^2)$ (which is indeed what happens in the ABJM model [33]). However a redefinition of g can have drastic dynamical consequences, for example it may radically change the strong coupling behavior of anomalous dimensions (ABJM is again a case in point.)

A second indication in favor of integrability of the SU(2, 1|2) sector comes from the AdS/CFT correspondence — at least, that is, for the subset of models that admit a string dual. The simplest $\mathcal{N}=2$ theories with a known string description are the orbifolds of $\mathcal{N}=4$ SYM by a discrete subgroup $\Gamma \subset SU(2) \subset SU(4)_R$, which are dual to the IIB backgrounds $AdS_5 \times S^5/\Gamma$ [34, 35]. These are quiver gauge theories with product gauge group $SU(N)^k$, where k is the order of Γ. The k gauge couplings are exactly marginal parameters. If all gauge couplings are equal, the spin chain (and the dual sigma model) is completely integrable [36, 37], but when they are different, integrability of the full chain

[9] In the context of $\mathcal{N}=4$ SYM, the SU(2, 1|2) sector can be regarded as a non-compact cousin of the SU(2|3) sector, whose Hamiltonian was determined up to three loops by Beisert [17] using symmetry arguments. The Hamiltonian of non-compact sectors is much harder to fix. Zwiebel's paper [18] represents the state of the art.

[10] We are postponing at this stage the harder questions about finite-size effects.

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is broken.[11] However, the situation is much better in the $SU(2,1|2)$ sector.[12] At strong coupling one can study the S-matrix of the $SU(2|2)$ excitations using the dual sigma model. Changing the relative gauge couplings is dual to twisted-sector deformations in the sigma model: to leading order in α' (tree level in the sigma model) they do not change the scattering of the $SU(2|2)$ excitations, which live in directions of the target space unaffected by the orbifold. So the n-body S-matrix still factorizes into two-body S-matrices. To be more precise, the only effect of the twisted deformation felt by the $SU(2|2)$ excitations is a renormalization of the string tension. For example, in the $\mathbb{Z}_2$ case, the relation between α' and the AdS radius R reads

$$\frac{R^4}{\alpha'} = \frac{2\lambda\check{\lambda}}{\lambda + \check{\lambda}}, \tag{5.2}$$

where λ and $\check{\lambda}$ are the two 't Hooft couplings. It would be very interesting to confirm this picture to next order in α', where the effect of the twisted deformation is non-trivial, by an explicit one-loop calculation of the sigma-model S-matrix. Recall that the two-body $SU(2|2)$ S-matrix is completely fixed by symmetry, so to really probe integrability one would have to study factorization of the n-body S-matrix or devise some other test.

In summary, the $SU(2,1|2)$ sector(s) of $\mathcal{N} = 2$ superconformal gauge theories have the same Hamiltonian as in $\mathcal{N} = 4$ SYM for small λ (to two-loop order, $O(\lambda^2)$); and in theories with AdS duals, the large λ limit of the Hamiltonian is also the same as in $\mathcal{N} = 4$ SYM, modulo a renormalization of the coupling. For example, in the $\mathbb{Z}_2$ quiver theory, it follows from (5.2) that for large λ and large $\check{\lambda}$ (with $\lambda/\check{\lambda}$ fixed) the dilation operator in the $SU(2,1|2)$ sector coincides with the one in $\mathcal{N} = 4$ SYM if one replaces $\lambda \to 2\lambda\check{\lambda}/(\lambda + \check{\lambda})$.[13] We are led to conjecture that this remains true for all intermediate values of the coupling, with the appropriate redefinition $\lambda \to f(\lambda)$ that matches the weak and strong coupling behaviors.

$\mathbf{SU(2,1|1)}$ and $\mathbf{SU(2,1)}$. In closing, it is tempting to entertain the natural extrapolations of this conjecture to $\mathcal{N} = 1$ and $\mathcal{N} = 0$ conformal gauge theories. Every $\mathcal{N} = 1$ superconformal gauge theory contains a closed $SU(2,1|1)$ sector, with letters belonging entirely to the $\mathcal{N} = 1$ vector multiplet,

$$SU(2,1|1) \text{ sector:} \qquad (\mathcal{D}_{+\dot{\alpha}})^n \{ \lambda_+, \mathcal{F}_{++} \}. \tag{5.3}$$

The diagrammatic arguments of [31] show again that in any $\mathcal{N} = 1$ superconformal theory the dilation operator in this sector coincides up to two loops with the restriction of the $\mathcal{N} = 4$ SYM dilation operator. (Of course this is a meaningful statement only for $\mathcal{N} = 1$

[11]For the simplest example of the $\mathbb{Z}_2$ orbifold, this phenomenon was studied in detail in [25, 31, 38], which focussed on the magnons transforming in the bifundamental representation of the $SU(N_c) \times SU(N_{\check{c}})$ gauge group, with $N_c \equiv N_{\check{c}}$. For $\lambda \neq \check{\lambda}$ their dispersion relation develops a gap. The form of their two-body S-matrix is fixed by symmetry, and fails to satisfy the Yang-Baxter equation except when $\lambda = \check{\lambda}$.

[12]There are actually k separate $SU(2,1|2)$ sectors, one for each of the $SU(N)$ vector multiplets.

[13]This correspondence is also precisely confirmed [39] by considering the strong coupling limit of the matrix model [40] that calculates the expectation value of the 1/2 BPS circular Wilson loop in the $\mathbb{Z}_2$ quiver theory, following [41, 42].

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SCFTs that have a weak coupling limit). Choosing the chiral vacuum $\mathrm{Tr}\,\lambda_+^k$, the asymptotic excitations on the chain are the massless magnons $\{\mathcal{D}_{+\dot{\alpha}}\}$, transforming as a doublet of $\mathrm{SU}(2_{\dot{\alpha}})$. This is not enough symmetry to completely fix the form of the two-body magnon S-matrix, which makes integrability of the $\mathrm{SU}(2,1|1)$ sector somewhat less compelling as a general conjecture. For models that admit string duals, some evidence for integrability comes again from the AdS/CFT correspondence. For example, while the generic Leigh-Strassler deformation of $\mathcal{N}=4$ SYM is not fully integrable (see [23] for a review), there is still hope for integrability in the $\mathrm{SU}(2,1|1)$ sector. Indeed, one can argue for integrability at strong coupling (to leading order): the deformation of the $AdS_5 \times S^5$ background that corresponds to the Leigh-Strassler deformation (whatever its explicit form may be) is not expected to affect the tree-level scattering of excitations in the $\mathrm{SU}(2,1|1)$ subsector, since those excitations live entirely in AdS_5.

It would be particularly interesting to explore this conjecture in $\mathcal{N}=1$ super QCD, in the conformal window $\frac{3}{2}N_c < N_f < 3N_c$. For fixed number of colors N_c and fixed number of flavors N_f, the theory flows in the IR to an isolated superconformal fixed point. It is possible however to define a systematic perturbative expansion near the upper edge of the conformal window, taking the Veneziano limit $N_c \to \infty$, $N_f \to \infty$ with $N_f/N_c = 3-\epsilon$. The dilation operator can be evaluated order by order in ϵ, and was indeed completely determined to leading order (one loop) in [27] following [43]. Similarly one can set up an expansion for the dilation operator of the magnetic Seiberg-dual theory, near the lower edge of the conformal window, with $N_f/N_c = \frac{3}{2} + \tilde{\epsilon}$. Seiberg duality implies that the resummation of the ϵ expansion in the electric theory must coincide with the resummation of the $\tilde{\epsilon}$ expansion in the magnetic theory. In the $\mathrm{SU}(2,1|1)$ sector, the dilation operator is the same as in $\mathcal{N}=4$ SYM, and thus obviously integrable, up to two loops in both expansions. The optimistic scenario is for the sector to remain integrable throughout the conformal window. It will be interesting to perform higher order checks in both ϵ and $\tilde{\epsilon}$. Integrability would offer the exciting prospect of much more quantitative tests of Seiberg duality than presently possible.

Finally, one may even consider purely bosonic conformal gauge theories, and hope for integrability of the $\mathrm{SU}(2,1)$ sector,

$$\mathrm{SU}(2,1)\ \text{sector:} \qquad (\mathcal{D}_{+\dot{\alpha}})^n\,\mathcal{F}_{++}\,. \tag{5.4}$$

Only isolated fixed points are known for non-supersymmetric theories in four dimensions. The simplest and most interesting case is QCD itself, in the Veneziano limit near the upper edge of the conformal window, $N_f/N_c = 11/2-\epsilon$. To leading order in ϵ (one loop) the dilation operator in the $\mathrm{SU}(2,1)$ sector is trivially the same as in $\mathcal{N}=4$ SYM, but unlike the supersymmetric cases, we are not aware of a diagrammatic argument that this agreement should persist to two loops. It would be very interesting to perform an explicit two-loop calculation and check integrability.

If our $\mathcal{N}=1$ and $\mathcal{N}=0$ speculations turn out to be valid, at least in some models, it will be because the integrability structures of $\mathcal{N}=4$ SYM, while generically broken, are sufficiently robust to survive deformations and RG flows in the special universal sectors that we have isolated. On the dual string side (when available) these sectors are captured

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entirely by the AdS_5 factor of the sigma model. Our conjectures may be phrased as "best case scenarios". It will be worth investigating them further.

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