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Scale without conformal invariance at three loops

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Abstract: We carry out a three-loop computation that establishes the existence of scale without conformal invariance in dimensional regularization with the MS scheme in unitary theories in $d = 4 - \epsilon$ spacetime dimensions. We also comment on the effects of scheme changes in theories with many couplings, as well as in theories that live on non-conformal scale-invariant renormalization group trajectories. Stability properties of such trajectories are analyzed, revealing both attractive and repulsive directions in a specific example. We explain how our results are in accord with those of Jack & Osborn on a c-theorem in $d = 4$ (and $d = 4 - \epsilon$) dimensions. Finally, we point out that limit cycles with turning points are unlike limit cycles with continuous scale invariance.

Keywords: Space-Time Symmetries, Renormalization Group, Conformal and W Symmetry

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Contents

1 Introduction

When it was first introduced in its modern form [1], the question *"Does unitarity and scale invariance imply conformal invariance?"* was mostly of academic interest. Recent work [2, 3] showed that scale-invariant theories display renormalization group (RG) flow recurrent behaviors and have novel implications for beyond the standard model phenomenology [4].[1] Thus, the existence of scale-invariant theories has deep consequences, especially with respect to the intuitive understanding of RG flows as the integrating out of degrees of freedom, and the c-theorem. *"Does unitarity and scale invariance imply conformal invariance?"* is therefore not simply a question of academic interest, and to answer it is of utmost importance.

[1] For other explorations of scale without conformal invariance see refs. [5–16].

In refs. [2, 3] it was shown that scale does not necessarily imply conformal invariance in a unitary quantum field theory (QFT) with enough scalars and fermions at two loops. However, no completely trustworthy examples have been discovered at this order. The failure to find concrete examples at two loops can be understood using the results of Osborn [17, 18] and Jack & Osborn [19]. In ref. [19] it is argued that, in the weak-coupling regime, RG flows are gradient flows at two loops. Hence, even though scale does not necessarily imply conformal invariance at two loops, the beta function monomials which could lead to concrete scale-invariant theories have coefficients that conspire to make all solutions conformal. Nothing forbids this from occurring order by order in perturbation theory. Therefore, either scale implies conformal invariance — and the coefficients of the beta function monomials are tightly constrained, forcing all would-be scale-invariant solutions to be conformal — or it does not — and recurrent behaviors exist. Either way, the answer to the original question leads to important implications (unexpected structure in the beta functions or the existence of recurrent behaviors) and the question deserves to be fully investigated.

In this paper we compute the necessary three-loop contributions to the beta functions to determine if the plausible scale-invariant solutions found in $d = 4 - \epsilon$ are eliminated at three loops in the MS scheme, i.e., within a well-defined renormalization scheme. Our results show that the scale-invariant solutions are robust at three loops, and thus open the door for a $d = 4$ scale-invariant example. Indeed, since scale implies conformal invariance in pure gauge theories at weak coupling [1, 19], the addition of gauge bosons in $d = 4$ should not qualitatively change the $d = 4 - \epsilon$ results. For example, the beta function monomials exhibited below, which lead to an obstruction to the gradient flow interpretation of the RG flow, are not modified in any way by the introduction of gauge bosons. However, to fully answer the question in $d = 4$, one needs the complete three-loop beta functions of theories with matter and gauge fields, a computation we hope to undertake soon.

It is important to point out that the c-theorem discussed in refs. [17–19], which leads to $dc/dt = -G_{IJ}\beta^I\beta^J$ with G_{IJ} positive-definite in the weak coupling regime, is too restrictive. Indeed, following Osborn [18], the all-loop proof of the c-theorem, which implies the existence of a monotonically decreasing c-function which is constant only at conformal fixed points, must be modified once spin-one operators of dimension three are taken into account. This is exactly the case for non-conformal scale-invariant theories, since the virial current is such an operator. Taking into account the virial current, the analysis is modified as described in ref. [18, section 3], and leads to $dc/dt = -(G_{IJ} + \cdots)\beta^I B^J$ where $B^I = \beta^I - \mathcal{Q}^I$ and $\beta^I = \mathcal{Q}^I$ for non-conformal scale-invariant theories. Thus, in its most general form the work of Osborn [17, 18] and Jack & Osborn [19] implies the existence of a c-function which is constant at conformal fixed points ($\beta^I = 0$) as well as on scale-invariant trajectories ($B^I = 0$). This is in accord with our three-loop results.

The paper is organized as follows: In section 2, we discuss the ϵ expansion in more detail, showing why the scale-invariant solutions can be destabilized at three loops. We then generate the most general three-loop beta function for the Yukawa coupling and determine which diagrams contribute to the virial current. We finally compute the beta function coefficients of the relevant diagrams and verify that the virial current does not vanish at three loops, thus demonstrating the existence of scale-invariant theories in $d = 4 - \epsilon$ in a well-defined renormalization scheme. Other plausible examples in $d = 4 - \epsilon$ spacetime dimensions exhibiting limit cycles are discussed and it is conjectured that limit cycles and ergodicity are generic in more general theories. In section 4, we examine scheme changes in theories with many couplings and also on scale-invariant solutions, showing that, as expected, physical parameters in $d = 4$ do not depend on the renormalization scheme. In section 5 we elucidate the stability properties of scale-invariant solutions and explicitly verify that the example of section 2 exhibits both attractive and repulsive directions. In section 6 we return to the arguments of Osborn [17, 18] and Jack & Osborn [19] and show that they are not in contradiction with our results. Finally, in section 7 we contrast our cyclic trajectories with the trajectories of ref. [20] which were recently discussed in connection with the c-theorem in ref. [21] (see also ref. [22]).

2 Establishing scale invariance

The results of refs. [2] were presented in an expansion in ϵ, similar in spirit to the expansion that reveals the Wilson-Fisher fixed point. Let us recall here how that works. We consider a model with real scalar fields ϕ_a and Weyl spinors ψ_i with quartic scalar self-couplings λ_{abcd} and Yukawa couplings $y_{a|ij}$. The equations for scale invariance are

$$\beta_{abcd}(\lambda, y) = \mathcal{Q}_{abcd} \equiv -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}\,, \tag{2.1a}$$

$$\beta_{a|ij}(\lambda, y) = \mathcal{P}_{a|ij} \equiv -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}\,, \tag{2.1b}$$

where $\beta_{abcd} = -d\lambda_{abcd}/dt$ and $\beta_{a|ij} = -dy_{a|ij}/dt$ are the beta functions for the coupling constants,[2] Q_{ab} is antisymmetric and P_{ij} anti-Hermitian. To proceed, we solve eqs. (2.1) for the coefficients of λ, y, Q and P in an ϵ expansion,

$$\lambda_{abcd} = \sum_{n\geq 1} \lambda^{(n)}_{abcd}\epsilon^n, \quad y_{a|ij} = \sum_{n\geq 1} y^{(n)}_{a|ij}\epsilon^{n-\frac{1}{2}}, \quad Q_{ab} = \sum_{n\geq 2} Q^{(n)}_{ab}\epsilon^n, \quad P_{ij} = \sum_{n\geq 2} P^{(n)}_{ij}\epsilon^n\,. \tag{2.2}$$

Scale-invariant solutions are solutions of eqs. (2.1) with non-vanishing Q and/or P.

2.1 Limit cycle in $d = 4 - \epsilon$: model with 2 scalars and 2 fermions

For the remainder of this section we will work with a theory of two real scalars and two Weyl fermions, canonical kinetic terms and interactions described by

$$V = \frac{1}{24}\lambda_1\phi_1^4 + \frac{1}{24}\lambda_2\phi_2^4 + \frac{1}{4}\lambda_3\phi_1^2\phi_2^2 + \frac{1}{6}\lambda_4\phi_1^3\phi_2 + \frac{1}{6}\lambda_5\phi_1\phi_2^3 + \Big(\frac{1}{2}y_1\phi_1\psi_1\psi_1 + \frac{1}{2}y_2\phi_2\psi_1\psi_1 + \frac{1}{2}y_3\phi_1\psi_2\psi_2 + \frac{1}{2}y_4\phi_2\psi_2\psi_2 + y_5\phi_1\psi_1\psi_2 + y_6\phi_2\psi_1\psi_2 + \text{h.c.}\Big)\,. \tag{2.3}$$

This is the simplest weakly-coupled unitary example in $d = 4 - \epsilon$ with a well-behaved bounded-from-below scalar potential. For this model Q is 2×2 antisymmetric and P 2×2

[2] With our conventions RG time increases as we flow to the IR, $t = \ln(\mu_0/\mu)$.

anti-Hermitian:

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} ip_1 & p_3 + ip_4 \\ -p_3 + ip_4 & ip_2 \end{pmatrix}, \tag{2.4}$$

where q and $p_{1,\ldots,4}$ are real. The two-loop beta functions for this model, formatted for use with *Mathematica*, can be found at http://het.ucsd.edu/misc/betas2s2f_D=4-eps.m.

2.1.1 The two-loop computation

To start our computation we solve eq. (2.1b) at order $\epsilon^{3/2}$. The result is used in eq. (2.1a) which is then solved at order ϵ^2. This is a system of coupled *nonlinear* equations and, as such, it has many solutions $y^{(1)}_{a|ij}$ and $\lambda^{(1)}_{abcd}$, some of them consistent with unitarity and boundedness of the scalar potential, while others not. Additionally, some of these solutions lead to conformal fixed points, while others allow for nonzero q, at least in principle.

At two-loop order solutions $y^{(1)}_{a|ij}$ and $\lambda^{(1)}_{abcd}$ of the previous order are used to solve eq. (2.1b) at order $\epsilon^{5/2}$, and eq. (2.1a) at order ϵ^3. This is now a system of coupled *linear* equations,[3] from which the unknowns $y^{(2)}_{a|ij}$ and $\lambda^{(2)}_{abcd}$ are determined. At this same order one can compute $q^{(2)}$. If the lower-order solution $y^{(1)}_{a|ij}$ and $\lambda^{(1)}_{abcd}$ corresponds to a fixed point, then the unknown $q^{(2)}$ is equal to zero. Surprisingly, this is also true at this order for solutions that correspond to scale-invariant trajectories (for which $q^{(3)} \neq 0$). This is somewhat of an accident. Suppose we replace the two-loop contribution to the actual beta functions, $\beta^{(\text{2-loop})}_{a|ij}$, by a linear combination of the same monomials that appear in $\beta^{(\text{2-loop})}_{a|ij}$, but with arbitrary coefficients. If we now use these in the computation of the second order corrections $y^{(2)}_{a|ij}$ and $\lambda^{(2)}_{abcd}$ and of $q^{(2)}$, then we find that only two terms in $\beta^{(\text{2-loop})}_{a|ij}$ contribute to $q^{(2)}$, namely,

$$(16\pi^2)^2 \beta^{(\text{2-loop})}_{a|ij} \supset b_1 y_{b|ik} y^*_{c|k\ell} y_{d|\ell j} \lambda_{abcd} + b_2 y_{b|ij} \lambda_{bcde} \lambda_{acde}, \tag{2.5}$$

and that these give (we omit the prefactor here since it is not relevant for the discussion),

$$q^{(2)} \propto b_1 + 24 b_2. \tag{2.6}$$

The accident we referred to above is the fact that the actual values of the coefficients are $b_1 = -2$ and $b_2 = \frac{1}{12}$, and hence $q^{(2)} = 0$. Moreover, for fixed points the proportionality constant in eq. (2.6) vanishes, so for fixed points the vanishing of $q^{(2)}$ is not the result of this particular cancelation. We also find that $p^{(2)}_4$ is undetermined, while $p^{(2)}_{1,2,3} = 0$. The freedom in the fermion part of the virial current is related to the enhanced symmetry

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.7}$$

It may be worth noting that, when using the generic form of couplings $y_{a|ij}$ and λ_{abcd}, there is a one-to-one correspondence between monomials in the beta functions $\beta_{a|ij}$ and

[3] For all higher orders in ϵ one only gets systems of coupled linear equations.

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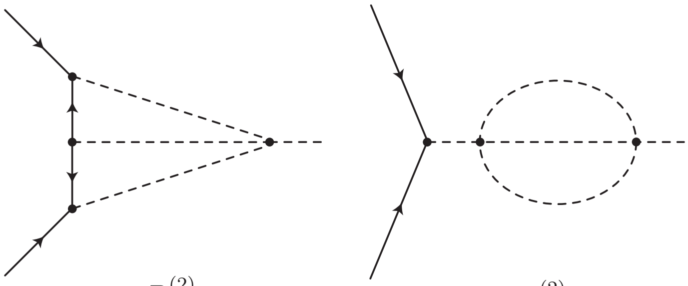

Figure 1. Diagrams that contribute to q at two-loop order.

β_{abcd} and multi-loop Feynman graphs. For example, the monomials in (2.5) correspond to the two loop Feynman diagrams in figure 1. This can be seen as follows. For a given topology we associate a coupling $y_{a|ij}$ or λ_{abcd} with each vertex. The indices correspond to each of the lines coming out of the vertex. Vertices are joined by internal propagators, which carry factors of δ_{ik} if joining two vertices with fermion lines labeled by i and k, or δ_{ae} if joining two vertices with scalar lines labeled by a and e. Finally, the free indices in the monomial (a, i and j for $\beta_{a|ij}$, or a, b, c and d for λ_{abcd}) correspond to external legs in the Feynman graph. This correspondence between graphs and monomials will prove useful in our three-loop analysis below.

As we already mentioned, the failure to find trustworthy non-conformal scale-invariant solutions at two loops can be explained by the gradient flow property of the RG flow at weak coupling described in ref. [19]. Note that here, contrary to the case of conformal fixed points, $q^{(3)} \neq 0$ at two-loop order. However, the three-loop contributions to the beta functions can very well conspire to set $q^{(3)} = 0$, and thus restore conformal invariance. (As we will demonstrate in the next subsection, this does not happen. Again, the fact that $q^{(2)} = 0$ is merely an accident.)

An interesting observation at this point is that if $q^{(2)} = 0$ were not an accident, then, as seen from eq. (2.6), that would directly imply that conformal symmetry somehow relates coefficients of beta-function monomials coming from vertex corrections and coefficients of beta-function monomials coming from wavefunction renormalizations. This is obvious from the fact that the first diagram in figure 1 contributes to the residue of the $1/\epsilon$ pole of Z_y, while the second to the residue of the $1/\epsilon$ pole of Z_ϕ. This would be reminiscent, e.g., of the Ward identity for charge conservation in QED.

A point on the candidate scale-invariant trajectory is given by

$$\begin{aligned}
\lambda_1 &= \frac{8(7087+357\sqrt{52953})}{102885}\pi^2\epsilon + \frac{2(490537743519+468277825\sqrt{52953})}{408605205375}\pi^2\epsilon^2 + \cdots\,,\\
\lambda_2 &= \frac{64(6346+9\sqrt{52953})}{102885}\pi^2\epsilon + \frac{17(11340943081+57223077\sqrt{52953})}{136201735125}\pi^2\epsilon^2 + \cdots\,,\\
\lambda_3 &= -\frac{272(\sqrt{52953}-57)}{102885}\pi^2\epsilon + \frac{291302437755-3043364867\sqrt{52953}}{817210410750}\pi^2\epsilon^2 + \cdots\,,
\end{aligned}$$

$$
\begin{aligned}
\lambda_4 &= \frac{32\sqrt{323(757-3\sqrt{52953})}}{102885}\pi^2\epsilon + \frac{13\sqrt{\begin{array}{r}190447787(139242697966441289257\,81\\ -4950945949443982653\,1)\sqrt{52953}\end{array}}}{5584352861166075\,0}\pi^2\epsilon^2 + \cdots\,,\\
\lambda_5 &= \frac{272\sqrt{323(757-3\sqrt{52953})}}{102885}\pi^2\epsilon + \frac{\sqrt{\begin{array}{r}571343361(6524747628672345183814\,07\\ -6636632190132526910\,17)\sqrt{52953}\end{array}}}{1970948068646850\,0}\pi^2\epsilon^2 + \cdots\,,\\
y_1 &= -y_3 = \frac{2\sqrt{10}}{5}\pi\epsilon^{1/2} + \frac{\sqrt{10}(175503+442\sqrt{52953})}{3249000}\pi\epsilon^{3/2} + \cdots\,,
\end{aligned}
\tag{2.8}
$$

where the remaining couplings vanish at this point, and only the real part of y_2 and y_4 are generated on the scale-invariant trajectory, with $\mathrm{Re}\,y_2 = -\,\mathrm{Re}\,y_4$. The fact that $\mathrm{Re}\,y_2$ and $\mathrm{Re}\,y_4$ run through zero, see figure 3, is what allows us to determine $q^{(2)}$ as in eq. (2.6).

One can check that eqs. (2.1) are satisfied on this scale-invariant trajectory with the help of the two-loop beta functions of ref. [23]. Since $q = \mathcal{O}(\epsilon^3)$ we need the three-loop Yukawa beta functions in order to establish that this solution is indeed a scale-invariant trajectory in dimensional regularization.

2.1.2 The three-loop computation

There is a large number of diagrams that contribute to $\beta_{a|ij}$ at three loops. (We use the *Mathematica* package `FeynArts` to automatically generate all required diagrams.) As we explained above, each diagram corresponds to a unique monomial in the beta function. From the diagrams we have generated we construct a linear combination of all the monomials that may appear in the three-loop beta function, with coefficients that remain to be computed. Using this representation of the beta function we compute $q^{(3)}$ (by inserting the two previous orders into eqs. (2.1)). We find that only a small fraction of monomials contribute to $q^{(3)}$. This is similar to the case at one lower order, where only two monomials contribute to $q^{(2)}$; cf. eq. (2.6). The monomials that contribute to $q^{(3)}$ are the ones that correspond to the Feynman graphs in figure 2.

Note that the diagrams in figure 2 are specific to the example of this subsection. More complicated models might involve more diagrams. However, we have checked that precisely the same diagrams contribute to $q^{(3)}$ in the model with two scalars and one Weyl spinor of ref. [2]. It is interesting to point out that very few of the ~ 200 diagrams in $\beta_{a|ij}$ contribute to $q^{(3)}$, and that the ones that do involve both Yukawa and quartic vertices. The same situation is encountered at two loops, and we conjecture that it holds to all orders in perturbation theory. This is also motivated by comments in ref. [24] regarding the "interference" between successive loop orders in the calculation of a potential for a gradient flow (see also section 6 below). It is also curious that the diagrams of figure 2 have an (obvious) topological relation to the diagrams of figure 1. Let us remark here that although only the diagrams of figure 2 contribute to scale without conformal invariance in the example of this section, we nevertheless have no clear physical understanding as to why this is the case. We have not succeeded in understanding why so few specific monomials (and corresponding diagrams) contribute to $q^{(3)}$. In particular we do not understand why, say, graphs with wave-function renormalization on external fermion legs or graphs whose computation gives Apéry's constant do not contribute to $q^{(3)}$.

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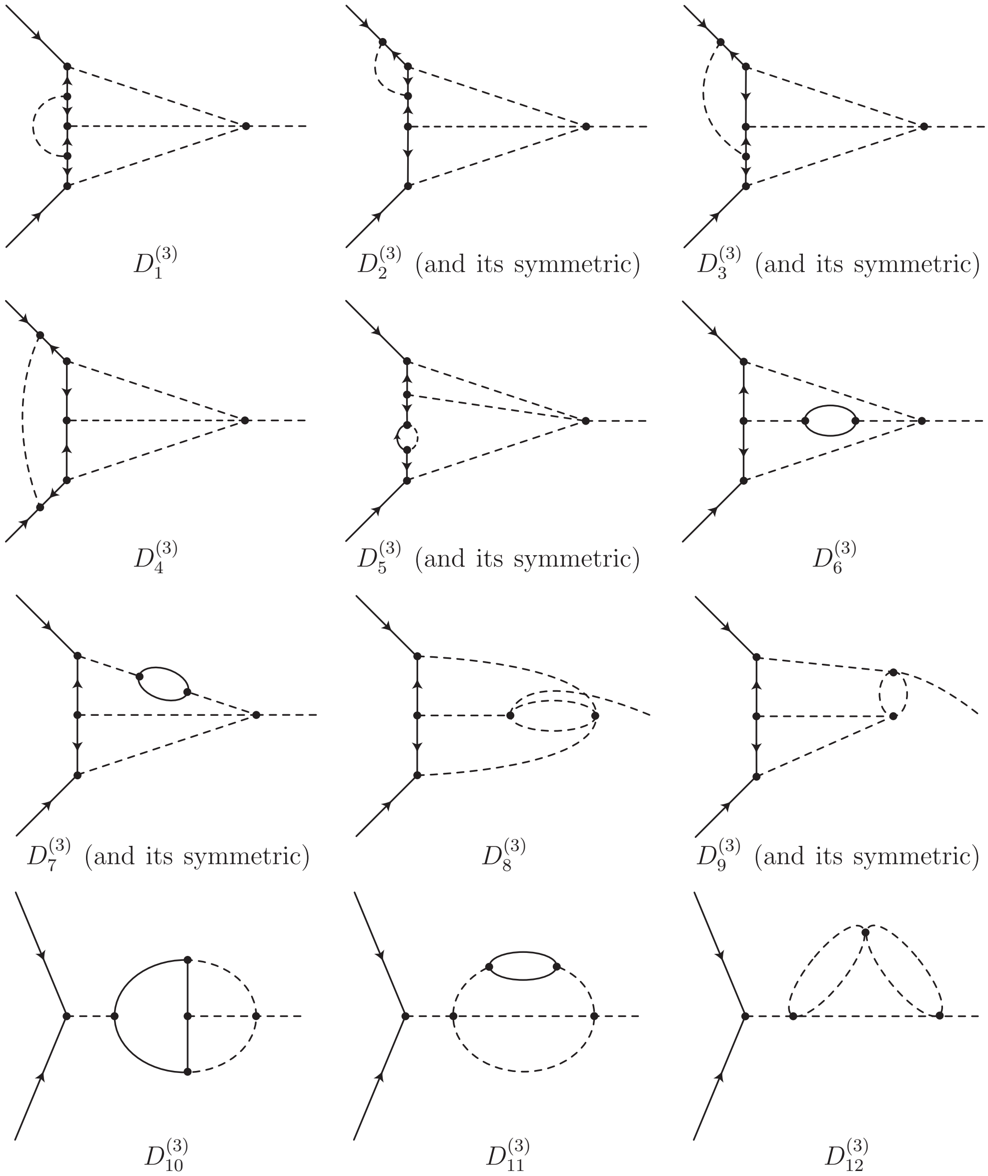

Figure 2. Diagrams that contribute to q at three-loop order.

The diagrams of figure 2 have simple poles in ϵ and so they contribute to the Yukawa beta function at three loops:

$$(16\pi^2)^3\beta^{(3\text{-loop})}_{a|ij} \supset c_1 y_{b|ik} y^*_{c|k\ell} y_{d|\ell m} y^*_{c|mn} y_{e|nj} \lambda_{abde} + \cdots + c_{12} y_{b|ij} \lambda_{bcde} \lambda_{cdfg} \lambda_{aefg}. \quad (2.9)$$

The three-loop analog of eq. (2.6) is then (again omitting the prefactor)[4]

$$q^{(3)} \propto -71+3(c_1+2c_2+2c_3+c_4+2c_5+4c_6+8c_7)+4(c_8+2c_9+3c_{10}+4c_{11}+58c_{12}), \quad (2.11)$$

[4]For the model of two scalars and one Weyl spinor of ref. [2] the expression for $q^{(3)}$ is

$$q^{(3)} \propto -219+12(4c_1+2c_2+2c_3+c_4+2c_5+2c_6+4c_7)+4(5c_8+10c_9+6c_{10}+10c_{11}+187c_{12}). \quad (2.10)$$

where the constant piece comes from contributions to $q^{(3)}$ from the previous order.

To compute these three-loop diagrams we implemented the algorithm of ref. [25].[5] There, IR divergences are regulated by introducing a spurious mass parameter through an exact decomposition of the massless propagator, and the calculation proceeds with properly choosing a loop momentum, regarding it as large, and expanding with respect to it the remaining two-loop subintegral, for which the chosen momentum is external. Remarkably, the authors of ref. [25] manage to construct explicit formulas for the pole parts of all three-loop scalar integrals. The implementation of their algorithm is straightforward, e.g., in *Mathematica*, but one must be very careful to take into account all required counterterms, including the ones introduced by the IR regulator. To test our implementation, we verified the two-loop result of ref. [23] for $\beta_{a|ij}$, and also part of the three-loop result for the beta function of the quartic coupling in a multi-flavor theory of scalars found in ref. [19]. We also performed explicit computations of a couple of diagrams.

From the diagrams of figure 2 we find

$$c_1 = 3, \quad c_2 = -1, \quad c_3 = 2, \quad c_4 = 5, \quad c_5 = \frac{1}{2}, \quad c_6 = \frac{3}{2}, \tag{2.12}$$

$$c_7 = \frac{1}{2}, \quad c_8 = \frac{3}{2}, \quad c_9 = \frac{1}{2}, \quad c_{10} = \frac{5}{8}, \quad c_{11} = -\frac{5}{32}, \quad c_{12} = -\frac{1}{16}. \tag{2.13}$$

Restoring the prefactor, then, eq. (2.11) gives[6]

$$q^{(3)} = \frac{\sqrt{323(757 - 3\sqrt{52953})}}{2057700} \approx 7 \times 10^{-5}. \tag{2.15}$$

Since $q^{(3)} \neq 0$ we have established the existence of theories that are scale but not conformally invariant! We expect that theories in $d = 4$ can also display scale without conformal invariance.

To summarize, it is important to emphasize that the distinction between scale-invariant and conformal solutions of eqs. (2.1) at the two-loop level is that, for the latter, $q^{(\geq 3)} = 0$ already at two loops. Higher loops are expected to slightly modify the critical values of the couplings, while preserving $q = 0$. But there are solutions for which $q^{(\geq 3)} \neq 0$ already at two loops. As a result, the nature of these solutions is uncertain, and a higher-loop calculation is needed. Even without that calculation, though, it should be clear that not all solutions to eqs. (2.1) can be declared conformal with the same confidence and the three-loop computation we present here shows that indeed non-conformal scale-invariant solutions exist.

Since there is only one oscillation frequency the scale-invariant trajectory is a limit cycle. The RG evolution of the couplings along the limit cycle is easily determined from eqs. (2.1) and is shown in figure 3 for $\epsilon = 0.01$. Notice that all phases can be rotated away

[5] We would like to thank M. Misiak for pointing us to this reference.

[6] For the model of two scalars and one Weyl spinor of ref. [2] we find

$$q^{(3)} = \frac{35\sqrt{34706(3601 + 6\sqrt{419802})}}{2489696256} \approx 2 \times 10^{-4}. \tag{2.14}$$

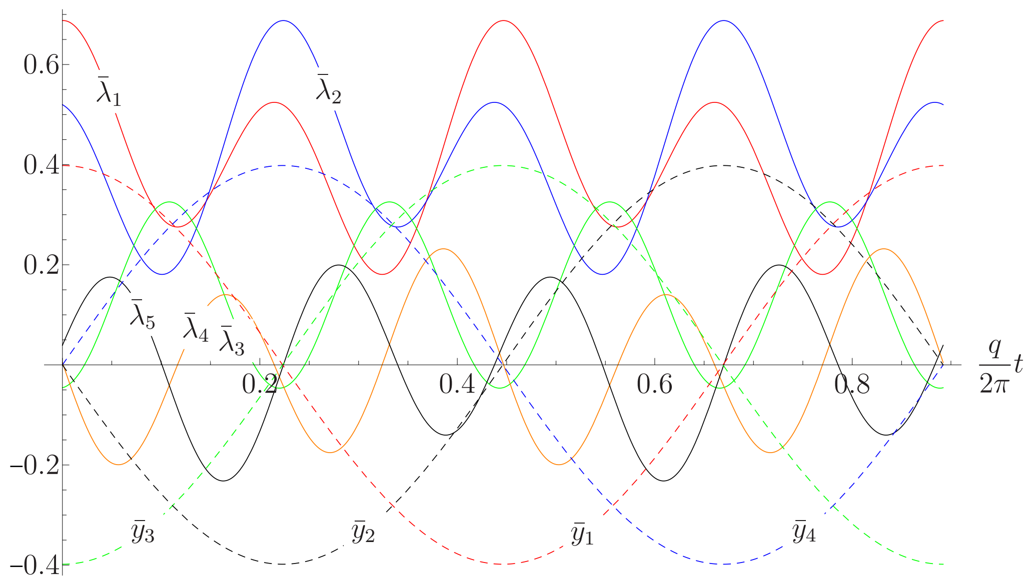

Figure 3. RG evolution of the couplings in the model with two real scalars and two Weyl fermions on a scale-invariant limit cycle as a function of RG time. Here $\epsilon = 0.01$.

and thus the model does not violate CP. Moreover the minimum of the scalar potential is located at the origin of field space. As expected, these statements (boundedness of the scalar potential, CP conservation, location of the vacuum in field space) are invariant along the limit cycle.

3 Other plausible examples

3.1 Limit cycle in $d = 4 - \epsilon$: model with 3 scalars and 2 fermions

The next simplest example in $d = 4 - \epsilon$ with a scalar potential which is bounded from below is described by a theory of three real scalars and two Weyl fermions, with canonical kinetic terms and interactions described by

$$\begin{aligned} V = {} & \frac{1}{24}\lambda_1\phi_1^4 + \frac{1}{24}\lambda_2\phi_2^4 + \frac{1}{24}\lambda_3\phi_3^4 + \frac{1}{4}\lambda_4\phi_1^2\phi_2^2 + \frac{1}{4}\lambda_5\phi_1^2\phi_3^2 + \frac{1}{4}\lambda_6\phi_2^2\phi_3^2 + \frac{1}{6}\lambda_7\phi_1^3\phi_2 \\ & + \frac{1}{6}\lambda_8\phi_1^3\phi_3 + \frac{1}{6}\lambda_9\phi_1\phi_2^3 + \frac{1}{6}\lambda_{10}\phi_2^3\phi_3 + \frac{1}{6}\lambda_{11}\phi_1\phi_3^3 + \frac{1}{6}\lambda_{12}\phi_2\phi_3^3 + \frac{1}{2}\lambda_{13}\phi_1^2\phi_2\phi_3 \\ & + \frac{1}{2}\lambda_{14}\phi_1\phi_2^2\phi_3 + \frac{1}{2}\lambda_{15}\phi_1\phi_2\phi_3^2 + \bigg(\frac{1}{2}y_1\phi_1\psi_1\psi_1 + \frac{1}{2}y_2\phi_2\psi_1\psi_1 + \frac{1}{2}y_3\phi_3\psi_1\psi_1 \\ & + \frac{1}{2}y_4\phi_1\psi_2\psi_2 + \frac{1}{2}y_5\phi_2\psi_2\psi_2 + \frac{1}{2}y_6\phi_3\psi_2\psi_2 + y_7\phi_1\psi_1\psi_2 + y_8\phi_2\psi_1\psi_2 + y_9\phi_3\psi_1\psi_2 + \text{h.c.}\bigg). \end{aligned} \tag{3.1}$$

Here the unknown parameters Q_{ab} and P_{ij} in the virial current are given by

$$Q = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1 & 0 & q_3 \\ -q_2 & -q_3 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} ip_1 & p_3 + ip_4 \\ -p_3 + ip_4 & ip_2 \end{pmatrix}, \tag{3.2}$$

where $q_{i=1,\ldots,3}$ and $p_{i=1,\ldots,4}$ are real. All the scalar quartic couplings, $\lambda_{1,\ldots,15}$, and two of the Yukawa couplings, y_1 and y_4, do not vanish on the scale-invariant trajectory. Due to its lengthy form we do not give here the explicit ϵ-expansion. Its exact knowledge does not lead to a better understanding of the physics and, moreover, the ϵ-expansion can easily be determined from the two-loop beta functions of ref. [23] and our new three-loop results. The non-vanishing virial current parameters on this scale-invariant trajectory are q_1, q_2 and p_4. The ϵ-expansion for q_1 and q_2 are distinct while, again, p_4 is undetermined and corresponds to the enhanced symmetry

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.3}$$

of the scale-invariant trajectory.

Since on this scale-invariant trajectory the oscillation frequencies are $\pm\sqrt{q_1^2+q_2^2+q_3^2}$ and 0, the scale-invariant trajectory is also a limit cycle. Again, the model has a bounded-from-below scalar potential, does not violate CP and has a minimum at the origin of field space.

3.2 Limit cycle and ergodicity in $d = 4 - \epsilon$: model with $N_S > 3$ scalars and $N_F > 2$ fermions

Up to now the models in $d = 4-\epsilon$ spacetime dimensions display scale-invariant trajectories that are limit cycles. Although the virial current has enough freedom to lead to several oscillation frequencies, in both models the non-trivial part of P_{ij} vanishes and thus the oscillation frequencies are solely obtained from Q_{ab}. For two and three real scalars it is thus impossible to get scale-invariant trajectories that exhibit ergodicity. Indeed, the eigenvalues of Q_{ab} are $\{\pm iq_1\}$ and $\{0, \pm i\sqrt{q_1^2+q_2^2+q_3^2}\}$ for two and three real scalars respectively, implying limit cycles. Eigenvalues of antisymmetric matrices with real entries always come in pairs $\pm i\omega$, except in the case where the dimensionality of the matrix is odd, where, in addition, there is a zero eigenvalue. Therefore, assuming $P_{ij} = 0$, four or more real scalars are necessary to obtain ergodic behaviors. For example, since the ϵ-expansion for the q_i are generically distinct it is expected that the model with four real scalars and two Weyl fermions will display both limit cycles and ergodic behavior as a function of ϵ.

We therefore conjecture that ergodic behavior in $d = 4-\epsilon$ spacetime dimensions occurs in models with $N_S > 3$ real scalars and $N_F > 1$ Weyl fermions. Unfortunately, due to the large number of couplings (for example the model with four real scalars and two Weyl fermions has $\frac{N_S(N_S+1)(N_S+2)(N_S+3)}{4!} + 2 \times N_S \frac{N_F(N_F+1)}{2} = 59$ real couplings) the computing time necessary to generate the three-loop beta functions becomes excessive and we have not pursued this direction further.

4 Renormalization-scheme changes

4.1 Scheme changes and conformal fixed points: the one-coupling case

Let us first review the effects of scheme changes in conformal theories. The simple case of a theory with only one coupling has been investigated long ago in ref. [26]. Under a scheme

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change, the coupling g and the wavefunction renormalization $Z(g)$ become

$$\begin{aligned} g &\to \tilde{g}(g) = g + \mathcal{O}(g^3), \\ Z^{1/2}(g) &\to \widetilde{Z}^{1/2}(\tilde{g}) = Z^{1/2}(g)F(g), \end{aligned} \tag{4.1}$$

where $F(g) = 1 + \mathcal{O}(g^2)$ and $F \neq 0$ for all g. In the new scheme $\tilde{g}$ is equal to g at lowest order since the coupling is unambiguous at the classical level. The same is true for the wavefunction renormalization as well. Therefore, since[7]

$$\begin{aligned} \beta(g) &= -\frac{dg}{dt}, \\ \gamma(g) &= -Z^{-1/2}(g)\frac{dZ^{1/2}(g)}{dt}, \end{aligned} \tag{4.2}$$

the new beta function and anomalous dimension are related to the old beta function and anomalous dimension through

$$\begin{aligned} \tilde{\beta}(\tilde{g}) &= \beta(g)\frac{\partial\tilde{g}}{\partial g}, \\ \tilde{\gamma}(\tilde{g}) &= \gamma(g) + F^{-1}(g)\beta(g)\frac{\partial F(g)}{\partial g}. \end{aligned} \tag{4.3}$$

Although the RG functions depend strongly on the renormalization scheme, properties that have physical consequences must be independent of the scheme. Such properties are:

(I) The existence of a conformal fixed point;

(II) The anomalous dimension at a conformal fixed point, which determines the scaling behavior of Green functions;

(III) The first derivative of the beta function at a conformal fixed point, which determines the sign[8] and rate of approach of the coupling to the conformal fixed point and thus modifies asymptotic formulae;

(IV) The first two coefficients in the beta function, which govern the UV or IR asymptotics of the coupling;

(V) The first coefficient in the anomalous dimension, which controls the scale factor of the field in the far UV or IR.

These properties all follow from eqs. (4.3) and the form of $\tilde{g}(g)$ and $F(g)$.

4.2 Scheme changes and conformal fixed points: the multi-coupling case

When the theory has more than one coupling, a scheme change still transforms the coupling vector[9] g^I and the wavefunction renormalization matrix $Z_I{}^J(g)$ as in (4.1) but, due to the

[7] We use $\phi_{\mathrm{B}} = Z^{1/2}(g)\phi_{\mathrm{R}}$.

[8] Note that the sign determines the character (attractive or repulsive) of the conformal fixed point.

[9] Capitalized indices run through all couplings. For matrices we use, e.g., $Q_I{}^J$ for both Q_{ab} and P_{ij}.

vector and matrix character of the coupling and wavefunction renormalization respectively, the new wavefunction renormalization is modified by a matrix ${F_I}^J(g)$ through

$$Z^{1/2}(g) \to \widetilde{Z}^{1/2}(\tilde{g}) = Z^{1/2}(g)F(g). \tag{4.4}$$

Thus, under a scheme change, one has

$$\tilde{\beta}^I(\tilde{g}) = \beta^J(g)\frac{\partial \tilde{g}^I}{\partial g^J}, \tag{4.5a}$$

$$\tilde{\gamma}_I{}^J(\tilde{g}) = \left[F^{-1}(g)\gamma(g)F(g)\right]_I{}^J + \left[F^{-1}(g)\beta^K(g)\frac{\partial F(g)}{\partial g^K}\right]_I{}^J. \tag{4.5b}$$

It is easy to see that, in the multi-coupling case, properties (I) and (V) are still scheme-independent. Property (II) is of course modified so that only $\operatorname{tr}\gamma$ and $\det\gamma$, and so the eigenvalues of γ, are scheme-independent. Property (III) is also modified since

$$\frac{\partial\tilde{\beta}^J(\tilde{g})}{\partial\tilde{g}^I} = \frac{\partial g^K}{\partial\tilde{g}^I}\frac{\partial\beta^L(g)}{\partial g^K}\frac{\partial\tilde{g}^J}{\partial g^L} + \frac{\partial g^K}{\partial\tilde{g}^I}\beta^L(g)\frac{\partial}{\partial g^L}\left(\frac{\partial\tilde{g}^J}{\partial g^K}\right), \tag{4.6}$$

such that at a conformal fixed point the eigenvalues of $\partial\beta^J(g)/\partial g^I$ are independent of the scheme. This is expected because $\partial\beta^J/\partial g^I = \gamma_I{}^J$, where $\gamma_I{}^J$ is the anomalous-dimension matrix of the operators sourced by the appropriate couplings. Therefore, eq. (4.6) can be seen as an extension of eq. (4.5b) with $F = \partial\tilde{g}/\partial g$.

Finally, if the one-loop beta function for one coupling depends on other couplings, property (IV) is no longer true [3] — only the first coefficient in the beta function is scheme-independent, although the UV or IR asymptotics of the couplings are the same in any scheme.

4.3 Natural scheme changes and scale-invariant trajectories

It is interesting to see how scale-invariant solutions behave under scheme changes.[10] Here we will distinguish between two types of scheme changes, which we dub natural and unnatural. A natural scheme change transforms the couplings as

$$\begin{aligned} \lambda_{abcd} &\to \tilde{\lambda}_{abcd} = \lambda_{abcd} + \eta_{abcd}, \\ y_{a|ij} &\to \tilde{y}_{a|ij} = y_{a|ij} + \xi_{a|ij}, \\ y^*_{a|ij} &\to \tilde{y}^*_{a|ij} = y^*_{a|ij} + \xi^*_{a|ij}, \end{aligned} \tag{4.7}$$

such that all couplings transform covariantly with respect to the symmetry group of the kinetic terms. MS and variants are examples of this — it occurs, e.g., every time one dresses a Feynman diagram topology with couplings. Unnatural scheme changes spoil the covariance of equations.

We can now show that entries of Q and P, which determine, e.g., the frequency on a cyclic trajectory, are scheme-independent for natural scheme changes. Indeed, if the

[10] The discussion of this subsection applies to scheme changes under which eqs. (2.1) transform covariantly. Since the analysis for gauge fields is straightforward, gauge fields are omitted for simplicity.

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scheme change is natural, then the time evolution of η and ξ on a scale-invariant trajectory is given by

$$\begin{aligned} \eta_{abcd}(t) &= (e^{Qt})_{a'a}(e^{Qt})_{b'b}(e^{Qt})_{c'c}(e^{Qt})_{d'd}\,\eta_{a'b'c'd'}(0),\\ \xi_{a|ij}(t) &= (e^{Qt})_{a'a}(e^{Pt})_{i'i}(e^{Pt})_{j'j}\,\xi_{a'|i'j'}(0), \end{aligned} \tag{4.8}$$

and so

$$\frac{d\eta_{abcd}}{dt} = Q_{a'a}\eta_{a'bcd} + \text{permutations}, \qquad \frac{d\xi_{a|ij}}{dt} = Q_{a'a}\xi_{a'|ij} + P_{i'i}\xi_{a|i'j} + P_{j'j}\xi_{a|ij'}. \tag{4.9}$$

On a scale-invariant trajectory eqs. (4.7) give

$$\tilde{\beta}_{abcd} = \mathcal{Q}_{abcd} - \frac{d\eta_{abcd}}{dt}, \qquad \tilde{\beta}_{a|ij} = \mathcal{P}_{a|ij} - \frac{d\xi_{a|ij}}{dt}, \tag{4.10}$$

and we can use eqs. (4.9) to obtain

$$\begin{aligned} \tilde{\beta}_{abcd} &= -Q_{a'a}\tilde{\lambda}_{a'bcd} + \text{permutations},\\ \tilde{\beta}_{a|ij} &= -Q_{a'a}\tilde{y}_{a'|ij} - P_{i'i}\tilde{y}_{a|i'j} - P_{j'j}\tilde{y}_{a|ij'}. \end{aligned} \tag{4.11}$$

Hence, Q and P are scheme-independent for natural scheme changes.

As a result of our analysis the existence of scale-invariant trajectories does not depend on the renormalization scheme. As expected, then, property (I) is easily extended to include non-conformal scale-invariant trajectories.

Focusing on scalar anomalous dimensions (the argument can be easily repeated for fermion anomalous dimensions), property (II) can also be generalized to scale-invariant theories. Indeed, for natural scheme changes on a scale-invariant trajectory eq. (4.5b) becomes

$$\tilde{\gamma}_{ab}(\tilde{g}) = \left[F^{-1}(g)\gamma(g)F(g)\right]_{ab} + \left\{F^{-1}(g)[Q, F(g)]\right\}_{ab} \tag{4.12}$$

since $-dF(g)/dt = [Q, F(g)]$. One can then immediately see that (using matrix notation)

$$\tilde{\gamma}(\tilde{g}) + Q = F^{-1}(g)[\gamma(g) + Q]F(g), \tag{4.13}$$

so that the eigenvalues of $\gamma + Q$ are scheme-independent. This is in accord with expectations: in ref. [4] it was shown that the behavior of two-point functions is determined by the eigenvalues of $\gamma + Q$, which are therefore expected to be scheme-independent.

Since property (II) can be generalized to scale-invariant theories, the same is expected for property (III) due to $\partial\beta^J/\partial g^I = \gamma_I{}^J$. Indeed, eq. (4.6) becomes

$$\frac{\partial\tilde{\beta}^J}{\partial\tilde{g}^I} = \left[F^{-1}(g)\frac{\partial\beta}{\partial g}F(g)\right]_I^{\;J} + \left\{F^{-1}(g)[Q, F(g)]\right\}_I^{\;J}, \tag{4.14}$$

where $F = \partial\tilde{g}/\partial g$, which gives (again using matrix notation)

$$\frac{\partial\tilde{\beta}}{\partial\tilde{g}} + Q = F^{-1}(g)\left[\frac{\partial\beta}{\partial g} + Q\right]F(g). \tag{4.15}$$

Therefore, the eigenvalues of $\partial\beta/\partial g + Q = \partial(\beta - \mathcal{Q})/\partial g$ (since $\mathcal{Q} = -gQ$) are scheme-independent. It is interesting to note that the eigenvalues of $\partial\beta/\partial g + Q$ are expected to determine the character (attractive, repulsive, etc.) of scale-invariant trajectories, and so one of them should be zero — that is indeed the eigenvalue corresponding to the (left) eigenvector $\beta^I = \mathcal{Q}^I$. This is because $\beta^I = \mathcal{Q}^I$ generates a motion *along* the scale-invariant trajectory, not away from it, as can be seen directly from

$$\beta^I \left[\frac{\partial\beta^J}{\partial g^I} + Q_I{}^J\right]_{\beta^I=\mathcal{Q}^I} = -\frac{d\mathcal{Q}^I}{dt} + \mathcal{Q}^I Q_I{}^J = 0. \tag{4.16}$$

Finally, properties (IV) and (V) in the multi-coupling case are trivially extended to scale-invariant theories since they do not rely on the existence of scale-invariant trajectories (or conformal fixed points).

To summarize, the scheme-independent properties (I–V) can be generalized to:

(I') The existence of conformal fixed points and scale-invariant trajectories;

(II') The eigenvalues of $\gamma + Q$ at conformal fixed points and scale-invariant trajectories;

(III') The eigenvalues of $\partial\beta/\partial g + Q$ at conformal fixed points and scale-invariant trajectories;

(IV') The first coefficient in the beta functions;

(V') The first coefficient in the anomalous-dimension matrix.

5 Stability properties

5.1 General discussion

It is of interest to study the stability of scale-invariant solutions under small deformations. Such an analysis determines the character of a particular scale-invariant solution, which can have (IR) attractive and/or repulsive deformations. In this section we will describe the properties of all possible scale-invariant solutions. The corresponding results for conformal fixed points are recovered by setting $Q = 0$ in the equations below. To simplify the equations, matrix notation is used throughout this section.

Since non-conformal scale-invariant solutions exhibit non-trivial RG flows, it is natural to disentangle the two contributions to the flow of the deformations, i.e., the expected contribution from the non-conformal scale-invariant solution, and the actual contribution from the deformations which we want to analyze. The appropriate quantity to study is thus $\delta g(t) = [g(t) - g_*(t)]e^{-Qt}$, where $g_*(t) = g_*(0)e^{Qt}$ is a scale-invariant solution, $\beta|_{g=g_*(t)} = \mathcal{Q}(t)$. The quantity $\delta g(t)$ determines the behavior of the deformations as a function of RG time in a "comoving frame", i.e., *modulo* the expected non-conformal scale-invariant solution RG flow. Note that, although for non-conformal scale-invariant solutions the choice of $g_*(0)$ in $\delta g(t) = g(t)e^{-Qt} - g_*(0)$ is arbitrary,[11] in order to study the behavior of small deformations one should first fix a $g_*(0)$.

[11] Any two points on a non-conformal scale-invariant trajectory are physically equivalent due to scale invariance.

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To proceed further it is necessary to Taylor expand the beta functions around the appropriate scale-invariant solution $g_*(t)$:

$$\beta(t) = \beta|_{g=g_*(t)} + [g(t) - g_*(t)] \left.\frac{\partial\beta}{\partial g}\right|_{g=g_*(t)} + \cdots = \mathcal{Q}(t) + \delta g(t) \left.\frac{\partial\beta}{\partial g}\right|_{g=g_*(0)} e^{Qt} + \cdots , \quad (5.1)$$

where the last equality follows since $-d(\partial\beta/\partial g)/dt = [Q, \partial\beta/\partial g]$ on the scale-invariant solution. Note that in order to disentangle the two contributions to the flow, the above Taylor expansion is RG-time dependent. It is now straightforward to write down, at lowest non-trivial order, the system of (linear) differential equations that the deformations must satisfy:

$$-\frac{d\,\delta g(t)}{dt} = [\beta(t) - \mathcal{Q}(t)]e^{-Qt} + \delta g(t)Q = \delta g(t)S + \cdots , \quad (5.2)$$

where

$$S = \left(\left.\frac{\partial\beta}{\partial g}\right|_{g=g_*(0)} + Q \right) \quad (5.3)$$

is the stability matrix. It is obvious that $\delta g(t)$ is the appropriate choice of variable that allows a separation of the RG flow contributions, for all RG-time dependence in eq. (5.2) comes solely from $\delta g(t)$. Note, moreover, that eq. (5.2) implies that the behavior of the deformations $\delta g(t)$ is dictated by the eigenvalues of S which, as we showed in the previous section, are scheme-independent (property (III')). The solution to the system of differential equations (5.2) is simply

$$\delta g(t) = \delta g(0)e^{-St} + \cdots \quad (5.4)$$

and one can easily see that positive (respectively, negative) eigenvalues of the stability matrix S correspond to IR attractive (respectively, repulsive) deformations. As usual, the fate of deformations related to vanishing eigenvalues cannot be determined from eq. (5.4) — for vanishing eigenvalues it is necessary to go to higher order in the Taylor expansion (5.2). However, as already mentioned, non-conformal scale-invariant solutions exhibit one special (left) eigenvector $\delta g(0) \propto \mathcal{Q}(0)$ with vanishing eigenvalue which represents a deformation along the scale-invariant solution. For this special deformation the full solution $\delta g(t) = [g_*(t \pm \delta t) - g_*(t)]e^{-Qt} = g_*(0)[e^{\pm Q\,\delta t} - 1] = \mp\mathcal{Q}(0)\,\delta t + \cdots$ is RG-time independent as expected, since it corresponds to a flow along the RG scale-invariant trajectory.

The previous analysis is a generalization of the similar analysis done for conformal solutions where $Q = 0$. Note that the special (left) eigenvector $\delta g(0) \propto \mathcal{Q}(0)$ does not exist for conformal fixed points, as expected since conformal solutions do not exhibit any non-trivial RG flow.

5.2 The example

We can now use the results discussed above to investigate the behavior of small deformations away from scale-invariant solutions. To this end it is natural to use an ϵ expansion for the stability matrix S and its eigenvalues x_m,

$$S = \sum_{n\geq 2} S^{(\frac{n}{2})}\epsilon^{\frac{n}{2}}, \qquad x_m = \sum_{n\geq 2} x_m^{(\frac{n}{2})}\epsilon^{\frac{n}{2}}. \quad (5.5)$$

The form of the expansion is dictated by the form of the beta functions in the stability matrix.

The eigenvalues of the stability matrix are the roots of the characteristic polynomial $\det(x\mathbb{1} - S)$ which can also be expanded in ϵ. To lowest order the characteristic polynomial simplifies and the eigenvalues are solutions of

$$\det(x^{(1)}\mathbb{1} - S^{(1)}) = 0. \tag{5.6}$$

Since there are only seven non-vanishing independent couplings ($\lambda_{1,\ldots,5}, y_{1,2}$ in (2.3)) at a generic point on the non-conformal scale-invariant solution described in section 2, eq. (5.6) for the corresponding couplings is

$$z(z-1)\left(z^5 - \frac{\sqrt{52953}}{57}z^4 + \frac{1894+\sqrt{52953}}{475}z^3 - \frac{240768-335\sqrt{52953}}{135375}z^2 \right.$$
$$\left. - \frac{421203-1573\sqrt{52953}}{225625}z + \frac{136(757\sqrt{52953}-158859)}{64303125}\right) = 0$$

which cannot be solved by factorization into radicals. (To avoid clutter we define $z = x^{(1)}$.) A numerical solution gives five positive, one negative and one vanishing eigenvalue:

$$z \approx 2.4, \quad z = 1, \quad z \approx 0.99, \quad z \approx 0.74, \quad z \approx 0.095, \quad z \approx -0.19, \quad z = 0. \tag{5.7}$$

The positive eigenvalues show that the scale-invariant solution is IR attractive in several directions. We thus expect that the limit cycle can be reached by an appropriate deformation of a theory defined at a UV conformal fixed point, although, to be certain, a more thorough analysis is necessary.

6 On the proof of the c-theorem at weak coupling

As discussed in the introduction, our three-loop results do not contradict the work of Osborn [17, 18] and Jack & Osborn [19]. Focusing on ref. [18], Osborn proved that RG flows are gradient flows at two loops in the weak coupling regime. Lifting the theory to curved space with spacetime-dependent couplings, Osborn showed that Weyl consistency conditions lead to

$$\frac{dc}{dt} = -\beta^I \frac{\partial c}{\partial g^I} = -G_{IJ}\beta^I\beta^J, \tag{6.1}$$

with G_{IJ} positive-definite in the weak coupling regime, thus forbidding the existence of recurrent behaviors at all loops. From the analysis of ref. [18] it would thus seem that scale-invariant trajectories are forbidden to all orders in perturbation theory. However, the analysis of ref. [18] leading to eq. (6.1) is too restrictive — it does not allow for spin-one operators of dimension three, i.e., it does not include the possibility of non-conformal scale-invariant theories.

The more general analysis, also performed by Osborn in ref. [18], includes possible spin-one operators of dimension three, which are related to the symmetry group of the

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kinetic terms. Such an analysis is done by promoting the related symmetry of the kinetic terms — for example the symmetry of the kinetic terms generated by the virial current, the natural spin-one operator of dimension three for scale-invariant theories — to a symmetry of the interacting theory. This is implemented by allowing the couplings to transform appropriately under a change generated by the spin-one operators of dimension three and by introducing background gauge fields to render the symmetry local. Then, assuming that the regularization procedure preserves local gauge invariance, Osborn's Weyl consistency conditions and current conservation show that

$$\frac{dc}{dt} = -\beta^I \frac{\partial c}{\partial g^I} = -(G_{IJ} + \cdots)\beta^I B^J, \tag{6.2}$$

where $B^I = \beta^I - \mathcal{Q}^I$. Note that $B^I = 0$ is precisely the condition for scale invariance. Thus, by allowing non-conformal scale-invariant theories from the start, the work of refs. [17–19] implies the existence of a c-function whose RG-time derivative vanishes at conformal fixed points as well as on scale-invariant trajectories. Note, moreover, that the c-function might not be monotonically decreasing due to the extra contributions to dc/dt represented by the ellipsis in eq. (6.2).

Note that, by promoting the symmetry of the spin-one operators of dimension three to a symmetry of the interacting theory, it is natural to demand regularization and renormalization schemes that satisfy the newly promoted symmetry. This also explains the special status of the natural renormalization schemes defined in the previous section.

Finally, it is interesting to see why the interference between quartic coupling one-loop beta functions and Yukawa coupling two-loop beta functions proposed by Wallace & Zia [24] as a possible obstruction to the gradient flow interpretation of the RG flow is circumvented by the introduction of the metric. Focusing on the problematic monomials in a possible c-function,

$$c \supset d_1 \operatorname{tr}(y_a^* y_b y_c^* y_d)\lambda_{abcd} + d_2 \operatorname{tr}(y_a^* y_b)\lambda_{acde}\lambda_{bcde}, \tag{6.3}$$

the related contributions to the beta functions at one and two loops respectively are

$$\begin{aligned} \frac{\partial c}{\partial \lambda_{abcd}} &\supset d_1 \operatorname{tr}(y_a^* y_b y_c^* y_d) + 2d_2 \operatorname{tr}(y_d^* y_e)\lambda_{abce} + \text{permutations},\\ \frac{\partial c}{\partial y_a} &\supset 2d_1 y_b y_c^* y_d \lambda_{abcd} + d_2 y_b \lambda_{acde}\lambda_{bcde}. \end{aligned} \tag{6.4}$$

Comparing with the true beta functions,

$$\begin{aligned} \beta_{abcd}^{(\text{1-loop})} &\supset -\frac{1}{16\pi^2} \operatorname{tr}(y_a^* y_b y_c^* y_d) + \frac{1}{16\pi^2}\frac{1}{6} \operatorname{tr}(y_d^* y_e)\lambda_{abce} + \text{permutations},\\ \beta_a^{(\text{2-loop})} &\supset -\frac{2}{(16\pi^2)^2}\, y_b y_c^* y_d \lambda_{abcd} + \frac{1}{(16\pi^2)^2}\frac{1}{12}\, y_b \lambda_{acde}\lambda_{bcde}, \end{aligned} \tag{6.5}$$

it is straightforward to see that the metric can account for the loop mismatch since $d_2/d_1 = -1/12$ for *both* beta functions, as pointed out in ref. [19]. Note that the conditions for a gradient flow interpretation of the RG flow introduced at higher orders are ever more constraining due to the large number of diagrams[12] and it is plausible that they are not

[12]This was already noticed in ref. [27].

satisfied, as our three-loop computation shows. The interference argument of Wallace & Zia [24] prevails at three loops, although for a complete investigation the knowledge of the full three-loop beta functions is necessary. Interestingly, the interference between the $(n-1)$-loop quartic-coupling beta function and the n-loop Yukawa beta function also explains why the n-loop quartic-coupling beta function is not necessary to argue for the existence of scale-invariant theories at n-th order in perturbation theory.

7 Cyclic trajectories and the c-theorem

It is important to note that the existence of recurrent behaviors in RG flows in $d = 4$ does not contradict all versions of the c-theorem.[13] In particular, the weak version of the c-theorem, where two conformal fixed points connected by an RG flow satisfy the inequality

$$a_{\mathrm{UV}} - a_{\mathrm{IR}} > 0 \tag{7.1}$$

with a the conformal anomaly (see, for example, ref. [28]),[14] is consistent with scale without conformal invariance. Even the stronger version of the c-theorem, where there exists a local function which is monotonically decreasing along non-trivial RG flows, is compatible with recurrent behaviors *as long as the c-function is constant on scale-invariant trajectories*. Only the strongest version of the c-theorem is violated by the existence of limit cycles and ergodicity; a gradient flow interpretation of RG flows is impossible for theories in which scale does not imply conformal invariance.

Since theories exhibiting limit cycles or ergodicity are scale-invariant, it is reasonable to expect the interpolating c-function to be constant on scale-invariant trajectories. Any such interpolating function is invariant under the symmetry group of the kinetic terms, i.e., it does not carry scalar or fermion indices. Thus, in a natural scheme, all the explicit RG-time dependence disappears on a scale-invariant trajectory. This is the behavior that is intuitively expected of the c-function, which should be some measure of the number of massless degrees of freedom of the theory. Therefore it must be constant on scale-invariant trajectories since any two points on such trajectories are physically equivalent.

This behavior is very different from that encountered on cyclic flows described in ref. [20] and recently discussed in association with the c-theorem in ref. [21] (see also ref. [22]). In ref. [21], the authors argue that monotonic RG flows can be simultaneously cyclic if one allows for a multi-valued interpolating c-function. This is fundamentally different from recurrent behavior with continuous scale invariance. As mentioned above, the interpolating c-function must be constant on scale-invariant trajectories. Moreover, the examples cited in ref. [21] exhibit one feature, turning points, which does not appear on continuously scale-invariant trajectories. Turning points are peculiar locations in coupling space: the beta functions vanish there, but the first derivative of the beta functions diverges. Consequently, RG flows can overshoot turning points. In contrast, all existing continuously scale-invariant examples are well-defined smooth weakly-coupled theories, and thus do not

[13] For a more extensive discussion see ref. [3].

[14] A claim for the proof of the inequality (7.1) appeared recently in ref. [29] (see also ref. [30]).

display turning points. The existence of turning points on cyclic flows is a reflection of the possibility of multi-valued c-functions which are monotonically decreasing along the flow. Here we want to stress that the physics of cyclic flows with turning points as described in ref. [21] is very different from that of recurrent behaviors with continuous scale invariance. It is therefore very unlikely that monotonically decreasing multi-valued c-functions exist on scale-invariant recurrent behaviors as suggested in ref. [21].

8 Conclusion

Does scale imply conformal invariance in unitary relativistic QFTs? The answer is negative in $d = 4-\epsilon$. Although a similarly conclusive statement in the $d = 4$ case cannot yet be made, we strongly believe that the answer there is also negative. There are no physical arguments on which one can rely to forbid non-conformal scale-invariant theories. Instead, one simply needs to compute the beta functions and explore the different regions in coupling space. That an example of a scale-invariant theory which is not conformal eluded the physics community for so long is easily explained by the complexity of the problem: to see non-conformal scale-invariant theories, one must go to three loops, and the beta functions at three loops in the most general QFT are not known.

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