# An integrand reconstruction method for three-loop amplitudes

**Simon Badger, Hjalte Frellesvig and Yang Zhang**

*Niels Bohr International Academy and Discovery Center, The Niels Bohr Institute,*
*University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

*E-mail:* badger@nbi.dk, hjf@nbi.dk, zhang@nbi.dk

ABSTRACT: We consider the maximal cut of a three-loop four point function with massless kinematics. By applying Gröbner bases and primary decomposition we develop a method which extracts all ten propagator master integral coefficients for an arbitrary triple-box configuration via generalized unitarity cuts. As an example we present analytic results for the three loop triple-box contribution to gluon-gluon scattering in Yang-Mills with adjoint fermions and scalars in terms of three master integrals.

## Contents

## 1 Introduction

Precise predictions for cross sections in collider experiments require knowledge of scattering amplitudes to a high loop order. Computations of this type, particularly in QCD, are still unfortunately out of reach using known techniques. Unitarity cuts have shown to be an enormously powerful tool for super-symmetric gauge and gravity theories with full amplitudes now computed up to four loops [1, 2]. At three-loops, steps beyond maximally super-symmetric amplitudes have also been considered in the recent computation of the UV behaviour of graviton-graviton scattering in $\mathcal{N} = 4$ supergravity [3]. For a recent review of state of the techniques see [4] and references therein.

The success of unitarity [5, 6] and generalised unitarity [7] in automating multi-loop one-loop corrections has sparked some recent interest in exploring the application of integrand reduction (OPP-like [8]) methods at two-loops [9–14]. Very recently the possibility of using computational algebraic geometry to overcome the traditional bottlenecks in amplitude computations has started to be explored [15, 16].

The traditional approach to a unitarity computation of a loop amplitude relies on the knowledge of a basis of known integral functions. At two loops it has been possible to derive sets of integration-by-parts identities [17, 18] that reduce amplitudes to a unitarity compatible basis [19, 20]. Using an integrand parametrisation constrained by the ideal generated from the propagators with help from Gram matrix identities we can also reduce a unitarity compatible form of the amplitude to master integrals using the well known Laporta algorithm [21] which is implemented in a number of public codes [22–25].

In this paper we consider the extension of the new integrand reduction techniques to three-loop amplitudes. We derive a complete reduction to ten-propagator master integrals
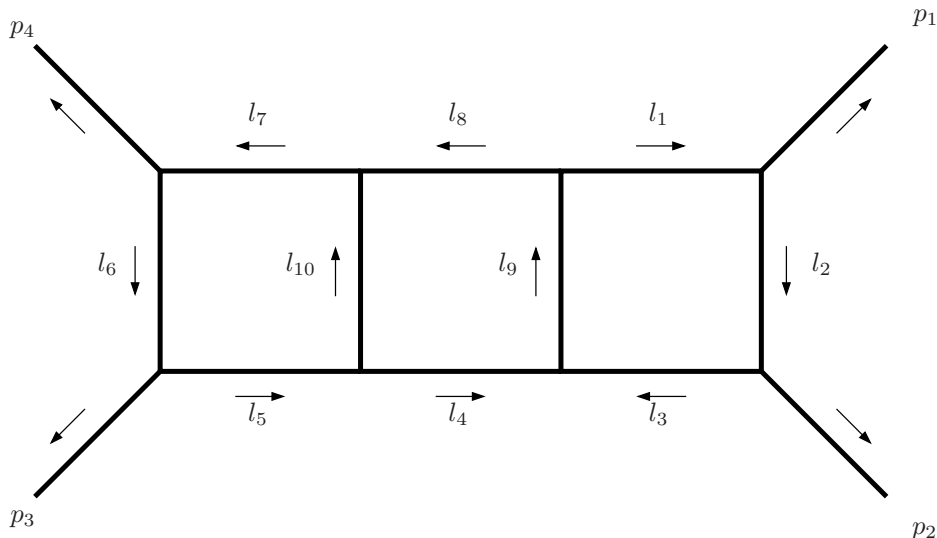
**Figure 1**. The momentum flow and propagator definitions for the three-loop planar triple box.

(MI) for the maximal cut of the three loop planar triple box with four massless external legs. As a first application of the technique we compute the ten-propagator MI coefficients for gluon scattering in Yang-Mills theory with adjoint fermions and scalars. The result applies in both $\mathcal{N} = 4, 2, 1$ and 0 super-symmetric theories.

## 2 Reduction of the massless triple box

In this paper we will study the three-loop planar triple box topology, shown in figure 1, defined by:

$$I_{10}\big[\tilde{N}_D(\epsilon, k_i, p_j)\big] = \int \frac{d^D k_1}{(2\pi)^D} \int \frac{d^D k_2}{(2\pi)^D} \int \frac{d^D k_3}{(2\pi)^D} \frac{\tilde{N}_D(\epsilon, k_i, p_j)}{\prod_{i=1}^{10} l_i^2}, \tag{2.1}$$

where the ten propagators $\{l_i\}$ are given by:

$$
\begin{aligned}
&l_1 = k_1\,, &&l_2 = k_1 - p_1\,, &&l_3 = k_1 - p_1 - p_2\,, &&l_4 = k_3 + p_1 + p_2\,, \\
&l_5 = k_2 + p_1 + p_2\,, &&l_6 = k_2 - p_4\,, &&l_7 = k_2\,, &&l_8 = k_3\,, \\
&l_9 = k_1 + k_3\,, &&l_{10} = k_2 - k_3\,. &&&&
\end{aligned}
\tag{2.2}
$$

The external momenta, $\{p_i\}$, and internal momenta, $\{l_i\}$, are considered massless. Though the integral needs to be dimensionally regulated in $D = 4 - 2\epsilon$ dimensions we will only consider generalized unitarity cuts in four dimensions and therefore reconstruct only the leading term of the numerator function, $\tilde{N}_D(\epsilon, k_i, p_j) = \tilde{N}(k_i, p_j) + \mathcal{O}(\epsilon)$.

We also consider the external momenta to be outgoing with the Mandelstam variables defined by:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 + p_4)^2, \qquad u = -s - t\,. \tag{2.3}$$

Following a similar approach taken at two loops in our previous work [11], we will proceed in three steps: firstly the on-shell constraints for the deca-cut will be solved.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $1-\frac{s}{t}\frac{1+\tau_1}{\tau_2}$ | 0 |
| 2 | 0 | 0 | $-\frac{u}{t}(1+\tau_1)$ | 0 |
| 3 | 0 | 0 | $\tau_2$ | 0 |
| 4 | 0 | 0 | $-\frac{u}{t}(1+\tau_1)$ | 0 |
| 5 | 0 | 0 | $\tau_2$ | 0 |
| 6 | 0 | 0 | $1$ | 0 |
| 7 | 0 | 0 | $\tau_1$ | 0 |

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $1+\tau_2$ | 0 |
| 2 | 0 | 0 | 0 | $-1+\frac{s}{u}\frac{1}{\tau_1}$ |
| 3 | 0 | 0 | 0 | $-1-\tau_1$ |
| 4 | 0 | 0 | 0 | $\tau_2$ |
| 5 | 0 | 0 | $1$ | 0 |
| 6 | 0 | 0 | $\tau_2$ | 0 |
| 7 | 0 | 0 | $\tau_2$ | 0 |

**Table 1**. Values of the coefficients in $l_2$ and $-l_6$ in (2.4).

Secondly, we invert a linear system to map the polynomial of the deca-cut defined by the on-shell constraints onto the general integrand basis. Finally the integrand is reduced to master integrals(MIs) using additional integration-by-parts(IBP) relations.

### 2.1 Solving the on-shell constraints

The parametrisation of the loop-momenta using two component Weyl spinors has been chosen as follows:

$$l_2 = x_1 p_1 + x_2 p_2 + x_3 \frac{\langle 23 \rangle}{\langle 13 \rangle} \frac{\langle p_1|\gamma^\mu|p_2]}{2} + x_4 \frac{\langle 13 \rangle}{\langle 23 \rangle} \frac{\langle p_2|\gamma^\mu|p_1]}{2},$$

$$-l_6 = y_1 p_3 + y_2 p_4 + y_3 \frac{\langle 41 \rangle}{\langle 31 \rangle} \frac{\langle p_3|\gamma^\mu|p_4]}{2} + y_4 \frac{\langle 31 \rangle}{\langle 41 \rangle} \frac{\langle p_4|\gamma^\mu|p_3]}{2},$$

$$-l_4 = z_1 p_2 + z_2 p_3 + z_3 \frac{\langle 34 \rangle}{\langle 24 \rangle} \frac{\langle p_2|\gamma^\mu|p_3]}{2} + z_4 \frac{\langle 24 \rangle}{\langle 34 \rangle} \frac{\langle p_3|\gamma^\mu|p_2]}{2}. \tag{2.4}$$

The ten cut-constraints $l_i^2 = 0$ form a set of ideals. Following the recent method proposed by Zhang [15], this system can be reduced using primary decomposition to find 14 independent solutions. These solutions come in complex conjugate[1] pairs and we label them 1–7 and $1'$–$7'$. We find that all 14 solutions have the same dimension and therefore can be parametrised with two variables which we call $\tau_1$ and $\tau_2$. For the solutions 1–7, explicit forms for the coefficients in (2.4) can be written as in tables 1 and 2.

It is worth mentioning that the parametrisation is Laurent-polynomial, so that no terms of the form $\frac{1}{1+\tau}$ appear. We will therefore be able to fit the integrand using an efficient discrete Fourier projection just as used succesfully at one-loop [26, 27].

The conjugate solutions can be easily constructed from the above expressions using,

$$x_1^{s'} = 0 \qquad x_2^{s'} = 0 \qquad x_3^{s'} = \frac{u}{t} x_4^s \qquad x_4^{s'} = \frac{t}{u} x_3^s$$

$$y_1^{s'} = 0 \qquad y_2^{s'} = 0 \qquad y_3^{s'} = \frac{u}{t} y_4^s \qquad y_4^{s'} = \frac{t}{u} y_3^s$$

$$z_1^{s'} = z_1^s \qquad z_2^{s'} = z_2^s \qquad z_3^{s'} = \frac{u}{s} z_4^s \qquad z_4^{s'} = \frac{s}{u} z_3^s. \tag{2.5}$$

Note that we have suppressed the functional dependence of the coefficients, e.g. $x_3^s \equiv x_3^s(\tau_1, \tau_2)$.

---

[1]Technically explicit complex conjugation is only valid for real external momenta. However, the solutions $1'$–$7'$ are also valid for complex external momenta.

| | $z_1$ | $z_2$ | $z_3$ | $z_4$ |
|---|---|---|---|---|
| 1 | $\frac{s}{t}\left(1+\frac{1}{\tau_1}\right)\left(1+\frac{1}{\tau_2}\right)$ | $-\frac{s}{t}\left(1+\frac{1}{\tau_1}\right)+\frac{\tau_2}{\tau_1}$ | $-\frac{u}{t}\left(1+\frac{1}{\tau_1}\right)$ | $Z_{41}$ |
| 2 | $\frac{s}{t}\left(1+\tau_2\right)\left(1-\frac{u}{s}\tau_1\right)$ | $\frac{s}{t}\left(1+\frac{1}{\tau_1}\right)\tau_2$ | $\left(\frac{u}{t}-\frac{s}{t}\frac{1}{\tau_1}\right)\tau_2$ | $-\frac{s}{t}(1+\tau_1)(1+\tau_2)$ |
| 3 | $\frac{s}{t}\left(1+\frac{1}{\tau_1}\right)$ | 0 | 0 | $\frac{s}{t}\left(\frac{s}{u}\frac{1}{\tau_1}-1\right)$ |
| 4 | 0 | $-\frac{s}{t}\left(1+\frac{1}{\tau_1}\right)$ | $\frac{s}{t}\frac{1}{\tau_1}-\frac{u}{t}$ | 0 |
| 5 | $\tau_1$ | 0 | 0 | $\frac{s}{u}(1+\tau_1)$ |
| 6 | 0 | $\tau_1$ | 0 | $\frac{s}{u}(1-\tau_1)$ |
| 7 | 0 | 0 | 0 | $\frac{s}{u}$ |

**Table 2**. Values of the coefficients in $-l_4$ in (2.4). $Z_{41} \equiv \frac{s^2}{ut}\left(1+\frac{1}{\tau_1}\right)\left(1+\frac{1}{\tau_2}\right)-\frac{s}{u}\frac{1+\tau_2}{\tau_1}$.

## 2.2 Fitting the integrand basis

The space of loop momenta is spanned by three external momenta, which we choose to be $\{p_1, p_2, p_4\}$, and an additional orthogonal direction $\omega$ which is defined by,

$$\omega \equiv \frac{1}{2s}\left(\langle 2|3|1]\langle 1|\gamma^\mu|2] - \langle 1|3|2]\langle 2|\gamma^\mu|1]\right). \tag{2.6}$$

The integrand numerator $N$, can be parametrised in terms of seven irreducible scalar products (ISPs) as follows:

$$N = \sum_{\alpha_i} c_{\alpha_1\ldots\alpha_7}(k_1\cdot p_4)^{\alpha_1}(k_2\cdot p_1)^{\alpha_2}(k_3\cdot p_4)^{\alpha_3}(k_3\cdot p_1)^{\alpha_4}$$
$$\times (k_1\cdot\omega)^{\alpha_5}(k_2\cdot\omega)^{\alpha_6}(k_3\cdot\omega)^{\alpha_7}. \tag{2.7}$$

We can use renormalizability conditions to restrict the maximum powers of $\alpha_i$ and complete the reduction by using polynomial division with respect to a Gröbner basis constructed from the propagators constraints. This procedure is implemented in the Mathematica package `BasisDet` and we refer to [15] for full details of the method. The resultant integrand naturally splits into 199 spurious (S) and 199 non-spurious (NS) terms of the form:

$$N^{\mathrm{NS}} = \sum_{\alpha_i}(k_1\cdot p_4)^{\alpha_1}(k_2\cdot p_1)^{\alpha_2}(k_3\cdot p_4)^{\alpha_3}(k_3\cdot p_1)^{\alpha_4}$$
$$\times \left(c^{\mathrm{NS}}_{\alpha_1\ldots\alpha_4 0} + c^{\mathrm{NS}}_{\alpha_1\ldots\alpha_4 1}(k_1\cdot\omega)(k_2\cdot\omega)\right), \tag{2.8}$$

$$N^{\mathrm{S}} = \sum_{\alpha_i}(k_1\cdot p_4)^{\alpha_1}(k_2\cdot p_1)^{\alpha_2}(k_3\cdot p_4)^{\alpha_3}(k_3\cdot p_1)^{\alpha_4}$$
$$\times \left(c^{\mathrm{S}}_{\alpha_1\ldots\alpha_4 0}(k_1\cdot\omega) + c^{\mathrm{S}}_{\alpha_1\ldots\alpha_4 1}(k_2\cdot\omega) + c^{\mathrm{S}}_{\alpha_1\ldots\alpha_4 2}(k_3\cdot\omega)\right). \tag{2.9}$$

The spurious terms will vanish after integration since they are linear in $k_i\cdot\omega$.

Inserting the expressions for the loop-momenta given in (2.4) into the integrand $N = N^{\mathrm{NS}} + N^{\mathrm{S}}$, defines 14 Laurent-polynomials of $\tau_1$ and $\tau_2$,

$$N|_s = \sum_{ij} d_{sij}\tau_1^i\tau_2^j, \tag{2.10}$$

$-4-$

where $s$ denotes one of the on-shell solutions either 1–7 or $1'$–$7'$. In total we find 622 terms in these expansions which we collect into a vector, $\mathbf{d}$. After collecting the 398 coefficients of (2.7) into a vector $\mathbf{c}$ we can define a Matrix $M$ such that,

$$\mathbf{d} = M\mathbf{c} \,. \tag{2.11}$$

After inverting this matrix we can re-construct an arbitrary integrand by using the Laurent expansion of the product of eight tree-level amplitudes to extract the values of the coefficients, $d_{sij}$.

Though in principle we could invert the matrix in its full $622 \times 398$ form this task is quite complicated in practice. However, it is straightforward to separate the problem into two pieces corresponding the spurious and non-spurious terms. We can show that,

$$k^s(\tau_1, \tau_2) \cdot p = k^{s'}(\tau_1, \tau_2) \cdot p \,, \qquad k^s(\tau_1, \tau_2) \cdot \omega = -k^{s'}(\tau_1, \tau_2) \cdot \omega \,. \tag{2.12}$$

Therefore when we combine the paired solutions we find,

$$N|_s + N|_{s'} = N^{\mathrm{NS}}|_s + N^{\mathrm{NS}}|_{s'} = 2N^{\mathrm{NS}}|_s \,, \tag{2.13}$$

$$N|_s - N|_{s'} = N^{\mathrm{S}}|_s - N^{\mathrm{S}}|_{s'} = 2N^{\mathrm{S}}|_s \,. \tag{2.14}$$

After defining two new vectors, $\mathbf{d}_\pm$, where

$$d_\pm^s = \frac{d^s \pm d^{s'}}{2} \,, \tag{2.15}$$

we find smaller matrices $311 \times 199$, $M_\pm$, satisfying

$$\mathbf{d}_+ = M_+ \mathbf{c}^{\mathrm{NS}}, \qquad \mathbf{d}_- = M_- \mathbf{c}^{\mathrm{S}}. \tag{2.16}$$

In this form it was possible to invert the matrices analytically using standard computer algebra packages.

## 2.3 Alternative branch-by-branch fitting

In this section we describe a technique that allows each branch of the on-shell solutions to be considered as a separate linear system, a natural extension of the strategy taken at the end of the previous section when we separated spurious and non-spurious terms. This will be achieved by first partially fitting the integrand on the 14 solutions to get 14 polynomials. Then the 14 polynomials are combined together to get the full integrand, by a Gröbner basis method. Some of the more mathematical details are described in the appendix A.

On each of the 14 solutions the integrand numerator $N$ can be reduced further to a polynomial $N_i$ with much fewer monomials, by the multivariate division towards the Gröbner basis of the corresponding branch. We note that $N_i \equiv N_i(\{\mathrm{ISPs}\})$ is distinct from $N|_i \equiv N|_i(\tau_1, \tau_2)$. The number of monomials of each $N_i$ is listed in table 3.

For example, divided by the Gröbner basis of the second branch, the integrand is reduced to a linear combination of 59 terms,

$$\begin{aligned}
\{ & x^4y^4,\ x^4y^3,\ x^3y^4,\ x^4z^2,\ x^4y^2,\ x^3z^3,\ x^3y^3,\ x^2z^4,\ x^2y^4,\ x^4z,\ x^4y,\ x^3z^2, \\
& x^3y^2,\ x^2z^3,\ x^2y^3,\ xz^4,\ xy^4,\ x^4,\ x^3z,\ x^3y,\ x^2z^2,\ x^2y^2,\ xz^3,\ xy^3,\ z^4,\ y^4,\ x^3, \\
& x^2z,\ x^2y,\ xz^2,\ xy^2,\ z^3,\ y^3,\ x^2,\ xz,\ xy,\ z^2,\ y^2,\ x,\ z,\ y,\ 1 \},
\end{aligned} \tag{2.17}$$

| | number of terms | | number of terms |
|---|---|---|---|
| 1 | 75 | $1'$ | 75 |
| 2 | 59 | $2'$ | 59 |
| 3 | 42 | $3'$ | 42 |
| 4 | 42 | $4'$ | 42 |
| 5 | 22 | $5'$ | 22 |
| 6 | 22 | $6'$ | 22 |
| 7 | 25 | $7'$ | 25 |

**Table 3.** Number of terms for reduced integrand in each solution.

where $x = k_1 \cdot p_4$, $y = k_2 \cdot p_1$, $z = k_3 \cdot p_4$. The coefficients are determined by polynomial fitting techniques, at the second solution.

$$\boldsymbol{d}^{(2)} = M^{(2)} \boldsymbol{c}^{(2)}, \tag{2.18}$$

where $\boldsymbol{c}^{(2)}$ is the list of the 59 coefficients and $\boldsymbol{d}^{(2)}$ contains the Laurent expansion coefficients at the second solution. $M^{(2)}$ is a $62 \times 59$ matrix, which is much smaller than $M$, so it is easy to compute $\boldsymbol{c}^{(2)}$ and the reduced integrand $N_2$.

Once all 14 $N_i$'s are obtained, we "merge" them together to get the integrand $N$. The step is equivalent to solving congruence equations in a polynomial ring. It is automatically done by a Macaulay2 program [28], based on a computation using Gröbner basis and ideal intersection. A more detailed mathematical description of the procedure is outlined in appendix A.

## 3 Results for gluon-gluon scattering in Yang Mills theories

The starting expression for the computation is the product of tree-level amplitudes:

$$N|_s = \sum_{f_i} \sum_{h_i} F_{f_1 \ldots f_5}(n_f, n_s) \, A_3^{(0)} \big( -l_{1,f_1}^{-h_1}, p_1^{\lambda_1}, l_{2,f_1}^{h_2} \big) \, A_3^{(0)} \big( -l_{2,f_1}^{-h_2}, p_2^{\lambda_2}, l_{3,f_1}^{h_3} \big)$$

$$\times A_3^{(0)} \big( -l_{3,f_1}^{-h_3}, -l_{4,f_2}^{-h_4}, l_{9,f_4}^{h_9} \big) \, A_3^{(0)} \big( l_{4,f_2}^{h_4}, -l_{5,f_3}^{-h_5}, l_{10,f_5}^{h_{10}} \big) \, A_3^{(0)} \big( l_{5,f_3}^{h_5}, p_3^{\lambda_3}, -l_{6,f_3}^{-h_6} \big)$$

$$\times A_3^{(0)} \big( l_{6,f_3}^{h_6}, p_4^{\lambda_4}, -l_{7,f_3}^{-h_7} \big) \, A_3^{(0)} \big( l_{7,f_3}^{h_7}, -l_{8,f_2}^{-h_8}, -l_{10,f_5}^{-h_{10}} \big) \, A_3^{(0)} \big( l_{8,f_2}^{h_8}, l_{1,f_1}^{h_1}, -l_{9,f_4}^{-h_9} \big), \tag{3.1}$$

where $l_i$ and $p_j$ are defined in (2.2). $\{h_i\}$ are the internal helicity states and $\{\lambda_i\}$ are the helicities of the external gluons. There are 34 distinct configurations of the internal flavours $\{f_i\}$ each associated with a number of fermion flavours, $n_f$ and complex scalar flavours, $n_s$. The explicit expressions for the flavour coefficients $F_{f_1 \ldots f_5}(n_f, n_s)$ are given in appendix B.

Complete expressions for all integrand coefficients have been found as functions of $n_f$ and $n_s$. Though each term is relatively compact, the full set of 398 coefficients makes the expression rather lengthy so we only present the result after further reduction to MIs. The full expressions for the integrands are available from the authors on request. We note however that all coefficients in the Laurent expansion (2.10) with $|i|$ or $|j| > 4$ vanish for all

$n_f$ and $n_s$ demonstrating the general renormalization conditions for this case are simpler than assumed when constructing the integrand in section 2.2.

IBP relations generated with `Reduze2` [24, 25] reduced the 199 non-vanishing integrals in the triple box integrand onto 3 Master Integrals, $I_{10}[1]$, $I_{10}[(k_1+p_4)^2]$ and $I_{10}[(k_3-p_4)^2]$. The analytic expression for the scalar integral $I_{10}[1]$ has been known already for some time [29]. We should be clear that to obtain the reduction of a full amplitude to all master integrals the procedure must be carried out in two steps. Firstly the full integrand, without applying IBPs, must be kept as a subtraction term for each lower propagator topology. Secondly IBP relations, again including all lower propagator MIs, should be applied to the complete amplitude.

In order to map the basis of 7 ISPs of (2.8) onto the 15 propagator[2] integral topology obeying IBP relations we must re-write the non-spurious term $(k_1 \cdot \omega)(k_2 \cdot \omega)$ in terms of the reducible quantity $k_1 \cdot k_2$. This is achieved straightforwardly using the Gram determinant identity generated by:

$$\det \begin{pmatrix} 1 & 2 & 4 & \omega & k_1 \\ 1 & 2 & 4 & \omega & k_2 \end{pmatrix} = 0 \,, \tag{3.2}$$

which is re-written as:

$$(k_1 \cdot \omega)(k_2 \cdot \omega) = -\frac{t^2}{4} + \frac{t}{2}\big((k_1 \cdot 4) + (k_2 \cdot 1)\big) + \frac{tu}{s}(k_1 \cdot k_2) + \frac{s+2t}{s}(k_1 \cdot 4)(k_2 \cdot 1) \,. \tag{3.3}$$

We note that this does not change the number of non-spurious integrals in the integrand basis.

We write the 4-point 3-loop primitive amplitude for this ladder topology as:

$$\begin{aligned} A_4^{(3)}\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) = &\; C_1\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) I_{10}[1] \\ &+ C_2\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) I_{10}\big[(k_1+p_4)^2\big] \\ &+ C_3\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) I_{10}\big[(k_3-p_4)^2\big] + \dots \end{aligned} \tag{3.4}$$

where the ellipses cover terms with less than ten propagators. We then define dimensionless coefficients $\hat{C}_k$ by,

$$C_k\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) = s^3 t A_4^{(0)}\big(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}\big) \hat{C}_k^{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t) \,, \tag{3.5}$$

where $A_4^{(0)}$ are the tree-level helicity amplitudes (expressions for the tree-level amplitudes are collected in appendix B for convenience). After following the procedure of computing the Laurent series of (3.1), reconstructing the integrand and applying the IBPs we obtain the following results for the three independent helicity configurations. The $--++$ turns out to be trivial,

$$\hat{C}_1^{--++}(s,t) = -1 \,, \tag{3.6}$$

$$\hat{C}_2^{--++}(s,t) = 0 \,, \tag{3.7}$$

$$\hat{C}_3^{--++}(s,t) = 0 \,. \tag{3.8}$$

---

[2]For the purposes of the IBPs we consider both positive and negative powers of the propagators in the topology hence the 15 propagators includes both the 10 denominators of eq. (2.1) and the 5 non-spurious ISPs.

The $-++-$ configuration is,

$$\hat{C}_1^{-++-}(s,t) = -1 - (4-n_f)(3-n_s)\frac{s(t+2s)}{2t^2} + (4-n_f)\frac{s(t+4s)}{2t^2}$$
$$- (1+n_s-n_f)\frac{s}{t^3}(2t^2+11st+10s^2)\,, \tag{3.9}$$

$$\hat{C}_2^{-++-}(s,t) = \frac{2}{t}(1-\hat{C}_1^{-++-})\,, \tag{3.10}$$

$$\hat{C}_3^{-++-}(s,t) = 0\,, \tag{3.11}$$

and finally the alternating $-+-+$,

$$\hat{C}_1^{-+-+}(s,t) = -1 + (4-n_f)\frac{st}{u^2} - 2(1+n_s-n_f)\frac{s^2t^2}{u^4}$$
$$+ \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2t(2t-s)}{4u^4}$$
$$- \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{st(t^2-4st+s^2)}{8u^4}\,, \tag{3.12}$$

$$\hat{C}_2^{-+-+}(s,t) = -(4-n_f)\frac{s}{u^2} + 2(1+n_s-n_f)\frac{s^2t}{u^4}$$
$$- \big(2(1-2n_s)+n_f\big)(4-n_f)\frac{s^2(2t-s)}{u^4}$$
$$+ \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{s(t^2-4st+s^2)}{2u^4}\,, \tag{3.13}$$

$$\hat{C}_3^{-+-+}(s,t) = +\big(2(1-2n_s)+n_f\big)(4-n_f)\frac{3s^2(2t-s)}{2u^4}$$
$$- \big(n_f(3-n_s)^2 - 2(4-n_f)^2\big)\frac{3s(t^2-4st+s^2)}{4u^4}\,. \tag{3.14}$$

It is easy to check that these coefficients correctly reproduce the known result in $\mathcal{N}=4$ super-symmetric Yang-Mills [30]. An expression valid for $\mathcal{N}>0$ super-symmetric generators can be found by setting $n_f = \mathcal{N}$ and $n_s = \mathcal{N}-1$. We also note that complicated flavour structures that only appear in the $-+-+$ coefficients vanish in the $\mathcal{N}=2$ theory,

$$n_f(3-n_s)^2 - 2(4-n_f)^2 \to (\mathcal{N}-2)(\mathcal{N}-4)^2, \tag{3.15}$$

$$2(1-2n_s)+n_f \to -3(\mathcal{N}-2)\,. \tag{3.16}$$

This shows that the integral basis for $\mathcal{N}=2$ is simpler than that of $\mathcal{N}=1$, a feature which is new for four-dimensional maximal cuts at three-loops. At one-loop both $\mathcal{N}=1$ and $\mathcal{N}=2$ have the same integral basis with no rational terms. At two-loops the two theories could differ in the lower propagator integrals but not in the maximal cut terms. We have also checked that by taking two-particle cuts the full integrand factorizes onto the two-loop results for the full integrand [11].

## 4  Conclusions

In this paper we have shown that an integrand reduction technique based on computational algebraic geometry techniques can be generalized to three-loop amplitudes. As a first step

in this direction we considered maximal cuts in four dimensions for the planar triple box topology contributing to massless $2 \to 2$ scattering.

By using primary decomposition the 14 independent branches of the 10 on-shell conditions could be found automatically. An explicit parametrisation of these solutions was found such that the integrand would take the form of a simple Laurent expansion in two free variables. The integrand on each solution factorises into a product of ten tree amplitudes which can be used to extract a set of 622 terms in the Laurent expansion.

Having derived a minimal parametrisation of the integrand in terms of 7 irreducible scalar products we were able to construct an invertible matrix to map the coefficients of the Laurent expansion to the coefficients of the ISPs. Though the matrix was quite large it was possible to invert the system efficiently by using a branch-by-branch reconstruction using Gröbner bases and the intersection of ideals.

Finally we were able to reduce the whole expression to master integrals using the `Reduze2` package for integration by parts identities. This whole procedure derives a complete reduction for an arbitrary process valid in any renormalizable gauge theory. As an application of the technique we computed the MI coefficients for gluon-gluon scattering in Yang-Mills theory with adjoint scalars and fermions.

Though a long way from a complete four-point amplitude in QCD the computation presented here can be seen as a small step in the right direction. We hope the techniques presented here will be useful when making the necessary generalizations to $D$-dimensional cuts and fewer particle cuts.

## Acknowledgments

## A  Branch-by-branch polynomial fitting method

Consider generalized unitarity for an $L$-loop diagram. Let $R$ be the polynomial ring of ISPs and $I$ be the ideal generated by cut equations.

By the integrand-level reduction algorithm, the original integrand numerator is a polynomial $\tilde{N} \in R$, while $[\tilde{N}]$ is its image in the quotient ring $R/I$. $[\tilde{N}]$ can be expanded over the *integrand basis*. The reduced integrand $N$, which is the simplest representative for $[N]$, ($[N] = [\tilde{N}]$ in $R/I$), can be obtained by dividing $\tilde{N}$ towards the Gröbner basis $G(I)$ of $I$ [15, 16] .

Often the cut solution have several branches. In other words, by *primary decomposition*, $I$ is decomposed to the intersection of several primary ideals [15],

$$I = \bigcap_{i=1}^{n} I_i \tag{A.1}$$

where $n$ is the number of branches.

Note that $I \subset I_i$, so $I_i$ contains more constraints than $I$ itself. Hence the integrand can be reduced further on each branch. We have the map,

$$p : R/I \to R/I_1 \oplus \ldots \oplus R/I_n,$$
$$[N] \mapsto ([N]_1, \ldots, [N]_n),$$

(A.2)

where $[\ldots]$ and $[\ldots]_i$ stand for the equivalence relations in $R/I$ and $R/I_i$. If $N$ is known, it is straightforward to get the simplest representative $N_i$ for $[N]_i$, by calculating the Gröbner basis for each $I_i$. In general, $N_i$ contains much fewer terms than $N$.

So it is much easier to fit each $N_i$ than $N$, from the product of tree amplitudes. After all $N_i$ are fitted, the goal is to determine $N$ from $N_i$'s. The existence of $N$ is guaranteed by the existence of the original numerator $\tilde{N}$. We just need the following uniqueness condition:

**Lemma 1.** *The map $p$ is injective. In other words, if $N$ exists, then $N_1, \ldots, N_n$ uniquely determine $[N]$.*

*Proof.* Assume that there are two polynomials $N$ and $N'$ such that $p(N) = p(N')$. So $[N]_i = [N']_i = [N_i]_i$, $\forall i$. This means that $N - N' \in I_i$, $\forall i$ and

$$N - N' \in \bigcap_{i=1}^n I_i = I,$$

(A.3)

so $[N] = [N']$ in $R/I$. $\qquad \square$

This is an analogy of *Chinese remainder theorem*, however we do not need the coprime condition since the existence of $N$ is guaranteed by physics. After $[N]$ is determined, it is straightforward to find its simplest representative $N$ by Gröbner basis $G(I)$ of $I$.

In practice, we present the following algorithm to get $N$ from $N_1, \ldots, N_n$. First, we consider the case $n = 2$,

1. For two ideals $I_1 = \langle f_1, \ldots, f_{m_1} \rangle$ and $I_2 = \langle h_1 \ldots h_{m_2} \rangle$, calculate the Gröbner basis $G(I_1 + I_2) = \{g_1, \ldots g_m\}$ for the ideal $I_1 + I_2$. Record the transform matrix,

$$g_i = \sum_{j=1}^{m_1} a_{ij} f_j + \sum_{k=1}^{m_2} b_{ik} h_k.$$

(A.4)

2. Divide $N_1 - N_2$ towards $G(I_1 + I_2)$, where $r$ is the remainder,

$$N_1 - N_2 = \sum_{i=1}^m \psi_i g_i + r.$$

(A.5)

3. If $r = 0$, rewrite

$$N_1 - N_2 = \sum_i \sum_j \psi_i a_{ij} f_j + \sum_i \sum_k \psi_i b_{ik} h_k \equiv F_1 + F_2,$$

(A.6)

where $F_1 \in I_1$ and $F_2 \in I_2$. Then $\hat{N} = N_1 - F_1$. If $r \neq 0$, print warning message and stop.

4. Calculate the intersection $I_1 \cap I_2$ and its Gröbner basis $G(I_1 \cap I_2)$. Divide $\hat{N}$ towards $G(I_1 \cap I_2)$ and the remainder is $N$.

The validity of this algorithm is self-evident since $\hat{N} = N_1 - F_1 = N_2 + F_2$ so $[\hat{N}]_1 = [N_1]_1$ and $[\hat{N}]_2 = [N_2]_2$. And as long as $N$ exists, $r$ must be zero.

For cases with more than 2 branches, we just need to repeat the above algorithm for $n - 1$ times,

1. Let $J = I_1$, $f = N_1$.

2. For $i = 1$ to $n - 1$

   - Carry out the 2-branch algorithm for polynomials $(f, N_{i+1})$ and the two ideals $(J, I_{i+1})$. Then redefine $f$ as the output polynomial.

   - $J = J \cap I_{i+1}$.

3. $N = f$.

The validity can be checked by induction. We realise this algorithm in a Macaulay2 program [28].

## B Flavour configurations in the triple box

The pre-factors of the 34 configurations appearing in (3.1) are given in table 4. We note that by combining these configurations together with different colour factors the results presented here would also be valid in QCD. The symmetry factor of $-1$ for each fermion loop is included in the pre-factor.

For completeness we also present the well-known tree level amplitudes used in this paper,

$$-iA_3^{(0)}(1_g^-, 2_g^-, 3_g^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \tag{B.1}$$

$$-iA_3^{(0)}(1_{\bar{q}}^-, 2_q^+, 3_g^-) = \frac{\langle 13 \rangle^2}{\langle 12 \rangle} \tag{B.2}$$

$$-iA_3^{(0)}(1_{\bar{s}}, 2_s, 3_g^-) = \frac{\langle 13 \rangle \langle 23 \rangle}{\langle 12 \rangle} \tag{B.3}$$

$$-iA_3^{(0)}(1_q^-, 2_q^+, 3_s) = \langle 12 \rangle \tag{B.4}$$

$$-iA_4^{(0)}(1_g^-, 2_g^-, 3_g^+, 4_g^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{B.5}$$

$$-iA_4^{(0)}(1_g^-, 2_g^+, 3_g^+, 4_g^-) = \frac{\langle 41 \rangle^3}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \tag{B.6}$$

$$-iA_4^{(0)}(1_g^-, 2_g^+, 3_g^-, 4_g^+) = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \tag{B.7}$$

– 11 –

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $F_{f_1\cdots f_5}$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $F_{f_1\cdots f_5}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $g$ | $g$ | $g$ | $g$ | $g$ | $1$ | $g$ | $g$ | $q$ | $g$ | $q$ | $-n_f$ |
| $g$ | $q$ | $g$ | $q$ | $q$ | $-n_f$ | $q$ | $g$ | $g$ | $q$ | $g$ | $-n_f$ |
| $g$ | $q$ | $q$ | $q$ | $g$ | $-n_f$ | $q$ | $q$ | $g$ | $g$ | $q$ | $-n_f$ |
| $q$ | $q$ | $q$ | $g$ | $g$ | $-n_f$ | $g$ | $g$ | $s$ | $g$ | $s$ | $n_s$ |
| $g$ | $s$ | $g$ | $s$ | $s$ | $n_s$ | $s$ | $g$ | $g$ | $s$ | $g$ | $n_s$ |
| $g$ | $s$ | $s$ | $s$ | $g$ | $n_s$ | $s$ | $s$ | $g$ | $g$ | $s$ | $n_s$ |
| $s$ | $s$ | $s$ | $g$ | $g$ | $n_s$ | $g$ | $s$ | $q$ | $s$ | $q$ | $-n_f n_s$ |
| $s$ | $s$ | $q$ | $g$ | $q$ | $-n_f n_s$ | $g$ | $q$ | $s$ | $q$ | $q$ | $-n_f n_s$ |
| $s$ | $q$ | $g$ | $q$ | $q$ | $-n_f n_s$ | $q$ | $s$ | $s$ | $q$ | $g$ | $-n_f n_s$ |
| $q$ | $s$ | $g$ | $q$ | $s$ | $-n_f n_s$ | $g$ | $q$ | $q$ | $q$ | $s$ | $-n_f n_s$ |
| $s$ | $q$ | $q$ | $q$ | $g$ | $-n_f n_s$ | $q$ | $q$ | $s$ | $g$ | $q$ | $-n_f n_s$ |
| $q$ | $q$ | $g$ | $s$ | $q$ | $-n_f n_s$ | $q$ | $q$ | $q$ | $g$ | $s$ | $-n_f n_s$ |
| $q$ | $q$ | $q$ | $s$ | $g$ | $-n_f n_s$ | $s$ | $g$ | $q$ | $s$ | $q$ | $-n_f n_s$ |
| $q$ | $g$ | $s$ | $q$ | $s$ | $-n_f n_s$ | $q$ | $g$ | $q$ | $q$ | $q$ | $n_f^2$ |
| $s$ | $g$ | $s$ | $s$ | $s$ | $n_s^2$ | $s$ | $q$ | $s$ | $q$ | $q$ | $-n_f n_s$ |
| $s$ | $q$ | $q$ | $q$ | $s$ | $-n_f n_s$ | $q$ | $q$ | $s$ | $s$ | $q$ | $-n_f n_s$ |
| $q$ | $q$ | $q$ | $s$ | $s$ | $-n_f n_s$ | $q$ | $s$ | $q$ | $q$ | $q$ | $2n_f n_s$ |

**Table 4**. The definitions of the flavour pre-factors in eq. (3.1).

# References

[1] Z. Bern, J. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The ultraviolet behavior of $N = 8$ supergravity at four loops*, *Phys. Rev. Lett.* **103** (2009) 081301 [arXiv:0905.2326] [inSPIRE].

[2] Z. Bern, J. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The complete four-loop four-point amplitude in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 82** (2010) 125040 [arXiv:1008.3327] [inSPIRE].

[3] Z. Bern, S. Davies, T. Dennen and Y.-t. Huang, *Absence of three-loop four-point divergences in $N = 4$ supergravity*, *Phys. Rev. Lett.* **108** (2012) 201301 [arXiv:1202.3423] [inSPIRE].

[4] J.J.M. Carrasco and H. Johansson, *Generic multiloop methods and application to $N = 4$ super-Yang-Mills*, *J. Phys.* **A 44** (2011) 454004 [arXiv:1103.3298] [inSPIRE].

[5] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226] [inSPIRE].

[6] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265] [inSPIRE].

[7] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103] [inSPIRE].

[8] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B 763** (2007) 147 [hep-ph/0609007] [inSPIRE].

[9] P. Mastrolia and G. Ossola, *On the integrand-reduction method for two-loop scattering amplitudes*, *JHEP* **11** (2011) 014 [arXiv:1107.6041] [inSPIRE].

[10] D.A. Kosower and K.J. Larsen, *Maximal unitarity at two loops*, *Phys. Rev.* **D 85** (2012) 045017 [arXiv:1108.1180] [InSPIRE].

[11] S. Badger, H. Frellesvig and Y. Zhang, *Hepta-cuts of two-loop scattering amplitudes*, *JHEP* **04** (2012) 055 [arXiv:1202.2019] [InSPIRE].

[12] K.J. Larsen, *Global poles of the two-loop six-point N = 4 SYM integrand*, arXiv:1205.0297 [InSPIRE].

[13] S. Caron-Huot and K.J. Larsen, *Uniqueness of two-loop master contours*, arXiv:1205.0801 [InSPIRE].

[14] R.H. Kleiss, I. Malamos, C.G. Papadopoulos and R. Verheyen, *Counting to one: reducibility of one- and two-loop amplitudes at the integrand level*, arXiv:1206.4180 [InSPIRE].

[15] Y. Zhang, *Integrand-level reduction of loop amplitudes by computational algebraic geometry methods*, arXiv:1205.5707 [InSPIRE].

[16] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Scattering amplitudes from multivariate polynomial division*, arXiv:1205.7087 [InSPIRE].

[17] K. Chetyrkin and F. Tkachov, *Integration by parts: the algorithm to calculate β-functions in 4 loops*, *Nucl. Phys.* **B 192** (1981) 159 [InSPIRE].

[18] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys.* **B 580** (2000) 485 [hep-ph/9912329] [InSPIRE].

[19] J. Gluza, K. Kajda and D.A. Kosower, *Towards a basis for planar two-loop integrals*, *Phys. Rev.* **D 83** (2011) 045012 [arXiv:1009.0472] [InSPIRE].

[20] R.M. Schabinger, *A new algorithm for the generation of unitarity-compatible integration by parts relations*, *JHEP* **01** (2012) 077 [arXiv:1111.4220] [InSPIRE].

[21] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys.* **A 15** (2000) 5087 [hep-ph/0102033] [InSPIRE].

[22] C. Anastasiou and A. Lazopoulos, *Automatic integral reduction for higher order perturbative calculations*, *JHEP* **07** (2004) 046 [hep-ph/0404258] [InSPIRE].

[23] A. Smirnov, *Algorithm FIRE — Feynman Integral REduction*, *JHEP* **10** (2008) 107 [arXiv:0807.3243] [InSPIRE].

[24] C. Studerus, *Reduze-Feynman integral reduction in C++*, *Comput. Phys. Commun.* **181** (2010) 1293 [arXiv:0912.2546] [InSPIRE].

[25] A. von Manteuffel and C. Studerus, *Reduze 2 — distributed Feynman integral reduction*, arXiv:1201.4330 [InSPIRE].

[26] C. Berger et al., *An automated implementation of on-shell methods for one-loop amplitudes*, *Phys. Rev.* **D 78** (2008) 036003 [arXiv:0803.4180] [InSPIRE].

[27] P. Mastrolia, G. Ossola, C. Papadopoulos and R. Pittau, *Optimizing the reduction of one-loop amplitudes*, *JHEP* **06** (2008) 030 [arXiv:0803.3964] [InSPIRE].

[28] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.

[29] V.A. Smirnov, *Analytical result for dimensionally regularized massless on shell planar triple box*, *Phys. Lett.* **B 567** (2003) 193 [hep-ph/0305142] [InSPIRE].

[30] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205] [InSPIRE].