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Three-loop matching coefficients for hot QCD: reduction and gauge independence

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Abstract: We perform an integral reduction for the 3-loop effective gauge coupling and screening mass of QCD at high temperatures, defined as matching coefficients appearing in the dimensionally reduced effective field theory (EQCD). Expressing both parameters in terms of a set master (sum-) integrals, we show explicit gauge parameter independence. The lack of suitable methods for solving the comparatively large number of master integrals forbids the complete evaluation at the moment. Taking one generic class of masters as an example, we highlight the calculational techniques involved. The full result would allow to improve on one of the classic probes for the convergence of the weak-coupling expansion at high temperatures, namely the comparison of full and effective theory determinations of the spatial string tension. Furthermore, the full result would also allow to determine one new contribution of order $\mathcal{O}(g^7)$ to the pressure of hot QCD.

Keywords: Thermal Field Theory, Strong Coupling Expansion, QCD

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1 Introduction

Thermal QCD at high temperatures (T) exhibits three different momentum scales. It has been known [1, 2] for a long time that the "soft" static color-electric modes $p \sim gT$, where g is the gauge coupling, are responsible for the slow convergence whereas the "ultra-soft" static color-magnetic modes $p \sim g^2T$ cause the well-known perturbative breakdown [3]. However, perturbation theory restricted to the "hard" scale $p \sim 2\pi T$ can be treated with conventional weak-coupling methods, while the soft and ultra-soft scales are only accessible through improved analytic methods or non-perturbatively via lattice simulations, as is especially the case for the ultra-soft g^2T scale. Here p denotes the characteristic momentum scale, g the gauge coupling and T the temperature. The infrared problems which cause the breakdown of perturbation theory can be isolated into a three-dimensional (3D) effective field theory called magnetostatic QCD (MQCD) and studied non-perturbatively with lattice simulations. Before computing various quantities in this framework a number of perturbative "matching" computations are necessary [4, 5], in order to relate the parameters of the effective theory with those of thermal QCD.

The plan of this paper is the following. In section 2 we review the most important facts of the dimensionally reduced effective field theory framework and show how to systematically determine the effective gauge coupling $g_{\rm E}$ and screening mass $m_{\rm E}$. In section 3 we explain some technical details about the integral reduction step, while in section 4 we discuss the structure of the explicit result for the one-, two-, and three-loop corrections, whose rather lengthy coefficients are detailed in the appendix. Section 5 contains the evaluation of a new class of master sum-integrals that appear in our result. We finally discuss possible applications of our results in section 6, before we conclude in section 7.

2 Effective gauge coupling and screening mass

We consider QCD at finite temperature with the gauge group SU($N_{\rm c}$) and $N_{\rm f}$ massless flavors of quarks. Before gauge fixing, the bare Euclidean Lagrangian in dimensional regularization reads

$$S_{QCD} = \int_0^{1/T} {\rm d}\tau \int {\rm d}^d x\, \mathcal{L}_{QCD}\,, \tag{2.1}$$

$$\mathcal{L}_{QCD} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar\psi \gamma_\mu D_\mu \psi, \tag{2.2}$$

where T is the temperature; $d = 3-2\epsilon$ denotes the number of spatial dimensions, such that Greek indices run as $\mu,\nu = 0,\dots,d$; $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ and $D_\mu = \mathbb{1}\partial_\mu - igA^a_\mu T^a$, where the T^a are hermitian generators of SU($N_{\rm c}$) with normalization ${\rm Tr}[T^a T^b] = \delta^{ab}/2$; we use hermitian Dirac matrices $\gamma^\dagger_\mu = \gamma_\mu$, $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$; g is the bare gauge coupling; and ψ carries Dirac, color, and flavor indices. For the group theory factors, we use the standard symbols $C_{\rm A} = N_{\rm c}$, $C_{\rm F} = (N_{\rm c}^2-1)/(2N_{\rm c})$.

At sufficiently high temperatures, the long-distance physics of eq. (2.2) can be described by a simpler, dimensionally reduced effective field theory [1, 2, 4, 5]:

$$S_{\rm EQCD} = \int {\rm d}^d x\, \mathcal{L}_{\rm EQCD}\,, \tag{2.3}$$

$$\mathcal{L}_{\rm EQCD} = \frac{1}{4} F^a_{ij} F^a_{ij} + {\rm Tr}[D_i, B_0]^2 + m_{\rm E}^2 {\rm Tr}[B_0^2] + \lambda_{\rm E}^{(1)} {\rm Tr}[B_0^2]^2 + \lambda_{\rm E}^{(2)} {\rm Tr}[B_0^4] + \dots\,, \tag{2.4}$$

where $i = 1,\dots,d$, $F^a_{ij} = \partial_i B^a_j - \partial_i B^a_j + g_{\rm E} f^{abc} B^b_i B^c_j$ and $D_i = \partial_i - ig_{\rm E} B_i$. The electrostatic gauge fields B^a_0 and magnetostatic gauge fields B^a_i appearing in the theory above can be related (up to normalization) to the zero modes of A^a_μ of thermal QCD in eq. (2.2).

The effective parameters in eq. (2.4), which we are ultimately interested in, can be obtained by matching. This means, we require the same result on the QCD and EQCD side within the domain of validity. A convenient way to perform the matching computation is to use a strict perturbation expansion in g^2. On both sides, the expansion is afflicted with infrared divergences. These divergences are screened by plasma effects and can be taken into account (at least for electrostatic gluons) by resumming an infinite set of diagrams. Screening of magnetostatic gluons is a completely non-perturbative effect. For the matching computation, it is not necessary to worry about the infrared divergences because the matching parameters are only sensitive to the effects of large momenta. All infrared divergences which occur can be removed by choosing a convenient infrared cutoff. It is essential to choose the same infrared cutoff in both theories.

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2.1 Relation for m_{E}^2

In order to establish a relation between the parameters of the theories eqs. (2.2),(2.4), consider the electric screening mass m_{el}, defined in the full theory[1] by the pole of the static A_0^a propagator,

$$0 = p^2 + \Pi_{00}(p^2)\big|_{p_0=0,\,\mathbf{p}^2=-m_{\mathrm{el}}^2} \,. \tag{2.5}$$

On the effective theory side, the electric screening mass is, equivalently, defined as the pole of the 3d adjoint scalar B_0 propagator,

$$0 = \mathbf{p}^2 + m_{\mathrm{E}}^2 + \Pi_{\mathrm{EQCD}}(\mathbf{p}^2)\big|_{\mathbf{p}^2=-m_{\mathrm{el}}^2} \,, \tag{2.6}$$

where Π_{EQCD} denotes the B_0 self-energy on EQCD side.

Noting that the self-energies start at one-loop order, the leading-order solutions for m_{el}^2 will be suppressed by the respective coupling parameters, such that p^2 is to be regarded perturbatively small, hence allowing for a Taylor expansion of the "on-shell" self-energies around zero. For eq. (2.5), one needs (let us write $\Pi_{\mathrm{E}} \equiv \Pi_{00}$ from now on)

$$\begin{aligned}\Pi_{\mathrm{E}}(-m_{\mathrm{el}}^2) &= \Pi_{\mathrm{E}}(0) - m_{\mathrm{el}}^2\Pi'_{\mathrm{E}}(0) + \ldots \\ &= \sum_{n=1}^{\infty} g^{2n}\,\Pi_{\mathrm{E}n}(0) - m_{\mathrm{el}}^2 \sum_{n=1}^{\infty} g^{2n}\,\Pi'_{\mathrm{E}n}(0) + \ldots \,,\end{aligned} \tag{2.7}$$

where in a second step we have introduced the n-loop self-energy coefficients $\Pi_{\mathrm{E}n}$. From eqs. (2.5) and (2.7), we can express the electric screening mass m_{el}^2 in terms of Taylor coefficients up to next-to-next to leading order (NNLO)

$$\begin{aligned} m_{\mathrm{el}}^2 = g^2\Pi_{\mathrm{E1}}(0) + g^4\left[\Pi_{\mathrm{E2}}(0) - \Pi'_{\mathrm{E1}}(0)\Pi_{\mathrm{E1}}(0)\right] + g^6\Big[\Pi_{\mathrm{E3}}(0) - \Pi'_{\mathrm{E1}}(0)\Pi_{\mathrm{E2}}(0) - \\ - \Pi'_{\mathrm{E2}}(0)\Pi_{\mathrm{E1}}(0) + \Pi''_{\mathrm{E1}}(0)\left(\Pi_{\mathrm{E1}}(0)\right)^2 + \Pi_{\mathrm{E1}}(0)\left(\Pi'_{\mathrm{E1}}(0)\right)^2\Big] + \mathcal{O}(g^8) \,. \end{aligned} \tag{2.8}$$

Diagrams contributing to the various orders of Π are depicted in figure 1.

To complete the matching computation for m_{el}^2, we have to compute Π_{EQCD} on the EQCD side in a strict perturbative expansion. Again treating the "on-shell" momentum $\mathbf{p}^2$ (as well as the tree-level mass m_{E}^2) as perturbatively small, due to the fact that the only scale in $\Pi_{\mathrm{EQCD}}(\mathbf{p}^2)$ is $\mathbf{p}^2$, after Taylor expansion the dimensionally regularized integrals (being scale-free) vanish identically.[2] From eq. (2.6) it hence follows that

$$m_{\mathrm{E}}^2 = m_{\mathrm{el}}^2 \,. \tag{2.9}$$

2.2 Relation for g_{E}^2

In order to relate the effective 3d gauge coupling g_{E}^2 to the parameters of the full theory, we can choose whether to go through a 3-point or a 4-point function, in addition to a 2-point

[1] In the presence of an infrared cut-off; otherwise, a non-perturbative definition is needed.

[2] Note that this is not the case for the coefficients of eq. (2.7), since those are vacuum sum-integrals in the full theory and hence know about the temperature scale T.

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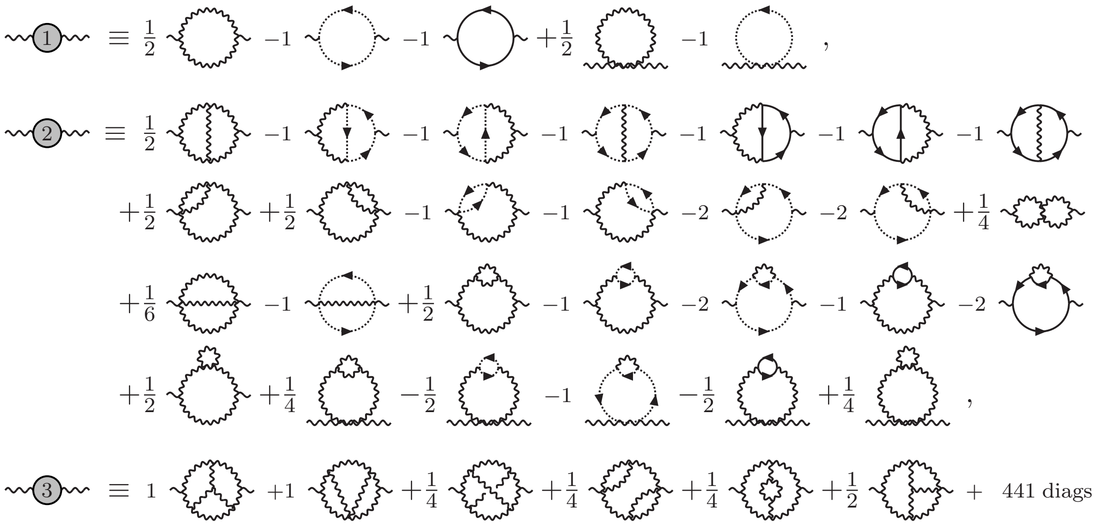

Figure 1. The 1-loop, 2-loop and some 3-loop self-energy diagrams in the background field gauge. Wavy lines represent gauge fields, dotted lines ghosts, and solid lines fermions.

function. However, it is further possible to simplify this task to a single 2-point calculation using the background field gauge method (see e.g. ref. [6]). Let us give the main argument here, closely following ref. [7].

The effective Lagrangian eq. (2.4) follows from integrating out the hard ($p \sim T$) scales which, symbolically, produces an expression of the form

$$\mathcal{L}_{\rm eff} \sim c_2(\partial B)^2 + c_3 g(\partial B)B^2 + c_4 g^2 B^4 + \dots\;, \tag{2.10}$$

where B denotes the background field potential and the coefficients $c_i = 1+\mathcal{O}\big(g^2\big)$. Redefining now the effective field as $B_{\rm eff}^2 \equiv c_2 B^2$, from $\mathcal{L}_{\rm eff} \sim (\partial B_{\rm eff})^2 + c_3 c_2^{-3/2} g(\partial B_{\rm eff})B_{\rm eff}^2 + c_4 c_2^{-2} g^2 B_{\rm eff}^4 + \dots$ we can read off the effective gauge coupling (considering the gauge invariant structure F^2) $g_{\rm eff} = c_3 c_2^{-3/2} g = c_4^{1/2} c_2^{-1} g$. Furthermore, since the effective action is gauge invariant with respect to both $B_{\rm eff}$ as well as B [6], we have $c_2 = c_3 = c_4$. Finally transforming to 3d notation, scaling the fields $B \to T^{1/2} B^2$ and comparing $\int_0^{1/T} {\rm d}\tau\, \mathcal{L}_{\rm QCD}$ with $\mathcal{L}_{\rm EQCD}$, it follows that

$$g_{\rm E} = T^{1/2}\, c_2^{-1/2}\, g\;. \tag{2.11}$$

Now we proceed in the same way with the effective gauge coupling $g_{\rm E}$ as for the screening mass $m_{\rm E}$. From eq. (2.11) we thus obtain

$$g_{\rm E}^2 = T\Big\{g^2 - g^4\Pi_{\rm T1}(0) + g^6\Big[\big(\Pi'_{\rm T1}(0)\big)^2 - \Pi'_{\rm T2}(0)\Big] + \\ + g^8\Big[2\,\Pi'_{\rm T1}(0)\Pi'_{\rm T2}(0) - \big(\Pi'_{\rm T1}(0)\big)^3 - \Pi'_{\rm T3}(0)\Big] + \mathcal{O}\big(g^{10}\big)\Big\}\;, \tag{2.12}$$

where $\Pi_{\rm T}$ denotes the transverse part of the (spatial part of the) self-energy

$$\Pi_{ij}(\mathbf{p}) \equiv \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2}\right)\Pi_{\rm T}(\mathbf{p}^2) + \frac{p_i p_j}{\mathbf{p}^2}\Pi_{\rm L}(\mathbf{p}^2)\,. \tag{2.13}$$

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To understand the split-up of $\Pi_{\mu\nu}$ in more detail, note that we can choose the external momentum p purely spatial, $p = (0, \mathbf{p})$, while the rest frame of the heat bath is time-like, with Euclidean four-velocity $u = (1, 0)$, such that $u \cdot u = 1, u \cdot p = 0$. In this case $\Pi_{\mu\nu}$ has three independent components (Π_{0i}, Π_{i0} vanish identically). The loop corrections to the spatially longitudinal part $\Pi_{\rm L}$ also vanish (which we will however explicitly check in our computations), such that only two non-trivial functions, $\Pi_{\rm E}$ and $\Pi_{\rm T}$, remain (recall $\Pi_{\rm E} = \Pi_{00}$).

Noting that the class of background field gauges still allows for a general gauge parameter ξ (we denote $(\xi)_{\rm here} = 1 - (\xi)_{\rm standard}$), we use the gauge field propagator

$$D^{ab}_{\mu\nu}(q) = \delta^{ab}\left[\frac{\delta_{\mu\nu}}{q^2} - \xi\frac{q_\mu q_\nu}{(q^2)^2}\right] \tag{2.14}$$

and verify gauge parameter cancellation in the end of our computations.

3 The reduction

After the Taylor expansion and decoupling of scalar products with external momentum, all integrals that contribute to the self-energies up to three-loop order that are needed for eqs. (2.8) and (2.12) can be written as

$$I^{\alpha,\beta,\gamma}_{a,b,c,d,e,f;\,c_1,c_2,c_3} \equiv \SumInt_{P_1P_2P_3} \frac{(P_1)_0^\alpha\,(P_2)_0^\beta\,(P_3)_0^\gamma}{[P_1^2]^a\,[P_2^2]^b\,[P_3^2]^c\,[(P_1-P_2)^2]^d\,[(P_1-P_3)^2]^e\,[(P_2-P_3)^2]^f}\,, \tag{3.1}$$

where $P_i^2 = (P_i)_0^2 + \mathbf{p}_i^2 = [(2n_i + c_i)\pi T]^2 + \mathbf{p}_i^2$ for $i \in \{1, 2, 3\}$ are bosonic (fermionic) loop momenta for $c_i = 0$ (1). The sum-integral symbol in eq. (3.1) is a shorthand for

$$\SumInt_P \to \mu^{2\epsilon} T \sum_{P_0} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,, \tag{3.2}$$

where μ is the minimal subtraction (MS) scheme scale parameter, and we take $d = 3 - 2\epsilon$.

An essential part of this work deals with the reduction of integrals of the type in eq. (3.1) to a small set of master integrals. We use the well-known integration by parts (IBP) identities and identities following from exchanges of integration variables. Both are implemented in a Laporta algorithm [8] using `FORM` [9, 10]. Compared to the well-established Laporta-type algorithms for zero-temperature reductions, one of the main differences here is that the IBP relations act only within the continuum (spatial) part of our sum-integrals. Another important difference is that in general, linear shifts or exchanges of integration momenta can cause a flip of bosonic and fermionic signature of the loop momenta, such that extra care must be taken for topology mapping. A precursor of this reduction algorithm had already been tested in ref. [7].

The main difference between the outcome of the 1-loop and 2-loop calculation on the one side and the 3-loop correction on the other side is that the former ones are expressible in terms of 1-loop tadpole sum-integrals which are known explicitly, see appendix A. This

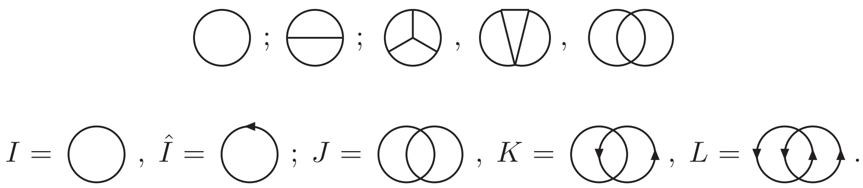

Figure 2. Top row: non-trivial vacuum topologies at 1-loop, 2-loop and 3-loop. Bottom row: types of bosonic and fermionic master integrals. Lines (arrow-lines) corresponds to bosonic (fermionic) propagators, respectively.

is no longer the case at 3-loop order. The mercedes- and spectacles topology shown on the first line of figure 2 can be expressed in terms of basketball-type sum-integrals as well as products of 1-loop tadpoles.

4 Structure of the result

After reduction, we can express all quantities as a sum of 1- and 3-loop master integrals (there are no master integrals at 2-loop order, see [5, 11]) of the generic types depicted on the second row of figure 2, the structure being

$$\Pi_3 = \sum_i a_i \, A_i + \sum_j b_j \, B_j \; , \tag{4.1}$$

$$\text{where} \;\; A_i = I \cdot I \cdot I \quad \text{with} \quad I \in \left\{ I^n_m, \hat{I}^n_m \right\} \tag{4.2}$$

$$\text{and} \;\; B_j = \text{basketball} \in \{J, K, L\} \; . \tag{4.3}$$

A detailed version is given in the appendix, cf. eqs. (C.14) and (C.15).

We have performed a number of cross-checks to confirm the validity of our results: the longitudinal parts of the self-energy vanish identically

$$\Pi_{\rm L3} = \Pi'_{\rm L3} = 0 \quad \text{for} \quad \xi^0, \ldots, \xi^6 \, , \tag{4.4}$$

and the specific combinations of (bare) self-energy coefficients that build up $m^2_{\rm E}$ (cf. eq. (2.8)) and $g^2_{\rm E}$ (cf. eq. (2.12)) are gauge-parameter independent up to three-loop order.

The one-, and two-loop calculations have already been performed in ref. [7] which we use as another serious cross-check of our independent calculation. We obtain full agreement when comparing our eqs. (C.1)–(C.4) and eqs. (C.7)–(C.10) with that reference.

There is considerable experience of how to calculate the genuine 3-loop integrals B_j up to the constant term (which can typically only be represented in terms of two-dimensional parameter integrals and evaluated numerically), see [12–14]. In section 5, we add to this available knowledge a specific class of 3-loop (basketball-type) sum-integrals which appear in our reduced expressions eqs. (C.14) and (C.15).

It turns out, however, that most of the pre-factors b_j are singular when expanded around $d = 3-2\epsilon$ dimensions. Hence, we need to expand the integrals B_j beyond their constant term (in fact, to $\mathcal{O}(\epsilon)$ for $\Pi_{\rm E}$ and to $\mathcal{O}(\epsilon^2)$ for $\Pi'_{\rm T}$). As the conventional techniques for

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computing these basketball-type integrals rely on a careful subtraction of sub-divergences on a case-by-case basis, it appears quite difficult to extend the known techniques in order to evaluate higher terms in the epsilon expansion.

To make progress, it might be advantageous to perform a change of basis, see e.g. [15], in order to avoid or at least reduce the number of divergent pre-factors. Due to the large number of integrals contained in our reduction tables, an algorithmic approach trying out all possible different combinations of basis elements might be somewhat involved, but certainly possible.

5 Evaluation of classes of master sum-integrals

After the successful reduction step, a number of non-trivial three-loop master sum-integrals will have to be evaluated. Noting that all bosonic and fermionic one-loop sum-integrals I_m^n and $\hat{I}_m^n$ that appear in eqs. (C.1)–(C.10) as well as in eqs. (C.14), (C.15) are known analytically (see appendix A), and noting that furthermore all 2-loop structures have been reduced to products of 1-loop integrals, let us tackle the first non-trivial sub-class of master integrals, the bosonic basketball

$$B_{N,M} \equiv I^{M,0,0}_{N,1,0,0,1,1;\,0,0,0} = \sum\!\!\!\!\!\!\!\int_{PQR} \frac{Q_0^M}{[Q^2]^N\,(P-Q)^2\,R^2\,(P-R)^2}\,, \tag{5.1}$$

with $N, M \geq 2$. After a careful subtraction of all UV and IR divergences (for more details see [12, 13, 16, 17]) we can write eq. (5.1) as

$$\begin{aligned} B_{N,M} &= \beta\left[A(N,\epsilon,1)\delta_{M,0} + \bar{\beta} I^M_{N-1+2\epsilon} + I^0_1\, I^M_{N+\epsilon}\right] + B^{IV}_{N,M} + \\ &\quad + 2I^0_1\, S(N,1,1;M,0) + \sum\!\!\!\!\!\!\!\int_{PQ} \frac{\Delta\Pi(P)\delta_{P_0}\delta_{Q_0}}{[Q^2]^N\,(P-Q)^2}\,\delta_{M,0} + B^{II}_N\delta_{M,0} + B^I_{N,M}\,, \end{aligned} \tag{5.2}$$

where $\beta \equiv G(1,1,d+1)$ stands for the 4d massless 1-loop bubble and $\bar{\beta} \equiv G(\frac{3-d}{2},1,d+1)$ is the 4d 1-loop propagator, where the function G reads ($s_{12} \equiv s_1 + s_2$ etc.)

$$G(s_1,s_2,d) \equiv (p^2)^{s_{12}-\frac{d}{2}} \int_q \frac{1}{[q^2]^{s_1}[(q-p)^2]^{s_2}} = \frac{\Gamma(\frac{d}{2}-s_1)\Gamma(\frac{d}{2}-s_2)\Gamma(s_{12}-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(s_1)\Gamma(s_2)\Gamma(d-s_{12})}\,, \tag{5.3}$$

and S stands for the two-loop tadpole at finite-temperature,

$$\begin{aligned} S(s_1,s_2,s_3;a_1,a_2) &\equiv \sum\!\!\!\!\!\!\!\int_{PQ} \frac{|Q_0|^{a_1}\,|P_0|^{a_2}}{[P^2]^{s_1}\,[Q^2]^{s_2}\,[(P-Q)^2]^{s_3}} \\ &= \sum_i I^0_i\, I^0_{s_{123}-a_{12}/2-i}\, e_i(s_1,s_2,s_3,a_1,a_2,d)\,, \end{aligned} \tag{5.4}$$

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where the coefficients e_i follow from IBP relations (for an example, see eq. (5.18) below). Furthermore, the abbreviation $A(s_1,s_2,s_3)$ stands for a specific 2-loop tadpole

$$A(s_1,s_2,s_3) \equiv \sum\!\!\!\!\!\!\!\int_{PQ} \frac{\delta_{Q_0}}{[Q^2]^{s_1}[P^2]^{s_2}[(P-Q)^2]^{s_3}} = \frac{2T^2\zeta(2s_{123}-2d)}{(2\pi T)^{2s_{123}-2d}}\, N(s_1,s_2,s_3)\,, \tag{5.5}$$

$$\begin{aligned} N(s_1,s_2,s_3) &\equiv \int_{pq} \frac{1}{[p^2+1]^{s_1}[q^2+1]^{s_2}[(p-q)^2]^{s_3}} \\ &= \frac{\Gamma(s_{13}-\frac{d}{2})\Gamma(s_{23}-\frac{d}{2})\Gamma(\frac{d}{2}-s_3)\Gamma(s_{123}-d)}{(4\pi)^d\Gamma(s_1)\Gamma(s_2)\Gamma(d/2)\Gamma(s_{1233}-d)}\,. \end{aligned} \tag{5.6}$$

In eq. (5.2) we make use of the one-loop subtracted quantities

$$\Delta\Pi(P) = \sum\!\!\!\!\!\!\!\int_R \frac{1}{R^2\,(R-P)^2} - \frac{\beta}{[P^2]^\epsilon} - \frac{2I_1}{P^2}\,, \tag{5.7}$$

$$\Delta\tilde\Pi(Q) = \sum\!\!\!\!\!\!\!\int_R \frac{1}{[R^2]^\epsilon\,(R-Q)^2} - \frac{\bar\beta}{[Q^2]^{2\epsilon-1}} - \frac{2I_1}{[Q^2]^\epsilon}\,, \tag{5.8}$$

as well as the three pieces

$$B^I_{N,M} = \sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q{}' \frac{\Delta\Pi(P)Q_0^M}{[Q^2]^N(P-Q)^2}\,, \quad B^{II}_N = \sum\!\!\!\!\!\!\!\int_P{}' \sum\!\!\!\!\!\!\!\int_Q \frac{\Delta\Pi(P)\delta_{Q_0}}{[Q^2]^N(P-Q)^2}\,, \tag{5.9}$$

$$B^{IV}_{N,M} = \beta \sum\!\!\!\!\!\!\!\int_Q{}' \frac{\Delta\tilde\Pi(Q)Q_0^M}{[Q^2]^N}\,, \tag{5.10}$$

where the primed sums denote $\sum'_n = \sum_{n\neq 0}$. It turns out, however, that B^{II}_N contains an additional IR divergence which can be taken into account either by means of IBP reduction [16] or by subtraction by hand [13, 17], adding the appropriate zeros (massless tadpoles which vanish in dimensional regularization). Performing a transformation to coordinate space in $d=3$ dimensions and evaluating the remaining sums give the 1d integral representations

$$B^n_{N,M}\big|_{\epsilon=0} = \frac{T^{6-2N}\,2^{N-1}}{\Gamma(N)\,(4\pi)^{2N}}\,(2\pi T)^M \int_0^\infty \mathrm{d}r\, \hat B^n_{N,M}(r)\,\Delta\pi(r)\,, \tag{5.11}$$

$$\hat B^I_{N,M}(r) = \sum_{i=0}^{N-2} c_{Ni}\, r^{N-3-i}\left\{\mathrm{Li}_{N-2+i-M}(e^{-2r}) + \coth(r)\,\mathrm{Li}_{N-1+i-M}(e^{-2r})\right\}\,, \tag{5.12}$$

$$\hat B^{II}_N(r) = -\sum_{n=0}^{N-2}\sum_{i=0}^{N-2+n} \frac{\Gamma(N)}{\Gamma(N+n)}\,\frac{a_{N,n}}{2^n}\,c_{N+n,i}\, r^{N-2+n-i}\,\mathrm{Li}_{N-2+i-n}(e^{-2r})\,, \tag{5.13}$$

$$\hat B^{IV}_{N,M}(r) = \sum_{i=0}^{N-2} c_{Ni}\, r^{N-3-i}\left\{\frac{N-1-i}{2r}\,\mathrm{Li}_{N-1+i-M}(e^{-2r}) - \frac{1}{2}\,\mathrm{Li}_{N-2+i-M}(e^{-2r})\right\}\,, \tag{5.14}$$

with $\Delta\pi(r) \equiv \coth(r) - \frac{1}{r} - \frac{r}{3}$ and where the $c_{N,i}$ are Fourier coefficients given by

$$\sqrt{\frac{2m}{\pi}}\,e^m K_{3/2-s}(m) = \sum_{n=0}^{\max(s-2,1-s)} \frac{c_{s,n}}{m^n} \tag{5.15}$$

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and $a_{N,n}$ can be obtained by IBP reduction of the inner sum-integral of eq. (5.9), see [16].

Putting all ingredients together for the special case $B_{3,2}$ (which is needed for $m_{\rm E}^2$, being the coefficient of α_4 in eq. (C.14)), evaluating the finite pieces numerically,

$$\begin{aligned} B^{I}_{3,2}\big|_{\epsilon=0} &= \frac{T^2}{2(4\pi)^4}\int_0^\infty {\rm d}r\,\Delta\pi(r)\left\{{\rm Li}_{-1}(e^{-2r})+\left(\coth(r)+\frac{1}{r}\right){\rm Li}_0(e^{-2r})+\frac{\coth(r)}{r}\,{\rm Li}_1(e^{-2r})\right\}\\ &\approx -\frac{T^2}{2\,(4\pi)^4}\times 0.02977967811050796716 8(1)\,, \end{aligned} \tag{5.16}$$

$$\begin{aligned} B^{IV}_{3,2}\big|_{\epsilon=0} &= \frac{T^2}{2\,(4\pi)^4}\int_0^\infty {\rm d}r\,\Delta\pi(r)\left\{-\frac{1}{2}\,{\rm Li}_{-1}(e^{-2r})+\frac{1}{2r}\,{\rm Li}_0(e^{-2r})+\frac{1}{2r^2}\,{\rm Li}_1(e^{-2r})\right\}\\ &\approx -\frac{T^2}{2\,(4\pi)^4}\times 0.0020065925001817061293(1)\,, \end{aligned} \tag{5.17}$$

and using (from IBP, see eq. (5.4))

$$e_i(3,1,1,2,0,d) = \frac{(d-4)^2}{(d-2)(d-5)(d-7)}\,\delta_{i,2}\,, \tag{5.18}$$

we obtain as final result for this new master integral (with $Z_1' \equiv \zeta'(-1)/\zeta(-1)$)

$$B_{3,2} = \frac{T^2\,(4\pi T^2)^{-3\epsilon}}{32\,(4\pi)^4\,\epsilon^2}\left[1+\left(\frac{41}{6}+\gamma_{\rm E}+2Z_1'\right)\epsilon+70.32026114816592109(1)\,\epsilon^2+\mathcal{O}(\epsilon^3)\right]. \tag{5.19}$$

For an important cross-check of this result, see appendix B.

6 Applications

To emphasize the necessity to pursue the matching computations as outlined in this note, let us briefly discuss two applications that would become relevant once full results are available.

The first immediate application involves the Debye screening mass $m_{\rm E}^2$ of section 2.1 and concerns higher-order perturbative contributions to basic thermodynamic observables, such as the pressure of hot QCD. In fact, once the quantity $\Pi_{\rm E3}(0)$ of eq. (C.14) has been fully determined, the mass term of EQCD (cf. eq. (2.4)) is available at NNLO, $m_{\rm E}^2 \sim g^2T^2[1+g^2+g^4+\mathcal{O}(g^6)\,]$, where g is the dimensionless gauge coupling of full QCD. Now, in the context of the effective theory setup for hot QCD, it turns out that the lowest-order EQCD contribution to the full pressure, coming from the quadratic part of $\mathcal{L}_{\rm EQCD}$, enters as $\sim Tm_{\rm E}^3$ [5], which translates to $T^4g^3[1+g^2+g^4+\mathcal{O}(g^6)\,]$, such that our 3-loop coefficient contributes to $\mathcal{O}(g^7)$ in the QCD pressure. According to the systematics of effective theory, due to the fact that there are typically large logarithms, a systematic g^6 evaluation of the pressure (almost completely known at present, only missing a well-defined perturbative 4-loop computation [5, 18, 19]) has actually been coined *physical leading order*, since it is the first order where all three physical scales (hard/soft/ultra-soft) have contributed. In this respect, the $\mathcal{O}(g^7)$ term would simply be next-to-leading order, and allow for a first serious investigation of convergence properties.

Leaving the incomplete $\mathcal{O}(g^6)$ (for which there exist numerical estimates, however, from comparisons with lattice data, see e.g. [20, 21]) aside for the moment, there are other sources of $\mathcal{O}(g^7)$ contributions, of course: from the the MQCD pressure plus NLO matching of the 3d MQCD gauge coupling $g_{\rm M}^2$; from the terms proportional to the quartic coupling $\lambda_{\rm E}$ in the 3-loop EQCD pressure; from the 5-loop EQCD pressure (at $\lambda_{\rm E} = 0$), which entails one of the conceptually simplest (3d, super-renormalizable, massive, vacuum-diagram) computations at the 5-loop level, for which techniques are presently developed by several groups; and from the leading terms of some higher-order operators in the EQCD Lagrangian, denoted by dots in eq. (2.4), but classified in [22]. All but the last two of these additional g^7 contributions are already known.

A second immediate application involves the 3d EQCD gauge coupling $g_{\rm E}^2$ of section 2.2 and concerns precision-tests of the dimensional reduction setup, such as for the spatial string tension σ_s, which parameterizes the large-area behavior of rectangular spatial Wilson loops. As has been demonstrated in ref. [7], it can be systematically determined, as a function of the temperature T, in the dimensionally reduced effective theory setup, and then compared to non-perturbative 4d lattice measurements. It turned out that the NLO result for $g_{\rm E}^2$ as obtained in [7] represents a considerable improvement over a 1-loop comparison — giving a sizable correction factor as well as a first estimate of (renormalization) scale dependence — while leaving room for NNLO effects, for which our 3-loop result for $\Pi'_{\rm T3}(0)$ of eq. (C.15) is the last missing building block.

7 Conclusions

We have successfully reduced the NNLO contributions to the matching parameters $m_{\rm E}^2$ and $g_{\rm E}^2$ to a sum of scalar sum-integrals. These matching parameters play an important role in higher-order evaluations of basic thermodynamic observables and in precision-tests of the dimensional reduction setup respectively, and hence are needed with high accuracy. Our result passes the non-trivial checks of transversality as well as gauge-parameter independence.

In a next step, a number of master integrals have to be evaluated. Although we managed to map all of them to the relatively simple class of basketball-type ones, the somewhat large number of masters that we need demand a semi-automated evaluation strategy, which still has to be developed. As a first and encouraging step towards this goal, we have demonstrated a systematic method to evaluate a certain class of such basketball-type sum-integrals.

Once full results for the matching coefficients discussed here become available, there are immediate applications to quantities of phenomenological interest, such as the pressure of hot QCD, or the spatial string tension, as discussed in section 6 above. However, these concrete applications will have to await progress in the art of sum-integration for now.

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A One- and two-loop vacuum sum-integrals

The one-loop bosonic tadpole is known analytically and reads

$$I_m^n \equiv \sum\!\!\!\!\!\!\int_P \frac{P_0^n}{(P^2)^m} = \frac{2\pi^{3/2}T^4}{(2\pi T)^{2m-n}} \left(\frac{\mu^2}{\pi T^2}\right)^{\epsilon} \frac{\Gamma\left(m-\frac{3}{2}+\epsilon\right)}{\Gamma(m)} \zeta(2m-n-3+2\epsilon)\,, \tag{A.1}$$

whereas the fermionic tadpole can be related to the corresponding bosonic one via

$$\hat{I}_m^n \equiv \sum\!\!\!\!\!\!\int_{\{P\}} \frac{P_0^n}{(P^2)^m} = (2^{2m-n-3+2\epsilon}-1) I_m^n\,. \tag{A.2}$$

As mentioned above, via integration-by-parts relations all two-loop integrals are expressible in terms of products of two one-loop tadpoles which means they are also available analytically up to arbitrary order in ϵ.

B Check of new sum-integrals

We can cross-check our new result given in section 5 using IBP reduction of the V-type topology which gives

$$V \equiv I^{0,0,0}_{1,1,1,1,1,0;\,0,0,0} = \frac{4}{3(d-3)^2}\left\{4B_{3,2} + \frac{3d^2-24d+47}{2(d-4)} B_{2,0}\right\}\,, \tag{B.1}$$

where V stands for the spectacles-type diagram given in [14]:

$$\begin{aligned} V &\equiv \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2\,Q^2\,(P-Q)^2\,R^2\,(P-R)^2} \\ &= -\frac{T^2\,(4\pi T^2 e^{\gamma_{\rm E}})^{-3\epsilon}}{4\,(4\pi)^4\,\epsilon^2}\left\{1 + AK_1\epsilon + AK_2\epsilon^2 + \mathcal{O}(\epsilon^3)\right\}\,, \end{aligned} \tag{B.2}$$

with $AK_1 = \frac{4}{3} + 4\gamma_{\rm E} + 2Z_1'$, while AK_2 is known only numerically. Writing the coefficients of our basketball-results, given in eq. (5.19) above as well as eq. (26) of [16], as

$$B_{3,2} = \frac{T^2\,(4\pi T^2)^{-3\epsilon}}{32(4\pi)^4\epsilon^2}\left[b_{320} + b_{321}\epsilon + b_{322}\epsilon^2 + \mathcal{O}(\epsilon^3)\right]\,, \tag{B.3}$$

$$B_{2,0} = \frac{T^2\,(4\pi T^2)^{-3\epsilon}}{8(4\pi)^4\epsilon^2}\left[1 + b_{21}\epsilon + b_{22}\epsilon^2 + \mathcal{O}(\epsilon^3)\right]\,, \tag{B.4}$$

to match the leading term of V it follows that the linear relations

$$b_{320} = 1\,, \quad b_{321} = b_{21} + 4\,, \quad b_{322} = b_{22} + 4b_{21} - 8 \tag{B.5}$$

have to be satisfied. Our results presented above do indeed confirm these relations, which we take as a nice check of our generic parameterizations. Eq. (B.5) provides a welcome check of our numerical constants.

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C Expansion coefficients up to three loops

For convenience, we here repeat the one- and two-loop coefficients that were already computed in [7], adding the second derivatives that are needed for eq. (2.8). The one-loop coefficients up to second derivative read

$$\Pi_{\rm T1}(0) = 0\,, \tag{C.1}$$

$$\Pi_{\rm E1}(0) = (d-1)\Big[C_{\rm A}(d-1)I_1^0 - 2N_{\rm f}\hat{I}_1^0\Big]\,, \tag{C.2}$$

$$\Pi'_{\rm T1}(0) = \frac{2N_{\rm f}}{3}\hat{I}_2^0 + \frac{C_{\rm A}}{6}(d-25)I_2^0\,, \tag{C.3}$$

$$\Pi'_{\rm E1}(0) = \frac{N_{\rm f}}{3}(d-1)\hat{I}_2^0 - C_{\rm A}\left[\frac{28-5d+d^2}{6} + (d-3)\xi\right] I_2^0\,, \tag{C.4}$$

$$\Pi''_{\rm T1}(0) = \frac{C_{\rm A}}{3}\left[\frac{41}{10} - \frac{1}{10}d + 2\,\xi - \frac{1}{4}\xi^2\right] I_3^0 - \frac{4N_{\rm f}}{15}\hat{I}_3^0\,, \tag{C.5}$$

$$\Pi''_{\rm E1}(0) = \frac{C_{\rm A}}{3}\left[\frac{23}{5} - \frac{7}{10}d + \frac{1}{10}d^2 + \xi\,(d-3) + \frac{\xi^2}{4}\,(d-6)\right] I_3^0 + \frac{N_{\rm f}}{15}(1-d)\hat{I}_3^0\,. \tag{C.6}$$

The two-loop coefficients up to first derivative are given by (see also [7])

$$\Pi_{\rm T2}(0) = 0\,, \tag{C.7}$$

$$\Pi_{\rm E2}(0) = (d-1)(d-3)\Big\{(1+\xi)\Big[2N_{\rm f}\hat{I}_1^0 - (d-1)C_{\rm A}\,I_1^0\Big]C_{\rm A}I_2^0 + 2\,N_{\rm f}C_{\rm F}\Big[I_1^0 - \hat{I}_1^0\Big]\hat{I}_2^0\Big\}\,, \tag{C.8}$$

$$\begin{aligned}
\Pi'_{\rm T2}(0) = {} & \frac{(d-3)(d-4)}{(d-7)(d-5)(d-2)d}\Big\{(-14-42d+8d^2)\,C_{\rm A}^2\,I_2^0\,I_2^0 \\
& - 4\Big[4C_{\rm F} + (1-6d+d^2)C_{\rm A}\Big]N_{\rm f}I_2^0\,\hat{I}_2^0 - \Big[\left(\frac{d^3}{2} - 6d^2 + \frac{39}{2}d - 6\right)C_{\rm A} \\
& - (-14+41d-12d^2+d^3)C_{\rm F}\Big]N_{\rm f}\hat{I}_2^0\,\hat{I}_2^0\Big\} \\
& + \frac{(d-1)}{3d(d-7)}\Big\{(144-31d+d^2)\Big[(1-d)C_{\rm A}I_1^0 + 2N_{\rm f}\hat{I}_1^0\Big]C_{\rm A}\,I_3^0 \\
& - 4(d-6)(d-1)C_{\rm F}N_{\rm f}\Big[I_1^0 - \hat{I}_1^0\Big]\hat{I}_3^0\Big\}\,,
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
\Pi'_{\rm E2}(0) = {} & \frac{(d-3)}{2(d-7)(d-5)(d-2)d}\Big\{\left(56+315d-231d^2+57d^3-5d^4\right)C_{\rm A}^2\,I_2^0\,I_2^0 \\
& + 2(d-4)(d-1)\Big[\left(2-5d+d^2\right)C_{\rm A} + 8C_{\rm F}\Big]N_{\rm f}\,I_2^0\,\hat{I}_2^0 \\
& + (d-1)\Big[\left(24-7d^2+d^3\right)C_{\rm A} - 2\left(28+2d-7d^2+d^3\right)C_{\rm F}\Big]N_{\rm f}\,\hat{I}_2^0\,\hat{I}_2^0\Big\} \\
& + \frac{(d-3)\,\xi}{24(d-2)}\Big[3\left(16-13d+3d^2\right)\xi - 4\left(44-29d+7d^2-d^3\right)\Big]C_{\rm A}^2 I_2^0\,I_2^0 \\
& - \frac{(d-3)(d-1)}{3}\,\xi\,C_{\rm A}N_{\rm f}\,I_2^0\,\hat{I}_2^0 + \frac{(d-1)}{6(d-7)d}\Big\{4\left(6+15d-10d^2+d^3\right)\times
\end{aligned}$$

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$$\times C_{\rm F} N_{\rm f}\left[\hat{I}_1^0 - I_1^0\right]\hat{I}_3^0 + \left[2\left(-72+42d-13d^2+d^3\right)+2(d-7)d^2\xi + (d-7)(d-6)d\xi^2\right]\left[(d-1)C_{\rm A}I_1^0 - 2N_{\rm f}\hat{I}_1^0\right]C_{\rm A}I_3^0\Bigg\}. \tag{C.10}$$

For presenting the outcome of the reduction procedure for the three-loop contributions, which constitutes the main result of this paper, we denote the master integrals as in figure (2), i.e. $I, \hat{I}$ for the 1-loop tadpoles of eqs. (A.1) and (A.2), and

$$J^{\alpha,\beta,\gamma}_{a,b,c,d,e,f} \equiv I^{\alpha,\beta,\gamma}_{a,b,c,d,e,f;\,0,0,0}\,, \tag{C.11}$$
$$K^{\alpha,\beta,\gamma}_{a,b,c,d,e,f} \equiv I^{\alpha,\beta,\gamma}_{a,b,c,d,e,f;\,0,0,1}\,, \tag{C.12}$$
$$L^{\alpha,\beta,\gamma}_{a,b,c,d,e,f} \equiv I^{\alpha,\beta,\gamma}_{a,b,c,d,e,f;\,1,1,0} \tag{C.13}$$

are 3-loop basketball-type integrals in a slightly more compact notation than eq. (3.1). The results needed for eqs. (2.8) and (2.12) then read

$$\begin{aligned}
\Pi_{\rm E3}(0) = C_{\rm A}^3\Big[&\alpha_1 J^{0,0,0}_{2,1,0,0,1,1} + \alpha_2 J^{0,0,2}_{2,2,0,0,1,1} + \alpha_3 J^{0,2,0}_{3,1,0,0,1,1} + \alpha_4 J^{2,0,0}_{3,1,0,0,1,1} + \alpha_5 J^{1,3,0}_{4,1,0,0,1,1}\\
&+ \alpha_6 J^{6,0,0}_{5,1,0,0,1,1} + \alpha_7 J^{6,4,0}_{5,3,0,0,1,1} + \alpha_8 J^{7,3,0}_{6,2,0,0,1,1} + \alpha_9 I_1^0 I_1^0 I_3^0 + \alpha_{10} I_1^0 I_2^0 I_2^0\Big]\\
+ C_{\rm A}^2 N_{\rm f}\Big[&\alpha_{11} K^{-0,0,0}_{1,1,0,0,2,1} + \alpha_{12} K^{-2,0,0}_{1,1,0,0,2,2} + \alpha_{13} K^{-1,1,0}_{1,1,0,0,3,1} + \alpha_{14} K^{-2,0,0}_{1,1,0,0,3,1}\\
&+ \alpha_{15} K^{-0,0,0}_{2,1,0,0,1,1} + \alpha_{16} K^{-0,2,0}_{2,1,0,0,2,1} + \alpha_{17} K^{-2,0,0}_{2,1,0,0,2,1} + \alpha_{18} K^{-3,1,0}_{2,1,0,0,3,1}\\
&+ \alpha_{19} K^{-0,0,2}_{2,2,0,0,1,1} + \alpha_{20} K^{-1,1,0}_{2,2,0,0,1,1} + \alpha_{21} K^{-2,0,0}_{2,2,0,0,1,1} + \alpha_{22} K^{-0,0,2}_{3,1,0,0,1,1}\\
&+ \alpha_{23} K^{-0,2,0}_{3,1,0,0,1,1} + \alpha_{24} K^{-1,1,0}_{3,1,0,0,1,1} + \alpha_{25} K^{-2,0,0}_{3,1,0,0,1,1} + \alpha_{26} K^{-1,3,0}_{3,1,0,0,2,1}\\
&+ \alpha_{27} K^{-0,4,0}_{3,2,0,0,1,1} + \alpha_{28} K^{-1,1,2}_{4,1,0,0,1,1} + \alpha_{29} K^{-1,3,0}_{4,1,0,0,1,1} + \alpha_{30} K^{-2,2,0}_{4,1,0,0,1,1}\\
&+ \alpha_{31} K^{-4,0,0}_{4,1,0,0,1,1} + \alpha_{32} K^{-6,0,0}_{4,2,0,0,1,1} + \alpha_{33} K^{-3,3,0}_{5,1,0,0,1,1} + \alpha_{34} K^{-4,0,2}_{5,1,0,0,1,1}\\
&+ \alpha_{35} K^{-5,1,0}_{5,1,0,0,1,1} + \alpha_{36} K^{-6,0,0}_{5,1,0,0,1,1} + \alpha_{37} K^{-7,1,0}_{6,1,0,0,1,1} + \alpha_{38} K^{-8,0,0}_{6,1,0,0,1,1}\\
&+ \alpha_{39} K^{-7,0,3}_{6,1,0,0,2,1} + \alpha_{40} K^{-9,0,1}_{6,1,0,0,2,1} + \alpha_{41} K^{-10,0,0}_{6,1,0,0,2,1} + \alpha_{42} K^{-7,3,0}_{6,2,0,0,1,1}\\
&+ \alpha_{43} K^{-8,2,0}_{6,2,0,0,1,1} + \alpha_{44} K^{-8,0,2}_{7,1,0,0,1,1} + \alpha_{45} K^{-8,2,0}_{7,1,0,0,1,1} + \alpha_{46} K^{-9,0,1}_{7,1,0,0,1,1}\\
&+ \alpha_{47} K^{-9,1,0}_{7,1,0,0,1,1} + \alpha_{48} K^{-10,0,0}_{7,1,0,0,1,1} + \alpha_{49} L^{0,0,0}_{2,1,0,0,1,1} + \alpha_{50} L^{2,0,0}_{3,1,0,0,1,1}\\
&+ \alpha_{51} \hat{I}_1^0 \hat{I}_1^0 I_3^0 + \alpha_{52} \hat{I}_1^0 I_2^0 \hat{I}_2^0 + \alpha_{53} \hat{I}_1^0 I_2^0 I_2^0 + \alpha_{54} \hat{I}_1^2 \hat{I}_2^0 I_3^0 + \alpha_{55} I_1^0 \hat{I}_1^0 \hat{I}_3^0\\
&+ \alpha_{56} I_1^0 \hat{I}_1^0 I_3^0 + \alpha_{57} I_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{58} I_1^0 I_1^0 \hat{I}_3^0 + \alpha_{59} I_1^0 I_2^0 \hat{I}_2^0 + \alpha_{60} I_1^2 \hat{I}_2^0 I_3^0\Big]\\
+ C_{\rm A} N_{\rm f}^2\Big[&\alpha_{61} L^{0,0,0}_{2,1,0,0,1,1} + \alpha_{62} L^{0,0,2}_{2,2,0,0,1,1} + \alpha_{63} L^{0,2,0}_{3,1,0,0,1,1} + \alpha_{64} L^{2,0,0}_{3,1,0,0,1,1}\\
&+ \alpha_{65} L^{1,3,0}_{4,1,0,0,1,1} + \alpha_{66} L^{6,0,0}_{5,1,0,0,1,1} + \alpha_{67} L^{6,4,0}_{5,3,0,0,1,1} + \alpha_{68} L^{7,3,0}_{6,2,0,0,1,1}\\
&+ \alpha_{69} \hat{I}_1^0 \hat{I}_1^0 \hat{I}_3^0 + \alpha_{70} \hat{I}_1^0 \hat{I}_1^0 I_3^0 + \alpha_{71} \hat{I}_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{72} \hat{I}_1^0 I_2^0 \hat{I}_2^0\Big]\\
+ N_{\rm f}^2 C_{\rm F}\Big[&\alpha_{73} L^{0,0,0}_{2,1,0,0,1,1} + \alpha_{74} L^{0,0,2}_{2,2,0,0,1,1} + \alpha_{75} L^{0,2,0}_{3,1,0,0,1,1} + \alpha_{76} L^{2,0,0}_{3,1,0,0,1,1}\\
&+ \alpha_{77} L^{6,0,0}_{5,1,0,0,1,1} + \alpha_{78} \hat{I}_1^0 \hat{I}_1^0 \hat{I}_3^0 + \alpha_{79} \hat{I}_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{80} \hat{I}_1^0 I_2^0 \hat{I}_2^0\Big]
\end{aligned}$$

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$$
\begin{aligned}
&+ N_{\rm f} C_{\rm F}^2 \Big[\alpha_{81} K^{0,0,0}_{1,1,0,0,2,1} + \alpha_{82} K^{2,0,0}_{1,1,0,0,2,2} + \alpha_{83} K^{1,1,0}_{1,1,0,0,3,1} + \alpha_{84} K^{0,0,0}_{2,1,0,0,1,1} \\
&\quad + \alpha_{85} K^{0,2,0}_{2,1,0,0,2,1} + \alpha_{86} K^{2,0,0}_{2,1,0,0,2,1} + \alpha_{87} K^{1,1,0}_{2,2,0,0,1,1} + \alpha_{88} K^{2,0,0}_{2,2,0,0,1,1} \\
&\quad + \alpha_{89} K^{0,2,0}_{3,1,0,0,1,1} + \alpha_{90} K^{1,1,0}_{3,1,0,0,1,1} + \alpha_{91} K^{2,0,0}_{3,1,0,0,1,1} + \alpha_{92} K^{1,3,0}_{3,1,0,0,2,1} \\
&\quad + \alpha_{93} K^{0,4,0}_{3,2,0,0,1,1} + \alpha_{94} K^{1,3,0}_{4,1,0,0,1,1} + \alpha_{95} K^{2,2,0}_{4,1,0,0,1,1} + \alpha_{96} K^{4,0,0}_{4,1,0,0,1,1} \\
&\quad + \alpha_{97} K^{6,0,0}_{4,2,0,0,1,1} + \alpha_{98} K^{3,3,0}_{5,1,0,0,1,1} + \alpha_{99} K^{5,1,0}_{5,1,0,0,1,1} + \alpha_{100} L^{0,0,0}_{2,1,0,0,1,1} \\
&\quad + \alpha_{101} L^{2,0,0}_{3,1,0,0,1,1} + \alpha_{102} \hat{I}_1^0 \hat{I}_1^0 \hat{I}_3^0 + \alpha_{103} \hat{I}_1^0 \hat{I}_1^0 I_3^0 + \alpha_{104} \hat{I}_1^0 \hat{I}_2^0 \hat{I}_2^0 \\
&\quad + \alpha_{105} \hat{I}_1^0 I_2^0 \hat{I}_2^0 + \alpha_{106} \hat{I}_1^2 \hat{I}_2^0 I_3^0 + \alpha_{107} I_1^0 \hat{I}_1^0 \hat{I}_3^0 + \alpha_{108} I_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{109} I_1^0 I_1^0 \hat{I}_3^0 \\
&\quad + \alpha_{110} I_1^0 I_2^0 \hat{I}_2^0 + \alpha_{111} I_1^2 \hat{I}_2^0 I_3^0 \Big] \\
&+ C_{\rm A} N_{\rm f} C_{\rm F} \Big[\alpha_{112} K^{0,0,0}_{1,1,0,0,2,1} + \alpha_{113} K^{2,0,0}_{1,1,0,0,2,2} + \alpha_{114} K^{0,0,0}_{2,1,0,0,1,1} + \alpha_{115} K^{0,2,0}_{2,1,0,0,2,1} \\
&\quad + \alpha_{116} K^{2,0,0}_{2,1,0,0,2,1} + \alpha_{117} K^{1,1,0}_{2,2,0,0,1,1} + \alpha_{118} K^{2,0,0}_{2,2,0,0,1,1} + \alpha_{119} K^{0,2,0}_{3,1,0,0,1,1} \\
&\quad + \alpha_{120} K^{1,1,0}_{3,1,0,0,1,1} + \alpha_{121} K^{2,0,0}_{3,1,0,0,1,1} + \alpha_{122} K^{1,3,0}_{3,1,0,0,2,1} + \alpha_{123} K^{0,4,0}_{3,2,0,0,1,1} \\
&\quad + \alpha_{124} K^{1,3,0}_{4,1,0,0,1,1} + \alpha_{125} K^{2,2,0}_{4,1,0,0,1,1} + \alpha_{126} K^{4,0,0}_{4,1,0,0,1,1} + \alpha_{127} K^{6,0,0}_{4,2,0,0,1,1} \\
&\quad + \alpha_{128} K^{3,3,0}_{5,1,0,0,1,1} + \alpha_{129} L^{0,0,0}_{2,1,0,0,1,1} + \alpha_{130} L^{2,0,0}_{3,1,0,0,1,1} + \alpha_{131} \hat{I}_1^0 \hat{I}_1^0 I_3^0 \\
&\quad + \alpha_{132} \hat{I}_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{133} \hat{I}_1^0 I_2^0 \hat{I}_2^0 + \alpha_{134} \hat{I}_1^2 \hat{I}_2^0 I_3^0 + \alpha_{135} I_1^0 \hat{I}_2^0 \hat{I}_2^0 + \alpha_{136} I_1^0 I_2^0 \hat{I}_2^0 \\
&\quad + \alpha_{137} I_1^2 \hat{I}_2^0 I_3^0 \Big] \,, \qquad \text{(C.14)}
\end{aligned}
$$

$$
\begin{aligned}
\Pi'_{\rm T3}(0) = {}& C_{\rm A}^3 \Big[\beta_1 J^{0,0,0}_{2,2,0,0,1,1} + \beta_2 J^{0,0,0}_{3,1,0,0,1,1} + \beta_3 J^{0,0,2}_{3,2,0,0,1,1} + \beta_4 J^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_5 J^{2,2,0}_{5,1,0,0,1,1} + \beta_6 J^{4,0,0}_{5,1,0,0,1,1} + \beta_7 J^{8,0,0}_{7,1,0,0,1,1} + \beta_8 J^{7,3,0}_{7,3,0,-1,1,1} \\
&\quad + \beta_9 J^{8,2,0}_{8,2,0,-1,1,1} + \beta_{10} I_1^0 I_1^0 I_4^0 + \beta_{11} I_1^0 I_2^0 I_3^0 + \beta_{12} I_2^0 I_2^0 I_2^0 \Big] \\
&+ C_{\rm A}^2 N_{\rm f} \Big[\beta_{13} K^{0,0,0}_{1,1,0,0,2,2} + \beta_{14} K^{0,0,0}_{1,1,0,0,3,1} + \beta_{15} K^{2,0,0}_{1,1,0,0,3,2} + \beta_{16} K^{1,1,0}_{1,1,0,0,4,1} \\
&\quad + \beta_{17} K^{2,0,0}_{1,1,0,0,4,1} + \beta_{18} K^{0,0,0}_{2,1,0,0,2,1} + \beta_{19} K^{0,2,0}_{2,1,0,0,3,1} + \beta_{20} K^{2,0,0}_{2,1,0,0,3,1} \\
&\quad + \beta_{21} K^{4,0,0}_{2,1,0,0,4,1} + \beta_{22} K^{0,0,0}_{2,2,0,0,1,1} + \beta_{23} K^{0,0,0}_{3,1,0,0,1,1} + \beta_{24} K^{0,2,0}_{3,1,0,0,2,1} \\
&\quad + \beta_{25} K^{1,1,0}_{3,1,0,0,2,1} + \beta_{26} K^{1,3,0}_{3,1,0,0,3,1} + \beta_{27} K^{4,0,0}_{3,1,0,0,3,1} + \beta_{28} K^{0,0,2}_{3,2,0,0,1,1} \\
&\quad + \beta_{29} K^{0,2,0}_{3,2,0,0,1,1} + \beta_{30} K^{1,1,0}_{3,2,0,0,1,1} + \beta_{31} K^{0,0,2}_{4,1,0,0,1,1} + \beta_{32} K^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_{33} K^{1,1,0}_{4,1,0,0,1,1} + \beta_{34} K^{2,0,0}_{4,1,0,0,1,1} + \beta_{35} K^{1,3,0}_{4,1,0,0,2,1} + \beta_{36} K^{0,4,0}_{4,2,0,0,1,1} \\
&\quad + \beta_{37} K^{4,0,0}_{4,2,0,0,1,1} + \beta_{38} K^{1,1,2}_{5,1,0,0,1,1} + \beta_{39} K^{1,3,0}_{5,1,0,0,1,1} + \beta_{40} K^{2,0,2}_{5,1,0,0,1,1} \\
&\quad + \beta_{41} K^{2,2,0}_{5,1,0,0,1,1} + \beta_{42} K^{3,1,0}_{5,1,0,0,1,1} + \beta_{43} K^{4,0,0}_{5,1,0,0,1,1} + \beta_{44} K^{3,3,0}_{6,1,0,0,1,1} \\
&\quad + \beta_{45} K^{4,2,0}_{6,1,0,0,1,1} + \beta_{46} K^{6,0,0}_{6,1,0,0,1,1} + \beta_{47} K^{8,0,0}_{6,2,0,0,1,1} + \beta_{48} K^{5,3,0}_{7,1,0,0,1,1} \\
&\quad + \beta_{49} K^{6,0,2}_{7,1,0,0,1,1} + \beta_{50} K^{7,1,0}_{7,1,0,0,1,1} + \beta_{51} K^{8,0,0}_{7,1,0,0,1,1} + \beta_{52} K^{8,2,0}_{7,2,0,0,1,1} \\
&\quad + \beta_{53} K^{9,1,0}_{7,2,0,0,1,1} + \beta_{54} K^{8,0,2}_{8,1,-1,0,2,1} + \beta_{55} K^{10,0,0}_{8,1,-1,0,2,1} + \beta_{56} K^{8,0,2}_{8,1,0,0,1,1} \\
&\quad + \beta_{57} K^{9,0,1}_{8,1,0,0,1,1} + \beta_{58} K^{9,1,0}_{8,1,0,0,1,1} + \beta_{59} K^{10,0,0}_{8,1,0,0,1,1} + \beta_{60} K^{8,1,1}_{8,2,-1,0,1,1} \\
&\quad + \beta_{61} K^{9,1,0}_{8,2,-1,0,1,1} + \beta_{62} L^{0,0,0}_{2,2,0,0,1,1} + \beta_{63} L^{0,0,0}_{3,1,0,0,1,1} + \beta_{64} L^{4,0,0}_{5,1,0,0,1,1} \\
&\quad + \beta_{65} \hat{I}_1^0 \hat{I}_1^0 I_4^0 + \beta_{66} \hat{I}_1^0 \hat{I}_2^0 I_3^0 + \beta_{67} \hat{I}_1^0 I_2^0 \hat{I}_3^0 + \beta_{68} \hat{I}_1^0 I_2^0 I_3^0 + \beta_{69} \hat{I}_1^2 \hat{I}_2^0 I_4^0
\end{aligned}
$$

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$$
\begin{aligned}
&\quad + \beta_{70}\hat{I}_1^2 I_2^0 \hat{I}_4^0 + \beta_{71}\hat{I}_1^2 I_3^0 \hat{I}_3^0 + \beta_{72}\hat{I}_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{73} I_1^0 \hat{I}_1^0 \hat{I}_4^0 + \beta_{74} I_1^0 \hat{I}_1^0 I_4^0 \\
&\quad + \beta_{75} I_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{76} I_1^0 \hat{I}_2^0 I_3^0 + \beta_{77} I_1^0 I_1^0 \hat{I}_4^0 + \beta_{78} I_1^0 I_2^0 \hat{I}_3^0 + \beta_{79} I_1^2 \hat{I}_2^0 I_4^0 \\
&\quad + \beta_{80} I_1^2 I_2^0 \hat{I}_4^0 + \beta_{81} I_1^2 I_3^0 \hat{I}_3^0 + \beta_{82} I_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{83} I_2^0 I_2^0 \hat{I}_2^0 \Big] \\
&+ C_{\mathrm{A}} N_{\mathrm{f}}^2 \Big[\beta_{84} L^{0,0,0}_{2,2,0,0,1,1} + \beta_{85} L^{0,0,0}_{3,1,0,0,1,1} + \beta_{86} L^{0,0,2}_{3,2,0,0,1,1} + \beta_{87} L^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_{88} L^{2,2,0}_{5,1,0,0,1,1} + \beta_{89} L^{4,0,0}_{5,1,0,0,1,1} + \beta_{90} L^{8,0,0}_{7,1,0,0,1,1} + \beta_{91} L^{7,3,0}_{7,3,0,-1,1,1} \\
&\quad + \beta_{92} L^{8,2,0}_{8,2,0,-1,1,1} + \beta_{93}\hat{I}_1^0 \hat{I}_1^0 \hat{I}_4^0 + \beta_{94}\hat{I}_1^0 \hat{I}_1^0 I_4^0 + \beta_{95}\hat{I}_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{96}\hat{I}_1^0 \hat{I}_2^0 I_3^0 \\
&\quad + \beta_{97}\hat{I}_1^0 I_2^0 \hat{I}_3^0 + \beta_{98}\hat{I}_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{99} I_2^0 \hat{I}_2^0 \hat{I}_2^0 \Big] \\
&+ N_{\mathrm{f}}^2 C_{\mathrm{F}} \Big[\beta_{100} L^{0,0,0}_{2,2,0,0,1,1} + \beta_{101} L^{0,0,0}_{3,1,0,0,1,1} + \beta_{102} L^{0,0,2}_{3,2,0,0,1,1} + \beta_{103} L^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_{104} L^{2,2,0}_{5,1,0,0,1,1} + \beta_{105} L^{4,0,0}_{5,1,0,0,1,1} + \beta_{106} L^{8,0,0}_{7,1,0,0,1,1} + \beta_{107}\hat{I}_1^0 \hat{I}_1^0 \hat{I}_4^0 \\
&\quad + \beta_{108}\hat{I}_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{109}\hat{I}_1^0 \hat{I}_2^0 I_3^0 + \beta_{110}\hat{I}_1^0 I_2^0 \hat{I}_3^0 + \beta_{111}\hat{I}_2^0 \hat{I}_2^0 \hat{I}_2^0 \Big] \\
&+ N_{\mathrm{f}} C_{\mathrm{F}}^2 \Big[\beta_{112} K^{0,0,0}_{1,1,0,0,2,2} + \beta_{113} K^{0,0,0}_{1,1,0,0,3,1} + \beta_{114} K^{1,1,0}_{1,1,0,0,4,1} + \beta_{115} K^{2,0,0}_{1,1,0,0,4,1} \\
&\quad + \beta_{116} K^{0,0,0}_{2,1,0,0,2,1} + \beta_{117} K^{0,2,0}_{2,1,0,0,3,1} + \beta_{118} K^{2,0,0}_{2,1,0,0,3,1} + \beta_{119} K^{0,0,0}_{2,2,0,0,1,1} \\
&\quad + \beta_{120} K^{0,0,0}_{3,1,0,0,1,1} + \beta_{121} K^{0,2,0}_{3,1,0,0,2,1} + \beta_{122} K^{1,1,0}_{3,1,0,0,2,1} + \beta_{123} K^{1,3,0}_{3,1,0,0,3,1} \\
&\quad + \beta_{124} K^{4,0,0}_{3,1,0,0,3,1} + \beta_{125} K^{0,2,0}_{3,2,0,0,1,1} + \beta_{126} K^{1,1,0}_{3,2,0,0,1,1} + \beta_{127} K^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_{128} K^{1,1,0}_{4,1,0,0,1,1} + \beta_{129} K^{2,0,0}_{4,1,0,0,1,1} + \beta_{130} K^{1,3,0}_{4,1,0,0,2,1} + \beta_{131} K^{0,4,0}_{4,2,0,0,1,1} \\
&\quad + \beta_{132} K^{4,0,0}_{4,2,0,0,1,1} + \beta_{133} K^{1,3,0}_{5,1,0,0,1,1} + \beta_{134} K^{2,2,0}_{5,1,0,0,1,1} + \beta_{135} K^{3,1,0}_{5,1,0,0,1,1} \\
&\quad + \beta_{136} K^{4,0,0}_{5,1,0,0,1,1} + \beta_{137} K^{3,3,0}_{6,1,0,0,1,1} + \beta_{138} K^{4,2,0}_{6,1,0,0,1,1} + \beta_{139} K^{6,0,0}_{6,1,0,0,1,1} \\
&\quad + \beta_{140} K^{8,0,0}_{6,2,0,0,1,1} + \beta_{141} K^{5,3,0}_{7,1,0,0,1,1} + \beta_{142} K^{7,1,0}_{7,1,0,0,1,1} + \beta_{143} L^{0,0,0}_{2,2,0,0,1,1} \\
&\quad + \beta_{144} L^{0,0,0}_{3,1,0,0,1,1} + \beta_{145} L^{4,0,0}_{5,1,0,0,1,1} + \beta_{146}\hat{I}_1^0 \hat{I}_1^0 \hat{I}_4^0 + \beta_{147}\hat{I}_1^0 \hat{I}_1^0 I_4^0 \\
&\quad + \beta_{148}\hat{I}_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{149}\hat{I}_1^0 \hat{I}_2^0 I_3^0 + \beta_{150}\hat{I}_1^0 I_2^0 \hat{I}_3^0 + \beta_{151}\hat{I}_1^2 \hat{I}_2^0 I_4^0 + \beta_{152}\hat{I}_1^2 I_3^0 \hat{I}_3^0 \\
&\quad + \beta_{153}\hat{I}_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{154} I_1^0 \hat{I}_1^0 \hat{I}_4^0 + \beta_{155} I_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{156} I_1^0 \hat{I}_2^0 I_3^0 + \beta_{157} I_1^0 I_1^0 \hat{I}_4^0 \\
&\quad + \beta_{158} I_1^0 I_2^0 \hat{I}_3^0 + \beta_{159} I_1^2 \hat{I}_2^0 I_4^0 + \beta_{160} I_1^2 I_3^0 \hat{I}_3^0 + \beta_{161} I_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{162} I_2^0 I_2^0 \hat{I}_2^0 \Big] \\
&+ C_{\mathrm{A}} N_{\mathrm{f}} C_{\mathrm{F}} \Big[\beta_{163} K^{0,0,0}_{1,1,0,0,2,2} + \beta_{164} K^{0,0,0}_{1,1,0,0,3,1} + \beta_{165} K^{2,0,0}_{1,1,0,0,3,2} + \beta_{166} K^{2,0,0}_{1,1,0,0,4,1} \\
&\quad + \beta_{167} K^{0,0,0}_{2,1,0,0,2,1} + \beta_{168} K^{0,2,0}_{2,1,0,0,3,1} + \beta_{169} K^{2,0,0}_{2,1,0,0,3,1} + \beta_{170} K^{0,0,0}_{2,2,0,0,1,1} \\
&\quad + \beta_{171} K^{0,0,0}_{3,1,0,0,1,1} + \beta_{172} K^{0,2,0}_{3,1,0,0,2,1} + \beta_{173} K^{1,1,0}_{3,1,0,0,2,1} + \beta_{174} K^{1,3,0}_{3,1,0,0,3,1} \\
&\quad + \beta_{175} K^{4,0,0}_{3,1,0,0,3,1} + \beta_{176} K^{0,2,0}_{3,2,0,0,1,1} + \beta_{177} K^{1,1,0}_{3,2,0,0,1,1} + \beta_{178} K^{0,2,0}_{4,1,0,0,1,1} \\
&\quad + \beta_{179} K^{1,1,0}_{4,1,0,0,1,1} + \beta_{180} K^{2,0,0}_{4,1,0,0,1,1} + \beta_{181} K^{1,3,0}_{4,1,0,0,2,1} + \beta_{182} K^{0,4,0}_{4,2,0,0,1,1} \\
&\quad + \beta_{183} K^{4,0,0}_{4,2,0,0,1,1} + \beta_{184} K^{1,3,0}_{5,1,0,0,1,1} + \beta_{185} K^{2,2,0}_{5,1,0,0,1,1} + \beta_{186} K^{3,1,0}_{5,1,0,0,1,1} \\
&\quad + \beta_{187} K^{4,0,0}_{5,1,0,0,1,1} + \beta_{188} K^{3,3,0}_{6,1,0,0,1,1} + \beta_{189} K^{4,2,0}_{6,1,0,0,1,1} + \beta_{190} K^{6,0,0}_{6,1,0,0,1,1} \\
&\quad + \beta_{191} K^{8,0,0}_{6,2,0,0,1,1} + \beta_{192} K^{5,3,0}_{7,1,0,0,1,1} + \beta_{193} L^{0,0,0}_{2,2,0,0,1,1} + \beta_{194} L^{0,0,0}_{3,1,0,0,1,1} \\
&\quad + \beta_{195} L^{4,0,0}_{5,1,0,0,1,1} + \beta_{196}\hat{I}_1^0 \hat{I}_1^0 I_4^0 + \beta_{197}\hat{I}_1^0 \hat{I}_2^0 \hat{I}_3^0 + \beta_{198}\hat{I}_1^0 \hat{I}_2^0 I_3^0 + \beta_{199}\hat{I}_1^0 I_2^0 \hat{I}_3^0 \\
&\quad + \beta_{200}\hat{I}_1^2 \hat{I}_2^0 I_4^0 + \beta_{201}\hat{I}_1^2 I_3^0 \hat{I}_3^0 + \beta_{202}\hat{I}_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{203} I_1^0 \hat{I}_1^0 \hat{I}_4^0 + \beta_{204} I_1^0 \hat{I}_2^0 \hat{I}_3^0
\end{aligned}
$$

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$$+ \beta_{205} I_1^0 \hat{I}_2^0 I_3^0 + \beta_{206} I_1^0 I_2^0 \hat{I}_3^0 + \beta_{207} I_1^2 \hat{I}_2^0 I_4^0 + \beta_{208} I_1^2 I_3^0 \hat{I}_3^0 + \beta_{209} I_2^0 \hat{I}_2^0 \hat{I}_2^0 + \beta_{210} I_2^0 I_2^0 \hat{I}_2^0 \Big] . \tag{C.15}$$

Looking at the master integrals that are needed for the above two lengthy expressions let us note that, while most of them have factors of P_0 etc. in the numerator, only eight of them (those multiplying $\beta_{\{8,9,54,55,60,61,91,92\}}$) contain irreducible scalar products in the numerator and hence need methods for their evaluation that go beyond those presented in appendix 5 (see, however, ref. [12, 23], where examples of such sum-integrals were treated). Also, some of the masters (such as e.g. those multiplying $\alpha_{\{7,8\}}, \beta_{\{8,9\}}$) involve somewhat large powers of propagators, which is a consequence of our ordering prescription. However, as was shown in appendix 5 in terms of the generic power N, this does not seem to be a particularly difficult obstacle.

We refrain from listing the coefficients $\alpha_{1...137}$ and $\beta_{1...210}$ here. They have the general form $\sum_n \xi^n \, p_n(d)/q_n(d)$, where ξ is the gauge parameter (see eq. (2.14)) and p, q are polynomials in d. The full expressions for eqs. (C.14) and (C.15) are provided in computer-readable form on [24].

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