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The energy of the analytic lump solution in SFT

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Abstract: In a previous paper a method was proposed to find exact analytic solutions of open string field theory describing lower dimensional lumps, by incorporating in string field theory an exact renormalization group flow generated by a relevant operator in a worldsheet CFT. In this paper we compute the energy of one such solution, which is expected to represent a D24 brane. We show, both numerically and analytically, that its value corresponds to the theoretically expected one.

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Contents

1 Introduction

In a recent paper, [1], which will be referred to as I, following an earlier suggestion of [35], a general method was described to obtain new exact analytic solutions in Witten's cubic open string field theory (OSFT) [2], and in particular solutions that describe inhomogeneous tachyon condensation. Let us recall that these solutions fill the gap left in the verification of the expectation that an OSFT defined on a particular boundary conformal field theory (BCFT) has classical solutions describing other boundary conformal field theories [3, 4]. The previous construction of analytic solutions describing the tachyon vacuum [5–18] and of those describing a general marginal boundary deformation of the initial BCFT [19–29], had added ground for this expectation.[1] In all these developments there was a missing element: the solutions describing inhomogeneous and relevant boundary deformations of the initial BCFT were not known, though their existence was predicted [3, 4, 32]. In I such solutions were put forward.

The method of I consists in translating an exact renormalization group (RG) flow, generated in a two-dimensional world-sheet theory by a relevant operator, to the language of OSFT. The so-constructed solution is a deformation of the Erler-Schnabl solution, [7], the latter being a solution that describes homogeneous tachyon condensation for the D25 brane. It was shown in I that, if the operator has suitable properties, such solution will describe tachyon condensation in specific space directions, thus representing the condensation of a lower dimensional brane. In this paper we will analyze a particular solution, generated by an exact RG flow analyzed first by Witten, [33]. In I it was concluded that, on the basis of the analysis carried out in the framework of 2D CFT in [34], this solution should describe a D24 brane, with the correct ratio of tension with respect to the starting D25 brane. Of course an important piece of evidence for this interpretation is a precise determination of the energy of the solution. This is our aim in this paper.

As it happens, the expression of the energy for our solution in the SFT language is very complicated and does not allow for a straightforward analytic evaluation. Nevertheless in this paper we will be able to determine it exactly via an indirect method. As one may suspect, the entire procedure is rather roundabout, so we would like to spend some time explaining it. We start from a solution ψ_u of SFT *on the perturbative vacuum* and our first aim is to show that its energy is finite. More precisely, the energy of such solution has an UV ($u = 0$) singularity, which originates from the infinite volume factor due to our normalization, and corresponds to the tachyon vacuum energy. Once we have subtracted it,

[1] See [30, 31] for reviews.

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the energy becomes finite and well defined. Specifically, the lump energy being determined by an integration over a real variable U from 0 to ∞, we show that asymptotically the integrand behaves like $1/U^2$, so that it is integrable. This is a piece of information that will be used throughout. Not only. Having pushed the analytic calculation as far as possible, we will continue with the numerical evaluation of the energy, obtaining finally a rather precise numerical result, which differs by about 1/3 from the expected theoretical value of a D24 brane tension. This teaches us that we should not expect to find the right lump energy in a solution based on the perturbative vacuum, whose energy functional depends on the UV subtraction. The true energy must be independent of the latter.

Although the previous numerical calculation is not satisfactory it will turn out instrumental in the sequel. But, in order to be able to access an analytic evaluation of the energy, we have to take a detour. To this end we introduce an ϵ regulator in the Schwinger representation we use to represent the solution. The so-obtained regularized solution is called ψ_ϵ, the original solution being identified with the $\epsilon = 0$ one. We will show that ψ_ϵ is a tachyon condensation vacuum solution, whose energy, after subtracting the UV contribution, is expected to be exactly 0. We will show that this is indeed the case to a great accuracy. This will allow us to evaluate the energy of the original solution to a great accuracy too, confirming in fact the previous numerical calculation.

At this point everything is ready for the exact calculation of the lump energy. The lump string field is not the initial ψ_u, but $\psi_u - \psi_\epsilon$, which is a solution to the SFT equation of motion *on the tachyon vacuum*. The analytic determination of its energy is almost elementary, is UV subtraction-independent and gives the expected theoretical result.

To start with let us briefly summarize the construction of I.

1.1 Review of the results from I

In I, to start with, the authors have enlarged the well-known K, B, c algebra defined by

$$K = \frac{\pi}{2} K_1^L |I\rangle, \qquad B = \frac{\pi}{2} B_1^L |I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{1.1}$$

in the sliver frame (obtained by mapping the UHP to an infinite cylinder C_2 of circumference 2, by the sliver map $\tilde{z} = \frac{2}{\pi}\arctan z$), by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle \tag{1.2}$$

with the properties

$$[c, \phi] = 0, \qquad [B, \phi] = 0, \qquad [K, \phi] = \partial\phi, \tag{1.3}$$

such that Q has the following action:

$$Q\phi = c\partial\phi + \partial c \delta\phi. \tag{1.4}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi) B c \partial c \tag{1.5}$$

does indeed satisfy the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0. \tag{1.6}$$

It is clear that (1.5) is a deformation of the Erler-Schnabl solution, see [7], which can be recovered for $\phi = 1$.

Much like in the Erler-Schnabl (ES) case, we can view this solution as a singular gauge transformation

$$\psi_\phi = U_\phi Q U_\phi^{-1}, \tag{1.7}$$

where

$$U_\phi = 1 - \frac{1}{K+\phi}\phi Bc, \qquad U_\phi^{-1} = 1 + \frac{1}{K}\phi Bc. \tag{1.8}$$

In order to prove that (1.5) is a solution, one demands that $(c\phi)^2 = 0$, which requires the OPE of ϕ at nearby points to be not too singular.

It is instructive to write down the kinetic operator around (1.5). With some manipulation, using the K, B, c, ϕ algebra one can show that

$$\mathcal{Q}_{\psi_\phi}\frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the homotopy-field $\frac{B}{K+\phi}$ is singular, the solution has trivial cohomology, which is the defining property of the tachyon vacuum [35, 36]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi - \delta\phi)$ should be well defined. Moreover, in order to be able to show that (1.5) satisfies the equation of motion, one needs $K+\phi$ to be invertible.

In full generality we thus have a new nontrivial solution if

1. $\frac{1}{K+\phi}$ is singular, but
2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular and
3. $\frac{1}{K+\phi}(K+\phi) = 1$.

Problems with the last equation and Schwinger representation are discussed in appendix D.

In [1] some sufficient conditions for ϕ to comply with the above requirements were determined. Let us parametrize the worldsheet RG flow, referred to above, by a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and rewrite ϕ as ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\,\phi_{tu}\left(\frac{z}{t}\right) \tag{1.9}$$

and, most important, that the partition function

$$g(u) \equiv Tr[e^{-(K+\phi_u)}] = \left\langle e^{-\int_0^1 ds\,\phi_u(s)}\right\rangle_{C_1}, \tag{1.10}$$

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satisfies the asymptotic finiteness condition

$$\lim_{u\to\infty}\left\langle e^{-\int_0^1 ds\,\phi_u(s)}\right\rangle_{C_1} = \texttt{finite}. \tag{1.11}$$

Barring subtleties, this satisfies the first two conditions above i.e. guarantees not only the regularity of the solution but also its 'non-triviality', in the sense that if this condition is satisfied, it cannot fall in the same class as the ES tachyon vacuum solution. It would seem that the last condition above cannot be satisfied in view of the first. But this is not the case. We will argue in appendix D that by suitably defining the objects involved the equation can indeed be satisfied.

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. This operator generates an exact RG flow studied by Witten in [33], see also [34], and is based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A), \tag{1.12}$$

where A is a constant first introduced in [35]. In C_1 we have

$$\phi_u(s) = u(X^2(s) + 2\ln Tu + 2A) \tag{1.13}$$

and on the unit disk D,

$$\phi_u(\theta) = u\Big(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A\Big). \tag{1.14}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\,\phi_u(s)}\rangle_{C_1} \tag{1.15}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{2\pi} d\theta\, u\Big(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\Big)}\rangle_D.$$

According to [33],

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)}, \tag{1.16}$$

where

$$Z(u) = \frac{1}{\sqrt{2\pi}}\sqrt{u}\Gamma(u)e^{\gamma u} \tag{1.17}$$

Requiring finiteness for $u\to\infty$ we get $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2u}\Gamma(2u)e^{2u(1-\ln(2u))} \tag{1.18}$$

and

$$\lim_{u\to\infty} g(u) = 1. \tag{1.19}$$

Moreover, as it turns out, $\delta\phi_u = -2u$, and so:

$$\phi_u - \delta\phi_u = u\partial_u\phi_u(s). \tag{1.20}$$

Therefore the ϕ_u just introduced satisfies all the requested properties and consequently $\psi_u \equiv \psi_{\phi_u}$ must represent a D24 brane solution.

In I it was proved that ψ_u can satisfy the closed string overlap condition and it was seen that the corresponding RG flow in BCFT reproduces the correct ratio of tension between D25 and D24 branes. Also, in I the energy functional in SFT was computed.

Let us now summarize the content of the paper. In section 2 we write down the energy functional of the solution ψ_u in the most convenient form for the calculation, by isolating the 'angular' integration variables. In section 3 we perform the integration over the latter, after which we are left with two infinite discrete summations and an integral from 0 to ∞ over the parameter U (an alias of the RG parameter in CFT). We next carry out analytically one of the discrete summations. The rest of the calculation we have been able to do only numerically. In section 4 we analyze the behaviour near $U = 0$ and describe our subtraction scheme for the UV singularity. In section 5 we study the behaviour as $U \to \infty$ and, with the help of some heuristics, we conclude that the energy integral converges in that region. In section 6, we carry out the numerical evaluation of the energy functional. In section 7 we introduce the regularized solution ψ_ϵ and in section 8 we proceed to the evaluation of its energy, which turns out to vanish after subtracting the UV singularity. In section 9 finally we compute the energy of $\psi_u - \psi_\epsilon$ and find the desired result.

2 The energy functional

In I the expression for the energy of the lump solution was determined by evaluating a three-point function on the cylinder C_T of circumference T in the arctan frame. It equals $-\frac{1}{6}$ times the following expression

$$\begin{aligned}\langle\psi_u\psi_u\psi_u\rangle = -\int_0^\infty & dt_1 dt_2 dt_3 \mathcal{E}_0(t_1,t_2,t_3) u^3 g(uT)\Bigg\{8\Bigg(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Bigg)^3 \\ & +4\Bigg(-\frac{\partial_{2uT}g(uT)}{g(uT)}\Bigg)\Bigg(G^2_{2uT}\Big(\frac{2\pi t_1}{T}\Big)+G^2_{2uT}\Big(\frac{2\pi(t_1+t_2)}{T}\Big)+G^2_{2uT}\Big(\frac{2\pi t_2}{T}\Big)\Bigg) \\ & +8G_{2uT}\Big(\frac{2\pi t_1}{T}\Big)G_{2uT}\Big(\frac{2\pi(t_1+t_2)}{T}\Big)G_{2uT}\Big(\frac{2\pi t_2}{T}\Big)\Bigg\},\end{aligned} \tag{2.1}$$

where $T = t_1 + t_2 + t_3$. Here $g(u)$ is given by

$$g(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2u}\,\Gamma(2u)e^{2u(1-\ln(2u))} \tag{2.2}$$

and represents the partition function of the underlying matter CFT on the boundary of the unit disk with suitable boundary conditions for $u \to \infty$, which will be discussed further on. $G_u(\theta)$ represents the correlator on the boundary, first determined by Witten, [33]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k+u} \tag{2.3}$$

where we have made the choice $\alpha' = 1$. Finally $\mathcal{E}_0(t_1, t_2, t_3)$ represents the ghost three-point function in C_T.

$$\mathcal{E}_0(t_1, t_2, t_3) = \langle Bc\partial c(t_1 + t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} = -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1 + t_2)}{T}\sin\frac{\pi t_2}{T}. \quad (2.4)$$

We change variables $(t_1, t_2, t_3) \to (T, x, y)$, where

$$x = \frac{t_1}{T}, \qquad y = \frac{t_2}{T}.$$

Then the matter part of (2.1) (before integration) can be written as $u^3 F(uT, x, y)$, where

$$\begin{aligned} F(uT, x, y) = g(uT)\Bigg\{ & 8\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)^3 + 8G_{2uT}(2\pi x)G_{2uT}(2\pi(x + y))G_{2uT}(2\pi y) \\ & +4\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x + y)) + G^2_{2uT}(2\pi y)\Big)\Bigg\}. \end{aligned}$$

while the ghost correlator becomes

$$\mathcal{E}_0(t_1, t_2, t_3) \equiv \mathcal{E}(x, y) = -\frac{4}{\pi}\sin\pi x\,\sin\pi y\,\sin\pi(x + y). \quad (2.5)$$

The ghost correlator only depends on x and y, which are scale invariant coordinates.

After the change

$$\int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 = \int_0^\infty dT\, T^2 \int_0^1 dx \int_0^{1-x} dy,$$

the energy becomes

$$\begin{aligned} E[\psi_u] &= -S[\psi_u] = -\frac{1}{6}\langle\psi_u\psi_u\psi_u\rangle \\ &= \frac{1}{6}\int_0^\infty dT\, T^2 \int_0^1 dx \int_0^{1-x} dy\, \mathcal{E}(x, y)\, u^3 F(uT, x, y). \end{aligned} \quad (2.6)$$

It is convenient to change further $x \to y$ and subsequently $y \to 1 - y$. The result is

$$E[\psi_u] = \tfrac{1}{6}\int_0^\infty dT\, T^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(1 - y, x)\, u^3 F(uT, 1 - y, x), \quad (2.7)$$

where

$$\mathcal{E}(1 - y, x) = \frac{4}{\pi}\sin\pi x\,\sin\pi y\,\sin\pi(x - y),$$

and

$$\begin{aligned} & F(uT, 1 - y, x) \quad (2.8) \\ & = g(uT)\Bigg\{8\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)^3 + 8G_{2uT}(2\pi x)G_{2uT}(2\pi(x - y))G_{2uT}(2\pi y) \\ & \qquad +4\Big(-\frac{\partial_{2uT} g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x - y)) + G^2_{2uT}(2\pi y)\Big)\Bigg\}. \end{aligned}$$

Summarizing

$$E[\psi_u] = \frac{1}{6}\int_0^\infty d(2uT)\,(2uT)^2 \int_0^1 dy \int_0^y dx\,\frac{4}{\pi}\sin\pi x \sin\pi y \sin\pi(x-y) \tag{2.9}$$
$$\cdot g(uT)\Bigg\{-\left(\frac{\partial_{2uT} g(uT)}{g(uT)}\right)^3 + G_{2uT}(2\pi x)G_{2uT}(2\pi(x-y))G_{2uT}(2\pi y)$$
$$-\frac{1}{2}\left(\frac{\partial_{2uT} g(uT)}{g(uT)}\right)\Big(G^2_{2uT}(2\pi x)+G^2_{2uT}(2\pi(x-y))+G^2_{2uT}(2\pi y)\Big)\Bigg\}.$$

As already stressed in I, the first important remark about this expression is its independence of u. In the original BCFT of Witten, u was the RG coupling running from 0 (the UV) to ∞ (the IR). In SFT u is simply a gauge parameter, with the exception of the extreme values $u=0$ and $u=\infty$.

The expression in (2.9) implies three continuous integrations and, in the most complicated case, three infinite discrete summations. At the best of our ability and knowledge, all these operations cannot be done analytically. Therefore the obvious strategy to evaluate (2.9) is to push as far as possible the analytic computations and bring the integral to a form accessible to numerical evaluation. This is what we will do in the sequel.

3 The angular integration

The first step in the evaluation of (2.9) consists in performing the 'angular' x, y integration. This will be done analytically. Let us consider for definiteness the most complicated term, the cubic one in G_U (from now on for economy of notation let us set $U = 2uT$). We represent G_U as the series (2.3) and integrate term by term in x and y. All these integrations involve ordinary integrals which can be evaluated by using standard tables, or, more comfortably, Mathematica. It is a lucky coincidence that most integrals are nonvanishing only for specific values of the integers k. We have, for instance,

$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x) = \frac{1}{8\pi(k^2-1)}, \quad k\neq 1$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi x) = \frac{3}{32\pi} \tag{3.1}$$

while the integral $\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi m y)$ vanishes for almost all k, m except $k = m, m\pm 1$ and $k = 1, m$ and $k, m = 1$. For example

$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi k y) = \frac{1}{16\pi(k^2-1)} \tag{3.2}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\cos(2\pi k x)\cos(2\pi(k+1)y) = -\frac{1}{32\pi k(k+1)}, \text{ etc.}$$

The integration with three cosines is of course more complicated, but it can nevertheless be done in all cases. The integrals mostly vanish except for specific values of the integers k, m, n inside the cosines. They are non-vanishing for $m = k$ with n generic, and $m = k, n = k, k \pm 1$:

$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y)) \tag{3.3}$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi n(x-y)) = \frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi k(x-y)) = -\frac{3(2k^2-1)}{16\pi(4k^2-1)}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi(k+1)(x-y)) = \frac{6k^3+9k^2+3k-1}{128\pi(2k+1)(k+1)k}$$
$$\int_0^1 dy \int_0^y dx \sin(\pi x)\sin(\pi y)\sin(\pi(x-y))$$
$$\cdot\cos(2\pi kx)\cos(2\pi ky)\cos(2\pi(k-1)(x-y)) = \frac{6k^3-9k^2+3k+1}{128\pi(2k-1)(k-1)k}$$

and so on. A delicate part of the program consists in finding all nonvanishing terms and identifying the nonoverlapping ranges of summation over k, m and n. Fortunately the triple infinite summation reduces to a finite number of double infinite summations. Mathematica knows how to do the summations over one discrete index, in general not over two.

Let us write down next the result of the angular integration, by considering the three different types of terms in (2.9) in turn.

3.1 The term without G_U

This is easy. We get

$$\frac{1}{6}\int_0^\infty dU\, U^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi} \sin\pi x \sin\pi y \sin\pi(x-y) g(U)\left[-\left(\frac{\partial_U g(U)}{g(U)}\right)^3\right] \tag{3.4}$$
$$= -\frac{1}{4\pi^2}\int_0^\infty dU\, U^2 g(U)\left[-\left(\frac{\partial_U g(U)}{g(U)}\right)^3\right].$$

3.2 The term quadratic in G_U

We have to compute

$$\frac{1}{6}\int_0^\infty d(U)\ (U)^2 \int_0^1 dy \int_0^y dx\, \frac{4}{\pi} \sin\pi x \sin\pi y \sin\pi(x-y) \tag{3.5}$$
$$\cdot\left(-\frac{1}{2}\right)\partial_U g(U)\Big(G_U^2(2\pi x) + G_U^2(2\pi(x-y)) + G_U^2(2\pi y)\Big)\Big\}.$$

Therefore the integrand of the quadratic term in G_U is made of the factor $-\frac{1}{12}U^2\partial_U g(U)$ multiplied by the factor

$$\left.\begin{aligned}&\frac{4}{\pi}\int_0^1 dy\int_0^y dx\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y)\\&\qquad\cdot\Big(G_U^2(2\pi x)+G_U^2(2\pi(x-y))+G_U^2(2\pi y)\Big)\end{aligned}\right\}. \tag{3.6}$$

After some work the latter turns out to equal

$$\begin{aligned}&\left.\begin{aligned}&\frac{4}{\pi}\int_0^1 dy\int_0^y dx\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y)\\&\qquad\cdot\Big(G_U^2(2\pi x)+G_U^2(2\pi(x-y))+G_U^2(2\pi y)\Big)\end{aligned}\right\} \\ &=-\frac{9}{2\pi^2}\frac{1}{U^2}+\frac{16}{\pi U}\left(\frac{9}{32\pi}\frac{1}{U+1}+\frac{3}{8\pi}\sum_{k=2}^{\infty}\frac{1}{k^2-1}\frac{1}{k+U}\right)\\ &\quad+\frac{48}{\pi}\Bigg(\frac{1}{8\pi}\sum_{\substack{k,n\\ n\neq k,k\pm 1}}^{\infty}\frac{n^2+k^2-1}{((n+k)^2-1)((n-k)^2-1)}\frac{1}{(n+U)(k+U)}\\ &\quad-\frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{3k^2-1}{4k^2-1}\frac{1}{(k+U)^2}\qquad\leftarrow R_1(U)\\ &\quad+\frac{1}{64\pi}\sum_{k=1}^{\infty}\frac{3k^2+3k+1}{k(k+1)}\frac{1}{(k+U)(k+U+1)}\qquad\leftarrow R_2(U)\\ &\quad+\frac{1}{64\pi}\sum_{k=2}^{\infty}\frac{3k^2-3k+1}{k(k-1)}\frac{1}{(k+U)(k+U-1)}\qquad\leftarrow R_3(U)\Bigg)\\ &\equiv E_1^{(2)}(U)+\frac{48}{\pi}\left(\sum_{p=2}^{\infty}RK(p,U)+R_1(U)+R_2(U)+R_3(U)\right),\end{aligned} \tag{3.7}$$

where

$$E_1^{(2)}(U)=-\frac{9}{2\pi^2}\frac{1}{U^2}+E_0^{(2)}(U) \tag{3.8}$$

and

$$E_0^{(2)}(U)=\frac{9}{2\pi^2}\frac{1}{U(U+1)}+\frac{3}{2\pi^2}\frac{1}{U(U^2-1)}\Big(3(U+1)-4\gamma-4\psi(2+U)\Big), \tag{3.9}$$

where ψ is the digamma function and γ the Euler-Mascheroni constant. To save space, we have introduced in (3.7) in a quite unconventional way the definitions of the quantities $R_i(U)$, $i=1,2,3$. Beside $R_1(U),R_2(U),R_3(U)$, we define

$$RK(k,n,U)=\frac{1}{8\pi}\frac{n^2+k^2-1}{((n+k)^2-1)((n-k)^2-1)}\frac{1}{(n+U)(k+U)} \tag{3.10}$$

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and

$$RK(p,U) = \sum_{k=1}^{\infty} RK(k,k+p,U) + \sum_{k=p+1}^{\infty} RK(k,k-p,U) \tag{3.11}$$

with the summation in (3.7) corresponding to: $\sum^{\infty}_{\substack{k,n \\ n\neq k,k\pm 1}} RK(k,n,U) = \sum_{p=2}^{\infty} RK(p,U)$.

3.2.1 Performing one discrete summation

As already pointed out it is possible to perform in an analytic way at least one of the two discrete summations above. To start with

$$\begin{aligned} R(U) \;=\; R_1(U)+R_2(U)+R_3(U) = \frac{1}{32\pi}\Bigg(&\frac{U^2(1+3U)-2(U+1)H(U)}{U(1+U)(U^2-1)} \\ &+\frac{4\left(1+4U(1-\gamma+U-\ln 4)-4U\psi(1+U)-2\left(1-7U^2+12U^4\right)\psi^{(1)}(1+U)\right)}{\left(1-4U^2\right)^2}\Bigg). \end{aligned} \tag{3.12}$$

Next

$$\begin{aligned} RK(p,U) = &\frac{1}{4p\left(-1+p^2\right)\pi(-1+p-2U)(1+p-2U)(-1+p+2U)(1+p+2U)} \\ &\cdot\Bigg(4p\left(-1+p^2\right)U\,H\left(\frac{p-1}{2}\right) \\ &-(-1+p+2U)(1+p+2U)\left(-1+p^2-2pU+2U^2\right)H(U) \\ &+(-1+p-2U) \\ &\cdot\left(-(-1+p)p(1+p+2U)+(1+p-2U)\left(-1+p^2+2pU+2U^2\right)H(p+U)\right)\Bigg), \end{aligned} \tag{3.13}$$

where $H(U) = \gamma + \psi(U+1)$ is the harmonic number function. It should be remarked that in both (3.12) and (3.13) there are zeros in the denominators, for positive values of U. These however do not correspond to real poles of $R(U)$ and $RK(p,U)$, because they are cancelled by corresponding zeroes in the numerator.

3.3 The term cubic in G_U

In (2.9) we have to compute

$$\frac{2}{3\pi}\int_0^{\infty}\! dU\,U^2\!\int_0^1\! dy\!\int_0^y\! dx\,\sin\pi x\,\sin\pi y\,\sin\pi(x-y)\,g(U)G_U(2\pi x)G_U(2\pi(x-y))G_U(2\pi y). \tag{3.14}$$

The most convenient form of the cubic term in G_U after angular integration is probably the following one (which must be multiplied by $\frac{1}{6}U^2 g(U)$)

$$\begin{aligned} &\frac{4}{\pi}\int_0^1 dy\int_0^y dx\,\sin(\pi x)\sin(\pi y)\sin(\pi(x-y))\left(\frac{1}{U}+2\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k+U}\right) \\ &\cdot\left(\frac{1}{U}+2\sum_{m=1}^{\infty}\frac{\cos(2\pi my)}{m+U}\right)\left(\frac{1}{U}+2\sum_{n=1}^{\infty}\frac{\cos(2\pi n(x-y))}{k+U}\right) \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&= -\frac{3}{2\pi^2}\frac{1}{U^3} + \frac{9}{4\pi^2}\frac{1}{U^2(U+1)} + \frac{3}{\pi^2}\frac{1}{U^2(U^2-1)}\left(-\gamma + \frac{3}{4}(U+1) - \psi(2+U)\right) \\
&\quad + \frac{3}{4\pi^2}\frac{1}{U(U+1)^2} - \frac{7}{2\pi^2}\frac{1}{U(U+1)(U+2)} + \frac{3}{4\pi^2}\frac{1}{U(U^2-1)^2}\cdot \\
&\quad \cdot\Big(3(1+U^2) - 8\gamma U + 6U - 8U\psi(2+U) + 4(U^2-1)\psi^{(1)}(2+U)\Big) \\
&\quad - \frac{1}{2\pi^2 U(U+1)(U^2-1)}\Big(17 + 5U - 12\gamma - 12\psi(3+U)\Big) - \frac{3}{2\pi^2 U^2(U^2-1)} \\
&\quad \cdot\Big(5 - 4\gamma + U - 2(U+1)\psi(2+U) + 2(U-1)\psi(3+U)\Big) \\
&\quad + \frac{32}{\pi}\Bigg[3\sum_{\substack{k,n\\ n\neq k,k\pm 1}} \frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}\,\frac{1}{(k+U)^2(n+U)} \leftarrow (S_4, S_5) \\
&\quad -3\sum_{\substack{k,n\\ n\neq k,k\pm 1,k-2}} \frac{n^2+k^2-k}{32\pi(k^2-n^2)((k-1)^2-n^2)}\,\frac{1}{(k+U)(k+U-1)(n+U)} \leftarrow (S_8) \\
&\quad -3\sum_{\substack{k,n\\ n\neq k,k\pm 1,k+2}} \frac{n^2+k^2+k}{32\pi(k^2-n^2)((k+1)^2-n^2)}\,\frac{1}{(k+U)(k+U+1)(n+U)} \leftarrow (S_9) \\
&\quad -3\sum_{k=1} \frac{2k^2-1}{16\pi(4k^2-1)}\,\frac{1}{(k+U)^3} \leftarrow (S_{10}) \\
&\quad +2\sum_{k=1} \frac{6k^3+9k^2+3k-1}{128\pi k(k+1)(2k+1)}\,\frac{1}{(k+U)^2(k+U+1)} \leftarrow (S_7) \\
&\quad +2\sum_{k=2} \frac{6k^3-9k^2+3k+1}{128\pi k(k-1)(2k-1)}\,\frac{1}{(k+U)^2(k+U-1)} \leftarrow (S_6) \\
&\quad -2\sum_{k=3} \frac{4k^2-8k+5}{64\pi(2k-1)(2k-3)}\,\frac{1}{(k+U)(k+U-1)(k+U-2)} \leftarrow (S_2) \\
&\quad +\sum_{k=2} \frac{6k^3-9k^2+3k-1}{128\pi k(k-1)(2k-1)}\,\frac{1}{(k+U)(k+U-1)^2} \leftarrow (S_{11}) \\
&\quad -2\sum_{k=2} \frac{4k^2+1}{64\pi(4k^2-1)}\,\frac{1}{(k+U)(k+U-1)(k+U+1)} \leftarrow (S_1) \\
&\quad +\sum_{k=1} \frac{6k^3+9k^2+3k+1}{128\pi k(k+1)(2k+1)}\,\frac{1}{(k+U)(k+U+1)^2} \leftarrow (S_{12}) \\
&\quad -2\sum_{k=1} \frac{4k^2+8k+5}{64\pi(2k+1)(2k+3)}\,\frac{1}{(k+U)(k+U+1)(k+U+2)} \leftarrow (S_3)\Bigg].
\end{aligned}
$$

The symbols S_i, $i = 1, \ldots, 12$ represents the corresponding terms shown in the formula and correspond to simple summations. As S_4, S_5, S_8, S_9 are shown in correspondence with double summations, they need a more accurate definitions. S_4 is the sum over k from 2 to ∞ of the corresponding term for $n = k+2$, while S_5 is the sum of the same term from 3 to ∞ for $n = k-2$; S_8 is the sum over k from 2 to ∞ of the corresponding term for $n = k+2$. S_9 is the sum over k from 3 to ∞ of the corresponding term for $n = k-2$.

The first line of the r.h.s. refers to the terms with one cosine, the next four lines to terms with 2 cosines and the remaining ones to terms with three cosines integrated over. In (3.15), $\psi^{(n)}$ is the n-th polygamma function and $\psi^{(0)} = \psi$. There are simple and quadratic poles at $U = 1$, but they are compensated by corresponding zeroes in the numerators. One can also see that all the summations are (absolutely) convergent for any finite U, including $U = 0$.

To proceed further let us define

$$\begin{aligned} SK0(k,n,U) &= \frac{n^2+k^2-1}{16\pi((n+k)^2-1)((n-k)^2-1)}\,\frac{1}{(k+U)^2(n+U)} \\ SK1(k,n,U) &= \frac{n^2+k^2-k}{32\pi(k^2-n^2)((k-1)^2-n^2)}\,\frac{1}{(k+U)(k+U-1)(n+U)} \\ SK2(k,n,U) &= \frac{n^2+k^2+k}{32\pi(k^2-n^2)((k+1)^2-n^2)}\,\frac{1}{(k+U)(k+U+1)(n+U)} \end{aligned}$$

and set

$$\begin{aligned} SK0_+(p,U) &= \sum_{k=1}^{\infty} SK0(k,k+p,U), \qquad SK0_-(p,U) = \sum_{k=p+1}^{\infty} SK0(k,k-p,U), \\ SK1_+(p,U) &= \sum_{k=2}^{\infty} SK1(k,k+p,U), \qquad SK1_-(p,U) = \sum_{k=p+1}^{\infty} SK1(k,k-p,U), \\ SK2_+(p,U) &= \sum_{k=1}^{\infty} SK2(k,k+p,U), \qquad SK2_-(p,U) = \sum_{k=p+1}^{\infty} SK2(k,k-p,U). \end{aligned}$$

Then the quantity within the square brackets in (3.15) corresponds to

$$\begin{aligned} &3\sum_{p=3}^{\infty}\Big(SK0_+(p,U)+SK0_-(p,U)-SK1_+(p,U)-SK1_-(p,U) \\ &\qquad -SK2_+(p,U)-SK2_-(p,U)\Big)+\sum_{i=1}^{12} S_i(U). \end{aligned} \tag{3.16}$$

3.3.1 Performing one discrete summation

Like in the quadratic term we can carry out in an analytic way one discrete summation. We have

$$\begin{aligned} &S(U) \qquad\qquad (3.17) \\ &=\sum_{i=1}^{12} S_i(U) = \frac{1}{256\pi(-2+U)U^2\left(-1-U+4U^2+4U^3\right)^3\left(18-9U-17U^2+4U^3+4U^4\right)^2} \\ &\cdot\Bigg(\frac{1}{3+U}\Big(12\gamma(1+U)(2+U)(3+U) \\ &\cdot\left(-324+13887U^2-48589U^4+72468U^6-44592U^8+11200U^{10}\right) \\ &+U^2\big((1+2U)^3(-845856+1192986U+1878099U^2-2889638U^3-2109474U^4 \\ &+3023246U^5+1453619U^6-1668346U^7-622980U^8+493352U^9 \end{aligned}$$

$$
\begin{aligned}
&+147696U^{10}-82016U^{11}-21440U^{12}+6016U^{13}+1536U^{14})\\
&-192(-2+U)(1+U)^3(3+U)\left(-2+U+U^2\right)^2\\
&\cdot\left(153-132U^2+112U^4+64U^6\right)\ln 4)\Big)\\
&+12(1+U)(2+U)\\
&\cdot\Big(\left(-324+13887U^2-48589U^4+72468U^6-44592U^8+11200U^{10}\right)\psi(1+U)\\
&+U\Big((-2+U)(-1+U)(1+U)(2+U)(-3+2U)(-1+2U)(1+2U)(3+2U)\\
&\cdot\left(9+138U^2-352U^4+160U^6\right)\psi^{(1)}(1+U)\\
&+2U\left(4-9U^2+2U^4\right)\left(-9+U^2\left(7-4U^2\right)^2\right)^2\psi^{(2)}(1+U)\Big)\Big)\Big). \qquad (3.18)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&SK(p,U) \qquad (3.19)\\
&\equiv\sum_{i=0}^{2}SKi_+(p,U)+SKi_-(p,U)\\
&=\frac{1}{32p^2\pi}\Bigg(\frac{2p}{(1+p)(1+p-2U)^2}+\frac{2p}{(1+p)^2(1+p-2U)}+\frac{p^2}{(2+3p+p^2)(2+p-2U)}\\
&+\frac{-1+p-p^2-2p^3}{(-1+p)^2(2+3p+p^2)(1+U)}+\frac{4+p(-3+p(2+(-2+p)p))}{(-2+p)(-1+p^2)^2(p+U)}+\frac{1+p(3+p)}{(1+p)^2(2+p)(1+p+U)}\\
&+\frac{p}{(-1+p)(2-p+2U)}-\frac{2p}{(1+p)^2(-1+p+2U)}+\frac{2\left(-2+p^2\right)}{(-1+p^2)(p+2U)}\\
&-\frac{2p}{(-1+p)(1+p+2U)^2}+\frac{4p}{(-1+p)^2(1+p)(1+p+2U)}-\frac{2(-1+p)p}{(-2+p)(1+p)(2+p+2U)}\\
&+2p\Bigg(-\frac{8\left(p-2p^3+p^5-16pU^4\right)\psi(\frac{1+p}{2})}{\left(\left(-1+p^2\right)^2-8\left(1+p^2\right)U^2+16U^4\right)^2}\\
&+\frac{8p\left(-4+p^2+4U^2\right)\psi(\frac{2+p}{2})}{(-2+p-2U)(p-2U)(2+p-2U)(-2+p+2U)(p+2U)(2+p+2U)}\\
&+\frac{1}{-1+p^2}\\
&\cdot\Big(2\frac{2-4p^4+21p^3U+6U^2-8U^4+p^2\left(2-38U^2\right)+pU\left(-9+28U^2\right)\psi(1+U)}{(-2+p-2U)(-1+p-2U)^2(p-2U)(1+p-2U)^2(2+p-2U)}\\
&-2\frac{\left(-2+4p^4+21p^3U-6U^2+8U^4+pU\left(-9+28U^2\right)+p^2\left(-2+38U^2\right)\right)\psi(p+U)}{(-2+p+2U)(-1+p+2U)^2(p+2U)(1+p+2U)^2(2+p+2U)}\\
&+\frac{\left(-1+p^2-2pU+2U^2\right)\psi^{(1)}(1+U)}{-1+p^2-4pU+4U^2}-\frac{\left(-1+p^2+2pU+2U^2\right)\psi^{(1)}(1+p+U)}{-1+p^2+4pU+4U^2}\Big)\Bigg)\Bigg).
\end{aligned}
$$

As explained above, in general we cannot proceed further with analytic means in performing the remaining summations and integrations. The strategy from now on consists therefore in making sure that summations and integrals converge (apart from the expected UV singularity, which has to be subtracted). Let us study first the behaviour at $U \approx 0$. We will proceed next to the behaviour at $U \to \infty$.

4 Behaviour near $U = 0$

Let us consider first the cubic term. We recall that all the summations are convergent at $U = 0$. In (3.14) the expression (3.15) is multiplied by $\frac{1}{6}U^2 g(U)$. Recalling that $g(U) \approx \frac{1}{2\sqrt{\pi U}}$ for $U \approx 0$, we see that the only term that produces a non-integrable singularity in U is the first term on the r.h.s. , which has a cubic pole in U. Altogether the UV singularity due to the cubic term is

$$-\frac{1}{8}\frac{1}{\pi^{\frac{5}{2}}U^{\frac{3}{2}}}. \tag{4.1}$$

As for the quadratic term, we have $\partial_U g(U) \approx -\frac{1}{4\sqrt{\pi}U^{\frac{3}{2}}}$. Once again all the discrete summations are convergent at $U = 0$. Therefore the only UV singular term corresponds to the first term at the r.h.s. of (3.7), i.e. $-\frac{9}{2\pi^2}\frac{1}{U^2}$. According to (3.5) we have to multiply this by $-\frac{1}{12}U^2\partial_U g(U)$. Therefore the contribution of the quadratic term to the UV singularity is

$$-\frac{3}{32}\frac{1}{\pi^{\frac{5}{2}}U^{\frac{3}{2}}}. \tag{4.2}$$

Finally for the last term, the one without G_U, we have

$$U^2 g(U)\left(\frac{\partial_U g(U)}{g(U)}\right)^3 \approx -\frac{1}{16\sqrt{\pi}U^{\frac{3}{2}}}.$$

Therefore altogether this term contributes

$$-\frac{1}{64}\frac{1}{\pi^{\frac{5}{2}}U^{\frac{3}{2}}}. \tag{4.3}$$

So the overall singularity at $U = 0$ is

$$-\frac{15}{64}\int_0 dU \frac{1}{\pi^{\frac{5}{2}}U^{\frac{3}{2}}} = \frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi U}}\bigg|_{U=0} = -\lim_{U\to 0}\frac{15}{8}\frac{1}{2\pi^2}\frac{1}{2\sqrt{\pi U}}. \tag{4.4}$$

In order to subtract this singularity we choose a function $f(U)$ that vanishes fast enough at infinity and such that $f(0) = 1$. For instance $f(U) = e^{-U}$. Then, if we subtract from the energy the expression

$$\frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi}}\int_0^\infty dU \frac{1}{\sqrt{U}}\left(f'(U) - \frac{1}{2U}f(U)\right) = \frac{15}{8}\frac{1}{4\pi^2\sqrt{\pi}}\int_0^\infty dU \frac{\partial}{\partial U}\left(\frac{1}{\sqrt{U}}f(U)\right) \tag{4.5}$$

the energy functional becomes finite, at least in the UV. What remains after the subtraction is the relevant energy.

Notice that the integral in (4.5) does not depend on the regulator f we use, provided it satisfies the boundary condition $f(0) = 1$ and decreases fast enough at infinity. As we shall see in section 9, the lump energy is anyhow thouroughly independ of such UV subtractions.

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5 The behaviour near $U = \infty$

The integrand in (3.4) behaves as $1/U^4$ at large U. Therefore the integral (3.4) converges rapidly in the IR.

5.1 The quadratic term as $U \to \infty$

With reference to (3.5) we remark first that for large U

$$U^2 \partial_U g(U) = -\frac{1}{12\sqrt{2}} + \mathcal{O}\left(\frac{1}{U}\right). \tag{5.1}$$

Therefore this factor does not affect the integrability at large U. The issue will be decided by the other factors. For large U we have

$$E_0^{(2)}(U) = \frac{9}{\pi^2}\frac{1}{U^3} - \frac{6}{\pi^2}\frac{\ln U}{U^4} + \cdots \tag{5.2}$$

and

$$R(U) = -\frac{3}{32\pi}\frac{1}{U} + \frac{1}{16\pi}\frac{1}{U^2} - \frac{3}{32\pi}\frac{\ln U}{U^3} + \cdots . \tag{5.3}$$

Moreover, again for large U,

$$RK(p,U) = \frac{1}{8\pi(p^2-1)}\frac{1}{U} - \frac{1}{8\pi(p^2-1)}\frac{1}{U^2} - \frac{1}{16\pi}\frac{\ln U}{U^3} + \cdots . \tag{5.4}$$

Since

$$\sum_{p=2}^{\infty} \frac{1}{8\pi(p^2-1)} = \frac{3}{32\pi}$$

the coefficient of $1/U$ in (5.3) cancels the corresponding coefficient of (5.4). The coefficient of $1/U^2$ equals $-1/(32\pi)$. This must be multiplied by $\frac{48}{\pi}$ and added to the term $-\frac{9}{2\pi^2}\frac{1}{U^2}$ in (3.7). This is anyhow an integrable term in the IR. This much takes care of the integrability of the $E_0^{(2)}(U)$, $R(U)$ and the first two terms in (5.4) in the IR. Let us now concentrate on the rest of $RK(p,U)$, that is

$$RK'(p,U) = \frac{1}{16\pi}\frac{\ln U}{U^3} + \cdots \tag{5.5}$$

(see a more complete asymptotic expansion in appendix A).

In order to estimate the integrability of this term, we can replace the infinite discrete sum with an integral over p, for large p. Now we evaluate the behaviour of $RK'(p,U)$ for any ray, departing from the origin of the (p,U) plane in the positive quadrant, when the rays approach infinity. We can parametrize a ray, for instance, as the line (aU,U). It is possible to find an analytic expression for this. We can compute the large U limit for any (positive) value of a. The behaviour is given by the following rule

$$RK'(aU,U) \approx \frac{B}{U^3} + \mathcal{O}(U^{-4}). \tag{5.6}$$

$a:$	45	7	1	$\frac{1}{13}$	$\frac{1}{150}$	$\frac{1}{15000}$
$B:$	-0.00001	-0.00048	0.00713	-0.04319	-0.09039	-0.18188

Table 1. Samples of a and B in eq. (5.6).

In table 1 are some examples (the output is numerical only for economy of space).

It is important to remark that very small values of a are likely not to give a reliable response in the table, because one is bound to come across to the forbidden value $p = 1$, which will give rise to an infinity (see (3.11). Apart from this, on a large range the values of $RK'(aU, U)$ are bounded in a.

It is even possible to find an analytic expression of B as a function of a in the large U limit. We have

$$B = \frac{1}{8a^3\left(-4+a^2\right)^2\pi}\Big(-a\left(16-8a+2a^3+a^4+8a^2(-1+\log 2)\right) \\ +8a^3\log a+2(-2+a)^2\left(2+2a+a^2\right)\log(1+a)\Big) \tag{5.7}$$

which is obviously integrable in the whole range of a. Since $dpdU{=}UdadU$, this confirms the integral behaviour of $\sum_{p=2}^{\infty} RK'(p,U)$ with respect to the U integration.

To study the integrability for large U and large p in a more systematic way, we divide the positive quadrant of the (p,U) plane in a large finite number N of small angular wedges. We notice that table 1 means that $RK'(p,U)$ varies slowly in the angular direction — it is actually approximately constant in that direction for large p and U. Therefore it is easy to integrate over such wedges from a large enough value of the radius $r = \sqrt{p^2+U^2}$ to infinity. The result of any such integration will be a finite number and a good approximation to the actual value (which can be improved at will). Their total summation will also be finite as a consequence of table 1, unless there are pathologies at the extremities. Looking at the asymptotic expansion for large p

$$RK(p,U) = \frac{1}{4\pi}\frac{\log p}{p^3} - \frac{1}{4\pi}(1+\psi(1+U))\frac{1}{p^3} + \cdots \tag{5.8}$$

and table 1 we see that also the integration for the very last wedge, a large, will be finite. The contribution of the very first wedge is more problematic for the above explained reason and is deferred to appendix A.

An additional support comes from a numerical analysis of $RK'(p,U)$. It turns out that, for large U, the leading coefficient of $\sum_{p=2}^{\infty} RK'(p,U)$ is

$$\sum_{p=2}^{\infty} RK'(p,U) \approx \frac{-0.0344761\ldots}{U^2} + \cdots . \tag{5.9}$$

This can be rewritten in the (probably exact) analytic way

$$\sum_{p=2}^{\infty} RK'(p,U) = \left(\frac{3}{32\pi} - \frac{1}{4\pi}\left(\gamma + \frac{1}{3}\log 2\right)\right)\frac{1}{U^2} + \cdots . \tag{5.10}$$

Finally, the numerical calculations of the next section further confirm our conclusion.

On the basis of that analysis and the above, we conclude that the quadratic term integrand in U, behaves in the IR in an integrable way, giving rise there to a finite contribution to the energy.

5.2 The cubic term as $U \to \infty$

To start with let us recall that for large U

$$U^2 g(U) = \frac{U^2}{\sqrt{2}} + \frac{U}{12\sqrt{2}} + \cdots . \tag{5.11}$$

Looking at (3.14) and (3.15), let us call

$$\begin{aligned} E_1^{(3)}(U) = & -\frac{3}{2\pi^2}\frac{1}{U^3} + \frac{9}{4\pi^2}\frac{1}{U^2(U+1)} + \frac{3}{\pi^2}\frac{1}{U^2(U^2-1)}\left(-\gamma + \frac{3}{4}(U+1) - \psi(2+U)\right) \\ & + \frac{3}{4\pi^2}\frac{1}{U(U+1)^2} - \frac{7}{2\pi^2}\frac{1}{U(U+1)(U+2)} + \frac{3}{4\pi^2}\frac{1}{U(U^2-1)^2} \cdot \\ & \cdot\left(3(1+U^2) - 8\gamma U + 6U - 8U\psi(2+U) + 4(U^2-1)\psi^{(1)}(2+U)\right) \\ & - \frac{1}{2\pi^2 U(U+1)(U^2-1)}\left(17 + 5U - 12\gamma - 12\psi(3+U)\right) - \frac{3}{2\pi^2 U^2(U^2-1)} \\ & \cdot\left(5 - 4\gamma + U - 2(U+1)\psi(2+U) + 2(U-1)\psi(3+U)\right). \end{aligned} \tag{5.12}$$

Then it is easy to prove that

$$\lim_{U\to\infty} U^3 E_1^{(3)}(U) = -\frac{3}{2\pi^2} \tag{5.13}$$

that is, the nonvanishing, nonintegrable, contribution comes solely from the first term on the r.h.s. of (5.12). Defining $E_0^{(3)}(U) = E_1^{(3)}(U) + \frac{3}{2\pi^2}\frac{1}{U^3}$, one finds

$$E_0^{(3)}(U) \approx \frac{3}{\pi^2}\frac{\ln U}{U^4} + \frac{3\gamma}{\pi^4}\frac{1}{U^4} + \cdots . \tag{5.14}$$

This corresponds to an integrable singularity at infinity in the U-integration. We expect the nonintegrable contribution coming from (5.13) to be cancelled by the three-cosine pieces. We will see that also the first terms in the r.h.s. of (5.14) gets cancelled.

Let us see the three-cosines pieces in eq. (3.15). The first contribution (3.17), for large U goes as follows

$$S(U) = \frac{5}{256\pi}\frac{1}{U^3} - \frac{2+\ln 8}{32\pi}\frac{1}{U^4} + \cdots . \tag{5.15}$$

The other contribution is given by $SK(p,U)$. We proceed as for $RK(p,U)$ above.

$$SK(p,U) = \frac{1}{32\pi p(p+2)}\frac{1}{U^3} - \frac{1 - p(p+1)\left(H\left(\frac{p-1}{2}\right) - H\left(\frac{p}{2}\right)\right)}{32\pi p(p+1)}\frac{1}{U^4} + \cdots . \tag{5.16}$$

$a:$	45	7	1	1/15	1/85	1/150
$B:$	-4×10^{-6}	-0.00017	-0.00497	-0.13988	-0.83567	-1.4822

Table 2. Samples of a and B in eq. (5.6).

Let us consider the first term in the r.h.s., which, from (3.15), must be multiplied by 3. The sum over p up to ∞ gives the following coefficient of $1/U^3$

$$3\sum_{p=3}^{\infty}\frac{1}{32\pi p(p+2)}=\frac{7}{256\pi}.$$

This must be added to the analogous coefficient in the r.h.s. of (3.17), yielding a total coefficient of $\frac{3}{64\pi}$. In eq. (3.15) this is multiplied by $\frac{32}{\pi}$, which gives $\frac{3}{2\pi^2}$. This cancels exactly the r.h.s. of (5.13). Therefore *in the integral (3.15) there are no contributions of order $1/U^3$ for large U.*

As already remarked for the quadratic term, the above takes care of the nonintegrable asymptotic behaviour of (5.13), (5.15), but it is not enough as far as the $SK(p,U)$ is concerned. We will proceed in a way analogous to the quadratic term. We will drop the first term in the r.h.s. of (5.16) (since we know how to deal exactly with the latter) and define

$$SK'(p,U)=SK(p,U)-\frac{1}{32\pi p(p+2)}\frac{1}{U^3}. \tag{5.17}$$

In order to estimate the integrability of this expression, we will replace, for large p, the infinite discrete sum with an integral over p. Next we evaluate the behaviour of $SK(p,U)$ for any ray departing from the origin of the (p,U) plane in the positive quadrant when the rays approach infinity, parametrizing a ray as the line (aU,U), a being some positive number. The behaviour is given in general by the following rule

$$SK(aU,U)\approx\frac{B}{U^5}+\cdots. \tag{5.18}$$

In table 2 are some examples (the output is numerical for economy of space): Also in this case we warn that it does not make sense to probe extremely small values of a.

On the other extreme, large p and fixed U, we have

$$SK(p,U)=\frac{-1+(1+U)\psi^{(1)}(1+U)}{16\pi(1+U)}\frac{1}{p^3}+\frac{-3-4U+4U(1+U)\psi^{(1)}(1+U)}{32\pi(1+U)}+\cdots. \tag{5.19}$$

This behaviour is of course integrable at $p=\infty$. One can also verify a behaviour in p similar to (5.18) and compute a table like table 2.

Next we study the problem of integrability for large U and large p following the same pattern as for the quadratic term. We divide the positive quadrant of the (p,U) plane in a large finite number N of small angle wedges. We notice that table 1 means that $U^2SK'(p,U)$ varies slowly in the angular direction — it is actually approximately constant in that direction for large p and U, see (5.20) below. Therefore it is easy to integrate

$U^2SK'(p,U)$ over such wedges from a large enough value of the radius $r = \sqrt{p^2+U^2}$ to infinity. The result of any such integration will be a finite number, including the integration for the very last wedge, a very large. To estimate the effectiveness of this approach one should consider the first wedge, which is the most problematic in view of what has been remarked above. But this point is very technical and we decided to postpone it to appendix B.

Additional evidence for convergence can be provided by a numerical analysis. One can see that the behaviour of $U^2SK'(p,U)$ for large p and U may be approximated by by

$$U^2SK'(p,U) \sim \frac{\log r}{r^3} \tag{5.20}$$

which is integrable. We can do better and compute, numerically, the asymptotic behaviour

$$U^2\sum_{p=3}^{\infty} SK'(p,U) \approx -\frac{0.0092Log(U)}{U^3} + \dots \tag{5.21}$$

The numerical calculations of the next section also confirm this. So we conclude that for the cubic term too, the integrand in U behaves in the IR in an integrable way, giving rise there to a finite contribution to the energy.

Finally, on the basis of the heuristic analysis of this section, we conclude that, once the UV singularity is suitably subtracted, the energy integral (2.9) is finite.

6 Numerical evaluation

This section is devoted to the numerical evaluation of (2.9) using the results of the previous sections.

The first step is subtracting the UV singularity. We have already illustrated the method in section 4. It remains for us to do it in concrete by choosing a regulator. Since we are interested in enhancing as much as possible the numerical convergence we will choose the following families of f's

$$f(v) = \begin{cases} e^{-\frac{v}{a^2-v^2}} & 0 \le v \le a \\ 0 & v \ge a \end{cases} \tag{6.1}$$

where a is a positive number. It equals 1 at $v=0$ and 0 at $v=a$. Therefore, for terms in the integrand of (2.9) that are singular in $v=0$, we will split the integral in two parts: from 0 to a, and from a to ∞. The part from 0 to a will undergo the subtraction explained in section 4.

We have checked the regulator for several values of a, $a = 0.01, 0.5, 1, 2, 10, 100, \dots$. Changing a may affect the fourth digit of the results below, which is within the error bars of our calculations. Therefore in the sequel we will make a favorable choice for the accuracy of the calculations: $a=1$.

Let us proceed to evaluate the three terms in turn.

6.1 The cubic term

In eq. (3.15) we have to pick out the term $-\frac{3}{2\pi^2}\frac{1}{U^3}$ and treat it separately. Let us consider it first in the range $0 \le U \le 1$ and subtract the UV divergence. In the range $1 \le U < \infty$

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instead, according to the discussion of the last section, we will combine it with the most divergent of the remaining terms. This will render the corresponding integrals convergent.

1) Let us start with the subtraction for $-\frac{3}{2\pi^2}\frac{1}{U^3}$. Proceeding as explained above the subtracted integrand (after multiplying by $\frac{1}{6}U^2 g(U)$) is

$$-\frac{1}{4\pi^2}\left(\frac{g(U)}{U}-\frac{1}{\sqrt{\pi U}}\frac{e^{\frac{U}{U^2-4}}\left(16+8U-8U^2+2U^3+U^4\right)}{2U\left(U^2-4\right)^2}\right). \tag{6.2}$$

This, integrated from 0 to 1, gives -0.0619767.

2) Now, let us consider the term (5.12). Leaving out the first term we get $E_0^{(3)}(U)$. When multiplied by $\frac{1}{6}U^2 g(U)$ the result has integrable singularity at $U=0$, therefore it can be directly integrated from 0 to ∞. The result is 0.109048.

3) Next we have the term $S(U)$. When multiplied by $\frac{1}{6}U^2 g(U)$, it is non-integrable at ∞. Thus we split $-\frac{3}{2\pi^2}\frac{1}{U^3}$ as $-\frac{32}{\pi}\frac{5}{256\pi}\frac{1}{U^3}-\frac{32}{\pi}\frac{7}{256\pi}\frac{1}{U^3}$. We add the first addend to $S(U)$ in the range $1\leq U<\infty$, so as to kill the singularity at infinity. Then we multiply the result by $\frac{32}{\pi}\frac{1}{6}U^2 g(U)$. The overall result is integrable both in 0 and at ∞. Finally we integrate from 0 to 1 and from 1 to ∞ the corresponding unsubtracted and subtracted integrands. The result is -0.0190537.

4) Now we are left with the $SK(p,U)$ terms. This must be summed over p from 3 to infinity. After summation this term must be multiplied by $\frac{96}{\pi}\frac{1}{6}U^2 g(U)$. The result is integrable in the UV, but not in the IR. In fact we must subtract the other piece of $-\frac{3}{2\pi^2}\frac{1}{U^3}$, more precisely we should add $-\frac{32}{\pi}\frac{7}{256\pi}\frac{1}{U^3}$ to (3.19) in the range $1\leq U\leq\infty$. The best way to do it is to split the integration in the intervals (0,1) and (1,∞), and to subtract from (3.19) the term $\frac{1}{32\pi p(p+2)}\frac{1}{U^3}$.

At this point we proceed numerically with Mathematica, both for the summation over p and the integration over U. The result is -0.029204, with possible errors at the fourth digit.

According to the above, the cubic term's overall contribution to the energy is -0.00118596.

6.2 The quadratic term

1) Also in this case, looking at (3.6), (3.7), we treat separately the term $-\frac{9}{2\pi^2}\frac{1}{U^2}$. This term must be multiplied by $-\frac{1}{12}U^2\partial_U g(U)$. We get as a result

$$s(U)=-\frac{3e^U e^{-(\frac{1}{2}+U)\ln U}\Gamma(U)(-1+2U\ln U-2U\psi(U))}{32\pi^{\frac{5}{2}}}. \tag{6.3}$$

The resulting term is regular in the IR but singular in the UV. We make the same subtraction as above and obtain

$$\mathfrak{s}(U)=s(U)+\frac{3}{16\pi^2}\frac{1}{\sqrt{\pi U}}\frac{e^{\frac{U}{U^2-4}}\left(16+8U-8U^2+2U^3+U^4\right)}{2U\left(U^2-4\right)^2}. \tag{6.4}$$

It is easy to see that this is now integrable also in the UV. Integrating it between 0 and 1 and $s(U)$ between 1 and infinity one gets 0.0379954.

2) Next comes the integration of the term containing $E_0^{(2)}(U)$, see (3.8) above. This must be multiplied also by $-\frac{1}{3}U^2\partial_U g(U)$. The result is a function regular both at 0 and ∞. One can safely integrate in this range and get 0.0156618.

3) The next term is $R(U)$, (3.12). This behaves like $\frac{1}{U}$ for large U, see (5.3). So we subtract the corresponding divergent term, knowing already that it cancels against the analogous behaviour of the RK piece (see also below). Therefore we define

$$R'(U) = R(U) + \frac{3}{32\pi}\frac{1}{U}. \tag{6.5}$$

This has the right behaviour in the IR, but not the UV. For multiplying by $-\frac{1}{12}U^2\partial_U g(U)$ one gets an ultraviolet singularity. The way out is to limit the subtraction (6.5) to the range $(1,\infty)$. This can be done provided we do the same with the RK term, see below. Finally, in the range (0,1) we will integrate the term containing $R(U)$ without correction, since it can safely be integrated there. In the range $(1,\infty)$ we will integrate the one containing $R'(U)$. The overall result is -0.00392332.

4) There remains the $RK(p,U)$ piece, see (3.13). Again we have to subtract the singularity at ∞ (knowing that it cancels against the previous one). So we define

$$RK'(p,U) = RK(p,U) - 1/(8\pi(p^2-1)U). \tag{6.6}$$

However, when multiplying by $-\frac{4}{\pi}U^2\partial_U g(U)$, this introduces an UV singularity, so in accordance with the previous subtraction, this subtraction has to be limited to the range $(1,\infty)$. Consequently we have also to split the integration. Both integrals from 0 to 1 and from 1 to ∞ are well defined. The numerical evaluation gives 0.000235065.

The overall contribution of the quadratic term is therefore 0.049969.

6.3 Last contribution

The last one is easy to compute. The integrand is

$$\frac{1}{4\pi^2}U^2 g(U)\left(\frac{\partial_U g(U)}{g(U)}\right)^3. \tag{6.7}$$

This converges very rapidly in the IR. The only problem is with the usual singularity in the UV, where (6.7) behaves like $-\frac{1}{64\pi^2}\frac{1}{\sqrt{\pi}U^{\frac{3}{2}}}$. To this end we will add to (6.7) the function

$$\frac{1}{32\pi^2}\frac{1}{\sqrt{\pi U}}\frac{e^{\frac{U}{U^2-4}}\left(16+8U-8U^2+2U^3+U^4\right)}{2U\left(U^2-4\right)^2}. \tag{6.8}$$

The sum of the two is now well behaved and can be integrated from 0 to ∞. The result is 0.0206096.

6.4 Overall contribution

In conclusion the total finite contribution to the energy is 0.0693926.

$$E^{(s)}[\psi_u] \approx 0.0693926, \tag{6.9}$$

where the superscript $^{(s)}$ means that we have subtracted away the UV singularity. This has to be compared with the expected D24 brane tension

$$T_{D24} = \frac{1}{2\pi^2} \approx 0.0506606. \tag{6.10}$$

This theoretical value is justified in appendix C. The two values (6.9) and (6.10) differ by about 27%.

6.5 Error estimate

All the numerical calculations of this paper have been carried out with Mathematica. Mathematica can be very precise when performing numerical manipulations. However in our case there are two main sources of error, beside the subtraction of the infinite D25-brane factor and the precision of Mathematica. The first is the summation over p of $RK(p, U)$ and especially $SK(p, U)$. The precision of this summation is probably limited by the computer capacity and seem to affect up to the fourth digit in item 4 of section 6.1 and especially 6.2. Another source of errors is the presence of zeroes in the denominators of the expressions of $RK(p, U)$ and $SK(p, U)$. As we have explained above, they do not correspond to poles, because they are canceled by corresponding zeroes in the numerators; but Mathematica, when operating numerically, is not always able (or we have not been able to use it properly) to smooth out the corresponding functions. This again may affect the fourth digit of item 4 of section 6.1 and especially 6.2.

It is not easy to evaluate these sources of error. A certain number of trials suggest that a possible error of 1% in the final figure (6.9) does not seem to be unreasonable. We shall see that actually the numerical result (6.9) we have obtained is more precise than that.

However it is clear from now that $E[\psi_u]$ *is not the lump energy* we are looking for. This may be a bit disconcerting at first sight, because, after all, we have subtracted from the energy the UV singularity, which corresponds to tachyon vacuum energy. However one must reflect on the circumstance that this subtraction contains an element of arbitrarness. In fact the subtraction is purely ad hoc, it is a subtraction on the energy functional alone, not a subtraction made in the framework of a consistent scheme. In order to make sure that our result is physical we have to render it independent of the subtraction scheme. This is what we will do in the next three sections. ψ_u is a (UV subtracted) solution to the SFT equation of motion *on the perturbative vacuum*; what we need is the solution corresponding to ψ_u *on the tachyon condensation vacuum*. As we shall see, the gap between (6.9) and (6.10) is the right gap between the (subtracted) energy of ψ_u and the energy of the lump above the tachyon condensation vacuum.

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7 The regularized solution

The solution to the puzzle came to us in a rather indirect way, from an early development of our research, when we thought a regularization of our solution was necessary. The Schwinger representation we use in our determination of the energy (2.9) looks, at a superficial inspection, singular and in need of a regularization. In fact it is not, as we show in appendix D. Instead, what actually happens when we regularize the Schwinger parametrization in ψ_u is that we turn it into the tachyon vacuum solution. But this results into a happy occurrence because it suggests the way to an analytic determination of the lump energy.

On a general ground it would seem necessary to regularize expressions like $1/K, 1/(K+1), 1/(K+\phi)$. The reason is the following. K is a vector in an infinite dimensional vector space. Therefore an expression like $1/K$ does not even make sense without a suitable specification. As for $1/(K+1)$, we can understand it as the power series expansion

$$\frac{1}{K+1} = 1 - K + K^2 - \dots,$$

but does the series converge? if it converges, in what topology should the convergence be understood? We consider two ways to answer such questions.

One way is to view $1/K$ as the action of the operator $1/K_1^L$ on the identity state $|I\rangle$. The operator K_L^1 is a hermitean operator. Its spectrum is necessarily real. It has been studied in the series of papers [37–40] and its spectrum extends over the full real axis.[2] Therefore not even the operator expressions $(K_1^L)^{-1}, (K_1^L+1)^{-1}$ make sense. However operator theory teaches us that we can write perfectly sensible expressions $(K_1^L+\epsilon)^{-1}$ and $(K_1^L+1+\epsilon)^{-1}$, provided ϵ has a non-vanishing imaginary part. For instance, an expression like $(K_1^L+\epsilon)^{-1}$ is analytic in the ϵ plane outside the real axis (see [41, 42]). We will call it the operator regularization.

The case $1/(K+\phi)$ is discussed in appendix D. It is very plausible that ϕ may in general play the role of a regulator. In any case one would not see any a priori harm in representing $1/(K+\phi)$ as $(K+\phi+\epsilon)^{-1}$ in the $\epsilon \to 0$ limit.

Another way of giving a meaning to expressions like $1/K, 1/(K+1), 1/(K+\phi)$ is by means of a Schwinger representation (an extended version of the Hille-Phillips-Yosida theorem, see [41, 42]). For instance

$$\frac{1}{K+1} = \int_0^\infty dt\, e^{-t(K+1)} \tag{7.1}$$

is a well-known example of regular representation (although it is not known if it is regular for all correlators). We will therefore give a meaning to such expressions as $1/K, 1/(K+\phi)$ by means of the regularized Schwinger representations

$$\frac{1}{K+\epsilon} = \int_0^\infty dt\, e^{-t(K+\epsilon)}, \qquad \frac{1}{K+\phi+\epsilon} = \int_0^\infty dt\, e^{-t(K+\phi+\epsilon)} \tag{7.2}$$

[2] In [38–40], in the ghost case, additional points of the spectrum were found outside the real axis, but only because the matrices G, A, B, C, D used to represent K_1^L are not hermitean.

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in the limit $\epsilon \to 0$. We remark that the Schwinger regularization usually converges for real $\epsilon > 0$, while the operator regularization requires $\Im(\epsilon) \neq 0$.

So far our argument has been classical (in the sense of classical operator theory) and one would not expect any harm from such regularizations, but in fact they are not innocuous in the quantum theory. From now on, in this section and the next, we will use the above regularized Schwinger representations and ϵ will be our regulator. We will show that the introduction of this innocent looking ϵ regulator actually changes the nature of our solution. Once regularized, it will represent the tachyon condensation vacuum solution and ϵ will turn out to be a gauge parameter.

7.1 Application to the lump solution

We proceed now to regularize our lump solution. At every step of our equations in section 2.2 of [1] or in the introduction of this paper, we replace ϕ with $\phi + \epsilon$, where ϵ is a small number we will take eventually to 0, and use the above Schwinger representations wherever we find inverted vectors.

Our lump solution becomes

$$\psi_\phi = c(\phi + \epsilon) - \frac{1}{K + \phi + \epsilon}(\phi + \epsilon - \delta\phi)Bc\partial c. \tag{7.3}$$

This is certainly a solution to the equation of motion since it is simply obtained by replacing ϕ with $\phi + \epsilon$ and it is certainly regular. Moreover

$$U_\phi = 1 - \frac{1}{K + \phi + \epsilon}(\phi + \epsilon)Bc, \qquad U_\phi^{-1} = 1 + \frac{1}{K}(\phi + \epsilon)Bc. \tag{7.4}$$

Therefore U_ϕ^{-1} remains singular, and the solution is non-gauge for any ϵ.

In proving that (7.3) is a solution we need

$$\frac{1}{K + \phi + \epsilon}(K + \phi + \epsilon) = 1. \tag{7.5}$$

This is certainly correct for any $\epsilon \neq 0$. Therefore we can assume, by continuity, that it is true also for $\epsilon = 0$. This can be confirmed in a weak sense as follows

$$\begin{aligned} &Tr\left(\frac{1}{K + \phi + \epsilon}(K + \phi + \epsilon)\right) = \int_0^\infty dt\, Tr\left(e^{-t(K+\phi+\epsilon)}(K + \phi + \epsilon)\right) = \\ &-\int_0^\infty dt\, \frac{\partial}{\partial t} Tr\left(e^{-t(K+\phi+\epsilon)}\right) = g(0) - Tr\left(e^{-t(K+\phi)}\right) e^{-t\epsilon}\Big|_{t=\infty} \\ &= g(0) - \lim_{t\to\infty} g(tu)e^{-t\epsilon}. \end{aligned} \tag{7.6}$$

The second term in the r.h.s. of (7.6) vanishes as long as $\Re\epsilon > 0$ (as we shall always assume). $g(0)$ is the expected response corresponding to 1 in the r.h.s. of (7.5). In fact

$$g(0) = \lim_{u\to 0} \frac{1}{2\sqrt{\pi u}} = \delta(0) = \frac{V}{2\pi} \tag{7.7}$$

The importance of this result should however not be overestimated because of the subtraction necessary in order to obtain a finite result on the r.h.s. , see in this regard appendix D.

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Remark 1. In the above integral (7.6) we are not allowed to exchange the $\epsilon \to 0$ limit with integration, because the function $g(tu)$ *is not integrable for large* t.

Let us see next the other conditions mentioned in the introduction. In the solution we find the expression

$$\frac{1}{K+\phi_u+\epsilon}(\phi_u+\epsilon-\delta\phi_u).$$

We have to check that it is regular. Again it is certainly well-defined for any ϵ with $\Im\epsilon \neq 0$. Using a Schwinger representation we choose $\Re\epsilon > 0$. The one-point correlator is

$$\begin{aligned}
&\langle\frac{1}{K+\phi_u+\epsilon}(\phi_u+\epsilon-\delta\phi_u)\rangle = \langle\frac{1}{K+\phi_u+\epsilon}(\epsilon+u\partial_u\phi_u)\rangle \\
&= \epsilon\int_0^\infty dt\, e^{-\epsilon t}\langle e^{-t(K+\phi_u)}\rangle - \int_0^\infty dt\, e^{-\epsilon t}\frac{u}{t}\partial_u\langle e^{-t(K+\phi_u)}\rangle \\
&= \epsilon\int_0^\infty \frac{dx}{u}\, g(x)e^{-\epsilon\frac{x}{u}} - \int_0^\infty dx\,\partial_x g(x)\, e^{-\epsilon\frac{x}{u}} \\
&= -\int_0^\infty dx\,\partial_x\left(g(x)e^{-\epsilon\frac{x}{u}}\right) = g(0) - \lim_{x\to\infty} g(x)e^{-\epsilon\frac{x}{u}},
\end{aligned} \tag{7.8}$$

where $x = tu$. As long as ϵ, u are kept finite, the above limit vanishes and we get

$$\lim_{\epsilon\to 0}\langle\frac{1}{K+\phi_u+\epsilon}(\phi_u+\epsilon-\delta\phi_u)\rangle = g(0). \tag{7.9}$$

If we take the limit $\epsilon \to 0$ first, we get instead

$$\langle\frac{1}{K+\phi_u}(\phi_u-\delta\phi_u)\rangle = g(0) - g(\infty). \tag{7.10}$$

Remark 2. This is another example in which we cannot exchange integration with $\epsilon \to 0$ limit, the reason being the usual one: $g(x)$ is not integrable for large x. Therefore the correct regularized result is given by (7.9). Such discontinuity of the $\epsilon \to 0$ limit will play a fundamental role in the sequel.

Let us consider next $\langle\frac{1}{K+\phi_u+\epsilon}\rangle$ which is expected to be singular. We have

$$\langle\frac{1}{K+\phi_u+\epsilon}\rangle = \int_0^\infty dt\, e^{-\epsilon t}\langle e^{-t(K+\phi_u)}\rangle = \int_0^\infty \frac{dx}{u} g(x)e^{-\epsilon\frac{x}{u}}. \tag{7.11}$$

The crucial region is at $x \to \infty$. Since $g(\infty) = \textit{finite}$ the behaviour of this integral is qualitatively similar to

$$\sim \frac{1}{\epsilon}e^{-\epsilon\frac{x}{u}}\Big|_M^\infty \sim \frac{e^{-\frac{\epsilon M}{u}}}{\epsilon} \tag{7.12}$$

for M a large number. The inverse of ϵ present in this expression makes the integral (7.11) divergent, as it is easy to verify also numerically. This tells us that homotopy operator corresponding to the regularized solution (see below) is well-defined, while if we set $\epsilon = 0$ it becomes singular.

As the above examples show, the $\epsilon \to 0$ limit, in general, is not continuous. This is true in particular for the energy, as we shall see in the next section.

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7.2 Other regularizations

The regularization we have considered so far in this paper (named ϵ-regularization) is far from unique. It consists in adding to ϕ_u the operator ϵI. However we are free to add suitable perturbing operators instead of the identity operator I and generate families of solutions. In particular we will consider in the following replacing $\phi_u(s)$ with $\phi_u(s) + \epsilon f(s) I$, where $f(s)$ is some function of s. It is easy to prove that these are all solutions to the equation of motion. Some of them are particularly important and simple to deal with, they are defined by the choices

$$f_1(s) = \theta(s - M) \tag{7.13}$$

$$f_2(s) = \theta(M - s) \tag{7.14}$$

where M is some finite number. The first choice gives rise to a regularization that dumps the IR, just as the ϵ regularization does. Therefore such a family of solutions (depending on M and ϵ) will be gauge equivalent to the tachyon vacuum solution. The second choice does not affect the IR and gives rise to a family of solutions which are gauge equivalent to the lump. These different regularizations will be discussed in a separate paper, [47].

8 Regulated energy

In this section we calculate the energy of the regularized solution. Our aim is to study the $\epsilon \to 0$ limit and verify whether it is continuous or not.

The regulated solution is

$$\psi_u^\epsilon = \lim_{\epsilon\to 0}\left(c(\phi_u + \epsilon) - \frac{1}{K + \phi_u + \epsilon}(\phi_u + \epsilon - \delta\phi_u)Bc\partial c\right). \tag{8.1}$$

The energy is proportional to

$$\begin{aligned}\langle \psi_u\psi_u\psi_u\rangle &= -\lim_{\epsilon\to 0}\left\langle\left(\frac{1}{K+\phi_u+\epsilon}(\phi_u+\epsilon+2u)BcKc\right)^3\right\rangle \\ &= -\lim_{\epsilon\to 0}\int_0^\infty dt_1dt_2dt_3\mathcal{E}_0(t_1,t_2,t_3)e^{-\epsilon T}\Big\langle(\phi_u(t_1+t_2)+\epsilon+2u) \\ &\qquad \times(\phi_u(t_1)+\epsilon+2u)(\phi_u(0)+\epsilon+2u)e^{-\int_0^T ds\phi_u(s)}\Big\rangle_{C_T},\end{aligned} \tag{8.2}$$

where $T = t_1 + t_2 + t_3$. We map the matter parts to the unit disc:

$$\begin{aligned}\langle \psi_u\psi_u\psi_u\rangle = &-\lim_{\epsilon\to 0}\int_0^\infty dt_1dt_2dt_3\mathcal{E}_0(t_1,t_2,t_3)e^{-2uT\left(\ln(\frac{uT}{2\pi})+A+\frac{\epsilon}{2u}\right)} \\ &\times u^3\Big\langle\left(X^2(\theta_{t_1+t_2})+2\left(\ln\left(\frac{uT}{2\pi}\right)+A+1+\frac{\epsilon}{2u}\right)\right) \\ &\times\left(X^2(\theta_{t_1})+2\left(\ln\left(\frac{uT}{2\pi}\right)+A+1+\frac{\epsilon}{2u}\right)\right) \\ &\times\left(X^2(0)+2\left(\ln\left(\frac{uT}{2\pi}\right)+A+1+\frac{\epsilon}{2u}\right)\right)e^{-\int_0^{2\pi}d\theta\frac{uT}{2\pi}X^2(\theta)}\Big\rangle_{\rm Disk}.\end{aligned} \tag{8.3}$$

Using appendix D of I and setting $A = \gamma - 1 + \ln 4\pi$, we obtain

$$\langle \psi_u \psi_u \psi_u \rangle = -\lim_{\epsilon \to 0} \int_0^\infty dt_1 dt_2 dt_3 \mathcal{E}_0(t_1, t_2, t_3) u^3 e^{-\epsilon T} g(uT) \tag{8.4}$$
$$\cdot \Bigg\{ 8 \left(\frac{h_{2uT}}{2} + \ln(2uT) + \gamma + \frac{\epsilon}{2u} \right)^3 + 8 G_{2uT}\left(\frac{2\pi t_1}{T}\right) G_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right) G_{2uT}\left(\frac{2\pi t_2}{T}\right)$$
$$+ 4 \left(\frac{h_{2uT}}{2} + \ln(2uT) + \gamma + \frac{\epsilon}{2u} \right) \left(G^2_{2uT}\left(\frac{2\pi t_1}{T}\right) + G^2_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right) + G^2_{2uT}\left(\frac{2\pi t_2}{T}\right) \right) \Bigg\}.$$

This can also be written as

$$\langle \psi_u \psi_u \psi_u \rangle = -\lim_{\epsilon \to 0} \int_0^\infty dt_1 dt_2 dt_3 \mathcal{E}_0(t_1, t_2, t_3) u^3 e^{-\epsilon T} g(uT) \Bigg\{ \left(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)} \right)^3 \tag{8.5}$$
$$+ 2 \left(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)} \right) \left(G^2_{2uT}\left(\frac{2\pi t_1}{T}\right) + G^2_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right) + G^2_{2uT}\left(\frac{2\pi t_2}{T}\right) \right)$$
$$+ 8 G_{2uT}\left(\frac{2\pi t_1}{T}\right) G_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right) G_{2uT}\left(\frac{2\pi t_2}{T}\right) \Bigg\}.$$

Let us make again a change of variables $(t_1, t_2, t_3) \to (T, x, y)$, where

$$x = \tfrac{t_1}{T}, \qquad y = \tfrac{t_2}{T}.$$

Then the matter part of the energy can be written as

$$u^3 e^{-\epsilon T} F_\epsilon(uT, x, y),$$

where

$$F_\epsilon(uT, x, y) = g(uT) \Bigg\{ \left(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)} \right)^3 \tag{8.6}$$
$$+ 8 G_{2uT}(2\pi x) G_{2uT}(2\pi(x+y)) G_{2uT}(2\pi y)$$
$$+ 2 \left(\frac{\epsilon}{u} - \frac{\partial_{uT} g(uT)}{g(uT)} \right) \Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x+y)) + G^2_{2uT}(2\pi y) \Big) \Bigg\}.$$

The ghost correlator has been given in the introduction. Making an additional change of coordinate $s = 2uT, x \to y \to 1-y$, yields finally

$$E_0[\psi_u] = \frac{1}{6} \lim_{\epsilon \to 0} \int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\frac{\epsilon s}{2u}} \mathcal{E}(1-y, x)\, F_\epsilon(s/2, 1-y, x) \tag{8.7}$$

with

$$F_\epsilon(s/2, 1-y, x) = \mathrm{g}(s) \Bigg\{ \left(\frac{\epsilon}{2u} - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)} \right)^3 \tag{8.8}$$
$$+ G_s(2\pi x) G_s(2\pi(x-y)) G_s(2\pi y)$$
$$+ \frac{1}{2} \left(\frac{\epsilon}{2u} - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)} \right) \Big(G^2_s(2\pi x) + G^2_s(2\pi(x-y)) + G^2_s(2\pi y) \Big) \Bigg\}.$$

where we have set $\mathrm{g}(s) \equiv g(s/2)$.

8.1 The energy in the limit $\epsilon \to 0$

Our purpose here is to study the energy functional $E_\epsilon[\psi_u^\epsilon]$. We notice that (8.7), (8.8) for generic ϵ can be obtained directly from (2.9) with the following exchanges: $U \to s$, $g(uT) \to e^{-\frac{\epsilon s}{2u}}\mathrm{g}(s) \equiv \tilde{\mathrm{g}}_\epsilon(s,u)$. Summarizing, we can write

$$E_\epsilon[\psi_u^\epsilon] = \frac{1}{6}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, \mathcal{E}(x,y)\, \tilde{\mathrm{g}}_\epsilon(s,u)\Bigg\{\Bigg(-\frac{\partial_s\tilde{\mathrm{g}}_\epsilon(s,u)}{\tilde{\mathrm{g}}_\epsilon(s,u)}\Bigg)^3 \\ +G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ +\frac{1}{2}\Bigg(-\frac{\partial_s\tilde{\mathrm{g}}_\epsilon(s,u)}{\tilde{\mathrm{g}}_\epsilon(s,u)}\Bigg)\Big(G_s^2(2\pi x)+G_s^2(2\pi(x-y))+G_s^2(2\pi y)\Big)\Bigg\}. \tag{8.9}$$

We are of course interested in the limit

$$E_0[\psi_u] = \lim_{\epsilon\to 0} E_\epsilon[\psi_u^\epsilon] \tag{8.10}$$

but we will see that in fact the energy functional $E_\epsilon[\psi_u^\epsilon]$ does not depend on ϵ.

The dependence on ϵ is continuous in the integrand, therefore a discontinuity in the limit $\epsilon \to 0$ may come only from divergent integrals that multiply ϵ factors. Now, looking at (8.8), we see that we have two types of terms. The first type is nothing but (2.9), with the only difference that the integrand of $d(2uT)$ is multiplied by $e^{-\frac{\epsilon s}{2u}}$. In the previous sections we have shown that, setting formally $\epsilon = 0$ everywhere in (8.7) and (8.8), or in (8.9), and subtracting the UV singularity, we get a finite integral, i.e. in particular the integrand has integrable behaviour for $s \to \infty$. Therefore this first type of term is certainly continuous in the limit $\epsilon \to 0$. However with the second type of terms the story is different. The latter are the terms linear, quadratic or cubic in $\frac{\epsilon}{u}$ in (8.8) (for convenience we will call them ϵ-terms). The factors that multiply such terms in the integrand may be more singular than the ones considered in the previous section. They may give rise to divergent integrals, were it not for the overall factor $e^{-\frac{\epsilon s}{2u}}$. In the $\epsilon \to 0$ limit these terms generate a (finite) discontinuity through a mechanism we shall explain in due course.

To proceed to a detailed proof we will split the s integration into three intervals: $0-m$, $m-M$ and $M-\infty$, where m and M are finite numbers, small (m) and large (M) enough for our purposes. It is obvious that, since possible singularities of the s-integral may arise only at $s=0$ or $s=\infty$, the integral between m and M is well defined and continuously dependent on ϵ, so for this part we can take the limit $\epsilon \to 0$ either before or after integration, obtaining the same result.

In the sequel we will consider the effect of the $\epsilon \to 0$ in the UV and in the IR, the only two regions where a singularity of the mentioned type can arise.

8.1.1 The ϵ-terms and the $\epsilon \to 0$ limit in the UV

Here we wish to check that the ϵ-terms do not affect the singularity in the UV, so that the subtraction in section 4 remains unaltered. It is enough to limit ourselves to the integral in the interval $(0,m)$, where m is a small enough number. Let us start from the term proportional to $\mathrm{g}(s)\left(\frac{\epsilon}{2u}\right)^3 e^{-\frac{\epsilon s}{2u}}$, coming from the first line of (8.8). To simplify the notation we will denote $\frac{\epsilon}{2u}$ simply by η. Since, near 0, $\mathrm{g}(s) \approx 1/\sqrt{s}$, this first term gives rise to the integral (for $s \approx 0$),

$$\int_0^m dss^{\frac{3}{2}}\eta^3 e^{-\eta s} \sim \left(-\eta\, e^{-\eta s}\frac{\sqrt{s}(3+2s\eta)}{2} + \sqrt{\eta}\frac{3\sqrt{\pi}Erf\left(\sqrt{\eta s}\right)}{4}\right)\Big|_0^m. \tag{8.11}$$

Since the error function $Erf(x) \approx x$ for small x, it is evident that this expression vanishes both at $s = 0$ and in the limit $\epsilon \to 0$.

The next term to be considered is $\eta^2 \partial_s \mathrm{g}(s)$, which leads to the integral

$$\int_0^m dss^{\frac{1}{2}}\eta^2 e^{-\eta s} \sim \left(-\eta\, e^{-\eta s}\sqrt{s} + \sqrt{\eta}\frac{\sqrt{\pi}Erf\left(\eta s\right)}{2}\right)\Big|_0^m \tag{8.12}$$

which again vanishes in the $\epsilon \to 0$ limit.

The following term leads to the integral

$$\int_0^m ds\frac{1}{\sqrt{s}}\,\eta\, e^{-\eta s} \sim \left(\sqrt{\pi\eta}Erf\left(\sqrt{\eta s}\right)\right)\Big|_0^m \tag{8.13}$$

which vanishes as well in the $\epsilon \to 0$ limit.

Finally the term linear in ϵ coming from the last two lines of (8.8) gives rise to an UV behaviour $\sim s^{-\frac{1}{2}}\eta$. Therefore the relevant UV integral is similar to (8.13) and we come to the same conclusion as above.

In conclusion the ϵ-terms do not affect the UV behaviour of the energy integral, and in the $\epsilon \to 0$ limit they yield evanescent contributions.

Finally let us consider what remains after discarding the ϵ-terms. From section 4 the behaviour for $\epsilon \approx 0$ is the following

$$\int_0^m ds\,\frac{e^{-\eta s}}{s^{\frac{3}{2}}} = \left(-2\frac{e^{-\eta s}}{\sqrt{s}} - 2\sqrt{\pi\eta}Erf\left(\sqrt{\eta s}\right)\right)\Big|_0^m. \tag{8.14}$$

In the limit $\epsilon \to 0$ the second term vanishes. The first term gives the expected UV singularity we have subtracted away in section 4.

8.1.2 The ϵ-terms and the $\epsilon \to 0$ limit in the IR

There is a chance, with ϵ-terms, that the corresponding integrals diverge or produce negative powers of ϵ, leading to finite or divergent contributions in the limit $\epsilon \to 0$.

Let us start again from the term proportional to $\mathrm{g}(s)\eta^3 e^{-\eta s}$, coming from the first line of (8.8). The integration in x, y gives a finite number. $\mathrm{g}(s)$ tends to a constant for $s \to \infty$. To appreciate qualitatively the problem we replace $\mathrm{g}(s)$ by a constant and integrate between M and infinity, M is chosen large enough so that $\mathrm{g}(s) = const$ is a good approximation. The integral is proportional to

$$\int_M^\infty ds\, s^2\eta^3 e^{-\eta s} \sim -e^{-\eta s}\left(2 + 2\eta s + \eta^2 s^2\right)\Big|_M^\infty = e^{-\eta M}\left(2 + 2\eta M + \eta^2 M^2\right) \tag{8.15}$$

which does not vanish in the limit $\epsilon \to 0$.

Let us notice that, if we consider an additional term in the asymptotic expression for $\mathrm{g}(s)$, say $\mathrm{g}(s) = a + \frac{b}{s} + \cdots$, the additional $\frac{1}{s}$ term contributes to the r.h.s. of (8.15) an additional term $\sim \eta\, e^{-\eta M}(1+\eta M)$, which vanishes in the $\epsilon \to 0$ limit. The more so for the next approximants. This is always the case in the following discussion, therefore considering the asymptotically dominant term will be enough for our purposes.

Let us consider next the term proportional to $\eta^2 \partial_s \mathrm{g}(s)$. For large s we have $\partial_s \mathrm{g}(s) \sim \frac{1}{s^2}$. Therefore the integral to be considered is

$$\int_M^\infty ds \eta^2 e^{-\eta s} \sim -e^{-\eta s}\eta\Big|_M^\infty = e^{-\frac{\epsilon M}{u}}\frac{\epsilon}{u} \tag{8.16}$$

which vanishes in the limit $\epsilon \to 0$.

The linear term in ϵ coming from the first line of (8.8), that is the term proportional to $\eta\frac{(\partial_s \mathrm{g}(s))^2}{\mathrm{g}(s)}$, leads to a contribution that can be qualitatively represented by the integral

$$\begin{aligned}&\int_M^\infty ds\, \frac{1}{s^2}\eta e^{-\eta s} \sim -\eta\left(\frac{e^{-\eta s}}{s} - \eta Ei\left(-\eta s\right)\right)\Big|_M^\infty \\ &= \eta e^{-\eta M} - \eta^2 Ei\left(-\eta M\right)\end{aligned} \tag{8.17}$$

which vanishes in the limit $\epsilon \to 0$, because the exponential integral function $Ei(-x)$ behaves like $\log x$ for small x.

Finally let us consider the term linear in ϵ coming from the last two lines of (8.8), i.e.

$$\mathrm{g}(s)\eta\left(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\right). \tag{8.18}$$

The integration of x, y of the G_s^2 terms in brackets gives a contribution behaving at infinity as $1/s^2$ (see section 5). Therefore the relevant s contribution for large s is

$$\int_M^\infty ds \eta e^{-\eta s} \sim -e^{-\eta s}\Big|_M^\infty = e^{-\eta M}$$

which is nonvanishing in the limit $\epsilon \to 0$.

Therefore we have found two nontrivial ϵ-terms, the first and the last ones above. Let us call them α and β, respectively. They do not vanish in the limit $\epsilon \to 0$, thus they may survive this limit and represent a finite difference between taking $\epsilon \to 0$ before and after the s-integration. It is therefore of utmost importance to see whether the overall contributions of these two terms survives. This turns out to be the case.

From section 5 one can check that the precise form of the first term in question is

$$-\frac{1}{4\pi^2}\lim_{\epsilon\to 0}\int_0^\infty ds\, s^2 \mathrm{g}(s)\eta^3 e^{-\eta s}. \tag{8.19}$$

If one knows the asymptotic expansion of the integrand for large s, it is very easy to extract the exact $\epsilon \to 0$ result of the integral. The asymptotic expansion of $\mathrm{g}(s)$ is $\mathrm{g}(s) \approx 1 + \frac{1}{24s} + \frac{1}{1152s^2} + \cdots$. Integrating term by term from M to ∞, the dominant one gives

$$-\frac{1}{4\pi^2}e^{-\eta M}(2 + 2M\eta + M^2\eta^2) \tag{8.20}$$

which, in the $\epsilon \to 0$ limit, yields $-\frac{1}{2\pi^2}$. The next term gives $\sim e^{-M\eta}(\eta(1+M\eta))$, which vanishes in the $\epsilon \to 0$ limit, and so on. So the net result of the integral (8.19) in the $\epsilon \to 0$ limit is $-\alpha$, where

$$\alpha \equiv \frac{1}{2\pi^2}. \tag{8.21}$$

For the β term (the one corresponding to (8.18)) we have

$$-\beta = \frac{1}{12} \lim_{\epsilon \to 0} \int_0^\infty ds\, s^2 \mathrm{g}(s) \eta e^{-\eta s} \left(-\frac{a}{s^2} + \cdots\right), \tag{8.22}$$

where ellipses denote terms that contribute vanishing contributions in the $\epsilon \to 0$ limit and a is the (overall) coefficient of the inverse quadratic term in (3.7). The problem is to compute the latter. With reference to the enumeration in section 6.2, the term 1) has the asymptotic expansion

$$\sim e^{-\eta s} \left(-\frac{3\eta}{8\pi^2} - \frac{\eta}{64 s \pi^2} - \cdots\right). \tag{8.23}$$

Integrating from M to ∞ and taking the $\epsilon \to 0$ limit, this gives $-\frac{3}{8\pi^2}$. Proceeding in the same way, term 2) of section 6.2 gives $\frac{3}{4\pi^2}$ and term 3) yields $\frac{1}{4\pi^2}$. So altogether we have $\frac{5}{8\pi^2}$ for the three terms contributing to (8.22) considered so far.

It remains term 4) of section 6.2. This corresponds to the contribution of $RK(p,U) - \frac{1}{8\pi(p^2-1)}$. One must explicitly sum over p in order to know the asymptotic expansion in U. This has not been possible so far analytically. However Mathematica can compute the coefficient of $1/U^2$ in the asymptotic expansion for large U to a remarkable accuracy. The coefficient turns out to be -0.064317, with an uncertainty only at the fifth digit. Within the same uncertainty this corresponds to the analytic value $-\frac{1}{4\pi}(\gamma + 1/3 \log 2)$. We therefore set

$$\beta = -\frac{5}{8\pi^2} + \frac{1}{\pi^2}\left(\gamma + \frac{1}{3}\log 2\right). \tag{8.24}$$

So the overall contribution of the ϵ-terms in the $\epsilon \to 0$ limit is

$$-\alpha - \beta = -\frac{1}{2\pi^2} + \frac{5}{8\pi^2} - \frac{1}{\pi^2}\left(\gamma + \frac{1}{3}\log 2\right) \approx -0.0692292, \tag{8.25}$$

which is accurate up to the fourth digit.

Let us consider now what remains apart from the ϵ-terms. The integrand takes the form

$$\int_0^\infty ds\, F(s)\, e^{-\eta s}, \tag{8.26}$$

where $F(s)$ represents the integrand when $\epsilon = 0$, i.e. the total integrand analyzed in section 5. We have already argued that the integration over s and the limit $\epsilon \to 0$ can be safely exchanged, which yields the already found value of 0.0693926.

Concluding we have

$$E_0^{(s)}[\psi_u] = \lim_{\epsilon \to 0} E_\epsilon^{(s)}[\psi_u^\epsilon] \approx 0.000163, \tag{8.27}$$

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$\eta:$	2	1	0.1	0.01	0.001
$E_\epsilon^{(s)}[\psi_u^\epsilon]:$	4×10^{-6}	8×10^{-6}	0.000603	0.001832	0.007360

Table 3. Samples of $E_\epsilon^{(s)}[\psi_u^\epsilon]$.

where again the superscript $^{(s)}$ means that the UV singularity has been subtracted away.

This teaches us two lessons. First, that the regularized solution is the tachyon vacuum solution. This is true not only in the limit $\epsilon \to 0$ but also for nonvanishing ϵ and, consequently, ϵ plays the role of a gauge parameter. This conclusion can be reached only via numerics, for the calculation with $\epsilon \neq 0$ cannot be analytical, and thus the result is less precise. But it is nevertheless significant to see the values $E_\epsilon^{(s)}[\psi_u^\epsilon]$ for various values of η in table 3. These values are close to 0, but with an accuracy that worsens for decreasing η. It is worth spending a few words on the numerical origin of this fact. For instance the quadratic ϵ-terms are characterized by an integrand consisting of two factors: the first is a sort of Gaussian, whose maximum increases in value and position like the inverse of η; on the contrary the other factor decreases, with the overall result that the integral in s varies slightly with η. This explains why the ϵ-terms, which are negative, kill completely the overall positive contribution coming from the other terms. The trouble with this scheme is that for smaller η's the integral must be evaluated over larger and larger intervals of s in order to approximate its true value, and this clashes inevitably with the computing capacities of Mathematica. This explains the worsening performance for decreasing η.

The second lesson we learn is that, since at this point we can assume the true value of $E_\epsilon^{(s)}[\psi_u]$ to be 0, and since the value (8.25) is much more accurate than (6.9), we can take for the latter the more reliable value

$$E^{(s)}[\psi_u] \approx 0.0692292 \tag{8.28}$$

which we can consider at this point to be exact (even though this will not play any role in the determination of the lump energy). It differs from (6.9) by 2 per mil. Therefore, after all, our numerical evaluation in section 6 was not so bad. Stated differently, the whole procedure of this section is nothing but a more reliable way to compute the energy functional (2.9).

We have already remarked that (8.28) differs from the theoretical value (6.10) of the lump energy by 27%. This is not the expected lump energy. But now we have everything at hand to explain the puzzle.

9 The lump and its energy

In the previous sections we have found various solutions to the equation of motion $Q\psi + \psi\psi = 0$ at the perturbative vacuum. One is ψ_u with UV-subtracted energy (8.28), the others are the ψ_ϵ's with generic ϵ and vanishing UV-subtracted energy. Using these we can construct a solution to the EOM *at the tachyon condensation vacuum*.

The equation of motion at the tachyon vacuum is

$$\mathcal{Q}\Phi + \Phi\Phi = 0, \quad \text{where} \quad \mathcal{Q}\Phi = Q\Phi + \psi_\epsilon\Phi + \Phi\psi_\epsilon. \tag{9.1}$$

We can easily show that

$$\Phi_0 = \psi_u - \psi_\epsilon \tag{9.2}$$

is a solution to (9.1). The action at the tachyon vacuum is

$$-\frac{1}{2}\langle \mathcal{Q}\Phi, \Phi\rangle - \frac{1}{3}\langle \Phi, \Phi\Phi\rangle. \tag{9.3}$$

Thus the energy is

$$E[\Phi_0] = -\frac{1}{6}\langle \Phi_0, \Phi_0\Phi_0\rangle = -\frac{1}{6}\big[\langle \psi_u, \psi_u\psi_u\rangle - \langle \psi_\epsilon, \psi_\epsilon\psi_\epsilon\rangle - 3\langle \psi_\epsilon, \psi_u\psi_u\rangle + 3\langle \psi_u, \psi_\epsilon\psi_\epsilon\rangle\big]. \tag{9.4}$$

Eq. (9.2) is the lump solution at the tachyon vacuum, therefore, this energy must be the energy of the lump.

We have already shown that $-\frac{1}{6}\langle \psi_u, \psi_u\psi_u\rangle^{(s)} = \alpha + \beta$ and that $\langle \psi_\epsilon, \psi_\epsilon\psi_\epsilon\rangle^{(s)} = 0$, after subtracting the UV singularity. It remains for us to compute the two remaining terms, which we will do in the next subsection. But, before, let us remark one important aspect of (9.4). The UV subtractions are the same in all terms, therefore they neatly cancel out.

9.1 Two more terms

The two terms $\langle \psi_\epsilon, \psi_u\psi_u\rangle$ and $\langle \psi_u, \psi_\epsilon\psi_\epsilon\rangle$ can be calculated in the same way as the other two, and we limit ourselves to writing down the final result:

$$\begin{aligned}\langle \psi_\epsilon, \psi_u\psi_u\rangle = &-\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s y}\, \mathrm{g}(s) \\ &\cdot\Bigg\{\left(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)\left(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^2 + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ &+\frac{1}{2}\left(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)G_s^2(2\pi x) + \frac{1}{2}\left(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)\Big(G_s^2(2\pi y) + G_s^2(2\pi(x-y))\Big)\Bigg\}.\end{aligned} \tag{9.5}$$

and

$$\begin{aligned}\langle \psi_u, \psi_\epsilon\psi_\epsilon\rangle = &-\int_0^\infty ds\, s^2 \int_0^1 dy \int_0^y dx\, e^{-\eta s}\mathcal{E}(1-y,x)\, e^{\eta s x}\, \mathrm{g}(s) \\ &\cdot\Bigg\{\left(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)^2\left(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right) + G_s(2\pi x)G_s(2\pi(x-y))G_s(2\pi y) \\ &+\frac{1}{2}\left(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)G_s^2(2\pi(x-y)) + \frac{1}{2}\left(\eta - \frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\right)\Big(G_s^2(2\pi x) + G_s^2(2\pi y)\Big)\Bigg\}.\end{aligned} \tag{9.6}$$

Although we believe that the results below holds for any value of ϵ, the calculation for generic ϵ is beyond our present means, therefore from now on in this section we will condider only the $\epsilon \to 0$ limit. A bonus of this limit is that it will give us analytic results. In this limit the factors $e^{\eta s y}$ and $e^{\eta s x}$, present in (9.5), (9.6), respectively, are irrelevant. In fact the integration over y, without this factor, is finite. Therefore we know from above that the integration over y with $e^{\eta s y}$ inserted back at its place is continuous in ϵ for $\epsilon \to 0$ (one can check that the subsequent integration over s does not lead to any complications). Therefore we can ignore these factors in the two integrals above.

The integrals are of the same type as those analyzed in the previous section. Of course they will have both the contribution that comes from setting $\epsilon = 0$, which is proportional to $\alpha + \beta$, like for $\langle \psi_\epsilon, \psi_\epsilon \psi_\epsilon \rangle$. But there are important differences as far as the ϵ terms are concerned. First of all we remark that, in both integrals, the first term in curly brackets does not contain the cubic term in ϵ. Therefore, according to the analysis in the previous section, the α contribution will not be present in either term. On the contrary the β contribution, which comes from the last line in both, will. To evaluate it there is no need of new explicit computations. Upon integrating over x, y one can easily realize that the three terms proportional to $G_s^2(2\pi x)$, $G_s^2(2\pi y)$ and $G_s^2(2\pi(x-y))$, give rise to the same contribution. So, when we come to ϵ-terms, each of them will contribute $\frac{1}{3}$ of the β contribution already calculated in the previous section. Summarizing: after subtracting the UV singularity, we will have

$$\langle \psi_\epsilon, \psi_u \psi_u \rangle^{(s)} = -\alpha - \beta + \frac{1}{3}\beta, \tag{9.7}$$

$$\langle \psi_u, \psi_\epsilon \psi_\epsilon \rangle^{(s)} = -\alpha - \beta + \frac{2}{3}\beta. \tag{9.8}$$

9.2 Final result

Let us now collect all the results in (9.4). The lump energy above the tachyon vacuum is

$$E[\Phi_0] = \alpha + \beta + 0 + 3\Big(-\alpha - \beta + \frac{1}{3}\beta\Big) - 3\Big(-\alpha - \beta + \frac{2}{3}\beta\Big) = \alpha = \frac{1}{2\pi^2}. \tag{9.9}$$

This coincides with the expected theoretical value (6.10).

As one can see there is no need to know the value of β, which, anyhow, we have explicitly computed in the previous section. We stress that the exact value (9.9) was computed analytically in the previous section, see eqs. (8.19), (8.20), and it is determined by the asymptotics of g(s). Moreover we recall another fundamental aspect of (9.9): the UV subtractions of the various terms in (9.4) exactly cancel out.

For completeness it should be added that the result (9.9) is based on the assumption we made in the last section (before eq. (8.28)) that $E_\epsilon^{(s)}[\psi_u] \equiv 0$. This was proved in part with numerical methods, but its validity is imposed by consistency. Finally a few words concerning the ϵ-regularization. It is evident from the above that lump energy comes from the asymptotic region in s. This is the region which is precisely suppressed by the $e^{-\eta s}$ factor produced by the ϵ-regularization. It is therefore not surprising that, modulo the UV subtraction, ψ_ϵ represents a tachyon condensation vacuum solution.

It is clear that ϵ plays the role of a gauge parameter, although the results we have derived are more easily obtained in the $\epsilon \to 0$ limit. There is however no doubt that the r.h.s. of (9.4) is independent of the value of ϵ. To show this let us start from the following remark. Using the equation of motion, the last two terms of the r.h.s. of (9.4) could be replaced by $3\langle \psi_\epsilon, Q\psi_u \rangle - 3\langle \psi_u, Q\psi_\epsilon \rangle$. Formally 'integrating by part' (that is moving Q from ψ_u to ψ_ϵ in the first term) this may seem to give 0. However such conclusion would be incorrect because the correlators in question are UV divergent and an integration by part is not allowed. One would have to regularize them first, but at that point the form (9.4) is much handier. However this remark leads us to an interesting conclusion. The obstruction to integrating by part is the UV divergence or the corresponding subtraction, which, as we have seen, are ϵ-independent. Therefore the value of $3\langle\psi_\epsilon, Q\psi_u\rangle - 3\langle\psi_u, Q\psi_\epsilon\rangle$ must be ϵ-independent as well. Since $\langle\psi_\epsilon, \psi_\epsilon\psi_\epsilon\rangle$ is also ϵ-independent as it is expected (and shown numerically in section 8), one is led to conclude that the r.h.s. of (9.4) does not depend on ϵ.

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A paper by T.Erler and C.Maccaferri, [45], will be posted simultaneously to the present paper. It deals with the same problem with a different method and leads to the same results as ours. We would like to thank the authors for keeping us informed of their progress. The critical remarks of C. Maccaferri in the course of our research have been also very helpful. One of us (L.B.) would like to thank the Yukawa Institute for Theoretical Physics, Kyoto, where a large part of this work has been carried out, for hospitality and financial support, and D.D.T. would like to thank SISSA for the kind hospitality during part of this research. The work of D.D.T. was supported by the Korean Research Foundation Grant funded by the Korean Government with grant number KRF 2009-0077423.

A Integration in the first wedge. Quadratic term

We examine here in more detail the integration of the term containing $RK(p,U)$ in the first wedge, as anticipated in section 6.1. The first wedge in the (p,U) plane is delimited by the U axis and by the ray $(\epsilon U, U)$, where ϵ is a finite small number. In this wedge the expansion of $RK(p,U)$ is given by:

$$\begin{aligned} RK'(p,U) = & -\frac{1}{16\pi}\frac{\ln U}{U^3} + \frac{-7+12p+7p^2+6\left(p^2-1\right)\psi\left(\frac{1+p}{2}\right)}{96\left(p^2-1\right)\pi}\frac{1}{U^3} \\ & -\frac{p\left(-1+2p+p^2\right)}{16\pi\left(p^2-1\right)}\frac{1}{U^4} - \frac{1+p^2}{32\pi}\frac{\ln U}{U^5} \\ & +\frac{-103-40p^2+240p^3+143p^4+60\left(-1+p^4\right)\psi\left(\frac{1+p}{2}\right)}{1920\pi\left(p^2-1\right)}\frac{1}{U^5} \\ & +\frac{p-2p^4-p^5}{16\pi\left(p^2-1\right)}\frac{1}{U^6} - \frac{3+10p^2+3p^4}{256\pi}\frac{\ln U}{U^7} + \cdots, \end{aligned} \tag{A.1}$$

where we have dropped the first two terms of $RK(p,U)$, because they have been dealt with exactly in section 5.1 (the first when summed over p contributes to cancel the dangerous $\frac{1}{U}$ dependence, the second can be summed exactly over p leading to a finite coefficient in front of $\frac{1}{U^2}$, giving an integrable term for large U). In order to integrate this over p and U

we will take the dominant terms (the potentially dangerous ones, as we will see) for large p,

$$RK'(p,U) \approx \frac{1}{8\pi}\Big(-\frac{1}{2}\frac{\ln U}{U^3}+\frac{1}{2}\ln p\frac{1}{U^3}-\frac{1}{2}\frac{p}{U^4} +\frac{1}{4}p^2\frac{\ln U}{U^5}+\frac{1}{4}p^2\ln p\frac{1}{U^5}-\frac{1}{2}\frac{p^3}{U^6}-\frac{3}{32}p^4\frac{\ln U}{U^7}+\cdots\Big). \tag{A.2}$$

Integrating in p up to ϵU for some small but finite number ϵ one gets,

$$\int^{\epsilon U} dpRK'(p,U) = \frac{1}{16\pi}\Big(-\epsilon\frac{\ln U}{U^2}+\epsilon(\ln(\epsilon U)-1)\frac{1}{U^2}-\frac{1}{2}\epsilon^2\frac{1}{U^2}+\frac{1}{6}\epsilon^3\frac{\ln U}{U^2} +\frac{1}{18}(3\ln(\epsilon U)-1)\frac{1}{U^2}-\frac{1}{4}\frac{\epsilon^4}{U^2}-\frac{3}{16}\frac{\epsilon^5}{5}\frac{\ln U}{U^2}+\cdots\Big). \tag{A.3}$$

It is evident that in the r.h.s. we have two numerical series, proportional to $\frac{1}{U^2}$ and $\frac{\ln U}{U^2}$, respectively, both strongly convergent because ϵ can be taken much smaller than 1. Integrating next over U we get a finite result, because both $\frac{1}{U^2}$ and $\frac{\ln U}{U^2}$ are integrable.

B Integration in the first wedge. Cubic term

We examine here in more detail the integration of the $SK(p,U)$ term in the first wedge as anticipated in section 5.2. The relevant expansion for $SK(p,U)$, for asymptotic p as well as U, is (discarding the first term in the r.h.s. of (5.16) as it is treated exactly in section 5.2)

$$SK'(p,U) = \frac{1}{32\pi}\left(-\frac{1}{p}\frac{1}{U^4}+\frac{1}{U^5}-\frac{p}{U^6}+\frac{p^2}{U^7}+\cdots\right). \tag{B.1}$$

Integrating over p up to ϵU:

$$\int^{\epsilon U} dp\, SK'(p,U) \approx \frac{1}{32\pi}\left(-\frac{\ln(\epsilon U)}{U^4}+\frac{\epsilon}{U^4}-\frac{\epsilon^2}{2U^4}+\frac{\epsilon^3}{3U^4}+\cdots\right) \tag{B.2}$$

Apart from the first term in the r.h.s., which will be discussed in a moment, we see that the r.h.s. is a numerical convergent series (because ϵ is a finite number which can be chosen to be much smaller than 1) multiplying the factor U^{-4}, i.e. a finite number times U^{-4}. When multiplied by $U^2 g(U)$ for large U, this produces an integrable term in U.

The first term in the right hand side of (B.2), which comes in the p-non-asymptotic form from the second term in the r.h.s. of (5.16), is a worrying term, because it increases as ϵ becomes small. This term is in connection with a $\frac{\ln U}{U^4}$ behavior. Such a behavior has been met previously only in the asymptotic expansion of the $E_0^{(3)}(U)$ term, see eq. (5.14). $E_0^{(3)}(U)$ contains the terms coming from the one-cosine and two-cosines angular integrations in the cubic term. Thus it is natural to search for terms corresponding to the first term in the r.h.s. of (B.2) among the one-cosine and two-cosines contributions. This will be rewarding because we will find an exact cancelation. To see this let us write again (only) the relevant part of the asymptotic expansion

$$\frac{96}{\pi}SK'(p,U) \sim -\frac{3}{\pi^2}\frac{1}{p}\frac{1}{U^4}+\cdots. \tag{B.3}$$

The corresponding terms coming from the one-cosine and two-cosines contribution are, respectively,

$$\text{one} - \text{cosine} \sim -\frac{3}{\pi^2}\frac{1}{p}\frac{1}{U^4} + \cdots \tag{B.4}$$

and

$$\text{two} - \text{cosines} \sim +\frac{6}{\pi^2}\frac{1}{p}\frac{1}{U^4} + \cdots . \tag{B.5}$$

They cancel exactly. On the other hand it is easy to see that there are no other contributions of the type $\frac{1}{p}\frac{1}{U^4}$ from the remaining three-cosines terms. This means, on one hand, that we do not have to worry about the first term in the right hand side of (B.2), on the other hand, that very likely the term proportional to $\frac{\ln U}{U^4}$ in (5.14), which is anyhow integrable, would not be there if all the summations could be done analytically down to the end.

C D-brane tension and normalization

In this appendix we will justify the value for the D24 brane tension in eq. (6.10). Let us start from the normalization conventions used in [5–7]. They are coherent with the conventions in Polchinski's book, vol.I, [43]. The result for the D25-brane tension (with $\alpha' = 1$)

$$T_{D25} = \frac{1}{2\pi^2} \tag{C.1}$$

was derived by Okawa in [44], appendix A, according to Polchinski's book's conventions. In the latter the normalization conventions for the delta function are given in eq. (4.1.15). In the one-dimensional case they are

$$\langle 0, k | 0, k' \rangle = 2\pi\delta(k - k') \tag{C.2}$$

which means in particular

$$\langle 0|0\rangle = 2\pi\delta(0) = \mathcal{V}, \tag{C.3}$$

where

$$\delta(k) = \int \frac{dx}{2\pi} e^{ipx}. \tag{C.4}$$

According to these conventions the D24 brane tension must be

$$\mathcal{T}_{D24} = \frac{1}{\pi}. \tag{C.5}$$

However in this paper and in I we have been using a different normalization, where it is understood that

$$\lim_{u\to 0} \frac{1}{2\sqrt{\pi u}} = \delta(0) = \int \frac{dx'}{2\pi} = \frac{V}{2\pi} = \langle 0|0\rangle. \tag{C.6}$$

where the prime in dx' means that length is measured in different units with respect to (C.4). Therefore there is a factor of 2π between such conventions for the volume, $V = 2\pi\mathcal{V}$, and consequently the inverse of this factor for the energy density.[3] It follows that for us the expected result for the D24 brane tension is

$$T_{D24} = \frac{1}{2\pi^2}. \tag{C.7}$$

D Problems with the equation of motion and the Schwinger representation

One of the equations used in I, in order to show that our solution ψ_ϕ satisfies the SFT equation of motion, is the following

$$\frac{1}{K+\phi}(K+\phi) = I. \tag{D.1}$$

Since this equation has been the source of a debate, we would like to use this appendix to explain our point of view in some detail. The problem dealt with in this appendix refers to [1] rather than the present paper, however it is only thanks to the results of this paper that some of the issues of [1] can be made clear.

With respect to eq. (D.1) a problem arises because, in the case $\phi = \phi_u$, introduced in section 2, we need a Schwinger representation in order to be able to compute correlators. When we use a Schwinger representation, the identity

$$\frac{1}{K+\phi_u}(K+\phi_u) = I, \tag{D.2}$$

would seem not to be satisfied. To illustrate the problem, let us calculate the overlap of both the left and the right hand sides of (D.2) with $Y = \frac{1}{2}\partial^2 c\partial cc$. The right hand side is trivial and, in our normalization, it is

$$\mathrm{Tr}(Y \cdot I) = \lim_{t\to 0}\langle Y(t)\rangle_{C_t}\langle 1\rangle_{C_t} = \frac{V}{2\pi}. \tag{D.3}$$

To calculate the left hand side we need the Schwinger representation

$$\mathrm{Tr}\big[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\big] = \int_0^\infty dt\mathrm{Tr}\big[Y \cdot e^{-t(K+\phi_u)}(K+\phi_u)\big] \tag{D.4}$$

To evaluate it one is naturally led to make the replacement

$$e^{-t(K+\phi_u)}(K+\phi_u) \to -\frac{d}{dt}e^{-t(K+\phi_u)} \tag{D.5}$$

[3] The subtraction of the infinite UV term in section 5 confirms this. In fact, in order to get a finite result, we have subtracted there an overall divergent term that can be written as follows

$$\frac{15}{8}\frac{1}{2\pi^2}\frac{V}{2\pi} = \frac{15}{8}\frac{V}{4\pi^3}$$

Apart from the renormalization factor $\frac{15}{8}$, this tells us that the D25 brane tension is, with our conventions, $\frac{1}{4\pi^3}$ instead of $\frac{1}{2\pi^2}$.

and obtain

$$\mathrm{Tr}\left[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\right] = g(0) - g(\infty) = \frac{V}{2\pi} - g(\infty), \tag{D.6}$$

which is different form (D.3) because $g(\infty)$ is nonvanishing. The latter relation is often written in a stronger form

$$\int_0^\infty dt\, e^{-t(K+\phi_u)}(K+\phi_u) = 1 - \Omega_u^\infty, \qquad \Omega_u^\infty = \lim_{\Lambda\to\infty} e^{-\Lambda(K+\phi_u)} \tag{D.7}$$

This (strong) equality, however, has to be handled with great care.

Before dealing with the contradiction between (D.3) and (D.6), it is useful to understand, on an independent ground, that eq. (D.1) must be true. Let us start from the observation that $K+\phi_u$ is a vector in an infinite dimensional space. Defining its inverse by means of a Schwinger representation is one possibility, but not the only one. In fact $K+\phi_u = (K_1^L+\phi_u)|I\rangle$, where $|I\rangle$ is the identity string field (and we remark that in our applications $\phi_u(\tilde{z})$ is always inserted in the left part of the string). Therefore the inverse of $K+\phi_u$ can also be obtained via the inverse of the operator $K_1^L+\phi_u$.

The operator $\mathcal{K}_u \equiv K_1^L+\phi_u$ is a self-adjoint operator. Therefore its spectrum lies on the real axis. To know more about it we would need a spectral analysis of $\mathcal{K}_u$, similar to what has been done for the operator K_1^L in [37–40]. The spectrum of the latter is the entire real axis. The spectrum of $\mathcal{K}_u$ is of course expected to be different, but we know on a general ground that it lies on the real axis. We can therefore define the resolvent of $\mathcal{K}_u$, $R(\kappa,\mathcal{K}_u)$, which is by definition the inverse of $\kappa-\mathcal{K}_u$. The resolvent is well defined (at least) for any non-real κ. We do not know what type of eigenvalue the $\kappa=0$ one is: discrete, continuous or residual. However, since $R(\kappa,\mathcal{K}_u)(\kappa-\mathcal{K}_u)=1$ is true for any κ outside the real axis, we can hold it valid also in the limit $\kappa\to 0$ by continuity.[4]

On a general ground we can therefore conclude that eq. (D.2) must be true. Then, how do we explain the discrepancy between (D.3) and (D.6)? These two equations are affected by an UV singularity and need a subtraction. Without this subtraction they are meaningless. We have seen in this paper (more examples can be found in [46]) that there is no canonical subtraction scheme for such singularities, and a physical meaning can be assigned only to quantities that are subtraction-independent. The difference between the r.h.s. of (D.3) and (D.6) is of course due to the use of a Schwinger representation utilized to derive the second, similar to what we have seen in the calculation of the energy. The Schwinger representation allows us to transplant in (D.6) a path integral result (the partition function $g(t)$ calculated by Witten). In view of this it would actually be surprising to find the same result in the r.h.s. of (D.3) and (D.6). Nevertheless (D.3) and (D.6) must represent the same thing. Since $g(\infty)$ is finite, we can figure out a subtraction scheme for (D.3) and another for (D.6) so that the subtracted quantities coincide (if we subtract an infinity we can also subtract an

[4]To the risk of pedantry, let us consider a simple model of the situation. In the complex z plane the expression $\frac{1}{z}$ is of course singular at $z=0$, but the expression $z\cdot\frac{1}{z}$ is 1 on the whole complex plane (not only for $z\neq 0$), in virtue of continuity. Naturally if one evaluates $\frac{1}{z}$ first at $z=0$ and then multiplies it by 0, one ends up with an indefinite expression, but this is the wrong way to proceed.

infinity plus something finite). If we need to represent the identity as in the l.h.s. of (D.3) we will use one subtraction, if instead we need to represent it as in the l.h.s. of (D.6) we will use another. Thus, in principle we can eliminate any contradiction between (D.3) and (D.6). However the next question is whether the use of two subtractions schemes is compatible with the validity of the equation of motion. We have not been able to prove or disprove this in a convincing way, therefore we prefer to take a conservative point of view.

D.1 A conservative viewpoint

The equation of motion for ψ_u is true on the basis of the previous general argument. As a consequence we have to conclude that the Schwinger representation may not be reliable when applied to the equation of motion and we will avoid it. But if the Schwinger representation is not essential (and perhaps inadequate) when applied to the equation of motion, it is essential in the energy calculation. Thus, how can we trust the result we have obtained in section 8 and 9? We will argue in this last part of the appendix that when applied to the energy calculation the Schwinger representation gives a consistent result, provided we drop the offending term proportional to Ω_u^∞ in (D.6).

To start with let us remark that our end result (9.9), obtained via a coherent procedure, is the expected one. This cannot be explained unless the calculation of the energy by means of the Schwinger representation, barring miraculous cancellations, is the correct one. Secondly, as we have remarked above, the ambiguity (if we can call it this way) of the Schwinger representation, that is (D.6) as compared to (D.3), involves the UV subtraction. But we have shown above that the energy calculation is independent of the subtraction scheme used. Therefore we conclude that our result must not be affected by the ambiguity.

The above arguments are, however, indirect. A more direct one is the following. From (D.7) we can write formally

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} + \frac{1}{K+\phi_u}\Omega_u^\infty \tag{D.8}$$

On the other hand from section 7 we have

$$\frac{1}{K+\phi_u+\epsilon} = \int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)} \tag{D.9}$$

Our solution is

$$\psi_u = c\phi_u - \frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)Bc\partial c \tag{D.10}$$

In computing the energy of ψ_u (section 2) we have utilized only the first part of the r.h.s. of (D.8) and disregarded the second part (the already mentioned Schwinger ambiguity). This is intuitively correct: we have just dropped the term that may violate the equation of motion. It is nevertheless a move not automatically inscribed in the formalism, therefore, as logical as it may seem, it needs some justification. To justify it we will show that the energy obtained in this way is the limit of a regularized version of the energy functional where the use of the Schwinger representation is justified beyond ambiguity.

To this end let us consider the following deformation of (D.10):

$$\psi_{u,\epsilon} = c\phi_u - \frac{1}{K+\phi_u+\epsilon}(\phi_u - \delta\phi_u)Bc\partial c \tag{D.11}$$

This is not a solution to the equation of motion, but nothing prevents us from using it as an auxiliary string field. Let us do the following exercise: let us replace everywhere in (9.4) ψ_u with $\psi_{u,\epsilon}$. In all the correlators in (9.4) we have to use (D.9). This induces some changes in the formulas of section 2 and 9.1, but is easy to see that in the limit $\epsilon \to 0$ the result coincides piece by piece. In the discussion in section 8 it has already been remarked that $\langle \psi_u, \psi_u \psi_u \rangle$ is the same as $\langle \psi_{u,\epsilon}, \psi_{u,\epsilon} \psi_{u,\epsilon} \rangle$ in the limit $\epsilon \to 0$ (see in particular the paragraph around eq. (8.26)). The same relation holds between $\langle \psi_\epsilon, \psi_{u,\epsilon} \psi_{u,\epsilon} \rangle$ and $\langle \psi_\epsilon, \psi_u \psi_u \rangle$ and between $\langle \psi_\epsilon, \psi_\epsilon \psi_{u,\epsilon} \rangle$ and $\langle \psi_\epsilon, \psi_\epsilon \psi_u \rangle$. The relevant correlators with $\psi_{u,\epsilon}$ instead of ψ_u are obtained from eqs. (9.5), (9.6) by suppressing the factors $e^{-\eta s y}$ and $e^{-\eta s x}$ in the latter. In the limit $\epsilon \to 0$ the equalities are thus established.

In conclusion, eq. (9.4) is the same whether we compute it by using for ψ_u the Schwinger representation (D.8) and dropping the Ω_u^∞ piece (as we have done throughout this paper), or we replace ψ_u with $\psi_{u,\epsilon}$ and use the regularized Schwinger representation (D.9) in the limit $\epsilon \to 0$. In other words we have achieved a representation of the energy functional where the use of Schwinger representation (D.9) does not suffer from any ambiguity and its use is absolutely legitimate. The outcome of this discussion is that the offending term proportional to Ω_u^∞ in (D.6), (D.8) must not be taken into consideration when computing the energy.

In conclusion, in this appendix, which is actually a prolongation of [1], we have seen that the Schwinger representation may not provide a faithful representation of $\frac{1}{K+\phi_u}(K+\phi_u) = 1$ and may affect the proof of the equation of motion (although we have not been able either to prove or disprove this point). We have however argued that the calculation of the energy by means of the Schwinger representation we have used throughout this paper is perfectly consistent.

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