

Maximal supergravity in $D = 10$: forms, Borcherds algebras and superspace cohomology

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ABSTRACT: We give a very simple derivation of the forms of $N = 2, D = 10$ supergravity theories from supersymmetry and $SL(2, \mathbb{R})$ (for IIB). Using superspace cohomology we show that, if the Bianchi identities for the physical fields are satisfied, the (consistent) Bianchi identities for all of the higher-rank forms must be identically satisfied, and that there are no possible gauge-trivial Bianchi identities (i.e. $dF = 0$) except for exact eleven-forms. We also show that the degrees of the forms can be extended beyond the spacetime limit, and that the representations they fall into agree with those predicted from Borcherds algebras. In IIA there are even-rank RR forms, including a non-zero twelve-form, while in IIB there are non-trivial Bianchi identities for thirteen-forms even though these forms are identically zero in supergravity. It is speculated that these higher-rank forms could be non-zero when higher-order string corrections are included.

KEYWORDS: Extended Supersymmetry, Superspaces, Supergravity Models

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Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | The forms of type II supergravity | 2 |
| 2.1 | IIB | 2 |
| 2.2 | IIA | 6 |
| 3 | Beyond the spacetime limit | 7 |
| 3.1 | IIB | 7 |
| 3.2 | IIA | 8 |
| 4 | The Borchers connection | 9 |
| 4.1 | IIB | 9 |
| 4.2 | IIA | 11 |
| 5 | Discussion | 12 |
| A | Borchers algebras | 13 |
| B | Superspace supergravity | 14 |

1 Introduction

An interesting feature of maximal supergravity theories is that they include p -form gauge fields of varying degrees. This set of fields includes some of the physical fields, their duals and further non-physical fields for which the potential forms have degree $(D - 1)$ and D . These sets of forms can be understood in terms of duality symmetries and supersymmetry alone, but they also have nice algebraic interpretations [1, 2]. It was subsequently shown that these can be understood in terms of (truncated, super) Borchers algebras [3, 4], and also that the spectrum of forms could be obtained from E_{11} [5–9]. In a recent article [10] it was shown that one can deduce the former from the latter, at least for forms with degrees that do not exceed the spacetime dimension. In this article we shall discuss these related topics in a superspace setting which has the advantage that there is no limit to the degrees of the forms that one can consider. A second advantage is that one can work in a manifestly covariant way in terms of the field strengths and this also allows one to see the algebraic structure in a rather direct fashion, along the lines of that proposed some time ago [1, 2].

The superspace approach is straightforward. One starts off with a set of physical forms, including the duals, and then asks how many further forms can be constructed that satisfy consistent Bianchi identities, of the type $dF = F^2$, and that also transform under appropriate representations of the duality group, when present. Here, consistency just means that

applying a second d must give zero on the left of the Bianchi identity and hence also on the right. One thus obtains an expression cubic in the F s on the right which must vanish. Thus, if we couple each field strength to an appropriate generator, the Bianchi identity will determine an antisymmetric product for these generators while the cubic consistency condition gives rise to the Jacobi identity which makes this algebraic structure into a Lie (super)-algebra.

In general, field strength forms with degree greater than $(D+1)$ vanish in supergravity, but this does not mean that these forms are not of interest. There are some examples of non-vanishing $(D+2)$ forms, including in IIA supergravity in $D=10$, while other forms may have interesting Bianchi identities of the form $dF = F^2$ where the two F s on the right-hand side do not vanish even though the left side does identically. An example of this occurs in the IIB theory where there are thirteen-forms whose Bianchi identities involve non-zero lower-degree forms on the right. More importantly, perhaps, it might be that these forms could become non-zero in the presence of higher-order string effects.

In this paper we restrict our attention to maximal supergravity in $D=10$. The full set of forms were constructed explicitly in components in [11] for IIA, in [12] for IIB, and in IIB superspace in [13]. In [14] a minor discrepancy in the earlier component results was sorted out and the IIA case given in superspace. We begin, in the next section, by re-examining these forms in a superspace setting and show that the derivations of these results can be considerably simplified by making use of superspace cohomology. In section three we discuss the Bianchi identities beyond the spacetime limit and explicitly exhibit the $SL(2, \mathbb{R})$ representations (for IIB) and their multiplicities up to degree fifteen. We also include a short discussion of the IIA case up to degree thirteen and show that there is a non-zero RR twelve-form field strength. In section four we briefly review the Borcherds algebras for IIA and IIB and show how the algebra of forms can be understood in terms of this algebraic framework. There are two appendices, on Borcherds algebras and superspace supergravity.

2 The forms of type II supergravity

2.1 IIB

The bosonic spectrum of the IIB theory consists of the graviton, two scalar fields, the dilaton and axion, a pair of two-form potentials and a four-form potential whose five-form field strength is self-dual. To these we can add their duals, a doublet of seven-form field strengths and a triplet of nine-forms. The latter are dual to the field strengths for the scalars and transform under the triplet representation of $SL(2, \mathbb{R})$ even though there are only two scalars. This can be achieved by means of a constraint on the field strength that ensures that there are only two dynamical dual eight-form potentials. This set can then be extended by a quadruplet and a doublet of eleven-forms, corresponding to the ten-form potentials studied in [12].¹ The set of forms is then $\{F_3^R, F_5, F_7^R, F_9^{RS}, F_{11}^{RST}, F_{11}^R\}$, together

¹The existence of these forms was known to the authors of [3]; see [15–17].

with the one-form field strengths for the scalars. The Bianchi identities for these forms are:

$$\begin{aligned}
 dF_3^R &= 0 \\
 dF_5 &= \varepsilon_{RS} F_3^R F_3^S \\
 dF_7^R &= F_3^R F_5 \\
 dF_9^{RS} &= F_3^{(R} F_7^{S)} \\
 dF_{11}^{RST} &= F_3^{(R} F_9^{ST)} \\
 dF_{11}^R &= \varepsilon_{ST} F_3^S F_9^{TR} + \frac{3}{4} F_5 F_7^R .
 \end{aligned}
 \tag{2.1}$$

The scalar potentials can be described by an element \mathcal{V}_r^R of $SL(2, \mathbb{R})$ modulo local $U(1)$ gauge transformations, where r is a local $SO(2)$ vector index. The Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1} = P + Q$, where Q is the $U(1)$ connection and P can be considered as the one-form field strength for the scalar potentials. It carries local $SO(2)$ indices and satisfies $DP = 0$, but we can convert these indices to global ones by multiplying by two factors of \mathcal{V} to form the $SL(2, \mathbb{R})$ triplet of one-forms $F_1^{RS} := \delta^{rt} P_t^s \mathcal{V}_r^R \mathcal{V}_s^S$. The Bianchi identity for F_1^{RS} is simply $dF_1^{RS} = 0$, and indeed one can solve it by setting $F_1^{RS} = \frac{1}{2} dM^{RS}$, where $M^{RS} := \delta^{rs} \mathcal{V}_r^R \mathcal{V}_s^S$.

It is a simple matter to check that the Bianchi identities (2.1) are indeed consistent. Furthermore, the full set is determined from the first two (for the physical fields) by consistency and $SL(2, \mathbb{R})$ symmetry. We shall now show that they can be solved straightforwardly, and that there are no further gauge-trivial identities ($dF = 0$). To do this we shall use superspace cohomology which we now briefly explain.

In superspace the tangent bundle T splits invariantly into even and odd parts, $T_0 \oplus T_1$, and it is therefore useful to consider forms with even and odd degrees. Thus the space of n -forms, Ω^n , splits into a sum of spaces of (p, q) -forms, $\Omega^{p,q}$, where $p + q = n$. In a similar way the exterior derivative splits into components with different bi-degrees:

$$d = d_0 + d_1 + t_0 + t_1, \tag{2.2}$$

where the bidegrees are $(1, 0)$, $(0, 1)$, $(-1, 2)$ and $(2, -1)$ respectively. The first two, d_0 and d_1 , are essentially even and odd differential operators, while the other two are algebraic operators formed with the dimension-zero and dimension three-halves torsion respectively. In particular,

$$(t_0 \omega_{p,q})_{a_2 \dots a_p \beta_1 \dots \beta_q} \propto T_{(\beta_1 \beta_2}{}^{a_1} \omega_{a_1 | a_2 \dots a_p | \beta_3 \dots \beta_{q+2})}, \tag{2.3}$$

where $T_{\alpha\beta}{}^c$ is the dimension-zero torsion which takes its flat space form, i.e. a gamma matrix, in supergravity.

The equation $d^2 = 0$ splits into various parts according to the bi-degrees amongst which one has

$$(t_0)^2 = 0 \tag{2.4}$$

$$t_0 d_1 + d_1 t_0 = 0 \tag{2.5}$$

$$d_1^2 + t_0 d_0 + d_0 t_0 = 0. \tag{2.6}$$

The first of these enables us to define the cohomology groups $H_t^{p,q}$, the space of t_0 -closed (p, q) -forms modulo the exact ones [18]. The other two then allow one to define the spinorial cohomology groups $H_s^{p,q}$ [19, 20]. These groups make use of a derivative d_s which is essentially d_1 acting on $H_t^{p,q}$, but they will not be needed here in this paper. In ten and eleven dimensions these cohomology groups are related to spaces of pure spinors and pure spinor cohomology respectively [21–23]. A key result, which we will make repeated use of below, is that, in $N = 2, D = 10$ supersymmetry, the groups $H_t^{p,q}$ vanish for $p > 1$ [24]. In IIB there are two non-trivial t_0 -closed $(1, 2)$ -forms which occur in the solution to the Bianchi identities for the three-forms, whereas in IIA there is only one such form. Using these forms one can construct elements of $H_t^{1,q}$ in terms of $(0, q - 2)$ forms.

Now suppose that we have a closed n -form whose lowest-dimensional non-vanishing component (i.e. the one with least even and greatest odd degree) is $\omega_{p,q}$ where $p + q = n$. The first three components of the equation $d\omega_n = 0$ are

$$\begin{aligned} t_0\omega_{p,q} &= 0 \\ d_1\omega_{p,q} + t_0\omega_{p+1,q-1} &= 0 \\ d_0\omega_{p,q} + d_1\omega_{p+1,q-1} + t_0\omega_{p-2,q-2} &= 0 \end{aligned} \tag{2.7}$$

The lowest component $\omega_{p,q}$ is therefore t_0 -closed, and hence, unless it is exact, will determine an element of the cohomology group $H_t^{p,q}$. As we shall see, when combined with the fact that these groups are zero for $p > 1$, this makes the analysis of the Bianchi identities rather simple.

We shall also need some elementary dimensional analysis. In geometrical units (of mass) the dimension of the purely even component of any field strength form F_n is $[F_{n,0}] = 1$ (excluding the basis forms) and this implies that $[F_{n-q,q}] = 1 - \frac{q}{2}$. In on-shell supergravity there are no scalar or tensor fields with negative dimensions and hence the only non-zero components of any F_n are $F_{n-2,2}$, $F_{n-1,1}$ and $F_{n,0}$ with dimensions $0, \frac{1}{2}$ and 1 respectively.

Let us write any of the Bianchi identities above in the form

$$I_{n+1}^X = dF_n^X - (FF)_{n+1}^X, \tag{2.8}$$

where X denotes the representation of $SL(2, \mathbb{R})$ under which F_n transforms. Even if a particular Bianchi is not satisfied, consistency means that $dI_{n+1}^X = 0$ modulo other lower-degree I s. Since there are no fields in supergravity that have negative dimensions, the lowest non-vanishing component of any I also has dimension zero and is given by $I_{n-3,4}^X$. This must be t_0 -closed and will therefore be t_0 -exact if $n \geq 5$. Now we know that the Bianchi identities are satisfied for all of the physical fields, so we can deduce from this that for all of the other forms, the physical duals and the non-physical eleven forms, the lowest components of the corresponding Bianchi identities are t_0 -exact.² Thus we will have $I_{n-3,4}^X = t_0 J_{n-2,2}^X$ for some $J_{n-2,2}^X$. But $J_{n-2,2}^X$ has precisely the same index structure as the lowest non-zero component of F_n , namely $F_{n-2,2}^X$, and hence setting $J_{n-2,2}^X = 0$ allows one to solve for $F_{n-2,2}^X$ in terms of the physical fields without imposing any further constraints. We can therefore do this

²This has to be done sequentially so that the lower-degree I s can be ignored.

and turn our attention to the next level, $I_{n-2,3}^X$, which is also t_0 -exact and which has the right number of components to allow us to solve for $F_{n-1,1}^X$. Going one step further in a similar fashion we see that we will be able to solve for $F_{n,0}^X$ and hence for the whole of F_n^X .

To make this more explicit we consider the example of the IIB seven-forms F_7^R . We have

$$I_8^R = dF_7^R - F_3^R F_5, \tag{2.9}$$

and we assume that the Bianchi identities for F_3^R and F_5 are satisfied. The lowest component of I_8^R that is not trivially satisfied on dimensional grounds is $I_{4,4}^R$. Since $dI_8^R = 0$ we have $t_0 I_{4,4}^R = 0$, and so, by the vanishing of $H_t^{4,4}$, we find

$$I_{4,4}^R = t_0 J_{5,2}^R. \tag{2.10}$$

Now

$$I_{4,4}^R = t_0 F_{5,2}^R - (F_3^R F_5)_{4,4}, \tag{2.11}$$

and because of (2.10) we know that the second term on the right is itself expressible as t_0 acting on some (5,2)-form which we can call $[F_3^R \cdot F_5]_{5,2}$. The (4,4) component of this Bianchi identity is clearly solved by setting $J_{5,2}^R = 0$ which means that we can choose $F_{5,2}^R$ to be given by

$$F_{5,2}^R = [F_3^R \cdot F_5]_{5,2}. \tag{2.12}$$

In principle we could add to this a t_0 -exact term of the form $t_0 G_{6,0}^R$, but this can be ignored as there are no dimension-zero gauge-invariant possibilities for $G_{6,0}^R$. In other words, the dimension-zero Bianchi identity determines the dimension-zero component of F_7^R , i.e. $F_{5,2}^R$, in terms of the dimension-zero components of the lower-degree forms which are known quantities. Having solved $I_{4,4}^R$ we can then apply similar arguments to $I_{5,3}^R$ and $I_{6,2}^R$ in order to solve for $F_{6,1}^R$ and $F_{7,0}^R$ in terms of the components of F_3^R and F_5 . At the next order, if $t_0 I_{7,1}^R = 0$ then automatically $I_{7,1}^R = 0$, and similarly for $I_{8,0}^R$. Thus the Bianchi identity I_8^R contains precisely enough information to solve for F_7^R in terms of F_3^R and F_5 and does not imply any additional constraints. We can then apply the same analysis to the higher-degree forms in sequence. The upshot of this cohomological analysis is that we do not have to solve for the components of any of the higher-degree forms explicitly in order to verify that solutions to their Bianchi identities are guaranteed to exist. Nevertheless, it is not difficult to find these solutions; they are given explicitly in a local $SO(2)$ basis in [13]. The dimension-zero components are constructed from Lorentz- and $SO(2)$ -invariant tensors, such as gamma-matrices, the dimension-one-half components are proportional to the physical fermion fields (dilatinos), while the dimension-one components can be physical field-strength tensors or bi-linears in the physical fermions.

To summarise, when the Bianchi identities for the physical fields are satisfied in on-shell IIB supergravity, there is no obstruction to their being solved for the higher-rank form fields provided that the Bianchi identities themselves are formally consistent. In addition, we can show that there are no gauge-trivial forms, i.e. with $dF = 0$ Bianchi identities, except with degree eleven. In IIB this could in principle be any of the duals or non-physical forms if it were to turn out that the right-hand sides of any of equations (2.1) could be set

to zero. We can see this by applying a similar argument to the one used for the Bianchi identities which implies that the lowest component of such an F , $F_{n-2,2}$, has to be exact, $F_{n-2,2} = t_0 G_{n-1,0}$, since $(n-2) > 2$. Iterating this one finds that F itself must be exact, $F_n = dG_{n-1}$, say. But the only non-zero component of G_{n-1} is $G_{n-1,0}$ which has to have dimension zero. This can only be some Lorentz-invariant tensor times a function of the scalars, and could therefore only be $\varepsilon_{a_1 \dots a_{10}}$, or $\varepsilon_{10,0}$ in form notation. Thus there is an exact gauge-trivial eleven-form that is a singlet under the duality group. This form is trivial in the sense that it is simply the exterior derivative of the bosonic volume form, and we shall not consider such forms as part of the set of forms that take part in the algebra. Note also that, in particular, this argument implies that there can be no gauge-trivial eleven-forms in the doublet representation of $SL(2, \mathbb{R})$ in agreement with the superspace discussion in [13] and the component discussion in [14].

2.2 IIA

The situation in IIA is similar, but there are two differences: there is no duality group and the forms can have both even and odd degree. The physical forms are the RR two- and four-forms, and the NS three-forms; their duals are RR six- and eight-forms and an NS seven-form, together with a nine-form which is dual to the one-form field strength of the dilaton. The RR Bianchi identities, including one for the ten-form, are

$$dG_{2n+2} = H_3 G_{2n} \quad \text{for } n = 0, 1, 2, 3, 4, \quad (2.13)$$

where G_0 is taken to be zero for the standard IIA theory. The Bianchi identities for the NS forms up to degree nine are

$$\begin{aligned} dH_3 &= 0, \\ dH_7 &= \frac{1}{2} G_4^2 - G_2 G_6, \\ dH_9 &= -H_3 H_7 + \frac{1}{2} G_4 G_6 - \frac{3}{2} G_2 G_8. \end{aligned} \quad (2.14)$$

Now consider the possible eleven-form field strengths. There are two allowable Bianchi identities that can be combined into one:

$$\begin{aligned} dH_{11} &= A \left(H_3 H_9 + \frac{3}{2} G_2 G_{10} - \frac{1}{4} G_6^2 \right) \\ &+ B \left(-G_2 G_{10} + G_4 G_8 - \frac{1}{2} G_6^2 \right), \end{aligned} \quad (2.15)$$

where A and B are real constants.

All of these Bianchi identities are consistent [14], and so we can use the same argument that we used in IIB to prove that they will all be satisfied given that the physical ones are (i.e. those for the two-, three- and four-forms). In IIA $H_t^{p,q} = 0$ if $p > 1$, and if $p = 1$ the basic non-trivial element is $\tilde{\Gamma}_{1,2} \in H_t^{1,2}$, where $\Gamma_{p,2}$ denotes a symmetric gamma-matrix with p spacetime indices and the tilde indicates the presence of a factor of Γ_{11} . One possibility for the eleven-forms is that both A and B are zero. In this case we have a gauge-trivial eleven-form, but by the same cohomological argument that we used in IIB, it is exact.

It is easy to verify that there are two gauge non-trivial eleven-forms by looking at the dimension-zero components. For

$$H_{9,2} = -iK e^{-2\phi} \Gamma_{9,2} \quad , \quad (2.16)$$

with K constant, we find that (2.15) is satisfied if $2A + 8B = K$, so that there are indeed two independent gauge non-trivial eleven-forms. The other possibility, namely $H_{9,2} \sim i\tilde{\Gamma}_{9,2}$ corresponds to the gauge-trivial case and so requires $A = B = 0$. It is exact, i.e. proportional to $t_0 \epsilon_{10,0}$.

The above analysis can easily be extended to the massive case. To do this one simply has to include a zero-form ‘‘RR’’ field strength $G_0 = m$ where m is the mass in the Romans deformation of IIA supergravity [25]. The Bianchi identities for the RR forms take the same form as in (2.13), but now

$$dG_2 = G_0 H_3 \quad , \quad (2.17)$$

while $dH_3 = 0$ as before. The remaining Bianchi identities also hold provided that one replaces the terms $G_2 G_{2n}$, $n \geq 3$, with $G_2 G_{2n} - G_0 G_{2n+2}$. This has been discussed previously in components in [26] for forms up to degree nine.

3 Beyond the spacetime limit

3.1 IIB

We start by considering the thirteen-forms in IIB supergravity. There are three possibilities corresponding to the five-, three- and one-dimensional representations of $SL(2, \mathbb{R})$:

$$\begin{aligned} dF_{13}^{RSTU} &= F_3^{(R} F_{11}^{STU)} \\ dF_{13}^{RS} &= \varepsilon_{UV} F_3^U F_{11}^{VRS} + \frac{8}{15} F_3^{(R} F_{11}^{S)} + \frac{2}{5} F_5 F_9^{RS} \\ dF_{13} &= \varepsilon_{RS} F_3^R F_{11}^S + \frac{3}{8} \varepsilon_{RS} F_7^R F_7^S \quad . \end{aligned} \quad (3.1)$$

Using (2.1) one can easily see that the first of these is consistent. The other two require a bit more work but turn out to be consistent for the given choice of constants. Any thirteen-form must be zero in supergravity since by dimensional analysis the only possible non-zero components are $F_{11,2}$, $F_{12,1}$ and $F_{13,0}$ and these all vanish by antisymmetry. However, all of the forms appearing on the right-hand side of the above equations are non-zero. Moreover, the dimension-zero component of any of the Bianchi identities has the form $I_{10,4}$ and so need not vanish identically. Nevertheless, since the Bianchi identities are consistent, it follows that $t_0 I_{10,4} = 0$. This implies that $I_{10,4} = 0$ because a cohomologically trivial $(10, 4)$ -form must vanish as the superspace has ten even dimensions. Therefore these Bianchi identities are guaranteed to be satisfied. The point of this discussion is that it might have been the case that these higher-degree forms should not be considered at all, but these examples give at least a minor indication that it does make sense to include them. One can show by direct computation that the right-hand sides of the $I_{10,4}$ identities vanish, but this is not trivially obvious unless one invokes the cohomological argument.

Moving on to the fifteen-forms, we find the following set of possibilities

$$dF_{15}^{RSTUV} = F_3^{(R} F_{13}^{STUV)} \quad (3.2)$$

$$dF_{15}^{RST} = a\varepsilon_{UV} F_3^U F_{13}^{V RST} + bF_3^{(R} F_{13}^{ST)} + cF_5 F_{11}^{RST} + dF_7^{(R} F_9^{ST)} \quad (3.3)$$

$$dF_{15}^R = e\varepsilon_{ST} F_3^S F_{13}^{TR} + fF_3^R F_{13} + gF_5 F_{11}^R + h\varepsilon_{ST} F_7^S F_9^{TR} . \quad (3.4)$$

Applying d to the second of these we find two constraints on the constants a, b, c, d coming from terms with $\varepsilon_{UV} F_3^U F_{13}^V F_{11}^{RST}$ and $F_5 F_3^{(R} F_9^{ST)}$ so that we can eliminate two of them, say c and d . We therefore find that there are two independent fifteen-forms in this representation whose Bianchi identities can be combined into

$$\begin{aligned} dF_{15}^{RST} = & a \left(\varepsilon_{UV} F_3^U F_{13}^{V RST} + \frac{5}{8} F_5 F_{11}^{RST} - \frac{5}{8} F_7^{(R} F_9^{ST)} \right) + \\ & + b \left(F_3^{(R} F_{13}^{ST)} - \frac{1}{2} F_5 F_{11}^{RST} + \frac{9}{10} F_7^{(R} F_9^{ST)} \right). \end{aligned} \quad (3.5)$$

For the doublet representation (3.4) we find three possible consistency conditions from terms with $\varepsilon_{ST} F_3^S F_3^T F_{11}^R$, $\varepsilon_{ST} F_3^S F_5 F_9^{TR}$ and $F_3^R \varepsilon_{ST} F_7^S F_7^T$. However, only two of them are independent and we therefore have two fifteen-forms in the doublet representation. Their Bianchi identities can be written

$$\begin{aligned} dF_{15}^R = & e \left(\varepsilon_{ST} F_3^S F_{13}^{TR} + \frac{2}{5} F_5 F_{11}^R \right) + \\ & + c \left(F_3^{(R} F_{13}^{ST)} - \frac{1}{2} F_5 F_{11}^R + \frac{1}{2} \varepsilon_{ST} F_7^S F_9^{TR} \right). \end{aligned} \quad (3.6)$$

The fifteen-forms vanish identically in supergravity, but not all of the forms on the right-hand side are zero. However, in this case the dimension-zero component of a Bianchi identity has the form $I_{12,4}$ and so must vanish by antisymmetry.

This analysis shows that at each level the number of representations that can arise increases, and that, from degree fifteen onwards, there are also multiplicities in some of the representations.

3.2 IIA

For the IIA case we observe first that the Bianchi identities (2.13) for the RR forms are consistent for any value of n . For the most part these are trivial in supergravity, but there is a non-zero RR twelve-form with dimension-zero component

$$G_{10,2} = -iK e^{-\phi} \Gamma_{10,2} \quad (3.7)$$

for some real non-zero constant K . In fact the Bianchi identity $dG_{12} = H_3 G_{10}$ is automatically soluble, by cohomology, the solution being given by the above expression. (The higher-dimensional components of G_{12} are identically zero.)

One can also have non-trivial thirteen-form Bianchi identities in IIA. The consistent ones turn out to be

$$dH_{13} = A \left(-\frac{5}{4} G_2 G_{12} - \frac{1}{4} G_4 G_{10} + \frac{1}{4} G_6 G_8 \right) + B \left(\frac{5}{2} G_2 G_{12} - \frac{3}{2} G_4 G_{10} + \frac{1}{2} G_6 G_8 \right) + H_3 H_{11} . \quad (3.8)$$

where A and B are the two constants that appear in (2.15). Since there are two H_{11} s depending on the choice of these, there are also two independent thirteen-form Bianchi identities.

As in the IIB case, the left-hand side of this equation is identically zero in supergravity, and the fact that the right-hand side vanishes as well, even though the individual forms that appear there do not, follows from cohomology.

4 The Borchers connection

The relation of the forms to Borchers algebras [27] was discussed in [3, 4]. In ten dimensions, it has been shown that the Borchers algebra for IIB is the same as one investigated earlier in a different context [28], while the IIA algebra is a superalgebra. Both algebras have 3×2 generators. There is also an intriguing relation with del Pezzo surfaces which is discussed in [3, 4]. In this section we shall relate these algebras to the algebras generated by the forms. The definition of a Borchers algebra can be found in appendix A.

4.1 IIB

The Borchers algebra for IIB is purely even since all of the field-strengths have odd degree. The Cartan matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.1}$$

so that the fundamental commutation relations between the generators are

$$\begin{aligned} [h_0, e_0] &= 0 & [h_1, e_0] &= -e_0 \\ [h_0, e_1] &= -e_1 & [h_1, e_1] &= 2e_1. \end{aligned} \tag{4.2}$$

There are two more generators $\{f_0, f_1\}$ (see appendix A for the full algebra), and we also have

$$(\text{ad } e_1)^2 e_0 = 0, \tag{4.3}$$

while $\{f_1, h_1, e_1\}$ forms a basis for $\mathfrak{sl}(2)$. The vectors e_0, e_1 are eigenvectors associated with the positive simple roots, α_0, α_1 , respectively.

It is clear from the discussion of the previous section that the algebra of forms is generated from the three-form field strengths. We shall associate a generator with each potential form, so the three-form generators will be denoted e_R^2 , the five-forms by e^4 and so on. In IIB these are all even generators. We write the sum of all the field-strengths as $\mathbb{F} = \sum (F_n^X e_X^{n-1})$ where X denotes the appropriate representation of $\mathfrak{sl}(2)$, so that all of the Bianchi identities can be combined into

$$d\mathbb{F} = \frac{1}{2} [\mathbb{F}, \mathbb{F}], \tag{4.4}$$

where the commutator denotes the commutator of the basis elements, and where due care has to be taken with signs. The generators e_R^2 form an $\mathfrak{sl}(2)$ doublet, so if we identify the lowest weight e_1^2 with e_0 , the second one can be obtained from it by the raising operator

| Level | Form degree | $\mathfrak{sl}(2)$ representation(s) |
|-------|-------------|--------------------------------------|
| 1 | 3 | 2 |
| 2 | 5 | 1 |
| 3 | 7 | 2 |
| 4 | 9 | 3 |
| 5 | 11 | 4+2 |
| 6 | 13 | 5+3+1 |
| 7 | 15 | 6+4(2)+2(2) |

Table 1. Low-lying levels in IIB.

e_1 , so $e_2^2 = [e_0, e_1] := e_{01}$. This cannot be raised any further so that we have the relation (4.3), $(\text{ad } e_1)^2 e_0 = 0$. To make further progress we investigate some of the states that are generated. For F_5 and F_7^R we have

$$\begin{aligned}
 e^4 &= [e_0, e_{01}] \\
 e_1^6 &= [e_0, [e_0, e_{01}]] \\
 e_2^6 &= [e_{01}, [e_0, e_{01}]] .
 \end{aligned}
 \tag{4.5}$$

Continuing in this way we find, for form degree $2n+1$, a state of the form $(\text{ad } e_0)^{n-1} e_{01}$, and this series can increase without limit. Moreover, it is clear that each state will be characterised by a corresponding root, although one should be aware that these can occur with multiplicities. Each $(2n+1)$ -form is associated with the roots $\alpha = n\alpha_0 + m\alpha_1$ where $m = 1, \dots, (n-1)$, although there can be multiplicities starting from $n = 5$. It is easy to see that one recovers the previously known results up to level 5, i.e. the forms of degree eleven that saturate the spacetime limit (ten-form potentials). However, the positive roots have been tabulated beyond this level [28], so that we can easily compare our results from section three to the Borchers prediction up to level 7, i.e. fifteen-forms. This is given in table 1 above. The form degrees here are those of the field strengths while the figures in brackets in the last entry in the third column indicate that these representations appear with multiplicity two. Comparing with the Bianchi identities in section three we find exact agreement including the correct multiplicities for the fifteen-forms. It is also easy to see that the roots are correctly given. For example, the vector $(\text{ad } e_0)^n e_1$ is the lowest weight state of the largest representation at level n corresponding to the root vector $n\alpha_0 + \alpha_1$. Indeed, this result is not surprising because it is clear that the algebra of forms must be isomorphic to the positive root algebra \mathcal{N}^+ modulo the one-dimensional space generated by e_1 .

It is also clear that the Borchers algebra determined by (4.2) is the smallest Borchers algebra that can accommodate the IIB form algebra. The existence of an $sl(2)$ subalgebra implies that $a_{11} = 2$ while (4.3) tells us that $a_{01} = -1$ (and hence, by symmetry, that $a_{10} = -1$). The fact that one can have arbitrary powers of $\text{ad } e_0$ means that a_{00} cannot be positive. If it was negative there would be a second $sl(2)$ subalgebra with infinite-

dimensional representations within the Borchers algebra, but this is not possible because there is only a finite number of forms of a given degree. So $a_{00} = 0$ and we are thus led to the Cartan matrix (4.2).

4.2 IIA

The Cartan matrix for IIA is given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{4.6}$$

The super-algebra has generators $\{f_0, f_1, h_0, h_1, e_0, e_1\}$, where e_1, f_1 are odd, which obey the basic commutation relations

$$\begin{aligned} [h_0, e_0] &= 0 \\ [h_0, e_1] &= -1 \\ [h_1, e_0] &= -1 \\ [h_1, e_1] &= 0. \end{aligned} \tag{4.7}$$

Since $a_{00} = a_{11} = 0$, the subalgebras associated with both sets of generators are of Heisenberg type.

The form algebra is generated from G_2 and H_3 so we shall associate elements of this algebra with them. For G_2 this is the odd element, e_1 , while for H_3 it is the even element e_0 . The first point to notice is that the e_0 component of (4.4) is

$$dH_3 e_0 = \frac{1}{2} G_2 G_2 [e_1, e_1]. \tag{4.8}$$

For this to agree with the correct identity, i.e. $dH_3 = 0$, we must have the relation $[e_1, e_1] = 0$. But this is also required from the general rules for a Borchers superalgebra in the appendix.

For the RR forms the situation is very simple. G_4 is associated with $[e_0, e_1] := e_{01}$, G_6 with $[e_0, e_{01}] := e_{001}$ and so on. For G_{2n} the element of the algebra is $(\text{ad } e_0)^{n-1} e_1$, and this series can increase without limit since $a_{00} = 0$.

For the NS forms the situation is slightly more complicated, but can be obtained directly from the Bianchi identities. For the seven-form one has only one possibility, namely $[e_{01}, e_{01}]$, while for the nine-form one has $[e_0, [e_{01}, e_{01}]]$. However, for the eleven-forms one finds two possibilities, $[e_0, [e_0, [e_{01}, e_{01}]]]$ and $[e_{001}, e_{001}]$. (Note that all of the vectors $e_{00\dots 1}$ for any number of zeroes are odd.) For the thirteen-forms there are again two possibilities, $(\text{ad } e_0)^3 [e_{01}, e_{01}]$ and $\text{ad } e_0 [e_{001}, e_{001}]$. These results are in agreement with those of section three. This pattern continues to higher levels, so that there is a series of terms of this type obtained by acting with $\text{ad } e_0$ on vectors of the form $[(\text{ad } e_0)^k e_1, (\text{ad } e_0)^k e_1]$. The situation can be summarised rather simply. For the RR forms, G_{2n} , $n \geq 1$, the roots are $(n-1)\alpha_0 + \alpha_1$, all with multiplicity one. For the NS forms, one has H_3 with root α_0 and two series (both with $n \geq 1$): the $(4n+3)$ -forms, which correspond to the roots $2n\alpha_0 + 2\alpha_1$, and which have multiplicity n , and the $(4n+5)$ -forms, which correspond

to the roots $(2n + 1)\alpha_0 + 2\alpha_1$ and which also have multiplicity n . These results for the multiplicities can be seen directly by inspection or by means of the Peterson formula for super-Borcherds algebras which is briefly summarised in appendix A.

In the IIA case the algebra of forms is clearly isomorphic to the positive part, \mathcal{N}^+ , of the Borcherds superalgebra. It is also clear that this Borcherds superalgebra is the smallest one that can accommodate the form algebra. Since $[e_1, e_1] = 0$ it follows that $a_{11} = 0$ while a_{00} must be zero for similar reasons to the IIB case. It is then possible to normalise the generators so that $a_{01} = a_{10} = -1$.

5 Discussion

In this article we have seen that the use of superspace techniques for maximal supergravity in $D = 10$ simplifies the discussion of the forms in the theory. Because the odd basis forms in superspace are commutative there is no limit to the degrees that forms can have. This means that we can avoid the use of potentials for the ten-forms, and that we can continue beyond the spacetime limit. The resulting formalism is manifestly covariant and automatically determines a Lie-(super)algebraic structure because the forms satisfy Bianchi identities of the form $dF = F^2$, which determines a graded antisymmetric multiplication, while the consistency conditions for the Bianchis are equivalent to the Jacobi identity. The algebras determined by the infinite set of possible forms are infinite-dimensional and are given by the positive elements of Borcherds algebras. In the IIB case the generator e_1 is associated to the axion [3, 4].

In [10] it was shown that these algebras, for all maximal supergravities, can be derived from E_{11} after suitable tensoring with an appropriate Grassmann algebra. However, it is not so clear how the higher-degree forms can be accommodated in this formalism. It might be that there is some extension of E_{11} that could account for them, or it might be the case that these forms should simply be disregarded.

In $N = 2, D = 10$ supergravity nearly all of the forms with degree greater than eleven are zero, the exception being the IIA RR twelve-form that was mentioned in section three. However, it might be that these forms could be non-zero when string-theory corrections are switched on. This is not an easy problem to investigate because one is faced with group-theoretic difficulties in whichever dimension one chooses to work with. In $D = 10$, for example, one could have α'^3 corrections in the $(0, 13)$ components of the thirteen-forms that would have to be linear in the dilatinos. The problem here is that one is faced with representations of the spin group involving the tensor product of thirteen spinors. On the other hand, in $D = 3$ there can be non-zero five-forms [29] whose lowest components would have to be α'^3 multiplied by dimension nine-halves functions of the fields, again not easy to analyse.

The fact that one can have forms with higher degrees than the spacetime dimension is implicit in the construction of the hierarchies of forms that appear in the context of gauged maximal supergravity theories in lower-dimensional spacetimes. Indeed, the possibility that the hierarchy could in principle be continued indefinitely was mentioned in [30]. However, it is only in the superspace context that the Bianchi identities for these forms re-

ally make sense, and only in superspace that such forms can actually be non-zero. Perhaps the simplest example of this occurs in the maximal $D = 3$ gauged theory where there are five-forms that are non-zero even in the lowest-order theory [29] and which are necessary in the gauged Bianchi identities.

The higher-degree forms would not seem to have any sort of brane interpretation. For example, the notion of an eleven-brane in ten-dimensional spacetime seems rather counter-intuitive. On the other hand, the fact that there is a non-zero RR twelve-form might lead one to ask whether it could have any geometrical significance beyond its algebraic rôle in the Borchers context.

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A Borchers algebras

The definition of a Borchers (or generalised Kac-Moody) (super)-algebra starts with a generalised symmetric Cartan matrix, (a_{ij}) , $i, j = 1 \dots N$, where some subset of the indices can be odd, which is non-degenerate and for which the following rules hold. The diagonal elements a_{ii} (no sum) can be positive, negative or zero, while the off-diagonal elements, a_{ij} , $i \neq j$, are less or equal to zero. In the case that $a_{ii} > 0$, then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}, \forall j$, while if i is also odd $\frac{a_{ij}}{a_{ii}} \in \mathbb{Z}, \forall j$.

The Borchers algebra \mathcal{A} associated with (a_{ij}) is then determined by $3N$ generators $\{h_i, e_i, f_i\}$, $i = 1 \dots N$, satisfying the following conditions:

$$[h_i, h_j] = 0 \tag{A.1}$$

$$[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}e_j, \quad [e_i, f_j] = \delta_{ij}h_i \tag{A.2}$$

$$(\text{ad } e_i)^{1 - \frac{2a_{ij}}{a_{ii}}} e_j = 0, \quad \text{for } a_{ii} > 0 \quad \text{and } i \neq j \tag{A.3}$$

$$[e_i, e_j] = 0 \quad \text{when } a_{ij} = 0, \tag{A.4}$$

with the last two conditions remaining valid if e_i, e_j are replaced by f_i, f_j . The generators h_i are even, and the generator f_i is even or odd if e_i is. If $a_{ii} > 0$ the integer $\frac{2a_{ij}}{a_{ii}}$ is negative, and if i is odd, it is also even.

In a Borchers algebra there is still a triangular decomposition of the form $\mathcal{A} = \mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+$, and it is still possible to define roots as in the Kac-Moody case. Furthermore, if $a_{ii} > 0$, the algebra generated by $\{f_i, h_i, e_i\}$ for i even, or by these together with $[f_i, f_i]$ and $[e_i, e_i]$ when i is odd, are isomorphic to $\mathfrak{sl}(2)$ or $\mathfrak{osp}(1|2)$, respectively, and the algebra can be decomposed into finite dimensional representations of these (super)algebras. When $a_{ii} < 0$, one has the same algebras but the Borchers algebra contains infinite-dimensional representations of them. In the case that $a_{ii} = 0$, the sub-algebra generated by $\{f_i, h_i, e_i\}$ is isomorphic to the Heisenberg (super)algebra.

The multiplicities of the roots for Borchers algebras may be computed using the Peterson formula [31, 32]. We found the discussion in [33] to be useful, especially for the super case. Let β be an element of the positive root lattice Q^+ , i.e. a linear combination of the positive simple roots (in both of the cases we discuss in the text, α_0 and α_1 .) with non-negative integral coefficients. And let \mathfrak{g}_β be the subalgebra of the Borchers algebra corresponding to a root β . The super-dimension of such a subalgebra is defined by $\text{sdim } \mathfrak{g}_\beta = (-1)^{\text{deg } \beta} \dim \mathfrak{g}_\beta$, where the degree is zero or one according to whether β is even or odd. The ordinary dimension is the multiplicity. The Peterson formula is

$$(\beta|\beta - 2\rho)c(\beta) = \sum (\beta'|\beta'')c(\beta')c(\beta''), \tag{A.5}$$

where the sum is over all elements such that $\beta = \beta' + \beta''$, ρ is a special combination of the simple roots and the quantities $c(\beta)$ are determined by the formula

$$c(\beta) = \sum_n \frac{1}{n} \text{sdim} \left(\frac{\beta}{n} \right), \tag{A.6}$$

n being a positive integer. The quantity ρ is determined by requiring that the left-hand-side of (A.5) should be zero for the positive simple roots; for IIB, $\rho = -\alpha_0$ while for IIA, $\rho = 0$. The round brackets denote the scalar product determined by the Cartan matrix, with $(\alpha_i|\alpha_j) = a_{ij}$ for the positive simple roots. In the sum in (A.6) the dimension of an element β of Q^+ that is not a root is zero, although this does not mean that the corresponding $c(\beta)$ vanishes because β may be a multiple of a root. Note that (A.6) only has more than one term if β is an integral multiple of a root. The $c(\beta)$ s, and hence the multiplicities, can be computed from these two formulae in an iterative fashion.

B Superspace supergravity

Although the details of the superspace descriptions of $N = 2, D = 10$ supergravities are not needed for the discussion in the main text we collect here a few basics and references.

The complete IIB supergravity (for the physical fields) was written down in superspace in [34], the component version having been given in [35]. The dual forms were added in [36, 37], and all of the forms up to degree eleven in [13]. The conventions we follow here are those of [13] although we have slightly changed the normalisations of some of the forms and written them with upper $\text{SL}(2, \mathbb{R})$ indices. One can transform from these to those of [13] by means of the ε -tensor. In the original paper a complex $\text{U}(1)$ notation was used for the spinors, but it is probably more convenient to use a real $\text{SO}(2)$ notation, as in [38], where the relation between the conventions of [34] and [13] can be found.

The dimension-zero torsion is given by

$$T_{\alpha_i \beta_j}{}^c = i \delta_{ij} (\gamma^c)_{\alpha\beta}, \tag{B.1}$$

where $i, j = 1, 2$ are $\text{SO}(2)$ spinor indices (we use r, s, \dots for vector indices). The geometric tensors cannot contain the scalar fields, as the formalism is $\text{SL}(2, \mathbb{R})$ covariant, as well as having a local $\text{U}(1)$ symmetry for which the gauge field is a composite constructed from the

scalars in the usual manner. The spin one-half fields are found in the dimension-one-half torsion, while the other physical field strengths arise at higher dimension, although the (bosonic) five-form does not appear directly.

In the text we gave the forms in an $SL(2, \mathbb{R})$ basis, but it is sometimes convenient to use the $SO(2)$ basis, the two being related by the scalar matrix \mathcal{V}_r^R . In this basis the Bianchi identities take the form

$$DF = FF - F \wedge P \tag{B.2}$$

where D is covariant with respect to $SO(2)$, FF denotes the same term that occurs in the $SL(2, \mathbb{R})$ basis, except that the indices are now lower case, and P denotes the matrix of one-forms in the representation appropriate to the form F on the left. One advantage of this basis is that the scalars cannot appear undifferentiated so that the dimension-zero components are simply given by products of (16×16) gamma-matrices and $SO(2)$ gamma-matrices, $(\tau^r)_{ij}$ (symmetric, traceless), δ_{ij} or ε_{ij} . We refer the reader to [13] for details.

The IIA theory was written down in components in [39] and in superspace in [40]. It was also derived by superspace dimensional reduction from $D = 11$ in [41].³ The version we use here was briefly outlined in [14]. We use thirty-two component Majorana spinors. The dimension-zero torsion is

$$T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta} . \tag{B.3}$$

The string frame is used, so that the dimension-zero component of H_3 has no factor of the dilaton, $H_{1,2} \propto \tilde{\Gamma}_{1,2}$, while the dimension-zero components of the RR forms all have a factor of $e^{-\phi}$, multiplied by appropriate gamma-matrices. Since $dG_{2n+2} = H_3 G_{2n}$, this implies that the dimension-zero components of the RR forms have a factor of Γ_{11} for n even, but not for n odd.

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³In appendix C of [41] equation (C.7) should read $\chi_\alpha = i(\Gamma_{11}\nabla)_\alpha\Phi$.

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