

Super-Yang-Mills and M5-branes

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ABSTRACT: We uplift 5-dimensional super-Yang-Mills theory to a 6-dimensional gauge theory with the help of a space-like constant vector η^M , whose norm determines the YM coupling constant. After the localization of η^M the 6D gauge theory acquires Lorentzian invariance as well as scale invariance. We discuss KK states, instantons and the flux quantization. The theory admits extended solutions like 1/2 BPS ‘strings’ and monopoles.

KEYWORDS: AdS-CFT Correspondence, M-Theory

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1 Introduction

Recent progress in formulating the holographic super-membrane theories [1]–[14], especially BLG tri-algebra based theories [1–4] and the ABJM Chern-Simons matter theories [9], has led to a renewed interest in understanding the mysteries behind so far unknown M5-brane theory [15–17]. The current understanding is that the dynamics of a single M5-brane is governed by an Abelian (2,0) super-conformal tensor theory having maximal supersymmetry in six-dimensions. The antisymmetric 2-rank tensor fields are natural objects to occur in six dimensions [19]. There is a reason to this; when a fundamental M2-brane ends on M5-brane the intersection produces a line defect on 6D world-volume of the M5-brane. These defects entirely live on the M5-brane world-volume and behave like extended ‘strings’. The belief is that these strings constitute the fundamental excitations on the M5-branes. The strings would naturally couple to 2-rank tensor field, $B_{\mu\nu}$, whose field strength is a 3-form, $H_{(3)} = dB_{(2)}$. There are already known self-dual string like solutions on M5-brane [20]. The tensor field, B , five scalars, X^I , and a spinor, Ψ , constitute what is known as (2,0) tensor super-multiplet in 6-dimensions [19]. The dynamical equations of this Abelian tensor theory are very simple and are given by

$$H_{(3)} \equiv dB_{(2)} = \star_6 H_{(3)}, \quad \partial_M \partial^M X^I = 0 = \not{\partial} \Psi \tag{1.1}$$

where \star_6 is the Hodge-dual operation in six dimensions, Ψ is the Majorana spinor. The massless scalars X^I correspond to five translational (Goldstone) modes, which an extended

M5-brane acquires when placed in a flat 11-dimensional spacetime. The (2,0) tensor theory is finite and superconformal, but the theory is trivial as it is noninteracting. However, it is being currently argued that all the states of a non-Abelian (2,0) tensor theory, once compactified on a circle, are probably contained in the 5-dimensional super-Yang-Mills (SYM) theory. Note, as such the 5D SYM theory is known to be powercounting nonrenormalizable and is strongly coupled in the UV region. But if 5D SYM indeed contains all the states of a 6D CFT without requiring any new UV degrees of freedom, then it should also be a finite theory by itself [17]. Although intuitive, but it is very difficult to directly check the finiteness of the 5D SYM. On the other hand not much is known about the non-Abelian construction of (2,0) tensor theory itself, which is supposed to describe an interacting theory living on a stack of multiple M5-branes. In analogy with YM fields, the tensor fields should be self-interacting just like Yang-Mills fields do. The non-Abelian 6D CFT should simply possess a $SU(N)$ gauge symmetry along with $SO(5)$ R-symmetry. These are some of the simple requirements which have so far eluded us in a meaningful construction of the M5-brane theory.¹

Our goal in this paper is rather different. We would like to construct a non-Abelian theory in six dimensions which could describe M5-branes and it need not be a tensor like theory. We shall introduce an overall constant space-like vector η^M whose norm will determine the Yang-Mills coupling constant. We then show that with the help of this vector the 5D Yang-Mills theory can be embedded into a 6D framework in a Lorentz covariant manner. After localization of η^M the theory recovers Lorentz invariance as well as scale invariance. The theory also carries correct counting of physical degrees of freedom such that we have maximal supersymmetry. The 6D non-Abelian gauge theory upon dimensional reduction gives rise to the usual 5D SYM.

The paper is organized as follows. In section-II we review the basics of $SU(N)$ super-Yang-Mills theory in five dimensions. In section-III we first introduce 6D Abelian gauge theory and discuss its solutions in detail. We find that the Abelian theory admits $\frac{1}{2}$ -BPS ‘strings’ and monopole solutions but there are no stable point like solutions. We then introduce a non-Abelian gauge theory with the help of space-like constant 6-vector. We also present the supersymmetry transformations and discuss straight forward dimensional reduction to 5D. In section-IV we restore the Lorentz invariance and replace the constant vector with an auxiliary Abelian field. It requires an introduction of a Lagrange multiplier 4-form potential. With this modification the 6D gauge theory is found to be scale invariant. We also discuss KK states, instantons and the quantization of the flux. The conclusion is given in section-V.

2 5D super Yang-Mills theory

Maximally supersymmetric Yang-Mills theories in $(p + 1)$ dimensions are well known to describe low energy dynamics on the stacks of multiple Dp -branes. For D4-branes the SYM

¹See some of the earlier developments in this field in the references [19, 21–24, 26–28].

action in five dimensions is given by

$$S_{\text{YM}} = \int d^5x \text{Tr} \left[-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu X^I D^\mu X^I + \frac{g^2}{4} ([X^I, X^J])^2 + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi - \frac{g}{2} \bar{\Psi} \Gamma^5 \Gamma^I [X^I, \Psi] \right] \quad (2.1)$$

where A_μ , ($\mu = 0, 1, \dots, 4$), is the gauge field and X^I , ($I = 6, 7, 8, 9, 10$), are five scalars and Ψ are the fermions. All the fields are in the $N \times N$ adjoint representation of the gauge group $SU(N)$. The commutators belong to the Lie algebra. The gauge field strength can be written as

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu]$$

and the covariant derivative is

$$D_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I]. \quad (2.2)$$

The supersymmetry variations are given by

$$\begin{aligned} \delta_s X^I &= i\bar{\epsilon} \Gamma^I \Psi \\ \delta_s A_\mu &= ig\bar{\epsilon} \Gamma_\mu \Gamma_5 \Psi \\ \delta_s \Psi &= \frac{1}{2g} F_{\mu\nu} \Gamma^{\mu\nu} \Gamma_5 \epsilon + D_\mu X^I \Gamma^\mu \Gamma^I \epsilon - \frac{i}{2} g [X^I, X^J] \Gamma^{IJ} \Gamma^5 \epsilon \end{aligned} \quad (2.3)$$

under which the action (2.1) closes on-shell. All spinors have 32 real components. The constant spinors in supersymmetry transformations satisfy the projection condition $\Gamma_{012345} \epsilon = \epsilon$ while spinor Ψ has opposite chirality, $\Gamma_{012345} \Psi = -\Psi$. Most of our notations match with [17].

The 5D super Yang-Mills theory is known to be powercounting nonrenormalizable as its coupling constant g is dimensionful with mass dimension $-\frac{1}{2}$. We know that in the strong coupling regime the string (bulk) theory becomes effectively 11-dimensional M-theory, hence from the AdS/CFT holography the boundary CFT should run to a strongly coupled conformal fixed point (in the UV regime) where the theory should rather be described by an stack of M5-branes instead of D4-branes. That is, an extra spatial world-volume direction should open up in 5D SYM theory and consequently it should become a 6-dimensional gauge/tensor theory. The precise mechanism how this would happen is not quite known yet. There are arguments which suggest that the natural objects which describe M5-branes are the 2-rank tensor fields instead of the gauge fields. Actually there are known string-like self-dual solitonic configurations which can live on M5-branes [20]. We shall discuss about these solutions in the next section. Note that the physical degrees of freedom contributed by a self-dual tensor fields in 6D add up to *three* only. We also know that these are precisely the d.o.f. also contributed by the Yang-Mills field in 5D. These d.o.f counts and the maximal supersymmetry are some of the evidences that the 5D SYM could perhaps be lifted to 6D gauge or a tensor-like theories. Thus if we could embed the 5D SYM in some suitable 6-dimensional framework we would be somewhat successful. We mention that writing down an explicit version of an interacting non-Abelian tensor

theory describing multiple M5-branes has remained an illusive goal so far. Although along these directions there have been some interesting attempts recently, e.g. introducing tri-Lie algebras in the M5-brane tensor theories [15, 18].

3 6D non-Abelian gauge theories

As stated above our modest aim is to uplift the cosmetic structure of the 5D SYM theory to six dimensions such that the theory looks like a gauge theory but gives 5D super-Yang-Mills upon dimensional reduction. In order to achieve this we first introduce a spacelike (coupling) constant vector η^M in six dimensions, which is normalized as

$$\eta^M \eta_M = g^2 > 0 \tag{3.1}$$

where g is a constant and will be related to 5D Yang-Mills coupling constant. Though out in this text $M, N = 0, 1, \dots, 5$ will represent 6-dimensional Lorentzian indices.

We have to understand first whether we need to work with vector fields in order to describe M5 branes. Factually, there are no apparent dynamical processes involving M5-branes to which we could assign the presence of vector fields. But, interestingly, we do include vector fields to work with Chern-Simons matter theories, ABJM or BLG, which are super-membrane theories [1–4, 9]. Under the same spirit, for M5-branes case here, although we do not apriori know what these 6D gauge fields may represent, but the idea is that the gauge fields on their own may also act like tensor fields and vice-versa.

3.1 Abelian gauge theory in 6D

First we discuss a simple example of an Abelian theory in 6D. It is rather straight forward to write down a 6-dimensional supersymmetric gauge action with the help of a fixed space-like vector η^M given in eq. (3.1). Correspondingly a covariant 6-dimensional action involving Abelian vector field is

$$S[A] \equiv \int d^6x \left[-\frac{1}{2 \cdot 3! \eta^4} (\eta_{[M} F_{NP])^2} - \frac{1}{2} (\partial_M X^I)^2 + \frac{i}{2} \bar{\Psi} \Gamma^M \partial_M \Psi \right] \tag{3.2}$$

where

$$\eta_{[M} F_{NP]} = \eta_M F_{NP} + \text{cyclic permutations of indices} \tag{3.3}$$

and the Abelian field strength is $F_{NP} = \partial_N A_P - \partial_P A_N$. Here X^I ($I = 6, 7, \dots, 10$) are five real scalars. These would correspond to the fact that M5-brane has five coordinates transverse to its world-volume. The bosonic equations of motion are

$$\begin{aligned} \partial_M \partial^M X^I &= 0 = \not{\partial} \Psi \\ \partial_M \eta^{[M} F^{NP]} &= 0 \end{aligned} \tag{3.4}$$

while the Bianchi identity is $dF = 0$. Thus equations of motion are covariant. But these equations are very different from those of Abelian tensor theory in (1.1) which involves a self-dual 3-form field strength. The supersymmetry variations can be written as

$$\delta_s X^I = i \bar{\epsilon} \Gamma^I \Psi$$

$$\begin{aligned}\delta_s A_M &= i\bar{\epsilon}\eta^N \Gamma_{MN} \Psi \\ \delta_s \Psi &= \frac{1}{3!\eta^2} \eta_{[M} F_{NP]} \Gamma^{MNP} \epsilon + \not{\partial} X^I \Gamma^I \epsilon\end{aligned}\tag{3.5}$$

under which the action (3.2) closes on-shell. Note that ϵ and Ψ are spinors of $SO(1, 10)$ and have opposite chiralities.

We notice that the gauge kinetic term in (3.2) is rather unusual. This axial form of gauge action helps us in maintaining the right field content in the theory. It can be seen as follows. The 6D gauge field has six off-shell degrees of freedom. Any preferential choice of vector η^M will break the covariance spontaneously. For example, if we locally choose $\eta^M = (\mathbf{0}, g)$, implying that the vector η^M is aligned along x^5 , it would turn the 5-th component of the gauge field, A_5 , essentially nondynamical and auxiliary. That is there would be no kinetic term involving A_5 in the action. The residual gauge symmetry of the theory fixes 2 more d.o.f.. Thus the actual on-shell gauge d.o.f. remain $(6 - 3) = 3$ only. These 3 gauge degrees of freedom and the 5 scalars, X^I , constitute in total 8 bosonic degrees of freedom. We recall that Ψ is a Majorana spinor, and it is chiral in nature, so it also contributes 8 fermionic d.o.f.. Thus our choice of covariant (axial) gauge field strength guarantees that the 6D Abelian theory (3.2) has the right physical (on-shell) content required for the maximal supersymmetry. (Had we taken a Maxwell type action $(F_{MN})^2$, it wouldn't have helped us, because the gauge field then would have contributed 4 physical d.o.f., one more than what is necessary.) The axial gauge action is however fully gauge invariant. (See also the discussion in the appendix.) Obviously the Lorentz invariance is compromised here. In section-4 we will show how η^M can be promoted to the status of a local $U(1)$ field where Lorentz invariance is ultimately regained.

We now list some of the static vacua of the Abelian theory in (3.2).

- i) We first write down the vacua which is an extended solitonic configuration, describing an M2-brane ending on M5-brane. (In the language of self-dual tensor theories [20], such a configuration is called a selfdual ‘string’.) We consider the case when $\eta^M = (\mathbf{0}, g)$ is aligned along x^5 coordinate, which we take to be an isometry direction. That is the soliton (string) is aligned along x^5 . We consider the ansatz

$$X^I(x^m) = \delta^{I8} \phi(x^m), \quad F_{0m} = \pm g \partial_m \phi.\tag{3.6}$$

This solitonic configuration is a solution of equations (3.4) provided

$$\phi(x^m) = \phi_0 + \sum_i \frac{2q_i}{|x - x_0^i|^2}\tag{3.7}$$

where fields depend upon all M5 world-volume coordinates x^m ($m = 1, 2, 3, 4$) except x^5 . Here x_0^i, q_i are the soliton parameters such as positions and charges. The supersymmetry (3.5) is preserved when

$$(1 \mp \Gamma^0 \Gamma^5 \Gamma^8) \epsilon = 0\tag{3.8}$$

Since one of the scalar fields X^8 representing a transverse coordinate is excited, we have a description in which M2-brane extended along x^5 - x^8 plane ends on M5-brane.

The intersection is along the common direction x^5 . Such a solitonic configuration will divide M5 world-volume into two halves along x^5 , with a bump on the brane. The electric field, $E_m \equiv F_{0m}$, due to string soliton dies off as $1/|x - x_0|^3$, while it is sharply peaked near its location at x_0 .

- ii) We also consider a magnetic configuration such as the monopole in [20]. We again take $\eta^M = (\mathbf{0}, g)$ aligned along x^5 as above but we also consider x^4 to be another isometry direction. The remaining spatial coordinates are denoted by x^a with index $a = 1, 2, 3$. Over this 3-dimensional Euclidean sub-space we have a magnetic monopole solution given by

$$F_{ab} = \mp g \epsilon_{abc} \partial_c \phi, \quad X^8(x^a) = \phi(x^a) = \phi_0 + \sum_i \frac{2p_i}{|x - x_0^i|} \quad (3.9)$$

which solves the equations of motion in (3.4). For the supersymmetry variations to vanish we require following condition on the constant spinors

$$(1 \mp \Gamma^0 \Gamma^4 \Gamma^8) \epsilon = 0. \quad (3.10)$$

Thus the 6D Abelian gauge theory admits $\frac{1}{2}$ -BPS ‘string’ and monopole like solutions, first discussed by [20] in the context of M5-branes. Also notice that each of these supersymmetric solutions have at least one isometry direction. It is also evident from our construction that there are no stable point-like solutions in the theory.

3.2 Uplifting of 5D super-Yang-Mills to 6D

From the Abelian exercise we learnt that it is possible to construct gauge theories in 6D which may well describe M5-brane. We now show that it is also possible to uplift 5D SYM to six dimensions with the help of a space-like fixed vector η^M in (3.1). We find that a 6-dimensional non-Abelian gauge action including the fermions can be written as

$$S_{M5}(A) \equiv \int d^6x \text{Tr} \left[-\frac{1}{12\eta^4} (\eta_{[M} F_{NP]})^2 - \frac{1}{2} (D_M X^I)^2 + \frac{1}{4} (\eta)^2 ([X^I, X^J])^2 + \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi - \frac{1}{2} \eta_M \bar{\Psi} \Gamma^M \Gamma^I [X^I, \Psi] \right] \quad (3.11)$$

where the field strength $F_{MN} = \partial_{[M} A_{N]} - i[A_M, A_N]$ is the Yang-Mills field strength. The scalar fields X^I ($I = 6, 7, \dots, 10$) are in the adjoint of $SU(N)$. These correspond to the fact that there are N parallel M5-branes. The covariant derivatives are

$$D_M X^I = \partial_M X^I - i[A_M, X^I], \quad D_M \Psi = \partial_M \Psi - i[A_M, \Psi]. \quad (3.12)$$

Interestingly the 5D supersymmetry variations (2.3) can also be lifted to a six-dimensional covariant form

$$\begin{aligned} \delta_s X^I &= i\bar{\epsilon} \Gamma^I \Psi, \quad \delta_s A_M = i\bar{\epsilon} \eta^N \Gamma_{MN} \Psi \\ \delta_s \Psi &= \frac{1}{3!} \frac{1}{\eta^2} \eta_{[N} F_{MP]} \Gamma^{NMP} \epsilon + D_M X^I \Gamma^M \Gamma^I \epsilon - \frac{i}{2} [X^I, X^J] \Gamma^{IJ} \eta_M \Gamma^M \epsilon \end{aligned} \quad (3.13)$$

under which the action (3.11) will close on-shell. Obviously the theory possesses a global SO(5) symmetry under which five scalars X^I are rotated, which is identical to the case of 5D SYM theory.

We now look at the gauge symmetry possessed by the 6D gauge action (3.11). The gauge transformations are

$$\begin{aligned} A_M &\rightarrow A'_M = U^{-1}A_MU - iU^{-1}\partial_MU \\ X^I &\rightarrow X'^I = U^{-1}X^IU, \quad \Psi \rightarrow \Psi' = U^{-1}\Psi U \end{aligned} \tag{3.14}$$

under which the action remains invariant. Here $U(x)$ is an $SU(N)$ element.

Notice that the gauge kinetic term in the action (3.11) has an axial form. Due to the presence of a constant vector η^M , the action is only covariant but not Lorentz invariant. Any preferential choice of vector η^M will break 6D covariance spontaneously.

3.3 Compactification to 5D SYM

We now check explicitly whether we get SYM theory upon compactification on a circle. For simplicity we will set fermions to zero here. We also separate the constant 6-vector η^M as

$$\eta^M = (\eta^\mu, \eta^5)$$

where indices $\mu, \nu = 0, 1, \dots, 4$. Without loss of generality let us now assume that $\eta^M = (\mathbf{0}, g)$, so that it is aligned along x^5 , which we are taking to be the direction of compactification. (We shall also discuss another case where it will not be so.) It is clear that A_5 is an auxiliary field. Consequently the constraint equation is

$$D_5X^I = \partial_5X^I - i[A_5, X^I] = 0 \tag{3.15}$$

So we can easily take all X^I 's to be independent of x^5 and set $A_5 = 0$. We also take $A_\mu(x^\mu, x^5) = A_\mu(x^\mu)$. From our ansatz we will have $\eta_{[5}F_{\mu\nu]} = \eta_5F_{\mu\nu} = gF_{\mu\nu}$, so the 6D action would reduce as

$$-\int d^6x \frac{1}{12\eta^4} (\eta_{[M}F_{NP]})^2 \rightarrow -\int d^5x \frac{1}{12\eta^4} 3(\eta_{[5}F_{\mu\nu]})^2 = -\int d^5x \frac{1}{4g^2} (F_{\mu\nu})^2$$

To be precise, upon compactification there would be a volume factor in 5D theory, for example radius of compactification R , which can be absorbed in the definition of 5D YM coupling as $R/g^2 \equiv 1/g_{\text{YM}}^2$ (see also eq. (4.10)), and in the rescaling of the fields as $X^I \rightarrow \sqrt{R}X^I$, $\Psi \rightarrow \sqrt{R}\Psi$. From this we determine that the action (3.11) indeed reduces to 5D SYM action (2.1) while the supersymmetry transformations (3.13) reduce to the transformations given in (2.3).

We now consider a slightly different situation where η^M is not be aligned along the isometry direction x^5 , along which we are compactifying. So we take $\eta^M = (n^\mu, 0)$ with $(n^\mu)^2 = g^2 > 0$. Here the gauge field components reduce as

$$A_5(x^\mu, x^5) = \phi(x^\mu), \quad A_\mu(x^\mu, x^5) = A_\mu(x^\mu) \tag{3.16}$$

So A_5 gives rise to a nontrivial scalar ϕ . Taking all fields $X^I(x^\mu, x^5)$ to be independent of x^5 , the gauge action (3.11) reduces to the following 5D action (bosonic)

$$S_{5D} = \int d^5x \text{Tr} \left[-\frac{1}{12n^4} (n_{[\mu} F_{\nu\lambda]})^2 - \frac{1}{4n^4} (n_{[\mu} D_{\nu]} \phi)^2 - \frac{1}{2} (D_\mu X^I)^2 + \frac{1}{4} n^2 ([X^I, X^J])^2 + \frac{1}{2} ([\phi, X^J])^2 \right] \quad (3.17)$$

Note the new scalar field ϕ also has a potential term of its own. This is an unfamiliar form for 5D Yang-Mills action, the difference is that it is written with the help of an axial-vector n^μ . We could easily see that the bosonic content of the theory (3.17) is the same as that of standard super-Yang-Mills (2.1). Only new thing is that with the help of n^μ we have been able to pull ϕ out of A_μ , so that only 2 d.o.f. are contributed by A^μ . It is again due to the fact that A_μ kinetic term has axial nature. To see this explicitly, we should first take n^μ to be aligned along some spatial direction, say x^4 , which will reduce A_4 component to an auxiliary field. In the next step we can take $A_4 = 0$ and all fields to be independent of x^4 coordinate. So that whole dynamics will get confined to $x_4 = 0$ hyper-plane. Effectively this can be viewed as if the 5D theory (3.17) is reduced or compactified to four dimensions where it would describe D3-branes. Note that ϕ along with X^I 's constitute six scalars of 4D SYM.

4 A local $\eta^M(x)$ and Lorentz invariance

In order to promote the 6-dimensional theory to the status of a conformal theory with Lorentz invariance it would be essential to lift the ‘constant’ vector η^M to the status of a local field $\eta^M(x)$. That is, η^M needs to behave like a local Abelian field. We can accomplish this task with the help of a Lagrange multiplier 4-form potential $C_{(4)}$. Let us consider the gauge action (3.11) (bosonic part only), especially in the following form where η^M appears outside the derivatives everywhere

$$S_1(A_M, X^I) \sim \int d^6x \text{Tr} \left[-\frac{1}{12\eta^4} (\eta_{[M} F_{NP]})^2 - \frac{1}{2} (D_M X^I)^2 + \frac{1}{4} \eta^2 ([X^I, X^J])^2 \right]. \quad (4.1)$$

We now replace η^M by a local field $\eta^M(x)$ and introduce a Lagrange multiplier term

$$S = S_1(\eta^M(x), A_M, X^I) - \int \eta_{(1)} \wedge dC_{(4)}. \quad (4.2)$$

where $\eta^M(x)$ appears without derivatives so it is an auxiliary field. As η^M is an Abelian field it is expected that the action (4.2) should have an invariance under gauge transformations like $\eta_M \rightarrow \eta_M + \partial_M \zeta$.² This local U(1) symmetry can be realized provided we appropriately replace $\eta^M(x)$ in the action (4.2) by a gauge invariant combination such as

$$\hat{\eta}^M \equiv (\eta^M - \partial^M a).$$

²Such a combination of an auxiliary u^M vector and 4-form potential as a multiplier field has been used in self-dual chiral field models also [23]. I am grateful to M. Tonin for this very helpful communication.

Here $a(x)$ is a new (axionic) scalar field. The complete 6D gauge action then could be written as

$$\begin{aligned} \hat{S}_{M5} &= \int d^6x \text{Tr} \left[-\frac{1}{12\hat{\eta}^4} (\hat{\eta}_{[M} F_{NP]})^2 - \frac{1}{2} (D_M X^I)^2 + \frac{1}{4} \hat{\eta}^2 ([X^I, X^J])^2 \right] - \int \eta_{(1)} \wedge dC_{(4)}. \\ &\equiv \hat{S}_1(\hat{\eta}^M(x), \dots) - \int \eta_{(1)} \wedge dC_{(4)}. \end{aligned} \quad (4.3)$$

The new local action \hat{S}_{M5} in (4.3) has U(1) invariance under

$$\delta \eta_M = \partial_M \zeta, \quad \delta a = \zeta. \quad (4.4)$$

Due to this local shift η^M can always eat up $a(x)$ such that it will altogether disappear from the action (4.3).

The $C_{(4)}$ variation of the action (4.3) provides the equation of motion

$$d\eta_{(1)} = 0. \quad (4.5)$$

Thus on-shell η^M will be a constant vector upto total derivative term. For example, a generic solution of (4.5) can be taken as $\eta_M = g_M + \partial_M \lambda$, where g_M being a constant 6-vector. With the help of shift symmetry (4.4) we fix a gauge where $a(x) = \lambda(x)$, so that the gauge invariant quantity $\hat{\eta}^M$ is a constant, i.e.

$$\hat{\eta}_M = g_M \quad (4.6)$$

However, we should clarify that it is not guaranteed from here that the norm $(\hat{\eta}_M \hat{\eta}^M)$ will always be positive definite, it may even be null or a negative quantity. But we shall always pick the vacua in which $\hat{\eta}^M$ is an spatial vector.

Scaling symmetry. The action (4.3) as usual has an in-built $SU(N)$ gauge symmetry. We now comment on the scaling symmetry possessed by this action. The scaling (mass) dimensions of the fields are assigned as

$$[\eta^M] = -1, \quad [A_M] = 1, \quad [X^I] = 2, \quad [C_{(4)}] = 6, \quad [a] = -2 \quad (4.7)$$

Note, there is no dimensionful parameter in the action (4.3). Thus this action will exhibit an invariance if spacetime coordinates are scaled homogeneously and the fields according to their scaling dimensions.

The vacua. Note that the $N \times N$ constant diagonal matrices

$$X^I = \text{diag}(x_1^I, x_2^I, \dots, x_N^I), \quad I = 6, 7, \dots, 10 \quad (4.8)$$

are the simplest solutions of the theory. These give rise to the moduli space $(\mathcal{R}^5)^N / S^N$ which corresponds to N M5-branes placed on a transverse flat space \mathcal{R}^5 .

KK states, 4D instantons and quantized flux. The variation of the action \hat{S} with respect to the field η^M implies an equation

$$\frac{\delta \hat{S}_1}{\delta \eta_M} = \frac{1}{4!} \epsilon^{MNPQRS} \partial_N C_{PQRS} \tag{4.9}$$

which is an important equation which relates η^M with $C_{(4)}$. It is also an important relation so far as the quantization of flux and the YM instantons is concerned. Let us consider an static configuration where all X^I 's are vanishing, and $\hat{\eta}^M = g_0 \delta_5^M$. We take x^5 to be the isometry direction and compactify it on a circle of radius R_5 . We will define

$$\frac{R_5}{g_0^2} = \frac{1}{g_{\text{YM}}^2}. \tag{4.10}$$

Note that, in our convention $\frac{R_5}{g_{\text{YM}}^2}$ is a dimensionless quantity and it is consistent with other expectations [16]. Consider now the YM instantons configuration

$$F_{ij} = \tilde{F}_{ij} = \frac{1}{2!} \epsilon_{ijkl} F^{kl}$$

living on the Euclidean space spanned by x^1, \dots, x^4 coordinates. Then eq. (4.9) implies

$$\frac{R_5}{8g_0^2} \int d^4x \text{Tr} F_{ij} F^{ij} = R_5 g_0 \int_{\Sigma} dK_{(3)} \tag{4.11}$$

using the relation eq. (4.10) we obtain

$$\frac{1}{8g_{\text{YM}}^2} \int d^4x \text{Tr} F_{ij} F^{ij} = (R_5)^{\frac{3}{2}} g_{\text{YM}} \int_{\Sigma} dK_{(3)}. \tag{4.12}$$

For the above static configuration we have introduced a 3-rank Euclidean tensor $K_{ijk} \simeq C_{ijk0}$ and the volume factor $\int dt$ cancels out on the both sides of the equation. Since the l.h.s. of (4.12) counts the instanton number so it would be quantized

$$\frac{1}{8g_{\text{YM}}^2} \int d^4x \text{Tr} F_{ij} F^{ij} = \frac{4\pi^2 k}{g_{\text{YM}}^2} \tag{4.13}$$

where index $k \in Z$ counts the instantons.³

We then obtain from (4.12)

$$\frac{1}{4\pi^2} \int_{\partial\Sigma} K_{(3)} = \frac{k}{(R_5 g_{\text{YM}}^2)^{\frac{3}{2}}}. \tag{4.14}$$

The equation (4.14) implies an existence of a quantized 4-form flux threading the Euclidean 4-fold Σ , having a boundary $\partial\Sigma$. The 4D Yang-Mills instanton number k has an interpretation as the momentum $p_5 = k/R'_5$ carried by the KK states of 5D SYM [16, 17]. Accordingly these KK states need to be taken into account in 5D SYM if the instanton number is nonvanishing. We have shown that the YM instantons for the 6D gauge theory compactified on S^1 , does imply a nonvanishing quantized flux.

³In order to make a better link with [16, 17] works, one may instead choose a different length parameter $R'_5 = g_{\text{YM}}^2/(4\pi^2)$ on the r.h.s. of (4.13). R'_5 can be identified with the radius of the circle in the 6D CFT compactification [17]. But we have not worried much about that here as our claim is to establish a quantized flux.

5 Conclusions

We have lifted 5D SYM theory to six dimensions keeping the non-Abelian structure intact. The whole procedure can be made Lorentz covariant provided we assume the existence of a space-like ‘coupling constant’ 6-vector η^M , such that $\eta^M \eta_M = g^2 > 0$. The norm of η^M manifests as the super-Yang-Mills coupling constant in five dimensions. The theory can also be made fully localized. The Lorentz invariance is regained with the help of a Lagrange multiplier 4-form field. It should nevertheless be explored further if our procedure leads to desired properties like conformal symmetry for these 6D gauge theories. We have shown that when we treat η^M as an auxiliary field the theory becomes scale invariant. We also find that the 6D gauge theory does admit 1/2-BPS ‘string’ and monopole like solutions, first obtained by [20] in the context of M5-branes. From our construction we find that there are no stable point-like solutions in our theory.

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A Yang-Mills theory in axial gauge

Let us take the case of pure Yang-Mills theory in 5 dimensions. The standard Yang-Mills action in axial gauge is written as

$$S_5 = \int d^5x \left(-\frac{1}{4g^2} \text{Tr}(F_{\mu\nu})^2 - \frac{\alpha}{2} \text{Tr}(\eta^\mu A_\mu)^2 \right) \tag{A.1}$$

where indices $\mu = 0, 1, \dots, 4$ and η^μ is a fixed 5-vector. The condition

$$\eta^\mu A_\mu = 0$$

is known as the axial gauge. The term $(\eta^\mu A_\mu)^2$ in the above action explicitly breaks the gauge invariance under the transformations

$$A_\mu \rightarrow A'_\mu = U^{-1} A_\mu U - iU^{-1} \partial_\mu U \tag{A.2}$$

The gauge transformations (A.2) have to be such that

$$\eta \cdot A = \eta \cdot A' \tag{A.3}$$

which puts following restriction on gauge functions

$$\eta^\mu U^{-1} \partial_\mu U = 0, \tag{A.4}$$

One however could choose a particular local frame in which $\eta^\mu = (0, \dots, 0, g)$ so that $A_4 = 0$, in that case U has to be independent of x^4 .

A distinct but less familiar Yang-Mills action involving a fixed vector can be written as

$$S = - \int d^5x \frac{1}{12(\eta)^4} \text{Tr}(\eta_{[\mu} F_{\nu\lambda]})^2 \tag{A.5}$$

where $\eta^\mu \eta_\mu > 0$. That is, vector η^μ is space-like. The important thing about action (A.5) is that it has explicit invariance under the gauge transformations (A.2) where U is unrestricted. Now if we locally take η^μ to be aligned along x^4 , i.e. $\eta^\mu = (\mathbf{0}, g)$, the A_4 component of the gauge field becomes auxiliary. An special case is $A_4 = 0$ and take A_α 's independent of x^4 coordinate. Then above gauge action (A.5) immediately reduces to the 4D Yang-Mills action

$$S = - \int d^4x \frac{1}{4g^2} \text{Tr}(F_{\alpha\beta})^2 \tag{A.6}$$

where $\alpha, \beta = 0, 1, 2, 3$. The special advantage with the action (A.5) is that it has explicit gauge invariance. While it contributes one less degree of freedom compared to the usual YM field. Thus the situation is completely different from that of the YM action in (A.1).

Let us comment on what will be the situation if η^μ is taken to be a time-like vector, $\eta^2 = -g^2$. We consider the case $\eta^\mu = (g, 0, 0, 0)$. One can see that the action (A.5) reduces to an Euclidean action in 4D

$$S_E = \int d^4x_E \frac{1}{4g^2} \text{Tr}(F_{ij})^2. \tag{A.7}$$

B Is there a tensor version of the 6D gauge theory?

We may wonder if 6D gauge theory could also be described in terms of tensors. For this we would need to introduce a tensor field B_{MN} in the adjoint representation of the $SU(N)$ gauge group. It is challenging how to first define a gauge invariant non-Abelian tensor field strength involving tensor fields such that it includes *self-interactions*, like the YM gauge fields do. Thus with the help of constant vector η^M at our disposal we define a non-Abelian tensor field strength as

$$H_{MNP} = \partial_{[M} B_{NP]} - \frac{i}{2} \eta^Q ([B_{QM}, B_{NP}] + \text{cycl. perm. of } M, N, P) \tag{B.1}$$

The main feature here is that there is a $[B, B]$ commutator as we encounter in the Yang-Mills case. But we can also introduce vector fields through the relation

$$\eta^N B_{NM} \equiv A_M \tag{B.2}$$

This way of introducing a vector fields is distinct in the sense that we will immediately have an axial gauge condition

$$(i) \eta^M A_M = 0, \tag{B.3}$$

and also then we can write

$$(ii) B_{MN} \equiv \frac{1}{(\eta)^2} \eta_{[M} A_{N]}. \tag{B.4}$$

Using (B.2) to (B.4) we find

$$\begin{aligned} H_{MNP} &\equiv \partial_{[M}B_{NP]} - \frac{i}{2}([A_M, B_{NP}] + \text{cycl. perm.}) \\ &\equiv \frac{1}{(\eta)^2}\eta_{[N}F_{MP]} \end{aligned} \tag{B.5}$$

where F_{MN} is usual YM field strength. Thus our definition (B.2) is analogous to including an axial gauge condition for the Yang-Mills fields A_M in 6D. That is in describing tensor field through a gauge field in (B.2) we would be in the axial gauge set up. We should keep in mind that writing the tensor field as in (B.4) means that it is not a fundamental 2-rank tensor field instead it is a kind of composite tensor and thus carries only partial degrees of freedom constituted primarily by the YM fields.⁴

The equation (B.1) suggests that the tensor field is indeed self-interacting but the last equality in (B.5) also suggests that, although we have defined a 3-form field strength in reality we are effectively dealing with Yang-Mills fields only in disguise. The 6D tensorial form of the action (3.11) can be written as

$$\begin{aligned} S_{M5}(B) = \int d^6x \text{Tr} &\left[-\frac{1}{2 \cdot 3!}(H_{MNP})^2 - \frac{1}{2}(D_M X^I)^2 + \frac{1}{4}(\eta_M[X^I, X^J])^2 \right. \\ &\left. + \frac{i}{2}\bar{\Psi}\Gamma^M D_M \Psi - \frac{1}{2}\eta_M \bar{\Psi}\Gamma^M \Gamma^I [X^I, \Psi] \right], \end{aligned} \tag{B.6}$$

The covariant derivatives can be defined as

$$D_M X^I = \partial_M X^I - i\eta^N [B_{NM}, X^I], \quad D_M \Psi = \partial_M \Psi - i\eta^N [B_{NM}, \Psi] \tag{B.7}$$

While the susy transformations can be written as

$$\begin{aligned} \delta_s X^I &= i\bar{\epsilon}\Gamma^I \Psi \\ \delta_s B_{MN} &= i\bar{\epsilon}\Gamma_{NM} \Psi \\ \delta_s \Psi &= \frac{1}{3!}H_{MNP}\Gamma^{MNP}\epsilon + D_M X^I \Gamma^M \Gamma^I \epsilon - \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\eta_M \Gamma^M \epsilon \end{aligned} \tag{B.8}$$

under which the action (B.6) closes on-shell.

The gauge transformations of the B -fields can be written as

$$B_{MN} \rightarrow B'_{MN} = U^{-1}B_{MN}U - ig^{-2}U^{-1}\eta_{[M}\partial_{N]}U. \tag{B.9}$$

while rest of the fields transform in the same way as in (3.14). This indicates that B_{MN} is not the fundamental 2-rank field. As discussed above these tensors contributes only 3 physical degrees of freedom.

⁴In alternative formulations like (2,0) tensor CFTs, one considers the self-dual tensor field strengths, $H_3 = \star H_3$, in order to halve the physical degrees of freedom carried by a tensor field. While, in our formulation here we are having a composite nature of the tensor field avoiding or circumventing the self-duality criterion altogether.

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