Published for SISSA by Springer

Received: *July 13, 2010*
Accepted: *August 5, 2010*
Published: *August 24, 2010*

JHEP08(2010)107

The superconformal index of the E_6 SCFT

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Abstract: We derive an integral representation for the superconformal index of the strongly-coupled $\mathcal{N} = 2$ superconformal field theory with E_6 flavor symmetry. The explicit expression of the index allows highly non-trivial checks of Argyres-Seiberg duality and of a class of S-dualities conjectured by Gaiotto.

Keywords: Supersymmetric gauge theory, Duality in Gauge Field Theories, Topological Field Theories

ArXiv ePrint: 1003.4244

doi:10.1007/JHEP08(2010)107

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1 Introduction

The paradigmatic S-duality of $\mathcal{N} = 4$ super Yang-Mills is the simplest instance of a much more general web of duality connections relating $\mathcal{N} = 2$ $4d$ superconformal field theories. This viewpoint has been emphasized by Gaiotto [1], who introduced a large class of $\mathcal{N} = 2$ SCFTs by compactifying the $(2, 0)$ $6d$ theory on a Riemann surfaces Σ with punctures. Different ways of cutting Σ into pairs of pants correspond to different S-duality frames for the $4d$ theory. A remarkable dictionary relates $4d$ gauge theory quantities with calculations in $2d$ conformal field theory on Σ. For example, the partition function of the gauge theory on S^4, or more generally the Nekrasov instanton partition function [2], is reproduced exactly by a Liouville or Toda correlation function on Σ [3, 4].

This dictionary was extended in [5] by considering the superconformal index [6], which can be viewed as a twisted partition function of the $4d$ gauge theory on $S^3 \times S^1$. The superconformal index counts the states of the $4d$ theory belonging to short multiplets, up to equivalent relations that set to to zero all sequences of short multiplets that may in principle recombine into long ones. By construction, the index is invariant under continuous deformations of the theory, and is also expected to be independent of the S-duality frame.

Assuming S-duality, it follows that the index must be computed by a topological QFT living on Σ. In [5] this TQFT structure was discussed for the generalized quiver gauge theories with $\mathrm{SU}(2)^k$ gauge group, which arise from compactifications on Σ of the A_1 (2,0) theory. Invariance of the index under S-duality translates into associativity of the operator algebra of the $2d$ TQFT. In turn, associativity holds thanks to a beautiful mathematical identity for an elliptic hypergeometric integral [7].

What distinguishes the A_1 theories from their counterparts with $A_{n\geq 2}$ is that in all duality frames they have a Lagrangian description. This makes it easy to compute their superconformal index explicitly and to identify the structure constants of the $2d$ TQFT [5]. The situation for the generalized quiver theories with higher rank gauge groups is qualitatively different: in some duality frames the quivers contain intrinsically strongly-coupled blocks with no Lagrangian description. The prototypical example of this phenomenon was discussed by Argyres and Seiberg [8]:[1] the SYM theory with SU(3) gauge group and $N_f = 6$ fundamental hypermultiplets has a dual description involving the strongly-coupled SCFT with E_6 flavor symmetry [10]. In the absence of a Lagrangian description for the E_6 SCFT, it seems difficult to compute its superconformal index and to define the TQFT structure for generalized quivers with SU(3) gauge groups.

We solve this problem in this paper. By demanding consistency with Argyres-Seiberg duality, we are able to write down an explicit integral expression for the index of the E_6 SCFT (equation (3.18)). Technically, this is possible thanks to a remarkable inversion formula for a class of integral transforms [11]. By construction, the resulting expression for the index is guaranteed to be invariant under an $\mathrm{SU}(6) \otimes \mathrm{SU}(2)$ subgroup of the E_6 flavor symmetry. The index is seen *a posteriori* to be invariant under the full E_6 symmetry, providing an independent check of Argyres-Seiberg duality itself.[2] We proceed to define a TQFT structure for generalized quivers with SU(3) gauge symmetries. We check associativity of the operator algebra, which is equivalent to a check of S-duality for Gaiotto's A_2 theories. Most of our checks are performed perturbatively, to several orders in an expansion in the chemical potentials that enter the definition of the index. Conversely, S-duality implies that associativity must hold exactly, so as a by-product of our analysis we conjecture new identities between integrals of elliptic Gamma functions.

The paper is organized as follows. In section 2 we set up the stage by briefly reviewing the definitions of the superconformal index and of the elliptic Gamma functions. In section 3.1 the index of $N_f = 6$ SU(3) theory is computed in the weakly-coupled frame and the usual S-duality invariance of this index is discussed. In section 3.2 we use Argyres-Seiberg duality to write down an explicit expression for the index of E_6 SCFT; we check perturbatively that the answer is E_6 covariant and that it is compatible with physical expectations about the Coulomb and Higgs branches of vacua. In section 4 we check invariance under S-duality of the superconformal index for the generalized SU(3) quiver theories, and we present the TQFT interpretation of this index. In section 5 we briefly discuss our results. Four appendices complement the text with technical details.

[1] See also [9] for more examples.

[2] For earlier checks of Argyres-Seiberg duality see [12] and [13].

2 Generalities

In this section we briefly review the definition of the superconformal index [6], and the relevant properties of elliptic Gamma functions.

2.1 The superconformal index

The superconformal index is defined as [6][3]

$$\mathcal{I} = \mathrm{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)}\,, \tag{2.1}$$

where we trace over the states of the theory on S^3 (in the usual radial quantization).[4] The chemical potentials t, y, and v keep track of various combinations of quantum numbers associated to the supercorformal algebra SU(2,2|2): E is the conformal dimension, (j_1, j_2) the $\mathrm{SU}(2)_1 \otimes \mathrm{SU}(2)_2$ Lorentz spins, and $(R\,, r)$ the quantum numbers under the $\mathrm{SU}(2)_R \otimes \mathrm{U}(1)_r$ R-symmetry.[5]

For a theory with a weakly-coupled description the index can be explicitly computed as a matrix integral,

$$\mathcal{I}(V,t,y,v) = \int [dU]\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{\mathcal{R}_j}(t^n, y^n, v^n) \cdot \chi_{\mathcal{R}_j}(U^n,\, V^n)\right)\,. \tag{2.2}$$

Here U is the matrix of the gauge group, V the matrix of the flavor group and $\mathcal{R}_j$ label representations of the fields under the flavor and gauge groups. The measure $[dU]$ is the invariant Haar measure, and it has the following property

$$\int [dU] \prod_{j=1}^{n} \chi_{\mathcal{R}_j}(U) = \#\text{of singlets in } \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_n\,. \tag{2.3}$$

The quantities $f^{\mathcal{R}_j}(t,y,v)$ are the single-letter partition functions for matter in representation $\mathcal{R}_j$. The "single letters" of an $\mathcal{N}=2$ gauge theory contributing to the index must obey $E - 2j_2 - 2R + r = 0$ [6] and are enumerated in table 1. The first block of table 1 shows the contributing letters from the $\mathcal{N}=2$ vector multiplet, including the equations of motion constraint. The second block shows the contributions from the half hypermultiplet (or $\mathcal{N}=1$ chiral multiplet). The last line shows the spacetime derivatives contributing to the index. Since each field can be hit by an arbitrary number of derivatives, the derivatives give a multiplicative contribution to the single-letter partition functions of the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (t^3 y)^m\, (t^3 y^{-1})^n = \frac{1}{(1-t^3y)(1-t^3y^{-1})}\,. \tag{2.4}$$

[3] See also [14].

[4] For definiteness we consider the "right-handed" Witten index $\mathcal{I}^{WR}$ of [6], which computes the cohomology of the supercharge $\bar{\mathcal{Q}}_{2+}$. We use the notations of [15] where the supercharges are denoted as $\mathcal{Q}^I_\alpha$, $\bar{\mathcal{Q}}_{I\dot\alpha}$, $\mathcal{S}_{I\alpha}$, $\bar{\mathcal{S}}^I_{\dot\alpha}$, with $I = 1, 2$ $\mathrm{SU}(2)_R$ indices and $\alpha = \pm$, $\dot\alpha = \pm$ Lorentz indices.

[5] Our normalization convention for the R-symmetry charges is as in [15] and differs from [6]: $R_{\text{here}} = R_{\text{there}}/2$, $r_{\text{here}} = r_{\text{there}}/2$.

Letters	E	j_1	j_2	R	r	$\mathcal{I}$
ϕ	1	0	0	0	-1	$t^2 v$
$\lambda^1_\pm$	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-t^3\, y,\ -t^3\, y^{-1}$
$\bar\lambda_{2+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-t^4/v$
$\bar F_{++}$	2	0	1	0	0	t^6
$\partial_{-+}\lambda^1_+ + \partial_{++}\lambda^1_- = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	t^6
q	1	0	0	$\frac{1}{2}$	0	$t^2/\sqrt{v}$
$\bar\psi_+$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-t^4\sqrt{v}$
$\partial_{\pm+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$t^3\, y,\ t^3\, y^{-1}$

Table 1. Contributions to the index from "single letters". We denote by $(\phi, \bar\phi, \lambda^I_\alpha, \lambda_{I\,\dot\alpha}, F_{\alpha\beta}, \bar F_{\dot\alpha\dot\beta})$ the components of the adjoint $\mathcal{N} = 2$ vector multiplet, by $(q, \bar q, \psi_\alpha, \bar\psi_{\dot\alpha})$ the components of the $\mathcal{N} = 1$ chiral multiplet, and by $\partial_{\alpha\dot\alpha}$ the spacetime derivatives. Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm$, $\dot\alpha = \pm$ Lorentz indices.

The single-letter partition functions of the $\mathcal{N} = 2$ vector and $\mathcal{N} = 1$ chiral multiplets are thus given by

$$\text{vector} \quad : \quad f^{\text{vect}}(t, y, v) = \frac{t^2 v - \frac{t^4}{v} - t^3(y + y^{-1}) + 2t^6}{(1 - t^3\, y)(1 - t^3 y^{-1})}\,, \tag{2.5}$$

$$\text{chiral} \quad : \quad f^{\text{chi}}(t, y, v) = \frac{\frac{t^2}{\sqrt{v}} - t^4\sqrt{v}}{(1 - t^3\, y)(1 - t^3 y^{-1})}\,. \tag{2.6}$$

Throughout this paper we will assume

$$0 < |t|^4 < |v| < 1\,. \tag{2.7}$$

2.2 Elliptic hypergeometric expressions for the index

As was observed by Dolan and Osborn [16] the expressions for the index can be recast in an elegant way in terms of special functions. First, recall the definition of the elliptic Gamma function,

$$\Gamma(z; p, q) \equiv \prod_{j,k\geq 0} \frac{1 - z^{-1}\, p^{j+1} q^{k+1}}{1 - z\, p^j q^k}\,. \tag{2.8}$$

For reviews of the elliptic Gamma function and of elliptic hypergeometric mathematics the reader can consult [17–20]. Throughout this paper we will use the standard condensed notations

$$\Gamma(z_1, \ldots, z_k; p, q) \equiv \prod_{j=1}^{k} \Gamma(z_j; p, q), \qquad \Gamma(z^{\pm 1}; p, q) \equiv \Gamma(z; p, q)\Gamma(1/z; p, q)\,. \tag{2.9}$$

Basic identities satisfied by the elliptic Gamma function that will be of use to us are

$$\Gamma\,(pq/z; p, q)\, \Gamma\,(z; p, q) \,=\, 1\,, \tag{2.10}$$

$$\lim_{z\to a}\, (1 - z/a)\, \Gamma(z/a; p, q) \,=\, \frac{1}{(p; p)(q; q)}\,, \tag{2.11}$$

with the bracket defined as

$$(a;\, b) \equiv \prod_{k=0}^{\infty} \left(1 - a\, b^k\right) . \tag{2.12}$$

From the definition (2.8), it is straightforward to show [16]

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{t^{2n} z^n - t^{4n} z^{-n}}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})}\right) = \Gamma(t^2\, z; p, q), \tag{2.13}$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{2t^{6n} - t^{3n}(y^n + y^{-n})}{(1-t^{3n}y^n)(1-t^{3n}y^{-n})} (z^n + z^{-n})\right) = -\frac{z}{(1-z)^2} \frac{1}{\Gamma(z^{\pm 1}; p, q)},$$

where

$$p = t^3 y, \qquad q = t^3 y^{-1} . \tag{2.14}$$

Using the above identities the basic building blocks of the superconformal index computation can be written as follows. The contribution to the integrand of (2.2) from hypers in a fundamental representation of an SU(n) gauge group is

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f^{\rm chi}\left(t^k,\, v^k,\, y^k\right) \left[\chi_f(U^k) + \chi_{\bar f}(U^k)\right]\right) = \prod_{i=1}^{n} \Gamma\left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1}; p, q\right) . \tag{2.15}$$

The contribution to the integrand of (2.2) from the vector multiplet of SU(n) is

$$\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f^{\rm vect}\left(t^k,\, v^k,\, y^k\right) \chi_{adj}(U^k)\right) = \frac{\left[\Gamma(t^2\, v; p, q)\, (p;p)(q;q)\right]^{n-1}}{\Delta(\mathbf{a})\Delta(\mathbf{a}^{-1})} \prod_{i\neq j} \frac{\Gamma(t^2\, v\, a_i/a_j; p, q)}{\Gamma(a_i/a_j; p, q)} . \tag{2.16}$$

We have defined the characters of the fundamental representation to be

$$\chi_f = \sum_{i=1}^{n} a_i, \qquad \chi_{\bar f} = \sum_{i=1}^{n} \frac{1}{a\, i}, \qquad \prod_{i=1}^{n} a_i = 1 . \tag{2.17}$$

The character of the adjoint representation is

$$\chi_{adj} = \chi_f\, \chi_{\bar f} - 1 = \sum_{i\neq j} a_i/a_j + n - 1 . \tag{2.18}$$

We have also defined

$$\Delta(\mathbf{a}) = \prod_{i\neq j} (a_i - a_j) . \tag{2.19}$$

The Haar measure is given by

$$\oint_{{\rm SU}(n)} d\mu(\mathbf{a}) f(\mathbf{a}) = \frac{1}{n!} \oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1} \frac{da_i}{2\pi i\, a_i} \Delta(\mathbf{a})\Delta(\mathbf{a}^{-1}) f(\mathbf{a})\Big|_{\prod_{i=1}^n a_i = 1} , \tag{2.20}$$

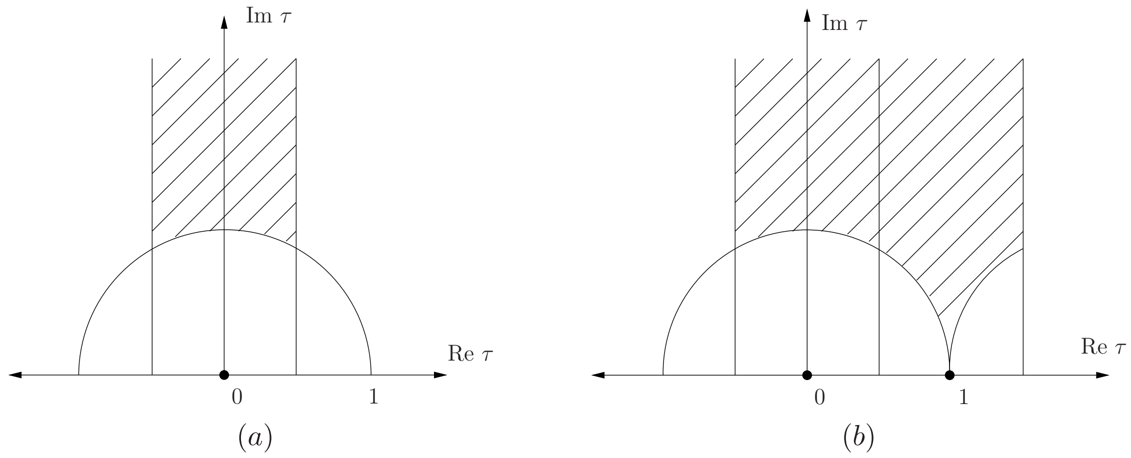

Figure 1. Moduli spaces for $\mathcal{N}=2$ SU(n) gauge theory with $2n$ flavors, (a) for $n=2$ and (b) for $n=3$ (in fact, for any $n>2$). The shaded region in (a) is $H/\mathrm{SL}(2,\mathbb{Z})$ while in (b) it is $H/\Gamma^0(2)$, where H is the upper half plane.

where $\mathbb{T}$ is the unit circle. Whenever we gauge a symmetry we have a vector multiplet associated to the integrated group and thus we will use the following notation

$$\mathcal{F}_{\mathbf{a}}\,\mathcal{G}^{\mathbf{a}} \equiv \frac{\left[2\Gamma(t^2\,v;p,q)\,\kappa\right]^{n-1}}{n!}\oint_{\mathbb{T}_{n-1}}\prod_{i=1}^{n-1}\frac{da_i}{2\pi i\,a_i}\prod_{i\neq j}\frac{\Gamma(t^2\,v\,a_i/a_j;p,q)}{\Gamma(a_i/a_j;p,q)}\mathcal{F}\left(\mathbf{a}\right)\mathcal{G}\left(\mathbf{a}^{-1}\right)\Bigg|_{\prod_{i=1}^{n}a_i=1}, \tag{2.21}$$

where $\kappa\equiv(p;p)(q;q)/2$. In what follows for the sake of brevity we will omit the parameters p and q from the elliptic Gamma function, i.e. $\Gamma(x)$ should always be understood as $\Gamma(x;\,p,q)$.

3 Argyres-Seiberg duality and the index of E_6 SCFT

The S-duality group of the $\mathcal{N}=2$ SU(2) gauge theory with four flavors is $\mathrm{SL}(2,\mathbb{Z})$. The action of this group on the gauge coupling is generated by $\tau\to\tau+1$ and $\tau\to-1/\tau$. In Gaiotto's description [1] this theory is constructed by compactification of the 6d $(2,0)$ theory on a sphere with four punctures of the *same* kind. Then, the S-duality group could be understood as the mapping class group of this Riemann surface. The moduli space of the gauge coupling is shown in figure 1 (a). We can see that a fundamental domain can be chosen such that nowhere in the moduli space does the coupling take an infinite value.

For the case of $\mathcal{N}=2$ SU(3) gauge theory with 6 flavors, however, the S-duality group is $\Gamma^0(2)$. The action of the S-duality on the complex coupling is generated by the transformations $\tau\to\tau+2$ and $\tau\to-1/\tau$. In Gaiotto's setup this theory is obtained by compactifying the $(2,0)$ theory on the sphere with two punctures of one type and two of another. The mapping class group of such a sphere is $\Gamma^0(2)$. The fundamental domain of this group is shown in the figure 1 (b) and, unlike the SU(2) case, this does

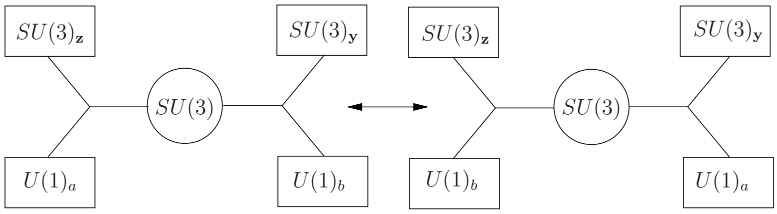

Figure 2. SU(3) SYM with $N_f = 6$. The U(6) flavor symmetry is decomposed as $\mathrm{SU}(3)_{\mathbf{z}} \otimes \mathrm{U}(1)_a \oplus \mathrm{SU}(3)_{\mathbf{y}} \otimes \mathrm{U}(1)_b$. S-duality $\tau \to -1/\tau$ interchanges the two U(1) charges.

unavoidably contain a point with infinite coupling. In [8], it was shown that this infinitely coupled cusp could be described in terms of an SU(2) gauge group weakly-coupled to a single hypermultiplet and a rank 1 interacting SCFT with E_6 flavor symmetry. Figure 3 describes this duality pictorially. The SU(2) subgroup of the flavor symmetry of the SCFT that is gauged commutes with the SU(6) subgroup of E_6. This SU(6) combined with SO(2) flavor symmetry of the single hypermultiplet generates the full U(6) flavor symmetry of the original SU(3) gauge theory. In other words, the SO(2) flavor symmetry of the single hypermultiplet corresponds to the baryon number of the original SU(3) gauge theory. Under this $\mathrm{U}(1)_B$, the quarks of the SU(3) theory have charges ± 1 while the quarks of the SU(2) theory have charges ± 3.

The E_6 SCFT has a Coulomb branch parametrized by the expectation value of a dimension 3 operator u which is identified with $\mathrm{Tr}\phi^3$ of the dual SU(3) theory, while the $\mathrm{Tr}\phi^2$ of the SU(3) theory corresponds to the Coulomb branch parameter of the SU(2) gauge theory. The E_6 CFT also has a Higgs branch parametrized by the expectation value of dimension 2 operators $\mathbb{X}$, which transform in the adjoint representation of E_6 (**78**). As shown in [13] the Higgs branch operators obey a Joseph relation at quadratic order which leaves a 22 complex dimensional Higgs branch. When coupled to the SU(2) gauge group, the resulting Higgs branch has complex dimension 20. The dual SU(3) theory also has a Higgs branch of complex dimension 20 and its Higgs operators can be easily constructed by combination of squark fields. See appendix C for more details.

The moduli space might contain also other infinitely coupled cusps which however are S-dual to the weakly-coupled cusp $\tau = i\infty$. This is the usual S-dualty mapping the $N_f = 6$ SU(3) gauge theory to itself with some of the U(1) flavor factors interchanged. This duality is represented in figure 2.

We proceed to compute the superconformal index of the SU(3) theory and, by using the Argyres-Seiberg duality, of the interacting E_6 SCFT.

3.1 Weakly-coupled frame

We take the chiral multiplets to be in the fundamental and antifundamental of the color and flavor. $\mathrm{U}(1)_B$ rotates them into each other. The vector multiplet is in the adjoint of the color. The SU(3) characters of the relevant representations are:

$$\chi_f = z_1 + z_2 + z_3 \quad \chi_{\bar{f}} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \quad \text{and} \quad \chi_{adj} = \chi_f \chi_{\bar{f}} - 1 \tag{3.1}$$

while writing down these characters, we have to impose $z_1 z_2 z_3 = 1$.

Let z's stand for the eigenvalues of the flavor group and x's be the eigenvalues of the color group. The $\mathrm{U}(1)_B$ charge is counted by the variable a. Let us write down the characters of the representation of the matter

$$\chi_{\text{hyp}} = \sum_{i=1}^{3}\sum_{j=1}^{3} a\, z_i\, x_j + \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{a\, z_i\, x_j}\,. \tag{3.2}$$

Using (2.15) the index contributed by the matter can be written in a closed form as

$$C_{a,\mathbf{x},\mathbf{y}} = \prod_{i=1}^{3}\prod_{j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\,(a\, x_i\, y_j)^{\pm 1}\right)\,. \tag{3.3}$$

The index for the SU(3) gauge theory with six hypermultiplets is then given by the following contour integral.

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = C_{b,\mathbf{y},\mathbf{x}}\, C_{a,\mathbf{z}}{}^{\mathbf{x}} = \tag{3.4}$$

$$\frac{2}{3}\kappa^2\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\, x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\,(b\, y_i\, x_j)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}.$$

By expanding this integral in t one can show that it is symmetric under interchanging the two U(1) factors (see appendix A),

$$a \quad \leftrightarrow \quad b\,. \tag{3.5}$$

Interchanging the two U(1)s is equivalent to performing a usual S-duality between a weakly-coupled and infinitely-coupled points of the moduli space and thus we expect the index to be invariant under this operation.[6]

One can analytically prove this statement in a special case. Notice that if $t = v$, the integral (3.4) is given by

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}\big|_{v=t} = I_{A_2}^{(2)}\left(1|\, t^{\frac{3}{2}}a^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}b\mathbf{y}; t^{\frac{3}{2}}a\mathbf{z}, t^{\frac{3}{2}}b^{-1}\mathbf{y}^{-1}\right)\,, \tag{3.6}$$

where [21]

$$I_{A_n}^{(m)}(Z|t_0,\ldots,t_{n+m+1};u_0,\ldots,u_{n+m+1};\,p,q) = \tag{3.7}$$

$$\frac{2^n}{n!}\kappa^n\oint_{\mathbb{T}^{n-1}}\prod_{i=1}^{n-1}\frac{dx_i}{2\pi i\, x_i}\left.\frac{\prod_{i=1}^{n}\prod_{j=0}^{m+n+1}\Gamma(t_j\, x_i,\; u_j/x_i;\, p,q)}{\prod_{i\neq j}\Gamma(x_i/x_j;\, p,q)}\right|_{\prod_{i=1}^{n} x_i = Z}.$$

[6]The integral (3.4) is an SU(3) generalization of the SU(2) integral in [5] for which the analogous statement to (3.5) has an analytic proof [7]. It is easy to generalize (3.4), (3.5) for SU(n) theories with arbitrary n, see appendix D.

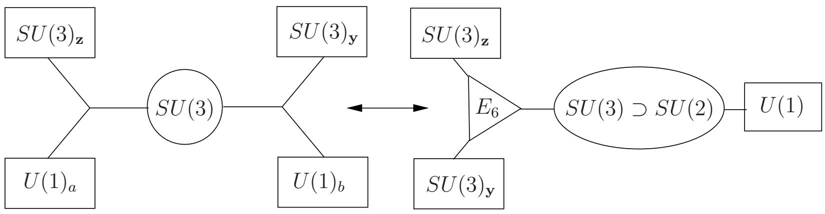

Figure 3. Argyres-Seiberg duality for SU(3) SYM with $N_f = 6$.

If the integral $I^{(m)}_{A_n}(Z|\dots t_i \dots ; \dots u_i \dots)$ satisfies the condition that $\prod_{i=1}^{m+n+2} t_i u_i = (pq)^{m+1}$ then due to [21], the following theorem holds

$$I^{(m)}_{A_n}(Z|\dots t_i \dots ; \dots u_i \dots) = I^{(n)}_{A_m}\left(Z|\dots \frac{T^{\frac{1}{m+1}}}{t_i} \dots ; \dots \frac{U^{\frac{1}{m+1}}}{u_i} \dots\right) \prod_{r,s=1}^{m+n+2} \Gamma(t_r u_s)\,, \quad (3.8)$$

where $T \equiv \prod_{r=1}^{m+n+2} t_r$ and $U \equiv \prod_{r=1}^{m+n+2} u_r$.[7] Coincidently, our integral (3.4) satisfies the above requirement and applying the theorem we can transform it into

$$I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}}b\mathbf{z}, t^{\frac{3}{2}}a^{-1}\mathbf{y}^{-1}; t^{\frac{3}{2}}b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}a\mathbf{y}\right) = I^{(2)}_{A_2}\left(1|t^{\frac{3}{2}}b^{-1}\mathbf{z}^{-1}, t^{\frac{3}{2}}a\mathbf{y}; t^{\frac{3}{2}}b\mathbf{z}, t^{\frac{3}{2}}a^{-1}\mathbf{y}^{-1}\right). \quad (3.9)$$

Note that the factor $\prod_{r,s=1}^{m+n+2}\Gamma(t_r u_s)$ in (3.8) reduces to 1 after pairwise cancelations using the property (2.11). What we have effectively achieved through this transformation is that we have exchanged the U(1) quantum numbers of the matter charged under the $\mathrm{SU}(3)^2$ flavor. This in particular implies that both the SU(3) flavor groups are on the same footing and are not associated with separate U(1)'s.

3.2 Strongly-coupled frame and the index of E_6 SCFT

In the strongly-coupled S-duality frame, figure 3, we have a fundamental hypermultiplet coupled to an SU(2) gauge theory. This gauge group is identified with an SU(2) subgroup of the E_6 flavor symmetry of a strongly-coupled rank one SCFT. We do not know the field content of the strongly-coupled rank 1 E_6 SCFT. This implies that we can not write down the "single letter" partition function for that theory and, a priori, can not directly compute its index. In what follows we will use the index computed in the weakly-coupled frame (3.4) and the above statements about Argyres-Seiberg duality to infer the index of the E_6 SCFT.

Let $C^{(E_6)}$ denote the index of rank 1 E_6 SCFT [10]. Consider the maximal subgroup $\mathrm{SU}(3)^3$ of E_6. Two among these three SU(3)'s are identified with the two SU(3) factors in the flavor group of the weakly-coupled theory, see figure 3. Let the additional SU(3) be denoted by $\mathbf{w}$. The fundamental representation of E_6 is decomposed under $\mathrm{SU}(3)_{\mathbf{w}} \otimes \mathrm{SU}(3)_{\mathbf{y}} \otimes \mathrm{SU}(3)_{\mathbf{z}}$ as,

$$\mathbf{27}_{E_6} = (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})\,. \quad (3.10)$$

[7]This identity was extensively used in [16] to show that certain theories related by Seiberg duality have equal superconformal indices [22]. In this context the authors of [23, 24] applied the elliptic hypergeometric techniques to a large class of Seiberg dualities.

Thus, the character of the E_6 fundamental fields is,

$$\chi_{\mathbf{27}} = \sum_{i,j=1}^{3} \left(\frac{w_i}{y_j} + \frac{z_i}{w_j} + \frac{y_i}{z_j} \right) , \qquad \prod_{i=1}^{3} y_i = \prod_{i=1}^{3} z_i = \prod_{i=1}^{3} w_i = 1 \,. \tag{3.11}$$

The index $C^{(E_6)}$ is thus a function of $\mathbf{w}$, $\mathbf{y}$, and $\mathbf{z}$. The S-duality picture suggests that we should decompose $\mathrm{SU}(3)_{\mathbf{w}}$ as $\mathrm{SU}(2)_e \otimes \mathrm{U}(1)_r$. This amounts to the change of variables $\{w_1, w_2, w_2\} \to \{er, \frac{r}{e}, \frac{1}{r^2}\}$, for which the character of the fundamental of E_6 becomes

$$\chi_{\mathbf{27}} = \left(er + \frac{r}{e} + \frac{1}{r^2} \right) \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) + \left(\frac{1}{er} + \frac{e}{r} + r^2 \right) (z_1 + z_2 + z_3) + \sum_{i,j=1}^{3} \frac{y_i}{z_j} . \tag{3.12}$$

Thus, the index of the E_6 SCFT can be denoted as $C^{(E_6)}\left((e,\, r), \mathbf{y}, \mathbf{z}\right)$. In the above notations the index of the additional hypermultiplet of the theory is

$$C_{s,\,e} = \Gamma \left(\frac{t^2}{\sqrt{v}} \, e^{\pm 1} \, s^{\pm 1} \right) . \tag{3.13}$$

Thus, one can write the superconformal index of the theory in the strongly-coupled frame as

$$\begin{aligned} \hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) &= C_s{}^e \, C^{(E_6)}_{(e,r),\mathbf{y},\mathbf{z}} = \\ &= \kappa \, \Gamma(t^2 v) \, \oint_{\mathbb{T}} \frac{de}{2\pi i \, e} \frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma\!\left(\frac{t^2}{\sqrt{v}} e^{\pm 1} \, s^{\pm 1} \right) \, C^{(E_6)}\left((e,\, r), \mathbf{y}, \mathbf{z}\right) . \end{aligned} \tag{3.14}$$

By Argyres-Seiberg duality we have to equate

$$\hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) = \mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} \,, \tag{3.15}$$

where $\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}}$ is given in (3.4), and we appropriately identify the U(1) charges,

$$s = (a/b)^{3/2}, \qquad r = (a\, b)^{-1/2} \,. \tag{3.16}$$

It so happens that the integral of equation (3.14) has special properties which allow us to invert it (see appendix B and [11] for the details). One can write the following

$$\kappa \oint_{C_w} \frac{ds}{2\pi i\, s} \frac{\Gamma\big(\frac{\sqrt{v}}{t^2} w^{\pm 1} \, s^{\pm 1}\big)}{\Gamma\big(\frac{v}{t^4}, \; s^{\pm 2}\big)} \, \hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) = \Gamma(t^2 v \, w^{\pm 2}) \, C^{(E_6)}\left((w,\, r), \mathbf{y}, \mathbf{z}\right) , \tag{3.17}$$

where the contour C_w is a deformation of the unit circle such that it encloses $s = \frac{\sqrt{v}}{t^2}\, w^{\pm 1}$ and excludes $s = \frac{t^2}{\sqrt{v}}\, w^{\pm 1}$ (for precise definition and details see appendix B and [11]). The above expression for the index $C^{(E_6)}$ does satisfy (3.14), but *a priori* does not uniquely follow from it. However, as we will explicitly see below, (3.17) is consistent with what is expected from E_6 SCFT. We will comment on this issue in the end of this section. We can thus use the Argyres-Seiberg duality (3.15) to write a closed form expression for the E_6 index

$$\boxed{\begin{aligned}C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) = \frac{2\,\kappa^3\Gamma(t^2v)^2}{3\,\Gamma(t^2v\,w^{\pm 2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma\big(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1}\big)}{\Gamma\big(\frac{v}{t^4},\;s^{\pm 2}\big)}\times\\ \times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac13}\,z_i}{x_j\,r}\right)^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac13}\,y_i\,x_j}{r}\right)^{\pm1}\right)\prod_{i\neq j}\Gamma\left(t^2v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\,.\end{aligned}}$$

One can rewrite the above expression without using the special integration contour. The integration contour C_w can be split into five pieces: a contour around the unit circle $\mathbb{T}$, two contours encircling the simple poles of $\Gamma\big(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1}\big)$ at $s=\frac{\sqrt{v}}{t^2}w^{\pm1}$, and two contours encircling in the opposite direction the simple poles of $\Gamma\big(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1}\big)$ at $\frac{t^2}{\sqrt{v}}w^{\pm1}$. Using the fact that elliptic Gamma function satisfies (2.11) we have

$$\begin{aligned}C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) &= \frac{\kappa}{\Gamma(t^2vw^{\pm2})}\oint_{\mathbb{T}}\frac{ds}{2\pi i\,s}\,\frac{\Gamma\big(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1}\big)}{\Gamma\big(\frac{v}{t^4},\;s^{\pm 2}\big)}\,\hat{\mathcal{I}}\left(s,r;\mathbf{y},\mathbf{z}\right)\\ &+\frac{1}{2}\frac{\Gamma(w^{-2})}{\Gamma(t^2vw^{-2})}\left[\hat{\mathcal{I}}\left(s=\frac{\sqrt{v}w}{t^2},r;\mathbf{y},\mathbf{z}\right)+\hat{\mathcal{I}}\left(s=\frac{t^2}{\sqrt{v}w},r;\mathbf{y},\mathbf{z}\right)\right]\\ &+\frac{1}{2}\frac{\Gamma(w^{2})}{\Gamma(t^2vw^{2})}\left[\hat{\mathcal{I}}\left(s=\frac{\sqrt{v}}{t^2w},r;\mathbf{y},\mathbf{z}\right)+\hat{\mathcal{I}}\left(s=\frac{t^2w}{\sqrt{v}},r;\mathbf{y},\mathbf{z}\right)\right]\,.\end{aligned}\tag{3.18}$$

The index (3.18) encodes some information about the matter content of the E_6 theory. To extract this information it is useful to expand the index (3.18) in the chemical potentials. We define an expansion in t as

$$C^{(E_6)}\equiv\sum_{k=0}^{\infty}a_k\,t^k\,.\tag{3.19}$$

The first several orders in this expansion have the following form

$$\begin{aligned}
a_0 =& 1 \\
a_1 t =& a_2 t^2 = a_3 t^3 = 0 \\
a_4 t^4 =& \frac{t^4}{v} \chi^{E_6}_{\mathbf{78}} \\
a_5 t^5 =& 0 \\
a_6 t^6 =& -t^6 \chi^{E_6}_{\mathbf{78}} - t^6 + t^6 v^3 \\
a_7 t^7 =& \frac{t^7}{v}\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} + \frac{t^7}{v}\left(y+\frac{1}{y}\right) - t^7 v^2 \left(y+\frac{1}{y}\right) \\
a_8 t^8 =& \frac{t^8}{v^2}\left(\chi^{E_6}_{sym^2(\mathbf{78})} - \chi^{E_6}_{\mathbf{650}} - 1\right) + t^8 v + t^8 v \\
a_9 t^9 =& -t^9\left(y+\frac{1}{y}\right)\chi^{E_6}_{\mathbf{78}} - 2t^9\left(y+\frac{1}{y}\right) + t^9 v^3\left(y+\frac{1}{y}\right) \\
a_{10} t^{10} =& -\frac{t^{10}}{v}(\chi^{E_6}_{\mathbf{78}}\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + \frac{t^{10}}{v}\left(y^2+1+\frac{1}{y^2}\right)\chi^{E_6}_{\mathbf{78}} + \\
&+ \frac{t^{10}}{v}\left(y+\frac{1}{y}\right)^2 - t^{10} v^2 \left(y+\frac{1}{y}\right)^2 \\
a_{11} t^{11} =& \frac{t^{11}}{v^2}\left(y+\frac{1}{y}\right)(\chi^{E_6}_{\mathbf{78}}\chi^{E_6}_{\mathbf{78}} - \chi^{E_6}_{\mathbf{650}} - 1) + t^{11} v\left(y+\frac{1}{y}\right) + t^{11} v\left(y+\frac{1}{y}\right) .
\end{aligned} \tag{3.20}$$

The adjoint representation of E_6 , **78**, decomposes in the following way in terms of its maximal $SU(3)^3$ subgroup

$$\mathbf{78} = (\mathbf{3},\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) + (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) , \tag{3.21}$$

and **650** of E_6 is composed as

$$\mathbf{650} = \mathbf{27} \times \overline{\mathbf{27}} - \mathbf{78} - \mathbf{1} . \tag{3.22}$$

The Higgs branch operators $\mathbb{X}$ of E_6 theory are in the adjoint (**78**) representation of E_6 flavor algebra. The terms of the index proportional to $\chi^{E_6}_{\mathbf{78}}$ are forming the following series,

$$\left[\frac{t^4}{v} - t^6 + \frac{t^7}{v}\left(y+\frac{1}{y}\right) - t^9\left(y+\frac{1}{y}\right) + \cdots\right]\chi^{E_6}_{\mathbf{78}} , \tag{3.23}$$

which is the index of a multiplet with $E=2$, $j_1=j_2=0$ and $r=0$ and of its derivatives (see appendix C.2 of [25]). Taken as a "letter" this multiplet has the following "single letter" partition function

$$\frac{t^4/v - t^6}{(1-t^3 y)(1-t^3/y)} , \tag{3.24}$$

which matches the quantum numbers of the Higgs branch operators on the weakly-coupled side of the Argyres-Seiberg duality if we follow the identifications listed in [13].

The E_6 singlet part of the index contains yet another series,

$$t^6 v^3 - t^7 v^2\left(y+\frac{1}{y}\right) + t^8 v + t^9 v^3\left(y+\frac{1}{y}\right) + \cdots . \tag{3.25}$$

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This series forms the index of a chiral multiplet with $E = 3$, $j_1 = j_2 = 0$ and $r = -3$ together with its derivatives (appendix C.1 of [25])

$$\frac{t^6 v^3 - t^7 v^2 \left(y + \frac{1}{y}\right) + t^8 v}{(1 - t^3 y)(1 - t^3/y)} . \tag{3.26}$$

Since the Coulomb branch operator, u, of E_6 theory (which is identified as $\mathrm{Tr}\phi^3$ of the dual SU(3) theory) has exactly the same quantum numbers, this multiplet is identified as the Coulomb branch operator.

The remaining singlet part of the index,

$$-t^6 + \frac{t^7}{v}\left(y + \frac{1}{y}\right) + t^8 v - 2t^9 \left(y + \frac{1}{y}\right) + \cdots , \tag{3.27}$$

is just the index of the stress tensor multiplet and its derivatives (appendix C.3 of [25])

$$\frac{-t^6 + \frac{t^7}{v}\left(y + \frac{1}{y}\right) + t^8 v - t^9 \left(y + \frac{1}{y}\right)}{(1 - t^3 y)(1 - t^3/y)} . \tag{3.28}$$

Besides the matter content, the index also provides possible constraints among operators. For example, it was argued [13] that the Higgs branch operators of the E_6 theory should obey the Joseph relations,

$$(\mathbb{X} \otimes \mathbb{X})|_{\mathcal{I}_2} = 0 , \tag{3.29}$$

where the representation $\mathcal{I}_2$ is defined as

$$sym^2(V(\mathbf{adj})) = V(2\mathbf{adj}) \oplus \mathcal{I}_2 . \tag{3.30}$$

For E_6, $\mathbf{adj} = \mathbf{78}$, $2\mathbf{adj} = \mathbf{2430}$ and then $sym^2(\mathbf{78}) = \mathbf{2430} \oplus \mathbf{650} \oplus \mathbf{1}$. Thus, in our case

$$\mathcal{I}_2 = \mathbf{650} \oplus \mathbf{1} . \tag{3.31}$$

The Joseph relation in E_6 theory reads,

$$(\mathbb{X} \otimes \mathbb{X})|_{\mathbf{650}\oplus\mathbf{1}} = 0 , \tag{3.32}$$

which means that these operators should not appear in the index. The index of $\mathbb{X}$ is t^4/v, then the index of $\mathbb{X} \otimes \mathbb{X}$ is t^8/v^2. (3.20) shows that our index is consistent with the Joseph relation.

Further constraints can also be derived from the higher order terms in (3.20). Let us consider the index at order t^{10}. The meaning of each term is clear. The first term corresponds to operators $\mathbb{X} \otimes (Q\mathbb{X})$ with the constraint $Q(\mathbb{X} \otimes \mathbb{X})_{\mathbf{650}+1} = 0$ which is a descendant of Joseph relation above (3.32). The last three terms are derivative descendants of $\frac{t^4}{v}\chi^{E_6}_{\mathbf{78}}$, $\frac{t^7}{v}\left(y + \frac{1}{y}\right)$ and $-t^7 v^2 \left(y + \frac{1}{y}\right)$ respectively. However, terms of the form

$$t^{10} v^2 \chi^{E_6}_{\mathbf{78}} , \tag{3.33}$$

which would be corresponding to the *Higgs* $\otimes$ *Coulomb* operators are absent. This fact implies the constraint

$$\mathbb{X} \otimes u = 0\,. \tag{3.34}$$

This is consistent with the fact that the E_6 theory has rank 1. The absence of $-\frac{t^{10}}{v}\chi^{E_6}_{\mathbf{78}}$ also implies the constraint

$$\mathbb{X} \otimes T = 0\,, \tag{3.35}$$

where T is the stress tensor. The structure of the index at order t^{11} is consistent with these two constraints.

Finally, let us comment on the uniqueness of our proposal. In principle, the index (3.18) produced by the construction of this section might differ from the true index of the E_6 SCFT: $C^{(E_6)}_{true}((e,\,r),\mathbf{y},\mathbf{z}) = C^{(E_6)}((e,\,r),\mathbf{y},\mathbf{z}) + \delta C((e,\,r),\mathbf{y},\mathbf{z})$, with δC satisfying

$$\oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma\left(\frac{t^2}{\sqrt{v}}\, e^{\pm 1} s^{\pm 1}\right)\Gamma(t^2 v\, e^{\pm 2})}{\Gamma(e^{\pm 2})}\, \delta C((e,\,r),\mathbf{y},\mathbf{z}) = 0\,. \tag{3.36}$$

At this stage we are not able to rigorously rule out such a possibility. However, the E_6 covariance of our proposal, its consistency with physical expectations about protected operators and the further S-duality checks performed in the following section, make us confident that we have identified the correct index of the E_6 SCFT.

Note that the expression for the index (3.18) is not explicitly given in terms of E_6 characters. However, as one learns from the perturbative expansion (3.20), the characters of $\mathrm{SU}(3)_{\mathbf{y}} \otimes \mathrm{SU}(3)_{\mathbf{z}} \otimes \mathrm{SU}(2)_w \otimes \mathrm{U}(1)_r$ always combine into E_6 characters. Essentially, since the weakly-coupled frame has really $\mathrm{SU}(6) \otimes \mathrm{U}(1)$ flavor symmetry we can write an expression for the E_6 index which has a manifest $\mathrm{SU}(6) \otimes \mathrm{SU}(2)$ symmetry,[8] but not the full E_6. The fact that just by assuming Argyres-Seiberg duality we obtain an index for a theory with an E_6 flavor symmetry and with a consistent spectrum of operators is a non-trivial check of Argyres-Seiberg duality.

4 S-duality checks of the E_6 index

In the previous section we have discussed the superconformal index of the $N_f = 6$ SU(3) theory and of its strongly-coupled dual. One can obtain this theory by compactifying a $(2,0)$ $6d$ theory on a sphere with four punctures, two U(1) punctures and two SU(3) punctures. The different S-duality frames are then given by the different degeneration limits of this Riemann surface. The weakly-coupled frames are obtained by bringing together one of the U(1) punctures and one of the SU(3) punctures, and the strongly-coupled frame is obtained by colliding the two SU(3) (U(1)) punctures. The coupling constant of the theory is related to the cross ratio of the four punctured sphere.

In [1] Gaiotto suggested to generalize this picture by considering general Riemann surfaces with an arbitrary numbers of punctures of different types (two types in case of the

[8] This is somewhat reminiscent of the construction of the E_6 symmetry using multi-pronged strings in [26]. It would be interesting to make an explicit connection between our expression of the index and the multi-pronged string language.

SU(3) theories). The claim is that all theories with the same number and type of punctures and same topology of the Riemann surface are related by S-dualities. The immediate consequence of this claim for the superconformal index is that all such theories have to have the same index as it is independent of the values of the coupling, i.e. the moduli of the Riemann surface. This implies that the superconformal index is a topological invariant of the punctured Riemann surface. It was claimed in [5] that the superconformal index can be actually interpreted as a correlator in a two dimensional topological quantum field theory. The structure constants of this TQFT are given by the index of the three punctured sphere and the contraction of indices (i.e. metric) is gauging of the flavor symmetries. The associativity of the algebra generated by the structure constants is equivalent to the invariance of the index of four punctured spheres under pair-of-pants decomposition into two three punctured spheres. The structure constants and the metric were constructed and the associativity was explicitly verified for the SU(2) case.

In this section we will make the same analysis for the SU(3) case. We have two types of punctures, associated to U(1) and SU(3) flavor symmetries. There are thus different three point functions one can construct (see figure 4). The index of the theory on a sphere with three SU(3) punctures, i.e. the index of the E_6 theory, is a structure constant which we will denote by $C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}}$ and it is just given by (3.18),

$$C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{z}} = C^{(E_6)}\left(\left(\sqrt{\frac{x_1}{x_2}}, \sqrt{x_1 x_2}\right), \mathbf{y}, \mathbf{z}\right) . \tag{4.1}$$

This vertex corresponds to the E_6 theory which has rank one, and thus we will refer to it as a rank $\mathbf{1}$ vertex. We will denote by $C^{(133)}_{\mathbf{x},\mathbf{y},a}$ the index of the sphere with two SU(3) punctures and one U(1) puncture. This is a free theory consisting of a hypermultiplet in fundamental of two SU(3) flavor groups and its value is given by (3.3),

$$C^{(133)}_{a,\mathbf{x},\mathbf{y}} = \prod_{i,j=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}}\,(a x_i y_j)^{\pm}\right) . \tag{4.2}$$

This vertex corresponds to a free, rank $\mathbf{0}$, theory and we will refer to it as rank zero structure constant. Later on we will define yet another three point function, formally associated to a sphere with two U(1) punctures and one SU(3) puncture. This vertex will have effective rank $-\mathbf{1}$. The metric of the model, $\eta^{\mathbf{x},\mathbf{y}}$, is defined as

$$\eta^{\mathbf{x},\mathbf{y}} = \frac{2}{3}\,\kappa^2\,\Gamma^2(t^2 v) \prod_{1\leqslant i<j\leqslant 3} \frac{\Gamma\left(t^2 v \left(\frac{x_i}{x_j}\right)^{\pm}\right)}{\Gamma\left(\left(\frac{x_i}{x_j}\right)^{\pm}\right)}\,\hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})\,, \tag{4.3}$$

where $\hat{\Delta}(\mathbf{x}^{-1},\mathbf{y})$ is a δ-function kernel defined by

$$\oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i}\,\hat{\Delta}(\mathbf{x},\mathbf{w})\, f(\mathbf{x}) = f(\mathbf{w})\,, \qquad \mathbf{w}\in\mathbb{T}^2 . \tag{4.4}$$

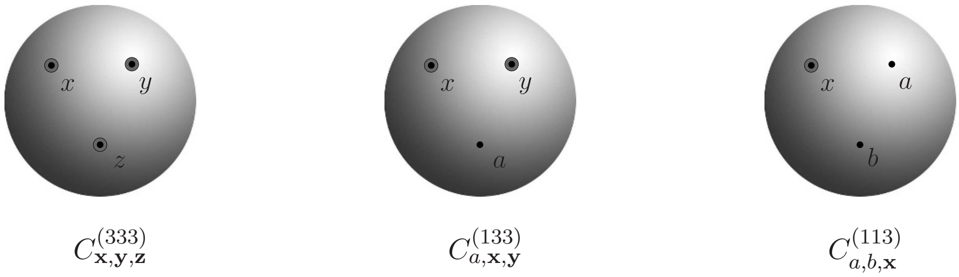

Figure 4. The three structure constants of the TQFT. The dots represent U(1) punctures and the circled dots SU(3) punctures.

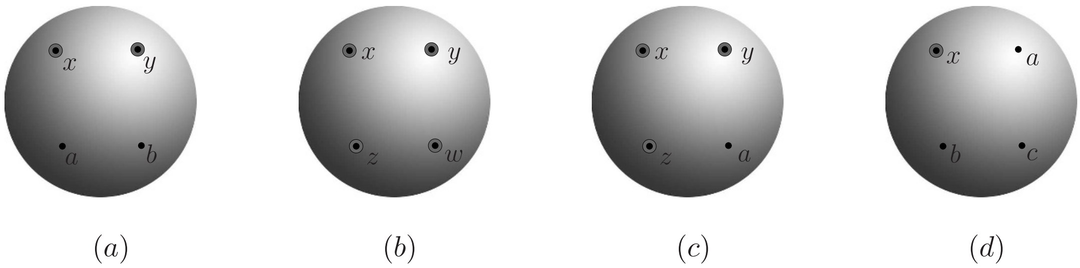

Figure 5. The relevant four-punctured spheres for A_2 theories. The three different degeneration limits of a four-punctured sphere correspond to different S-duality frames. For example, in (a) two of the degeneration limits (when a U(1) puncture collides with an SU(3) puncture) correspond to the weakly-coupled $N_f = 6$ SU(3) theory, the third limit (when two like punctures collide) corresponds to the Argyres-Seiberg theory. In (d) the degeneration limits correspond to the different duality frames of SU(2) SYM with $N_f = 4$ theory plus a decoupled hypermultiplet.

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The indices are contracted as follows

$$A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots} \equiv \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{du_i}{2\pi i u_i} A^{\cdots\mathbf{u}\cdots}B_{\cdots\mathbf{u}\cdots}\Big|_{\prod_{i=1}^{3} u_i=1} . \tag{4.5}$$

Following these definitions the superconformal indices of all the SU(3) generalized quivers are obtained by contracting the structure constants in different ways.

For the S-duality to hold, and subsequently for the structure constants to have a TQFT interpretation, the algebra generated by these objects has to be associative (see figure 5). We proceed to verify this fact.

(333) − (333) associativity. Let us consider the generalized quiver with genus zero and four SU(3) punctures. The index should be invariant under the permutation of the four SU(3) characters,

$$\mathcal{I}_{3333}(\mathbf{x},\mathbf{y};\mathbf{w},\mathbf{z}) = C^{(333)}_{\mathbf{x},\mathbf{y},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{z},\mathbf{w}} = C^{(333)}_{\mathbf{x},\mathbf{z},\mathbf{u}}\eta^{\mathbf{u},\mathbf{v}}C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{w}} . \tag{4.6}$$

At order $O(t^4)$ we find,

$$\mathcal{I}_{3333} \sim t^4 \left[\frac{1}{v}(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + v^2\right] , \tag{4.7}$$

and at order $O(t^6)$,

$$\mathcal{I}_{3333} \sim t^6 \left[-(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + \chi_{\mathbf{8}}(\mathbf{w})) + 3v^3\right] . \tag{4.8}$$

These expressions are symmetric under the exchange $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z} \leftrightarrow \mathbf{w}$. The associativity can be checked to hold to higher orders as well.

(333) − (331) associativity. Let us consider the generalized quiver with genus zero, three SU(3) punctures and one U(1) puncture. The index should be invariant under permutations of the three SU(3) characters

$$\mathcal{I}_{3331}(a, \mathbf{x}; \mathbf{y}, \mathbf{z}) = C^{(133)}_{a,\mathbf{x},\mathbf{u}} \eta^{\mathbf{uv}} C^{(333)}_{\mathbf{v},\mathbf{y},\mathbf{z}} = C^{(133)}_{a,\mathbf{y},\mathbf{u}} \eta^{\mathbf{uv}} C^{(333)}_{\mathbf{v},\mathbf{x},\mathbf{z}} . \tag{4.9}$$

We also expand the integrand in t around $t = 0$. The first non-trivial check is for the coefficient of $\mathcal{I}_{3331}$ at order $O(t^4)$,

$$\mathcal{I}_{3331} \sim t^4 \left[\frac{1}{v}\left(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + 1\right) + v^2\right] , \tag{4.10}$$

which is indeed symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. At order $O(t^6)$,

$$\begin{aligned}\mathcal{I}_{3331} \sim\ & \frac{t^6}{v^{3/2}}\left(a^{-3} + a^{-1}\chi_{\bar{\mathbf{3}}}(\mathbf{x})\chi_{\bar{\mathbf{3}}}(\mathbf{y})\chi_{\bar{\mathbf{3}}}(\mathbf{z}) + a\chi_{\mathbf{3}}(\mathbf{x})\chi_{\mathbf{3}}(\mathbf{y})\chi_{\mathbf{3}}(\mathbf{z}) + a^3\right) \\ & -t^6\left(\chi_{\mathbf{8}}(\mathbf{x}) + \chi_{\mathbf{8}}(\mathbf{y}) + \chi_{\mathbf{8}}(\mathbf{z}) + 1\right) + 2t^6 v^3 ,\end{aligned} \tag{4.11}$$

which is also symmetric under $\mathbf{x} \leftrightarrow \mathbf{y} \leftrightarrow \mathbf{z}$. Again, we can perform systematic checks to arbitrary high order in t.

The (311) three point function and (311) − (331) associativity. The index of the $N_f = 6$ SU(3) theory in the strongly-coupled frame is given in terms of an integral over an SU(2) character. Thus, we can not write it using the structure constants and the metric we defined in the beginning of this section. The strongly-coupled frame is obtained when two U(1) punctures collide and thus in what follows we will formally define a structure constant with two U(1) characters and an SU(3) character such that when contracted with the E_6 structure constant using the metric above it will produce the index of the strongly-coupled frame.

Let us rewrite the index in the strongly-coupled frame,

$$\hat{\mathcal{I}}\,(s, r; \mathbf{y}, \mathbf{z}) = \kappa\, \Gamma(t^2 v) \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma\left(\frac{t^2}{\sqrt{v}} e^{\pm}\, s^{\pm}\right)}{\Gamma(e^{\pm 2})} \Gamma(t^2 v\, e^{\pm 2})\, C\left((e,\, r), \mathbf{y}, \mathbf{z}\right) , \tag{4.12}$$

as rank one (E_6) (333) and rank −1 (113) vertices contracted

$$\begin{aligned}\hat{\mathcal{I}}\,(a, b; \mathbf{y}, \mathbf{z}) &= C^{(113)}_{a,b,\mathbf{x}}\, \eta^{\mathbf{x},\mathbf{x}'}\, C^{(333)}_{\mathbf{x}',\mathbf{y},\mathbf{z}} = \\ &= \frac{2}{3}\kappa^2\, \Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \prod_{i\neq j} \frac{\Gamma(t^2 v\, x_i/x_j)}{\Gamma(x_i/x_j)}\, C^{(113)}\left(a, b, \mathbf{x}^{-1}\right) C^{(333)}\left(\mathbf{x}, \mathbf{y}, \mathbf{z}\right) .\end{aligned} \tag{4.13}$$

For this we define

$$C^{(113)}\left(a,b,\mathbf{x}^{-1}\right) = \frac{3}{2\kappa\Gamma(t^2v)} \oint_{\mathbb{T}} \frac{de}{2\pi i\, e} \frac{\Gamma\left(\frac{t^2}{\sqrt{v}}\, e^{\pm 1}\, s^{\pm 1}\right)\Gamma(t^2 v\, e^{\pm 2})}{\Gamma(e^{\pm 2})} \prod_{i\neq j} \frac{\Gamma(x_i/x_j)}{\Gamma(t^2 v\, x_i/x_j)}\, \hat{\Delta}(\mathbf{x},\mathbf{w})\,. \tag{4.14}$$

Here, $\mathbf{w} = (e,\, r)$ with e an SU(2) character and r a U(1) character. The U(1) charges are related as in (3.16), $s = (a/b)^{3/2}$ and $r = (a\,b)^{-1/2}$. $\hat{\Delta}(\mathbf{x},\mathbf{w})$ is a δ-function kernel defined in (4.4). The (113) vertex has effective rank $-\mathbf{1}$. Using the above definition the TQFT algebra is well defined with all the contractions being SU(3) integrals.

The associativity of (311) vertex contracted with a (333) vertex is achieved by construction: remember that we obtained the index of E_6 SCFT by requiring this property. Let us check the associativity of (331) contracted with (113)

$$\begin{aligned}
\mathcal{I}(a,b;\,c,\mathbf{y}) &= C^{(113)}_{a,b,\mathbf{x}}\,\eta^{\mathbf{x},\mathbf{x}'}\,C^{(331)}_{\mathbf{x}',\mathbf{y},c} = \qquad\qquad (4.15)\\
&\frac{2}{3}\,\kappa^2\,\Gamma(t^2v)^2 \oint \prod_{i=1}^{2} \frac{dx_i}{2\pi i\, x_i} \prod_{i\neq j} \frac{\Gamma(t^2 v\, x_i/x_j)}{\Gamma(x_i/x_j)}\, C^{(113)}\left(a,b,\mathbf{x}^{-1}\right) \prod_{i,j} \Gamma\left(\frac{t^2}{\sqrt{v}}\,(c\, x_i\, y_j)^{\pm 1}\right).\\
&= \prod_{i=1}^{3} \Gamma\left(\frac{t^2}{\sqrt{v}} \left(\frac{c\, y_i}{r^2}\right)^{\pm 1}\right) \times\\
&\qquad \kappa\Gamma(t^2v) \oint \frac{de}{2\pi i\, e} \frac{\Gamma(t^2 v\, e^{\pm 2})}{\Gamma(e^{\pm 2})} \Gamma\left(\frac{t^2}{\sqrt{v}}\, s^{\pm 1}\, e^{\pm 1}\right) \Gamma\left(\frac{t^2}{\sqrt{v}}\,(c\, r\, y_i)^{\pm 1}\, e^{\pm 1}\right)\,.
\end{aligned}$$

This is exactly the index of SU(2) $N_f = 4$ (the fourth line in (4.15)) with a decoupled hypermultiplet in the fundamental of an SU(3) flavor (the third line in (4.15)). Remembering (3.16) and the results of [5, 7] it is easy to show that there is a permutation symmetry between the three U(1) punctures a, b and c,

$$a \quad\leftrightarrow\quad b \quad\leftrightarrow\quad c\,. \tag{4.16}$$

Using the definition (4.14) the index of a sphere with four U(1) punctures is singular. However, we do not have a physical interpretation of this surface and it does not appear in any decoupling limit of a physical theory. Thus, making sense of this surface is not essential.

We have shown that the structure constants define an associative algebra and thus define a TQFT. In particular the superconformal index of theories with equal genus and equal number/type of punctures is the same in agreement with S-duality.

5 Discussion

In this paper we have obtained an explicit expression for the superconformal index of the strongly-coupled SCFT with an E_6 flavor symmetry [10]. The strategy is to use the Argyres-Seiberg duality, which relates a weakly-coupled theory, index of which can be easily obtained through the Lagrangian description of the theory, and E_6 SCFT with part of the global symmetry gauged. The index of the two theories should be the same. Thus, one

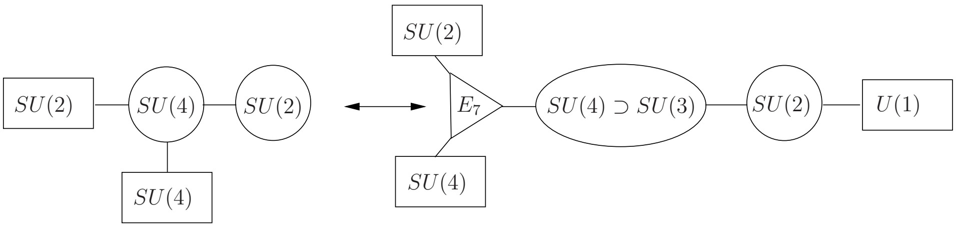

Figure 6. An Argyres-Seiberg duality relating a Lagrangian theory (left quiver) with a theory containing a strongly-coupled E_7 piece (right quiver).

obtains the index of the E_6 theory by "*inverting*" the gauging, see (3.18). Upon gauging a flavor symmetry one looses information about the theory by projecting on gauge invariant states. However, what allows us to "*invert*" the gauging in our case is the fact that additional matter is coupled to the SU(2) gauge group along with the E_6 SCFT, and thus effectively preserves enough information to reconstruct the complete index of E_6 SCFT. We do not have a physical interpretation of the expression for the index (3.18) and it would be very interesting to find one.

In principle one can try to use the same techniques to obtain the superconformal index for other strongly-coupled SCFTs of [1]. However, the generalization is not completely straightforward. Let us discuss the case of the E_7 theory [8, 27, 28] as an example. To obtain the E_7 SCFT we can apply Argyres-Seiberg duality to a Lagrangian theory with $\mathrm{SU}(4) \otimes \mathrm{SU}(2)$ gauge group, with a single hypermultiplet in the bi-fundamental representation and six hypermultiplets in the fundamental representation of SU(4). The Argyres-Seiberg dual of this theory involves an E_7 strongly-coupled piece, with an SU(3) subgroup of E_7 gauged. The theory has a second gauge group factor SU(2) and two hypermultiplets: one in the fundamental of SU(2) and the in bi-fundamental of the two gauge groups. See figure 6. The index of the weakly-coupled theory can be easily written down,

$$\begin{aligned}\mathcal{I}_{\text{weak}} = \kappa\,\Gamma(t^2 v)\, &\oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})} \times \\ &\frac{1}{3}\kappa^3\,\Gamma(t^2 v)^3 \oint_{\mathbb{T}^3} \prod_{i=1}^{3} \frac{du_i}{2\pi i\, u_i} \prod_{i\neq j} \frac{\Gamma\big(t^2 v \frac{u_i}{u_j}\big)}{\Gamma\big(\frac{u_i}{u_j}\big)} \Gamma\Big(\frac{t^2}{\sqrt{v}}(e^{\pm 1}\, u_i\, a)^{\pm 1}\Big) \times \\ &\prod_{i=1}^{4}\prod_{j=1}^{4} \Gamma\Big(\frac{t^2}{\sqrt{v}}(y_j\, u_i\, b)^{\pm 1}\Big) \prod_{i=1}^{4}\prod_{j=1}^{2} \Gamma\Big(\frac{t^2}{\sqrt{v}}(z_j\, u_i\, c)^{\pm 1}\Big)\,.\end{aligned} \tag{5.1}$$

The index of the dual theory is given by

$$\begin{aligned}\mathcal{I}_{\text{strong}} = \kappa\,\Gamma(t^2 v) & \oint_{\mathbb{T}} \frac{de}{2\pi i\, e}\frac{\Gamma(t^2 v e^{\pm 2})}{\Gamma(e^{\pm 2})}\Gamma\left(\frac{t^2}{\sqrt{v}} e^{\pm 1}\, s^{\pm 1}\right) \times \\ & \frac{2}{3}\kappa^2\,\Gamma(t^2 v)^2 \oint_{\mathbb{T}^2} \prod_{i=1}^{2}\frac{du_i}{2\pi i\, u_i}\prod_{i\neq j}\frac{\Gamma\left(t^2 v \frac{u_i}{u_j}\right)}{\Gamma\left(\frac{u_i}{u_j}\right)}\prod_{i=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}(e^{\pm 1}\, u_i\, m)^{\pm 1}\right) \times \\ & C^{(E_7)}\left((u_i,\, r)_{\text{SU}(4)}, \mathbf{y}_{\text{SU}(4)}, \mathbf{z}_{\text{SU}(2)}\right)\,.\end{aligned} \tag{5.2}$$

One can invert the SU(2) integral by the same techniques we used for the E_6 index, but there is no simple inversion formula known to us for the SU(3) integral. To obtain a closed form for the index of the strongly-coupled CFTs appearing in higher rank theories one has to learn how to "invert the superconformal tails".

The superconformal index of the generalized quiver theories can be built from a small number of building blocks, the structure constants and the metric of section 4. We have explicitly shown, at least in perturbation theory in the chemical potential t, that the superconformal index of these theories is consistent with S-duality. These structure constants and metric can be interpreted as defining a $2d$ topological quantum field theory, generalizing to A_2 the construction given in [5] for A_1. It would be very interesting to obtain a Lagrangian description for these TQFTs, perhaps by direct dimensional reduction of the twisted $(2,0)$ theory on $S^3 \times S^1$.

Finally, from a pure mathematics viewpoint, we have seen that S-duality implies a number of identities that must be obeyed by integrals of elliptic Gamma functions and that we have checked perturbatively. We collect these identities in appendix D. It would be nice to find analytic proofs.

Acknowledgments

We thank Davide Gaiotto and Yuji Tachikawa for useful discussions. This work was supported in part by DOE grant DEFG-0292-ER40697 and by NSF grant PHY-0653351-001. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

A t expansion in the weakly-coupled frame

We expand the index (3.4) in t as

$$\mathcal{I}_{a,\mathbf{z};b,\mathbf{y}} = \sum_{k=0}^{\infty} b_k\, t^k. \tag{A.1}$$

The first few orders are

$$
\begin{aligned}
b_0 &= 1, \\
b_1 &= b_2 = b_3 = 0, \\
b_4 &= \frac{1}{v}\chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} + \frac{1}{v} + v^2, \\
b_5 &= -v\left(y + \frac{1}{y}\right), \\
b_6 &= \frac{1}{v^{3/2}}\chi^{\mathrm{SU}(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) - \chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} + v^3 - 1, \\
b_7 &= \frac{1}{v}\left(y + \frac{1}{y}\right)\chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} + \frac{2}{v}\left(y + \frac{1}{y}\right), \\
b_8 &= \frac{1}{v^2}\chi^{\mathrm{SU}(6)}_{sym^2\mathbf{35}} + v\chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} - \frac{1}{\sqrt{v}}\chi^{\mathrm{SU}(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) + v^4 - v\left(y + \frac{1}{y}\right)^2 + 2v, \\
b_9 &= -2\left(y + \frac{1}{y}\right)\chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} + \frac{1}{v^{3/2}}\left(y + \frac{1}{y}\right)\chi^{\mathrm{SU}(6)}_{\mathbf{20}}\left(\left(\frac{a}{b}\right)^{3/2} + \left(\frac{b}{a}\right)^{3/2}\right) - 2\left(y + \frac{1}{y}\right).
\end{aligned}
\tag{A.2}
$$

In the above equation we decomposed $\mathrm{SU}(6) \supset \mathrm{SU}(3)_z \otimes \mathrm{SU}(3)_{y^{-1}} \otimes \mathrm{U}(1)$. The branching of **35** and **20** of SU(6) is given by (see [29]),

$$
\begin{aligned}
\mathbf{35} &= (\mathbf{1},\mathbf{1})_0 + (\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{8})_0 + (\bar{\mathbf{3}},\mathbf{3})_2 + (\mathbf{3},\bar{\mathbf{3}})_{-2}\,, \\
\mathbf{20} &= (\mathbf{1},\mathbf{1})_3 + (\mathbf{1},\mathbf{1})_{-3} + (\bar{\mathbf{3}},\mathbf{3})_{-1} + (\mathbf{3},\bar{\mathbf{3}})_1\,.
\end{aligned}
\tag{A.3}
$$

For example, the character of the adjoint is

$$
\begin{aligned}
\chi^{\mathrm{SU}(6)}_{\mathbf{35},adj} = &\left[(a\,b)^{1/2}(z_1+z_2+z_3) + (a\,b)^{-1/2}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}\right)\right] \times \\
&\times \left[(a\,b)^{-1/2}\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) + (a\,b)^{1/2}(y_1+y_2+y_3)\right] - 1\,.
\end{aligned}
\tag{A.4}
$$

We conclude that the U(1) charge in SU(6) can be identified as $(a\,b)^{-1/2}$.

B Inversion theorem

In this appendix we quote the inversion theorem [11], which we use in section 3.2 to obtain the index of the E_6 theory. Define

$$
\delta(z,\, w; T) \equiv \frac{\Gamma(T\, z^{\pm 1}\, w^{\pm 1};\, p, q)}{\Gamma(T^2,\, z^{\pm 2};\, p, q)}. \tag{B.1}
$$

If T, p and q are such that

$$
|\max(p,\, q)| < |T| < 1\,, \tag{B.2}
$$

then the following theorem holds true. For fixed w on the unit circle we define a contour C_w (see figure 7) in the annulus $\mathbb{A} = \{|T| - \epsilon < |z| < |T|^{-1} + \epsilon\}$ with small but finite $\epsilon \in \mathbb{R}^+$,

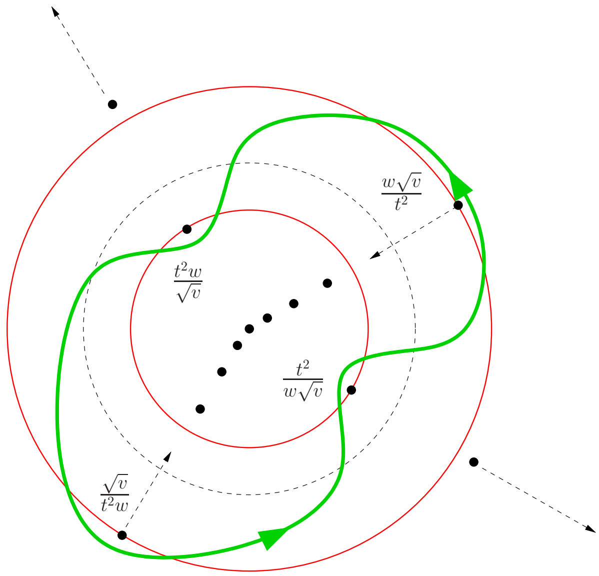

Figure 7. The integration contour C_w (green). The dashed (black) circle is the unit circle $\mathbb{T}$. Black dots are poles of $\Gamma\left(\frac{\sqrt{v}}{t^2}\, w^{\pm 1}\, z^{\pm 1}\right)$. There are four sequences of poles: two sequences starting at $\frac{\sqrt{v}}{t^2} w^{\pm 1}$ and converging to $z = 0$, and two sequences starting at $\frac{t^2}{\sqrt{v}} w^{\pm 1}$ and converging to $z = \infty$. The contour encloses the two former sequences.

such that the points $T^{-1} w^{\pm 1}$ are in its interior and $C_w = C_w^{-1}$ (i.e. an inverse of the point in the interior of C_w is in the exterior of C_w). Let $f(z) = f(z^{-1})$ be a holomorphic function in $\mathbb{A}$. Then for $|T| < |x| < |T|^{-1}$,

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{dz}{2\pi i\, z}\, \delta(z, w; , T^{-1})\, f(z) \longrightarrow f(x) = \kappa \oint_{\mathbb{T}} \frac{dw}{2\pi i\, w}\, \delta(w, x; , T)\, \hat{f}(w)\,. \quad \text{(B.3)}$$

Our expression for the index in the strongly-coupled frame (3.14) is of the form of the right hand side of (B.3). Thus, to use the inversion theorem to obtain the index of E_6 theory we *assume* that this index can be written as

$$\Gamma(t^2 v\, w^{\pm 2})\, C^{(E_6)}\left((w,\, r), \mathbf{y}, \mathbf{z}\right) = \kappa \oint_{C_w} \frac{ds}{2\pi i\, s}\, \frac{\Gamma\left(\frac{\sqrt{v}}{t^2} w^{\pm 1}\, s^{\pm 1}\right)}{\Gamma\left(\frac{v}{t^4},\, s^{\pm 2}\right)}\, F\left(s, r; \mathbf{y}, \mathbf{z}\right)\,, \quad \text{(B.4)}$$

for some function F. The theorem (B.3) then implies that $F\left(s, r; \mathbf{y}, \mathbf{z}\right) = \hat{\mathcal{I}}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ with $\mathcal{I}\left(s, r; \mathbf{y}, \mathbf{z}\right)$ given in (3.14).

C The Coulomb and Higgs branch operators of E_6 SCFT

We collect here a few facts about the Coulomb and the Higgs branches of E_6 SCFT, following the analysis of [13]. Argyres-Seiberg duality can be used to determine the quantum numbers of protected operators of E_6 theory if their dual operators in the dual SU(3) theory are known. The Coulomb branch operator u of the E_6 theory (the operator whose vev parametrized the Coulomb branch) is identified as $\mathrm{Tr}\,\phi^3$ in the SU(3) theory. Since ϕ has quantum numbers $(E, j_1, j_2, R, r) = (1, 0, 0, 0, -1)$, u should have quantum numbers $(3, 0, 0, 0, -3)$ and contribute to the superconformal index as $t^6 v^3$.

The operator $\mathbb{X}$ whose vev parametrized the Higgs branch transforms in the adjoint representation of E_6. Under the $\mathrm{SU}(2) \otimes \mathrm{SU}(6)$ subgroup of E_6 it decomposes as

$$X^i_j, \quad Y^{[ijk]}_\alpha, \quad Z_{\alpha\beta}\,, \tag{C.1}$$

where $i, j, k = 1, \ldots, 6$ are the SU(6) indices, and $\alpha, \beta = 1, 2$ are the SU(2) indices. At the same time, the SU(2) gauge theory provides the quarks q_α, $\tilde{q}_\alpha$ and the F-term constraint

$$Z_{\alpha\beta} + q_{(\alpha}\tilde{q}_{\beta)} = 0\,. \tag{C.2}$$

Thus the gauge-invariant operators are

$$(q\tilde{q}), \quad X^i_j, \quad (Y^{ijk}q), \quad (Y_{ijk}\tilde{q})\,. \tag{C.3}$$

On the SU(3) side, the Higgs branch is parameterized by gauge invariant operators

$$M^i_j = Q^i_a \tilde{Q}^a_j, \quad B^{ijk} = \epsilon^{abc} Q^i_a Q^j_b Q^k_c, \quad \tilde{B}_{ijk} = \epsilon_{abc} \tilde{Q}^a_i \tilde{Q}^b_j \tilde{Q}^c_k\,, \tag{C.4}$$

where Q^i_a and $\tilde{Q}^a_i$ are the squark fields, $i = 1, \ldots, 6$ are flavor indices, and $a = 1, 2, 3$ the color indices.

The duality of the two sides suggests the following identification

$$\mathrm{Tr} M \leftrightarrow (q\tilde{q}), \qquad \hat{M}^i_j \leftrightarrow X^i_j, \tag{C.5}$$

$$B^{ijk} \leftrightarrow (Y^{ijk}q), \qquad \tilde{B}_{ijk} \leftrightarrow (Y_{ijk}\tilde{q}) \tag{C.6}$$

where $\hat{M}^i_j$ is the traceless part of M^i_j. Since the quantum numbers of Q are $(1, 0, 0, 1/2, 0)$, the quantum numbers of $\mathbb{X}$ should be $(2, 0, 0, 1, 0)$, and contribute to the index as t^4/v.

D Identities from S-duality

In this appendix we summarize identities of integrals of elliptic Gamma functions implied by S-duality of the SU(3) quiver theories.

Generalization of [7]. We define

$$\mathcal{I}^{(n)}\left(a\,, \mathbf{z}_{\mathrm{SU}(n)}; b\,, \mathbf{y}_{\mathrm{SU}(n)}\right) \equiv \frac{2^{n-1}}{n!}\kappa^{n-1}\Gamma(t^2 v)^{n-1} \times \tag{D.1}$$

$$\oint_{\mathbb{T}^{n-1}} \prod_{i=1}^{n-1} \frac{dx_i}{2\pi i\, x_i} \frac{\prod_{i=1}^n \prod_{j=1}^n \Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{a z_i}{x_j}\right)^{\pm 1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(b\, y_i\, x_j\right)^{\pm 1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\,\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\Bigg|_{\prod_{j=1}^n x_j = 1}.$$

The claim is that

$$\mathcal{I}^{(n)}\left(a\,, \mathbf{z}_{\mathrm{SU}(n)}; b\,, \mathbf{y}_{\mathrm{SU}(n)}\right) = \mathcal{I}^{(n)}\left(b\,, \mathbf{z}_{\mathrm{SU}(n)}; a\,, \mathbf{y}_{\mathrm{SU}(n)}\right)\,. \tag{D.2}$$

For SU(2) this identity was proven in [7], and for SU(3) we have performed perturbative checks. The usual S-duality of $N_f = 2n$ $\mathrm{SU}(n)$ theories implies that this identity should be true for any n. Note that for $t = v$ this is a special case of identities discussed in [21].

E_6 integral. We define

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) \equiv \frac{2\,\kappa^3\Gamma(t^2 v)^2}{3\,\Gamma(t^2 v\,w^{\pm 2})}\oint_{C_w}\frac{ds}{2\pi i\,s}\,\frac{\Gamma\big(\frac{\sqrt{v}}{t^2}w^{\pm1}\,s^{\pm1}\big)}{\Gamma\big(\frac{v}{t^4},\;s^{\pm2}\big)}\times \tag{D.3}$$

$$\times\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i\,x_i}\frac{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{\frac{1}{3}}\,z_i}{x_j\,r}\right)^{\pm1}\right)\Gamma\left(\frac{t^2}{\sqrt{v}}\left(\frac{s^{-\frac{1}{3}}\,y_i\,x_j}{r}\right)^{\pm1}\right)\prod_{i\neq j}\Gamma\left(t^2 v\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\Gamma\left(\frac{x_i}{x_j}\right)}\,.$$

This integral has manifest symmetry under $\mathrm{SU}(2)_w\otimes\mathrm{SU}(6)$, where the SU(6) has been decomposed as $\mathrm{SU}(3)_{\mathbf{z}}\otimes\mathrm{SU}(3)_{\mathbf{y}^{-1}}\otimes\mathrm{U}(1)_r$. The identification with the index of the E_6 SCFT implies that there must be a symmetry enhancement $\mathrm{SU}(2)_w\otimes\mathrm{SU}(6)\to E_6$. Two properties that are sufficient to guarantee E_6 covariance are: first,

$$C^{(E_6)}\left((w,\,r),\mathbf{y},\mathbf{z}\right) = C^{(E_6)}\left(\left(\frac{w^{1/2}}{r^{3/2}},\,\frac{1}{w^{1/2}\,r^{1/2}}\right),\mathbf{y},\mathbf{z}\right)\,, \tag{D.4}$$

which is the statement that (w,r) combine into a character of SU(3) (which we shall denote by $\mathbf{w}$); second,

$$C^{(E_6)}(\mathbf{w},\mathbf{y},\mathbf{z}) = C^{(E_6)}(\mathbf{y},\mathbf{w},\mathbf{z})\,. \tag{D.5}$$

We presented perturbative evidence for the full E_6 symmetry in the text.

S-dualities of SU(3) quivers. Define

$$\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right) \equiv \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2 v x_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)C^{(E_6)}\left(\mathbf{u},\mathbf{s},\mathbf{x}^{-1}\right)\,, \tag{D.6}$$

$$\mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right) \equiv \oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}\frac{\Gamma\left(t^2 v x_i/x_j\right)}{\Gamma\left(x_i/x_j\right)}C^{(E_6)}\left(\mathbf{y},\mathbf{z},\mathbf{x}\right)\prod_{i,j=1}^{3}\Gamma\left(\frac{t^2}{\sqrt{v}}\left(a\,x_i^{-1}\,u_j\right)^{\pm}\right)\,.$$

The S-dualities of the SU(3) quivers imply

$$\begin{aligned}\mathcal{I}_{3333}\left(\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{s}\right) &= \mathcal{I}_{3333}\left(\mathbf{y},\mathbf{u},\mathbf{z},\mathbf{s}\right)\,,\\ \mathcal{I}_{3331}\left(\mathbf{y},\mathbf{z},\mathbf{u},a\right) &= \mathcal{I}_{3331}\left(\mathbf{y},\mathbf{u},\mathbf{z},a\right)\,.\end{aligned} \tag{D.7}$$

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