

Quasi-normal modes of extremal BTZ black holes in TMG

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ABSTRACT: We study the spectrum of tensor perturbations on extremal BTZ black holes in topologically massive gravity for arbitrary values of the coefficient of the Chern-Simons term, μ . Imposing proper boundary conditions at the boundary of the space and at the horizon, we find that the spectrum contains quasi-normal modes.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence, Black Holes

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1 Introduction

Topologically Massive Gravity (TMG) [1, 2] in three dimensions has attracted a lot of attention recently owing to its rich structure. Its wide range of solutions and also its interesting properties at special values of its coupling constant has opened several directions and venues for studying quantum gravity in the framework of a toy model.

Recall that Einstein gravity in three dimensions, even with a negative cosmological constant, is too trivial a theory at a first glance. This is because the constraint equations are so strict that do not allow any local propagating degrees of freedom. Yet, the theory contains the famous BTZ black holes [3] as well as the Brown-Henneaux boundary gravitons [4]. This theory is conjectured to be dual to a, yet to be found, conformal field theory that lives on the boundary of the space. The most trivial solution of the theory, AdS_3 vacuum, is conjectured to be dual to the classical limit of a highest weight state in the CFT whereas the boundary gravitons are taken in correspondence with the descendant states that constitute a Verma module on the highest weight.

Probably the most challenging question in this context is to give a clear account for the microscopic origin of the BTZ black holes' entropy. Perhaps the closest answer to this question has been given in [5] where the BTZ black holes are interpreted as the classical limit of highest weight states in the CFT. This CFT is conjectured to have a global symmetry group that acts on the highest weight states and the dimension of the representation determines the microscopic degeneracy of the black hole states.¹

¹For some issues related to the quantum behaviour of three dimensional gravity, see e.g. [6–8].

Propagating degrees of freedom can appear if higher derivative corrections are added to the above system. In TMG these corrections are provided by the gravitational Chern-Simons action

$$I_{\text{TMG}} = \frac{1}{16\pi G} \left(I_{\text{EH}} + \frac{1}{\mu} I_{\text{CS}} \right), \quad (1.1)$$

with

$$I_{\text{EH}} = \int d^3x \sqrt{-g} \left(R + \frac{2}{\ell^2} \right), \quad I_{\text{CS}} = \frac{1}{2} \int d^3x \sqrt{-g} \epsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^{\rho} \left(\partial_{\mu} \Gamma_{\lambda\sigma}^{\rho} + \frac{2}{3} \Gamma_{\mu\tau}^{\sigma} \Gamma_{\nu\rho}^{\tau} \right), \quad (1.2)$$

where ℓ determines the cosmological constant through $\Lambda = -1/\ell^2$ and μ is a coupling constant.

TMG contains all the solutions of Einstein gravity including the AdS_3 vacuum. Linearized equations of motion around this vacuum first showed the appearance of a local propagating massive normal mode ψ^M as well as the usual left and right moving boundary gravitons ψ^L and ψ^R [9]. It was then shown that either the massive mode or the BTZ black holes have a negative mass unless the coupling constant takes the value $\mu = 1/\ell$ [9]. At this point ψ^M and ψ^L become degenerate with zero energy and can be removed. This is because exactly at this point gauge symmetry is enhanced to include all the left moving Virasoro generators. What remains is the tower of right moving boundary gravitons which are massless and of course the whole spectrum of BTZ black holes that have non negative masses. The point $\mu = 1/\ell$ was thus called the chiral point and it was conjectured that at this point we are left with a chiral unitary theory.

Soon after, it was shown in [10] that the linearized equations of motion at the critical point have a solution which may be interpreted as a left moving excitation.² However it is worth mentioning that this new mode which has the same asymptotic behavior as AdS wave solutions [21, 22] does not obey Brown-Henneaux boundary conditions [4]. Therefore if we restrict ourselves to solutions which satisfy the Brown-Henneaux conditions one may still have a chiral theory at least classically. On the other hand if we relax the boundary conditions the theory will not be chiral and indeed it was conjectured in [10] and proved in [23, 24] that the dual theory (in the sense of AdS/CFT correspondence [25]) could be a logarithmic CFT (LCFT) [26].³

To explore some features of TMG we study quasi-normal modes (QNM's) of the tensor perturbations in the model. Having put the subject in the context of AdS/CFT correspondence [25], the study of QNM's of asymptotically AdS black holes becomes more interesting as they are giving us information about the behavior of the dual thermal CFT that is living on the boundary of the space time [35, 36]. For usual BTZ black holes in a theory of pure gravity, gravitational tensor QNM's cannot exist as there are no local propagating degrees

²Whether at the critical point the model is really chiral or not has been further investigated in several papers including [11–19]. See also [20] for a rigorous definition of chiral CFT.

³Such a behavior has also been appeared in NMG [27–30], Born-Infeld gravity [31–33] as well as bigravity [34].

of freedom in the bulk. Therefore in such theories, the focus has only been on scalar, fermion or vector perturbations on BTZ black holes [37].⁴

However, in TMG, as well as other higher derivative three dimensional gravities, gravitons can propagate and hence gravitational QNM's become relevant. Such an analysis has recently been done for non-extremal BTZ black holes in TMG [40, 41] (for scalar perturbation in TMG see [42]). In the following we extend these results to the case of extremal BTZ black holes in TMG which requires an independent analysis and which demonstrate some exclusive behaviors (for scalar and fermion perturbation on extremal BTZ see [43]). For a recent review on QNM's see [44].

The paper is organized as follows. In section 2 we will analyze gravitons on extremal BTZ black holes in TMG by making use of the linearized equations of motion. Using the result of section 2 we shall study the QNM's of the tensor perturbations in section 3. the last section is devoted to discussions.

2 Gravitons on extremal BTZ black hole

In this section we make a detailed analysis of gravitons on extremal BTZ black holes in TMG for arbitrary values of $\mu\ell$. We choose the Gaussian Normal coordinate [47, 48] in which the BTZ black hole is given by (for more information see appendix A)

$$ds^2 = \ell^2 \left[L^+ du^2 + L^- dv^2 + d\rho^2 - (e^{2\rho} + L^+ L^- e^{-2\rho}) dudv \right], \quad (2.1)$$

where $u = t/\ell - \phi$, $v = t/\ell + \phi$ and

$$L^\pm = \frac{(r_\pm \pm r_-)^2}{4\ell^2} \quad (2.2)$$

To get the AdS_3 solution one needs to set $r_+^2 = -1$ and $r_- = 0$, while the extremal BTZ black hole is given in the limit of $r_+ = r_-$ where the corresponding metric in the Gaussian Normal coordinate becomes [20, 43]

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = r_{ex}^2 du^2 - \ell^2 e^{2\rho} dudv + \ell^2 d\rho^2 \quad (2.3)$$

Our aim is to study the spectrum of linear perturbations around this background. To proceed, consider the following perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (2.4)$$

Since the background is stationary and static one may start with a general ansatz for the excitations, ψ , as follows (see however [45])

$$\psi_{\mu\nu}(u, v, \rho) = e^{-ihu - i\bar{h}v} F_{\mu\nu}(\rho), \quad (2.5)$$

⁴Quasi-normal modes of quantum corrected BTZ black hole have been studied in [38]. See also [39] for early discussions on BTZ perturbations and quasi-normal modes.

whose real part represents the excitations of the metric; $\text{Re}(\psi_{\mu\nu}) = h_{\mu\nu}$. It should also be noted that since the direction ϕ is periodic, the corresponding momentum along this direction must be quantized. In other words one has

$$h - \bar{h} = k \in \mathbb{Z}. \quad (2.6)$$

By making use of this ansatz the linearized equations of motion reduce to two first order coupled differential equations⁵

$$F'_{vv} = -(\mu - 1)F_{vv} - i\bar{h}F_{\rho v}, \quad F'_{\rho v} = -(\mu + 1)F_{\rho v} - i\bar{h}F_{\rho\rho}, \quad (2.7)$$

together with the following four algebraic equations⁶

$$\begin{aligned} (\mu + 1)F_{\rho v}e^{2\rho} &= 2i(hF_{uv} - \bar{h}F_{uu}), & \mu F_{\rho\rho}e^{4\rho} &= 2i(hF_{\rho v} - \bar{h}F_{\rho u})e^{2\rho} + 4F_{vv}r_{ex}^2, \\ (\mu - 1)F_{\rho v}e^{2\rho} &= 2i(hF_{vv} - \bar{h}F_{uv}), & F_{\rho\rho}e^{4\rho} &= 4(r_{ex}^2 F_{vv} + e^{2\rho} F_{uv}), \end{aligned} \quad (2.8)$$

From these equations it is straightforward to find that F_{vv} satisfies the following second order differential equation

$$F''_{vv} + 2F'_{vv} + [4(\bar{h}h + \bar{h}^2 r_{ex}^2 e^{-2\rho})e^{-2\rho} - (\mu - 1)(\mu - 3)] F_{vv} = 0. \quad (2.9)$$

It is evident from the above differential equation that for the case of $\bar{h} = 0$ the equation has no propagating solutions and therefore we will assume that $\bar{h} \neq 0$. In this case it is useful to define a new variable⁷

$$z = 2i\bar{h}r_{ex}e^{-2\rho} \quad (2.10)$$

by which the equation (2.9) can be recast to a familiar equation, Whittaker equation, as follows

$$\frac{d^2 F_{vv}}{dz^2} + \left[-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right] F_{vv} = 0, \quad (2.11)$$

where $\lambda = \frac{h}{2ir_{ex}}$ and $m = \pm(\frac{\mu}{2} - 1)$. Since the equation is symmetric under the sign of m , in the rest of the paper we only consider the case of $m = \frac{\mu}{2} - 1$.

The most general solution of the above differential equation which is suitable for all ranges of m [46] is

$$F_{vv} = C_1 W_{\lambda, m}(z) + C_2 W_{-\lambda, m}(-z), \quad (2.12)$$

where

$$W_{\lambda, m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - \lambda)} M_{\lambda, m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m - \lambda)} M_{\lambda, -m}(z), \quad (2.13)$$

⁵From now on we set $\ell = 1$.

⁶This is consistent with the fact that the massive graviton in D -dimensional spacetime has $\frac{(D+1)(D-2)}{2}$ degrees of freedom.

⁷The asymmetry of equations under $h \leftrightarrow \bar{h}$ is a consequence of the absence of vv component in the background metric.

and

$$M_{\lambda,m}(z) = z^{m+1/2} e^{-z/2} {}_1F_1\left(m - \lambda + \frac{1}{2}, 1 + 2m; z\right), \quad (2.14)$$

with ${}_1F_1$ being the Kummer confluent hypergeometric function.

Having found F_{vv} , the other components can be obtained from equations (2.8) and (2.7) as follows

$$\begin{aligned} F_{v\rho} &= \frac{i}{\hbar} [(\mu - 1)F_{vv} + F'_{vv}] \\ F_{uv} &= -\frac{1}{2\hbar^2} [((\mu - 1)^2 e^{2\rho} - 2h\bar{h})F_{vv} + (\mu - 1)e^{2\rho} F'_{vv}] \\ F_{\rho\rho} &= -\frac{2}{\hbar^2} [((\mu - 1)^2 - 2\bar{h}^2 r_{ex}^2 e^{-4\rho} - 2h\bar{h}e^{-2\rho})F_{vv} + (\mu - 1)F'_{vv}] \\ F_{u\rho} &= -\frac{i}{\hbar^3} [(\mu(\mu - 1)^2 e^{2\rho} - 2\bar{h}^2 r_{ex}^2 (\mu - 1)e^{-2\rho} + h\bar{h}(1 - 3\mu))F_{vv} + (\mu(\mu - 1)e^{2\rho} - h\bar{h})F'_{vv}] \\ F_{uu} &= \frac{1}{2\bar{h}^4} [(\mu(\mu + 1)(\mu - 1)^2 e^{4\rho} - 4\mu^2 h\bar{h}e^{2\rho} + 2(h^2 - r_{ex}^2(\mu^2 - 1))\bar{h}^2) F_{vv} \\ &\quad + \mu((\mu^2 - 1)e^{2\rho} - 2h\bar{h}) e^{2\rho} F'_{vv}] \end{aligned} \quad (2.15)$$

3 Quasi-normal modes

Black holes as thermodynamic systems, can be studied under small perturbations. The decay of these perturbations are described by quasi-normal modes. In fact the relaxation time for the decay of the black hole perturbation is determined by the imaginary part of the lowest quasi-normal mode. In this section we would like to study the QNM in the extremal black hole.

3.1 Direct derivation

In order to determine QNM's, one needs to solve the wave equation in the black hole background with the specific boundary conditions at horizon and the conformal boundary. Usually one assumes that the wave function vanishes at the boundary while it should be an ingoing wave at the horizon. The appearance of QNM's is the reflection of the fact that these boundary conditions lead to frequencies with a non-zero imaginary part. To see whether we have QNM's for the extremal back hole in TMG one needs to impose the above boundary conditions on the solution that we have found in the previous section.

By looking at the behavior of the Whittaker's function at $z \rightarrow \infty$ (at horizon)

$$W_{\lambda,m}(z) \sim e^{-z/2} z^\lambda, \quad W_{-\lambda,m}(-z) \sim e^{z/2} z^{-\lambda}, \quad (3.1)$$

and from the equation (2.12) one can see that at the horizon one may have two different modes as follows

$$\psi_{vv}(t, \rho) \sim e^{-i(h+\bar{h})t} W_{\pm\lambda,m}(\pm z) \sim e^{-i[(h+\bar{h})t \pm \bar{h}r_{ex}e^{-2\rho}] \mp 2\lambda\rho}. \quad (3.2)$$

It is then evident that for $\text{Re}(1 + h/\bar{h}) > 0$, the solution (2.12) gives an ingoing wave at the horizon provided $C_1 = 0$. As we will see, since h is purely imaginary, the condition

$\text{Re}(1 + h/\bar{h}) > 0$ is always satisfied. On the other hand taking into account that

$$F'_{vv} \sim e^{-2\rho} F_{vv} \quad \text{and} \quad F_{vv} \sim e^{i\bar{h}r_{ex}e^{-2\rho}} \quad \text{as} \quad \rho \rightarrow -\infty, \quad (3.3)$$

and from the equation (2.15), we find that if F_{vv} is ingoing at the horizon, then all other components will be ingoing as well.

Having imposed the condition at the horizon one needs to be sure that the solution vanishes as we approach the boundary. Therefore we should impose the condition that the dominant component of the solution (2.15) is zero at the boundary, $z = 0$.

To proceed it is worth noting that the near boundary behavior of the Whittaker's function, where $z \rightarrow 0$, is given by [46]

$$W_{-\lambda, m}(z) \sim \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m + \lambda)} z^{\frac{1}{2}+m} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m + \lambda)} z^{\frac{1}{2}-m}. \quad (3.4)$$

Therefore, whatever the dominant component would be, the possible poles one might have could come from either the condition $\frac{1}{2} + m + \lambda = 0, -1, \dots$ or $\frac{1}{2} - m + \lambda = 0, -1, \dots$.

Case 1. Consider the case where the poles are given by $\frac{1}{2} + m + \lambda = -n$, in which

$$h = -ir_{ex}(2n + \mu - 1), \quad \text{for } n = 0, 1, 2, \dots. \quad (3.5)$$

In this case the dominant component at the boundary will be F_{vv} (see appendix B) which has the following near boundary behavior

$$F_{vv} \sim \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m + \lambda)} z^{\frac{1}{2}+m}. \quad (3.6)$$

We thus find that this component vanishes as $z \rightarrow 0$, if $m + \frac{1}{2} > 0$ or $\mu > 1$. Note also that the lowest mode ($n = 0$) also decays in time if $\mu > 1$. Therefore in this case we get non-trivial QNM's for $\mu > 1$ whose frequencies are given by

$$\omega_n = k - 2ir_{ex}(2n + \mu - 1), \quad \text{for } n = 0, 1, 2, \dots. \quad (3.7)$$

Case 2. If $\frac{1}{2} - m + \lambda = -n$, for which

$$h = -ir_{ex}(2n + 3 - \mu), \quad \text{for } n = 0, 1, 2, \dots, \quad (3.8)$$

it is evident from (2.15), and also from our discussions in the appendix B, that the dominant component is F_{uu} . The behavior of this component at the boundary is now given by

$$F_{uu} \sim e^{4\rho} F_{vv} \sim \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m + \lambda)} z^{-\frac{3}{2}-m}, \quad (3.9)$$

which vanishes for $m + \frac{3}{2} < 0$ or $\mu < -1$. On the other hand for the lowest mode to be a decaying mode in time one needs to have $\mu < 3$ which is automatically satisfied. So for $\mu < -1$ we get QNM's whose frequencies are given by

$$\omega_n = k - 2ir_{ex}(2n + 3 - \mu), \quad \text{for } n = 0, 1, 2, \dots. \quad (3.10)$$

To summarize, we find QNM's for tensor perturbations on extremal BTZ black holes in TMG when $\mu l < -1$ or $\mu l > 1$. The cases of $\mu l = \pm 1$ will be discussed later. This result is particularly interesting because QNM's are usually associated with thermal properties of the horizon. For an extremal horizon, a priori, it is not obvious why we should still have QNM's. Before making speculations on this point, we would like to compare our results with those in the literature by rederiving QNM's using a certain procedure that has recently been used in [40], namely, by imposing the so called *chiral highest weight condition*. This is the subject of next subsection.

3.2 Chiral highest weight condition

In this subsection we would like to compare our procedure to those in the literature (see [40]). The main observation is that the QNM's we have found in the previous subsection are given in terms of the Whittaker's functions which satisfy a certain recursive relation [46]

$$z \frac{d}{dz} W_{\lambda,m}(z) = \left(\lambda - \frac{z}{2} \right) W_{\lambda,m}(z) - \left[m^2 - \left(\lambda - \frac{1}{2} \right)^2 \right] W_{\lambda-1,m}(z). \quad (3.11)$$

In terms of the ρ coordinate, this relation may be recast to the following form

$$\frac{1}{2r_{ex}} \left(-r_{ex} \partial_\rho - i\hbar - i\bar{\hbar} r_{ex}^2 e^{-2\rho} \right) W_{-\lambda,m}(-z) = \left(\frac{1}{2} + m + \lambda \right) \left(\frac{1}{2} - m + \lambda \right) W_{-\lambda-1,m}(-z). \quad (3.12)$$

Restricting our attention to the lowest mode, in which either $\frac{1}{2} + m + \lambda$ or $\frac{1}{2} - m + \lambda$ is zero, and taking into account that the u and v dependence of the wave function is given by (2.5), the above equation reduces to

$$-\frac{1}{2r_{ex}} \left(\partial_u + 2r_{ex}^2 e^{-2\rho} - r_{ex} \partial_\rho \right) e^{-ihu - i\bar{\hbar}v} F_{vv} = 0. \quad (3.13)$$

Now consider the following operators

$$\begin{aligned} L_0 &= -\frac{1}{2r_{ex}} \partial_u, \\ L_1 &= -\frac{e^{2r_{ex}u}}{2r_{ex}} \left[\partial_u + 2r_{ex}^2 e^{-2\rho} \partial_v - r_{ex} \partial_\rho \right], \\ L_{-1} &= -\frac{e^{-2r_{ex}u}}{2r_{ex}} \left[\partial_u + 2r_{ex}^2 e^{-2\rho} \partial_v + r_{ex} \partial_\rho \right]. \end{aligned} \quad (3.14)$$

These operators satisfy the $SL(2, R)$ algebra and are identified with the Killing vectors of the background (2.3). We therefore find out that the recursive relations for Whittaker functions require our lowest QNM to be annihilated by L_1 . That is, the lowest mode satisfies a chiral highest weight condition.

Motivated by the above observation, one may turn around the argument as follows; start by imposing the chiral highest weight condition on the lowest mode, solve the equations of motion and then impose the proper boundary conditions. That is, we start with

$$L_1 \psi^{(0)}_{\mu\nu} = 0. \quad (3.15)$$

Using the notation of the previous subsection the above condition can be solved easily,

$$\begin{aligned}
F_{vv} &= E_1 z^{-\lambda} e^{z/2}, \\
F_{v\rho} &= \left(E_2 - \frac{iE_1}{\hbar} z \right) z^{-\lambda} e^{z/2}, \\
F_{\rho\rho} &= \left(E_3 - 2\frac{iE_2}{\hbar} z - \frac{E_1}{\hbar^2} z^2 \right) z^{-\lambda} e^{z/2}, \\
F_{uv} &= r_{ex} (2i\bar{h}E_4 z^{-1} + E_2) z^{-\lambda} e^{z/2}, \\
F_{u\rho} &= r_{ex} \left(\frac{E_2}{i\hbar} z + (E_3 + 2E_4) + 2i\bar{h}E_5 z^{-1} \right) z^{-\lambda} e^{z/2}, \\
F_{uu} &= r_{ex}^2 (E_3 + 4i\bar{h}E_5 z^{-1} - 4\bar{h}^2 E_6 z^{-2}) z^{-\lambda} e^{z/2},
\end{aligned} \tag{3.16}$$

where E_i 's are constants to be determined by the equations of motion. In fact from the linearized equations of motion, $D^M \psi_{\mu\nu} = 0$, one finds

$$\begin{aligned}
E_2 &= \frac{h + ir_{ex}(\mu - 1)}{\bar{h}r_{ex}} E_1, \\
E_3 &= 2 \frac{(h + ir_{ex}(\mu - 1))(h + 2ir_{ex})}{(\bar{h}r_{ex})^2} E_1, \\
E_4 &= \frac{(h + ir_{ex}(\mu - 1))(h + 2ir_{ex})}{2(\bar{h}r_{ex})^2} E_1, \\
E_5 &= \frac{(h + ir_{ex}(\mu - 1))(h + 2ir_{ex})(h + 3ir_{ex})}{(\bar{h}r_{ex})^3} E_1, \\
E_6 &= \frac{(h + ir_{ex}(\mu - 1))(h + 2ir_{ex})(h + 3ir_{ex})(h + 4ir_{ex})}{(\bar{h}r_{ex})^4} E_1.
\end{aligned} \tag{3.17}$$

Moreover the highest weight, h , is found to be either $h = -ir_{ex}(\mu - 1)$ or $h = ir_{ex}(\mu - 3)$, which are indeed the poles of the lowest mode obtained in the previous subsection. In particular for the case of $h = -ir_{ex}(\mu - 1)$ the solution reads

$$F_{\mu\nu} = e^{-i\left(\frac{h}{r_{ex}}\rho - \bar{h}r_{ex}e^{-2\rho}\right)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2r_{ex}e^{-2\rho} \\ 0 & 2r_{ex}e^{-2\rho} & 4r_{ex}^2 e^{-4\rho} \end{pmatrix}, \tag{3.18}$$

which has to be compared with (B.6) obtained from the direct computations. This represents a QNM if $\mu > 1$ with the frequency $\omega = k - 2ir_{ex}(\mu - 1)$, in agreement with (3.7).

On the other hand for the other case, $h = ir_{ex}(\mu - 3)$, the solution is given by

$$\begin{aligned}
E_2 &= \frac{2i}{\hbar}(\mu - 2)E_1, \\
E_3 &= \frac{-4}{\hbar^2}(\mu - 2)(\mu - 1)E_1, \\
E_4 &= \frac{-1}{\hbar^2}(\mu - 2)(\mu - 1)E_1, \\
E_5 &= \frac{-2i}{\hbar^3}\mu(\mu - 2)(\mu - 1)E_1, \\
E_6 &= \frac{1}{\hbar^4}\mu(\mu - 2)(\mu^2 - 1)E_1,
\end{aligned} \tag{3.19}$$

which represents a QNM if $\mu < -1$ with the frequency $\omega = k - 2ir(3 - \mu)$ in agreement with (3.10).

Note that what we have found in this subsection is the lowest QNM in two different cases given by the highest weights $h = -ir_{ex}(\mu - 1)$ or $h = ir_{ex}(\mu - 3)$. This confirms our calculations in the previous section.

It is important to also note that imposing the chiral highest weight condition guarantees that the solutions we find are ingoing at the horizon. Moreover from the above considerations one would expect that the higher QNM's can be obtained by apply L_{-1} on the lowest mode. This is indeed the case if we use another recursive relation,

$$W_{-\lambda+1,m}(-z) = (-1/2z + \lambda)W_{\lambda,m}(-z) - z \frac{d}{dz} W_{-\lambda,m}(-z). \quad (3.20)$$

Writing the n th mode as

$$\psi_{\mu\nu}^{(n)}(u, v, \rho) = e^{-ih_n u - i\bar{h}_n v} W_{-\lambda_n, m}(-z), \quad (3.21)$$

where $\bar{h}_n = h_n - k$, $h_n = 2ir_{ex}\lambda_n$ and $\lambda_n = -(\frac{1}{2} \pm m + n)$, the $(n + 1)$ th mode is found by using (3.20) and the fact that two consecutive λ 's are related by $\lambda_{n+1} = \lambda_n - 1$,

$$\psi_{\mu\nu}^{(n+1)}(u, v, \rho) = e^{-2r_{ex}u - 2r_{ex}v} [-z/2 + \lambda_n - z\partial_z] \psi_{\mu\nu}^{(n)}(u, v, \rho). \quad (3.22)$$

This can be written in a more suggestive form as

$$\psi_{\mu\nu}^{(n+1)}(u, v, \rho) = -e^{-2r_{ex}v} L_{-1} \psi_{\mu\nu}^{(n)}(u, v, \rho), \quad (3.23)$$

where L_{-1} is given in (3.14). We therefore find that

$$\psi_{\mu\nu}^{(n)}(u, v, \rho) = (-e^{-2r_{ex}v} L_{-1})^n \psi_{\mu\nu}^{(0)}(u, v, \rho). \quad (3.24)$$

Comparing this with the non-extremal case, the factor of $-e^{-2r_{ex}v}$ is all that remains from \bar{L}_{-1} .

3.3 QNM's for $\mu = \pm 1$

We now turn to the question of whether QNM's persist to exist at the special values of the coupling constant, $\mu = \pm 1$.

Looking at the solutions we found in the previous sections at the special value of , say, $\mu = 1$, we find that they do not fall off at the boundary, and besides, the lowest frequency has no imaginary part. This might lead to the conclusion that we no longer have QNM's at this point. But we should note that exactly at this point, the solutions we have found degenerate with the left moving boundary gravitons and are no longer propagating. However, a new propagating mode appears at this point which is logarithmic and which is given by

$$\psi_{\mu\nu}^{\text{new}} = \left. \frac{d\psi_{\mu\nu}}{d\mu} \right|_{\mu=1} = -i \left(u \frac{dh}{d\mu} + v \frac{d\bar{h}}{d\mu} \right)_{\mu=1} \psi_{\mu\nu} + e^{-ihu - i\bar{h}v} F_{\mu\nu}^{\text{new}}, \quad (3.25)$$

where $F_{\mu\nu}^{\text{new}} = \left. \frac{dF_{\mu\nu}}{d\mu} \right|_{\mu=1}$. At $\mu = 1$ with $h = -2ir_{ex}n$, where the dominant component is F_{vv} , one finds

$$\psi_{vv}^{\text{new}} = (-2r_{ex}tF_{vv} + F_{vv}^{\text{new}}) e^{-ik(t-\phi)-4nr_{ex}t}. \quad (3.26)$$

Using the explicit expressions for F_{vv} one observes that for $n = 0$ as we approach the boundary one gets

$$\psi_{\mu\nu}^{\text{new}} \rightarrow \infty, \quad (3.27)$$

whereas for $n \geq 1$ we have

$$\psi_{\mu\nu}^{\text{new}} \rightarrow \text{finite}. \quad (3.28)$$

Therefore we get no QNM's at $\mu = 1$.

At $\mu = -1$ with $h = -2ir_{ex}(2+n)$, where the dominant component is F_{uu} , we have

$$\psi_{uu}^{\text{new}} = (2r_{ex}tF_{uu} + F_{uu}^{\text{new}}) e^{-ik(t-\phi)-4(n+2)r_{ex}t}. \quad (3.29)$$

Plugging the solutions in this expression we find that we have QNM's in this case even for $n = 0$ with frequencies given by (3.10) with $\mu = -1$.

3.4 Standing waves

To complete our analysis of the tensor perturbations, we will consider the case of $\bar{h} = 0$. Note that the results we have found so far were based on the assumption that $\bar{h} \neq 0$ (see equation (2.10)). Setting $\bar{h} = 0$ one encounters two possibilities depending on whether h is zero or not. Of course in both cases, the perturbations result in standing waves.

In the first case where $h \neq 0$ and $\mu \neq -1, -2$, the equations (2.7) and (2.8) can be solved to give

$$\begin{aligned} F_{vv}(\rho) &= C_1(\mu - 1)e^{-(\mu-1)\rho} \\ F_{\rho v}(\rho) &= 2ihC_1e^{-(\mu+1)\rho} \\ F_{\rho\rho}(\rho) &= \frac{4C_1}{\mu}[(\mu - 1)r_{ex}^2 - h^2]e^{-(\mu+3)\rho} \\ F_{uv}(\rho) &= -\frac{C_1}{\mu}[(\mu - 1)^2r_{ex}^2 + h^2]e^{-(\mu+1)\rho} \\ F_{\rho u}(\rho) &= -\frac{2ihC_1}{\mu(\mu + 1)}[(\mu - 1)^2r_{ex}^2 + h^2]e^{-(\mu+3)\rho} \\ F_{uu}(\rho) &= \frac{C_1}{\mu(\mu + 1)(\mu + 2)}[(\mu + 1)^2r_{ex}^2 + h^2][(\mu - 1)^2r_{ex}^2 + h^2]e^{-(\mu+3)\rho} + C_2e^{(\mu+1)\rho}. \end{aligned} \quad (3.30)$$

As we anticipated, the above solution is not well defined for $\mu = -1$ and $\mu = -2$. Indeed for the latter case where F_{uu} is singular, one finds that the solution is modified to one with a logarithmic behavior. More precisely in this case F_{uu} is given by

$$F_{uu}(\rho) = [-C_1(h^2 + 9r_{ex}^2)(h^2 + r_{ex}^2)\rho + C_2]e^{-\rho}. \quad (3.31)$$

The other components can be read off from (3.30) with $\mu = -2$. For the case of $\mu = -1$, the equations (2.7) and (2.8) put a constrain on the parameters $h^2 + 4r_{ex}^2 = 0$. The corresponding solution is

$$\begin{aligned} F_{uv} &= 0, & F_{vv} &= C_1 e^{2\rho}, & F_{\rho\rho} &= 4C_1 r_{ex}^2 e^{-2\rho}, & F_{\rho v} &= -iC_1 h, & F_{\rho u} &= C_2 e^{-2\rho}, \\ F_{uu} &= C_3 + \frac{i h C_2}{2} e^{-2\rho}. \end{aligned} \tag{3.32}$$

On the other hand for $\bar{h} = h = 0$, from (2.7) and (2.8) we observe that μ must be either $\mu = 1$ or $\mu = -1$. For the former case one finds

$$F_{\rho u} = F_{uv} = 0, \quad F_{vv} = C_1, \quad F_{\rho\rho} = 4C_1 r_{ex}^2 e^{-4\rho}, \quad F_{\rho v} = C_2 e^{-2\rho}, \quad F_{uu} = C_3 e^{2\rho}. \tag{3.33}$$

while for $\mu = -1$ we get

$$F_{\rho v} = 0, \quad F_{\rho u} = C_1 e^{-2\rho}, \quad F_{vv} = C_2 e^{2\rho}, \quad F_{uv} = -2r_{ex}^2 C_2, \quad F_{\rho\rho} = -4C_2 r_{ex}^2 e^{-2\rho}, \quad F_{uu} = C_3. \tag{3.34}$$

We note, however, that at $\mu = \pm 1$, the model develops logarithmic solutions. In our notation we get

$$\begin{aligned} \text{for } \mu = 1, & \quad F_{uu}^{\text{new}} = \left. \frac{dF_{uu}}{d\mu} \right|_{\mu=1} \sim \rho e^{2\rho}, \\ \text{for } \mu = -1, & \quad F_{uu}^{\text{new}} = \left. \frac{dF_{uu}}{d\mu} \right|_{\mu=-1} \sim \rho, \end{aligned} \tag{3.35}$$

which are the logarithmic solutions found in [10].

4 Discussions

In this paper we have considered tensor perturbations on the extremal BTZ black hole in TMG. We have found that, imposing proper boundary conditions at the boundary of the space and at the horizon, one finds QNM's in the spectrum. These modes vanish for $r_{ex} = 0$, where we have $M = J = 0$, or $T_L = T_R = 0$ from the boundary CFT point of view.

We note that our considerations do not have a counterpart in pure Einstein gravity in three dimensions. This is due to the fact the gravity in three dimensions even with a negative cosmological constant does not have propagating degrees of freedom. Higher derivative terms allow for such propagating modes.

Since QNM's are usually associated with thermal properties of the black holes, it is quite interesting that the model we are considering supports QNM's even for extremal black holes. This is indeed one of the special features of TMG.

Recall that for generic values of μ , TMG contains negative masses either in perturbative states or in the spectrum of BTZ black holes. Even at the chiral points $\mu = \pm 1$, the appearance of logarithmic modes render the system non-unitary unless a strict Brown-Henneaux boundary condition is imposed. Whether or not such an assumption remains valid at a quantum level, allowing for a chiral unitary sector to decouple at the chiral

points, remains to be verified. A natural venue for such an analysis is the AdS/CFT correspondence where one can study the n -point functions of the dual field theory and see whether the above decoupling is possible. Another direction is to calculate the gravitational partition function of TMG around any of its vacua.

The non-unitarity of TMG might explain the appearance of QNM's on extremal black holes. Since we have allowed logarithmic boundary conditions for perturbations, these modes persist to exist at the chiral point. It would be interesting to understand the consequences of the result from the dual CFT.

As a final remark we recall that the boundary condition we have imposed was to have an ingoing wave as we approach the horizon and all of our results rely on this assumption. Another possibility is to impose a vanishing flux at the horizon and at the boundary. Using the properties of the Whittaker's function, one can see that this assumption requires $\bar{h} = 0$ which, as we have seen, results in standing modes.

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A Gaussian normal coordinates

The metric of BTZ black holes is usually written as

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2(d\phi + N^\phi dt)^2, \tag{A.1}$$

with

$$N^2 = -8MG + \frac{r^2}{\ell^2} + \frac{16G^2 J^2}{r^2} \quad N^\phi = -\frac{4GJ}{r^2}. \tag{A.2}$$

If we define

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+ r_-}{4G\ell}, \tag{A.3}$$

then the metric takes the form

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 \ell^2} dt^2 + \frac{r^2 \ell^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{r^2 \ell} dt \right)^2. \tag{A.4}$$

By the following change of coordinates

$$r^2 = r_+^2 \cosh^2(\rho - \rho_0) - r_-^2 \sinh^2(\rho - \rho_0), \quad e^{2\rho_0} = \frac{r_+^2 - r_-^2}{4\ell^2}, \tag{A.5}$$

we go to the Gaussian Normal coordinates [47, 48]

$$ds^2 = \ell^2 [L^+ du^2 + L^- dv^2 + d\rho^2 - (e^{2\rho} + L^+ L^- e^{-2\rho}) dudv], \tag{A.6}$$

where $u = t/\ell - \phi$, $v = t/\ell + \phi$ and

$$L^\pm = \frac{(r_+ \pm r_-)^2}{4\ell^2}. \tag{A.7}$$

Changing the variables as

$$\frac{e^{2\rho}}{\sqrt{L^+L^-}} = e^{2\hat{\rho}}, \tag{A.8}$$

we obtain

$$ds^2 = \ell^2 [L^+ du^2 + L^- dv^2 + d\hat{\rho}^2 - 2\sqrt{L^+L^-} \cosh(2\hat{\rho}) dudv]. \tag{A.9}$$

Pure AdS₃ is obtained by setting

$$r_+^2 = -1, \quad r_- = 0. \tag{A.10}$$

If we make a second coordinate transformation as

$$\hat{u} = 2u\sqrt{L^+}, \quad \hat{v} = 2v\sqrt{L^-}, \tag{A.11}$$

we have

$$ds^2 = \ell^2 \left[\frac{1}{4} d\hat{u}^2 + \frac{1}{4} d\hat{v}^2 + d\hat{\rho}^2 - \frac{1}{2} \cosh(2\hat{\rho}) d\hat{u} d\hat{v} \right], \tag{A.12}$$

which is the BTZ black hole with unit mass and zero angular momentum.

In the extremal limit, $r_+ = r_- = r_{ex}$, the transformation (A.8) is not well defined. Starting from

$$ds^2 = -\frac{(r^2 - r_{ex}^2)^2}{r^2 \ell^2} dt^2 + \frac{r^2 \ell^2}{(r^2 - r_{ex}^2)^2} dr^2 + r^2 \left(d\phi - \frac{r_{ex}^2}{r^2 \ell} dt \right)^2, \tag{A.13}$$

the transformation

$$e^{2\rho} = \frac{r^2 - r_{ex}^2}{\ell^2}, \tag{A.14}$$

takes us to

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = r_{ex}^2 du^2 - \ell^2 e^{2\rho} dudv + \ell^2 d\rho^2, \tag{A.15}$$

where u and v are defined as before. In this coordinate $\rho = -\infty$ corresponds to the location of horizon, $r = r_{ex}$, and $\rho = \infty$ corresponds to $r = \infty$.

B Dominant component

In this appendix we show that in the first case where the poles are given by $\frac{1}{2} + m + \lambda = 0, -1, -2, \dots$, the dominant component is F_{vv} , as we approach the boundary. To proceed we note that the Whittaker's function obeys the following recursive relation [46]

$$z \frac{d}{dz} W_{\lambda, m}(z) = \left(\lambda - \frac{z}{2} \right) W_{\lambda, m}(z) - \left[m^2 - \left(\lambda - \frac{1}{2} \right)^2 \right] W_{\lambda-1, m}(z). \tag{B.1}$$

Therefore setting

$$\frac{1}{2} + m + \lambda = -n \quad \text{with} \quad n = 0, 1, 2, \dots, \quad (\text{B.2})$$

one gets

$$F'_{vv} = (2\lambda - z)C_2W_{-\lambda,m}(-z) + 2n \left(\lambda - m + \frac{1}{2} \right) C_2W_{-\lambda-1,m}(-z). \quad (\text{B.3})$$

On the other hand, at the poles (B.2), by making use of the asymptotic behavior of the Whittaker's function one finds

$$F_{vv} = C_2W_{-\lambda,m}(-z) = C_2 \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m + \lambda)} (-z)^{\frac{1}{2}+m} \left(1 - \frac{\lambda}{2m+1}z + \mathcal{O}(z^2) \right), \quad (\text{B.4})$$

$$F'_{vv} = C_2 \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m + \lambda)} (-z)^{\frac{1}{2}+m} \left(2(\lambda + n) - \frac{2\lambda^2 + 2n(\lambda + 1) + 2m + 1}{2m + 1}z + \mathcal{O}(z^2) \right),$$

as $z \rightarrow 0$. Plugging these expressions in (2.15) we arrive at

$$\begin{aligned} F_{uv} &= -\frac{1}{\hbar} [ir_{ex}(2m+1)(2\lambda+2m+1+2n)z^{-1} + \mathcal{O}(1)] F_{vv}, \\ F_{u\rho} &= -\frac{2i}{\hbar^2} [ir_{ex}(2m+2)(2m+1)(2\lambda+2m+1+2n)z^{-1} + \mathcal{O}(1)] F_{vv}, \\ F_{uu} &= -\frac{2}{\hbar^2} [r_{ex}^2(2m+3)(2m+2)(2m+1)(2\lambda+2m+1+2n)z^{-2} \\ &\quad - 2r_{ex}^2(2m+2)(2m\lambda+2m+5\lambda+3)(2\lambda+2m+1+2n)z^{-1} + \mathcal{O}(1)] F_{vv}, \end{aligned}$$

which shows that the leading terms evaluated at the poles (B.2) are zero. Therefore all components, at most, could be in the same order as F_{vv} . As a result to find the QNM's it is enough to impose the boundary condition on F_{vv} .

It is worth mentioning that since we are evaluating the expressions at the poles (B.2), one has

$$\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m + \lambda)} = (-1)^n (2m+1)_n \quad \text{where} \quad (a)_n = a(a+1)(a+2)\dots(a+n), \quad (\text{B.5})$$

which shows that at the points $\mu = 2 - k$, $k = 1, 2, \dots, n$, the dominant term in the above expansion starts from $\mathcal{O}(z^{m+\frac{1}{2}+k})$. Nevertheless since we are interested in the case where $\mu > 1$, these points do not affect the validity of our conclusion.

Note that for the lowest mode, where $h = -ir_{ex}(\mu - 1)$, we get

$$\begin{aligned} F_{uv} &= -\frac{1}{2ir_{ex}\bar{h}^2} (h + ir_{ex}(\mu - 1)) [(\mu - 1)e^{2\rho} - 2i\bar{h}r_{ex}] F_{vv}, \\ F_{u\rho} &= -\frac{1}{r_{ex}\bar{h}^3} (h + ir_{ex}(\mu - 1)) [\mu(\mu - 1)e^{2\rho} - \bar{h}(h - 2\mu ir_{ex}) - 2\bar{h}^2 r_{ex}^2 e^{-2\rho}] F_{vv}, \\ F_{uu} &= \frac{1}{2ir_{ex}\bar{h}^4} (h + ir_{ex}(\mu - 1)) [\mu(\mu^2 - 1)e^{4\rho} - 2\bar{h}(h + ir_{ex}(\mu + 1))(\mu e^{2\rho} + \bar{h}r_{ex})] F_{vv}, \end{aligned}$$

which are all zero. This leads to the following expression for the perturbations

$$F_{\mu\nu} = F_{vv} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2r_{ex}e^{-2\rho} \\ 0 & 2r_{ex}e^{-2\rho} & 4r_{ex}^2e^{-4\rho} \end{pmatrix}. \quad (\text{B.6})$$

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