

Skyrmions, instantons and holography

Paul Sutcliffe

*Department of Mathematical Sciences, Durham University,
Durham DH1 3LE, U.K.*

E-mail: p.m.sutcliffe@durham.ac.uk

ABSTRACT: Some time ago, Atiyah and Manton observed that computing the holonomy of Yang-Mills instantons yields good approximations to static Skyrme solutions of the Skyrme model. This paper provides an extension and explanation of this result, by proving that instanton holonomies produce exact solutions of a BPS Skyrme model, in which the Skyrme field is coupled to a tower of vector mesons. Neglecting any (or indeed all) of the vector mesons breaks the scale invariance and removes the BPS property of the Skyrmions. However, it is shown that a truncation of the BPS Skyrme theory, in which only the first vector meson is included, already moves the Skyrme model significantly closer to the BPS system. A theory that is close to a BPS system is required to reproduce the experimental data on binding energies of nuclei. A zero-mode quantization of the Skyrme model is performed in the truncated BPS theory and the results are compared to the physical properties of the nucleon. The approach is an analogue in five-dimensional Minkowski spacetime of a recent holographic construction of a Skyrme model by Sakai and Sugimoto, based on a string theory derivation of a Yang-Mills-Chern-Simons theory in a curved five-dimensional spacetime.

KEYWORDS: Solitons Monopoles and Instantons, Sigma Models

Contents

1	Introduction	1
2	Skyrmions and instantons	2
3	An abelian prototype	5
4	The Skyrme model from Yang-Mills theory	9
5	Including the first vector meson	10
6	The Skyrmion and its instanton approximation	12
7	Quantization of the Skyrmion	16
8	Conclusion	20

1 Introduction

The Skyrme model [1] is a nonlinear theory of pions with topological soliton solutions, called Skyrmions, that are identified as baryons. In this paper, three outstanding Skyrmion issues are drawn together and progress is made by introducing an extended Skyrme model, obtained from five-dimensional Yang-Mills theory. The three outstanding issues that the new extended theory addresses are as follows.

Firstly, the Skyrme model is not a BPS (Bogomolny-Prasad-Sommerfield) theory, in the sense that the soliton solutions do not attain the topological lower bound on the energy. In fact, the single Skyrmion exceeds the topological energy bound by 23% in the case of massless pions. This energy excess allows the possibility of a significant classical binding energy for higher charge Skyrmions, and indeed this is the case [2]: for example, the energy of the baryon number two Skyrmion exceeds the topological bound by 18%, which is already 5% lower than the single Skyrmion. Such binding energies are much greater than those observed experimentally in nuclei, where binding energies are typically less than 1%. A BPS Skyrme model would therefore appear to be a better starting point for obtaining more realistic binding energies, since a small perturbation away from a BPS theory is likely to produce the required small binding energies. Motivated by this application a BPS Skyrme model is introduced, in which the usual Skyrme model is extended by the inclusion of an infinite tower of vector mesons. Furthermore, neglecting some of the vector mesons provides a natural way to perturb away from the BPS system.

Secondly, Atiyah and Manton [3, 4] have shown that computing the holonomy of Yang-Mills instantons yields good approximations to static Skyrmion solutions of the Skyrme

model. It is not obvious why this approximation turns out to be so successful, or if instanton holonomies give exact solutions of some modified Skyrme model. This paper provides an explanation of the Atiyah-Manton procedure, by proving that the holonomy of Yang-Mills instantons yields exact solutions of the BPS Skyrme model. The BPS Skyrme model reverts to the usual Skyrme model if all the vector mesons are neglected, and this explains the accuracy of the Atiyah-Manton approximation. Furthermore, it is shown that including only the first vector meson already significantly improves the accuracy of the instanton approximation of the Skyrme field and allows an excellent approximation to the vector meson field to be extracted from the instanton.

Thirdly, the usual Skyrme model includes pion degrees of freedom, but neglects all the other mesons. There is a long history of attempts to include other mesons, particularly the ρ meson [5, 6], but there are difficulties because of the large number of coupling constants that need to be determined: although some progress has been made using the ideas of hidden local symmetry and vector meson dominance [7–9]. In the BPS Skyrme model, all parameters are uniquely determined once the energy and length units are fixed. Truncating the BPS theory, for example by including only the first vector meson, allows the usual Skyrme model to be extended without the introduction of further unknown parameters. The Skyrmion is studied in this truncated BPS theory, including its zero-mode quantization, and the results are compared with the physical properties of the nucleon.

The techniques used in this paper, to derive the BPS Skyrme model and its connection to instanton holonomies, are inspired by the work of Sakai and Sugimoto [10]. Using a string theory construction and holographic methods, they were able to derive a Skyrme model coupled to an infinite tower of massive vector mesons from a Yang-Mills-Chern-Simons theory in a curved five-dimensional spacetime. The Skyrme field of their extended Skyrme model corresponds to the holonomy of the curved space instanton, though unfortunately this instanton solution has not yet been determined, even numerically. The Skyrmion in the truncated version of the extended Skyrme model, which includes only the pion and ρ meson degrees of freedom, has been investigated [11], though its quantization has not. A collective coordinate quantization of the instanton has been performed [12, 13], but only by approximating the true curved space Yang-Mills-Chern-Simons instanton by the flat space Yang-Mills instanton.

In some respects, the work in the present paper may be regarded as a five-dimensional Minkowski spacetime analogue of the five-dimensional curved spacetime theory of Sakai and Sugimoto [10], with the advantage that the instanton, and various other ingredients, can be found explicitly. Of course, a disadvantage of this work is that there is no AdS/CFT correspondence to justify the approach, even though several results are qualitatively similar to those of the Sakai-Sugimoto theory, suggesting that there are some merits in considering this theory.

2 Skyrmions and instantons

In the Skyrme model [1] the pion degrees of freedom are encoded into an $SU(2)$ -valued Skyrme field U . In the massless pion approximation, the static energy of the Skyrme model

is

$$E_{\text{Sky}} = \int \left(-\frac{f_\pi^2}{4} \text{Tr}(R_i R_i) - \frac{1}{32e^2} \text{Tr}([R_i, R_j]^2) \right) d^3x, \quad (2.1)$$

where $R_i = \partial_i U U^{-1}$ is the $su(2)$ -valued current. In the above, e is the dimensionless Skyrme parameter and f_π may be interpreted as the pion decay constant. Note that there are differing conventions, related by a factor of 2, for the pion decay constant. In this paper the convention is chosen so that the physical value is $f_\pi = 92.6 \text{ MeV}$, which agrees with the convention in [10] and related papers.

The parameters f_π and e , whose values are to be fixed by comparison with experimental data, merely set the energy and length units and can be scaled away. Explicitly, if energy units of $f_\pi/2e$ and length units of $1/e f_\pi$ are used, then in dimensionless Skyrme units the energy becomes

$$E_{\text{Sky}} = \int \left(-\frac{1}{2} \text{Tr}(R_i R_i) - \frac{1}{16} \text{Tr}([R_i, R_j]^2) \right) d^3x. \quad (2.2)$$

This dimensionless form is used in the remainder of this section.

The Skyrme field is required to tend to a constant element of $SU(2)$ at spatial infinity (usually chosen to be the identity matrix) and this compactifies space to S^3 . A given Skyrme field therefore has an associated integer topological charge $B \in \mathbb{Z} = \pi_3(SU(2))$ given explicitly by

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \text{Tr}(R_i R_j R_k) d^3x. \quad (2.3)$$

It is this topological charge that is to be identified with baryon number [14, 15]. The Skyrmion of charge B is the field U that is the global minimum of the energy (2.2) for all fields in the given topological charge sector.

The Faddeev-Bogomolny bound [16] states that

$$E_{\text{Sky}} \geq 12\pi^2 |B|, \quad (2.4)$$

and it is easy to prove that this bound cannot be attained for non-zero B .

Recall that BPS solitons may be defined as solutions in which a topological energy bound is saturated and therefore, in this sense, Skyrmions are not BPS solitons. Skyrmion solutions can only be obtained numerically and, as mentioned in the previous section, the energy of the $B = 1$ Skyrmion is $12\pi^2 \times 1.23$ and the energy of the $B = 2$ Skyrmion is $24\pi^2 \times 1.18$. Numerical Skyrmion solutions have been obtained up to reasonably large baryon numbers [17, 18] and reveal that the energies for larger values of B are significantly closer to the bound than for these low charge Skyrmions: for example, the $B = 17$ Skyrmion has an energy less than $17 \times 12\pi^2 \times 1.08$. Computations based on periodic Skyrme fields [19] predict the limiting value $E_{\text{Sky}}/B \rightarrow 12\pi^2 \times 1.036$, as $B \rightarrow \infty$. The fact that the energy of the single Skyrmion is much further from the bound than for larger values of B implies binding energies that are much greater than those found experimentally for nuclei, which are typically less than 1%. The small binding energies of nuclei therefore motivate the search for a BPS Skyrme model, in which binding energies would vanish, allowing the possibility that a small perturbation of the BPS system might result in realistic nuclear binding energies.

Although Skyrmions can only be obtained numerically, there are two analytic methods that produce Skyrme fields which are excellent approximations to the true Skyrme solutions. One approach is the rational map approximation [20], but the method of interest in this paper is that of Atiyah and Manton [3, 4] in which a Skyrme field is generated from the holonomy of a Yang-Mills instanton in \mathbb{R}^4 . This is briefly reviewed below.

Let A_I be the components of an $SU(2)$ Yang-Mills instanton in \mathbb{R}^4 , where uppercase latin indices run over all four space coordinates $I = 1, 2, 3, 4$. The Skyrme field is defined to be the holonomy of this instanton computed along lines parallel to the x_4 -axis. Explicitly,

$$U(\mathbf{x}) = \pm \mathcal{P} \exp \int_{-\infty}^{\infty} A_4(\mathbf{x}, x_4) dx_4, \tag{2.5}$$

where \mathcal{P} denotes path ordering and $\mathbf{x} = (x_1, x_2, x_3)$ are the Cartesian coordinates in the remaining $\mathbb{R}^3 \subset \mathbb{R}^4$. As A_4 takes values in the Lie algebra $su(2)$ its exponential is group-valued, so that $U(\mathbf{x}) : \mathbb{R}^3 \mapsto SU(2)$, as required for a static Skyrme field. The \pm factor in (2.5) is because the holonomy should really be defined on a closed loop on S^4 and the sign may be required to account for the transition function that connects $-\infty$ to ∞ , corresponding to the same point on S^4 .

As shown by Atiyah and Manton [3, 4], the baryon number of this Skyrme field is equal to the instanton number of the gauge field, that is, $B = N$ where

$$N = -\frac{1}{16\pi^2} \int \text{Tr}(F_{IJ} {}^*F_{IJ}) d^4x, \tag{2.6}$$

and the dual field strength is defined by ${}^*F_{IJ} = \frac{1}{2}\epsilon_{IJKL}F_{KL}$.

The Yang-Mills theory is conformally invariant and hence the instanton field includes an arbitrary scale. This construction does not provide an exact solution of the Skyrme model for any instanton, but for each N a suitable choice of instanton, including its scale, provides a remarkably good approximation to the static Skyrme field with baryon number N . The energy of the approximate Skyrme field is typically around a percent higher than that of the numerical Skyrme field and correctly reproduces the symmetry of the Skyrme field for a range of highly symmetric cases studied to date. For example, instantons have been constructed that correspond to the $B = 1$ spherically symmetric and $B = 2$ axially symmetric Skyrme fields [3, 4], tetrahedral and cubic Skyrme fields with $B = 3$ and $B = 4$ [21], icosahedrally symmetric Skyrme fields with $B = 7$ and $B = 17$ [22, 23], and the triply periodic Skyrme crystal [24]. There is therefore significant evidence to support the correspondence between Skyrme fields and instanton holonomies, though a deeper understanding of this connection and its remarkable accuracy is desirable.

Recently, the representation of a Skyrme field as an instanton holonomy has reappeared in the context of five-dimensional theories in compactified and/or curved spacetimes [10, 25–27]. Although these approaches certainly have the flavour of the Atiyah-Manton construction, none of them actually involve exact self-dual Yang-Mills instantons in \mathbb{R}^4 . However, as the Sakai-Sugimoto construction [10] is a main motivation for the present paper, and the techniques used here have some similarities to that work, it is perhaps useful to give a brief overview of the relevant aspects of this model.

The Sakai-Sugimoto theory is based on a holographic approach to QCD in the limit of a large number of colours, using the AdS/CFT correspondence to map to a dual string theory consisting of probe D8-branes in a background of D4-branes. The action on the probe D8-branes leads to a Yang-Mills-Chern-Simons theory in a five-dimensional curved spacetime. The spacetime involves a warped product of (3+1)-dimensional Minkowski spacetime and an additional holographic direction. The static soliton in this theory is interpreted as the baryon, and has a fixed size determined by the ratio of the Chern-Simons coefficient to the curvature associated with the holographic direction. Unfortunately, the soliton in this theory has not been determined, even numerically. The expectation is that for a sufficiently small Chern-Simons coupling the soliton size will be small enough that the soliton can be approximated by a soliton of the flat space theory, which is simply a self-dual Yang-Mills instanton in \mathbb{R}^4 , with a particular small scale. In fact, all work to date on this theory has used the flat space Yang-Mills instanton approximation, see for example [12, 13]. The dual theory in (3+1)-dimensional Minkowski spacetime is obtained by performing an expansion of the five-dimensional theory in terms of Kaluza-Klein modes of the holographic direction. These modes contain the Skyrme field plus an infinite tower of massive vector mesons and the associated theory is an extended Skyrme model, which reverts to the usual Skyrme model if the massive vector mesons are ignored. In summary, the Sakai-Sugimoto theory provides a correspondence between Yang-Mills-Chern-Simons instantons on a curved four-manifold and an extended Skyrme model. One of the results of the present paper is to produce an analogous correspondence where the curved four-manifold is replaced by \mathbb{R}^4 and the Chern-Simons term is absent. This allows a direct connection to be made to the Atiyah-Manton construction and also provides a natural extension of this method, together with an understanding of the remarkable accuracy of this approach.

It should be noted that a realization of the Atiyah-Manton construction has been proposed [28] in which instantons appear as domain wall Skyrmons in a five-dimensional Yang-Mills-Higgs theory. This is certainly different from the approach discussed in the present paper and does not involve vector mesons. It also implies that terms involving higher derivatives than the Skyrme term should appear, which is not the case here, though it might be possible that some connection could be made between the two approaches by integrating out the vector mesons.

3 An abelian prototype

As the details of the derivation for the full non-abelian gauge theory are quite cumbersome, it is useful to first consider a similar approach in a prototype abelian gauge theory, where the formulae are more manageable.

Consider an abelian gauge theory in five-dimensional Minkowski spacetime with coordinates t, x_I , where $I = 1, 2, 3, 4$. For notational convenience define $z = x_4$ and let lowercase latin indices run over the three remaining spatial coordinates $i = 1, 2, 3$. The real-valued gauge potential has components a_t, a_i, a_z . As most of this paper will be concerned with

static fields with $a_t = 0$, attention may be restricted to the static Yang-Mills energy

$$E = \frac{1}{4} \int f_{IJ} f_{IJ} d^3x dz, \tag{3.1}$$

where $f_{IJ} = \partial_I a_J - \partial_J a_I$.

As mentioned in the previous section, a crucial ingredient of the Sakai-Sugimoto construction [10] is the expansion of the gauge potential in terms of Kaluza-Klein modes in the holographic direction. In flat Euclidean space a replacement needs to be found for the curved space Kaluza-Klein modes. Simply taking the zero curvature limit is not suitable as the modes then degenerate to Fourier modes, which are not appropriate on the infinite line. The required modes must form a complete orthonormal basis for square integrable functions on the real line with unit weight function (this is necessary to obtain canonical kinetic terms for the vector mesons). This problem is familiar to numerical analysts using spectral methods [29] and the recognized solution is provided by Hermite functions $\psi_n(z)$, where n is a non-negative integer and

$$\psi_n(z) = \frac{(-1)^n}{\sqrt{n! 2^n \sqrt{\pi}}} e^{\frac{1}{2}z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{3.2}$$

In a gauge in which $a_I \rightarrow 0$ as $|z| \rightarrow \infty$, the components of the gauge potential can be expanded in terms of Hermite functions as

$$a_z(\mathbf{x}, z) = \sum_{n=0}^{\infty} \alpha^n(\mathbf{x}) \psi_n(z), \quad a_i(\mathbf{x}, z) = \sum_{n=0}^{\infty} \beta_i^n(\mathbf{x}) \psi_n(z). \tag{3.3}$$

Consider a gauge transformation $a_I \mapsto \tilde{a}_I = a_I - \partial_I h$, for which $\tilde{a}_z = 0$. Clearly, this requires that $\partial_z h = a_z$, and hence h is given by

$$h(\mathbf{x}, z) = \int_{-\infty}^z a_z(\mathbf{x}, \xi) d\xi = \sum_{n=0}^{\infty} \left(\alpha^n(\mathbf{x}) \int_{-\infty}^z \psi_n(\xi) d\xi \right). \tag{3.4}$$

Hermite functions satisfy

$$\psi'_n(z) = \sqrt{\frac{n}{2}} \psi_{n-1}(z) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(z), \tag{3.5}$$

where prime denotes differentiation with respect to z . This implies that

$$\int_{-\infty}^z \psi_{2p+1}(\xi) d\xi = \sum_{m=0}^p \gamma_{2p+1}^m \psi_{2m}(z), \tag{3.6}$$

$$\int_{-\infty}^z \psi_{2p}(\xi) d\xi = \gamma_{2p}^+ \psi_+(z) + \sum_{m=0}^{p-1} \gamma_{2p}^m \psi_{2m+1}(z), \tag{3.7}$$

where γ_{2p}^+ and γ_n^m are non-zero constants.

The additional function $\psi_+(z)$ has been introduced and is defined by

$$\psi_+(z) = \frac{1}{\sqrt{2\pi^{\frac{1}{4}}}} \int_{-\infty}^z \psi_0(\xi) d\xi = \frac{1}{2} + \frac{1}{2} \operatorname{erf}(z/\sqrt{2}) \tag{3.8}$$

with $\text{erf}(z)$ the usual error function

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi. \quad (3.9)$$

The normalization of $\psi_+(z)$ has been chosen so that $\psi_+(-\infty) = 0$ and $\psi_+(\infty) = 1$.

The gauge transformation (3.4) can now be written in terms of the basis functions $\psi_+(z), \psi_n(z)$ as

$$h(\mathbf{x}, z) = u(\mathbf{x}) \psi_+(z) + \sum_{n=0}^{\infty} h^n(\mathbf{x}) \psi_n(z). \quad (3.10)$$

As $\psi_n(\infty) = 0$ and $\psi_+(\infty) = 1$, then $u(\mathbf{x})$ is identified as the holonomy

$$u(\mathbf{x}) = h(\mathbf{x}, \infty) = \int_{-\infty}^{\infty} a_z(\mathbf{x}, \xi) d\xi. \quad (3.11)$$

In the new gauge, where $\tilde{a}_z = 0$, then

$$\tilde{a}_i = a_i - \partial_i h = -\partial_i u(\mathbf{x}) \psi_+(z) + \sum_{n=0}^{\infty} (\beta_i^n(\mathbf{x}) - \partial_i h^n(\mathbf{x})) \psi_n(z). \quad (3.12)$$

After defining the vector fields $v_i^n(\mathbf{x}) = \beta_i^n(\mathbf{x}) - \partial_i h^n(\mathbf{x})$ this becomes

$$\tilde{a}_i = -\partial_i u(\mathbf{x}) \psi_+(z) + \sum_{n=0}^{\infty} v_i^n(\mathbf{x}) \psi_n(z). \quad (3.13)$$

In this gauge the holonomy appears in the boundary condition $\tilde{a}_i \rightarrow -\partial_i u$ as $z \rightarrow \infty$.

Using (3.13) and (3.5) the components of the field strength are

$$\begin{aligned} \tilde{f}_{zi} &= -\partial_i u \psi'_+(z) + \sum_{n=0}^{\infty} v_i^n \psi'_n(z) \\ &= \left(-\frac{1}{\pi^{\frac{1}{4}}} \partial_i u + v_i^1 \right) \frac{\psi_0(z)}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(v_i^{n+1} \sqrt{n+1} - v_i^{n-1} \sqrt{n} \right) \frac{\psi_n(z)}{\sqrt{2}} \end{aligned} \quad (3.14)$$

and

$$\tilde{f}_{ij} = \sum_{n=0}^{\infty} (\partial_i v_j^n - \partial_j v_i^n) \psi_n(z). \quad (3.15)$$

Using the orthonormality of the Hermite functions

$$\int_{-\infty}^{\infty} \psi_m(z) \psi_n(z) dz = \delta_{mn} \quad (3.16)$$

to perform the integration over z , the abelian Yang-Mills energy (3.1) becomes

$$E = \int \left(\frac{1}{4\sqrt{\pi}} (\partial_i u)^2 - \frac{1}{2\pi^{\frac{1}{4}}} v_i^1 \partial_i u + \sum_{n=0}^{\infty} \left\{ \frac{1}{4} (\partial_i v_j^n - \partial_j v_i^n)^2 + \frac{1}{2} m_n^2 (v_i^n)^2 - \frac{1}{2} q_n v_i^n v_i^{n+2} \right\} \right) d^3 x, \quad (3.17)$$

where the coefficients are $m_n^2 = n + \frac{1}{2}$ and $q_n = \sqrt{(n+1)(n+2)}$.

The second term in (3.17) may seem a little strange, but it is simply the analogue in the prototype abelian theory of the familiar mixing between the Skyrme field and the lightest axial vector meson that arises in coupling the Skyrme model to vector mesons [6].

This approach has produced a correspondence between abelian Yang-Mills theory in \mathbb{R}^4 and a field theory in \mathbb{R}^3 containing an infinite tower of vector mesons, plus a scalar field related to the holonomy of the gauge potential. Note that m_n are not the meson masses because $q_n \neq 0$, hence the associated mass matrix is not diagonal. A truncated theory can be defined by including only the first \mathcal{N} vector mesons and setting $v_i^n \equiv 0$ for all $n \geq \mathcal{N}$. The mass matrix is then diagonalized by an $\text{SO}(\mathcal{N})$ rotation of the remaining vector mesons and the meson masses determined from the eigenvalues of the $\mathcal{N} \times \mathcal{N}$ mass matrix. In the extreme case, $\mathcal{N} = 1$, where only the first vector meson remains, no rotation is required and the mass of this meson is obviously $m_0 = 1/\sqrt{2}$.

Perhaps it is worth making a comparison between the above approach and the more common techniques of holographic QCD. In holographic QCD the curvature of the extra dimension induces a discrete spectrum and fields are then expanded in terms of the associated Kaluza-Klein modes. In the current situation the extra dimension is flat and therefore the spectrum is continuous. A discrete spectrum must be identified in order to mimic the holographic construction. The traditional approach to this problem in flat space is to compactify the extra dimension to produce a discrete spectrum. The continuous spectrum is then recovered in the limit of decompactification. However, for the application in the present paper it is not appropriate to compactify the extra dimension, because a modification of space means that the connection to the instanton in \mathbb{R}^4 is then lost. Furthermore, the identification of the holonomy with the non-normalizable mode is no longer obvious in a compact extra dimension. The above Hermite truncation selects a discrete spectrum without the need to modify spacetime, and the continuous spectrum is recovered in the limit $\mathcal{N} \rightarrow \infty$. In this respect it is a crucial feature that the associated mass matrix is not diagonal. Increasing \mathcal{N} not only adds an additional mode but also shifts the masses of all the previous modes, allowing the continuous spectrum to reappear as $\mathcal{N} \rightarrow \infty$.

In principle, the eigenfunctions of any Sturm-Liouville operator on the line might be used as basis functions. However, a significant restriction is imposed by the requirement that the basis functions are orthonormal with respect to the unit weight function, which is needed to obtain the standard kinetic terms for the vector mesons. Hermite functions are a canonical choice and possess additional desirable features, such as the associated Sturm-Liouville operator involving only second and not first order derivatives. The precise combination of conditions that need to be imposed to uniquely arrive at the Hermite functions has not been investigated.

In the next section a similar approach will be applied to $\text{SU}(2)$ Yang-Mills theory, where it is shown that the $\text{SU}(2)$ -valued scalar field that arises from the instanton holonomy is the Skyrme field, with energy function precisely that of the Skyrme model.

4 The Skyrme model from Yang-Mills theory

Consider $SU(2)$ Yang-Mills gauge theory in \mathbb{R}^4 with the $su(2)$ -valued gauge potential A_I , with $I = 1, 2, 3, 4$, and again for convenience set $z = x_4$. The Yang-Mills energy is given by

$$E = -\frac{1}{8} \int \text{Tr}(F_{IJ}F_{IJ}) d^3x dz, \tag{4.1}$$

where the factor of $\frac{1}{8}$ is due to the normalization of the $su(2)$ generators as $-\text{Tr}(T_a T_b) = 2\delta_{ab}$.

Starting with a gauge in which $A_I \rightarrow 0$ as $|z| \rightarrow \infty$, the gauge $A_z = 0$ is obtained by applying the gauge transformation

$$A_I \mapsto gA_I g^{-1} - \partial_I g g^{-1}, \tag{4.2}$$

with

$$g(\mathbf{x}, z) = \mathcal{P} \exp \int_{-\infty}^z A_z(\mathbf{x}, \xi) d\xi. \tag{4.3}$$

The holonomy is

$$U(\mathbf{x}) = g(\mathbf{x}, \infty) = \mathcal{P} \exp \int_{-\infty}^{\infty} A_z(\mathbf{x}, z) dz, \tag{4.4}$$

and in the new gauge the holonomy appears in the boundary condition for A_i , since now $A_i \rightarrow -\partial_i U U^{-1}$ as $z \rightarrow \infty$.

As in the abelian case (3.13), the gauge field can be expanded in terms of Hermite functions as

$$A_i = -\partial_i U U^{-1} \psi_+(z) + \sum_{n=0}^{\infty} W_i^n(\mathbf{x}) \psi_n(z). \tag{4.5}$$

The emergence of the Skyrme model can be seen by first neglecting the vector fields W_i^n . With this truncation the components of the field strength are

$$F_{zi} = -\partial_i U U^{-1} \psi'_+ = -R_i \frac{\psi_0}{\sqrt{2\pi^{\frac{1}{4}}}}, \quad F_{ij} = [R_i, R_j] \psi_+ (\psi_+ - 1). \tag{4.6}$$

Substituting these expressions into the Yang-Mills energy (4.1), and performing the integration over z , yields the energy of the Skyrme model

$$E_S = \int \left(-\frac{c_1}{2} \text{Tr}(R_i R_i) - \frac{c_2}{16} \text{Tr}([R_i, R_j]^2) \right) d^3x, \tag{4.7}$$

where

$$c_1 = \frac{1}{4\sqrt{\pi}}, \quad c_2 = \int_{-\infty}^{\infty} 2\psi_+^2 (\psi_+ - 1)^2 dz = 0.198. \tag{4.8}$$

This is the Skyrme model in dimensionless units, but it is not in Skyrme units because the constants c_1 and c_2 are not unity. In these units the Faddeev-Bogomolny energy bound (2.4) becomes

$$E_S \geq 12\pi^2 \sqrt{c_1 c_2} |B| = 2.005 \pi^2 |B|. \tag{4.9}$$

This bound should be compared with the energy bound that derives from the full Yang-Mills theory, namely

$$E \geq 2\pi^2 |B|, \tag{4.10}$$

which is attained by the instanton solutions. This shows that the two bounds are remarkably close, but that the Faddeev-Bogomolny bound is stricter by $\frac{1}{4}\%$. Of course, the Faddeev-Bogomolny bound only applies to the usual Skyrme model, whereas the bound (4.10) is equally valid if some, or indeed all, of the vector mesons are included.

The Skyrme model is not scale invariant, in contrast to the Yang-Mills theory, and hence the scale of the Skyrme model has emerged because of the truncation that ignores the vector mesons. Perhaps it is useful to think of this in terms of the theory flowing to a conformal theory as all the vector mesons are included. The issue of Skyrmion and instanton scales also appears in the Sakai-Sugimoto derivation [10] of the Skyrme model. In that case the Yang-Mills theory in curved space with a Chern-Simons term is not scale invariant and the instanton has a fixed small size, of the order of the string scale. The size of the Skyrmion in the usual Skyrme model is not related to the size of the instanton, but as more vector mesons are included the size of the Skyrmion in the extended Skyrme model must tend to the small size of the instanton.

The energy of a single Skyrmion exceeds the Faddeev-Bogomolny bound by about 23% and the energy of a Skyrme field generated from the holonomy of a single instanton exceeds the Faddeev-Bogomolny bound by about 24%, for an optimal choice of the instanton scale. When combined with the above bounds, this reveals that, for the optimal instanton scale, the result of neglecting all the vector mesons is to raise the energy by less than 25%. The vector meson terms in the expansion (4.5) have trivial topology and therefore the holonomy term captures all the topological features of the instanton (and hence the Skyrmion), but the above results demonstrate that it also captures most of the energetic properties too.

Including the infinite tower of vector mesons produces a BPS Skyrme model, since the model is simply equivalent to Yang-Mills theory with one extra dimension. An infinite sequence of extended Skyrme models exist that interpolate between the usual Skyrme model and the BPS Skyrme model, as the number of included vector mesons ranges from zero to infinity. The remainder of this paper is devoted to a detailed analysis of the first member of this sequence that extends the usual Skyrme model.

5 Including the first vector meson

This section considers the extension of the Skyrme model obtained by including only the first vector meson, which physically corresponds to coupling the pion field to the ρ meson.

The expansion (4.5) is not convenient once the vector mesons are included, because the fields do not have a definite parity. It is first necessary to perform an additional gauge transformation to obtain an expansion in terms of parity eigenstates.

Given the holonomy U , define the $SU(2)$ -valued field S to be its positive square root, so that $S^2 = U$. After a gauge transformation by $g = S^{-1}$, the expansion (4.5) takes the

form

$$A_i = -\frac{1}{2}P_i\psi_\star + \frac{1}{2}Q_i + \sum_{n=0}^{\infty} V_i^n \psi_n \quad (5.1)$$

where $\psi_\star = \text{erf}(z/\sqrt{2})$ and

$$P_i = S^{-1}\partial_i S + \partial_i S S^{-1}, \quad Q_i = S^{-1}\partial_i S - \partial_i S S^{-1}. \quad (5.2)$$

The vector meson V_i^n is simply the previous vector meson W_i^n in the new gauge.

Including only the first vector meson V_i^0 (and dropping the superscript 0 for notational convenience) gives

$$F_{zi} = -P_i \frac{\psi_0}{\sqrt{2}\pi^{\frac{1}{4}}} - V_i \frac{\psi_1}{\sqrt{2}} \quad (5.3)$$

and

$$\begin{aligned} F_{ij} = & -[P_i, P_j] \frac{1}{4} (1 - \psi_\star^2) + (\partial_i V_j - \partial_j V_i) \psi_0 + [V_i, V_j] \psi_0^2 \\ & - ([P_i, V_j] + [V_i, P_j]) \frac{1}{2} \psi_\star \psi_0 + ([Q_i, V_j] + [V_i, Q_j]) \frac{1}{2} \psi_0. \end{aligned} \quad (5.4)$$

Substituting these expressions into the Yang-Mills energy and integrating over z produces the energy

$$E = E_S + E_V + E_I, \quad (5.5)$$

where E_S is the earlier Skyrme energy (4.7) and E_V is the vector meson energy

$$E_V = \int -\text{Tr} \left\{ \frac{1}{8} (\partial_i V_j - \partial_j V_i)^2 + \frac{1}{4} m^2 V_i^2 + c_3 (\partial_i V_j - \partial_j V_i) [V_i, V_j] + c_4 [V_i, V_j]^2 \right\} d^3x, \quad (5.6)$$

with mass $m = \frac{1}{\sqrt{2}}$ and constants

$$c_3 = \int_{-\infty}^{\infty} \frac{1}{4} \psi_0^3 dz = \frac{1}{2\sqrt{6}\pi^{\frac{1}{4}}}, \quad c_4 = \int_{-\infty}^{\infty} \frac{1}{8} \psi_0^4 dz = \frac{1}{8} \sqrt{\frac{1}{2\pi}}. \quad (5.7)$$

In most phenomenological approaches to including the ρ meson, the energy E_V is taken to be that of a massive Yang-Mills field. However, E_V only has this form if $2c_3^2/c_4$ is equal to unity, whereas $2c_3^2/c_4 = 2\sqrt{2}/3 = 0.94$, and hence there is a slight difference from a massive Yang-Mills theory.

The interaction energy E_I is

$$\begin{aligned} E_I = \int -\text{Tr} \left\{ & -c_5 [P_i, P_j] (\partial_i V_j - \partial_j V_i) - c_6 [P_i, P_j] [V_i, V_j] - c_5 [P_i, P_j] [Q_i, V_j] \right. \\ & + \frac{1}{4} [Q_i, V_j] (\partial_i V_j - \partial_j V_i) + c_3 [Q_i, V_j] [V_i, V_j] + c_7 ([P_i, V_j] + [V_i, P_j])^2 \\ & \left. + \frac{1}{32} ([Q_i, V_j] + [V_i, Q_j])^2 \right\} d^3x, \end{aligned} \quad (5.8)$$

where the constants are

$$\begin{aligned} c_5 &= \int_{-\infty}^{\infty} \frac{1}{16} (1 - \psi_\star^2) \psi_0 dz = \frac{\pi^{\frac{1}{4}}}{12\sqrt{2}}, \quad c_6 = \int_{-\infty}^{\infty} \frac{1}{16} (1 - \psi_\star^2) \psi_0^2 dz = 0.049 \\ c_7 &= \int_{-\infty}^{\infty} \frac{1}{32} \psi_\star^2 \psi_0^2 dz = \frac{1}{32} - \frac{1}{2} c_6 = 0.007. \end{aligned} \quad (5.9)$$

The energy (5.5) corresponds to an extended Skyrme model in which both the pion and ρ meson degrees of freedom are included. Models similar to this have been investigated in the past [5, 6, 8, 9] but because of the difficulty in determining the many possible interaction coefficients some of these terms have not been included, and/or simple relations between the coefficients have been imposed, for example by generating the theory using principles of hidden local symmetry [7]. An advantage of the current derivation is that all interaction coefficients are uniquely fixed from the higher-dimensional Yang-Mills theory.

A similar truncation of the Sakai-Sugimoto theory produces an energy of precisely the same form as that found here, but with different values for the coefficients, and the classical single Skyrmion has been studied numerically [11]. The results show that the classical energy of the Skyrmion in that case is reduced by about 10% in comparison to the usual Skyrmion, but no quantization of the Skyrmion has been performed. In a later section it will be shown that the energy of a single Skyrmion in the theory (5.5) is even lower than this, and moreover a quantization of the Skyrmion reveals that it is vital to take into account the quantum spin energy of the Skyrmion when attempting to match to the physical properties of the nucleon.

6 The Skyrmion and its instanton approximation

As the extended model (5.5) is obtained directly from Yang-Mills theory then good approximations to the energy minimizing fields of this model will be obtained by extracting the appropriate components of the instanton, in terms of the basis expansion used for the truncation. In this section this extraction is performed for the case of a single Skyrmion through a detailed calculation of the decomposition of the instanton in terms of the expansion (5.1).

The single Skyrmion and the single instanton both have $SO(3)$ symmetry, associated with spherical symmetry in \mathbb{R}^3 . It is therefore useful to introduce the following symmetric tensors

$$X_{ia} = \delta_{ia} - \hat{x}_i \hat{x}_a, \quad Y_{ia} = \hat{x}_i \hat{x}_a, \quad Z_{ia} = \epsilon_{ija} \hat{x}_j, \quad (6.1)$$

where $\hat{x}_i = x_i/|\mathbf{x}|$.

The $N = 1$ instanton located at the origin in \mathbb{R}^4 is given by $A_I = iA_{Ia}\tau_a$ where τ_a are the Pauli matrices and

$$A_{ia} = \eta(X_{ia} + Y_{ia}) + \zeta Z_{ia}, \quad A_{za} = \zeta Y_{ia}, \quad (6.2)$$

with the functions η and ζ defined to be

$$\eta(r, z) = \frac{z}{\lambda^2 + r^2 + z^2}, \quad \zeta(r, z) = -\frac{r}{\lambda^2 + r^2 + z^2}. \quad (6.3)$$

Here λ is the arbitrary scale of the instanton.

The gauge $A_z = 0$ is obtained after the gauge transformation

$$g(\mathbf{x}, z) = \exp(iF(r, z)\hat{x}_a\tau_a), \quad (6.4)$$

where

$$F(r, z) = \int_{-\infty}^z \zeta(r, \xi) d\xi = -\frac{\pi r}{\sqrt{\lambda^2 + r^2}} \left\{ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{z}{\sqrt{\lambda^2 + r^2}} \right) \right\}. \quad (6.5)$$

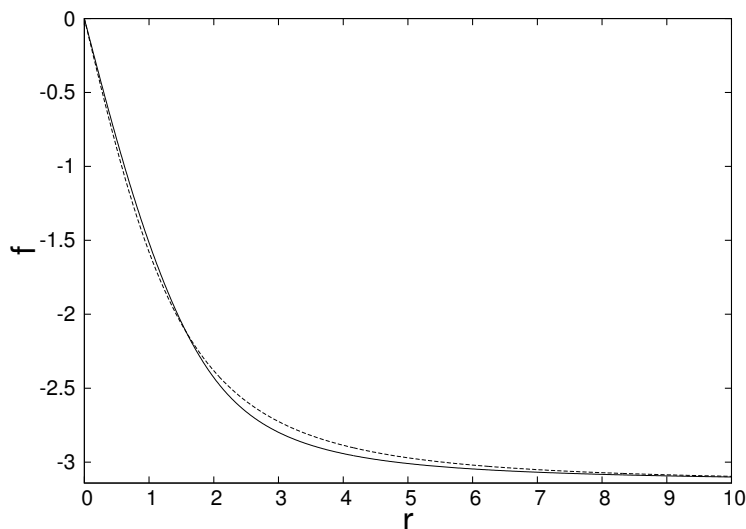


Figure 1. The Skyrme field profile function minimizing the energy E_S . The solid curve is the numerical solution and the dashed curve is the instanton approximation.

The associated holonomy has the standard hedgehog form

$$U = \exp(if(r)\hat{x}_a\tau_a), \tag{6.6}$$

with profile function

$$f(r) = F(r, \infty) = -\frac{\pi r}{\sqrt{\lambda^2 + r^2}}. \tag{6.7}$$

The Skyrme field must tend to a constant element of $SU(2)$ as $r \rightarrow \infty$ and this is usually taken to be the identity matrix, corresponding to the boundary condition $f(\infty) = 0$. In this section it is slightly more convenient to take the non-standard choice that $U \rightarrow -1$ as $r \rightarrow \infty$, with the associated profile function boundary conditions $f(0) = 0$ and $f(\infty) = -\pi$, as satisfied by the profile function in (6.7). This is not an important change but does mean that some additional factors of π do not need to be introduced and carried throughout the following calculation.

First, consider restricting to the usual Skyrme energy E_S , which will reproduce the results of Atiyah and Manton [3, 4].

For a Skyrme field of the hedgehog form (6.6) the usual Skyrme energy (4.7) reduces to the expression

$$E_S = 4\pi \int_0^\infty \left\{ c_1 \left(f'^2 + \frac{2 \sin^2 f}{r^2} \right) + c_2 \frac{\sin^2 f}{r^2} \left(2f'^2 + \frac{\sin^2 f}{r^2} \right) \right\} r^2 dr. \tag{6.8}$$

A numerical minimization of E_S yields an energy of $E_S = 1.236 \times 2\pi^2$, with the associated profile function $f(r)$ displayed as the solid curve in figure 1.

Restricting to the instanton approximation (6.7), the usual Skyrme energy $E_S/2\pi^2$ is plotted as a function of the instanton scale λ as the dashed curve in figure 2. This computation reveals that within the instanton approximation E_S is minimized by an instanton

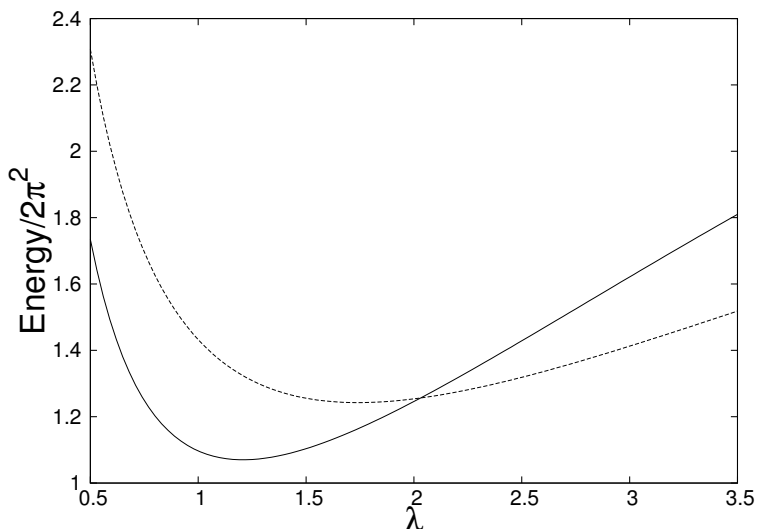


Figure 2. The usual Skyrme energy $E_S/2\pi^2$ (dashed curve) and the energy in the extended model $E/2\pi^2$ (solid curve) as a function of the instanton scale λ .

with scale $\lambda = 1.72$, which has an energy $E_S = 1.246 \times 2\pi^2$. For comparison, the instanton generated profile function with minimizing scale is displayed as the dashed curve in figure 1. The instanton is found to generate a good approximation to the Skyrme field, with energy only 1% above that of the numerical solution.

The next step is to extend these calculations to determine the vector meson fields hidden inside the instanton. The energy minimizing Skyrme field and vector meson field are then computed in the extended Skyrme model (5.5) and compared with those from the instanton approximation.

Performing the gauge transformation (6.4) yields the gauge $A_z = 0$, and in this gauge the remaining components are

$$A_{ia} = X_{ia} \left(\eta \cos 2F - \left(\zeta + \frac{1}{2r} \right) \sin 2F \right) + Y_{ia} \left(\eta - \partial_r F \right) + Z_{ia} \left(\eta \sin 2F + \left(\zeta + \frac{1}{2r} \right) \cos 2F - \frac{1}{2r} \right). \quad (6.9)$$

Now perform the additional gauge transformation with

$$g = S^{-1} = \exp\left(-\frac{i}{2} f \hat{x}_a \tau_a\right), \quad (6.10)$$

to obtain

$$A_{ia} = X_{ia} \left(\eta \cos H - \left(\zeta + \frac{1}{2r} \right) \sin H \right) + Y_{ia} \left(\eta - \frac{1}{2} \partial_r H \right) + Z_{ia} \left(\eta \sin H + \left(\zeta + \frac{1}{2r} \right) \cos H - \frac{1}{2r} \right), \quad (6.11)$$

where

$$H = -\frac{2r}{\sqrt{\lambda^2 + r^2}} \tan^{-1} \left(\frac{z}{\sqrt{\lambda^2 + r^2}} \right). \quad (6.12)$$

The gauge potential (6.11) now has the correct parity properties to be compared with the expansion (5.1) in terms of parity eigenstates. An immediate comparison yields that

$$P_{ia} = X_{ia} \frac{1}{r} \sin f + Y_{ia} f', \quad \text{and} \quad Q_{ia} = Z_{ia} \frac{1}{r} (\cos f - 1), \quad (6.13)$$

where $P_i = iP_{ia}\tau_a$ etc. Then, by definition of the terms in the expansion (5.1), this gives that

$$V_{ia}^n = k_1^n X_{ia} + k_2^n Y_{ia} + k_3^n Z_{ia}, \quad (6.14)$$

where the profile functions $k_i^n(r)$ are given by the integrals

$$\begin{aligned} k_1^n(r) &= \int_{-\infty}^{\infty} \left\{ \eta \cos H - \left(\zeta + \frac{1}{2r} \right) \sin H + \frac{1}{2r} \sin f \psi_\star \right\} \psi_n dz \\ k_2^n(r) &= \int_{-\infty}^{\infty} \left\{ \eta - \frac{1}{2} \partial_r H + \frac{1}{2} f' \psi_\star \right\} \psi_n dz \\ k_3^n(r) &= \int_{-\infty}^{\infty} \left\{ \eta \sin H + \left(\zeta + \frac{1}{2r} \right) \cos H - \frac{1}{2r} \cos f \right\} \psi_n dz. \end{aligned} \quad (6.15)$$

For the even vector mesons it is easy to show that $k_1^{2n}(r) = k_2^{2n}(r) = 0$ due to the symmetry $\psi_{2n}(-z) = \psi_{2n}(z)$, and for the odd vector mesons $k_3^{2n+1}(r) = 0$ due to the symmetry $\psi_{2n+1}(-z) = -\psi_{2n+1}(z)$. This is the correct parity associated with the fact that V_i^n is a vector meson for even n and an axial vector meson for odd n .

For the first vector meson $n = 0$, and again dropping the superscript on V_i^0 , the above results reduce to

$$V_{ia} = \rho(r) Z_{ia}, \quad (6.16)$$

where the profile function $\rho(r)$ is

$$\rho(r) = \int_{-\infty}^{\infty} \left\{ \eta \sin H + \left(\zeta + \frac{1}{2r} \right) \cos H - \frac{1}{2r} \cos f \right\} \psi_0 dz. \quad (6.17)$$

The profile function $\rho(r)$ satisfies the boundary conditions $\rho(0) = \rho(\infty) = 0$.

In terms of arbitrary profile functions $f(r)$ and $\rho(r)$ appearing in the spherical ansatz (6.6) and (6.16), the additional terms in the energy $E = E_S + E_V + E_I$ become

$$E_V = 4\pi \int_0^\infty \left\{ \rho'^2 + \frac{3\rho^2}{r^2} + \frac{2\rho\rho'}{r} + m^2\rho^2 + c_3 16 \frac{\rho^3}{r} + c_4 16\rho^4 \right\} r^2 dr, \quad (6.18)$$

and

$$\begin{aligned} E_I = 4\pi \int_0^\infty \left\{ -16c_5 \frac{\sin f}{r} \left(f' \left(\rho' + \frac{\rho}{r} \right) + \frac{\rho \sin f \cos f}{r^2} \right) + 4 \frac{\rho^2}{r^2} (\cos f - 1) \right. \\ \left. -16c_6 \frac{\rho^2 \sin^2 f}{r^2} + 16c_3 \frac{\rho^3}{r} (\cos f - 1) + 32c_7 f'^2 \rho^2 + 2 \frac{\rho^2}{r^2} (\cos f - 1)^2 \right\} r^2 dr. \end{aligned} \quad (6.19)$$

A numerical minimization of the extended energy E yields the value $E = 1.060 \times 2\pi^2$. This shows that the energy of a Skyrmion in this theory is significantly closer to the topological lower energy bound (4.10) than in the usual Skyrme model. This result reveals

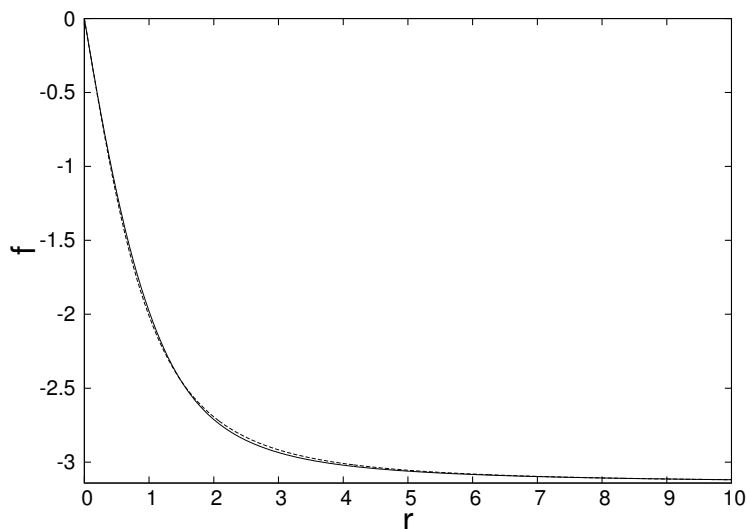


Figure 3. The Skyrme field profile function minimizing the energy E . The solid curve is the numerical solution and the dashed curve is the instanton approximation.

that the truncation of the BPS Skyrme theory, in which only the first vector meson is included, already moves the usual Skyrme model significantly closer to the BPS theory. The numerically determined Skyrme profile function $f(r)$ is displayed as the solid curve in figure 3 and the vector meson profile function $\rho(r)$ as the solid curve in figure 4.

Applying the instanton approximation, with profile functions (6.7) and (6.17), the energy $E/2\pi^2$ is plotted, as a function of the instanton scale λ , as the solid curve in figure 2. The minimizing instanton scale is $\lambda = 1.20$ at which the energy is $E = 1.071 \times 2\pi^2$. This demonstrates that the instanton generated fields provide an excellent approximation to both the Skyrme field and the vector meson field, with an energy only 1% above that of the numerical solution. For comparison to the numerical solutions, the minimizing instanton profiles are displayed as the dashed curves in figure 3 and figure 4. Comparing figure 1 with figure 3 confirms that, as expected, the instanton approximation to the Skyrme field profile function is much closer to the numerical solution in the extended Skyrme theory than in the usual Skyrme model: in fact, it is difficult to distinguish the two curves in figure 3.

7 Quantization of the Skyrmion

It is well-known that fixing the parameters of the Skyrme model in the meson sector does not produce good results in the baryon sector. It is common practice to treat (at least some of) the meson constants as free parameters which are then fixed by comparison to selected baryon properties [5, 30–34]. It has been suggested [35] that the meson parameters could be interpreted as renormalized constants in the baryon sector, that result from known quantum effects not addressed within a simple zero-mode quantization. As demonstrated below, the same situation persists in the extended Skyrme model.

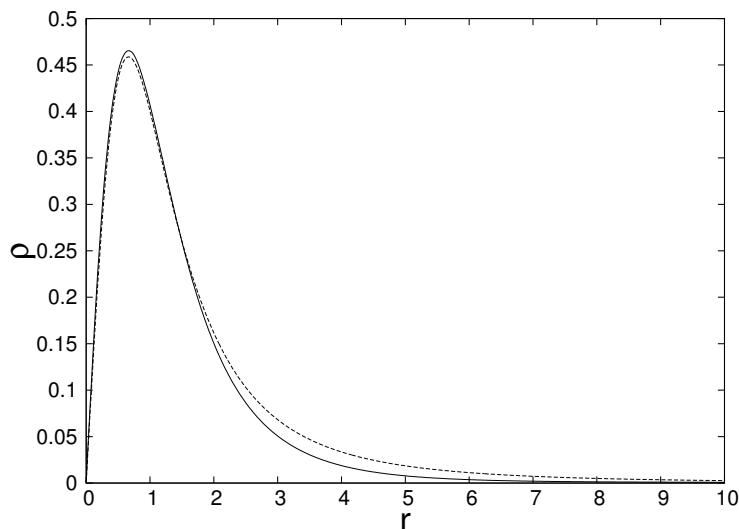


Figure 4. The vector meson profile function ρ minimizing the energy E . The solid curve is the numerical solution and the dashed curve is the instanton approximation.

Given the dimensionless formulation of the theory (5.5), the energy unit ε and the length unit l are related to the pion decay constant f_π and ρ meson mass m_ρ by

$$\varepsilon = f_\pi^2 \sqrt{2\pi} / m_\rho, \quad l = 1 / (\sqrt{2} m_\rho). \quad (7.1)$$

This can be seen by performing the scaling $E \mapsto \varepsilon E$, $x_i \mapsto x_i / l$, $V_i \mapsto V_i \sqrt{l / \varepsilon}$, after which the dimensionless Skyrme energy (4.7) takes the standard form (2.1) with Skyrme parameter

$$e = \frac{m_\rho}{f_\pi \sqrt{2c_2 \pi^{1/4}}}. \quad (7.2)$$

The vector meson energy (5.6) becomes

$$E_V = \int -\text{Tr} \left\{ \frac{1}{8} (\partial_i V_j - \partial_j V_i)^2 + \frac{1}{4} m_\rho^2 V_i^2 + \tilde{c}_3 (\partial_i V_j - \partial_j V_i) [V_i, V_j] + \tilde{c}_4 [V_i, V_j]^2 \right\} d^3x, \quad (7.3)$$

where

$$\tilde{c}_3 = \frac{m_\rho}{2\sqrt{6\pi} f_\pi}, \quad \tilde{c}_4 = \frac{m_\rho^2}{8\pi\sqrt{2} f_\pi^2}. \quad (7.4)$$

For completeness, the additional coefficients in the interaction energy E_I remain unchanged except for the replacement

$$c_5 \mapsto \tilde{c}_5 = \sqrt{\frac{\pi}{2}} \frac{f_\pi}{12m_\rho}. \quad (7.5)$$

The classical Skyrminion energy is $E = \varepsilon M$, where $M = 1.06 \times 2\pi^2$ is the dimensionless static energy. Taking the physical values $f_\pi = 92.6 \text{ MeV}$ and $m_\rho = 776 \text{ MeV}$, to set the units (7.1), gives a classical Skyrminion energy $E = 580 \text{ MeV}$, which is far too low in comparison to the nucleon mass of 939 MeV . There is also a quantum contribution to

the energy associated with the spin of the nucleon, but even if this contribution raised the total energy to that of the nucleon mass this would still not be an acceptable result, since physically the spin contribution needs to provide only a small contribution to the total energy. The quantum spin energy is calculated later in this section and reveals that taking the physical values for the meson parameters f_π and m_ρ yields a quantum spin contribution of 1039 MeV, which is almost twice that of the classical energy and gives a total energy which is far too large. This shows that taking the physical values for the meson parameters does not produce acceptable results for the baryon. Confirmation of this is provided by calculating the size of the baryon, as follows.

The physical value of the nucleon isoscalar root mean square radius is $\sqrt{\langle r^2 \rangle} = 0.72$ fm. For the Skyrmion its dimensionless form R is calculated from the radial baryon density \mathcal{B} as

$$R^2 = \int_0^\infty r^2 \mathcal{B} dr = - \int_0^\infty r^2 \frac{2}{\pi} f' \sin f dr = 0.82. \tag{7.6}$$

Inserting the length unit gives

$$\sqrt{\langle r^2 \rangle} = \frac{R}{\sqrt{2}m_\rho}. \tag{7.7}$$

Taking the physical value for m_ρ , and using the fact that in natural units $\text{MeV}^{-1} = 197$ fm, yields $\sqrt{\langle r^2 \rangle} = 0.16$ fm, which is far too small. This is the origin of the excessive quantum spin energy mentioned above, when physical values are taken for the meson parameters. The baryon is far too small and hence so is its moment of inertia, which occurs in the denominator of the quantum spin energy.

From now on the common practice is adopted of treating the meson parameters of the theory (in this case f_π and m_ρ) as free parameters that are to be fixed by comparison to physical properties in the baryon sector.

The choice made in this paper is to fix the energy and length units by matching to the physical values of the nucleon mass and the isoscalar root mean square radius. Matching the latter, using equation (7.7), yields $m_\rho = 176$ MeV, which is therefore only around a quarter of its physical value. To determine f_π it is first necessary to calculate the quantum spin contribution to the nucleon mass, which is presented below.

So far in this paper the discussion has been restricted to static fields. It is a simple matter to obtain the relevant Lagrangians from the static energies presented earlier by applying the obvious relativistic generalization. From the dimensionless form (5.5) of the static energy of the extended Skyrme model, the associated dimensionless kinetic energy is $T = T_S + T_V + T_I$ where

$$T_S = \int -\text{Tr} \left\{ \frac{c_1}{2} R_0^2 + \frac{c_2}{8} [R_0, R_i]^2 \right\} d^3x, \tag{7.8}$$

$$T_V = \int -\text{Tr} \left\{ \frac{1}{4} (\partial_0 V_i - \partial_i V_0)^2 + \frac{1}{4} m^2 V_0^2 + 2c_3 (\partial_0 V_i - \partial_i V_0) [V_0, V_i] + 2c_4 [V_0, V_i]^2 \right\} d^3x, \tag{7.9}$$

$$\begin{aligned}
T_I = \int -\text{Tr} \left\{ -2c_5[P_0, P_i](\partial_0 V_i - \partial_i V_0) - 2c_6[P_0, P_i][V_0, V_i] - c_5[P_0, P_i]([Q_0, V_i] + [V_0, Q_i]) \right. \\
+ \frac{1}{4}([Q_0, V_i] + [V_0, Q_i])(\partial_0 V_i - \partial_i V_0) + c_3([Q_0, V_i] + [V_0, Q_i])[V_0, V_i] \\
\left. + 2c_7([P_0, V_i] + [V_0, P_i])^2 + \frac{1}{16}([Q_0, V_i] + [V_0, Q_i])^2 \right\} d^3x. \quad (7.10)
\end{aligned}$$

The zero-mode quantization involves the rigid rotor ansatz

$$S = e^{\frac{1}{2}\Omega t} \tilde{S} e^{-\frac{1}{2}\Omega t}, \quad V_i = e^{\frac{1}{2}\Omega t} \tilde{V}_i e^{-\frac{1}{2}\Omega t}, \quad V_0 = e^{\frac{1}{2}\Omega t} \tilde{V}_0 e^{-\frac{1}{2}\Omega t}, \quad (7.11)$$

where $\Omega = i\Omega_a \tau_a$ is a constant element of $su(2)$ determining the rotation frequency and axis, and \tilde{S} , \tilde{V}_i are the earlier static fields that minimize the classical static energy E . For time-dependent fields \tilde{V}_0 can no longer be set to zero, as this is not consistent with the Gauss law for this system, that is, the field equation for V_0 . The Gauss law requires that \tilde{V}_0 has the same tensorial structure as $[\tilde{V}_i, [\tilde{V}_i, \Omega]]$. This determines the form of $\tilde{V}_0 = i\tilde{V}_{0a} \tau_a$ to be [5]

$$\tilde{V}_{0a} = \chi_1 \Omega_a + \chi_2 \hat{x}_a \hat{x}_b \Omega_b, \quad (7.12)$$

where $\chi_1(r), \chi_2(r)$ are two additional radial profile functions with boundary conditions $\chi_1'(0) = \chi_1(\infty) = \chi_2(0) = \chi_2(\infty) = 0$. The kinetic energy takes the form $T = \frac{1}{2}I|\Omega|^2$, where $|\Omega|^2 = -\frac{1}{2}\text{Tr}(\Omega^2)$ and $I = I_S + I_V + I_I$ is the moment of inertia, which after a tedious calculation is found to be

$$I_S = \frac{16\pi}{3} \int_0^\infty \left\{ c_1 + c_2 \left(f'^2 + \frac{\sin^2 f}{r^2} \right) \right\} \sin^2 f r^2 dr, \quad (7.13)$$

$$\begin{aligned}
I_V = \frac{4\pi}{3} \int_0^\infty \left\{ 4\rho^2 + 4\frac{\chi_2^2}{r^2} + 3\chi_1'^2 + 2\chi_1'\chi_2' + \chi_2'^2 + m^2(3\chi_1^2 + 2\chi_1\chi_2 + \chi_2^2) \right. \\
\left. + 32c_3\rho \left(2\rho\chi_1 + \rho\chi_2 + \frac{\chi_2^2}{r} \right) + 64c_4\rho^2(2\chi_1^2 + 2\chi_1\chi_2 + \chi_2^2) \right\} r^2 dr, \quad (7.14)
\end{aligned}$$

$$\begin{aligned}
I_I = \frac{4\pi}{3} \int_0^\infty \left\{ -32c_5 \sin f \left(\frac{\sin f}{r} \left(\rho - \frac{\chi_2}{r} \right) + f'\chi_1' \right) - 64c_6\rho \frac{\sin^2 f}{r} \chi_1 \right. \\
- 8\sin^2 \frac{f}{2} \left(-8c_5 \frac{\sin^2 f}{r} \left(\rho + \frac{\chi_1}{r} \right) + \rho^2 + 2\rho \frac{\chi_1}{r} + \frac{\chi_2^2}{r^2} + 8c_3\rho \left(2\frac{\chi_1^2}{r} + 2\frac{\chi_1\chi_2}{r} + \frac{\chi_2^2}{r} + \chi_1\rho \right) \right) \\
+ 64c_7 \left(f'^2 \chi_1^2 + \sin^2 f \left(\left(\rho - \frac{\chi_1}{r} \right)^2 + \frac{(\chi_1 + \chi_2)^2}{r^2} \right) \right) \\
\left. + 8\sin^4 \frac{f}{2} \left(\left(\rho + \frac{\chi_1}{r} \right)^2 + \frac{(\chi_1 + \chi_2)^2}{r^2} \right) \right\} r^2 dr. \quad (7.15)
\end{aligned}$$

The functions χ_1, χ_2 are determined by the V_0 field equation and this is equivalent to the minimization of $I_V + I_I$, given the profile functions f and ρ . The minimizing profile functions are presented in figure 5 and the associated moment of inertia is computed to be $I = I_S + (I_V + I_I) = 13.73 + 1.96 = 15.69$

In terms of the spin $J = I|\Omega|$ the dimensionless quantum spin energy is

$$E_Q = \frac{J^2}{2I}, \quad (7.16)$$

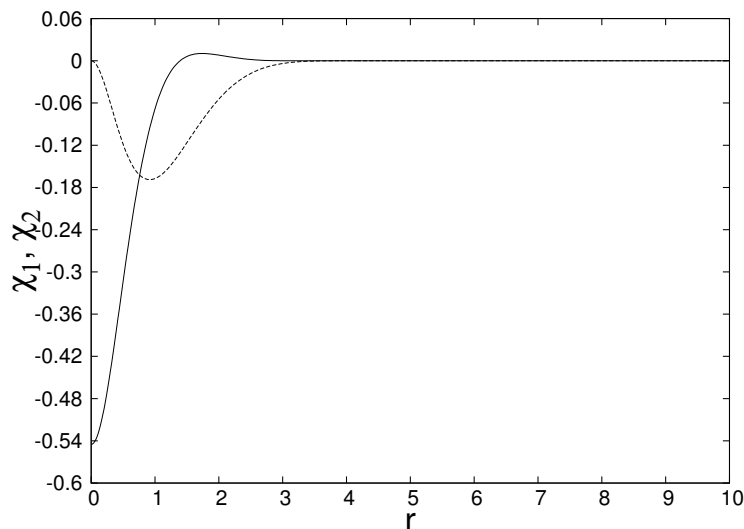


Figure 5. The profile functions χ_1 (solid curve) and χ_2 (dashed curve).

where $J^2 = j(j + 1)$ and the quantum spin number $j = \frac{1}{2}$ for the nucleon. The moment of inertia I has units of εl^2 and the quantum spin energy has units which are the reciprocal of this. The mass of the nucleon is therefore

$$M_N = \varepsilon M + \frac{1}{\varepsilon l^2} E_Q = \frac{f_\pi^2}{m_\rho} \sqrt{2\pi} M + \frac{m_\rho^3}{f_\pi^2} \sqrt{\frac{2}{\pi}} \frac{3}{8I}. \quad (7.17)$$

Taking the previously determined value $m_\rho = 176$ MeV and requiring that (7.17) reproduces the nucleon mass $M_N = 939$ MeV yields $f_\pi = 55.1$ MeV, which is around 60% of its physical value. With these parameter values the classical and quantum spin contributions to the nucleon mass are 905 MeV and 34 MeV respectively, which is an acceptable split.

By equation (7.2), this set of parameter values gives a Skyrme parameter $e = 3.81$. It is interesting to note that these parameters are reasonably close to those suggested in the usual Skyrme model by fitting to the properties of the lithium-6 nucleus, which give $f_\pi = 37.6$ MeV and $e = 3.26$ [33].

8 Conclusion

Inspired by methods of holographic QCD, a sequence of extended Skyrme models has been introduced that interpolate between the usual Skyrme model and a BPS Skyrme model. This provides an explanation and extension of the Atiyah-Manton construction of Skyrme fields from instanton holonomies, as this construction produces exact solutions of the BPS Skyrme model.

The first extended Skyrme model is a nonlinear theory of pions coupled to the ρ meson and this has been investigated in some detail. The results reveal that this model is significantly closer to a BPS theory than the usual Skyrme model, and it has been demonstrated that an extension of the Atiyah-Manton construction provides an

excellent approximation to both the pion and ρ meson fields. This is encouraging as the experimental data on nuclear binding energies reveals that they are typically less than 1%, suggesting that a model close to a BPS theory is required.

There are several avenues for future research that follow from the work presented in this paper. An obvious next step is to investigate multi-Skyrmions in the extended theory, to confirm and evaluate the reduced binding energies. This could be performed either using numerical full field simulations, similar to those applied in the usual Skyrme model [17, 18], or using the instanton approximation. A study of the extended Skyrme models that include additional vector mesons would also be of interest: in particular it would be useful to compute the change in binding energies as more vector mesons are introduced.

Many aspects of the Sakai-Sugimoto theory have been investigated since its introduction and it would be interesting to attempt similar studies for the flat space analogue introduced here. Examples of aspects to study include finite baryon density and temperature [36–41], the determination of interaction coupling constants and form factors [42–45], together with an analysis of the nuclear force [46].

The Skyrme models considered in this paper are all applicable to massless pions, but it has been shown that in the usual Skyrme model there are significant differences in the massive pion theory, and the differences are encouraging in respect to comparisons with the properties of nuclei [34, 47, 48]. A pion mass term could simply be included in the extended Skyrme models, although the connection to a BPS Skyrme theory would then be lost. However, it is still to be expected that binding energies would be reduced in comparison to the usual Skyrme model, as the resulting increase in energy applies to Skyrmions of all baryon numbers. The Atiyah-Manton construction does not produce Skyrme fields with asymptotic fields appropriate to massive pions, but a modification of this construction has been introduced for massive pions [49], based on a connection to hyperbolic Skyrmions. It would be interesting to see if this modified construction can be understood in terms of the techniques introduced in the present paper, and perhaps this might lead to an extended BPS Skyrme theory for massive pions.

Finally, during the preparation of this manuscript a preprint appeared [50] introducing novel Skyrmions in a different BPS Skyrme model. The BPS Skyrme model in question is of the type introduced some time ago [51], involving only a pion mass term and a term of sixth-order in the derivatives of the Skyrme field, and is therefore quite different from the one considered in the present paper. The novel Skyrmions presented in [50] are examples of compactons, that is, they have compact support. Such Skyrmions are trivially BPS solitons, since compactons placed far enough apart do not interact at all. However, there is an interesting mathematical structure, associated with an infinite dimensional symmetry, that for example allows the single Skyrmion to be obtained as a solution of a first order equation.

Acknowledgments

Many thanks to Nick Manton, Kasper Peeters and Marija Zamaklar for useful comments. I acknowledge the STFC for support under the rolling grant ST/G000433/1.

References

- [1] T.H.R. Skyrme, *A unified field theory of mesons and baryons*, *Nucl. Phys.* **31** (1962) 556 [[SPIRES](#)].
- [2] N.S. Manton and P.M. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004) [[SPIRES](#)].
- [3] M.F. Atiyah and N.S. Manton, *Skyrmions from instantons*, *Phys. Lett. B* **222** (1989) 438 [[SPIRES](#)].
- [4] M.F. Atiyah and N.S. Manton, *Geometry and kinematics of two skyrmions*, *Commun. Math. Phys.* **153** (1993) 391 [[SPIRES](#)].
- [5] G.S. Adkins, *ρ mesons in the Skyrme model*, *Phys. Rev. D* **33** (1986) 193 [[SPIRES](#)].
- [6] U.G. Meissner and I. Zahed, *Skyrmions in the presence of vector mesons*, *Phys. Rev. Lett.* **56** (1986) 1035 [[SPIRES](#)].
- [7] M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, *Is ρ meson a dynamical gauge boson of hidden local symmetry?*, *Phys. Rev. Lett.* **54** (1985) 1215 [[SPIRES](#)].
- [8] M. Bando, T. Kugo and K. Yamawaki, *Nonlinear realization and hidden local symmetries*, *Phys. Rept.* **164** (1988) 217 [[SPIRES](#)].
- [9] M. Harada and K. Yamawaki, *Hidden local symmetry at loop: a new perspective of composite gauge boson and chiral phase transition*, *Phys. Rept.* **381** (2003) 1 [[hep-ph/0302103](#)] [[SPIRES](#)].
- [10] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843 [[hep-th/0412141](#)] [[SPIRES](#)].
- [11] K. Nawa, H. Suganuma and T. Kojo, *Baryons in holographic QCD*, *Phys. Rev. D* **75** (2007) 086003 [[hep-th/0612187](#)] [[SPIRES](#)].
- [12] H. Hata, T. Sakai, S. Sugimoto and S. Yamato, *Baryons from instantons in holographic QCD*, *Prog. Theor. Phys.* **117** (2007) 1157 [[hep-th/0701280](#)] [[SPIRES](#)].
- [13] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Chiral dynamics of baryons from string theory*, *Phys. Rev. D* **76** (2007) 061901 [[hep-th/0701276](#)] [[SPIRES](#)].
- [14] E. Witten, *Global aspects of current algebra*, *Nucl. Phys. B* **223** (1983) 422 [[SPIRES](#)].
- [15] E. Witten, *Current algebra, baryons and quark confinement*, *Nucl. Phys. B* **223** (1983) 433 [[SPIRES](#)].
- [16] L.D. Faddeev, *Some comments on the many dimensional solitons*, *Lett. Math. Phys.* **1** (1976) 289 [[SPIRES](#)].
- [17] R.A. Battye and P.M. Sutcliffe, *Solitonic fullerenes*, *Phys. Rev. Lett.* **86** (2001) 3989 [[hep-th/0012215](#)] [[SPIRES](#)].
- [18] R.A. Battye and P.M. Sutcliffe, *Skyrmions, fullerenes and rational maps*, *Rev. Math. Phys.* **14** (2002) 29 [[hep-th/0103026](#)] [[SPIRES](#)].
- [19] R.A. Battye and P.M. Sutcliffe, *A Skyrme lattice with hexagonal symmetry*, *Phys. Lett. B* **416** (1998) 385 [[hep-th/9709221](#)] [[SPIRES](#)].
- [20] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, *Rational maps, monopoles and skyrmions*, *Nucl. Phys. B* **510** (1998) 507 [[hep-th/9705151](#)] [[SPIRES](#)].

- [21] R.A. Leese and N.S. Manton, *Stable instanton generated Skyrme fields with baryon numbers three and four*, *Nucl. Phys. A* **572** (1994) 575 [SPIRES].
- [22] M.A. Singer and P.M. Sutcliffe, *Symmetric instantons and Skyrme fields*, *Nonlinearity* **12** (1999) 987 [hep-th/9901075] [SPIRES].
- [23] P. Sutcliffe, *Instantons and the buckyball*, *Proc. Roy. Soc. Lond. A* **460** (2004) 2903 [hep-th/0309157] [SPIRES].
- [24] N.S. Manton and P.M. Sutcliffe, *Skyrme crystal from a twisted instanton on a four torus*, *Phys. Lett. B* **342** (1995) 196 [hep-th/9409182] [SPIRES].
- [25] C.T. Hill, *Topological solitons from deconstructed extra dimensions*, *Phys. Rev. Lett.* **88** (2002) 041601 [hep-th/0109068] [SPIRES].
- [26] D.T. Son and M.A. Stephanov, *QCD and dimensional deconstruction*, *Phys. Rev. D* **69** (2004) 065020 [hep-ph/0304182] [SPIRES].
- [27] A. Pomarol and A. Wulzer, *Stable skyrmions from extra dimensions*, *JHEP* **03** (2008) 051 [arXiv:0712.3276] [SPIRES].
- [28] M. Eto, M. Nitta, K. Ohashi and D. Tong, *Skyrmions from instantons inside domain walls*, *Phys. Rev. Lett.* **95** (2005) 252003 [hep-th/0508130] [SPIRES].
- [29] J.P. Boyd, *Chebyshev and Fourier spectral methods*, Springer-Verlag, Germany (1989).
- [30] G.S. Adkins, C.R. Nappi and E. Witten, *Static properties of nucleons in the Skyrme model*, *Nucl. Phys. B* **228** (1983) 552 [SPIRES].
- [31] G.S. Adkins and C.R. Nappi, *The Skyrme model with pion masses*, *Nucl. Phys. B* **233** (1984) 109 [SPIRES].
- [32] V.B. Kopeliovich, A.M. Shunderuk and G.K. Matushko, *Mass splittings of nuclear isotopes in chiral soliton approach*, *Phys. Atom. Nucl.* **69** (2006) 120 [*Yad. Fiz.* **69** (2006) 124] [nucl-th/0404020] [SPIRES].
- [33] N.S. Manton and S.W. Wood, *Reparametrising the Skyrme model using the lithium-6 nucleus*, *Phys. Rev. D* **74** (2006) 125017 [hep-th/0609185] [SPIRES].
- [34] R.A. Battye, N.S. Manton, P.M. Sutcliffe and S.W. Wood, *Light nuclei of even mass number in the Skyrme model*, *Phys. Rev. C* **80** (2009) 034323 [arXiv:0905.0099] [SPIRES].
- [35] R.A. Battye, S. Krusch and P.M. Sutcliffe, *Spinning skyrmions and the Skyrme parameters*, *Phys. Lett. B* **626** (2005) 120 [hep-th/0507279] [SPIRES].
- [36] N. Horigome and Y. Tanii, *Holographic chiral phase transition with chemical potential*, *JHEP* **01** (2007) 072 [hep-th/0608198] [SPIRES].
- [37] O. Bergman, G. Lifschytz and M. Lippert, *Holographic nuclear physics*, *JHEP* **11** (2007) 056 [arXiv:0708.0326] [SPIRES].
- [38] K.-Y. Kim, S.-J. Sin and I. Zahed, *The chiral model of Sakai-Sugimoto at finite baryon density*, *JHEP* **01** (2008) 002 [arXiv:0708.1469] [SPIRES].
- [39] M. Rozali, H.-H. Shieh, M. Van Raamsdonk and J. Wu, *Cold nuclear matter in holographic QCD*, *JHEP* **01** (2008) 053 [arXiv:0708.1322] [SPIRES].
- [40] O. Aharony, J. Sonnenschein and S. Yankielowicz, *A holographic model of deconfinement and chiral symmetry restoration*, *Annals Phys.* **322** (2007) 1420 [hep-th/0604161] [SPIRES].

- [41] K. Peeters, J. Sonnenschein and M. Zamaklar, *Holographic melting and related properties of mesons in a quark gluon plasma*, *Phys. Rev. D* **74** (2006) 106008 [[hep-th/0606195](#)] [[SPIRES](#)].
- [42] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Dynamics of baryons from string theory and vector dominance*, *JHEP* **09** (2007) 063 [[arXiv:0705.2632](#)] [[SPIRES](#)].
- [43] K.-Y. Kim and I. Zahed, *Electromagnetic baryon form factors from holographic QCD*, *JHEP* **09** (2008) 007 [[arXiv:0807.0033](#)] [[SPIRES](#)].
- [44] D.K. Hong, M. Rho, H.-U. Yee and P. Yi, *Nucleon form factors and hidden symmetry in holographic QCD*, *Phys. Rev. D* **77** (2008) 014030 [[arXiv:0710.4615](#)] [[SPIRES](#)].
- [45] K. Hashimoto, T. Sakai and S. Sugimoto, *Holographic baryons: static properties and form factors from gauge/string duality*, *Prog. Theor. Phys.* **120** (2008) 1093 [[arXiv:0806.3122](#)] [[SPIRES](#)].
- [46] K. Hashimoto, T. Sakai and S. Sugimoto, *Nuclear force from string theory*, *Prog. Theor. Phys.* **122** (2009) 427 [[arXiv:0901.4449](#)] [[SPIRES](#)].
- [47] R. Battye and P. Sutcliffe, *Skyrmions with massive pions*, *Phys. Rev. C* **73** (2006) 055205 [[hep-th/0602220](#)] [[SPIRES](#)].
- [48] R. Battye, N.S. Manton and P. Sutcliffe, *Skyrmions and the alpha-particle model of nuclei*, *Proc. Roy. Soc. Lond. A* **463** (2007) 261 [[hep-th/0605284](#)] [[SPIRES](#)].
- [49] M. Atiyah and P. Sutcliffe, *Skyrmions, instantons, mass and curvature*, *Phys. Lett. B* **605** (2005) 106 [[hep-th/0411052](#)] [[SPIRES](#)].
- [50] C. Adam, J. Sanchez-Guillen and A. Wereszczynski, *A Skyrme-type proposal for baryonic matter*, *Phys. Lett. B* **691** (2010) 105 [[arXiv:1001.4544](#)] [[SPIRES](#)].
- [51] K. Arthur, G. Roche, D.H. Tchrakian and Y.S. Yang, *Skyrme models with selfdual limits: $D = 2, 3$* , *J. Math. Phys.* **37** (1996) 2569 [[SPIRES](#)].