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Conformally related vacuum gravitational waves and their symmetries

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Abstract: A special conformal transformation which carries a vacuum gravitational wave into another vacuum one is built by using Möbius-redefined time. It can either transform a globally defined vacuum wave into a vacuum sandwich wave, or carry the gravitational wave into itself. The first type, illustrated by linearly and circularly polarised vacuum plane gravitational waves, permutes the symmetries and the geodesics. Our second type is a pp wave with conformal $\mathrm{O}(2,1)$ symmetry. An example inspired by molecular physics which seems to have escaped attention so far is an anisotropic generalisation of the familiar inverse-square profile and is reminiscent of Aichelburg-Sexl ultraboosts. The particle can escape, or perform circular periodic motion, or fall into the singularity.

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Contents

1 Introduction

We consider special class of gravitational wave (GW) space-times whose metric is written in Brinkmann coordinates [1] as,

$$ds^2 = dX^2 + dY^2 + 2dUdV - 2H(U, \mathbf{X})\, dU^2 \,. \tag{1.1}$$

Here U and V are light-cone coordinates, X and Y represent the transverse plane and $H(U, \mathbf{X})$ is the wave profile. Brinkmann space-times are smooth Lorentz manifolds endowed with a covariantly constant null Killing vector field $\xi = \partial_V$ [2]. Such structures arised before in the study of the one-parameter central extension of the Galilei group [3] called the *Bargmann group* [4]. In the proposed Kaluza-Klein-type "Bargmann" framework [2, 4–6] the motions in ordinary space are obtained by projecting out the "vertical" null direction along the coordinate V and identifying the other null coordinate, U, with Newtonian time. The profile $H(U, \mathbf{X})$ is the Newtonian potential [2, 5, 6].

An insight is provided by the so-called memory effect [7–11] which amounts to studying test particles initially at rest by using the symmetries of the corresponding background space-time. It is particularly convenient to use conformal symmetries [12–17], generated by conformal Killing vectors (CKV),

$$\mathcal{L}_W g_{ab} = 2\psi g_{ab} \,, \tag{1.2}$$

where ψ is a smooth function of the coordinates [18–21, 23]. In flat Minkowski space-time, the conformal Lie algebra of CKVs is 15 dimensional. The same number of dimensions arises for conformally flat space-times, which include, in addition to Minkowski space-time, also that for oscillator and for a linear potential [2, 5, 24–28, 60] . The maximal number of symmetries of a non-conformally-flat space-time depends on the signature of the metric. In the usual case of $(-,+,+,+)$ it is 7 [20–23].[1] The research of symmetries is simplified when the space-time is conformally related to one whose symmetries are known, and therefore the CKVs are interchanged. This happens, in particular, for the time redefinition (2.1) below, proposed in [23, 29].

In this paper we study special time-redefined conformal transformations of simple rational form (2.7) referred to as Möbius transformations. They can (i) either interchange *two vacuum GWs* (as illustrated by linearly polarized plane GWs (LPP) and circularly polarized vacuum plane GWs (CPP) [30–32], or (ii) leave the wave form-invariant (sections 3 and 4) as illustrated by a wave inspired by the anisotropic polar molecule [34–36].

The U-dependence brought in by Möbius transformation is "mild", though, because of the simple rational form of (2.7). Realistic GWs are however "sandwich waves" [38] with a short wave zone $[U_i, U_j]$ outside of which the space-times is flat [14, 15, 38–44]. Their study is postponed to a next publication.

Some entertaining historical facts are recounted in section 6.

2 Conformal transformations of gravitational waves

The GW space-time, ds^2 in (1.1), can be transformed into another GW, $d\widetilde{s}^2$, by a special conformal transformation with redefined time $f(\widetilde{U})$ [23, 29],

$$U = f(\tilde{U}), \qquad \mathbf{X} = \sqrt{\overset{\circ}{f}}\,\widetilde{\mathbf{X}}, \qquad V = \tilde{V} - \frac{1}{4}\frac{\overset{\circ\circ}{f}}{\overset{\circ}{f}}\widetilde{\mathbf{X}}^2 \,, \tag{2.1}$$

where $\overset{\circ}{(\cdot)}$ means $d/d\widetilde{U}$. The corresponding conformal relation is,

$$ds^2 = \Omega^2 d\widetilde{s}^2 = \overset{\circ}{f}\, d\widetilde{s}^2 \,, \tag{2.2}$$

$$d\widetilde{s}^2 = d\widetilde{\mathbf{X}}^2 + 2d\widetilde{U}d\widetilde{V} - 2\widetilde{H}(\widetilde{U},\widetilde{\mathbf{X}})d\widetilde{U}^2 \,, \tag{2.3}$$

$$\widetilde{H}(\widetilde{U},\widetilde{\mathbf{X}}) = \overset{\circ}{f}\, H\left[\widetilde{\mathbf{X}}\sqrt{\overset{\circ}{f}},\, f(\widetilde{U})\right] + \frac{1}{4}\mathcal{S}_{\widetilde{U}}(f)\,\widetilde{\mathbf{X}}^2, \tag{2.4}$$

where $\mathcal{S}_{\widetilde{U}}(f)$ is the Schwarzian derivative,

$$\mathcal{S}_{\widetilde{U}}(f) = \frac{\overset{\circ\circ\circ}{t}}{\overset{\circ}{t}} - \frac{3}{2}\left(\frac{\overset{\circ\circ}{t}}{\overset{\circ}{t}}\right)^2 . \tag{2.5}$$

The vacuum condition for a pp-wave space-time (1.1) requires the Ricci tensor to vanish, $R_{\mu\nu} = 0$, which implies that

$$\Delta H = H_{,XX} + H_{,YY} = 0\,. \tag{2.6}$$

[1]In the complexified, or split signature setup the maximal number of symmetries is 9, with the metric given by the anti-self-dual pp wave with constant ASD Weyl tensor $g = dwdx + dzdy + y^2dw^2$. See [22] for details.

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This condition involves only the spatial behavior of the wave profile. Assuming that H is that of a vacuum, $\widetilde{H}$ in (2.4) will satisfy also the vacuum condition if the Schwarzian derivative term vanishes [28, 32], which yield in turn a *special Möbius conformal transformation* (SMCT),[2]

$$U = f(\widetilde{U}) = \frac{A\widetilde{U} + B}{C\widetilde{U} + D}\,, \tag{2.7a}$$

$$V = \widetilde{V} + \frac{1}{2}\frac{C}{C\widetilde{U} + D}\widetilde{\mathbf{X}}^2\,, \quad \mathbf{X} = \Omega\,\widetilde{\mathbf{X}}\,, \quad \text{where} \quad \Omega = \frac{\sqrt{AD - BC}}{C\widetilde{U} + D}\,. \tag{2.7b}$$

A, B, C and D here are arbitrary constants such that $AD - BC \neq 0$. Under such a redefinition the metric (2.3) becomes,

$$ds^2 = \Omega^2\, d\tilde{s}^2 = d\widetilde{\mathbf{X}}^2 + 2d\widetilde{U}d\widetilde{V} - 2\Omega^2 H\left(f(\widetilde{U})\,, \Omega\,\widetilde{\mathbf{X}}\right) d\widetilde{U}^2\,. \tag{2.8}$$

The new wave profile is in general different from the initial one. An initially U-independent profile (as e.g. for Brdicka (3.10)) becomes indeed U-dependent. Examples will be seen in section 3.

However it might happen also that the wave is *invariant* under Möbius redefinition — i.e., (2.7) acts as a *symmetry* studied in some detail in section 4 and illustrated by the pp wave inspired by molecular physics and studied in section 5.

3 Conformally related vacuum gravitational waves

Hence we focus our attention at vacuum plane GWs with line element

$$ds^2 = dX^2 + dY^2 + 2dUdV - \Big[\alpha(U)(X^2 - Y^2) + 2\gamma(U)XY\Big]\, dU^2, \tag{3.1}$$

where the arbitrary functions $\alpha(U)$ and $\gamma(U)$ correspond to the "+" and "×" polarization modes. These waves are taken conformally into an approximate sandwich form (2.8) by (2.7a)–(2.7b) [14, 15, 41, 43, 45] with new profile function

$$-2\widetilde{H} = \Omega^4\left[\widetilde{\alpha}(\widetilde{U})(\widetilde{X}^2 - \widetilde{Y}^2) + 2\widetilde{\gamma}(\widetilde{U})\widetilde{X}\widetilde{Y}\right], \tag{3.2}$$

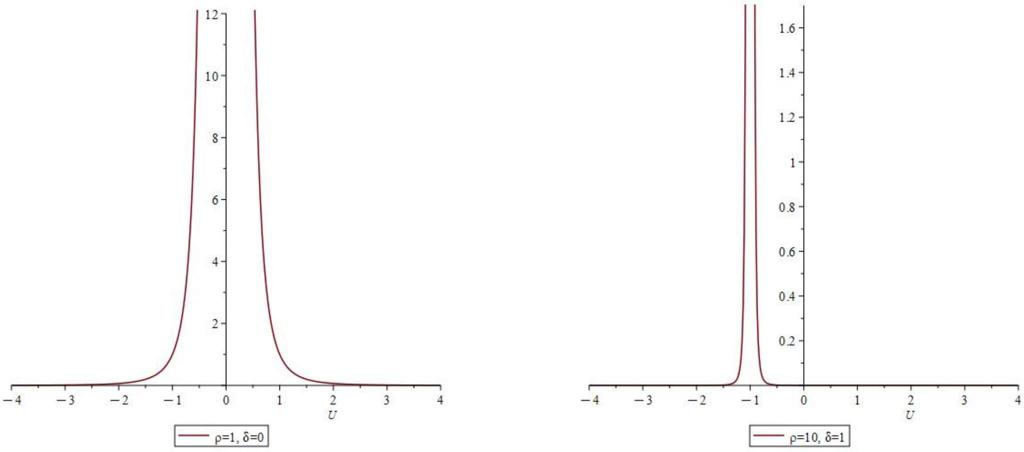
where $\widetilde{\alpha}(\widetilde{U}) = \alpha[f(\widetilde{U})]$ and $\widetilde{\gamma}(\widetilde{U}) = \gamma[f(\widetilde{U})]$. The new GWs include two classes which correspond to different choices of the coefficients A, B, C and D. $C = 0$ means a dilation and an U-translation of the original GWs which does not bring any new insight and will therefore not considered further.

$C \neq 0$ introduces in turn a new, rationally-redefined scale factor. In terms of the redefined parameters $\rho = C/\sqrt{AD - BC}$ and $\delta = D/C$ which determine the width and the center of the new GW shown in figure 1, the conformal factor can be presented as

$$\Omega^4 = \frac{1}{[\rho(U + \delta)]^4}\,. \tag{3.3}$$

Apart of focusing and shifting, the parameters ρ and δ do not change the trajectory. Choosing

[2] The Möbius approach goes actually back to a short note of Donald Lynden-Bell [33].



(a) (b)

Figure 1. The conformal factor (3.4) determines the width and position of the wave (3.5): 1(a) is for parameters $\rho = 1$ and $\delta = 0$ and figure 1(b) is for $\rho = 10$ and $\delta = 1$, respectively.

$\rho = 1$ and $\delta = 0$ for the sake of simplicity,

$$\Omega^4(U) = \frac{1}{U^4} \tag{3.4}$$

generates the special rational transformation [23],

$$U = -\frac{1}{\widetilde{U}}\,, \quad \mathbf{X} = \frac{\widetilde{\mathbf{X}}}{\widetilde{U}}\,, \quad V = \widetilde{V} + \frac{\widetilde{\mathbf{X}}^2}{2\widetilde{U}}\,. \tag{3.5}$$

Eqs. # (2.10) and # (2.11) of Andrzejewski and al. [45, 46] are also similar, except for that their profiles tend, unlike ours, to a Dirac delta.

The Möbius mapping SMCT (2.7a)–(2.7b) shown in figure 1 shrinks a globally defined GW into one concentrated around a single point which then behaves as an approximate sandwich wave [14, 15, 41, 43, 45].

Eq. (3.5) is in fact the oscillator counterpart of the conformal transformation applied to planetary motion with a time-dependent gravitational constant, proposed by Dirac [2, 47].

Keane and Tupper [23] noted in particular that (3.5) allows us to obtain a conformally related "dual" space-time. Our Möbius-redefined time and SMCT (2.7a)–(2.7b) have this property also, since the inverse transformation is identical to the original one.

The general vacuum GWs in (3.1) admit Killing vectors of the form

$$\widehat{\beta} = \boldsymbol{\beta}\,\partial_{\mathbf{X}} - \dot{\beta}_i X^i \partial_V, \tag{3.6}$$

where the two-vector $\boldsymbol{\beta} = (\beta_i)$ satisfies the vectorial Sturm-Liouville equation [48, 49]:

$$\ddot{\beta}_i(U) = K_{ij}\beta_j(U), \quad \text{with} \quad K_{ij} = \begin{pmatrix} \alpha(U) & \gamma(U) \\ \gamma(U) & -\alpha(U) \end{pmatrix}, \tag{3.7}$$

where $(\dot{\cdot})$ means d/dU. The transformation (2.7) then carries the Killing vector (3.6) into:

$$\widetilde{\widehat{\beta}} = \widetilde{\boldsymbol{g}}(\widetilde{U})\partial_{\widetilde{\mathbf{X}}} - \widetilde{\mathbf{X}}\cdot \mathring{\widetilde{\boldsymbol{g}}}\,\partial_{\widetilde{V}}\,, \quad \text{where} \quad \widetilde{\boldsymbol{g}}(\widetilde{U}) = \Omega^{-1}\boldsymbol{g}\left(f(\widetilde{U})\right). \tag{3.8}$$

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$\widetilde{\boldsymbol{g}}$ here satisfies the redefined Sturm-Liouville equation,

$$\overset{\circ\circ}{\widetilde{g}}_j + \widetilde{K}_{ij}\widetilde{g}^i, \quad \text{with} \quad \widetilde{K}_{ij} = \Omega^4 \begin{pmatrix} \widetilde{\alpha}(\widetilde{U}) & \widetilde{\gamma}(\widetilde{U}) \\ \widetilde{\gamma}(\widetilde{U}) & -\widetilde{\alpha}(\widetilde{U}) \end{pmatrix}. \tag{3.9}$$

Below we illustrate our point by two vacuum GWs, one linearly polarized, and the other circularly polarized. Both are globally defined and have a 7-dimensional symmetry algebra. Then we study how their symmetries and geodesics change under the Möbius transformation (3.5).

3.1 Conformally related linearly polarized vacuum GWs

1. The simplest globally defined *linearly polarized vacuum* GW (LPP) of Brdicka [51], whose metric is,

$$ds^2 = dX^2 + dY^2 + 2dUdV - (X^2 - Y^2)\, dU^2 \,. \tag{3.10}$$

Its CKVs are obtained by solving the conformal Killing equations,

$$W_L = \eta\partial_U + \left(2\rho V + \epsilon - \boldsymbol{X}\cdot\frac{d\boldsymbol{g}_L}{dU}\right)\partial_V + (\rho\boldsymbol{X} + \boldsymbol{g}_L)\cdot\partial_{\boldsymbol{X}} \,, \tag{3.11}$$

where

$$\mathbf{g}_L(U) = (\delta_1\sin U + \beta_1\cos U)\mathbf{e}_X + (\delta_2\cosh U - \beta_2\sinh U)\mathbf{e}_Y \,. \tag{3.12}$$

Here η, ϵ, ρ, δ_i and β_i are arbitrary constants which generate time-translations, $\widehat{E}$, vertical-translations, $\widehat{N}$, dilations, $\widehat{D}$, space-translations, $\widehat{P}_i$, and boosts $\widehat{G}_i$, respectively. These symmetries span the 7-dimensional homothetic algebra $\mathcal{E}_7$,

$$\begin{aligned} &[\widehat{P}_i,\widehat{P}_j]=0, && [\widehat{G}_i,\widehat{G}_j]=0, && [\widehat{P}_i,\widehat{G}_j]=\delta_{ij}\varpi\widehat{N}, && [\widehat{D},\widehat{N}]=-2\widehat{N},\\ &[\widehat{D},\widehat{G}_i]=-\widehat{G}_i, && [\widehat{D},\widehat{P}_j]=-\widehat{P}_j, && [\widehat{D},\widehat{E}]=0, && [\widehat{E},\widehat{G}_i]=-\widehat{P}_i,\\ &[\widehat{E},\widehat{P}_1]=\widehat{G}_1, && [\widehat{E},\widehat{P}_2]=-\widehat{G}_2\,. \end{aligned} \tag{3.13}$$

The $\boldsymbol{g}_L$-terms in (3.11) can be collected into

$$\boldsymbol{g}_L\cdot\partial_{\boldsymbol{X}} - \dot{\boldsymbol{g}}_L\boldsymbol{X}\cdot\partial_V \,, \tag{3.14}$$

which is (3.6).

2. The conformal transformation (3.5) carries the Brdicka wave (3.10) into a *rational LPP* with damped profile,

$$d\widetilde{s}^2 = d\widetilde{X}^2 + d\widetilde{Y}^2 + 2d\widetilde{U}d\widetilde{V} - \frac{1}{\widetilde{U}^4}(\widetilde{X}^2 - \widetilde{Y}^2)d\widetilde{U}^2, \tag{3.15}$$

whose CKVs can be obtained either directly or by the conformal transformation (3.5),

$$\widetilde{W}_L = \eta\widetilde{U}^2\partial_{\widetilde{U}} + \left(2\rho\widetilde{V} + \epsilon - \eta\frac{1}{2}\widetilde{\boldsymbol{X}}^2 - \widetilde{\boldsymbol{X}}\cdot\overset{\circ}{\widetilde{\mathbf{g}}}_L\right)\partial_{\widetilde{V}} + \left(\rho\widetilde{\boldsymbol{X}} + \eta\widetilde{U}\widetilde{\boldsymbol{X}} + \widetilde{\mathbf{g}}_L\right)\cdot\partial_{\widetilde{\boldsymbol{X}}}\,, \tag{3.16}$$

where

$$\widetilde{\mathbf{g}}_L(\widetilde{U}) = \widetilde{U}\left(-\delta_1\sin\frac{1}{\widetilde{U}} + \beta_1\cos\frac{1}{\widetilde{U}}\right)\mathbf{e}_X + \widetilde{U}\left(\delta_2\cosh\frac{1}{\widetilde{U}} + \beta_2\sinh\frac{1}{\widetilde{U}}\right)\mathbf{e}_Y\,. \tag{3.17}$$

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Note for further reference that the $\tilde{\boldsymbol{g}}_L$-terms in (3.16) combine into a solution of (3.8).

The parameters represent the same symmetries as in the Brdicka case except for η, which becomes a special Killing vector (SCKV) identified as an expansion $\widehat{K}$,

$$\widehat{K} = U^2\partial_U - \frac{1}{2}\mathbf{X}^2\partial_V + U\mathbf{X}\cdot\partial_{\mathbf{X}}\,, \tag{3.18}$$

which acts as a redefined-time translation $\widehat{E} = \partial_U$. The commutation relations are,

$$\begin{aligned}
&[\widehat{P}_i, \widehat{P}_j] = 0, \qquad [\widehat{G}_i, \widehat{G}_j] = 0, \qquad [\widehat{P}_i, \widehat{G}_j] = \delta_{ij}\bar{\omega}\widehat{N}, \quad [\widehat{D}, \widehat{N}] = -2\widehat{N}\,,\\
&[\widehat{D}, \widehat{G}_i] = -\widehat{G}_i, \quad [\widehat{D}, \widehat{P}_j] = -\widehat{P}_j, \quad [\widehat{D}, \widehat{K}] = 0, \qquad [\widehat{K}, \widehat{G}_i] = -\widehat{P}_i\,,\\
&[\widehat{K}, \widehat{P}_1] = \widehat{G}_1, \quad [\widehat{K}, \widehat{P}_2] = -\widehat{G}_2\,.
\end{aligned} \tag{3.19}$$

Thus the algebra $\mathcal{E}_7 \supset \mathcal{G}_6$ for the Brdicka GW (3.10) is transformed, for the rational-time LPP GW (3.15), into

$$\mathcal{S}_7 \supset \mathcal{E}_6 \supset \mathcal{G}_5\,. \tag{3.20}$$

Here $\mathcal{S}$, $\mathcal{E}$, $\mathcal{G}$ are the special conformal algebra, homothetic algebra and isometric algebra generators, respectively. The subscripts indicate the dimension of the algebra. The commutation relations do not change even if the CKVs do [23, 52].

3. The *circularly polarized (CPP) GW* has line element

$$ds^2 = dX^2 + dY^2 + 2dUdV - \Big[\cos(2\omega U)(X^2 - Y^2) + 2\sin(2\omega U)XY\Big]dU^2, \tag{3.21}$$

where ω is an arbitrary constant frequency. The corresponding CKVs were studied, e.g., in [16, 31]:

$$W_C = \eta\left[\partial_U + \omega\left(X\partial_Y - Y\partial_X\right)\right] + \left(2\rho V + \epsilon - \boldsymbol{X}\cdot\dot{\boldsymbol{g}}_C\right)\partial_V + \left(\rho\boldsymbol{X} + \boldsymbol{g}_C\right)\cdot\partial_{\boldsymbol{X}}\,, \tag{3.22}$$

where

$$\mathbf{g}_C(U) = g_{C1}(U)\,\mathbf{e}_X + g_{C2}(U)\,\mathbf{e}_Y\,, \tag{3.23}$$

$$\begin{aligned}
g_{C1}(U) = {}& \beta_2\mathcal{D}_U\left(\sin\omega U\cdot\sin\omega_- U\right) + \delta_2\mathcal{D}_U\left(\sin\omega U\cdot\cos\omega_- U\right)\\
& + \beta_1\mathcal{D}_U\left(\cos\omega U\cdot\sin\omega_+ U\right) - \delta_1\mathcal{D}_U\left(\cos\omega U\cdot\cos\omega_+ U\right),
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
g_{C2}(U) = {}& -\beta_2\mathcal{D}_U\left(\cos\omega U\cdot\sin\omega_- U\right) - \delta_2\mathcal{D}_U\left(\cos\omega U\cdot\cos\omega_- U\right)\\
& + \delta_1\mathcal{D}_U\left(\sin\omega U\cdot\cos\omega_+ U\right) - \beta_1\mathcal{D}_U\left(\sin\omega U\cdot\sin\omega_+ U\right),
\end{aligned} \tag{3.25}$$

where $\omega_\pm = \sqrt{\omega^2 \pm 1}$ and $\mathcal{D}_U$ is the bilinear derivative $\mathcal{D}_U(f\cdot g) = g\frac{df}{dU} - f\frac{dg}{dU}$. These formulae represent also analytic geodesics in the CPP GW space-time, as said before.

The parameters η, ρ, ϵ, δ_i and β_i generate "screw" symmetries $\widehat{S}$ [31], namely dilations $\widehat{D}$, vertical-translations $\widehat{N}$, space-translations $\widehat{P}_i$, and boosts $\widehat{G}_i$, respectively, span a 7-d homothetic algebra $\mathcal{E}_7 \supset \mathcal{G}_6$ [16].

4. Inserting (3.5) into (3.21) yields the *rational CPP GW* whose line element is,

$$
\begin{aligned}
d\tilde{s}^2 &= d\widetilde{X}^2 + d\widetilde{Y}^2 + 2d\widetilde{U}d\widetilde{V} \\
&\quad - \frac{1}{\widetilde{U}^4}\left[\cos\left(\frac{2\omega}{\widetilde{U}}\right)(\widetilde{X}^2-\widetilde{Y}^2) + 2\sin\left(\frac{2\omega}{\widetilde{U}}\right)\widetilde{X}\widetilde{Y}\right]d\widetilde{U}^2. \qquad (3.26)
\end{aligned}
$$

Its CKVs are obtained as in the rational-time LPP case,

$$
\begin{aligned}
\widetilde{W}_C &= \eta\left[\partial_{\widetilde{U}} + \widetilde{U}\widetilde{\boldsymbol{X}}\cdot\partial_{\widetilde{\boldsymbol{X}}} + \omega\left(\widetilde{X}\partial_{\widetilde{Y}} - \widetilde{Y}\partial_{\widetilde{X}}\right)\right] \\
&\quad + \left(2\rho\widetilde{V} + \epsilon - \widetilde{\boldsymbol{X}}\cdot\mathring{\tilde{\boldsymbol{g}}}_C\right)\partial_{\widetilde{V}} + (\rho\widetilde{\boldsymbol{X}} + \tilde{\boldsymbol{g}}_C)\cdot\partial_{\widetilde{\boldsymbol{X}}}\,, \qquad (3.27)
\end{aligned}
$$

where

$$
\tilde{\mathbf{g}}_C(\widetilde{U}) = \tilde{g}_{C1}(\widetilde{U})\mathbf{e}_{\widetilde{X}} + \tilde{g}_{C2}(\widetilde{U})\mathbf{e}_{\widetilde{Y}}, \qquad (3.28)
$$

$$
\begin{aligned}
\tilde{g}_{C1} &= -\beta_2\frac{\widetilde{U}}{\omega_+}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\frac{\omega_+}{\widetilde{U}}\cdot\widetilde{U}\cos\frac{\omega}{\widetilde{U}}\right) - \delta_2\frac{\widetilde{U}}{\omega_+}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\sin\frac{\omega_+}{\widetilde{U}}\cdot\widetilde{U}\cos\frac{\omega}{\widetilde{U}}\right) \\
&\quad + \beta_1\frac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\frac{\omega_-}{\widetilde{U}}\cdot\widetilde{U}\sin\frac{\omega}{\widetilde{U}}\right) - \delta_1\frac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\sin\frac{\omega_-}{\widetilde{U}}\cdot\widetilde{U}\sin\frac{\omega}{\widetilde{U}}\right), \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}_{C2} &= -\beta_2\frac{\widetilde{U}}{\omega_+}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\frac{\omega_+}{\widetilde{U}}\cdot\widetilde{U}\sin\frac{\omega}{\widetilde{U}}\right) - \delta_2\frac{\widetilde{U}}{\omega_+}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\sin\frac{\omega_+}{\widetilde{U}}\cdot\widetilde{U}\sin\frac{\omega}{\widetilde{U}}\right) \\
&\quad + \beta_1\frac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\frac{\omega_-}{\widetilde{U}}\cdot\widetilde{U}\cos\frac{\omega}{\widetilde{U}}\right) - \delta_1\frac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\sin\frac{\omega_-}{\widetilde{U}}\cdot\widetilde{U}\cos\frac{\omega}{\widetilde{U}}\right), \qquad (3.30)
\end{aligned}
$$

are also analytical geodesics in the rational CPP GW space-time.

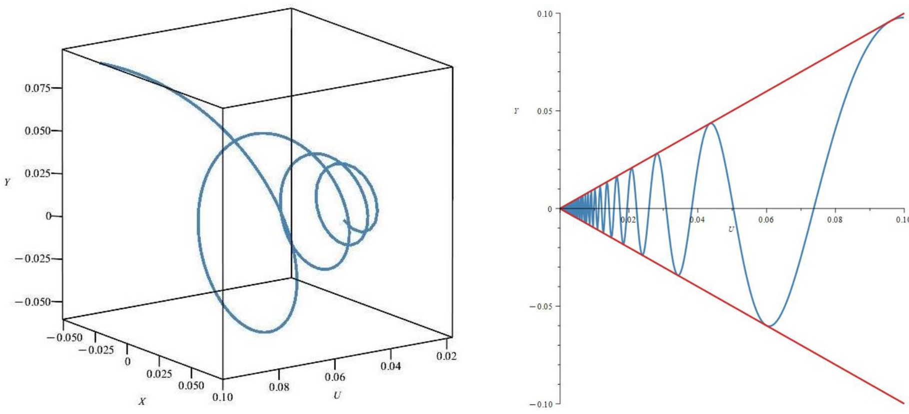
Here the parameters ρ, ϵ, δ_i, β_i represent the same symmetries as for the CPP wave, — except for η, which is a new special symmetry denoted by $\widehat{S}_K$,

$$
\widehat{S}_K = \widetilde{U}^2\partial_{\widetilde{U}} - \frac{1}{2}\widetilde{\mathbf{X}}^2\partial_{\widetilde{V}} + \widetilde{U}\widetilde{\mathbf{X}}\cdot\partial_{\widetilde{\mathbf{X}}} + \omega(\widetilde{X}\partial_{\widetilde{Y}} - \widetilde{Y}\partial_{\widetilde{X}})\,, \qquad (3.31)
$$

which corresponds to eq. # (147) of Keane and Tupper in ref. [23]. Its geometric meaning is obtained by integrating the Killing vector (3.31). Its space part,

$$
\begin{cases}
X = -\frac{U}{U_0}\left[X_0\cos\left(\frac{\omega(U-U_0)}{UU_0}\right) + Y_0\sin\left(\frac{\omega(U-U_0)}{UU_0}\right)\right] \\
Y = \frac{U}{U_0}\left[Y_0\cos\left(\frac{\omega(U-U_0)}{UU_0}\right) - X_0\sin\left(\frac{\omega(U-U_0)}{UU_0}\right)\right]
\end{cases}, \qquad (3.32)
$$

where X_0, Y_0 and U_0 are initial positions, describes a "growing screw" whose size increases linearly with U while its frequency decreases as shown in figure 2. A similar "screw" has also been found for planetary motions with for time-dependent gravitational constant in Newtonian gravity [2, 47].



(a) (b)

Figure 2. 2(a): the "screw" (3.32) of the rational CCP GW (3.26) expands linearly with U. Figure 2(b) shows its projection onto the $Y - U$ plane.

3.2 Gedesics, found numerically and analytically

Analytic solutions are readily derived from the results in section 3. Our clue is that the Sturm-Liouville equation (3.7) for *symmetries* is indeed *identical* to the *equations of motion* satisfied by the transverse coordinates $\mathbf{X}(U)$ [50],

$$\ddot{X}_i(U) = K_{ij} X_j(U), \tag{3.33}$$

(while the 3rd component $V(U)$ is then obtained by horizontal lift [2, 4]).

Thus once we know the Killing vectors (3.6), we get the geodesic for free and vice versa. Below we derive the analytic formulae by spelling out this remarkable correspondence. The numerical solutions shown in figure 3 for the *LPP* GW of Brdicka, (3.10), and for the *rational LPP*, (3.15), are matched by the analytic solutions deduced from (3.12) and a piecewise continuous solutions deduced from (3.17),

$$\widetilde{X}(\widetilde{U}) = \begin{cases} \widetilde{U}\sin\widetilde{U}^{-1} & \widetilde{U} < 0 \\ \widetilde{U}\cos\widetilde{U}^{-1} & \widetilde{U} > 0\,, \end{cases} \tag{3.34}$$

$$\widetilde{Y}(\widetilde{U}) = \begin{cases} 0, & \widetilde{U} \leq 0\,, \\ \widetilde{U}\left(\sinh\widetilde{U}^{-1} - \cosh\widetilde{U}^{-1}\right)\,, & \widetilde{U} > 0. \end{cases} \tag{3.35}$$

These analytical solutions are plotted in figure 4.

The geodesics of both the CPP GW (3.21) and the rational-time CPP GW (3.26) perform screw-like motions. Figure 5 compares these two numerically-obtained geodesics. Eq. (3.23) is an analytically found geodesic in the *CPP GW* space-time (3.21) which, choosing the parameters as $\omega = 1.5$, $\delta_1 = 0$, $\delta_2 = 0$, $\beta_1 = 0$ and $\beta_2 = 5$, matches the numerical one in figure 5(a).

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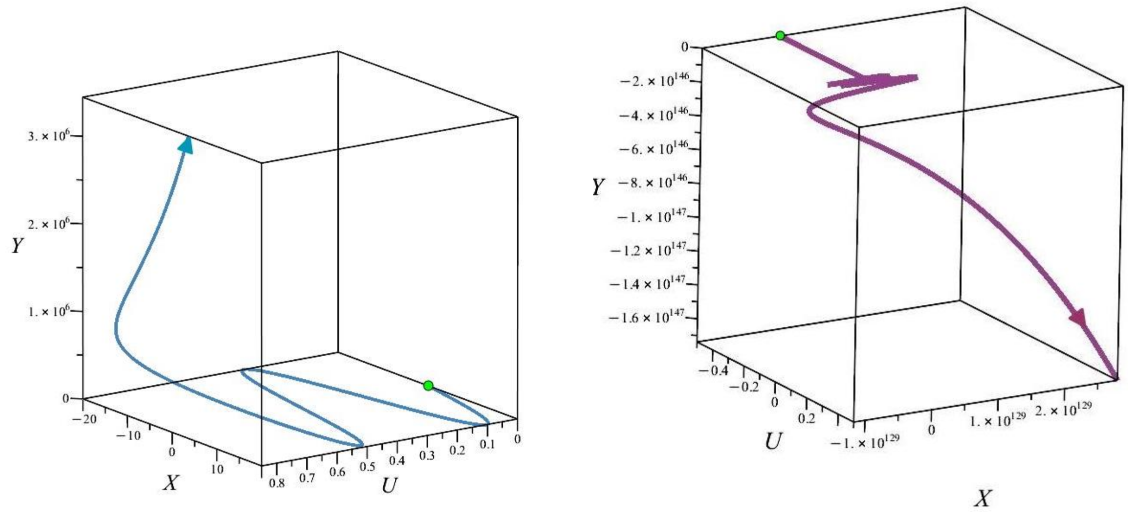
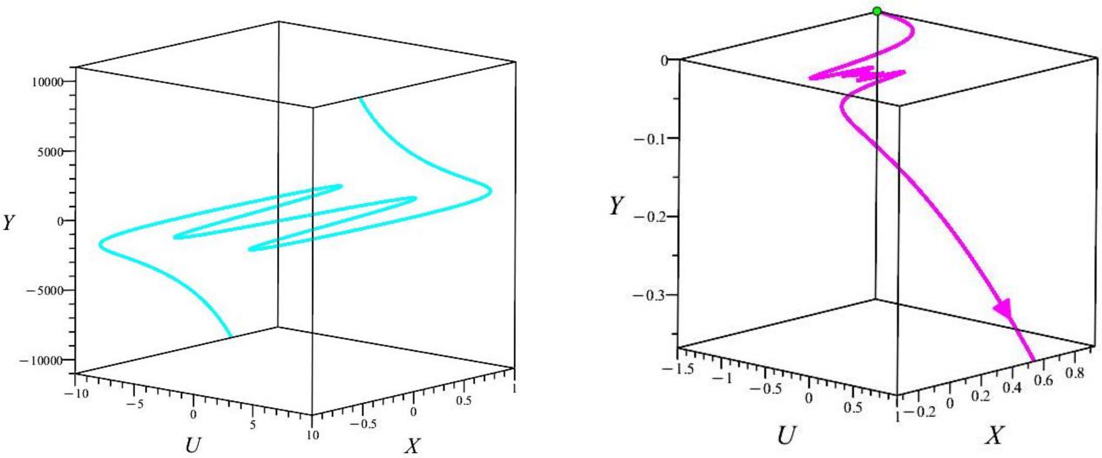


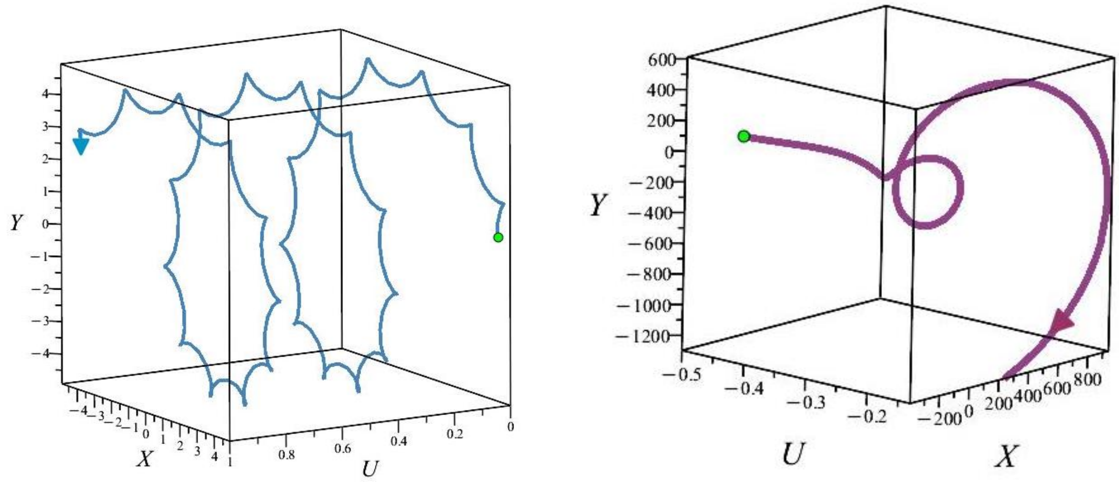
(a) (b)

Figure 3. 3(a): a particle in the LPP GW space-time (3.10) of Brdicka (drawn in steel blue) oscillates. It should be compared with what happens in the rational LPP GW (3.15), obtained by squeezing the wave as in (3.5) and drawn in dark orchid in figure 3(b), for which the particle initially in rest is shaken by the GW and then escapes with straightened-out velocity due to the damping factor U^{-1} after the wave has passed.



(a) (b)

Figure 4. 4(a) shows analytically found geodesics for the LPP (Brdicka) (3.10), and 4(b) for the rational LPP in (3.34)–(3.35), metric respectively. These plots should be compared with the numerical ones in figure 3.



(a) (b)

Figure 5. 5(a): in the usual CPP GW (depicted in steel blue) the particle performs a "gear wheel-like" motion. 5(b): in the rational CPP GW (in dark orchid) the particle which is at rest before the GW arrives escapes along an expanding screw after the GW has passed. For large U its velocity becomes approximately constant due to the damping factor $\widetilde{U}^{-4}$ in (3.26).

The *rational CPP GW* (3.26) admits special piecewise solutions,

$$\widetilde{X}(\widetilde{U}) = \begin{cases} 0, \quad \widetilde{U} \leq 0, \\ \dfrac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\dfrac{\omega_-}{\widetilde{U}} \cdot U\sin\dfrac{\omega}{\widetilde{\widetilde{U}}}\right), \quad \widetilde{U} > 0. \end{cases} \tag{3.36}$$

$$\widetilde{Y}(\widetilde{U}) = \begin{cases} 0, \quad \widetilde{U} \leq 0, \\ \dfrac{\widetilde{U}}{\omega}\mathcal{D}_{\widetilde{U}}\left(\widetilde{U}\cos\dfrac{\omega_-}{\widetilde{U}} \cdot \widetilde{U}\cos\dfrac{\omega}{\widetilde{\widetilde{U}}}\right), \quad \widetilde{U} > 0\,, \end{cases} \tag{3.37}$$

plotted in figure 6 which should be compared with the numerical solution in figure 5(b). Figure 7 shows the variations of the velocities in figure 6 on X and Y directions.

4 O(2, 1)-conformally invariant gravitational waves

In the previous section we discussed vacuum GWs that are carried into another vacuum GW by the special Möbius transformation (2.7a)–(2.7b). In this section we consider a special vacuum GWs which are *invariant*.

We start by completing (2.7a) by the well-known ξ-preserving conformal transformations of the conformal Killing equations in the free Minkowski metric in $2+1$ dimensions, (1.1)

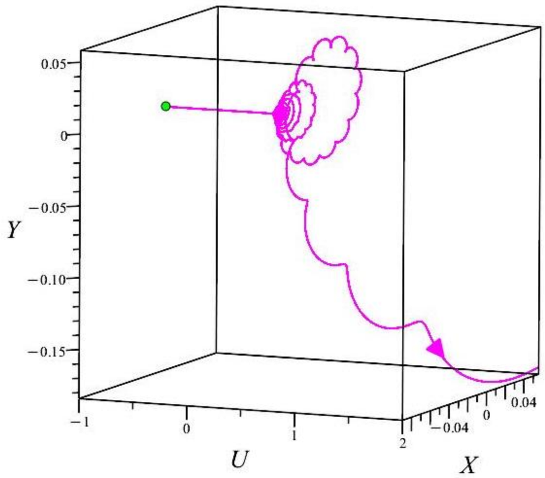
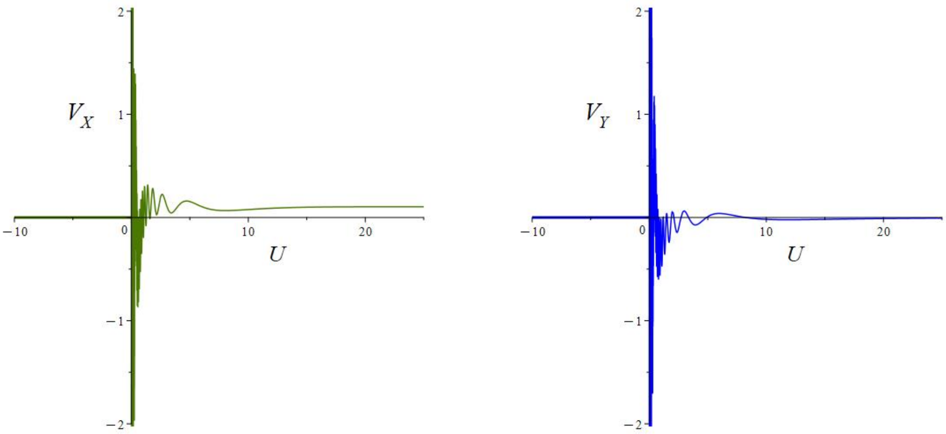


Figure 6. The analytic rational CPP solution (3.36)–(3.37), to be compared with the numerically found one in 5(b).



(a) (b)

Figure 7. For the rational CPP GW (3.26) the velocities become approximately constant after the wave has passed due to the damping factor $\widetilde{U}^{-4}$: we get the velocity effect [9, 16].

with $H = 0$. We get three special transformations, namely,

$$\text{time-translation}: \; U = \widetilde{U} + \epsilon, \quad \mathbf{X} = \widetilde{\mathbf{X}}\,, \quad V = \widetilde{V}\,, \tag{4.1a}$$

$$\text{dilatation}: \; U = e^{2\delta}\widetilde{U}, \quad \mathbf{X} = e^{\delta}\widetilde{\mathbf{X}}\,, \quad V = \widetilde{V}, \tag{4.1b}$$

special conformal transformation :

$$U = \frac{\widetilde{U}}{1+\kappa\,\widetilde{U}}\,, \qquad \mathbf{X} = \frac{\widetilde{\mathbf{X}}}{1+\kappa\,\widetilde{U}}\,, \qquad V = \widetilde{V} + \frac{\kappa}{2(1+\kappa\,\widetilde{U})}\widetilde{\mathbf{X}}^2\,, \tag{4.1c}$$

where ϵ, δ and κ are arbitrary real constants. The corresponding infinitesimal generators,

$$\text{time}-\text{translation}: \qquad \widehat{E} = \partial_U, \tag{4.2a}$$

$$\text{dilation}: \qquad \widehat{D} = U\partial_U + \frac{1}{2}\left(X\partial_X + Y\partial_Y\right), \tag{4.2b}$$

$$\text{expansion}: \qquad \widehat{K} = U^2\partial_U + U(X\partial_X + Y\partial_Y) - \frac{1}{2}(X^2+Y^2)\partial_V\,, \tag{4.2c}$$

span an $o(2,1)$ algebra,

$$[\widehat{D},\widehat{E}] = -\widehat{E}, \quad [\widehat{D},\widehat{K}] = \widehat{K}, \quad [\widehat{E},\widehat{K}] = 2\widehat{D} \tag{4.3}$$

which generate an $\mathrm{O}(2,1)$ conformal group.

Systems with $\mathrm{O}(2,1)$ symmetry were considered in various physical instances:

- For a free particle [2, 4, 53–55] or in Chern-Simons field theory [56–59] it extends the Galilei to the Schrödinger algebra [53–55]. All Schrödinger-symmetric systems are derived, in $d \geq 3$ space dimensions, from the vanishing of the Weyl [59] or in $d = 1$ from that of the Cotton tensor [60], respectively.
- An inverse-square potential could be added [4, 53–63]. Applications include the interaction of a polar molecule with an electron [34, 36] (which will be discussed further in subsection 5), the Efimov effect [36, 64], near-horizon fields of black holes [35, 36, 65] and the vacuum AdS/CFT correspondence [36, 66, 67];
- A Dirac-monopole and a magnetic vortex [68, 69] .

Hence we focus our attention at vacuum gravitational waves with $\mathrm{O}(2,1)$ symmetry. For symplicity, we focus our investigations to the planar case with coordinates X, Y. Substituting the three vectors in (4.2a)–(4.2c) into the conformal Killing equation (1.2) for the Brinkmann metric (1.1) leaves us with,

$$\text{time}-\text{translation}: \qquad H_{,U} = 0\,, \tag{4.4a}$$

$$\text{dilatations}: \qquad UH_{,U} + XH_{,X} + YH_{,Y} + 2H = 0\,, \tag{4.4b}$$

$$\text{special conformal transformation}: \qquad 2UH_{,U} + XH_{,X} + YH_{,Y} + 2H = 0\,. \tag{4.4c}$$

Note that (4.4b) and (4.4c) differ only in the coefficients of their first terms — which involves the generator of time translation symmetry, (4.4a).

Solving these equations with the vacuum condition (2.6) yields, for an exact plane wave, the line element,

$$ds^2_{O21} = dX^2 + dY^2 + 2dUdV - 2\left(\frac{C_1(X^2-Y^2) + 2C_2XY}{R^4}\right)dU^2\,, \tag{4.5}$$

where $R^2 = X^2 + Y^2$; C_1 and C_2 are arbitrary constants. The proof follows at once from that dilatation symmetry (4.4b), combined with time-translation-invariance (4.4a) imply indeed, by Euler's formula, that the potential is homogeneous of order (-2).

The potential (4.5) breaks the rotational symmetry, however still allows for the conformal $\mathrm{O}(2,1)$ symmetry of the inverse-square potential [2, 4, 62] to the anisotropic case. It should be compared to the statement [59, 60] which says that the profiles of the only Bargmann manifolds with *Schrödinger* symmetry correspond, in 3+1 dimensions, to an (i) isotropic oscillator, to an (ii) inverse-square potential with constant coefficient, to a (iii) uniform force field.

The special GW (4.5) satisfies, for an arbitrary linear combination of $\widehat{E}$, $\widehat{D}$, $\widehat{K}$ in (4.2a)–(4.2c), the conformal Killing equations (1.2) with,

$$W = a\widehat{E} + b\widehat{D} + c\widehat{K} = (a + bU + cU^2)\partial_U - c\frac{1}{2}\mathbf{X}^2\partial_V + \left(cU + \frac{b}{2}\right)\mathbf{X}\cdot\partial_{\mathbf{X}}\,, \tag{4.6}$$

where a, b and c are arbitrary constants. By integrating the U component of (4.6), the associated SKV reduces to the Möbius-redefined time (2.7a) with $\mathbf{X}$, A, etc replaced by, $\tilde{\mathbf{X}}$ and by,

$$\tilde{A} = -\left[\frac{b}{2} + \frac{\sqrt{4ac-b^2}}{2\tan\left(\frac{\eta\sqrt{4ac-b^2}}{2}\right)}\right], \quad \tilde{B} = -c, \quad \tilde{C} = a, \quad \tilde{D} = -\left[\frac{\sqrt{4ac-b^2}}{2\tan\left(\frac{\eta\sqrt{4ac-b^2}}{2}\right)}\right], \tag{4.7}$$

where η is the parameter of the integral curve. In conclusion, the special gravitational wave (4.5) is form-invariant under the SMCT (2.7b).

The metric (4.5) is conveniently presented in cylindrical coordinates (R,θ),

$$ds^2_{O21} = dR^2 + R^2 d\theta^2 + 2dUdV - 2\left(\frac{C_1\cos 2\theta + C_2\sin 2\theta}{R^2}\right)dU^2\,, \tag{4.8}$$

reminiscent of the potential for the interaction between a *polar molecule and an electron* [34, 35],

$$H \equiv H(r,\theta) = \frac{C\cos\theta}{r^2}\,, \tag{4.9}$$

where the constant C is proportional to the product of the electric charge and the dipole momentum, and θ is the polar angle in the $X-Y$ plane.

5 A molecular physics-inspired gravitational wave

In this section, we study a vacuum spacetime inspired by polar molecules represented by the anisotropic inverse-square potential [34, 35],

$$H = \frac{C_1\cos 2\theta + C_2\sin 2\theta}{R^2}\,, \tag{5.1}$$

where C_1 and C_2 are real constants, cf. (4.8). Postponing the 3-dimensional problem to further study, we limit our attention at the plane. For simplicity, we put also the NR mass $M=1$. The conformal Killing vectors in (4.2a)–(4.2b)–(4.2c) preserve the vertical vector $\xi = \partial_V$ and therefore project to conformal symmetries of the underlying non-relativistic system providing us with three conserved quantities [4, 53–55, 61, 62],

$$\widehat{E} \quad \to \quad \mathcal{E} = \frac{\mathbf{P}^2}{2} + \frac{C_1 \cos 2\theta + C_2 \sin 2\theta}{R^2}\,, \tag{5.2a}$$

$$\widehat{D} \quad \to \quad \mathcal{D} = 2\mathcal{E}U - \mathbf{P}\cdot\mathbf{X}\,, \tag{5.2b}$$

$$\widehat{K} \quad \to \quad \mathcal{K} = -\mathcal{E}U^2 + \mathcal{D}U + \frac{1}{2}R^2\,. \tag{5.2c}$$

To explain in simple terms what happens, consider first dilations, (4.2b), which leave the Lagrange density $L_0 dU$ of a free NR particle invariant provided the time scales with the square of the factor as the position does [53–55]. Then adding a potential H changes the Lagrange density by $-HdU$, which is also invariant if H is inverse-square in the radius [4, 62, 63].

However dilations act only on the radial variable, therefore the potential (5.1) is left invariant. Then an easy calculation shows that the two other transformations in (4.2) remain also unbroken. Remarkably, the associated "Noether" quantities were found by Jacobi [61] ...60 years *before* Emmy Noether was born!

The Casimir operator of $\mathrm{O}(2,1)$ is,

$$\mathcal{C}^2 = \mathcal{R}^2 - \mathcal{G}_-^2 - \mathcal{G}_+^2, \tag{5.3}$$

where

$$\mathcal{R} = \frac{1}{2}\left(\frac{1}{\tau}\widehat{K} + \tau\widehat{E}\right), \quad \mathcal{G}_- = \frac{1}{2}\left(\frac{1}{\tau}\widehat{K} - \tau\widehat{E}\right), \quad \mathcal{G}_+ = \widehat{D} \tag{5.4}$$

generate a compact $\mathrm{SO}(2)$ group of rotations, augmented with two non-compact two dimensional boosts. Here τ is a positive fixed parameter with the dimension of time. See ref. [69] for details. The Casimir operator can also be written as,

$$\mathcal{C}^2 = J^2 + 2\left(C_1 \cos 2\theta + C_2 \sin 2\theta\right), \tag{5.5}$$

where $J = \mathbf{R}\times\mathbf{V}$ is the orbital angular momentum. (The angular momentum in 2 dimensions is just a scalar, namely the 3rd component of the 3-dimensional one, J_z. The conserved quantity generated by translations along the V coordinate and interpreted as the mass of the underlying non-relativistic system [2, 4, 6] was scaled to unity).

A lightlike particle in the special GW background (4.5) (viewed, in the Bargmann framework, as a massive non-relativistic particle in one dimension less) moves along null geodesics. In cylindrical coordinates,

$$\frac{d^2R}{dU^2} - R\left(\frac{d\theta}{dU}\right)^2 - \frac{2\left[C_1 \cos 2\theta + C_2 \sin 2\theta\right]}{R^3} = 0\,, \tag{5.6}$$

$$\frac{d^2\theta}{dU^2} + \frac{2}{R}\frac{dR}{dU}\frac{d\theta}{dU} - \frac{2\left[C_1 \sin 2\theta - C_2 \cos 2\theta\right]}{R^4} = 0\,. \tag{5.7}$$

Let us assume, for simplicity, that $C_1 = 0$ so that the planar metric (4.8) has only one polarization state,

$$ds^2 = dR^2 + R^2 d\theta^2 + 2dUdV - 2\left(\frac{C_2 \sin 2\theta}{R^2}\right) dU^2\,, \tag{5.8}$$

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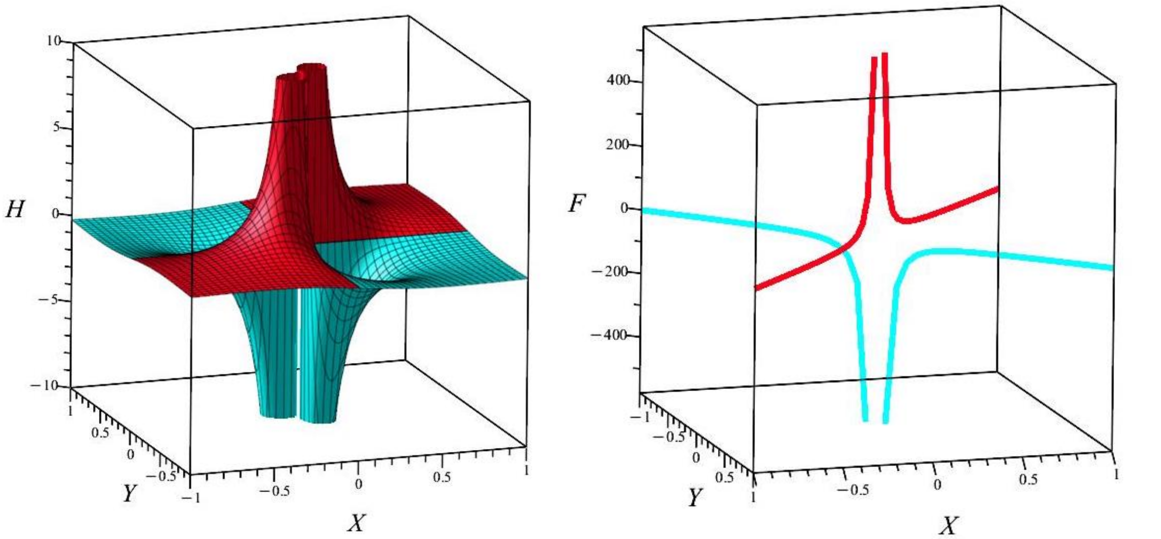

(a) (b)

Figure 8. 8(a): the potential (5.8) alternates between **repulsive (NE-SW)** and **attractive (NW-SE)**, changing sign at every quadrant. The apparent doubling of the "chimneys and sinks" in figure 8a are computer artifacts as confirmed by figure 8b: the only singularity is at the origin.

For $C_2 = 0$ we get Minkowski-space which has no interest for us. Then $C_2 > 0$ can be achieved by shifting θ. Henceforth we set $C_2 = 1$.

The metric (5.8) is the "Bargmannian" form [2, 4, 6] of the anisotropic version of a NR particle in an inverse-square potential

$$H(R, \theta) = \frac{\sin 2\theta}{R^2}\,, \tag{5.9}$$

shown in figure 8. Its anisotropy is manifest by realizing that for fixed $R = R_0$, $H(R, \theta)$ is proportional to $\sin 2\theta$. A long-distance view is shown in figure 9.

The nature of the potential (5.8) is determined by the sign of the coefficient of dU^2 — the potential of the underlying non-relativistic dynamics [2, 4] — which alternates at every quadrant. Its behavior is conveniently studied by plotting the force, figure 10: It is **repulsive** for $0 < \theta < \pi/2$ and for $\pi < \theta < 3\pi/2$, and **attractive** for $\pi/2 < \theta < \pi$ and for $3\pi/2 < \theta < 2\pi$. The force is maximal on the separation "crosslines" at $\theta = k\pi/2$, $k = 0, 1, 2, 3$, where the repulsive potential becomes attractive and vice versa, cf. (5.9). It is obviously symmetric w.r.t. $\theta \to \theta + \pi$.

A qualitative insight into the possible motions can be obtained by using the conformal o(2, 1) symmetry. For simplicity we restrict our attention at what happens to a particle that we simply put at $U = U_0$ to some position (R_0, θ_0) with vanishing initial velocity. Then the conserved quantities (5.2) generated by o(2, 1) reduce, putting $M = 1$, $C_1 = 0$, $C_2 = 1$

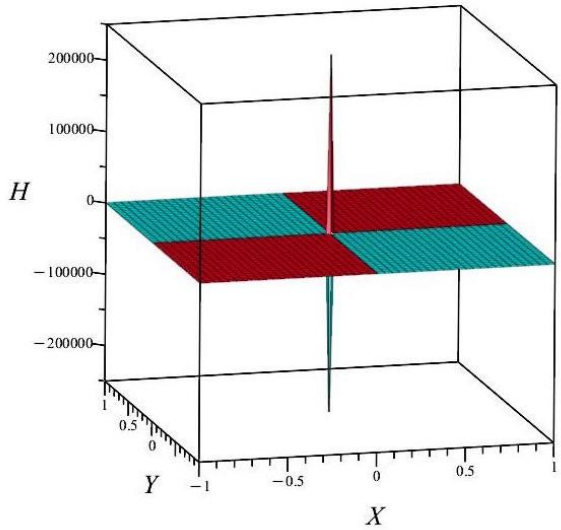

Figure 9. A long-distance view of the wave (5.8) shows a "spike" whose sign alternates at every quarter of the circle.

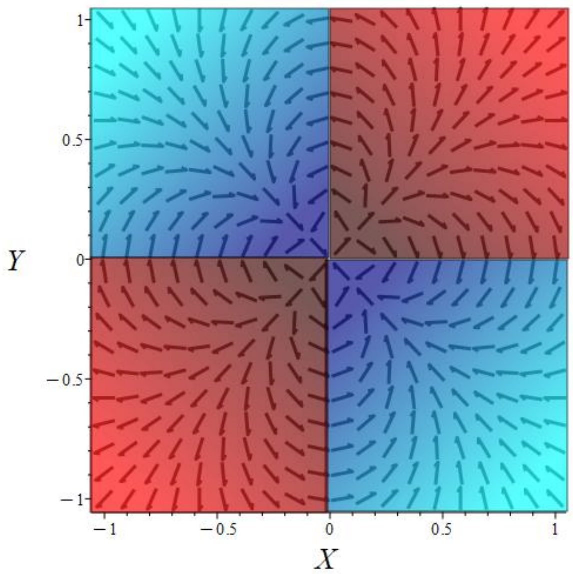

Figure 10. The force $-\boldsymbol{\nabla} H$ alternates at every quarter-of-circle between **repulsive (NE – SW)** and **attractive (NW – SE)** regions. The force is maximally repulsive along the "crests" at $\pi/4$ and $5\pi/4$ and maximally attractive in the "valley bottoms" at $3\pi/4$ and $7\pi/4$, respectively.

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for simplicity, to

$$\mathcal{E}_0 = \frac{\sin 2\theta_0}{R_0^2}\,, \tag{5.10a}$$

$$\mathcal{D}_0 = 2\mathcal{E}_0 U_0\,, \tag{5.10b}$$

$$\mathcal{K}_0 = -\mathcal{E}_0 U_0^2 + \mathcal{D}_0 U_0 + \frac{1}{2} R_0^2\,. \tag{5.10c}$$

From (5.10a) we deduce that the conserved energy, which is initially just the potential, may be **positive**, **negative** or **zero**, corresponding to the repulsive or attractive quadrant or to the separation line between them, as depicted in figures 8 and 10.

1. In the repulsive quadrants $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$ the energy is positive,

$$\mathcal{E} = \mathcal{E}_0 = \frac{\mathbf{P}^2}{2} + \frac{\sin 2\theta}{R^2} > 0 \Rightarrow \frac{\mathbf{P}^2}{2} > \left|\frac{\sin 2\theta}{R^2}\right|. \tag{5.11}$$

   Thus the motion is outgoing. When the particle crosses the separation line and enters into the attractive area, the absolute value of negative potential is less than that of the initial potential: the particle will be pushed out to infinity.

2. In the attractive quadrants, $\pi/2 < \theta < \pi$ or $3\pi/2 < \theta < 2\pi$ the energy is negative,

$$\mathcal{E} = \mathcal{E}_0 = \frac{\mathbf{P}^2}{2} + \frac{\sin 2\theta}{R^2} < 0 \Rightarrow \frac{\mathbf{P}^2}{2} < \left|\frac{\sin 2\theta}{R^2}\right|. \tag{5.12}$$

   Thus the kinetic energy is dominated by the potential energy, and we get incoming motion with the particle falling into the hole.

3. An intermediate behaviour is observed for vanishing energy when the initial position is on one of the a **separation line** between repulsive and attractive quadrants, i.e., for $\theta_k = k\frac{\pi}{2}$, $k = 0, 1, 2, 3$: by (5.10a) and (5.10b) we have,

$$\mathcal{E} = \mathcal{E}_0 = 0 \quad \text{and} \quad \mathcal{D} = \mathcal{D}_0 = -\mathbf{P}\cdot\mathbf{X} = 0, \tag{5.13}$$

   so that (5.10c) implies that

$$R = R_0 = \text{const.} \quad \text{and} \quad \mathbf{P} \perp \mathbf{X}\,. \tag{5.14}$$

   In conclusion, a particle put on the "rim" will follow a circular trajectory inside the attractive region. Moreover, the vanishing of the energy,

$$2\mathcal{E}_0 = \mathbf{P}^2 + 2\frac{\sin 2\theta}{R^2} = 0, \tag{5.15}$$

   implies that the particle oscillates between the "rims" of the attractive quadrants,

$$\frac{\pi}{2} \le \theta \le \pi \quad \text{or} \quad \frac{3\pi}{2} \le \theta \le 2\pi\,. \tag{5.16}$$

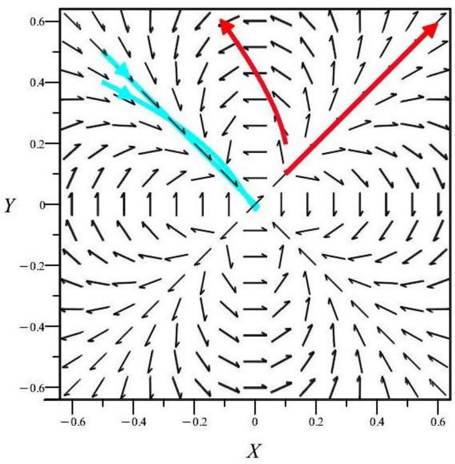
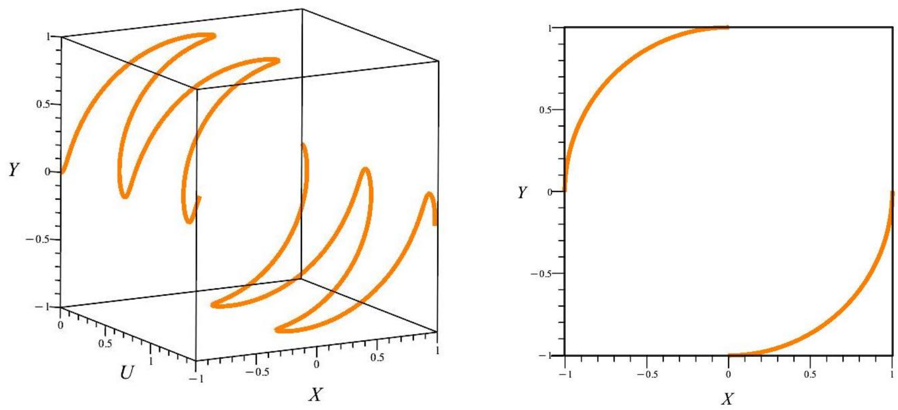


Figure 11. Particles which start in the repulsive zone are pushed to infinity both in the repulsive quadrant, and, after crossing over, also in the attractive quadrant. Particles which start from the attractive zone are in turn sucked into the hole. This behavior corresponds to the sign of the non-relativistic energy (5.10a).



(a) (b)

Figure 12. Numerically obtained periodic trajectories 12(a) in the $o(2,1)$ symmetric but non-isotropic gravitational wave (5.8). 12(b) shows their projections onto the $X-Y$ plane, as seen also in figure 13. The curves show two particles which start from $(1,0)$ resp. at $(-1,0)$ with zero initial velocity. The trajectories oscillate along quarters-of-a-circle.

Numerical investigations indicate that the eqs. (5.6)–(5.7) admit all three types of outgoing/infalling/bounded solutions. The first two are shown in figure 11, and the circularly oscillating one in figure 12. The general behavior is summarised in figure 13.

Analytic solutions can be found also.

We first inquire about radial motions. Putting $\theta=\theta_0=\text{const.}$ into (5.6)–(5.7) yields,

$$\frac{d^2R}{dU^2}-\frac{2\sin 2\theta_0}{R^3}=0 \quad \text{and} \quad \frac{2\cos 2\theta_0}{R^4}=0\,.$$

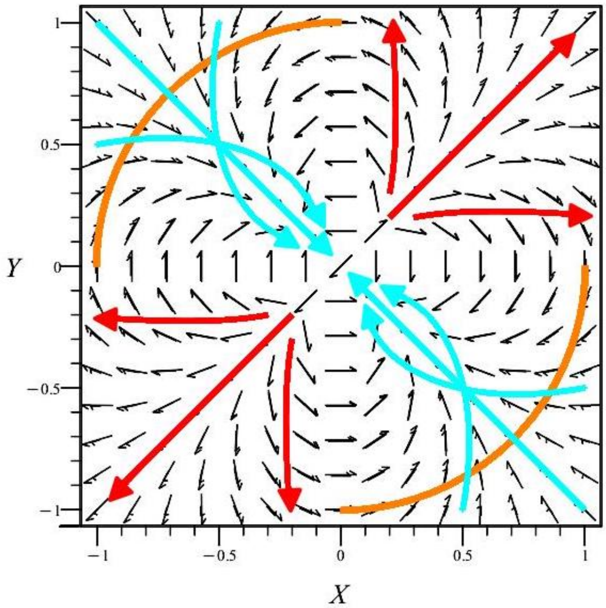


Figure 13. In the **NE** and **SW** quadrants the particle is pushed outwards to infinity whereas it is sucked into the origin in the **NW** and **SE** quadrants. **Bounded** zero-energy motions arise which oscillate in the attractive quadrant between the separation lines of the attractive and repulsives zones.

The 2nd eq. implies,

$$\theta_0 = (2\ell+1)\frac{\pi}{4}\,, \qquad \ell = 0,\, 1,\, 2,\, 3\,, \tag{5.17}$$

leaving us with the familiar inverse-square-potential equation,

$$\frac{d^2R}{dU^2} = \pm\frac{2}{R^3}\,, \tag{5.18}$$

where the sign is positive in the repulsive, $\ell = 0, 2$ case and is negative in the attractive, $\ell = 1, 3$ one. Thus for ℓ pair the particle is repulsed to infinity along the "crest", and for ℓ odd it is sucked into the origin along the "valley bottom" which correspond to the maximally repulsive or maximally attractive directions in figures 8 and 10.

For motion along the diagonals the solution is [4, 59, 62, 63],

$$R(U) = \sqrt{(V_0U + R_0)^2 \pm \frac{2U^2}{R_0^2}}\,, \tag{5.19}$$

where $R_0 > 0$ and V_0 are the initial position and velocity at $U = 0$, respectively. We choose $V_0 = 0$ for simplicity. Then starting in the **repulsive quadrants** with $\theta = \pi/4$ or $\theta = 5\pi/4$ we have the plus sign and

$$R(U) \geq \sqrt{R_0^2 + \frac{2U^2}{R_0^2}} \geq R_0 \tag{5.20}$$

increasing with U: the particle is expelled.

In the **attractive quadrants** with $\theta = 3\pi/4$ or $\theta = 7\pi/4$ we have the minus sign and the motion is directed towards the origin:

$$R(U) = \sqrt{R_0^2 - \frac{2U^2}{R_0^2}} \leq R_0\,, \tag{5.21}$$

which says that the particle moves inwards, however after the critical value

$$U_{\text{crit}} = \frac{R_0^2}{\sqrt{2}} \tag{5.22}$$

$R(U)$ would become imaginary, indicating that the particle has fallen into the hole.

The equations (5.6)–(5.7) admit also exact *circular, analytic solutions*. Let us indeed fix the radius, $R(U) = R_0 =$ const. which reduces (5.6)–(5.7) to,

$$\left(\frac{d\theta}{dU}\right)^2 + \frac{2}{R_0^4}\sin 2\theta = 0\,, \qquad \frac{d^2\theta}{dU^2} + \frac{2}{R_0^4}\cos 2\theta = 0\,. \tag{5.23}$$

Deriving the first eq. by U we get $\frac{d\theta}{dU}\left(\frac{d^2\theta}{dU^2} + \frac{2C_2}{R_0^4}\cos 2\theta\right) = 0$, which is an identity when the 2nd equation is satisfied. The first equation in (5.23) then implies that

$$\frac{d\theta}{dU} = \left(\frac{2}{R_0^2}\right)^{1/2}\sqrt{-\sin 2\theta} = 0\,, \tag{5.24}$$

which admits real solutions when the sin is negative i.e. in the quadrants $\pi/2 \leq \theta \leq \pi$ and $3\pi/2 \leq \theta \leq 2\pi$ and is then solved in terms of elliptic integrals [70],

$$\theta(U) = -\frac{1}{2}\arcsin\left\{\text{JacobiCN}^2\left[\frac{2}{R_0^2}(U+D), \frac{\sqrt{2}}{2}\right]\right\}\,, \tag{5.25}$$

where D is an integration constant. This formula can also be verified directly and is plotted in figure 14 (to be compared with the numerical solution in figure 12).

This solution has zero-energy. Conversely [71], for vanishing energy $\mathcal{E} = 0$ the conserved quantity generated by dilations, (5.10b) implies $R = R_0 =$ const., (5.14). Then (5.2a) becomes (5.15) which for $R = R_0$ is (5.24) that we have just solved. In conclusion, the $\mathrm{o}(2,1)$ symmetry implies, for zero energy, motion on (part of) a circle.

Restoring the radius in the equations shows that the period increases proportionally to the its square, R_0^2,

$$\Delta U = R_0^2 \frac{4K}{\sqrt{2C_2}} \;\Rightarrow\; \Delta U \propto R_0^2\,, \tag{5.26}$$

as seen in figure 15. Inserting $\theta(U)$ from (5.25) into the conserved Casimir (5.5) we get, for $C_1 = 0$,

$$J^2 = 2\,\text{JacobiCN}^2\left[\frac{2}{R_0^2}(U+C), \frac{\sqrt{2}}{2}\right] + \text{const.} \tag{5.27}$$

On the other hand, the angular momentum for (5.25) is

$$J = \vec{R} \times \vec{V} = \text{JacobiCN}\left[\frac{2}{R_0^2}(U+C), \frac{\sqrt{2}}{2}\right]\,, \tag{5.28}$$

whose square fixes the constant in (5.27) to vanish. The length of (5.28) thus oscillates as shown in figure 16, consistently with the breaking of the axial symmetry.

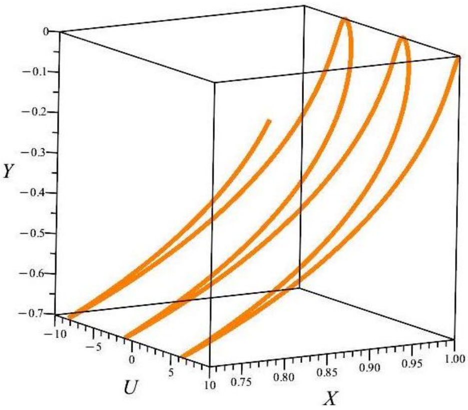
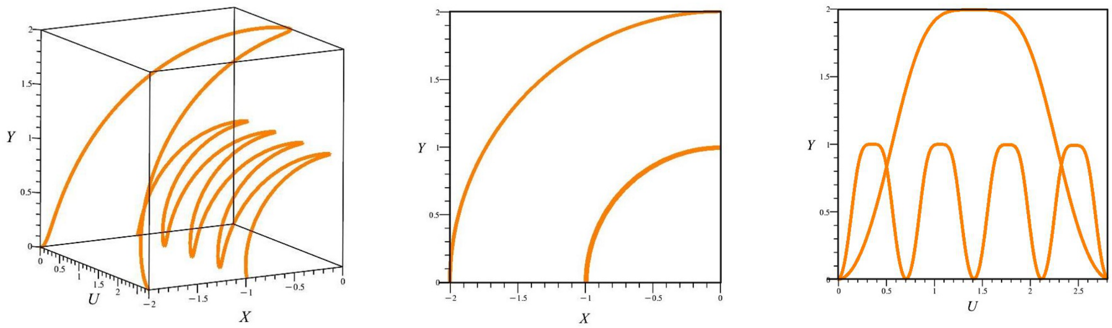


Figure 14. The analytic solution obtained in terms of elliptic integrals describes periodic motion along a circular arc confined into the attractive quadrant $\pi/2 < \theta < \pi$, consistently with the numerical solutions in figures 12 and 15.



(a) (b) (c)

Figure 15. Figure 15(a) shows the trajectories of two particles initially at rest on the separation line of the repulsive and attractive quadrants at $(\mathbf{1}, \mathbf{0})$ and at $(\mathbf{2}, \mathbf{0})$, respectively. The projections in figure 15(b) into the $X - Y$ plane follow quarter-of-circle arcs with radiuses $R_0 = 1$ and $R_0 = 2$. The projection into the $Y - U$ plane in figure 15(c), shows that the period for $R_0 = 2$ is four times that for $R_0 = 1$, consistently with (5.26).

6 History: from Arnold through Newton, back to Galilei

The zero-energy motions which oscillate along quarter-of-circles in the attractive zone between the separation lines of the attractive and repulsive quarters have indeed quite remarkable ancestry. Let's proceed backwards in time.

We start with noting that our equations (5.23) are reminiscent of the study of planetary motion by making use of the Bohlin-Arnold duality between harmonic oscillators and the Kepler problem [72, 73]. It is worth noting that the Bohlin-Arnold duality can be put directly in the projective geometry framework [74].

Eqs. #(8.3) in [75] which *assume* circular trajectories are consistent when the *force is inversely proportional to the fifth power of the distance from the sun*,

$$\text{force} \;\propto\; -\frac{1}{r^5}\,. \tag{6.1}$$

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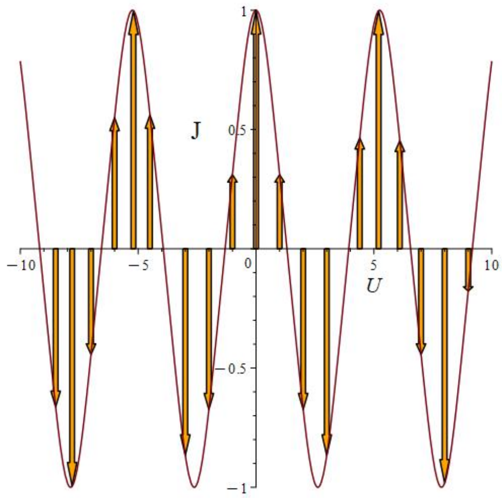

Figure 16. In the anisotropic metric (5.8) in figure 8, the orbital angular momentum J in (5.28) is not conserved: for the circular periodic motion found for zero energy, for example, its length oscillates. The direction of the oscillations seen in figures 12 and 15 corresponds to the sign of the angular momentum, with the turning point corresponding to the zeros of the angular momentum.

This fact was known already by Newton, who, in his *Principia*, inquired: – *What force laws do allow for circular trajectories?* — and he found, using geometrical techniques that in addition to r^{-2} one can have also (6.1), see [76] vol. I Proposition VII. Problem II, where the proof is left as an exercise.

Yet another intriguing feature is that both our circular solution in section 5 and the parabolic trajectory of the 1680 comet (discussed by Newton in Book III Proposition XLI, Problem XXI of [76]), has also *zero energy*. These solutions separate bounded and unbounded motions.

Even more incredibly, figure4 in Galilei's *Dialogo* [77], written before Newton was even born, suggests circular motion which would pass through the center of the Earth.

Returning to our circularly oscillating motions found in section 5 we note that they do *not* enter into the Bohlin-Arnold framework. Let us explain. The Bohlin-Arnold trick [72, 73] is based on a *duality* between two central potentials proportional to r^a and r^A, respectively, which are dual when the constraint

$$\left(1+\frac{a}{2}\right)\left(1+\frac{A}{2}\right)=1 \tag{6.2}$$

is satisfied; then motion in the r^a and in the r^A potentials can be swapped into each other.

The newtonian potential corresponds, for example, to $a=-1$; its dual has therefore $A=2$ i.e., is an isotropic harmonic oscillator.

The duality swaps also the celebrated dynamical symmetries of the oscillator with the Runge-Lenz vector-induced one of planetary motion. Working for simplicity in the plane using complex coordinates, $\zeta=\xi+i\eta$ for the oscillator and $z=x+iy$ for the Kepler problem, the corresponding Levi-Civita-Bohlin-Arnold map [72–75, 78, 79],

$$z=\left(\zeta+\frac{1}{\zeta}\right)^2 \tag{6.3}$$

interchanges also those two (oscillator and Kepler) dynamical symmetries, as discussed in this context [80].

The potential of the inverse-5 force (6.1) is in turn self-dual, $a = A = -4$.

However the *inverse square* potential, which is precisely what we are interested in this paper, has no Bohlin-Arnold dual: the constraint (6.2) can not be satisfied for $a = -2$. It is therefore a remarkable *tour de force* that Sundaram et al. [71] could extend the Bohlin-Arnold duality to that case.

7 Summary and discussions

In this paper we study conformally related vacuum gravitational waves and their associated symmetries by using a special Möbius conformal transformation (2.7a)–(2.7b). The vacuum condition is preserved by eliminating the additional non-vacuum oscillator term (2.4) [28, 32]. The resulting GW is in general different from the original one. The transformation (2.7a)–(2.7b) carries a global GW into an (approximate) sandwich wave, as illustrated by LPP GW and CPP GW which exemplify also the memory effect [7–15].

A vacuum GW can also be invariant under the special Möbius conformal transformation (2.7a)–(2.7b) when it has an $\mathrm{O}(2,1)$ symmetry. The remarkable efficiency of this symmetry comes from that its generators act on the radial variable only, therefore they apply equally well to anisotropic systems.

The particularly interesting example originating in molecular physics [34] but applied here in the gravitational context by using the Bargmann framework [2, 4, 6] is studied in some detail. It has the form of an anisotropic inverse-square potential [62].

For the polar-molecular application, (5.1), the familiar rotational symmetry is broken by an angle-dependent coefficient which makes it anisotropic: it alternates between repulsive and attractive at every quarter-of-a circle, see (5.8). The particle is accordingly being pushed out to infinity or attracted towards the singularity at the origin, depending on the sign of the energy of the underlying non-relativistic problem. Bounded motion arise in the attractive quadrant, with the particle oscillates along quarter-of-circle between the lines which separate the attractive and repulsive quadrants. Their behavior is reminiscent of that in the Kepler problem where the bounded (elliptical) and unbounded (hyperbolic) motions with negative or positive energy are separated by zero-energy parabolic motions.

Analytic solutions were found also for escaping or incoming radial motion along the "crests" or "valley bottoms" which corresponds to the usual inverse-square potential with repulsive or attractive sign.

The anisotropy breaks the rotational symmetry: the length of the angular momentum (5.28) oscillates, as shown in figure 16 in the periodic case.

The periodic motions in the attractive zone show remarkable historical analogies, recounted in section 6 by proceeding backwards in time.

The rôle played by the inverse-square potential in black-hole physics has been noticed before [65] for the *isotropic* Reissner-Nordström solution [35]. The anisotropic metric (5.8), which seems to have escaped attention so far, is a pp wave which resembles that near the "Dirac String" in the Lorentzian Taub-NUT metric [81–83].

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Replacing the trigonometric functions of θ in (4.8) or in (5.8) by a constant, we would recover the familiar inverse-square profile

$$ds^2 = \left(dR^2 + R^2 d\theta^2 + 2dUdV\right) - \frac{2}{R^2}dU^2, \tag{7.1}$$

which is reminiscent of Aichelburg-Sexl ultraboosts [84–86],

$$ds^2 = \left(dr^2 + r^2 d\theta^2 + 2dudv\right) - 8\,\delta(u)\log r\, du^2, \quad -\pi < \theta < \pi \tag{7.2}$$

which describes the gravitational field of a massless particle which moves with the velocity of light. It can be considered as an approximation of the gravitational field of a photon [86]. The metric (7.2) is indeed the impulsive limit of the axisymmetric Gaussian pulse

$$ds^2 = \left(dr^2 + r^2 d\theta^2 + 2dudv\right) - \frac{4a\log r}{\pi(1+a^2u^2)}du^2 \tag{7.3}$$

when $a \to \infty$.

The substantial difference between our inverse-square (7.1) and the Aichelburg-Sexl metric (7.2) is that the latter is a vacuum wave outside the origin because of $\Delta(\log r) = \delta(r)$, while (7.1) and our anisotropic generalisation (5.8) are merely pp waves.

The relation of the inverse-square metric with that of Aichelburg and Sexl can be enlightend by putting (7.1) first into a Gaussian envelope,

$$ds^2_{\text{Gauss}} = (dX^2 + dY^2 + 2dUdV) - \frac{2}{\lambda}\exp\left[-\frac{U^2}{\lambda^2}\right]\frac{\sin 2\theta}{R^2}\,dU^2\,. \tag{7.4}$$

The parameter λ rules the width of Gaussian bell. For $\lambda \to \infty$ we recover the U-independent profile (5.8), and $\lambda \to 0$ is the impulsive limit it shrinks to $\delta(U)$ with alternating sign which depends on the quadrant. The metric (7.4) is still a pp-wave however the U-dependent pre-factor breaks the $\mathrm{O}(2,1)$ symmetry.

Then taking the impulsive limit $\lambda \to 0$ yields an anisotropic analog of the Aichelburg-Sexl metric (7.2),

$$ds^2 = \left(dR^2 + R^2 d\theta^2 + 2dUdV\right) - \delta(U)\frac{2\sin 2\theta}{R^2}dU^2. \tag{7.5}$$

Another difference is that our (4.8) is $\mathrm{o}(2,1)$-symmetric, while the Aichelburg-Sexl ultraboost is not: $\log r$ is not scale-invariant which makes the discussion more elaborate.

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