

Large N analytical functional bootstrap. Part I. 1D CFTs and total positivity

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ABSTRACT: We initiate the analytical functional bootstrap study of conformal field theories with large N limits. In this first paper we particularly focus on the 1D $O(N)$ vector bootstrap. We obtain a remarkably simple bootstrap equation from the $O(N)$ vector crossing equations in the large N limit. The numerical conformal bootstrap bound is saturated by the generalized free field theories, while its extremal functional actions do not converge to any non-vanishing limit. We study the analytical extremal functionals of this crossing equation, for which the total positivity of the $SL(2, \mathbb{R})$ conformal block plays a critical role. We prove the $SL(2, \mathbb{R})$ conformal block is totally positive in the limits with large Δ or small $1 - z$ and show that the total positivity is violated below a critical value $\Delta_{\text{TP}}^* \approx 0.32315626$. The $SL(2, \mathbb{R})$ conformal block forms a surprisingly sophisticated mathematical structure, which for instance can violate total positivity at the order 10^{-5654} for a normal value $\Delta = 0.1627$! We construct a series of analytical functionals $\{\alpha_M\}$ which satisfy the bootstrap positive conditions up to a range $\Delta \leq \Lambda_M$. The functionals $\{\alpha_M\}$ have a trivial large M limit. However, due to total positivity, they can approach the large M limit in a way consistent with the bootstrap positive conditions for arbitrarily high Λ_M . Moreover, in the region $\Delta \leq \Lambda_M$, the analytical functional actions are consistent with the numerical bootstrap results, therefore it clarifies the positive structure in the crossing equation analytically. Our result provides a concrete example to illustrate how the analytical properties of the conformal block lead to nontrivial bootstrap bounds. We expect this work paves the way for large N analytical functional bootstrap in higher dimensions.

KEYWORDS: Nonperturbative Effects, Field Theories in Lower Dimensions, $1/N$ Expansion, Global Symmetries

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1 Introduction

The conformal bootstrap [1, 2] has been revived since the breakthrough work [3], which shows that strong constraints on the parameter space of general conformal field theories (CFTs) can be obtained merely from few consistency conditions. This approach has led to remarkable successes in studying the strongly coupled critical phenomena, see [4, 5] for comprehensive reviews. It is followed by an immediate question: *how can such strong results be obtained from such few inputs?* The bootstrap results, or the “numerical experiments” indicate certain mysterious mathematical structures in conformal theories which can play key roles in determining the CFT landscape. Since the ingredients in conformal bootstrap are just unitarity and the conformal blocks of the conformal group $SO(D + 1, 1)$, the unreasonable effectiveness of the bootstrap method is likely related to certain properties

of the $SO(D + 1, 1)$ conformal blocks. The goal of this work is to explore such presumed mathematical structures. We will focus on 1D large N conformal bootstrap for which the extremal bootstrap functional can be studied analytically. Moreover, we will clarify the key mathematical property which makes our construction possible.

We will start with the 1D $O(N)$ vector bootstrap in the large N limit. Under suitable conditions the $O(N)$ vector crossing equations are reduced to one of the simplest bootstrap equations

$$\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 z^{-2\Delta_{\phi}} G_{\Delta}(z) - \sum_{\mathcal{O} \in T} \lambda_{\mathcal{O}}^2 (1 - z)^{-2\Delta_{\phi}} G_{\Delta}(1 - z) = 0, \quad (1.1)$$

where $G_{\Delta}(z)$ is the 1D SL conformal block. The numerical bootstrap bound on the scaling dimension of the lowest operator in the $O(N)$ traceless symmetric sector (T) is saturated by the generalized free field theories, while its extremal functional actions do not have a stable non-vanishing limit and decrease with higher numerical precision. We will attempt to construct the analytical functionals for the above crossing equation (1.1). The extremal analytical functional is expected to produce the spectrum of the 1D generalized free field theories and also satisfy the bootstrap positive conditions. In our construction, we firstly require the analytical functionals can generate the spectrum of the generalized free field theories. Then we find such functionals can also satisfy the bootstrap positive conditions, due to the *total positivity* of the SL conformal block. This is the key mathematical property for the crossing equation (1.1) to be saturated by the generalized free field theories

Total positivity of the SL conformal block \longrightarrow Large N bootstrap bound.

We will show that the total positivity of the SL conformal block is true in the limits with large Δ or small $1 - z$, but for small Δ 's, it relates to a surprisingly delicate mathematical structure. Based on the total positivity of the SL conformal block, we show that a series of analytical functionals can satisfy the bootstrap positive conditions up to arbitrarily high scaling dimension. In particular, this family of analytical functionals can generate the spectrum of generalized free field theories but do not converge to any stable non-vanishing limit, remarkably consistent with the numerical bootstrap results.

Our interests in the large N CFTs and their bootstrap studies are motivated by several reasons.

The large N CFTs play fundamental roles in the AdS/CFT correspondence [6–8]. In the large N limit, the conformal correlation functions are dominated by the generalized free field theories, and they provide pivotal solutions to the conformal crossing equations [9–11]. Perturbative CFT data can be obtained by expanding the solutions to the crossing equations near generalized free field theories [12–14]. The role of large N CFTs in holography has been extensively studied, e.g. [15, 16]. The generalized free field theories also provide nice examples for the harmonic analysis of the Euclidean conformal group [17]. In this work, we will show that the generalized free field theories are not just pivotal solutions to the crossing equations, but can also saturate the bootstrap bounds. This indicates a special positive structure in the generalized free field theories which restricts any dynamical corrections to the bounded parameters are either vanishing or negative. Decoding this positive structure is an interesting problem for bootstrap studies.

We use $O(N)$ vector bootstrap to study the large N CFTs. The $O(N)$ vector bootstrap plays a special role in conformal bootstrap with global symmetries. Due to novel algebraic relations between crossing equations with different global symmetries [18, 19], the non- $O(N)$ vector crossing equations can be mapped to those of $O(N)$ vector's, and their bootstrap bounds are identical or weaker than the $O(N)$ vector bootstrap bounds. On the physics side, the $O(N)$ vector bootstrap bounds have close relation to several interesting theories. For instance, the 3D $O(N)$ vector bootstrap bounds have two types of kinks [20, 21]. The type I kinks with an $O(N)$ vector scalar ϕ near a free boson $\Delta_\phi = \frac{1}{2}$ are related to the critical $O(N)$ vector model [20], while the type II kinks with Δ_ϕ near free fermion bilinears also appear in general dimensions and show close relation with conformal gauge theories [18, 21].

The analytical construction of the extremal bootstrap functional [22] provides a substantial approach to uncover the positive structure in conformal bootstrap. Analytical extremal functionals have been firstly constructed in [23–25] for a 1D conformal bootstrap problem, in which the bootstrap bound is saturated by the generalized free fermion theory [26]. Analytical functionals for higher dimensional conformal bootstrap have been studied in [27–29]. In [28] a family of functional basis dual to the generalized free field spectrum has been constructed, which shows close relation to the conformal dispersion relation [30]. Nevertheless, it is quite challenging to construct extremal analytical functionals whose actions can satisfy positive conditions required by the conformal bootstrap. In this work, we aim to answer this critical question for the large N analytical functional bootstrap in a simplified laboratory, the 1D $O(N)$ vector bootstrap. In 1D CFTs, there is only one conformal invariant cross ratio (z) and the spectrum does not depend on spin. Interestingly, although the crossing equations have been simplified notably in 1D, the bootstrap bounds show similar patterns as their higher dimension analogues. Therefore we expect the 1D analytical functional bootstrap studies are instructive for similar studies in higher dimensions.

In addition, the 1D conformal bootstrap with global symmetries also corresponds to many interesting physics problems. A large set of 1D CFTs are given by the line defects of higher dimensional CFTs. Two typical examples are provided by the monodromy line defect in the 3D Ising model [26, 31] and the Wilson lines in the 4D $N = 4$ SYM [32–36]. The 1D CFTs can also be realized as boundary theories of quantum field theories in AdS_2 background [37, 38]. Recently, there are growing interests in the applications of 1D $O(N)$ symmetric CFTs in the celestial holography [39, 40]. Conformal bootstrap in 1D has been a powerful approach to extract dynamical information in above theories.

Total positivity of the 1D SL conformal block will play a key role in constructing the analytical functionals of the 1D large N bootstrap. In mathematics the total positivity has been extensively studied since the early of 20th century. It has deep connections to quantum field theories, see e.g. [41–44]. The possible role of total positivity in conformal bootstrap has been proposed in [45], in which the authors focused on the geometrical configuration supported by the SL conformal blocks. Due to total positivity, the 1D bootstrap equation (without an $O(N)$ global symmetry) admits a cyclic polytope structure which can lead to nontrivial constraints on the CFT data, see also [46, 47]. In this work, we will focus on

a new 1D bootstrap equation (1.1) with a different approach, but we will reach a similar conclusion that the total positivity of the SL conformal block can play a key role for the bootstrap constraints.

This paper is organized as follows. In section 2 we study the 1D $O(N)$ vector numerical bootstrap with large N . We discuss similarities and differences between the 1D and higher dimensional $O(N)$ vector bootstrap. We obtain a simplified crossing equation, which determines the first part of the $O(\infty)$ vector bootstrap bound and provides an ideal example for analytical functional bootstrap study. In section 3 we study total positivity of the SL conformal block, which will be important to construct the analytical functionals. In section 4 we construct the analytical functionals for the 1D $O(\infty)$ vector bootstrap which is saturated by the generalized free field theory. We firstly review the functional basis for 1D conformal block obtained from the dispersion relation. Then we explain how the total positivity of the conformal block function can play a key role to construct the analytical functionals satisfying the bootstrap positivity conditions. This work initiates a series of analytical functional bootstrap studies of the large N CFTs and their holographic duals, for which we briefly discuss in section 5.

2 Large N numerical conformal bootstrap in 1D

In this section we study 1D $O(N)$ vector numerical conformal bootstrap in the large N limit. The 1D conformal bootstrap has been studied in [26, 34, 36, 38, 48–51]. The 1D analytical functional bootstrap with global symmetries has also been studied in [50]. Our interest in the 1D $O(N)$ vector bootstrap is from the observation that the 1D bootstrap bounds share several key properties of the $O(N)$ vector bootstrap bounds in higher dimensions, thus it can provide a drastically simplified while still representative example to study the underlying mathematical structures in conformal bootstrap. The numerical bootstrap results provide insightful bases for analytical functional bootstrap study in section 4.

2.1 $O(N)$ vector crossing equations in 1D

Let us consider an operator ϕ_i which forms a vector representation of the $O(N)$ global symmetry. Its four point correlation function is given by

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \mathcal{G}_{ijkl}(z), \tag{2.1}$$

where the variables x_i are the 1D coordinates, $x_{ij} = x_i - x_j$ and the conformal invariant cross-ratio z is defined as

$$z = \frac{x_{12}x_{34}}{x_{13}x_{24}}. \tag{2.2}$$

When the external operators $\phi_i(x_i)$ are in the ordered configuration $x_1 < x_2 < x_3 < x_4$, the cross-ratio stays in the range $z \in (0, 1)$. The stripped correlation function $\mathcal{G}_{ijkl}(z)$ in (2.1) can be analytically continued in the complex plane except the branch points at $z = 0, 1, \infty$, which correspond to coincidences of two operators. The $\mathcal{G}_{ijkl}(z)$ is a holomorphic function with two branch cuts at $(-\infty, 0]$ and $[1, +\infty)$. In the s-channel (12)(34) limit with $z \rightarrow 0$,

the conformal correlation function $\mathcal{G}_{ijkl}(z)$ can be expanded in terms of the four point invariant tensors of $O(N)$ singlet (S), traceless symmetric (T) and anti-symmetric (A) representations

$$\mathcal{G}_{ijkl}(z) = \delta_{ij}\delta_{kl}\mathcal{G}^S(z) + \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}\right)\mathcal{G}^T(z) + (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\mathcal{G}^A(z), \quad (2.3)$$

in which \mathcal{G}^R denotes the series expansion

$$\mathcal{G}^R(z) = \sum_{\mathcal{O}_R} \lambda_{\mathcal{O}_R}^2 G_{\Delta}(z) \quad (2.4)$$

of the s-channel SL conformal block [52]

$$G_{\Delta}(z) = z^{\Delta} {}_2F_1(\Delta, \Delta, 2\Delta; z). \quad (2.5)$$

Alternatively, one can expand the same correlation functions (2.3) in the t-channel (23)(41) limit, which can be formally written as $\mathcal{G}_{jkli}(1-z)$. The crossing symmetry of the correlation function (2.1) identifies the s- and t-channel expansions

$$z^{-2\Delta_{\phi}}\mathcal{G}_{ijkl}(z) = (1-z)^{-2\Delta_{\phi}}\mathcal{G}_{jkli}(1-z). \quad (2.6)$$

Together with (2.3), above crossing equation leads to following independent equations

$$z^{-2\Delta_{\phi}}\left(\mathcal{G}^T(z) - \mathcal{G}^A(z)\right) = (1-z)^{-2\Delta_{\phi}}\left(\mathcal{G}^T(1-z) - \mathcal{G}^A(1-z)\right), \quad (2.7)$$

$$z^{-2\Delta_{\phi}}\left(\mathcal{G}^S(z) - \frac{2}{N}\mathcal{G}^T(z)\right) = (1-z)^{-2\Delta_{\phi}}\left(\mathcal{G}^T(1-z) + \mathcal{G}^A(1-z)\right). \quad (2.8)$$

Note to derive above crossing equations, we do not assume the statistical property of the external operator ϕ_i , so they can be applied to both fermions and bosons in 1D.

A family of unitary solutions to the $O(N)$ vector crossing equations are provided by

$$\mathcal{G}^S(z) = 1 + \frac{2}{N}\mathcal{G}^T(z), \quad (2.9)$$

$$\mathcal{G}^T(z) = \frac{1}{2}z^{2\Delta_{\phi}}\left((1-z)^{-2\Delta_{\phi}} - \lambda\right), \quad (2.10)$$

$$\mathcal{G}^A(z) = \frac{1}{2}z^{2\Delta_{\phi}}\left((1-z)^{-2\Delta_{\phi}} + \lambda\right), \quad (2.11)$$

in which $\lambda = \pm 1$ give the $O(N)$ symmetric generalized free fermion and boson theories. Above correlation functions can be decomposed into the 1D conformal blocks

$$\mathcal{G}^R(z) = \sum_{n=0}^{\infty} c_n^R G_{2\Delta_{\phi}+n}(z), \quad (2.12)$$

where

$$c_n^T = \frac{(2\Delta_{\phi})_n^2}{2n!(4\Delta_{\phi}+n-1)_n}(1 - (-1)^n\lambda), \quad (2.13)$$

$$c_n^A = \frac{(2\Delta_{\phi})_n^2}{2n!(4\Delta_{\phi}+n-1)_n}(1 + (-1)^n\lambda). \quad (2.14)$$

For different λ 's ($|\lambda| < 1$), the correlation functions $\mathcal{G}^{S/T/A}$ contain the same spectrum $\Delta_n = 2\Delta_\phi + n$, $n \in \mathbb{N}$. An interesting question in CFT studies is that given the whole spectrum of a CFT, can we determine the theory uniquely? The correlation function (2.9)–(2.11) provides a counter example for this question.

The $O(N)$ vector crossing equations (2.7), (2.8) have the same algebraic structure as those in higher dimensions [20, 53]. However, in higher dimensions, there are spin selection rules in different $O(N)$ representations due to the boson symmetry of the external scalars. The correlation function is invariant under permutation $(i, x_1) \leftrightarrow (j, x_2)$, which leads to

$$\lambda_{\phi\phi\mathcal{O}_S} = c_S(-1)^\ell \lambda_{\phi\phi\mathcal{O}_S}, \quad \lambda_{\phi\phi\mathcal{O}_T} = c_T(-1)^\ell \lambda_{\phi\phi\mathcal{O}_T}, \quad \lambda_{\phi\phi\mathcal{O}_A} = c_A(-1)^\ell \lambda_{\phi\phi\mathcal{O}_A}, \quad (2.15)$$

where $c_S = c_T = 1$, $c_A = -1$ are the signs from $O(N)$ indices when permuting two ϕ 's. Therefore only even (odd) spins can appear in the S/T (A) representations. While there is no spin in 1D, do we have similar selection rules in different $O(N)$ representations? The answer is yes and it relates to the so-called \mathcal{S} -parity symmetry [26, 31].

The action of the \mathcal{S} -parity is

$$\mathcal{S}: \quad x \rightarrow -x, \quad \mathcal{S} \mathcal{O}(x) \mathcal{S} = (-1)^{S_{\mathcal{O}}} \mathcal{O}(-x). \quad (2.16)$$

In 1D, the continuous part of the conformal symmetry preserves the cyclic order of the three point function $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle$ with $x_1 < x_2 < x_3$. However, the cyclic order can be modified by the \mathcal{S} transformation

$$\mathcal{S}: \quad \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \rightarrow (-1)^{S_{\mathcal{O}_1} + S_{\mathcal{O}_2} + S_{\mathcal{O}_3}} \langle \mathcal{O}_3(-x_3) \mathcal{O}_2(-x_2) \mathcal{O}_1(-x_1) \rangle. \quad (2.17)$$

Therefore in an \mathcal{S} -parity invariant theory, we have

$$\lambda_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} = (-1)^{S_{\mathcal{O}_1} + S_{\mathcal{O}_2} + S_{\mathcal{O}_3}} \lambda_{\mathcal{O}_2 \mathcal{O}_1 \mathcal{O}_3}. \quad (2.18)$$

In the 1D $O(N)$ vector bootstrap, if the external operators ϕ_i are scalars, the boson symmetry between the two ϕ 's requires

$$\lambda_{\phi\phi\mathcal{O}_S} = (-1)^{S_{\mathcal{O}_S}} \lambda_{\phi\phi\mathcal{O}_S}, \quad \lambda_{\phi\phi\mathcal{O}_T} = (-1)^{S_{\mathcal{O}_T}} \lambda_{\phi\phi\mathcal{O}_T}, \quad \lambda_{\phi\phi\mathcal{O}_A} = -(-1)^{S_{\mathcal{O}_A}} \lambda_{\phi\phi\mathcal{O}_A}, \quad (2.19)$$

which leads to

$$S_{\mathcal{O}_S} = 1, \quad S_{\mathcal{O}_T} = 1, \quad S_{\mathcal{O}_A} = -1. \quad (2.20)$$

While for the external fermions, the \mathcal{S} -parity charges in the $O(N)$ representations are opposite

$$S_{\mathcal{O}_S} = -1, \quad S_{\mathcal{O}_T} = -1, \quad S_{\mathcal{O}_A} = 1. \quad (2.21)$$

In the generalized free boson theory, the \mathcal{S} -parity of the double-trace operators $\mathcal{O}_n = \phi \partial^n \phi$ is $S_n = (-1)^n$. According to the \mathcal{S} -parity charges in (2.20), the generalized free boson theory has spectrum \mathcal{O}_{2n} with $\Delta = 2\Delta_\phi + 2n$ in the S/T sectors and spectrum \mathcal{O}_{2n+1} with $\Delta = 2\Delta_\phi + 2n + 1$ in the A sector. The spectra in S/T and A sectors are switched in the generalized free fermion theory due to the \mathcal{S} -parity charges (2.21).

The $O(N)$ vector bootstrap plays a special role in bootstrapping CFTs with general global symmetries. In [18, 19] it has been verified that for a large variety of symmetries \mathbf{G}

and representations \mathbf{R} , the crossing equations of the four point correlator $\langle \mathbf{R}\bar{\mathbf{R}}\mathbf{R}\bar{\mathbf{R}} \rangle$ and $\langle \mathbf{R}\mathbf{R}\bar{\mathbf{R}}\bar{\mathbf{R}} \rangle$ can be linearly mapped into the $O(N)$ symmetric form (2.7), (2.8) through a transformation $\mathcal{T}_{\mathbf{R}}$ which is consistent with positivity conditions in the bootstrap algorithm. The transformation $\mathcal{T}_{\mathbf{R}}$ is purely algebraic so can also be applied in 1D conformal bootstrap. In consequence, the bootstrap bound on the lowest \mathbf{G} singlet scalar coincides with the bound on the $O(N)$ singlet scalar, while the $O(N)$ vector bootstrap bound on the lowest T scalar can be interpreted as the bound on the lowest \mathbf{G} non-singlet scalar appearing in the bootstrap equations of \mathbf{R} .

2.2 $O(N)$ vector bootstrap bounds in the large N limit

In the large N limit, the $O(N)$ vector crossing equations (2.7), (2.8) become

$$\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ E_{\Delta}(z) \end{pmatrix} + \sum_{\mathcal{O} \in T} \lambda_{\mathcal{O}}^2 \begin{pmatrix} F_{\Delta}(z) \\ -E_{\Delta}(1-z) \end{pmatrix} + \sum_{\mathcal{O} \in A} \lambda_{\mathcal{O}}^2 \begin{pmatrix} -F_{\Delta}(z) \\ -E_{\Delta}(1-z) \end{pmatrix} = 0, \quad (2.22)$$

where

$$E_{\Delta}(z) = z^{-2\Delta_{\phi}} G_{\Delta}(z), \quad (2.23)$$

$$F_{\Delta}(z) = E_{\Delta}(z) - E_{\Delta}(1-z). \quad (2.24)$$

Their bootstrap bound on the lowest non-unit $O(N)$ singlet operator goes to infinity. A solution to such bound is given by the correlation function (2.9)–(2.11) with $N = \infty$, in which the only $O(N)$ singlet operator is the unit operator, while all the double-trace singlets have vanishing OPE coefficients and are decoupled in the crossing equation. Moreover, without extra assumptions on the spectrum, there is no upper bound on the lowest operator in the T or A representation. To show this, let us consider the bootstrap bound on the scaling dimension of the lowest operator in the T sector, denoted Δ_T^* . A solution to the crossing equation (2.22) can be constructed as follows. Given a four point correlator $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \sim \mathcal{G}^*$ which satisfies:

$$z^{-2\Delta_{\phi}} \mathcal{G}^*(z) - (1-z)^{-2\Delta_{\phi}} \mathcal{G}^*(1-z) = 0, \quad (2.25)$$

the $O(N)$ vector correlation functions

$$\mathcal{G}^A = \mathcal{G}^S = \mathcal{G}^*, \quad \mathcal{G}^T = 0 \quad (2.26)$$

satisfy the crossing equation (2.22). In this solution the T sector is empty therefore corresponding to an infinity high upper bound $\Delta_T^* = \infty$. Due to the same logic there is no upper bound on Δ_A^* either. Note the unit operator is an indispensable ingredient in the OPEs of the correlation functions \mathcal{G}^* and $\mathcal{G}^{S/A}$. It seems the 1D large N bootstrap is too simplified to capture nontrivial dynamics and is not insightful for higher dimensional bootstrap. However, this is not the case.

As discussed before, different $O(N)$ representations carry different \mathcal{S} -parity charges, similar to the spin selection rules in higher dimensional bootstrap. With different \mathcal{S} -parity charges it is expected that the spectra in different sectors are notably different. Specifically,

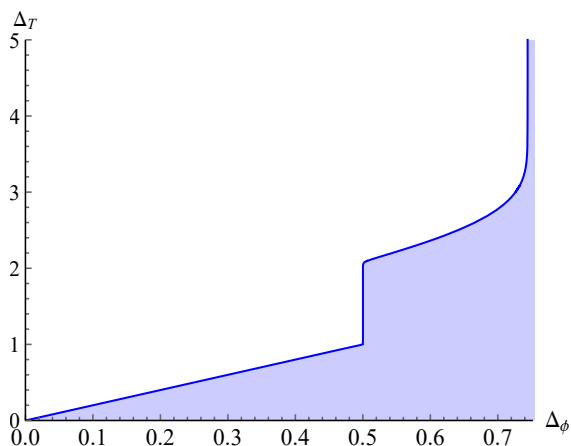


Figure 1. 1D $O(\infty)$ vector bootstrap bound on the scaling dimension of the lowest operator in the T sector. Gap assumption $\Delta_A > 1.0$ in the A sector. $\Lambda = 20$.

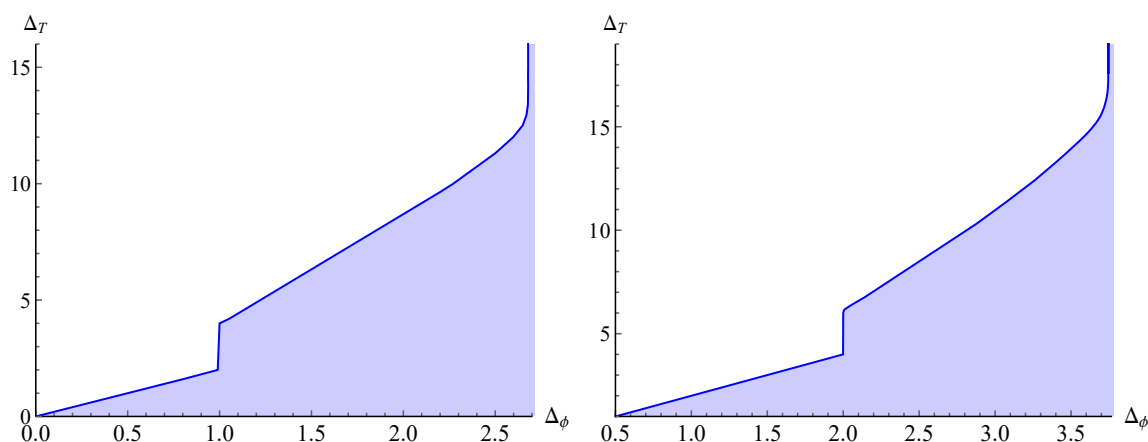


Figure 2. 2D (left) and 3D (right) $O(\infty)$ vector bootstrap bounds on the scaling dimensions of the lowest scalars in the T sector. No gaps. $\Lambda = 31$.

in the $O(N)$ vector bootstrap, to bound Δ_T^* , we may expect a non-trivial gap for the lowest operator in the A sector which has opposite \mathcal{S} -parity. In the bootstrap studies of defect CFTs, such gaps can be justified by the physical spectrum [26, 51].

With a gap assumption on the A sector spectrum, the bootstrap bound on Δ_T^* can be modified drastically. The result is shown in figure 1, in which we have introduced an assumption that the lowest operator in the A sector satisfies $\Delta_A \geq \Delta_c = 1$.¹ In the range $\Delta_\phi \in (0, \Delta_c/2)$ the bootstrap bound on Δ_T^* is given by $\Delta_T^* = 2\Delta_\phi$. It is followed by a sharp kink at $\Delta_\phi = \Delta_c/2$, where the bound on Δ_T^* jumps to $\Delta_T^* = 2\Delta_\phi + 1$. The solution (2.26) requires a unit operator in the A sector, therefore is excluded by the gap assumption. The bootstrap bound on Δ_T^* disappears near $\Delta_\phi = 0.744$, which suggest an *end of the scalar bootstrap constraints*.²

¹Bootstrap bounds with different gaps Δ_c are qualitatively similar to figure 1.

²Note the bound on Δ_T^* provides the strongest constraint among the scalar bootstrap with global symmetries.

The 1D $O(\infty)$ vector bootstrap bound is remarkably similar to the higher dimensional $O(\infty)$ vector bootstrap bounds shown in figure 2. It has been known since [20] that in 3D, the $O(N)$ vector bootstrap bounds show sharp kinks (type I) which are saturated by the 3D critical $O(N)$ vector models. Moreover, in [21] the author observed that besides the type I kinks, the 3D $O(N)$ vector bootstrap bounds also show another family of kinks (type II) which approach the free fermion bilinear in the large N limit. The type II kinks appear in general dimensions [18], and the kink in figure 1 at $\Delta_\phi = \Delta_c/2$ could be considered as their dimensional continuation in 1D. In higher dimensions the type II kinks at finite N are conjectured to be related to the conformal gauge theories, while mixed with the bootstrap bound coincidences due to a positive algebraic structure in the four point crossing equations [19]. The numerical bootstrap results of the type II kinks are affected by the numerical convergence issue and it is hard to evaluate the CFT data numerically.

One of the motivations of this work is to develop an analytical functional bootstrap method to study the kinks in the $O(N)$ vector bootstrap bounds and clarify their putative connections to the conformal gauge theories. The higher dimensional bootstrap equations relate to conformal blocks with two cross ratios z, \bar{z} and spins, which make the analytical functional bootstrap more intricate. Here our results suggest that similar bootstrap bounds can also be realized in 1D conformal bootstrap, with a drastically simplified bootstrap setup. Therefore the 1D large N bootstrap can provide a key to unlock the large N analytical functional bootstrap in higher dimensions.

2.3 Extremal solutions and the simplified bootstrap equation

We focus on the 1D large N bootstrap bound in the range $\Delta_\phi \in (0, \Delta_c/2)$. Spectrum of the theory saturating the bootstrap bound can be obtained from the extremal functionals [22], which are shown in figure 3 for $\Delta_\phi = 0.1, 0.3$. In the S sector the spectrum is trivial with only one first order zero at $\Delta = 0$, corresponding to the unit operator. Surprisingly, the extremal functional in the T sector shows a first order zero at $\Delta = 2\Delta_\phi$, and double zeros at $\Delta = 2\Delta_\phi + n$, $n \in \mathbb{N}^+$. Therefore the extremal spectrum is not from generalized free boson or fermion alone, but is given by the correlation functions (2.9)–(2.11) with $|\lambda| < 1$. Furthermore, action of the extremal functional in the A sector is the same as that of T sector up to numerical errors! In the A sector, we only introduced the positivity constraint above the gap $\Delta_A \geq \Delta_c$, while the extremal solution automatically satisfies the positivity condition down to $\Delta > 2\Delta_\phi$!

Let us go back to the $O(\infty)$ vector crossing equation (2.22) and check what does it mean by two “almost” identical actions in T and A sectors. Consider a linear functional $\vec{\alpha}$ for the $O(\infty)$ vector crossing equation (2.22)

$$\vec{\alpha} \cdot \vec{V}_T \equiv \alpha_1 \cdot F_\Delta(z) - \alpha_2 \cdot E_\Delta(1-z), \tag{2.27}$$

$$\vec{\alpha} \cdot \vec{V}_A \equiv -\alpha_1 \cdot F_\Delta(z) - \alpha_2 \cdot E_\Delta(1-z). \tag{2.28}$$

The observation $\vec{\alpha} \cdot \vec{V}_T = \vec{\alpha} \cdot \vec{V}_A$ suggests $\alpha_1 \rightarrow 0$! That is to say, to get the upper bound in figure 1 for $\Delta_\phi < \Delta_c/2$, the first row in the crossing equation (2.22) is not necessary! The extremely small α_1 has been verified in our numerical bootstrap results.

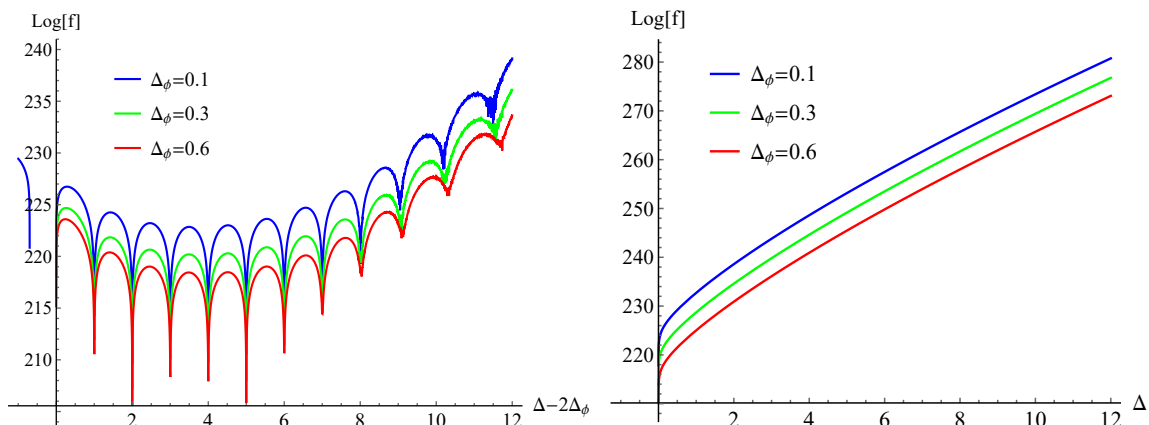


Figure 3. Extremal functional spectra in the $O(N)$ T sector (left) and S sector (right) at $\Lambda = 48$.

Without the first row of (2.22), the conformal blocks in the T and A sectors are the same and the positivity constraint in the A sector

$$\vec{\alpha} \cdot \vec{V}_A \geq 0, \quad \forall \Delta > \Delta_c, \tag{2.29}$$

is substituted by the positivity constraint in the T sector

$$\vec{\alpha} \cdot \vec{V}_T \geq 0, \quad \forall \Delta > \Delta_T^*, \tag{2.30}$$

given $\Delta_T^* < \Delta_c$. While for $\Delta_T^* \geq \Delta_c$, the positivity constraints between the two sectors are switched and the bootstrap bound in figure 1 suggests the first row of (2.22) becomes important. The bootstrap constraints have a transition at $\Delta_T^* = \Delta_c$, corresponding to the jump of the bootstrap bound at $\Delta_\phi = \Delta_c/2$ in figure 1. We leave a detailed study of the bootstrap bound with $\Delta_\phi \geq \Delta_c/2$ for future work.

The correlation functions (2.9)–(2.11) with different $|\lambda| < 1$ have the same spectrum while different OPE coefficients (2.13), (2.14). The extremal OPE coefficients c_n^* are given by the generalized free boson or fermion theories when $|\lambda| \rightarrow 1$

$$c_n^* = \frac{(2\Delta_\phi)_n^2}{n!(4\Delta_\phi + n - 1)_n}. \tag{2.31}$$

We have checked that our bootstrap bounds on the OPE coefficients of low-lying spectrum $\Delta = 2\Delta_\phi + n$ are well consistent with (2.31) up to $n = 9$.

Remarkably, according to our numerical bootstrap results, the 1D large N extremal functional actions $f \equiv \vec{\alpha} \cdot \vec{V}_T$ do not converge to any stable limit to the best of our current numerical precision. Instead, the magnitudes of the extremal functional actions keep on decreasing with increasing numerical precision, as shown in figure 4. This suggests a critical property of the extremal functionals that it can generate the higher spectrum of the generalized free field theories, but the functional action does not admit a stable non-vanishing limit. In section 4 we will study the large N crossing equation (2.22) analytically and show that the numerical results in figure 4 are nicely consistent with a positive structure of the $SL(2, \mathbb{R})$ conformal block.

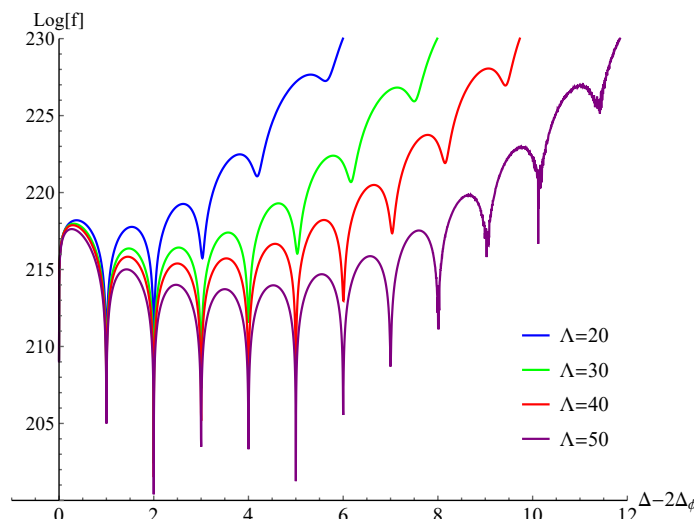


Figure 4. Actions $f \equiv \vec{\alpha} \cdot \vec{V}_T$ of the extremal functional on the $O(N)$ T sector conformal block at $\Delta_\phi = 0.3$ with different Λ s. For comparison we have vertically shifted the numerical functional actions. The functional actions with different Δ_ϕ 's are qualitatively similar.

To summarize, the $O(\infty)$ vector bootstrap leads to a rather simple crossing equation

$$\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 z^{-2\Delta_\phi} G_{\Delta}(z) - \sum_{\mathcal{O} \in T} \lambda_{\mathcal{O}}^2 (1-z)^{-2\Delta_\phi} G_{\Delta}(1-z) = 0. \quad (2.32)$$

Bootstrap bound on Δ_T^* from above crossing equation is given by $\Delta_T^* = 2\Delta_\phi$ for general Δ_ϕ , and its extremal spectrum is the same as those in figure 3. The $O(\infty)$ vector bootstrap equation (2.22) is reduced to (2.32) in the range $\Delta_\phi < \Delta_c/2$. For $\Delta_\phi \geq \Delta_c/2$, the bootstrap bound from (2.32) stays in the line $\Delta_T^* = 2\Delta_\phi$, see e.g. the extremal spectrum at $\Delta_\phi = 0.6$ in figure 3, while the bound from (2.22) goes differently as the first row in (2.22) starts to play a role. The rest part of this work aims to construct analytical functionals for the crossing equation (2.32).

We would like to add comments on $O(\infty)$ vector bootstrap in higher dimensions [54]. In the range between free boson and free fermion bilinear: $\Delta_\phi \in \left(\frac{D-2}{2}, D-1\right)$, the bootstrap bound on Δ_T^* is also saturated by the generalized free theory and the $O(\infty)$ vector bootstrap equations are reduced to the higher dimensional form of (2.32)

$$\begin{aligned} \sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 (z\bar{z})^{-\Delta_\phi} G_{\Delta,\ell}(z, \bar{z}) - \sum_{\mathcal{O} \in T} \lambda_{\mathcal{O}}^2 ((1-z)(1-\bar{z}))^{-\Delta_\phi} G_{\Delta,\ell}(1-z, 1-\bar{z}) \\ - \sum_{\mathcal{O} \in A} \lambda_{\mathcal{O}}^2 ((1-z)(1-\bar{z}))^{-\Delta_\phi} G_{\Delta,\ell}(1-z, 1-\bar{z}) = 0, \end{aligned} \quad (2.33)$$

where $G_{\Delta,\ell}(z, \bar{z})$ are the $SO(D+1,1)$ conformal blocks [52, 55]. Considering the close relation between the $O(\infty)$ vector bootstrap in 1D and higher dimensions, the analytical functional for 1D $O(\infty)$ vector bootstrap constructed in this work will be instructive to construct analytical functionals in higher dimensions [54].

3 SL conformal block and total positivity

The 1D large N bootstrap provides an ideal example to decode the underlying mathematical structures of conformal bootstrap. Considering there are only few ingredients in the bootstrap crossing equation (2.32), it is expected that the presumed mathematical structures should be certain properties of the SL conformal block. In section 4 we will construct the analytical functionals for the crossing equation (2.32) and show that the answer to this riddle is *total positivity*. In this section we provide a brief explanation of *total positivity* and study its relation to the conformal block $G_{\Delta}(z)$.

3.1 Total positivity: definition and theorems

Definition. A two-variable function $K(x, y)$ defined on $I \times J$ with $I, J \subset \mathbb{R}$ is totally positive of the order k , if for all $1 \leq m \leq k$, and arbitrary ordered variables $x_1 < \dots < x_m, y_1 < \dots < y_m, x_i \in I, y_j \in J$, the following determinants are positive

$$\|K(x, y)\|_m \equiv K \begin{pmatrix} x_1, \dots, x_m \\ y_1, \dots, y_m \end{pmatrix} = \det \begin{bmatrix} K(x_1, y_1) & \dots & K(x_1, y_m) \\ \vdots & & \vdots \\ K(x_m, y_1) & \dots & K(x_m, y_m) \end{bmatrix} > 0. \quad (3.1)$$

We are interested in the totally positive functions of the order infinity, which will be assumed implicitly in the following part. For the finite sets I, J , the two-variable functions $K(x, y)$ are reduced to the matrices $K(x, y) \rightarrow K_{i,j}$. In this case, the definition (3.1) for totally positive matrices becomes that all the minors of the matrix K are positive.

From the definition (3.1), it is straightforward to show following rules for totally positive functions:

- If $g(x)$ and $h(y)$ are positive functions defined on I and J , respectively, and $K(x, y)$ is totally positive, then so is the function $g(x)K(x, y)h(y)$.
- If $g(x) \in I$ and $h(y) \in J$ are defined on $x \in U$ and $y \in V$, and monotone in the same direction, and if $K(x, y)$ is totally positive on $I \times J$, then the function $K(g(x), h(y))$ is totally positive on $U \times V$.

An important tool to study total positivity is the so-called “basic composition formula”. It shows how to construct a new totally positive function from two such functions and provides a powerful method to prove total positivity of certain functions.

Basic composition formula. Let K, L, M be two-variable functions which satisfy

$$M(x, y) = \int K(x, z)L(z, y)d\sigma(z), \quad (3.2)$$

where $\sigma(z)$ is a σ -finite measure and the integral converges absolutely, then the basic composition formula suggests

$$M \begin{pmatrix} x_1, \dots, x_m \\ y_1, \dots, y_m \end{pmatrix} = \int \dots \int_{z_1 < \dots < z_m} K \begin{pmatrix} x_1, \dots, x_m \\ z_1, \dots, z_m \end{pmatrix} L \begin{pmatrix} z_1, \dots, z_m \\ y_1, \dots, y_m \end{pmatrix} d\sigma(z_1) \dots d\sigma(z_m). \quad (3.3)$$

A proof of this formula is sketched in [56].

The convolution (3.2) of two kernels $K(x, z)$ and $L(z, y)$ can be considered as a continuous version of the standard matrix product, then above basic composition formula (3.3) is an extension of the Cauchy-Binet formula in matrix multiplication which expands subdeterminants of M_{ij} in terms of those of K_{im} and L_{mj} . The basic composition formula (3.3) directly leads to following theorem.

Theorem. The convolution (3.2) of two totally positive kernels is also totally positive.

Variation Diminishing Property. Consider a function $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$. The number of sign changes of f on I , denoted $S_I^+(f)$, is defined as the maximum number of sign changes in a finite sequence $\{f(x_1), f(x_2), \dots, f(x_m)\}$, $x_i \in I$, $x_1 < \dots < x_m$.³ Assume $Z_I(g)$ gives the number of zeros, counting multiplicity, of f in I . An important property of the totally positive functions is given by [56]:

Theorem. For $I, J \subset \mathbb{R}$, consider a totally positive kernel $K : I \times J \rightarrow \mathbb{R}$ which is Borel-measurable. Let $\sigma(y)$ be a regular σ -finite measure on J and $f : J \rightarrow \mathbb{R}$ be a bounded and Borel-measurable function on J , so that the convolution of $f(y)$ converges absolutely

$$g(x) = \int_J K(x, y)f(y)d\sigma(y). \tag{3.4}$$

Then the number of sign changes of $g(x)$ on I is not larger than that of $f(y)$ on J :

$$S_I^+(g) \leq Z_I(g) \leq S_J^+(f). \tag{3.5}$$

Moreover, if $S_I^+(g) = S_J^+(f)$, then the two functions $f(y)$ and $g(x)$ should have the same arrangement of signs.

The *variation diminishing property* of totally positive functions will play a critical role to construct analytical functionals of 1D large N bootstrap.

3.2 Total positivity of the Gauss hypergeometric function

The Gauss hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ appears in the SL conformal block $G_\Delta(z)$. There is numerical evidence indicating that the function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ is indeed totally positive in the region $z \in (0, 1)$, $\Delta > 0$ [45]. We have also numerically verified the total positivity of this function using a large set of data. While it is hard to obtain a complete proof for the total positivity of ${}_2F_1(\Delta, \Delta, 2\Delta, z)$, we can get promising evidence for this observation beyond the numerical checks.

Total positivity of ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ in the large Δ limit. In the large Δ limit, the hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ has a much simpler asymptotic form, for which the total positivity can be proved easily. Let us consider the integral formula of the hypergeometric function

$${}_2F_1(\Delta, \Delta, 2\Delta, z) = \frac{1}{B(\Delta, \Delta)} \int_0^1 x^{\Delta-1}(1-x)^{\Delta-1}(1-zx)^{-\Delta} dx, \tag{3.6}$$

³The zeros in the sequence are discarded when counting the number of sign changes.

where $B(\Delta, \Delta) = \frac{\Gamma(\Delta)^2}{\Gamma(2\Delta)}$ is the Euler Beta function. In the large Δ limit above integration can be solved using the method of steepest descent:

$$\int_0^1 x^{\Delta-1}(1-x)^{\Delta-1}(1-zx)^{-\Delta} dx = \int_0^1 \frac{1}{x(1-x)} e^{-\Delta \log\left[\frac{1-xz}{x(1-x)}\right]} dx, \quad (3.7)$$

which has a single stationary point $x = \frac{1-\sqrt{1-z}}{z}$ in the region $x \in (0, 1)$. Then the integration (3.6) is approximately given by

$${}_2F_1(\Delta, \Delta, 2\Delta, z)|_{\Delta \rightarrow \infty} \approx \frac{1}{B(\Delta, \Delta)} \sqrt{\frac{\pi}{\Delta}} (1-z)^{-\frac{1}{4}} \left(1 + \sqrt{1-z}\right)^{1-2\Delta}. \quad (3.8)$$

We find above approximation is reasonably good even for $\Delta = 5$.

It is straightforward to prove the total positivity of the right hand side of (3.8). Since the positive factors depending solely on z or Δ have no effect on the total positivity, the only relevant factor in the approximated formula is

$$\left(1 + \sqrt{1-z}\right)^{-2\Delta} = \rho(z)^{2\Delta}, \quad (3.9)$$

where $\rho(z) = (1 + \sqrt{1-z})^{-1}$ is a monotone increasing function in $z \in (0, 1)$. Therefore the asymptotic formula (3.8) has the same total positivity as the function z^Δ , which has been proved in appendix A.1.

A sufficient condition for the total positivity of ${}_2F_1(\Delta, \Delta, 2\Delta, z)$. Both the large Δ approximation and numerical tests with small Δ 's suggest the hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ is totally positive. Here we discuss a sufficient condition which, if true, can prove the total positivity of ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ for general $\Delta > 0$.

The hypergeometric function has a series expansion

$${}_2F_1(\Delta, \Delta, 2\Delta, z) = \sum_{i=0}^{\infty} \frac{(\Delta)_i^2}{(2\Delta)_i} \frac{z^i}{i!}, \quad (3.10)$$

where $(a)_i$ is the Pochhammer symbol. Above expansion can be considered as a convolution of $K(\Delta, i) \equiv (\Delta)_i^2/(2\Delta)_i$ and $f(i, z) \equiv z^i/i!$ with a discrete σ -measure in (3.2). Therefore according to the basic composition formula (3.3), the hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ is totally positive if both of the two functions $K(\Delta, i)$ and $f(i, z)$ are totally positive. The function z^i has been shown to be totally positive, see the appendix A.1.

The total positivity of the function $K(\Delta, i)$ requires

$$\|K(\Delta, i)\|_m = K \begin{pmatrix} \Delta_1, \dots, \Delta_m \\ i_1, \dots, i_m \end{pmatrix} = \det \begin{bmatrix} \frac{(\Delta_1)_{i_1}^2}{(2\Delta_1)_{i_1}} & \dots & \frac{(\Delta_m)_{i_1}^2}{(2\Delta_m)_{i_1}} \\ \vdots & & \vdots \\ \frac{(\Delta_1)_{i_m}^2}{(2\Delta_1)_{i_m}} & \dots & \frac{(\Delta_m)_{i_m}^2}{(2\Delta_m)_{i_m}} \end{bmatrix} > 0, \quad (3.11)$$

with $0 \leq \Delta_1 < \dots < \Delta_m$, $0 \leq i_1 < \dots < i_m$, $\Delta_k \in \mathbb{R}$, $i_k \in \mathbb{N}$ for any integer m . A compact formula for above determinants with general m is not known. Here we show for small m , above determinants are indeed positive.

Consider the determinant $\|K(\Delta, i)\|_{m=2}$ for general Δ_k and i_k in the domain of definition

$$\begin{aligned} \|K(\Delta, i)\|_2 &= \det \begin{bmatrix} \frac{(\Delta_1)_{i_1}^2}{(2\Delta_1)_{i_1}} & \frac{(\Delta_2)_{i_1}^2}{(2\Delta_2)_{i_1}} \\ \frac{(\Delta_1)_{i_2}^2}{(2\Delta_1)_{i_2}} & \frac{(\Delta_2)_{i_2}^2}{(2\Delta_2)_{i_2}} \end{bmatrix} \\ &= \frac{(\Delta_2)_{i_1}^2 (\Delta_1)_{i_2}^2}{(2\Delta_2)_{i_1} (2\Delta_1)_{i_2}} \left(\prod_{k=0}^{i_2-i_1-1} \frac{(\Delta_2 + i_1 + k)^2 (2\Delta_1 + i_1 + k)}{(\Delta_1 + i_1 + k)^2 (2\Delta_2 + i_1 + k)} - 1 \right). \end{aligned} \quad (3.12)$$

For each term in the product with $k \geq 0$, we have

$$\begin{aligned} (\Delta_2 + i_1 + k)^2 (2\Delta_1 + i_1 + k) - (\Delta_1 + i_1 + k)^2 (2\Delta_2 + i_1 + k) = \\ (\Delta_2 - \Delta_1) (2\Delta_2 \Delta_1 + \Delta_1 i_1 + \Delta_2 i_1 + \Delta_1 k + \Delta_2 k) > 0, \end{aligned} \quad (3.13)$$

and consequently

$$\frac{(\Delta_2 + i_1 + k)^2 (2\Delta_1 + i_1 + k)}{(\Delta_1 + i_1 + k)^2 (2\Delta_2 + i_1 + k)} > 1. \quad (3.14)$$

Therefore the right hand side of (3.12) is positive.

With higher m 's the determinant formula $\|K(\Delta, i)\|_m$ is too complicated for a general study. By choosing a specific set of i_k 's one can evaluate the determinants explicitly. For instances, taking $i_k = k$, the determinants $\|K(\Delta, k)\|_m$ are given by

$$\begin{aligned} \|K(\Delta, k)\|_{m=3} = \\ (\Delta_2 - \Delta_1) (\Delta_3 - \Delta_1) (\Delta_3 - \Delta_2) \Delta_1 \Delta_2 \Delta_3 \frac{(\Delta_1 \Delta_2 + \Delta_3 \Delta_2 + \Delta_1 \Delta_3 + 2\Delta_1 \Delta_2 \Delta_3)}{16 (2\Delta_1 + 1) (2\Delta_2 + 1) (2\Delta_3 + 1)} \end{aligned} \quad (3.15)$$

for $m = 3$ and

$$\begin{aligned} \|K(\Delta, k)\|_{m=4} = \\ (\Delta_2 - \Delta_1) (\Delta_3 - \Delta_1) (\Delta_3 - \Delta_2) (\Delta_4 - \Delta_1) (\Delta_4 - \Delta_2) (\Delta_4 - \Delta_3) \Delta_1 \Delta_2 \Delta_3 \Delta_4 \\ \times (3\Delta_1 \Delta_2 \Delta_3 (9\Delta_3 + \Delta_1 (2\Delta_2 + 3)) (2\Delta_3 + 3) + \Delta_2 (6\Delta_3 + 9) + 13) + \\ (9\Delta_3 + \Delta_1 (2\Delta_2 + 3)) (2\Delta_3 + 3) + \Delta_2 (6\Delta_3 + 9) + 13 (3\Delta_2 \Delta_3 + \Delta_1 (3\Delta_3 + \Delta_2 (8\Delta_3 + 3))) \Delta_4 \\ + (2\Delta_1 + 3) (2\Delta_2 + 3) (2\Delta_3 + 3) (\Delta_2 \Delta_3 + \Delta_1 (\Delta_3 + \Delta_2 (2\Delta_3 + 1))) \Delta_4^2 \\ / 64 (2\Delta_1 + 1) (2\Delta_1 + 3) (2\Delta_2 + 1) (2\Delta_2 + 3) (2\Delta_3 + 1) (2\Delta_3 + 3) (2\Delta_4 + 1) (2\Delta_4 + 3) \end{aligned} \quad (3.16)$$

for $m = 4$, both of which are obviously positive for ordered Δ_i 's. In all similar checks we find the results are well consistent with the total positivity. We conjecture this function is totally positive at infinity order for general $\Delta \geq 0$.

3.3 Total positivity of the 1D SL conformal block

Now we study the total positivity of the fundamental ingredient in 1D conformal bootstrap, the SL conformal block $G_\Delta(z) = z^\Delta {}_2F_1(\Delta, \Delta, 2\Delta, z)$, which is a product (but not convolution) of two totally positive factors. However, it is *not* guaranteed that the product of two totally positive functions is also totally positive, and indeed, the function $G_\Delta(z)$ loses

its total positivity in the region with small Δ_i . This surprising fact was firstly observed in [45],⁴ which found that the total positivity of $G_\Delta(z)$ can be violated with small Δ 's. Nevertheless, the total positivity of $G_\Delta(z)$ for general Δ and z is still not studied yet, and the possible critical value of Δ below which the total positivity is violated remains unclear. In this section, we study the total positivity of $G_\Delta(z)$ from several aspects. Specifically we will show that in the large Δ or small $1 - z \rightarrow 0^+$ limit, the asymptotic behavior of the conformal block $G_\Delta(z)$ is indeed totally positive. We will provide a careful study of the total positivity of the function $G_\Delta(z)$ with small Δ at the order 3, while a comprehensive study for the total positivity of $G_\Delta(z)$ at higher orders is left for future work.

Total positivity of $G_\Delta(z)$ in the large Δ limit. Using the asymptotic formula (3.8) of the Gauss hypergeometric function, the large Δ limit of the 1D conformal block is given by

$$G_\Delta(z)|_{\Delta \rightarrow \infty} \approx \frac{1}{B(\Delta, \Delta)} \sqrt{\frac{\pi}{\Delta}} (1-z)^{-\frac{1}{4}} z^\Delta (1 + \sqrt{1-z})^{1-2\Delta}. \quad (3.17)$$

The total positivity of above formula is determined by the factors depending on both z and Δ :

$$z^\Delta (1 + \sqrt{1-z})^{-2\Delta} = \tilde{\rho}(z)^\Delta, \quad (3.18)$$

where $\tilde{\rho}(z) = z(1 + \sqrt{1-z})^{-2}$,⁵ like $\rho(z)$ in (3.9), is a monotone increasing function in $z \in (0, 1)$. Thus the 1D conformal block function is totally positive for sufficiently large Δ . However, for small Δ , the large Δ approximation (3.17) fails and it cannot say anything about the total positivity of $G_\Delta(z)$ with small Δ .

Total positivity of $G_\Delta(z)$ in the limit $z \rightarrow 1^-$. In the limit $z \rightarrow 1^-$, the hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ approaches to its branch cut at $z \in [1, +\infty)$ with an asymptotic formula

$$\lim_{z \rightarrow 1^-} {}_2F_1(\Delta, \Delta, 2\Delta, z) = -\frac{1}{B(\Delta, \Delta)} \ln(1-z). \quad (3.19)$$

In this limit, the $SL(2, \mathbb{R})$ conformal block $G_\Delta(z)$ becomes

$$\lim_{z \rightarrow 1^-} G_\Delta(z) = -\frac{1}{B(\Delta, \Delta)} \ln(1-z) z^\Delta. \quad (3.20)$$

Both $\frac{1}{B(\Delta, \Delta)}$ and $-\ln(1-z)$ are positive factors of Δ and z , therefore the total positivity of the above asymptotic formula of $G_\Delta(z)$ follows the total positivity of the function z^Δ , which has been proved in the appendix A.1.

A “fixed point” of the 1D conformal block $G_\Delta(z)$. We show an interesting property of the conformal block $G_\Delta(z)$, though its physical correspondence is not clear yet.

The conformal blocks $G_\Delta(z)$ with different Δ 's are plotted in figure 5. A surprising fact is that all these functions intersect near $\Delta \in (0.62, 0.64)$ with $G_\Delta(z) \simeq 1$. This tiny

⁴The author would like to thank Nima Arkani-Hamed for the inspiring discussion on this problem.

⁵Interestingly, the variable $\tilde{\rho}$ in (3.18) is just the variable $\rho(z)$ in [57] motivated by different reasons.

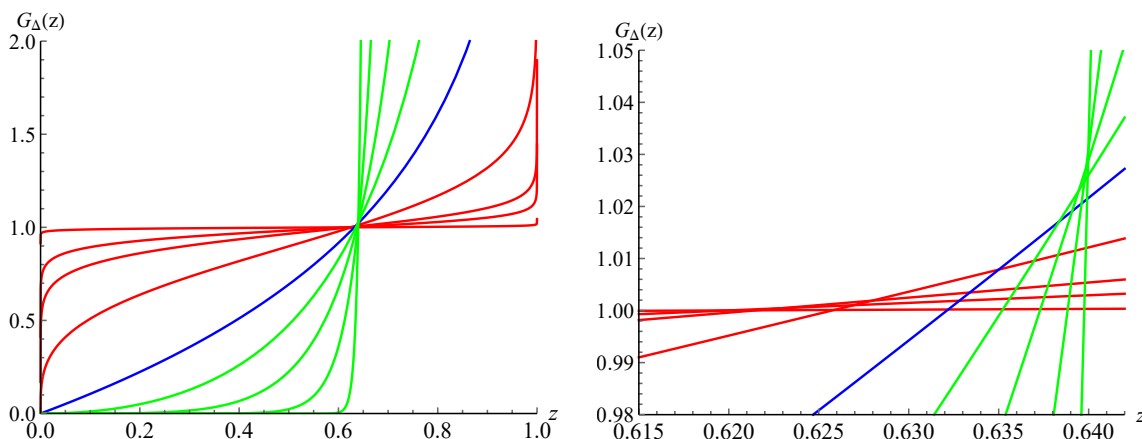


Figure 5. Plots for the conformal block functions $G_\Delta(z)$ with $\Delta = 0.005, 0.05, 0.1, 0.3$ (red curves), $\Delta = 1$ (blue curve) and $\Delta = 2, 4, 10, 50$ (Green curves).

intersection region looks like a “fixed point” of the conformal block $G_\Delta(z)$, besides another trivial “fixed point” at $z = 0$. Why?

Let us first consider the large Δ approximation of $G_\Delta(z)$ (3.17). The dominating part of $G_\Delta(z)$ in this limit is

$$G_\Delta(z)|_{\Delta \rightarrow \infty} \sim \frac{1}{B(\Delta, \Delta)} \left(\frac{z}{(1 + \sqrt{1-z})^2} \right)^\Delta \approx 4^\Delta \left(\frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}} \right)^\Delta. \quad (3.21)$$

Here we have used the Stirling’s formula for the Gamma function which gives $B(\Delta, \Delta) \sim 4^{-\Delta}$. From (3.21) it is clear that in the large Δ limit, the equation $G_\Delta(z) = 1$, or $\log(G_\Delta(z)) = 0$ has a Δ -independent solution at $z = 0.64$. Contributions from extra factors are exponentially suppressed.

Then let us go to the small Δ limit. With a small Δ the Gauss hypergeometric function is simplified to

$${}_2F_1(\Delta, \Delta, 2\Delta, z)|_{\Delta \ll 1} \approx 1 - \frac{\Delta}{2} \log(1-z) + O(\Delta^3) \quad (3.22)$$

and the conformal block function $G_\Delta(z)$ becomes

$$G_\Delta(z)|_{\Delta \ll 1} \approx 1 + \Delta \log \left(\frac{z}{\sqrt{1-z}} \right) + O(\Delta^2), \quad (3.23)$$

in which the equation $G_\Delta(z) = 1$ is solved by $z = (\sqrt{5} - 1)/2 \approx 0.618$.

So both in the large and small Δ limits, the equation $G_\Delta(z) = 1$ has a solution independent of Δ . The solution *walks* slowly from $z \approx 0.618$ near $\Delta = 0$ to $z = 0.64$ near $\Delta = \infty$. Such a “fixed point” shows an interesting interplay between the factors z^Δ and the hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ in $G_\Delta(z)$. As will be shown below, the factor z^Δ also changes the total positivity of $G_\Delta(z)$ with small Δ_i .

Loss of total positivity of $G_\Delta(z)$ with small Δ s. We show the total positivity is violated by the 1D conformal block $G_\Delta(z)$ with small $\Delta_i \ll 1$ at the order 3. For sufficiently

small Δ it is convenient to take the lower order expansion (3.23) of $G_\Delta(z)$. Up to the order Δ^2 , it is given by

$$G_\Delta(z)|_{\Delta \ll 1} \approx 1 + \Delta \log\left(\frac{z}{\sqrt{1-z}}\right) - \Delta^2 \log(z) \tanh^{-1}(1-2z) + O(\Delta^3), \quad (3.24)$$

Let us consider the determinant of $G_\Delta(z)$ at the third order

$$\begin{aligned} \|G_\Delta(z)\|_3 &= \det \begin{bmatrix} G_{\Delta_1}(z_1) & G_{\Delta_1}(z_2) & G_{\Delta_1}(z_3) \\ G_{\Delta_2}(z_1) & G_{\Delta_2}(z_2) & G_{\Delta_2}(z_3) \\ G_{\Delta_3}(z_1) & G_{\Delta_3}(z_2) & G_{\Delta_3}(z_3) \end{bmatrix} \\ &= \frac{(\Delta_2 - \Delta_1)(\Delta_3 - \Delta_1)(\Delta_3 - \Delta_2)}{4} \left(\log(z_1) \log\left(\frac{z_1}{1-z_1}\right) \log\left(\frac{z_3^2(1-z_2)}{z_2^2(1-z_3)}\right) \right. \\ &\quad \left. + \log(z_3) \log\left(\frac{z_3}{1-z_3}\right) \log\left(\frac{z_2^2(1-z_1)}{z_1^2(1-z_2)}\right) + \log(z_2) \log\left(\frac{z_2}{1-z_2}\right) \log\left(\frac{z_1^2(1-z_3)}{z_3^2(1-z_1)}\right) \right), \end{aligned} \quad (3.25)$$

in which the Δ factors are positive for the ordered Δ_i . However, the z_i -dependent factor is not definitely positive. Considering $z_i = 1 + (i-4)\delta$ with a small variable δ , at the leading order the z -dependent factor in (3.25) is

$$\left(\log \frac{4}{3} \log \delta + \log 2 \log 3 \right) \delta + O(\delta^2), \quad (3.26)$$

which is negative for $0 < \delta < 2^{-\frac{\log 3}{\log \frac{4}{3}}}$. A nonperturbative plot of the whole z -dependent factor in (3.25) is shown in figure 6. This confirms that the total positivity is violated by the function $G_\Delta(z)$ with sufficiently small Δ_i and $1-z$.⁶ Note that we have shown the asymptotic formula of $G_\Delta(z)$ in the limit $z \rightarrow 1^-$ is indeed totally positive. This suggests the two limits with small $\Delta \rightarrow 0$ or $z \rightarrow 1^-$ do not commute for the total positivity of $G_\Delta(z)$.

Using the same approach one can compute the determinant at the fourth order $\|G_\Delta(z)\|_4$, and its limit with small Δ and $1-z$ is actually positive

$$\|G_\Delta(z)\|_4 \approx 0.024 \delta^2 \prod_{i < j} (\Delta_j - \Delta_i). \quad (3.27)$$

Non-positive determinants appear again at the fifth order. The Δ -expansion of $G_\Delta(z)$ at the order $O(\Delta^4)$ is rather complicated and similar analytical results for $\|G_\Delta(z)\|_5$ are not available yet. Numerically one can show that for $\Delta_i = 0.1 * i$, $z_j = 1 + 0.001(j-6)$, $\|G_\Delta(z)\|_5 \approx -8.7 \times 10^{-26}$.

Critical value Δ_{TP}^* for the total positivity of $G_\Delta(z)$. Above computations show the total positivity of $G_\Delta(z)$ is violated with small Δ and $1-z$. In contrast, in the large Δ limit the 1D conformal block is totally positive. The non-positive determinants $\|G_\Delta(z)\|_m$ should disappear above certain threshold value $\Delta > \Delta_{\text{TP}}^*$. To determine Δ_{TP}^*

⁶In contrast, the small Δ expansion of the Gauss hypergeometric function ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ does not generate such negative determinants and is always totally positive.

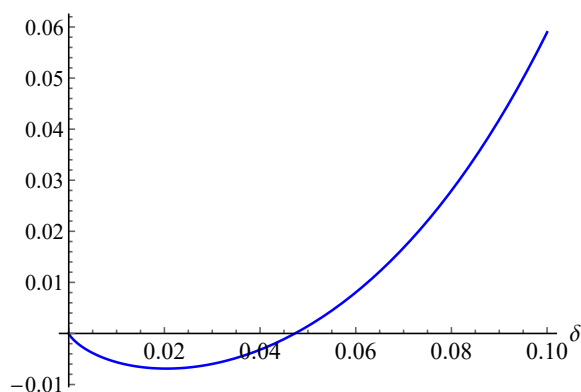


Figure 6. The z -dependent factor in (3.25) is negative in a small range $0 < \delta < 0.047$.

m	3	5	7	9	11	13	15
ϵ_0	10^{-5000}	10^{-3000}	10^{-2500}	10^{-1200}	10^{-1000}	10^{-800}	10^{-600}
$\Delta_{\text{TP}}^c \approx$	0.1627	0.1321	0.1080	0.0886	0.076	0.066	0.060

Table 1. Smallest Δ_{TP}^c to have positive $\|G_{\Delta}(z)\|_m > 0$ in the range $1 - z > \epsilon_0$ for any $\Delta > \Delta_{\text{TP}}^c$. The numerical precision is 10^{-8888} .

Δ_{TP}	0.01	0.1	0.14	0.15	0.16	0.162	0.1626	0.16264	0.1627
$\epsilon^c \approx$	10^{-2}	10^{-4}	10^{-9}	10^{-15}	10^{-67}	10^{-245}	10^{-1357}	10^{-1949}	10^{-5647}

Table 2. Critical value ϵ^c for a given Δ_{TP} with which $\|G_{\Delta}(z)\|_{m=3} < 0$ for $\epsilon < \epsilon^c$.

is of critical importance for conformal bootstrap study. Moreover, it uncovers a surprising mathematical structure of the 1D conformal block.

We numerically evaluate the determinants $\|G_{\Delta}(z)\|_m$ using the exact formula of $G_{\Delta}(z)$ (2.5). We firstly adopt evenly distributed data to compute $\|G_{\Delta}(z)\|_m$ and the determinants with general data will be studied later:

$$\Delta_i = \Delta_{\text{TP}} * i, \quad z_j = 1 + \epsilon(j - m - 1), \quad 1 \leq i, j \leq m. \quad (3.28)$$

Numerical results show that the negative determinants appear for odd m with $\Delta_{\text{TP}} < \Delta_{\text{TP}}^c$ and $\epsilon < \epsilon^c$, in which the threshold values depend on m . The threshold value Δ_{TP}^c for $\|G_{\Delta}(z)\|_m > 0$ decreases with increasing m , and the maximum estimation $\Delta_{\text{TP}}^c \approx 0.1627$ is obtained with $m = 3$, see table 1.

In table 2 we show the range of ϵ^c below which the third order determinant $\|G_{\Delta}(z)\|_{m=3}$ becomes negative for a given Δ_{TP} . Near the threshold value $\Delta_{\text{TP}}^* \approx 0.1627$ the range of variable $|1 - z| = \epsilon < \epsilon^c$ for negative determinant becomes extremely small! At $\Delta_{\text{TP}} = 0.1627$ the positivity of the determinant $\|G_{\Delta}(z)\|_{m=3}$ is only violated by a tiny factor at the order

$$\|G_{\Delta}(z)\|_{m=3} \approx -1.5939 \times 10^{-5654} |_{\Delta_{\text{TP}}=0.1627}. \quad (3.29)$$

The total positivity of 1D conformal block $G_\Delta(z)$ is so sophisticated that for a normal parameter with four effective digits at the order 10^{-1} , it is merely violated by a negative 3×3 determinant at the order 10^{-5654} ! Such a “hierarchy” naturally arising from the total positivity of the 1D conformal block has a span of 5653 orders, drastically larger than the famous hierarchy problem between the electroweak scale (10^2 GeV) and the Planck scale (10^{19} GeV)! A not directly relevant but curious question is that could the hierarchy problem in particle physics be related to certain positive structure in quantum field theories?

Behavior of $\|G_\Delta(z)\|_{m=3}$ near the threshold value Δ_{TP}^c can be studied analytically. Let us consider the small ϵ expansion of the conformal block $G_\Delta(1 - \epsilon)$

$$\frac{\Gamma(\Delta)^2}{\Gamma(2\Delta)} G_\Delta(1 - \epsilon) = \tag{3.30}$$

$$- 2\psi^{(0)}(\Delta) - \log(\epsilon) - 2\gamma - \epsilon \Delta(\Delta - 1) \left(2 \left(\psi^{(0)}(\Delta) + \gamma \right) + \log(\epsilon) - 2 \right) + O(\epsilon^2),$$

where γ is the Euler constant and $\psi^{(0)}$ is the zeroth order Polygamma function. With the data set (3.28), the sign of the determinant $\|G_\Delta(z)\|_3$ is given by

$$\|G_\Delta(z)\|_3 \propto \epsilon (\mathbf{P}(\Delta_{\text{TP}}) - \mathbf{Q}(\Delta_{\text{TP}}) \log(\epsilon)) + O(\epsilon^2), \tag{3.31}$$

where

$$\mathbf{Q}(x) = 2x \log(4/3) \times \tag{3.32}$$

$$\left(\psi^{(0)}(x) - 2\psi^{(0)}(2x) + \psi^{(0)}(3x) - x \left(5\psi^{(0)}(x) - 8\psi^{(0)}(2x) + 3\psi^{(0)}(3x) \right) \right).$$

In the limit $\epsilon \rightarrow 0$, $\log(\epsilon) \rightarrow -\infty$, the sign of $\|G_\Delta(z)\|_3$ is determined by the factor $\mathbf{Q}(\Delta_{\text{TP}})$ in (3.31). It is straightforward to check that the function $\mathbf{Q}(x)$ has a unique positive root at $x^c \approx 0.1627316$ and is always positive for $x > x^c$. Therefore in the small ϵ limit, the determinant $\|G_\Delta(z)\|_3$ with arguments (3.28) is positive for $\Delta > \Delta_{\text{TP}}^c \approx 0.1627316$, beautifully consistent with our high precision numerical results in tables 1 and 2.

Now let us consider the determinant $\|G_\Delta(z)\|_3$ with more general Δ_i and z_j :

$$\Delta_i = \Delta_{\text{TP}} w_i, \quad z_j = 1 + \epsilon y_j, \quad 1 = w_1 < w_2 < w_3, \quad y_1 < y_2 < y_3 = -1. \tag{3.33}$$

In the small ϵ limit the determinant $\|G_\Delta(z)\|_3$ has similar formula as (3.31). In particular its dominating part is also given by the term proportional to $\epsilon \log(\epsilon)$ and the function \mathbf{Q} is modified to

$$\mathbf{Q}(x) = 2x (y_1 \log(y_2/y_3) + y_2 \log(y_3/y_1) + y_3 \log(y_1/y_2)) \times \tag{3.34}$$

$$\left((w_3 - w_2) \psi^{(0)}(x) - (w_3 - 1) \psi^{(0)}(xw_2) + (w_2 - 1) \psi^{(0)}(xw_3) + \right.$$

$$\left. x \left((w_2^2 - w_3^2) \psi^{(0)}(x) - (w_2^2 - 1) \psi^{(0)}(xw_3) + (w_3^2 - 1) \psi^{(0)}(xw_2) \right) \right),$$

in which the y_i -dependent term is always positive. Note the non-even distribution of the variables z_j has trivial effect on the sign of $\mathbf{Q}(x)$, as in the small $\epsilon \rightarrow 0$ limit, the factors y_i are decoupled from the dominating term $\propto \log(\epsilon)$. The equation $\mathbf{Q}(x) = 0$ can be solved

numerically or using small $w_2 - 1$, $w_3 - 1$ expansion. The equation has a unique solution x^c above which $\mathbf{Q}(x) > 0$. The root x^c depends on w_2, w_3 and it reaches the maximum value in the limit $w_2, w_3 \rightarrow 1$, which is given by the equation

$$2\psi^{(1)}(x) + (1 - 2x)\psi^{(2)}(x) = 0. \tag{3.35}$$

The solution to this equation is $x^* \approx 0.32315626$. It gives the maximum value of Δ_{TP} to have a negative $\|G_\Delta(z)\|_3$ for general data set $\{\Delta_i, z_j\}$.

Let us summarize what we have obtained so far. From the evenly distributed data $\{\Delta_i, z_j\}$, the numerical results suggest $\|G_\Delta(z)\|_m$ can have negative values below a threshold value $\Delta_{\text{TP}} < \Delta_{\text{TP}}^c$ for small $1 - z$. The Δ_{TP}^c decreases with larger m and obtains its maximum value at $m = 3$:

$$\Delta_{\text{TP}}^c|_{m=3} > \Delta_{\text{TP}}^c|_{m>3}. \tag{3.36}$$

From a careful analysis for $\|G_\Delta(z)\|_m$ with general $\{\tilde{\Delta}_i, \tilde{z}_j\}$ at $m = 3$, we find the Δ_{TP}^c reaches its maximum value near the cusp $0 < w_2 - 1 < w_3 - 1 \ll 1$. If the inequality (3.36) is also true for non-evenly distributed data $\{\tilde{\Delta}_i, \tilde{z}_j\} \cup \{\Delta_{i>3}, z_{j>3}\}$, then our results suggest the solution x^* is optimal and the 1D SL conformal block is totally positive for any $\Delta > \Delta_{\text{TP}}^* \approx 0.32315626$; otherwise, the total positivity could be violated with larger $\Delta > x^*$ at higher orders. We hope this puzzle could be addressed by a more comprehensive study of the total positivity of $G_\Delta(z)$ in the near future.

Total positivity of the linear functional action on $G_\Delta(z)$. In section 4 we will study the functional α'_i whose action is given by

$$S(\Delta, i) \equiv \alpha'_i[G_\Delta] = \int_0^1 x^{i+a} G_\Delta(x) dx. \tag{3.37}$$

and it is important to know the total positivity of the function $S(\Delta, i)$.

With sufficiently large Δ , the total positivity of the function $S(\Delta, i)$ can be proved using the basic composition formula (3.3). Since the function z^{i+a} and $G_\Delta(z)$ with large Δ are totally positive, their convolution is also totally positive. Note the total positivity of $G_\Delta(z)$ is a sufficient but not necessary condition for $S(\Delta, i)$ being totally positive, and it could be totally positive with small Δ_i though this is not the case for $G_\Delta(z)$. The integration (3.37) can be evaluated using series expansion of $G_\Delta(z)$:

$$S(\Delta, i) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(\Delta)_k^2}{(2\Delta)_k} \frac{1}{k+i+a+1}. \tag{3.38}$$

In the above formula, the function $\frac{1}{k+i+a+1}$ is the modified Cauchy's matrix which is totally positive, see appendix A.2. For another relevant factor $\frac{(\Delta)_k^2}{(2\Delta)_k}$, we have provided promising evidence for its total positivity before. Therefore the linear functional action $S(\Delta, i)$ is also expected to be totally positive for $\Delta \geq 0$, $i \in \mathbb{N}$.

4 Analytical functionals for the 1D $O(N)$ vector bootstrap bound

In this section we construct the analytical functionals for the 1D large N bootstrap bound with $\Delta \in (0, \Delta_c/2)$, which is saturated by the generalized free field theory with spectrum $\Delta_n = 2\Delta_\phi + n$, $n \in \mathbb{N}$ in the T sector. By constructing the analytical functionals for this simple while representative bootstrap problem, we want to study the critical question in conformal bootstrap: what is the mathematical structure responsible for the nontrivial bootstrap constraints? To construct the analytical functionals, we utilize the functional basis dual to the spectrum of generalized free field theories [28], for which we review in the first part of this section.

4.1 Analytical functional basis

In section 2, the linear functionals are constructed based on the derivatives of variable z at the crossing symmetric point $z = \frac{1}{2}$: $\alpha = \sum_{i \leq \Lambda} c_i \partial_z^i \cdot |_{z=\frac{1}{2}}$. These functionals are convenient for numerical computations. Nevertheless, due to the singularities at $z = 0, 1$ of the conformal block $G_\Delta(z)$, the series expansion of $G_\Delta(z)$ only converges in the range $|z - \frac{1}{2}| < \frac{1}{2}$. To construct functionals more effectively, it needs new basis which contains information of the singularities of $G_\Delta(z)$, namely the analytical functional basis [23–25].

The analytical functional basis is dual to the function basis in terms of which the conformal correlation functions can be expanded. The function basis can be provided by the s- and t-channel conformal blocks

$$G_n^s \equiv z^{-2\Delta_\phi} G_{\Delta_n}(z), \quad G_n^t \equiv (1-z)^{-2\Delta_\phi} G_{\Delta_n}(1-z), \quad (4.1)$$

and their derivatives, associated with the spectrum of generalized free field theories, e.g., the generalized free boson $\Delta_n = 2\Delta_\phi + 2n$ or fermion $\Delta_n = 2\Delta_\phi + 2n + 1$ [24, 25]. In this work, inspired by the extremal functional spectrum in figure 3, we adopt a different function basis for the conformal correlation function, which is given by the conformal blocks G_n^s, G_n^t without their derivatives, associated with the spectrum $\Delta_n = 2\Delta_\phi + n$, $n \in \mathbb{N}$. Consider a correlation function $\mathcal{G}(z)$ which is superbounded in the u-channel Regge limit $|z| \rightarrow \infty$:⁷

$$|\mathcal{G}(z)| < |z|^{-\epsilon} \quad (4.2)$$

with $\epsilon > 0$, it admits a unique expansion in terms of the above function basis

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} \lambda_n^s G_n^s + \sum_{n=0}^{\infty} \lambda_n^t G_n^t \equiv \mathcal{G}^s(z) + \mathcal{G}^t(z). \quad (4.3)$$

The basis G_n^s is holomorphic away from $z \in [1, +\infty)$, so is $\mathcal{G}^s(z)$. Likewise, the function $\mathcal{G}^t(z)$ is holomorphic away from $z \in (-\infty, 0]$. The functional basis $\alpha_m^{s,t}$ dual to the above function basis satisfies

$$\alpha_m^s \cdot G_n^s = \delta_{mn}, \quad \alpha_m^s \cdot G_n^t = 0, \quad (4.4)$$

$$\alpha_m^t \cdot G_n^t = \delta_{mn}, \quad \alpha_m^t \cdot G_n^s = 0, \quad (4.5)$$

⁷Here the correlation function $\mathcal{G}(z)$ is the correlation function $\mathcal{G}(z)$ in (2.1) dressed with a factor $z^{-2\Delta_\phi}$.

based on which the coefficients $\lambda_n^{s/t}$ in (4.3) can be extracted from the Regge superbounded conformal correlator

$$\lambda_n^s = \alpha_n^s \cdot \mathcal{G}, \quad \lambda_n^t = \alpha_n^t \cdot \mathcal{G} \tag{4.6}$$

and the expansion (4.3) can be formally rewritten as

$$\mathcal{G}(z) = \mathcal{G}^s(z) + \mathcal{G}^t(z) = \sum_{n=0}^{\infty} (\alpha_n^s \cdot \mathcal{G}) G_n^s + (\alpha_n^t \cdot \mathcal{G}) G_n^t. \tag{4.7}$$

Above formula has close relation with the dispersion relation of conformal correlation functions [28, 58]. Here we sketch the main idea. Consider the Cauchy's integral formula for the conformal correlation function $\mathcal{G}(z)$:

$$\mathcal{G}(z) = \oint \frac{dw}{2\pi i} \frac{1}{w-z} \mathcal{G}(w), \tag{4.8}$$

in which the contour encircles $w = z$ but does not contact the branch cuts $(-\infty, 0]$ and $[1, +\infty)$. The contour can be deformed into contours wrapping the two branch cuts, denoted C_{\mp} and the arcs at infinity. For the Regge superbounded correlation functions which satisfy $\mathcal{G}(w) = O(|w|^{-\epsilon})$ in the Regge limit $|w| \rightarrow \infty$, contributions from infinity vanish and the integral of $\mathcal{G}(z)$ consists of two parts

$$\mathcal{G}(z) = - \int_{C_-} \frac{dw}{2\pi i} \frac{1}{w-z} \mathcal{G}(w) + \int_{C_+} \frac{dw}{2\pi i} \frac{1}{w-z} \mathcal{G}(w) \equiv \mathcal{G}^t(z) + \mathcal{G}^s(z), \tag{4.9}$$

in which the $\mathcal{G}^t(z)$ and $\mathcal{G}^s(z)$ are holomorphic away from $z \in (-\infty, 0]$ and $z \in [(1, +\infty)$, respectively. The holomorphicity of the two terms in (4.9) suggests they can be decomposed into the function basis of G_n^t and G_n^s , as in (4.3). Such decomposition can be alternatively fulfilled with the expansion of the integral kernel $\frac{1}{w-z}$

$$\frac{1}{w-z} = \sum_{n=0}^{\infty} \Theta_n(w) z^{-2\Delta_\phi} G_{\Delta_n}(z) \tag{4.10}$$

for integral along the contour C_+ and

$$\frac{1}{w-z} = - \sum_{n=0}^{\infty} \Theta_n(1-w) (1-z)^{-2\Delta_\phi} G_{\Delta_n}(1-z) \tag{4.11}$$

for integral along the contour C_- , in which

$$\Theta_n(w) = \frac{(-1)^n (2\Delta_\phi)_n^2}{n! (4\Delta_\phi + n - 1)_n} \frac{1}{w} {}_3F_2 \left(1, -n, 4\Delta_\phi + n - 1; 2\Delta_\phi, 2\Delta_\phi; \frac{1}{w} \right). \tag{4.12}$$

The integrals in (4.9) turn into

$$\mathcal{G}^s(z) = \int_{C_+} \frac{dw}{2\pi i} \frac{1}{w-z} \mathcal{G}(w) = \sum_{n=0}^{\infty} \left(\int_{C_+} \frac{dw}{2\pi i} \Theta_n(w) \mathcal{G}(w) \right) G_n^s, \tag{4.13}$$

$$\mathcal{G}^t(z) = - \int_{C_-} \frac{dw}{2\pi i} \frac{1}{w-z} \mathcal{G}(w) = \sum_{n=0}^{\infty} \left(\int_{C_-} \frac{dw}{2\pi i} \Theta_n(1-w) \mathcal{G}(w) \right) G_n^t. \tag{4.14}$$

Comparing with the expansion (4.7), it gives explicit formulas for the actions of the functional basis

$$\alpha_n^s \cdot \mathcal{G} = \int_{C_+} \frac{dz}{2\pi i} \Theta_n(z) \mathcal{G}(z), \quad (4.15)$$

$$\alpha_n^t \cdot \mathcal{G} = \int_{C_-} \frac{dz}{2\pi i} \Theta_n(1-z) \mathcal{G}(z). \quad (4.16)$$

By deforming the contours it is clear that the actions $\alpha_n^s \cdot G_n^t = \alpha_n^t \cdot G_n^s = 0$.

There are constraints the analytical functionals need to satisfy [24, 25, 59]. Here $\Theta_n(z) \sim O(|z|^{-1})$ in the Regge limit $|z| \rightarrow \infty$, therefore the integrals (4.15), (4.16) only converge for the functions $\mathcal{G}(z) \sim O(|z|^{-\epsilon})$ in this limit. The most general conformal correlation functions have Regge limit $\mathcal{G}(z) \rightarrow |z|^0$ for which the integrals (4.15), (4.16) do not converge. In this case one can use the subtracted functionals [28]

$$\bar{\alpha}_n^r = \alpha_n^r - \frac{(-1)^n (2\Delta_\phi)_n^2}{n! (4\Delta_\phi + n - 1)_n} \alpha_0^r, \quad r = s, t, \quad (4.17)$$

which correspond to new integral kernels with Regge behavior $\bar{\Theta}_n(z) \sim O(|z|^{-2})$.

Actions of the functional basis on conformal blocks. The dual relations (4.4), (4.5) show the actions of functional basis on the conformal blocks with $\Delta = 2\Delta_\phi + n$. For the actions on conformal blocks with general Δ 's, we need to evaluate the integrals (4.15), (4.16) with $\mathcal{G} = z^{-2\Delta_\phi} G_\Delta(z)$ and $\mathcal{G} = (1-z)^{-2\Delta_\phi} G_\Delta(1-z)$

$$\alpha_n^s \cdot \left(z^{-2\Delta_\phi} G_\Delta(z) \right) \equiv \mathbf{S}(\Delta, n) = \int_1^\infty \frac{dz}{2\pi i} \text{Disc}[\Theta_n(z) z^{-2\Delta_\phi} G_\Delta(z)], \quad (4.18)$$

$$\alpha_n^s \cdot \left((1-z)^{-2\Delta_\phi} G_\Delta(1-z) \right) \equiv \mathbf{T}(\Delta, n) = \int_1^\infty \frac{dz}{2\pi i} \text{Disc}[\Theta_n(z) (1-z)^{-2\Delta_\phi} G_\Delta(1-z)]. \quad (4.19)$$

The function $\Theta_n(z)$ is regular along $z \in [1, +\infty)$, while the conformal blocks acquire discontinuities between the two sides of the branch cut $[1, +\infty)$

$$\frac{1}{2\pi i} \text{Disc}[z^{-2\Delta_\phi} G_\Delta(z)] = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} z^{-2\Delta_\phi - \Delta + 1} {}_2F_1(1 - \Delta, 1 - \Delta, 1, 1 - z),$$

$$\frac{1}{2\pi i} \text{Disc}[(1-z)^{-2\Delta_\phi} G_\Delta(1-z)] = -\frac{\sin(\pi(\Delta - 2\Delta_\phi))}{\pi} (z-1)^{\Delta - 2\Delta_\phi} {}_2F_1(\Delta, \Delta, 2\Delta, 1-z).$$

Applying above two formulas in (4.18) and (4.19) it gives

$$\begin{aligned} \mathbf{S}(\Delta, n) &= \frac{(-1)^n \Gamma(2\Delta) \Gamma(n + 2\Delta_\phi)^2}{\Gamma(n+1) \Gamma(\Delta)^2 \Gamma(-\Delta + 2\Delta_\phi + 1) \Gamma(\Delta + 2\Delta_\phi) (n + 4\Delta_\phi - 1)_n} \\ &\quad \times {}_3F_2(1, -n, n + 4\Delta_\phi - 1; -\Delta + 2\Delta_\phi + 1, \Delta + 2\Delta_\phi; 1), \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \mathbf{T}(\Delta, n) &= \frac{\sin(\pi(\Delta - 2\Delta_\phi))}{\pi} \sum_{i=0}^n \frac{(-1)^{i+n+1} \Gamma(2\Delta) \Gamma(\Delta - 2\Delta_\phi + 1) \Gamma(2\Delta_\phi + i) (2\Delta_\phi + i)_{n-i}^2}{\Gamma(n-i+1) (4\Delta_\phi + n + i - 1)_{n-i}} \\ &\quad \times {}_3\tilde{F}_2(\Delta, \Delta, \Delta - 2\Delta_\phi + 1; 2\Delta, \Delta + i + 1; 1), \end{aligned} \quad (4.21)$$

where ${}_3\tilde{F}_2$ is the regularized generalized hypergeometric function. For $\Delta = 2\Delta_\phi + m$, $m \in \mathbb{N}$, above formulas agree with the dual relation (4.4). Actions of the linear functionals α_n^t on the s- and t-channel conformal blocks can be obtained from (4.20) and (4.21) through $z \leftrightarrow 1 - z$ transformation.

For $\Delta_\phi = \frac{1}{2}$ the kernel $\Theta_n^s(z)$ in (4.4) is drastically simplified

$$\Theta_n(z) = \frac{1}{z} \frac{(-1)^n \Gamma(n+1)^2}{\Gamma(2n+1)} {}_2F_1\left(-n, n+1; 1; \frac{1}{z}\right). \quad (4.22)$$

Its actions on conformal blocks are reduced to

$$\mathbf{S}(\Delta, n)|_{\Delta_\phi=\frac{1}{2}} = \frac{\Gamma(2\Delta)\Gamma(n+1)^2 \sin(\pi(\Delta - n - 1))}{\pi\Gamma(\Delta)^2(\Delta - n - 1)(\Delta + n)\Gamma(2n+1)}, \quad (4.23)$$

and

$$\begin{aligned} \mathbf{T}(\Delta, n)|_{\Delta_\phi=\frac{1}{2}} &= \frac{(-1)^n (n!)^2 \Gamma(2\Delta) \Gamma(\Delta)^2 \sin(\pi\Delta)}{\Gamma(2n+1) \pi} \\ &\times {}_4\tilde{F}_3(\Delta, \Delta, \Delta, \Delta; 2\Delta, \Delta - n, \Delta + n + 1; 1). \end{aligned} \quad (4.24)$$

Above formulas provide necessary ingredients to construct analytical functionals.

4.2 Analytical functionals for Regge superbounded conformal correlator

In section 2 we have shown that in the range $\Delta < \Delta_c$, the $O(\infty)$ vector bootstrap bound on the scaling dimension of the lowest operator in the T sector Δ_T^* is determined by the crossing equation

$$\sum_{\mathcal{O} \in S} \lambda_{\mathcal{O}}^2 z^{-2\Delta_\phi} G_\Delta(z) - \sum_{\mathcal{O} \in T} \lambda_{\mathcal{O}}^2 (1-z)^{-2\Delta_\phi} G_\Delta(1-z) = 0, \quad (4.25)$$

which is saturated by $\Delta_T^* = 2\Delta_\phi$ with extremal spectrum $2\Delta_\phi + n$, $n \in \mathbb{N}$. Consequently the extremal functional α^* should satisfy following positive conditions

$$\alpha^* \cdot \left(z^{-2\Delta_\phi} G_\Delta(z) \right) = 0, \quad \text{for } \Delta = 0, \quad (4.26)$$

$$\alpha^* \cdot \left(z^{-2\Delta_\phi} G_\Delta(z) \right) > 0, \quad \forall \Delta > 0, \quad (4.27)$$

$$-\alpha^* \cdot \left((1-z)^{-2\Delta_\phi} G_\Delta(1-z) \right) = 0^1, \quad \text{for } \Delta = 2\Delta_\phi, \quad (4.28)$$

$$-\alpha^* \cdot \left((1-z)^{-2\Delta_\phi} G_\Delta(1-z) \right) = 0^2, \quad \text{for } \Delta = 2\Delta_\phi + n, n \in \mathbb{N}^+, \quad (4.29)$$

$$-\alpha^* \cdot \left((1-z)^{-2\Delta_\phi} G_\Delta(1-z) \right) > 0, \quad \forall \Delta > 2\Delta_\phi \ \& \ \Delta \neq 2\Delta_\phi + n, n \in \mathbb{N}^+, \quad (4.30)$$

in which the notation 0^i refers to the zeros of order i . In (4.29) the zeros 0^{2i} with $i > 1$ also satisfy the positive condition, however, we will show that there are only second order zeros in (4.29).

For the Regge superbounded conformal correlators, e.g., correlation functions (2.9)–(2.11) with $\lambda = 0$, the bootstrap functional α can be expanded in terms of the functional basis

$$\alpha = \sum_{n=0}^{\infty} c_n \alpha_n^s + d_n \alpha_n^t. \quad (4.31)$$

Considering the actions of the functional basis on $G_n^{s,t}$ (4.4), (4.5), the positive condition (4.27) of the extremal functional α^* requires

$$c_n > 0 \quad \forall n \in \mathbb{N}. \quad (4.32)$$

Moreover, the conditions (4.28), (4.29) suggest

$$d_n = 0 \quad \forall n \in \mathbb{N}. \quad (4.33)$$

The positive coefficients c_n in $\alpha^* = \sum_{n=0}^{\infty} c_n \alpha_n^s$ should be arranged so that the extra positive conditions can be satisfied.

It is easy to verify that the positive condition (4.26) is satisfied by the functional $\alpha^* = \sum_{n=0}^{\infty} c_n \alpha_n^s$. For each α_n^s , its action is

$$\alpha_n^s \cdot z^{-2\Delta_\phi} = \int_{C_+} \frac{dz}{2\pi i} \Theta_n(z) z^{-2\Delta_\phi}, \quad (4.34)$$

in which the integrand has a pole at $z = 0$ and is holomorphic for $z \in [1, +\infty)$ enclosed by the contour C_+ . Therefore the action vanishes, as required by (4.26).

The critical constraints to solve c_n are from (4.28), (4.29). In the action (4.21) of functional basis α_n^s , the factor $\sin(\pi(\Delta - 2\Delta_\phi))$ generates single zeros at $\Delta = 2\Delta_\phi + n$ with $n \in \mathbb{N}$. To further form double zeros for $n \in \mathbb{N}^+$, the coefficients c_n should satisfy

$$\sum_{i=0}^{\infty} \mathcal{T}(2\Delta_\phi + n, i) \cdot \tilde{c}_i = \delta_{n0}, \quad \forall n \in \mathbb{N} \quad (4.35)$$

in which the coefficients \tilde{c}_i is

$$\tilde{c}_i \equiv c_i (-1)^i \frac{\Gamma(i+1)^2}{\Gamma(2i+1)}. \quad (4.36)$$

and \mathcal{T} is the stripped action

$$\mathbf{T}(\Delta, i) = (-1)^{i+1} \frac{\sin(\pi(\Delta - 2\Delta_\phi))}{\pi} \mathcal{T}(\Delta, i) \frac{\Gamma(i+1)^2}{\Gamma(2i+1)}. \quad (4.37)$$

One may expect the coefficients c_n can be obtained by solving the whole infinite equation group (4.35), like the remarkable work [23]. However, the solutions to the whole equation group (4.35) lead to a trivial functional. We demonstrate this point using an example with $\Delta_\phi = \frac{1}{2}$.

4.2.1 Solution of the infinite equation group

Solution to the linear equation group (4.35) is given by the inverse of the infinite matrix $\mathcal{T}(2\Delta_\phi + n, i)$. For general Δ_ϕ the matrix $\mathcal{T}(2\Delta_\phi + n, i)$ is quite complicated and hard to solve directly. The formulas are notably simplified with $\Delta_\phi = \frac{1}{2}$, see (4.22)–(4.24). In this case it is convenient to take the variable transformation $x = 1 - z^{-1}$ [23]. The kernel $\Theta_n(z)$ degenerates to the Legendre polynomials $P_n(2x - 1)$ and the stripped action $\mathcal{T}(\Delta, n)$ becomes

$$\mathcal{T}(\Delta, n) = \int_0^1 dx x^{\Delta-1} {}_2F_1(\Delta, \Delta, 2\Delta, x) P_n(2x - 1). \quad (4.38)$$

Since the Legendre polynomials $P_n(2x - 1)$ are orthogonal under the inner product

$$\int_0^1 dx P_n(2x - 1)P_m(2x - 1) = \frac{1}{2n + 1} \delta_{n,m}, \tag{4.39}$$

the matrix $\mathcal{T}(m, n)$ can be interpreted as the coefficients of the following expansion

$$x^{m-1} {}_2F_1(m, m, 2m, x) = \sum_{n=0}^{\infty} (2n + 1) \mathcal{T}(m, n) P_n(2x - 1). \tag{4.40}$$

Thus the inverse matrix \mathcal{T}^{-1} is given by the inverse expansion

$$P_n(2x - 1) = \frac{1}{2n + 1} \sum_{m=1}^{\infty} \mathcal{T}^{-1}(n, m) x^{m-1} {}_2F_1(m, m, 2m, x). \tag{4.41}$$

Let us compare the constant term on both sides for each n . The Legendre polynomial satisfies $P_n(2x - 1)|_{x=0} = (-1)^n$. While the function $x^{m-1} {}_2F_1(m, m, 2m, x)$ is equal to 1 at $x = 0$ for $m = 1$ and vanishes at $x = 0$ for $m > 1$. Therefore the elements $\mathcal{T}^{-1}(n, 1)$, as well as the coefficients \tilde{c}_n in (4.35) can be solved from (4.41):

$$\mathcal{T}^{-1}(n, 1) = \tilde{c}_n = (-1)^n (2n + 1), \tag{4.42}$$

which gives

$$c_n = \frac{\Gamma(2n + 2)}{\Gamma(n + 1)^2}. \tag{4.43}$$

A comment is that all the c_n 's are positive, as needed to satisfy the positive condition (4.27).

The whole extremal functional α^* is given by

$$\alpha^* \cdot \mathcal{G}(z) = \int_{C_+} \frac{dz}{2\pi i} \Theta^*(z) \mathcal{G}(z) \tag{4.44}$$

with a kernel

$$\Theta^*(z) = \sum_{n=0}^{\infty} c_n \Theta_n(z) \propto \sum_{n=0}^{\infty} (-1)^n (2n + 1) P_n(2x - 1)|_{x=\frac{z-1}{z}} \equiv \bar{\Theta}^*(x). \tag{4.45}$$

Using the generating function of Legendre polynomial

$$\frac{1}{\sqrt{t^2 - 2tx + 1}} = \sum_{n=0}^{\infty} P_n(x) t^n, \tag{4.46}$$

one can show

$$\bar{\Theta}^*(x) = \frac{1 - t^2}{(4tx + (1 - t)^2)^{3/2}} \Big|_{t \rightarrow 1}. \tag{4.47}$$

In the limit $t \rightarrow 1$, we have

$$\bar{\Theta}^*(x)|_{x \neq 0} = 0. \tag{4.48}$$

While with $x = 0$ the function $\bar{\Theta}^*(x)$ has a pole at $t = 1$. Such “extremal” kernel behaves like a Dirac $\delta(x)$. The action of functional α^* on the t-channel conformal block $-(1-z)^{-2\Delta_\phi} G_\Delta(1-z)$ with $\Delta_\phi = \frac{1}{2}$ becomes

$$-\alpha^* \cdot \left((1-z)^{-1} G_\Delta(1-z) \right) = \frac{\sin(\pi(\Delta-1))}{\pi} \int_0^1 dx \frac{1-t^2}{(4tx + (1-t)^2)^{3/2}} x^{\Delta-1} {}_2F_1(\Delta, \Delta, 2\Delta, x) \Big|_{t \rightarrow 1}. \quad (4.49)$$

Due to the pole at $x = 0$ with $t = 1$, above integral only gives a nonzero value for $\Delta = 2\Delta_\phi = 1$, while vanishes for $\Delta > 1$. Together with the factor $\sin(\pi(\Delta-1))$, the whole action vanishes for all $\Delta \geq 1$. On the other hand, its action on the s-channel conformal block

$$\alpha^* \cdot \left(z^{-1} G_\Delta(z) \right) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \int_0^1 dx \frac{1-t^2}{(4tx + (1-t)^2)^{3/2}} {}_2F_1(\Delta, 1-\Delta, 1, x) \Big|_{t \rightarrow 1} \quad (4.50)$$

vanishes at $\Delta = 0$ and is always positive for $\Delta > 0$.

To summarize, the functional constructed from the whole infinity set of equations (4.35) is actually trivial due to its vanishing action on the t-channel conformal block, or the $O(N)$ T sector in (4.25). We expect this is the case for general Δ_ϕ .

4.2.2 Solutions of the finite subset of equation group

Although the inverse of the whole equation group (4.35) leads to a degenerated functional, surprisingly the inverse of the finite subset of the equation group can produce functionals which satisfy all the positive conditions (4.26)–(4.30) within a range $\Delta \leq \Lambda_N$. This allows us to construct a series of functionals with arbitrarily high Λ_N !

Instead of constructing a functional whose action on t-channel conformal block has double zeros at $\Delta = 2\Delta_\phi + n$ for all $n \in \mathbb{N}^+$, we would like to relax the restriction on the double zeros. Specifically, we consider the functional

$$\alpha'_N = \sum_{n=0}^N c_n \alpha_n^s, \quad (4.51)$$

whose action has a single zero at $\Delta = 2\Delta_\phi$ and double zeros at $\Delta = 2\Delta_\phi + n$ for each integer $0 < n \leq N$. This amounts to the following constraints

$$\sum_{i=0}^N \mathcal{T}(2\Delta_\phi + n, i) \cdot \tilde{c}_i = \delta_{n0}, \quad 0 \leq n \leq N. \quad (4.52)$$

It is straightforward to solve above equations for small N 's. Taking $N = 4$, the matrix $\mathcal{T}(n+1, i)$ is⁸

$$\begin{pmatrix} \frac{\pi^2}{6} & \frac{1}{6}(12 - \pi^2) & \frac{1}{6}(\pi^2 - 9) & \frac{1}{18}(31 - 3\pi^2) & \frac{1}{72}(12\pi^2 - 115) \\ 12 - \pi^2 & 5\pi^2 - 48 & 129 - 13\pi^2 & \frac{1}{3}(75\pi^2 - 739) & \frac{1}{12}(4859 - 492\pi^2) \\ 5(\pi^2 - 9) & 5(129 - 13\pi^2) & 5(73\pi^2 - 720) & \frac{5}{3}(7492 - 759\pi^2) & \frac{5}{12}(7932\pi^2 - 78283) \\ \frac{70}{9}(31 - 3\pi^2) & \frac{70}{9}(75\pi^2 - 739) & \frac{70}{9}(7492 - 759\pi^2) & \frac{70}{9}(4335\pi^2 - 42784) & \frac{35}{18}(679939 - 68892\pi^2) \\ \frac{35}{4}(12\pi^2 - 115) & \frac{35}{4}(4859 - 492\pi^2) & \frac{35}{4}(7932\pi^2 - 78283) & \frac{35}{4}(679939 - 68892\pi^2) & 35(99003\pi^2 - 977120) \end{pmatrix},$$

⁸The cautious readers may notice that the matrix is symmetric up to certain numerical factors. It can be proved that the matrix $M_{m,n} \equiv \frac{\Gamma(m+1)^2}{\Gamma(2m+2)} \mathcal{T}(m+1, n)$ is indeed symmetric.

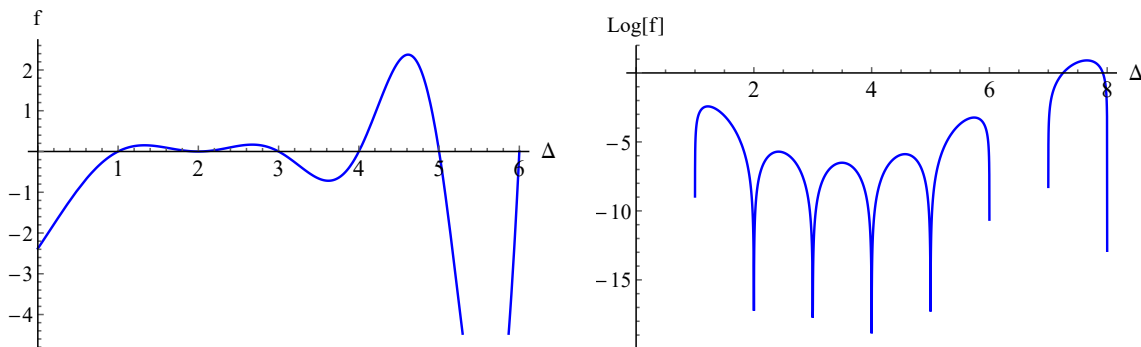


Figure 7. Actions (denoted f) of the functionals α'_N on the t-channel conformal block with $N = 1$ (left) and $N = 4$ (right). Note in the right plot, the y -axis is $\log(f)$ which has no real value for negative f .

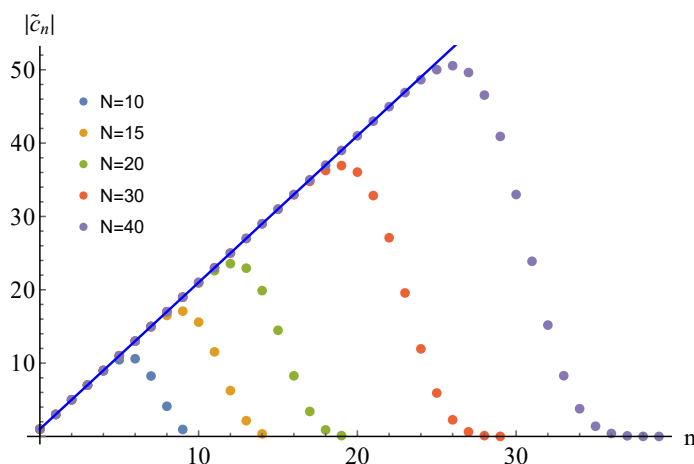


Figure 8. $|\tilde{c}_n|$ solved from (4.52) with finite N 's. The straight line is $|\tilde{c}_n| = 2n + 1$.

which corresponds to the coefficients \tilde{c}_i

$$\tilde{c}_i \approx (0.999753, -2.97534, 4.5689, -4.39935, 1.88092). \tag{4.53}$$

It is impressive that the first few elements are close to the limit $\tilde{c}_n = (-1)^{-n}(2n + 1)$ even for $N = 4$. In figure 8 we show more solutions of \tilde{c}_n with larger N 's.⁹ From these examples, the coefficients $|\tilde{c}_n|$ with $n \leq N/2$ are close to the limit $|\tilde{c}_n| = (2n + 1)$, while for larger n 's, $|\tilde{c}_n|$ deviates the straight line and decreases exponentially. Nevertheless, for all n 's the coefficients \tilde{c}_n have signs $\tilde{c}_n \propto (-1)^n$. This suggests the coefficients $c_n \propto (-1)^n \tilde{c}_n$ are all positive, therefore satisfying the positive condition (4.27) up to $\Delta < N + 1$.

In figures 9 and 10 we show the actions (denoted f) of the functional α'_N on the t-channel and s-channel conformal blocks, respectively. The actions on the t-channel conformal block have a single zero at $\Delta = 1$ and double zeros at $\Delta = n + 1$ for integer $0 < n < N$.

⁹To solve \tilde{c}_n from (4.52) with large N , it is necessary to adopt high numerical precision, reminiscent of the numerical conformal bootstrap with SDPB [60, 61].

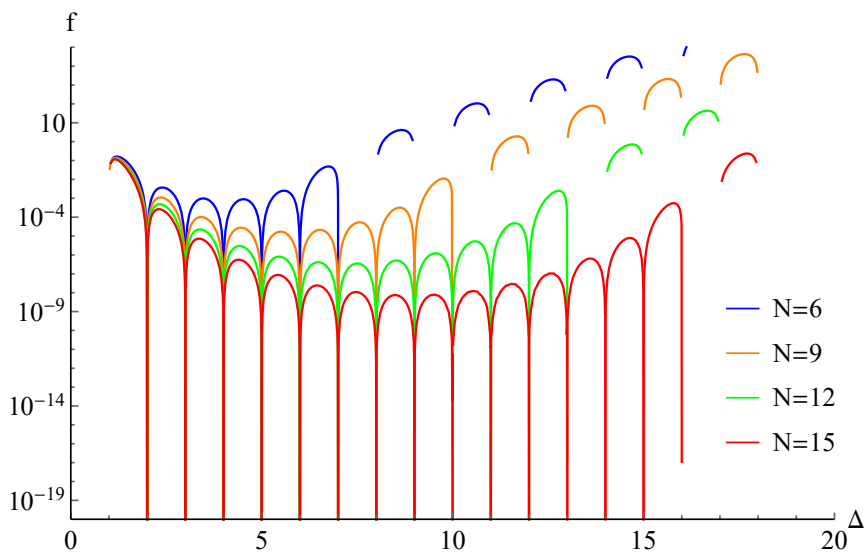


Figure 9. Actions of the functionals α'_N on the t-channel conformal block.

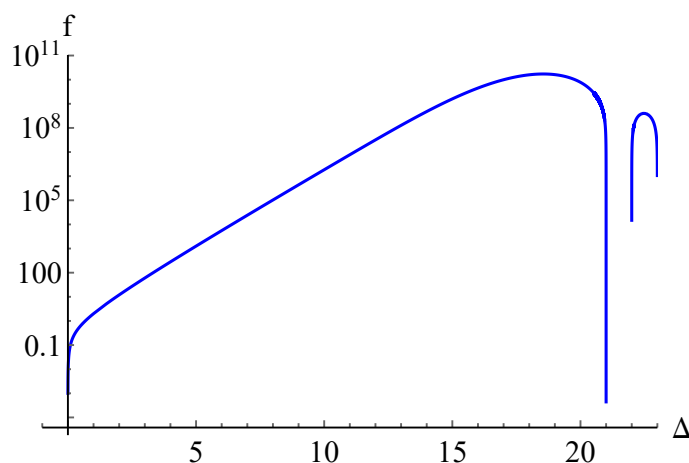


Figure 10. Action of the functional α'_{20} on the s-channel conformal block.

For $1 < \Delta < N + 1$, the actions f decrease with larger N , and to produce higher spectrum of the generalized free field theories, the functional actions do not converge to any stable non-vanishing limit, in agreement with the behavior of the numerical extremal functionals shown in figure 4. This is also consistent with our previous results that with $N = \infty$, the functional α^* becomes trivial: $f = 0$ for $\Delta \geq 2\Delta_\phi$. The action on the s-channel conformal block ($N = 20$) shown in figure 10 has a single zero at $\Delta = 0$, corresponding to the unit operator, and is positive for $0 < \Delta < N + 1$. The fact $f(n) |_{n>20} = 0$, $n \in \mathbb{N}$ is expected since there is no $\alpha^s_{n>20}$ in $\alpha'_{N=20}$. The numerical solutions of the equations (4.52) do satisfy all the positive conditions (4.26)–(4.30) up to $\Delta < N + 1$ for finite N .

4.2.3 Positivity from total positivity

Our goal is to generalize previous results to arbitrarily large $N \in \mathbb{N}^+$ with general Δ_ϕ . However, it requires highly nontrivial conditions to guarantee the strong positive constraints (4.26)–(4.30) for large Δ . The reason why our previous examples can satisfy the positive conditions for $\Delta \leq \Lambda_n$ is due to a simple fact: in the action

$$f(\Delta) \propto \sin(\pi(\Delta - 2\Delta_\phi)) \sum_{n=0}^N \tilde{c}_n \mathcal{T}(\Delta, n), \quad (4.54)$$

the sign of the function $\sum_{n=0}^N \tilde{c}_n \mathcal{T}(\Delta, n)$ oscillates in phase with $\sin(\pi(\Delta - 2\Delta_\phi))$ in the range $2\Delta_\phi < \Delta < 2\Delta_\phi + N$. This is equivalent to the following properties of the function $\sum_{n=0}^N \tilde{c}_n \mathcal{T}(\Delta, n)$ with general N :

- The matrix $\mathcal{T}(2\Delta_\phi + n, i)|_{0 \leq n, i \leq N}$ is non-degenerate and invertible.
- All the zeros at $n = 1, \dots, N$ are of order 1 or higher odd numbers.
- There are no other zeros besides $n = 1, \dots, N$.

Any violations of above properties will necessarily break the positive conditions and invalid the functionals constructed from (4.52). Surprisingly, above three properties are closely related to the total positivity of the SL conformal block for which we have studied in section 3.

Consider the equation (4.52) for general Δ_ϕ . The left part corresponds to

$$g(\Delta) \equiv \sum_{n=0}^N \tilde{c}_n \mathcal{T}(\Delta, n) = \int_0^1 dx (1-x)^{2\Delta_\phi-1} x^{-2\Delta_\phi} G_\Delta(x) \sum_{n=0}^N \tilde{c}_n {}_3F_2(1, -n, 4\Delta_\phi + n - 1; 2\Delta_\phi, 2\Delta_\phi; 1-x), \quad (4.55)$$

where

$$\tilde{c}_n = c_n \frac{(-1)^n (2\Delta_\phi)_n^2}{n!(n + 4\Delta_\phi - 1)_n}. \quad (4.56)$$

In (4.55), the total positivity of the factor $(1-x)^{2\Delta_\phi-1} x^{-2\Delta_\phi} G_\Delta(x)$ follows the total positivity of the conformal block $G_\Delta(x)$ for $\Delta > \Delta_{\text{TP}}^*$. Therefore we have the *Variation Diminishing Property* (3.5) that the sign changes and zeros of the functions $g(\Delta)$ and $\bar{\Theta}(x) = \sum \tilde{c}_n {}_3F_2$ in (4.55) should satisfy the relation

$$S^+(g) \leq Z(g) \leq S^+(\bar{\Theta}) \leq N, \quad (4.57)$$

i.e., the number of sign changes and zeros of $g(\Delta)$ in $\Delta \in (2\Delta_\phi, \infty)$ ¹⁰ is not larger than the number of sign changes of $\bar{\Theta}(x)$ in $x \in (0, 1)$! The third inequality in (4.57) is due to the fact that $\bar{\Theta}(x)$ is a polynomial of order N

$$\bar{\Theta}(x) = \sum_{n=0}^N a_n x^n, \quad (4.58)$$

¹⁰In general the lower bound of Δ should be the critical value Δ_{TP}^* where $G_\Delta(x)$ loses its total positivity. Here we assume $2\Delta_\phi > \Delta_{\text{TP}}^*$ and the problem with smaller Δ_ϕ will be studied later.

consequently including multiplicity, there are at most N zeros of $\bar{\Theta}(x)$. Only the odd order zeros relate to sign changes of $\bar{\Theta}(x)$, therefore the polynomial $\bar{\Theta}(x)$ can have at most N sign changes, corresponding to N first order zeros. Moreover, in the extremal case $S^+(\bar{\Theta}) = N$, according to the *Descartes' rule of signs*, the number of positive roots of $\bar{\Theta}(x)$ is at most the number of sign changes in the sequence of its coefficients $\{a_n\}$, therefore there has to be N sign changes in $\{a_n\}$, which requires $a_n \propto a_0(-1)^n$.

The inequality (4.57) provides substantial restrictions on the solutions of the equation group (4.52)! The function $g(\Delta)$ has at most N zeros including multiplicity. Therefore the N zeros specified in the equations (4.52) at $\Delta = 2\Delta_\phi + n$, $n = 1, 2, \dots, N$ are all the zeros allowed by the inequality (4.57). Moreover, they are single zeros! With $S^+(g) = S^+(\bar{\Theta})$, the *Variation Diminishing Property* requires the function $\bar{\Theta}(x)$ should have the same sign arrangement in $x \in (0, 1)$ as $g(\Delta)$ in $\Delta \in (2\Delta_\phi, \infty)$, which is in the order $\{+, -, \dots\}$ according to the equation (4.52). This suggests $a_0 > 0$ and consequently, $a_n \propto a_0(-1)^n \propto (-1)^n$. In particular, we have

$$c_N \propto (-1)^N \tilde{c}_N \propto (-1)^N a_N > 0. \tag{4.59}$$

For small Δ_ϕ , e.g. $\Delta_\phi = 0.1$, the conformal block $G_\Delta(x)$ is not totally positive at $\Delta = 2\Delta_\phi$, and the sign inequality (4.57) only works for $\Delta \in (\Delta_{\text{TP}}^*, \infty)$. At the order 3 we have $\Delta_{\text{TP}}^*|_{m=3} \approx 0.32315626 < 1$, if the higher order total positivity is not drastically modified, then it implies all the zeros of $g(\Delta)$ specified in (4.52): $\Delta = 2\Delta_\phi + n$, $1 \leq n \leq N$ are above Δ_{TP}^* and the inequalities (4.57) can prove that these zeros are of first order and no other zeros in $(\Delta_{\text{TP}}^*, \infty)$. Besides, one needs to show that there are no extra zeros of $g(\Delta)$ between $(2\Delta_\phi, \Delta_{\text{TP}}^*)$. We do not have a strict proof for this statement but have verified it by numerically solving (4.52) with small Δ_ϕ 's.

We also need to prove that there are indeed N zeros in $g(\Delta)$, e.g., the equations (4.52) have non-trivial solutions. This is equivalent to the statement that the matrix $\mathcal{T}(2\Delta_\phi + n, i)|_{0 \leq n, i \leq N}$ is invertible for any N , which can be proved within two steps based on the assumed total positivity of the function $f(\Delta, i) = (\Delta)_i^2 / (2\Delta)_i$. Firstly it can be shown that the integral

$$\mathcal{P}(\Delta, k) = \int_0^1 dx (1-x)^{2\Delta_\phi-1} x^{-2\Delta_\phi} G_\Delta(x) x^k \tag{4.60}$$

is totally positive, similar to (3.38). Therefore the determinant of its sub-matrices are always nonzero. The determinant $||\mathcal{T}(\Delta, k)||_N$ is related to $||\mathcal{P}(\Delta, k)||_N$ through a non-degenerate basis transformation $\{x^k\} \rightarrow \{\Theta_k\}$ and is also nonzero.

Therefore based on the total positivity of the SL conformal block, previous three questions on the equation group (4.52) can be nicely addressed. It suggests that for general N , the equations (4.52) always have nontrivial solutions, and the related function $g(\Delta)$ only has single zeros at $\Delta = 2\Delta_\phi + n$, $n = 1, \dots, N$. This guarantees the positive conditions on the t-channel conformal block (4.28)–(4.30) for $\Delta \leq 2\Delta_\phi + N$ with arbitrary positive integer N .

An interesting question is whether the matrix $\mathcal{T}(\Delta, n)$ is also totally positive. If true, then it can prove that the coefficients c_n solved from (4.52) are always positive. The

Variation Diminishing Property tells us the signs of a_n in (4.58) with basis $\{x^n\}$, which can be used to verify the positive sign of c_N (4.59). In future studies, it would be important to provide quantitative estimations of \tilde{c}_n 's for general N and explain the non-monotonic shapes in figure 8.

Here we summarize the properties of the analytical functional $\alpha'_N = \sum_{n=0}^N c_n \alpha_n^s$ constructed through the equations (4.52): for a given positive integer N , the functional α'_N can produce the spectrum consistent with the numerical bootstrap results up to $\Delta \leq 2\Delta_\phi + N$. It gives the unit operator in the $O(N)$ singlet sector and double trace operators in the $O(N)$ T sector below $2\Delta_\phi + N$. The actions of the functional satisfy the positive conditions in both singlet and T sectors for $\Delta < 2\Delta_\phi + N$. Based on the total positivity of the SL conformal block, such functionals exist for any finite N . The large N limit of the functional $\lim_{N \rightarrow \infty} \alpha'_N = \alpha^*$ is trivial, which produces zero action on the t-channel conformal block. In contrast, the truly nontrivial point here is the way how the series of functionals $\{\alpha'_N\}$ approach the limit α^* : for any given large cutoff Λ_N , one can construct α'_N so that its action satisfies the required positive conditions for any $\Delta < \Lambda_N$. This explains, for the Regge superbounded correlators, the numerical bootstrap bound of the crossing equation (2.32), or the bound with $\Delta_\phi < \Delta_c/2$ in figure 1.

4.3 Analytical functionals for general conformal correlators

The functional constructed in the last section scales as $O(|z|^{-1})$ in the Regge limit and only works for the superbounded conformal correlators, e.g. (2.9)–(2.11) with $\lambda = 0$. For more general correlators, such as (2.9)–(2.11) with $\lambda \neq 0$, one can construct functionals using the subtracted basis $\bar{\alpha}_n^s$ (4.17):

$$\bar{\alpha}'_N = \sum_{n=1}^N c_n \bar{\alpha}_n^s. \tag{4.61}$$

Above functional should satisfy the same positive conditions (4.26)–(4.30). Following the same reasons for (4.52) we can get a similar equation group

$$\sum_{i=1}^N \bar{\mathcal{T}}(2\Delta_\phi + n, i) \cdot \tilde{c}_i = \delta_{n0}, \quad \text{for } n = 0, 1, \dots, N-1, \tag{4.62}$$

in which

$$\bar{\mathcal{T}}(\Delta, i) = \mathcal{T}(\Delta, i) - \frac{(-1)^i (2\Delta_\phi)_i^2}{i!(4\Delta_\phi + i - 1)_i} \mathcal{T}(\Delta, 0) \tag{4.63}$$

is invertible.

The solutions to the equations (4.62) and (4.52) are closely related with each other. Note the equation (4.62) can be rewritten into a form similar to (4.52)

$$\sum_{i=0}^N \mathcal{T}(2\Delta_\phi + n, i) \cdot \tilde{c}_i = \delta_{n0}, \quad \text{for } n = 0, 1, \dots, N-1, \tag{4.64}$$

in which the coefficients \tilde{c}_i are subjected to an extra constraint

$$\sum_{i=0}^N \frac{(-1)^i (2\Delta_\phi)_i^2}{i!(4\Delta_\phi + i - 1)_i} \tilde{c}_i = 0. \tag{4.65}$$

For any given $N = N_0$, the unique solution \tilde{c}_i to the equation (4.62) can be constructed from the solutions to the equation (4.52) as follows. Assume the equations (4.52) with $N = N_0 - 1$ and $N = N_0$ are solved by $\tilde{c}_i = \tilde{c}_i^1$ and $\tilde{c}_i = \tilde{c}_i^2$, respectively. Then both \tilde{c}_i^1 and \tilde{c}_i^2 satisfy the equation (4.64) with $N = N_0$ but not (4.65)¹¹

$$\sum_{i=0}^{N_0} \frac{(-1)^i (2\Delta_\phi)_i^2}{i!(4\Delta_\phi + i - 1)_i} \tilde{c}_i^a \equiv I_a \neq 0, \quad a = 1, 2. \quad (4.66)$$

Now it is easy to see the linear combination of the two solutions

$$\tilde{c}_i^0 \equiv \frac{I_1 \tilde{c}_i^2 - I_2 \tilde{c}_i^1}{I_1 - I_2} \quad (4.67)$$

can satisfy both the equation (4.64) and (4.65), thus it provides the unique solution to the equation (4.62) for $N = N_0$.

Since \tilde{c}_i^0 is just a linear combination of the two solutions $\tilde{c}_i^{1,2}$ of the equation (4.52), one may expect in the large N limit, the solution \tilde{c}_i^0 also leads to a degenerated functional, like $\tilde{c}_i^{1,2}$. Nevertheless, its strict proof requires a quantitative estimation of $I_{1,2}$, which corresponds to the tails of the points in figure 8 and is beyond our current scope. Numerically we have checked that the solution to the equation (4.62) indeed show similar pattern to that of the equation (4.52), i.e., its large N limit leads to a degenerated functional.

Now a critical question is whether the rather complicated solution (4.67) can satisfy the bootstrap positive conditions (4.26)–(4.30) up to $\Delta = 2\Delta_\phi + N - 1$. The equation (4.62) is related to the function \bar{g}

$$\begin{aligned} \bar{g}(\Delta) &\equiv \sum_{n=1}^N \bar{T}(\Delta, n) \tilde{c}_n = \\ &\int_0^1 dx (1-x)^{2\Delta_\phi-1} x^{-2\Delta_\phi} G_\Delta(x) \sum_{n=1}^N \tilde{c}_n ({}_3F_2(1, -n, 4\Delta_\phi + n - 1; 2\Delta_\phi, 2\Delta_\phi; 1-x) - 1). \end{aligned} \quad (4.68)$$

Again we want to bound the number of zeros of the function $\bar{g}(\Delta)$ by the number of sign changes in the sequence of the polynomial $\tilde{\Theta}(x) = \sum \tilde{c}_n ({}_3F_2 - 1)$. However, the function $\bar{g}(\Delta)$ is expected to have $N - 1$ zeros at $\Delta = 2\Delta_\phi + n$ with $n = 1, \dots, N - 1$, while $\tilde{\Theta}(x)$ remains an order N polynomial of x , which in principle could have N single zeros and sign changes, indicating the function $\bar{g}(\Delta)$ could have, including multiplicity, an extra zero besides the $N - 1$ single zeros specified in (4.62). Solution to this puzzle is that the lowest term of $\tilde{\Theta}(x)$, when expanded as an order N polynomial of $(1-x)$, is linear in $(1-x)$, while the constant term has been canceled in the subtraction (4.17) for better Regge behavior. Therefore this linear term can be factorized and $\bar{g}(\Delta)$ becomes

$$\bar{g}(\Delta) = \int_0^1 dx (1-x)^{2\Delta_\phi} x^{-2\Delta_\phi} G_\Delta(x) \sum_{n=0}^{N-1} \lambda_n x^n. \quad (4.69)$$

The function $(1-x)^{2\Delta_\phi} x^{-2\Delta_\phi} G_\Delta(x)$ remains totally positive while the order of the polynomial $\tilde{\Theta}(x)$ is reduced to $N - 1$, which can have at most $N - 1$ zeros and sign changes.

¹¹Here we set $\tilde{c}_{N_0}^1 = 0$.

Thus according to the *Variation Diminishing Property* (3.5), the function $\bar{g}(\Delta)$ can have at most $N - 1$ zeros. This confirms the function $\bar{g}(\Delta)$ has no other zeros besides $2\Delta_\phi + n$, $n = 1, \dots, N - 1$. Moreover, they are single zeros. It can be verified numerically that the coefficients c_n solved from (4.62) are all positive, corresponding to the positive actions of $\bar{\alpha}'_N$ on the s-channel conformal block for $\Delta \in (0, 2\Delta_\phi + N)$, similar to figure 10.

The upshot is that the subtraction (4.17) is consistent with *Variation Diminishing Property* and the functionals $\bar{\alpha}'_N$ for the general conformal correlators have similar positive properties as the functionals α'_N for the Regge superbounded correlators.

To summarize, we have shown that the bootstrap crossing equation (4.25) admits a family of analytical functionals $\{\bar{\alpha}'_N\}$ which satisfy the bootstrap positive conditions (4.26)–(4.30) and can generate extremal spectrum of the 1D generalized free field theories up to a range $\Delta < 2\Delta_\phi + N$. We have provided evidence indicating the $SL(2, \mathbb{R})$ conformal block is totally positive for large Δ , as a result, the analytical functionals $\{\bar{\alpha}'_N\}$ can be constructed for arbitrarily large N . The analytical functional bootstrap study suggests the crossing equation (4.25) is saturated by the 1D generalized free field theories while with extremal functional action $\log[f]$ unbounded from below. This is nicely in agreement with the numerical bootstrap results shown in figures 3 and 4. At finite N (for analytical functional bootstrap) and Λ (for numerical bootstrap), the difference is that the analytical functional $\bar{\alpha}'_N$ has precise spectrum and satisfies the positive conditions up to the range $\Delta \leq 2\Delta_\phi + N$, while the numerical bootstrap extremal functional α_Λ satisfies the positivity for general large Δ but there are errors in the spectrum as compensation. In the region with finite $\Delta < 2\Delta_\phi + N$, both the analytical and numerical extremal functionals show the spectrum of generalized free field theories and the functional actions decrease with increasing precision, suggesting their behaviors are dominated by the same positive structure of the crossing equation (4.25).

5 Conclusion and outlook

We have studied the 1D $O(N)$ vector bootstrap in the large N limit. We obtained a remarkably simple bootstrap equation (2.32), whose numerical conformal bootstrap bound is saturated by the generalized free field theories. We also showed that the numerical extremal functional actions do not converge to a stable non-vanishing limit, but keep on decreasing with increasing numerical precision, see figure 3 and 4. The most interesting part of this work is the construction of analytical functionals (4.51) and (4.61) for the large N bootstrap equation (2.32). By requiring the analytical functionals can generate the spectrum of the generalized free field theories, we obtained an infinite set of equations (4.52) or (4.62) for the analytical functionals. Using a particular example we illustrated that the solutions to the infinite set of equations lead to degenerated analytical functionals for the crossing equation (2.32). Nevertheless, after truncating the equations to finite ranks M , we obtained a series of analytical functionals $\{\alpha'_M\}$ whose actions on the crossing equations can satisfy the bootstrap positive conditions for $\Delta < \Lambda_M = 2\Delta_\phi + M$. Although the large M limit of the functionals $\{\alpha'_M\}$ becomes trivial, they can approach the limit in a particular way so that the bootstrap positive conditions can be satisfied up to arbitrarily high Λ_M . Below

Λ_M the extremal spectrum and the large M behavior of the analytical functionals $\{\alpha'_M\}$ are well consistent with the numerical conformal bootstrap results, therefore it clarifies the underlying positive structure of the crossing equation (2.32).

In our construction the total positivity of the SL conformal block plays a critical role for the analytical functionals to satisfy the bootstrap positive conditions. We showed the SL conformal block is totally positive in the limits with large Δ or small $1 - z$. With small Δ it relates to a sophisticated mathematical structure: at the order 3 the total positivity is violated for $\Delta < \Delta_{\text{TP}}^* \approx 0.32315626$. Numerically we did not find violations of total positivity for $\Delta > 1$, but a solid conclusion needs to be studied further. This work provides a concrete example to demonstrate the mathematical structure in conformal bootstrap and the intriguing connections between mathematics and quantum field theories.

We believe this work opens the door towards more systematical studies for many fascinating problems in quantum field theories and their connections to mathematics. Part of these problems are explained below.

- The most fundamental question is the total positivity of the SL conformal block $G_\Delta(z)$, which provides the key ingredient in bootstrap studies. We have proved the SL conformal block is totally positive with large Δ and showed the total positivity is violated below a threshold value $\Delta_{\text{TP}}^* \approx 0.32315626$. We have provided numerical evidence indicating this estimation could be optimal but a strict proof is not available yet. Moreover, we have observed that total positivity of the conformal block relates to a special mathematical structure which can naturally generate a huge hierarchy in the parameter space. It would be exciting to improve our understanding of this mathematical structure and its applications in quantum field theories.
- In this work we have constructed the analytical functional for the first part of the 1D large N bootstrap bound before the kink in figure 1, which is saturated by the generalized free field theory and the bootstrap equations are reduced to a simple form (2.32). It is tempting to know the theories saturating the second part of the bootstrap bound and construct the analytical functionals. Furthermore, the bootstrap bound almost disappears after $\Delta_\phi > 0.75$ in figure 1. Similar phenomenon also appears in higher dimensions, see figure 2. It would be interesting to know the reasons which dissolve the bootstrap restrictions.
- Conformal field theories with large N limits have close relation to the quantum field theories in the AdS spacetime. Constraints on the CFT side can lead to nontrivial restrictions on the theories in AdS, see e.g. [37, 38, 62–64]. It would be interesting to explore the constraints of the analytical functional constructed in this work on the S-matrices in AdS₂. In particular, how does the total positivity affect the scattering process in AdS₂? Do the AdS analogs of the conformal blocks, the Witten diagrams also satisfy total positivity? The role of total positivity in the 4D amplitude in flat spacetime has been extensively studied recently [41–44]. Our results on the 1D CFTs suggest that the AdS₂ could provide another interesting and technically tractable laboratory to explore the role of (total) positivity in quantum field theories. We

hope to report the applications of analytical functionals and total positivity on AdS physics in another work.

- Total positivity is powerful to analyze positivity of analytical functionals. In our construction, the positivity of bootstrap functionals can be established based on the total positivity of the SL conformal block while without solving the equation groups (4.52), (4.62) explicitly. Nevertheless, it would be interesting to know more concrete information on the analytical functionals $\{\alpha'_M\}$, such as the curves of the coefficients $|\tilde{c}|_n$ shown in figure 8. One may wonder if the equation groups (4.52), (4.62) are easier to solve in Mellin space [65–67].
- The 1D large N vector bootstrap provides insights to study higher dimensional $O(N)$ vector bootstrap. There are solid evidence for close relations between the two problems. Firstly their bootstrap bounds have similar patterns, as shown in figures 1 and 2. Moreover, for the bootstrap bounds saturated by the generalized free field theories, the $O(N)$ vector bootstrap equations degenerate to similar forms in 1D and higher dimensions. The functional basis dual to higher dimensional generalized free field spectrum has been constructed in [28] and their relation to dispersion relation has been studied in [29], see also [27, 68]. However, a crucial question is how to organize the functional basis in order to satisfy the positive conditions. The method developed in this work can be useful to construct analytical functionals with suitable positive properties in higher dimensions. We leave this problem for future work [54].
- A more challenging problem along this direction is to construct the analytical functionals for the $O(N)$ vector bootstrap bounds with large but finite N . This was one of the motivations for the author to start this work. In this case we need to go back to the whole $O(N)$ vector crossing equations (2.7), (2.8) and take the $1/N$ terms into account. These $1/N$ terms and the crossing equation (2.7) will necessarily introduce new ingredients responsible for the $1/N$ interactions in the underlying theories. The related analytical functionals could provide a new nonperturbative frame to study CFTs with large N limits, including the 3D critical $O(N)$ vector models and the conformal gauge theories in general dimensions.
- The series of analytical functionals $\{\alpha'_N\}$ constructed in this work are sensitive to the large Δ spectrum. Associated with total positivity, they can be employed to detect non-unitarity in the large Δ region, which relates to the high energy dynamics in AdS. We hope more systematical studies of the large N analytical functionals can provide solid conclusions for some widely interested questions on the large Δ spectrum of large N unitary CFTs.

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A Examples of the totally positive functions

In this appendix we show some classical examples of the totally positive functions. Some of the results in this part have been applied in our study of the total positivity of the Gauss hypergeometric functions ${}_2F_1(\Delta, \Delta, 2\Delta, z)$ and SL conformal block functions $G_\Delta(z)$.

A.1 Example 1: $f(\Delta, x) = x^\Delta$

The determinant formula (3.1) of the function $f(\Delta, x) = x^\Delta$ is given by

$$\|f(\Delta, x)\|_m \equiv f \begin{pmatrix} \Delta_1, \dots, \Delta_m \\ x_1, \dots, x_m \end{pmatrix} = \det \begin{bmatrix} x_1^{\Delta_1} & \dots & x_1^{\Delta_m} \\ \vdots & & \vdots \\ x_m^{\Delta_1} & \dots & x_m^{\Delta_m} \end{bmatrix}. \tag{A.1}$$

Taking $\Delta_i = i - 1$, above determinant goes back to the Vandermonde determinant, which is given by

$$\|f(\Delta, x)\|_m = \prod_{i>j} (x_i - x_j) \tag{A.2}$$

and is positive for the ordered variables $0 < x_1 < \dots < x_m$. Then to prove the total positivity of the function $f(\Delta, x)$, one only needs to show that its determinant can never be zero, which can be done by induction [45, 69].

The statement $\|f(\Delta, x)\| \neq 0$ is equivalent to the claim that for a given set of $c_i \in \mathbb{R}$, the equation

$$h_m(x) = \sum_{i=1}^m c_i x^{\Delta_i} \tag{A.3}$$

cannot have m solutions in the region $x > 0$. For $n = 1$, $h_1(x) = c_1 x^{\Delta_1}$ and there is no positive solution for h . Assume above statement is true for $h_i(x)$ with $i < n$. If $h_n(x)$ has n positive solutions, then according to Rolle’s theorem, the following function

$$(x^{-\Delta_1} h_n(x))' = \sum_{i=2}^n (\Delta_i - \Delta_1) c_i x^{\Delta_i - \Delta_1} \sim h_{n-1}(x) \tag{A.4}$$

can have $n - 1$ positive zeros, which is inconsistency with our previous induction assumption that $h_{n-1}(x)$ cannot have $n - 1$ positive solutions. Therefore the function $h_n(x)$ should have

positive solutions less than n . This completes the proof that the determinant $\|f(\Delta, x)\|_m$ can never be zero.

From the total positivity of the function x^Δ , one can show a family of totally positive functions. For instances, the function $e^{xy} = (e^x)^y$ is also totally positive.

A.2 Example 2: $f(x, y) = \frac{1}{x+y}$

The determinant formula (3.2) for the function $f(x, y)$ is

$$\|f(\Delta, x)\|_m \equiv f \begin{pmatrix} x_1, \dots, x_m \\ y_1, \dots, y_m \end{pmatrix} = \det \begin{bmatrix} \frac{1}{x_1+y_1} & \dots & \frac{1}{x_1+y_m} \\ \vdots & & \vdots \\ \frac{1}{x_m+y_1} & \dots & \frac{1}{x_m+y_m} \end{bmatrix}. \tag{A.5}$$

Above determinant can be solved in a compact form, i.e., the Cauchy formula

$$\|f(\Delta, x)\|_m = \frac{\prod_{i>k} (x_i - x_k) \prod_{i>k} (y_i - y_k)}{\prod_{i,k=1}^m (x_i + y_k)}, \tag{A.6}$$

which is positive for the ordered variables $x_1 < \dots < x_m, y_1 < \dots < y_m$.

The total positivity of $f(x, y)$ can be alternatively proved using the basic composition formula (3.3). The function can be rewritten as

$$\frac{1}{x+y} = \int_0^\infty e^{-(x+y)t} dt = \int_0^1 u^x u^y d(\log(u)). \tag{A.7}$$

Due to the basic composition formula, the total positivity of above integral follows the total positivity of the function u^x .

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