A free field perspective of λ-deformed coset CFT's

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Abstract: We continue our study of λ-deformed σ-models by setting up a $\frac{1}{k}$ perturbative expansion around the free field point for cosets, in particular for the λ-deformed $SU(2)/U(1)$ coset CFT. We construct an interacting field theory in which all deformation effects are manifestly encoded in the interaction vertices. Using this we reproduce the known β-function and the anomalous dimension of the composite operator perturbing away from the conformal point. We introduce the λ-dressed parafermions which have an essential Wilson-like phase in their expressions. Subsequently, we compute their anomalous dimension, as well as their four-point functions, as exact functions of the deformation and to leading order in the k expansion. Correlation functions with an odd number of these parafermions vanish as in the conformal case.

Contents

1 Introduction

A class of integrable two-dimensional field theories having an explicit action realization, were systematically constructed in recent years [1–10]. They typically represent the effective action corresponding to the deformation of one or more WZW model current algebra theories at levels k_i by current bilinears. They are called λ-deformed σ-models due to the preferred letter used to denote the deformation parameters. In this context, a research avenue is the systematic study of various aspects of the corresponding two-dimensional quantum field theories. Roughly speaking, these fall into two general categories. In the first, belong studies concerning directly the coupling constants in these theories. In particular, the computation of their running under the renormalization group flow (RG) (β-functions) has been exhaustively studied [6, 8, 11, 12, 14–17, 21]. In addition, the geometrical aspects of the space of couplings in these theories have been elucidated [18, 19] and the C-function capturing the number of the degrees of freedom along the RG flow has been

evaluated [20, 21]. The second category consists of works aiming at discovering how the operators of the CFT respond to the deformation. That includes the computation of the anomalous dimension they acquire [18, 22–25] as well as their dressing induced by the deformation of the original CFT [19]. The above works utilized a combination of CFT and gravitational techniques together with symmetry arguments in the couplings space of these models. The results that were obtained are valid to all orders in the deformation parameters and to leading order at the level k.

Recently a new purely field theoretic approach to the study of λ-deformed theories was initiated in [26]. This method was applied to the isotropic case having a single coupling λ. The resulting theory is a theory of free fields having certain interaction terms with all the coupling constants depending on λ, in a specific manner. Similarly, the dressed operators of the theory, elementary as well composite ones, can be expressed, building also on results of [19], in terms of the free fields with specific couplings. In this approach all computations are organized around a free field theory and not around the conformal point. The advantage is that all information about the deformation parameter λ is encoded in the coupling constants appearing in the action and in the various coefficients in the expressions of the operators which are given in terms of the free fields. This approach delivered results for the β-function, correlation functions and anomalous dimensions in complete agreement with the previous methods [18, 19], however with much less effort.

Most of the studies in this direction were done for the λ-deformed WZW current algebra CFTs. However, λ-deformations can be constructed for coset CFTs which have an action realization based on gauged WZW models. In particular, λ-deformed models have been constructed based on the SU(2)/U(1) coset CFT in [1], for more general symmetric spaces in [2] and the AdS$_5 \times S^5$ superstring in [3]. In the present paper we will construct and utilize the field theory based on free fields, analog of the construction for the group case of [19] for the λ-deformed SU(2)/U(1) coset CFT. With our approach we will be able to compute the anomalous dimension of parafermionic fields which are seemingly very hard to do by means of other methods. In particular, it is much more difficult to apply conformal perturbation theory when the underlying theory is a coset CFT instead of a current algebra CFT. Essentially, this is due to the fact that in the former case parafermions are involved instead of currents and they have more complicated correlation functions [27].

The plan of the paper is as follows: in section 2, the perturbative expansion of the action around the free field point will be performed. In this way we will construct an interacting field theory by keeping terms up to sixth order in the fields which suffices for our purposes. In the process we will freely use various field redefinitions in order to simplify the final form of the action. In section 3, we will use this action to compute the β-function of the σ-model using standard heat-kernel techniques. In section 4, we will introduce the λ-dressed parafermions which contain an important Wilson-like phase in their expressions. Next, we compute the anomalous dimension of the composite operator which is bilinear in the parafermions and which drives the model away from the conformal point. We also compute the anomalous dimension of a single parafermion. In this case, the presence of the non-trivial Wilson-like phase plays the important rôle. In addition, we compute all four-point functions of parafermions. Finally, in section 5 we will present our conclusions and

future directions of this work. Last but not least, two appendices follow. In appendix A we calculate the anomalous dimension of variations of the composite operator deforming the CFT while in appendix B a list of integrals and the regularization scheme we employ is considered.

2 Constructing the interacting theory

In this section the interacting field theory corresponding to the λ-deformed coset CFT for $SU(2)/U(1)$ will be constructed. Our approach follows in spirit that of [26]. In that work the interacting theory, based on free fields, for the isotropic single λ-deformed σ-model deformation of the WZW model CFT was constructed.

2.1 Expansion around the free point

Even though we are interested in the coset case, it is convenient to start with the general λ-deformed σ-model action for the group case given by [1]

$$S_{k,\lambda}(g) = S_{\text{WZW,k}}(g) + \frac{k}{\pi} \int \mathrm{d}^2\sigma \, R_+^a (\lambda^{-1} - D^T)_{ab}^{-1} L_-^b \,, \tag{2.1}$$

where

$$S_{\text{WZW},k}(g) = -\frac{k}{2\pi} \int \mathrm{d}^2\sigma \, \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \frac{k}{12\pi} \int \mathrm{Tr}(g^{-1}\mathrm{d}g)^3 \,, \tag{2.2}$$

is the WZW action for a group element g of a semi-simple Lie group G at level k and

$$R_+^a = -i\,\mathrm{Tr}(t_a \partial_+ g g^{-1})\,, \quad L_-^a = -i\,\mathrm{Tr}(t_a g^{-1}\partial_- g)\,, \quad D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1})\,. \tag{2.3}$$

The t_a's are representation matrices satisfying

$$[t_a, t_b] = i f_{abc} t_c \,, \quad \mathrm{Tr}(t_a t_b) = \delta_{ab}\,, \quad a = 1, 2, \ldots, \dim \mathrm{G}\,, \tag{2.4}$$

where the f_{abc}'s are the algebra structure constants which are taken to be real. The coupling matrix λ_{ab} parametrizes the deviation from the conformal point at which the currents R_+ and L_- are chirally, respectively anti-chirally, conserved.

The λ-deformed action for the $SU(2)/U(1)$ coset CFT can be obtained from the above action specialized to the $SU(2)$ case with $t_a = \sigma_a/\sqrt{2}$, where the σ_a's are the Pauli matrices, and $\lambda_{ab} = \mathrm{diag}(\lambda, \lambda, \lambda_3)$ where λ_3 corresponds to the $U(1)$ subgroup of $SU(2)$ via a limiting procedure [1]. We review this by first parameterizing the group element as

$$g = \mathrm{e}^{i(\phi+\phi_1)\sigma_3/2} \mathrm{e}^{i(\pi/2-\theta)\sigma_2} \mathrm{e}^{i(\phi-\phi_1)\sigma_3/2}\,, \tag{2.5}$$

where the range of values of the Euler angles are

$$\theta \in \left[0, \frac{\pi}{2}\right], \quad \phi \in [0, 2\pi]\,, \quad \phi_1 \in [0, 2\pi]\,. \tag{2.6}$$

Inserting the above into (2.1) and taking the limit $\lambda_3 \to 1$ we obtain that

$$\begin{aligned} S_{k,\lambda}(g) = \frac{k}{\pi} \int \mathrm{d}^2\sigma \, \Bigg(&\frac{1-\lambda}{1+\lambda}(\partial_+\theta\partial_-\theta + \tan^2\theta\partial_+\phi\partial_-\phi) \\ &+ \frac{4\lambda}{1-\lambda^2}(\cos\phi\partial_+\theta - \sin\phi\tan\theta\partial_+\phi)(\cos\phi\partial_-\theta - \sin\phi\tan\theta\partial_-\phi)\Bigg)\,. \end{aligned} \tag{2.7}$$

We note that the coordinate ϕ_1 has decoupled at the level of the action and the remaining two fields are θ and ϕ. However, this angle will play a very important rôle, as we shall see in the course of the paper. Its presence will be instrumental in determining the form of the λ-dressed parafermions and, as a consequence, of their anomalous dimensions.

The action (2.7) is invariant under the following two symmetries [13, 14]

$$
\begin{aligned}
\text{I}: & \qquad \lambda \to \lambda^{-1}, \quad k \to -k, \\
\text{II}: & \qquad \lambda \to -\lambda, \quad \phi \to \phi + \frac{\pi}{2}.
\end{aligned}
\tag{2.8}
$$

In what follows we will see that these symmetries will be manifest in the expressions for the physical quantities of this theory.

Next we proceed with our construction by zooming around $\theta = 0$ by first setting

$$
\theta = \frac{\rho}{\sqrt{2k}}
\tag{2.9}
$$

and defining two new fields as

$$
y_1 = \sqrt{\frac{1+\lambda}{1-\lambda}}\, \rho \cos\phi, \qquad y_2 = \sqrt{\frac{1-\lambda}{1+\lambda}}\, \rho \sin\phi.
\tag{2.10}
$$

Then, in the large k expansion the action becomes

$$
\begin{aligned}
S_{k,\lambda} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \Bigg(& \partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + \frac{g_{11}}{k} y_2^2\, \partial_+ y_1 \partial_- y_1 \\
& + \frac{g_{22}}{k} y_1^2\, \partial_+ y_2 \partial_- y_2 + \frac{g_{12}}{k} y_1 y_2\, (\partial_+ y_1 \partial_- y_2 + \partial_+ y_2 \partial_- y_1) \\
& + \frac{y_2^2}{k^2} (h_{11} y_2^2 + \tilde{h}_{11} y_1^2)\, \partial_+ y_1 \partial_- y_1 + \frac{y_1^2}{k^2} (h_{22} y_1^2 + \tilde{h}_{22} y_2^2)\, \partial_+ y_2 \partial_- y_2 \\
& + \frac{y_1 y_2}{k^2} (h_{12} y_1^2 + \tilde{h}_{12} y_2^2)\, (\partial_+ y_1 \partial_- y_2 + \partial_+ y_2 \partial_- y_1) \Bigg) + \cdots,
\end{aligned}
\tag{2.11}
$$

where we have kept terms up to quartic order in the fields. Note that, this is an expansion in the number of fields. Retaining appropriate powers of $\frac{1}{k}$, is just for book keeping as we could rescale the fields by a factor of $\sqrt{k}$ and have k as an overall coefficient in the action. The various couplings are given by

$$
\begin{aligned}
g_{11} &= \frac{1}{3}\frac{1+\lambda}{1-\lambda}, & g_{22} &= \frac{1}{3}\frac{1-\lambda}{1+\lambda}, & g_{12} &= -\frac{1}{3}\frac{1+\lambda^2}{1-\lambda^2}, \\
h_{11} &= \frac{17(1+\lambda)^2}{180(1-\lambda)^2}, & \tilde{h}_{11} &= \frac{17+14\lambda+17\lambda^2}{180(1+\lambda)^2}, & \\
h_{22} &= \frac{17(1-\lambda)^2}{180(1+\lambda)^2}, & \tilde{h}_{22} &= \frac{17-14\lambda+17\lambda^2}{180(1-\lambda)^2}, & \\
h_{12} &= -\frac{17-10\lambda+17\lambda^2}{180(1+\lambda)^2}, & \tilde{h}_{12} &= -\frac{17+10\lambda+17\lambda^2}{180(1-\lambda)^2}.
\end{aligned}
\tag{2.12}
$$

Note that (2.11) with the above couplings is invariant under the transformations

$$
\begin{aligned}
\text{I}: & \qquad \lambda \to \frac{1}{\lambda}, \qquad k \to -k, \\
\text{II}: & \qquad \lambda \to -\lambda, \qquad (y_1, y_2) \to (-y_2, y_1).
\end{aligned}
\tag{2.13}
$$

which of course correspond to (2.8) when the above zoom-in limit is taken.

Next we perform the following field redefinitions

$$y_1 = \left(1 + \frac{a_1}{k}x_2^2 + \frac{b_1}{k^2}x_2^4 + \frac{c_1}{k^2}x_1^2 x_2^2\right)x_1, \quad y_2 = \left(1 + \frac{a_2}{k}x_1^2 + \frac{b_2}{k^2}x_1^4 + \frac{c_2}{k^2}x_1^2 x_2^2\right)x_2. \quad (2.14)$$

Choosing the coefficients as

$$
\begin{aligned}
a_1 &= -\frac{g_{11}}{2}, & b_1 &= \frac{3g_{11}^2}{8} - \frac{h_{11}}{2}, & c_1 &= \frac{g_{11}g_{22}}{6} + \frac{g_{12}g_{22}}{3} - \frac{g_{22}^2}{6} - \frac{\tilde{h}_{11}}{6}, \\
a_2 &= -\frac{g_{22}}{2}, & b_2 &= \frac{3g_{22}^2}{8} - \frac{h_{22}}{2}, & c_2 &= \frac{g_{22}g_{11}}{6} + \frac{g_{12}g_{11}}{3} - \frac{g_{11}^2}{6} - \frac{\tilde{h}_{22}}{6},
\end{aligned}
\quad (2.15)
$$

we obtain the simpler action

$$
\begin{aligned}
S_{k,\lambda} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \Big(& \partial_+ x_1 \partial_- x_1 + \partial_+ x_2 \partial_- x_2 + \frac{\hat{g}_{12}}{k} x_1 x_2 \left(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1\right) \\
& + \frac{x_1 x_2}{k^2}\left(\hat{h}_{12}x_1^2 + \hat{\tilde{h}}_{12}x_2^2\right)\left(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1\right) \Big) + \cdots,
\end{aligned}
\quad (2.16)
$$

where the new couplings are denoted by a hat and are given by

$$
\begin{aligned}
\hat{g}_{12} &= g_{12} - g_{11} - g_{22} = -\frac{1 + \lambda^2}{1 - \lambda^2}, \\
\hat{h}_{12} &= \frac{1}{3}\left(g_{11}g_{22} - g_{12}g_{22} + 2g_{22}^2 + 3h_{12} - 6h_{22} - \tilde{h}_{11}\right) = -\frac{1}{6}\left(\frac{1-\lambda}{1+\lambda}\right)^2, \\
\hat{\tilde{h}}_{12} &= \frac{1}{3}\left(g_{22}g_{11} - g_{12}g_{11} + 2g_{11}^2 + 3\tilde{h}_{12} - 6h_{11} - \tilde{h}_{22}\right) = -\frac{1}{6}\left(\frac{1+\lambda}{1-\lambda}\right)^2.
\end{aligned}
\quad (2.17)
$$

Obviously, the action (2.16) with the above couplings is invariant under (2.13), where for the symmetry II the y_a's should be replaced accordingly by the x_a's.

2.2 Computational QFT conventions

We would like to set up a perturbative expansion around the free theory and perform quantum computations. Passing to the Euclidean regime we have the following basic propagators which are consistent with our normalizations

$$\langle x_a(z_1, \bar{z}_1) x_b(z_2, \bar{z}_2)\rangle = -\delta_{ab}\ln|z_{12}|^2, \qquad a = 1, 2, \quad (2.18)$$

where $z_{12} = z_1 - z_2$. Note that the above propagator implies that

$$\langle \partial x_a(z_1)\bar{\partial}x_b(\bar{z}_2)\rangle = \pi\,\delta_{ab}\delta^{(2)}(z_{12}), \quad (2.19)$$

inducing a coupling of the holomorphic and anti-holomorphic sectors which will be very important in the calculations that follow, in particular in subsubsections 4.4.1 & 4.4.5.

We will see later that it will be most convenient to define two complex conjugate bosons as

$$x_\pm = x_1 \pm i x_2. \quad (2.20)$$

In this complex basis, the only non-vanishing two-point function is

$$\langle x_+(z_1, \bar{z}_1) x_-(z_2, \bar{z}_2) \rangle = -2 \ln |z_{12}|^2 \ . \tag{2.21}$$

Finally, for the free theory the holomorphic energy-momentum tensor is given by

$$T = -\frac{1}{2}\Big((\partial x_1)^2 + (\partial x_2)^2 \Big) = -\frac{1}{2} \partial x_+ \partial x_- \ , \tag{2.22}$$

with a similar expression for the anti-holomorphic one.

3   The β-function

In this section we use the field theory action (2.16) in order to compute the β-function for the deformation parameter λ. The analogous computation for the group case was performed in [26]. We will find precisely the result of [14] obtained by gravitational methods.

In order to obtain the β-function for λ we will employ the background field heat kernel method, so that as an initial step we need the equations of motion for the fields x_1 and x_2 derived from the action (2.16). We obtain, up to $\mathcal{O}(\frac{1}{k^2})$, that

$$
\begin{aligned}
\partial_+ \partial_- x_1 + \frac{x_1 x_2}{k} &\left(\hat{g}_{12} + \frac{\hat{h}_{12} x_1^2 + \hat{\bar{h}}_{12} x_2^2}{k} \right) \partial_+ \partial_- x_2 \\
&+ \frac{x_1}{k} \left(\hat{g}_{12} + \frac{\hat{h}_{12} x_1^2 + 3\hat{\bar{h}}_{12} x_2^2}{k} \right) \partial_+ x_2 \partial_- x_2 = 0 \,, \\
\partial_+ \partial_- x_2 + \frac{x_1 x_2}{k} &\left(\hat{g}_{12} + \frac{\hat{h}_{12} x_1^2 + \hat{\bar{h}}_{12} x_2^2}{k} \right) \partial_+ \partial_- x_1 \\
&+ \frac{x_2}{k} \left(\hat{g}_{12} + \frac{3\hat{h}_{12} x_1^2 + \hat{\bar{h}}_{12} x_2^2}{k} \right) \partial_+ x_1 \partial_- x_1 = 0 \ .
\end{aligned}
\tag{3.1}
$$

For large k we may simplify them by solving for $\partial_+ \partial_- x_a$, $a = 1, 2$. Keeping terms up to $\mathcal{O}(\frac{1}{k^2})$, we find that

$$
\begin{aligned}
\partial_+ \partial_- x_1 + \frac{\hat{g}_{12}}{k} x_1 \partial_+ x_2 \partial_- x_2 & \\
&+ \frac{x_1}{k^2} \left(\left(\hat{h}_{12} x_1^2 + 3\hat{\bar{h}}_{12} x_2^2 \right) \partial_+ x_2 \partial_- x_2 - \hat{g}_{12}^2 x_2^2 \, \partial_+ x_1 \partial_- x_1 \right) = 0 \,, \\
\partial_+ \partial_- x_2 + \frac{\hat{g}_{12}}{k} x_2 \partial_+ x_1 \partial_- x_1 & \\
&+ \frac{x_2}{k^2} \left(\left(\hat{\bar{h}}_{12} x_2^2 + 3\hat{h}_{12} x_1^2 \right) \partial_+ x_1 \partial_- x_1 - \hat{g}_{12}^2 x_1^2 \, \partial_+ x_2 \partial_- x_2 \right) = 0 \ .
\end{aligned}
\tag{3.2}
$$

From (3.2) one can derive the equations which the fluctuations δx_a of the coordinates x_a obey. The fluctuations will be taken around a classical solution which we will still denote by x_a. They will be casted in the form

$$\hat{D}_{ab} \, \delta x_b = 0 \,, \tag{3.3}$$

where the operator $\hat{D}$ is a certain second order in the worldsheet derivatives that will also depend on the classical solution around which we expand. We will present its explicit expression after performing the Euclidean analytic continuation and passing to momentum space. In the conventions of [8], we replace (∂_+, ∂_-) by $\frac{1}{2}(\bar{p}, p) \equiv (p_+, p_-)$. Then after dividing by $p_+ p_-$ we obtain for $\hat{D}$ the result

$$\hat{D}_{ab} = \delta_{ab} + \frac{1}{k}\left(\hat{F}_2 + \hat{F}_2' + \frac{1}{k}\hat{F}_4\right)_{ab},\tag{3.4}$$

where the matrices are given by

$$\hat{F}_2 = \frac{\hat{g}_{12}}{p_+ p_-}\begin{pmatrix} \partial_+ x_2 \partial_- x_2 & 0 \\ 0 & \partial_+ x_1 \partial_- x_1 \end{pmatrix},$$

$$\hat{F}_2' = \hat{g}_{12}\begin{pmatrix} 0 & \frac{x_1 \partial_+ x_2}{p_+} + \frac{x_1 \partial_- x_2}{p_-} \\ \frac{x_2 \partial_+ x_1}{p_+} + \frac{x_2 \partial_- x_1}{p_-} & 0 \end{pmatrix},\tag{3.5}$$

and

$$\hat{F}_4 = \frac{1}{p_+ p_-}\begin{pmatrix} (\hat{F}_4)_{11} & (\hat{F}_4)_{12} \\ (\hat{F}_4)_{21} & (\hat{F}_4)_{22} \end{pmatrix},$$

$$(\hat{F}_4)_{11} = 3\big(\hat{h}_{12}x_1^2 + \hat{\bar{h}}_{12}x_2^2\big)\partial_+ x_2 \partial_- x_2 - \hat{g}_{12}^2 x_2^2\, \partial_+ x_1 \partial_- x_1,\tag{3.6}$$

$$(\hat{F}_4)_{22} = 3\big(\hat{\bar{h}}_{12}x_2^2 + \hat{h}_{12}x_1^2\big)\partial_+ x_1 \partial_- x_1 - \hat{g}_{12}^2 x_1^2\, \partial_+ x_2 \partial_- x_2.$$

Note that we have not provided the expressions for $(\hat{F}_4)_{12}$ and $(\hat{F}_4)_{21}$, since their form will be irrelevant for the discussion that follows. Integrating out the fluctuations, gives the effective Lagrangian of our model which reads

$$-\mathcal{L}_{\text{eff}} = \mathcal{L}_{k,\lambda}^{(0)} + \int^\mu \frac{\mathrm{d}^2 p}{(2\pi)^2} \ln(\det \hat{D})^{-1/2},\tag{3.7}$$

where $\mathcal{L}_{k,\lambda}^{(0)}$ is the Lagrangian (2.16) on the classical solution. This integral is logarithmically divergent with respect to the UV mass scale μ. The logarithmic term is isolated by performing the large momentum expansion of the integrand and keeping terms proportional to $\frac{1}{|p|^2}$, where $|p|^2 = p\bar{p}$. Next we use the fact that

$$\ln(\det \hat{D}) = \frac{1}{k}\text{Tr}\hat{F}_2 + \frac{1}{k^2}\left(\text{Tr}\hat{F}_4 - \frac{1}{2}\text{Tr}\hat{F}_2'^2\right) + \dots,\tag{3.8}$$

where we have included only terms potentially contributing to the logarithmically divergent term. Then calculating the traces individually one obtains

$$\text{Tr}\hat{F}_2 = \frac{\hat{g}_{12}}{p_+ p_-}\big(\partial_+ x_1 \partial_- x_1 + \partial_+ x_2 \partial_- x_2\big),$$

$$\text{Tr}\hat{F}_4 = \frac{1}{p_+ p_-}\Big[\big((3\hat{\bar{h}}_{12} - \hat{g}_{12}^2)x_2^2 + 3\hat{h}_{12}x_1^2\big)\partial_+ x_1 \partial_- x_1$$

$$+ \big((3\hat{h}_{12} - \hat{g}_{12}^2)x_1^2 + 3\hat{\bar{h}}_{12}x_2^2\big)\partial_+ x_2 \partial_- x_2\Big],\tag{3.9}$$

$$\text{Tr}\hat{F}_2'^2 = 2\hat{g}_{12}^2\frac{x_1 x_2}{p_+ p_-}\big(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1\big),$$

where again we have not included terms which will vanish upon the angular integration that follows. Using polar coordinates, i.e. $p = re^{i\omega}$, $\bar{p} = re^{-i\omega}$, in which the integration measure is $\mathrm{d}^2 p = r \mathrm{d}r \mathrm{d}\omega$, we evaluate the effective action from (3.7) to be (we return back to the Lorentzian regime)

$$
\begin{aligned}
S_{\text{eff}} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \Bigg[& \left(1 - \frac{\hat{g}_{12}}{k} \ln \mu^2 \right) \left(\partial_+ x_1 \partial_- x_1 + \partial_+ x_2 \partial_- x_2 \right) \\
& + \frac{\hat{g}_{12}}{k} x_1 x_2 \left(1 + \frac{\hat{g}_{12}}{k} \ln \mu^2 \right) \left(\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1 \right) \\
& - \frac{\ln \mu^2}{k^2} \Big[\left(3\hat{h}_{12} x_1^2 + (3\hat{\bar{h}}_{12} - \hat{g}_{12}^2) x_2^2 \right) \partial_+ x_1 \partial_- x_1 \\
& + \left(3\hat{\bar{h}}_{12} x_2^2 + (3\hat{h}_{12} - \hat{g}_{12}^2) x_1^2 \right) \partial_+ x_2 \partial_- x_2 \Big] \Bigg] + \cdots .
\end{aligned}
\tag{3.10}
$$

The wavefunction renormalization and field redefinition

$$
\begin{aligned}
x_1 &= \left(1 + \frac{\hat{g}_{12}}{2k} \ln \mu^2 \right) \left(1 + \frac{\hat{h}_{12} \hat{x}_1^2 + (3\hat{\bar{h}}_{12} - \hat{g}_{12}^2) \hat{x}_2^2}{2k^2} \ln \mu^2 \right) \hat{x}_1 , \\
x_2 &= \left(1 + \frac{\hat{g}_{12}}{2k} \ln \mu^2 \right) \left(1 + \frac{\hat{\bar{h}}_{12} \hat{x}_2^2 + (3\hat{h}_{12} - \hat{g}_{12}^2) \hat{x}_1^2}{2k^2} \ln \mu^2 \right) \hat{x}_2 ,
\end{aligned}
\tag{3.11}
$$

puts the kinetic term into a canonical form and (3.10) becomes

$$
\begin{aligned}
S_{\text{eff}} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \Bigg[& \partial_+ \hat{x}_1 \partial_- \hat{x}_1 + \partial_+ \hat{x}_2 \partial_- \hat{x}_2 \\
& + \frac{1}{k} \left(\hat{g}_{12} + \frac{\hat{g}_{12}^2 + 3\hat{h}_{12} + 3\hat{\bar{h}}_{12}}{k} \ln \mu^2 \right) \hat{x}_1 \hat{x}_2 \left(\partial_+ \hat{x}_1 \partial_- \hat{x}_2 + \partial_+ \hat{x}_2 \partial_- \hat{x}_1 \right) \Bigg] + \cdots .
\end{aligned}
\tag{3.12}
$$

Demanding now that the action (3.10) is μ-independent, i.e. $\frac{\mathrm{d}\mathcal{L}_{\text{eff}}}{\mathrm{d}\ln \mu^2} = 0$, and keeping in mind that we should keep for consistency the leading term in the $\frac{1}{k}$ expansion we obtain

$$
\frac{\mathrm{d}\hat{g}_{12}}{\mathrm{d}\ln \mu^2} = -\frac{1}{k} \left(\hat{g}_{12}^2 + 3\hat{h}_{12} + 3\hat{\bar{h}}_{12} \right) .
\tag{3.13}
$$

Using the specific expression for the couplings (2.17) we finally get that

$$
\beta^\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln \mu^2} = -\frac{\lambda}{k} .
\tag{3.14}
$$

We have, thus, reproduced the result of [14].

4 Parafermion correlators

The objects naturally arising in the $SU(2)/U(1)$ coset CFT are the parafermions [27]. These will be denoted by $\Psi_\pm$ and $\bar{\Psi}_\pm$ for the holomorphic and anti-holomorphic sectors, respectively. In this section, we will calculate correlation functions, namely two and four-point functions, for the λ-deformed versions of the CFT parafermions. At the CFT point

all correlation functions with an odd number of parafermions vanish. We will show that the same statement is true for the deformed versions of the correlators, as well. The first question we address in subsection 4.1 is how the deformation dresses the chirally conserved SU(2)/U(1) CFT parafermions which have a classical form in terms of the fields θ and ϕ [30] . Equipped with their correct form we proceed to calculate in subsection 4.2 the anomalous dimension of the bilinear in the parafermions operator that perturbs the theory away from the free point. Our result is in complete agreement with the one obtained earlier in [28] by the use of methods involving the geometry in the coupling space. In the next two subsections, we calculate the anomalous dimension of the single λ-dressed parafermion and all four-point correlation functions involving parafermionic fields. We will develop methods to deal with Wilson-line factors present in the expressions for the parafermion fields. All our results will respect the symmetries (2.8) of the action.

4.1 λ-dressed parafermionic fields

In the σ-model (2.1), derived in [1] throughout a certain gauging procedure, the classical equations of motion for the gauge fields assume the following form

$$A_+ = i(\lambda^{-T} - D)^{-1}R_+ , \quad A_- = -i(\lambda^{-T} - D^T)^{-1}L_- . \tag{4.1}$$

As discussed in [19, 22], these λ-dependent fields are the counterparts of the chiral and anti-chiral currents of the conformal point to which they reduce, up to overall scales, after taking the limit $\lambda \to 0$. Hence they provide the correct form of the operator generalizing the chiral and anti-chiral currents in the presence of λ. The above form was essential for obtaining the correct anomalous dimensions and correlation functions of currents in the free field based approach of [26] and this will be the case here, as well.

The aim of this work is to apply the approach of [26] to the case where the deformed theory is not a deformation of a group based CFT but of a coset CFT, and more specifically of the SU(2)/U(1) coset CFT. In this case the natural chiral objects of the CFT are not currents but parafermions. As it happens for the currents in the group case, the parafermions are dressed when the deformation parameter λ is turned on. The way the parafermions are dressed should be derived from the same limiting procedure we used to obtain the action. In order to proceed, recall that we have parametrized the group element of SU(2) as in (2.5) and that we have chosen the deformation matrix to take the form $\lambda_{ab} = \text{diag}(\lambda, \lambda, \lambda_3)$, where λ_3 corresponds to the Abelian subgroup U(1) in SU(2). In order to obtain the σ-model action (2.7) one should set the parameter $\lambda_3 = 1$. In this procedure the Euler angle ϕ_1 appearing in the group element (2.5) drops out of the σ-model which instead of being three-dimensional becomes two-dimensional (2.7). We should follow the same limiting procedure for the gauge fields (4.1) as well. In doing so, we firstly and most importantly realize that the gauge fields $A_\pm^a$, $a = 1, 2, 3$ retain an explicit dependence on the angle ϕ_1 which therefore at that level does not decouple. In fact, we will see that this is a desired feature for the classical description of the parafermions even at the CFT point. In particular, we find that, after passing to the Euclidean regime, the gauge fields

projected, via the above limiting procedure, to the coset take the form

$$A_+^1 = \frac{\lambda}{\sqrt{2}}\Big(\Psi_+ - \Psi_-\Big)\,, \qquad A_+^2 = -i\frac{\lambda}{\sqrt{2}}\Big(\Psi_+ + \Psi_-\Big)\,,$$
$$A_-^1 = -\frac{\lambda}{\sqrt{2}}\Big(\bar{\Psi}_+ - \bar{\Psi}_-\Big)\,, \qquad A_-^2 = i\frac{\lambda}{\sqrt{2}}\Big(\bar{\Psi}_+ + \bar{\Psi}_-\Big)\,, \tag{4.2}$$

where the Ψ's, as we will see, will be the λ-deformed parafermions. They are given in terms of the angles parametrizing the group element g, as follows after the analytic continuation to the Euclidean regime

$$\Psi_+ = \frac{1}{1-\lambda^2}\left((\mathrm{e}^{-i\phi} + \lambda\mathrm{e}^{i\phi})\partial\theta - i(\mathrm{e}^{-i\phi} - \lambda\mathrm{e}^{i\phi})\tan\theta\partial\phi\right)\mathrm{e}^{-i\phi_1}\,,$$
$$\Psi_- = \frac{1}{1-\lambda^2}\left((\mathrm{e}^{i\phi} + \lambda\mathrm{e}^{-i\phi})\partial\theta + i(\mathrm{e}^{i\phi} - \lambda\mathrm{e}^{-i\phi})\tan\theta\partial\phi\right)\mathrm{e}^{i\phi_1}\,,$$
$$\bar{\Psi}_+ = \frac{1}{1-\lambda^2}\left((\mathrm{e}^{i\phi} + \lambda\mathrm{e}^{-i\phi})\bar{\partial}\theta + i(\mathrm{e}^{i\phi} - \lambda\mathrm{e}^{-i\phi})\tan\theta\bar{\partial}\phi\right)\mathrm{e}^{-i\phi_1}\,,$$
$$\bar{\Psi}_- = \frac{1}{1-\lambda^2}\left((\mathrm{e}^{-i\phi} + \lambda\mathrm{e}^{i\phi})\bar{\partial}\theta - i(\mathrm{e}^{-i\phi} - \lambda\mathrm{e}^{i\phi})\tan\theta\bar{\partial}\phi\right)\mathrm{e}^{i\phi_1}\,. \tag{4.3}$$

The components of the gauge fields along the subgroup U(1) turn out to be

$$A_+^3 = \frac{i}{\sqrt{2}}\left(\partial\phi_1 + J\right)\,, \qquad A_-^3 = \frac{i}{\sqrt{2}}\left(\bar{\partial}\phi_1 - \bar{J}\right)\,, \tag{4.4}$$

where the angle ϕ_1 has become imaginary after the analytic continuation to the Euclidean regime and J and $\bar{J}$ are given by

$$J = \frac{\left(\left(1 - 2\lambda\cos\phi + \lambda^2\right)\tan\theta\,\partial\phi\ - 2\lambda\sin 2\phi\,\partial\theta\right)\tan\theta}{1-\lambda^2}\,,$$
$$\bar{J} = \frac{\left(\left(1 - 2\lambda\cos\phi + \lambda^2\right)\tan\theta\,\bar{\partial}\phi - 2\lambda\sin 2\phi\,\bar{\partial}\theta\right)\tan\theta}{1-\lambda^2}\,. \tag{4.5}$$

Under the non-perturbative symmetries (2.8) the Ψ's transform as

$$\text{I}: \qquad \Psi_\pm \to -\lambda\mathrm{e}^{\mp 2i\phi_1}\Psi_\mp\,, \qquad \bar{\Psi}_\pm \to -\lambda\mathrm{e}^{\mp 2i\phi_1}\bar{\Psi}_\mp\,,$$
$$\text{II}: \qquad \Psi_\pm \to \mp i\,\Psi_\pm\,, \qquad \bar{\Psi}_\pm \to \pm i\,\bar{\Psi}_\pm\,. \tag{4.6}$$

In addition, the gauge fields satisfy the following equations of motion [2]

$$\partial A_-^{g/h} = -[A_-^{g/h}, A_+^h]\,, \qquad \bar{\partial}A_+^{g/h} = -[A_+^{g/h}, A_-^h]\,,$$
$$\partial A_-^h - \bar{\partial}A_+^h = \lambda^{-1}[A_+^{g/h}, A_-^{g/h}]\,,$$
$$A_\pm^h = A_\pm^3 t_3\,, \qquad A_\pm^{g/h} = A_\pm^\alpha t_\alpha\,, \qquad \alpha = 1, 2\,. \tag{4.7}$$

In order to elucidate the expressions for the Ψ's we firstly consider the conformal limit $\lambda = 0$. Then, these become the standard classical parafermions of the coset SU(2)/U(1) CFT that are given by [30]

$$\Psi_\pm = (\partial\theta \mp i\tan\theta\partial\phi)\,\mathrm{e}^{\mp i(\phi+\phi_1)}\,, \quad \bar{\Psi}_\pm = \left(\bar{\partial}\theta \pm i\tan\theta\bar{\partial}\phi\right)\mathrm{e}^{\pm i(\phi-\phi_1)}\,. \tag{4.8}$$

Using (4.2) and (4.7), we find that the gauge fixing conditions $A_\pm^3 = 0$ imply that the parafermions $\Psi_\pm$ and $\bar\Psi_\pm$ are on-shell chirally and anti-chirally conserved, respectively[1]

$$\bar\partial\Psi_\pm = 0, \qquad \partial\bar\Psi_\pm = 0. \tag{4.9}$$

In addition, the constraints $A_\pm^3 = 0$ enforces ϕ_1 to satisfy the following equations

$$\begin{aligned}\partial\phi_1 &= -J_0, & \bar\partial\phi_1 &= \bar J_0, \\ J_0 &= \tan^2\theta\,\partial\phi, & \bar J_0 &= \tan^2\theta\,\bar\partial\phi.\end{aligned} \tag{4.10}$$

Note that J_0 and $\bar J_0$ can be obtained from (4.5) after setting $\lambda = 0$. Moreover, using (4.10) one can show that ϕ_1 satisfies on-shell the compatibility condition

$$\left(\partial\bar\partial - \bar\partial\partial\right)\phi_1 = \partial\bar J_0 + \bar\partial J_0 = 0. \tag{4.11}$$

The above conservation law results from the U(1) isometry of the SU(2)/U(1) coset CFT. Given (4.11), one can solve (4.10) to express the angle ϕ_1 as a line integral[2]

$$\phi_1(z, z_0) = \phi_1(z) - \phi_1(z_0) = \int_C \left(\partial\phi_1\mathrm{d}z + \bar\partial\phi_1\mathrm{d}\bar z\right) = \int_C \left(-J_0\mathrm{d}z + \bar J_0\mathrm{d}\bar z\right), \tag{4.12}$$

where C is a curve connecting the arbitrary base point $(z_0, \bar z_0)$ with the end point $(z, \bar z)$. Note that, given the above, the phase ϕ_1 is independent from the choice of C but depends only the base and end points. However, correlation functions should not depend on the arbitrary base point. In the following sections, we will see that this is indeed the case.

However, in the case of non-zero deformation $\lambda \neq 0$ the conditions $A_\pm^3 = 0$ can no longer be imposed, as can be seen from the second equation in (4.7). Nevertheless, one can impose an alternative gauge fixing condition, namely

$$A_+^3 = -\frac{i}{\sqrt{2}}F, \quad A_-^3 = \frac{i}{\sqrt{2}}\bar F, \tag{4.13}$$

where F is at the moment an arbitrary function which will be later specified and $\bar F$ is its complex conjugate. Equivalently, using (4.4), we find

$$\partial\phi_1 = -J_F, \quad \bar\partial\phi_1 = \bar J_F, \tag{4.14}$$

with $J_F = J + F$. The equations (4.14) uniquely determines ϕ_1 on-shell provided that the following consistency condition is satisfied

$$\left(\partial\bar\partial - \bar\partial\partial\right)\phi_1 = \partial\bar J_F + \bar\partial J_F = 0. \tag{4.15}$$

---

[1]The second of (4.7) seems singular in the $\lambda = 0$ limit. However, it is also satisfied since $A_\pm^{g/h} \sim \lambda$, so that $\lambda^{-1}[A_+^{g/h}, A_-^{g/h}] \sim \lambda$, as $\lambda \to 0$.

[2]In a closed curve it would identically vanish due to Stokes theorem and (4.11)

$$\frac{i}{2}\oint_C \left(-\mathrm{d}z\,J_0 + \mathrm{d}\bar z\,\bar J_0\right) = \int_S \mathrm{d}^2z \left(\partial\bar J_0 + \bar\partial J_0\right) = 0, \quad C = \partial S.$$

Equivalently, the last equation implies that F and $\bar{F}$ should satisfy the relation

$$\partial\bar{F} + \bar{\partial}F = \frac{2\lambda}{1-\lambda^2}\left((\partial\theta\bar{\partial}\theta - \partial\phi\bar{\partial}\phi\tan^2\theta)\sin 2\phi + (\partial\theta\bar{\partial}\phi + \partial\phi\bar{\partial}\theta)\tan\theta\cos 2\phi\right). \quad (4.16)$$

Similarly to (4.12), we can solve (4.14) and (4.15) and express ϕ_1 as a line integral through

$$\phi_1(z, z_0) = \phi_1(z) - \phi_1(z_0) = \int_C \left(-J_F\mathrm{d}z + \bar{J}_F\mathrm{d}\bar{z}\right). \quad (4.17)$$

In what follows, we shall expand the parafermions (4.3), as well as the phase (4.14), for $k \gg 1$ and keep terms up to order $\frac{1}{k}$. This is a straightforward calculation which can be summarized in the following steps. Firstly, we zoom θ around zero as in (2.9). Then we express the variables (ρ, ϕ) in terms of (y_1, y_2) by using (2.10). Subsequently, we perform the field redefinition of (2.14) keeping terms up to order $\frac{1}{k}$. Finally, we rewrite all expressions in terms of chiral coordinates, namely $x_\pm = x_1 \pm ix_2$. The end result is given by the following expressions (we dismiss an overall factor of $\frac{1}{\sqrt{2k(1-\lambda^2)}}$)

$$\Psi_+ = \left(\partial x_- + \frac{F_+}{8k(1-\lambda^2)}\right)\mathrm{e}^{-i\phi_1}, \qquad \Psi_- = \left(\partial x_+ - \frac{F_-}{8k(1-\lambda^2)}\right)\mathrm{e}^{i\phi_1},$$
$$\bar{\Psi}_+ = \left(\bar{\partial}x_+ - \frac{\bar{F}_+}{8k(1-\lambda^2)}\right)\mathrm{e}^{-i\phi_1}, \qquad \bar{\Psi}_- = \left(\bar{\partial}x_- + \frac{\bar{F}_-}{8k(1-\lambda^2)}\right)\mathrm{e}^{i\phi_1}, \quad (4.18)$$

where

$$\begin{aligned}
F_+ &= (x_+^2 - x_-^2)\left((1+\lambda^2)\partial x_+ - 2\lambda\partial x_-\right), \\
F_- &= (x_+^2 - x_-^2)\left((1+\lambda^2)\partial x_- - 2\lambda\partial x_+\right), \\
\bar{F}_+ &= (x_+^2 - x_-^2)\left((1+\lambda^2)\bar{\partial}x_- - 2\lambda\bar{\partial}x_+\right), \\
\bar{F}_- &= (x_+^2 - x_-^2)\left((1+\lambda^2)\bar{\partial}x_+ - 2\lambda\bar{\partial}x_-\right).
\end{aligned} \quad (4.19)$$

Furthermore, the currents (4.5) have the following large k expansion

$$J = \frac{i}{4k(1-\lambda^2)}\left(((1+\lambda^2)x_+ - 2\lambda x_-)\partial x_- - ((1+\lambda^2)x_- - 2\lambda x_+)\partial x_+\right),$$
$$\bar{J} = \frac{i}{4k(1-\lambda^2)}\left(((1+\lambda^2)x_+ - 2\lambda x_-)\bar{\partial}x_- - ((1+\lambda^2)x_- - 2\lambda x_+)\bar{\partial}x_+\right). \quad (4.20)$$

Note that the condition (4.14) determines ϕ_1 provided that F satisfies on-shell the condition

$$\partial\bar{F} + \bar{\partial}F = -\frac{i\lambda}{k(1-\lambda^2)}(\partial x_+\bar{\partial}x_+ - \partial x_-\bar{\partial}x_-). \quad (4.21)$$

Using the equation of motion (3.2), we find that to $\mathcal{O}(\frac{1}{k})$ (4.21) is solved by[3]

$$F = -\frac{i\lambda}{2k(1-\lambda^2)}(x_+\partial x_+ - x_-\partial x_-), \quad (4.22)$$

---

[3]To $\mathcal{O}(\frac{1}{k})$ the equations of motion (3.2) read $\partial\bar{\partial}x_\pm = 0$.

Employing the latter and (4.20), we find that

$$J_F = \frac{i}{4k} \frac{1 + \lambda^2}{1 - \lambda^2} \left(x_+ \partial x_- - x_- \partial x_+ \right) ,$$

$$\bar{J}_F = \frac{i}{4k} \frac{1 + \lambda^2}{1 - \lambda^2} \left(x_+ \bar{\partial} x_- - x_- \bar{\partial} x_+ \right) ,$$

(4.23)

which generate ϕ_1 through (4.17). Let us note that the parafermions in (4.18), with the phase factor ϕ_1 of (4.17) with (4.23), are not on-shell chirally and anti-chirally conserved for $\lambda \neq 0$, since

$$\bar{\partial} \Psi_\pm = \mp i \, \partial x_\mp \, \bar{F} \, \mathrm{e}^{\mp i\phi_1} , \quad \partial \bar{\Psi}_\pm = \pm i \, \bar{\partial} x_\pm \, F \, \mathrm{e}^{\mp i\phi_1} . \tag{4.24}$$

We close this subsection by noticing that in this set of variables the two symmetries of (2.8) are mapped respectively to

$$
\begin{aligned}
\text{I}: \quad & \lambda \to \lambda^{-1} , \quad && k \to -k , \\
\text{II}: \quad & \lambda \to -\lambda , \quad && x_\pm \to \pm i x_\pm ,
\end{aligned}
\tag{4.25}
$$

under which the parafermions (4.18) transform as

$$
\begin{aligned}
\text{I}: \quad & \Psi_\pm \to \Psi_\pm , \quad && \bar{\Psi}_\pm \to \bar{\Psi}_\pm , \\
\text{II}: \quad & \Psi_\pm \to \mp i \, \Psi_\pm , \quad && \bar{\Psi}_\pm \to \pm i \, \bar{\Psi}_\pm ,
\end{aligned}
\tag{4.26}
$$

while (4.20) and (4.23) remain intact. In the computations which follow we shall make use of (4.18), with the phase factor ϕ_1 of (4.17), (4.23), keeping terms up to order $\mathcal{O}(\frac{1}{k})$.

4.2 The anomalous dimension of parafermion bilinear

The aim of this subsection is to calculate the anomalous dimension of the operator that perturbs the SU(2)/U(1) CFT. This operator is bilinear in the parafermions and its form is given by the classical parafermions bilinear [1]

$$\mathcal{O} = \Psi_+ \bar{\Psi}_- + \Psi_- \bar{\Psi}_+ , \tag{4.27}$$

where we note that the phase factors cancel, as one may readily verify using (4.8).

Taking the large k-limit we obtain that the leading term of $\mathcal{O}$ is given by $2(\partial x_1 \bar{\partial} x_1 - \partial x_2 \bar{\partial} x_2)$. Had we kept the leading correction, a bilinear in the parafermions operator would have been of the generic form (we ignore the irrelevant factor of 2)

$$\mathcal{O} = \partial x_1 \bar{\partial} x_1 - \partial x_2 \bar{\partial} x_2 + \frac{1}{k} \sum_{i=1}^{4} c_i \mathcal{O}_i + \dots , \tag{4.28}$$

for some coefficients c_i and a basis of operators with engineering dimension $(1, 1)$ $\mathcal{O}_i$.[4] However, notice that the overlap of the leading and subleading term in (4.28) is zero. This

---

[4]Such a basis is

$$\mathcal{O}_1 = x_1 x_2 \partial x_1 \bar{\partial} x_2 , \quad \mathcal{O}_2 = x_1 x_2 \partial x_2 \bar{\partial} x_1 , \quad \mathcal{O}_3 = \frac{1}{2} x_1^2 \partial x_2 \bar{\partial} x_2 , \quad \mathcal{O}_4 = \frac{1}{2} x_2^2 \partial x_1 \bar{\partial} x_1 .$$

means that one can safely ignore the subleading term in (4.28), at least up to $\mathcal{O}(\frac{1}{k})$ which is the order we are working in the present paper.

In order to proceed, we will also need the interaction terms of the action (2.16) up to order $\mathcal{O}(\frac{1}{k})$. There is a single such term which in the Euclidean regime reads

$$
\begin{aligned}
S_{\text{int}} &= \frac{\hat{g}_{12}}{2\pi k} \int \mathrm{d}^2 z \, x_1 x_2 (\partial x_1 \bar{\partial} x_2 + \partial x_2 \bar{\partial} x_1) \\
&= -\frac{\hat{g}_{12}}{16\pi k} \int \mathrm{d}^2 z \, (x_+^2 - x_-^2)(\partial x_+ \bar{\partial} x_+ - \partial x_- \bar{\partial} x_-) \, .
\end{aligned}
\tag{4.29}
$$

The two-point function then reads

$$
\begin{aligned}
\langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle &= \frac{2}{|z_{12}|^4} - \frac{\hat{g}_{12}}{2\pi k} \int \mathrm{d}^2 z \, \langle \left(\partial x_1(z_1) \bar{\partial} x_1(\bar{z}_1) - \partial x_2(z_1) \bar{\partial} x_2(\bar{z}_1) \right) \\
&\qquad \times \left(\partial x_1(z_2) \bar{\partial} x_1(z_2) - \partial x_2(z_2) \bar{\partial} x_2(\bar{z}_2) \right) \\
&\qquad \times \left(x_1 x_2 (\partial x_1 \bar{\partial} x_2 + \partial x_2 \bar{\partial} x_1) \right)(z, \bar{z}) \rangle \\
&= \frac{2}{|z_{12}|^4} + 2 \frac{\hat{g}_{12}}{2\pi k} (I_1 + I_2) \, ,
\end{aligned}
\tag{4.30}
$$

where the minus sign in the second term above, corresponding to a single insertion of the interaction term, is due to the fact that in the Euclidean regime we have the term $e^{-S_{\text{int}}}$ in the correlators. By inspecting (4.30) it is straightforward to see that we need two kinds of contractions, namely

$$
\begin{aligned}
I_1 &= \int \mathrm{d}^2 z \, \langle \partial x_1(z_1) \bar{\partial} x_1(\bar{z}_1) \partial x_1(z) x_1(z, \bar{z}) \rangle \langle \partial x_2(z_2) \bar{\partial} x_2(\bar{z}_2) \bar{\partial} x_2(\bar{z}) x_2(z, \bar{z}) \rangle \\
&= \int \frac{\mathrm{d}^2 z}{(z - z_1)^2 (z - z_2)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)^2}
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
I_2 &= \int \mathrm{d}^2 z \, \langle \partial x_1(z_1) \bar{\partial} x_1(\bar{z}_1) \bar{\partial} x_1(\bar{z}) x_1(z, \bar{z}) \rangle \langle \partial x_2(z_2) \bar{\partial} x_2(\bar{z}_2) \partial x_2(z) x_2(z, \bar{z}) \rangle \\
&= \int \frac{\mathrm{d}^2 z}{(z - z_1)(z - z_2)^2 (\bar{z} - \bar{z}_1)^2 (\bar{z} - \bar{z}_2)} \, .
\end{aligned}
\tag{4.32}
$$

Using (B.4) twice we find that

$$
\langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle = \frac{2}{|z_{12}|^4} \left(1 + \frac{2\hat{g}_{12}}{k} \left(1 + \ln \frac{\varepsilon^2}{|z_{12}|^2} \right) \right) \, ,
\tag{4.33}
$$

from which we read the anomalous dimension of the parafermion bilinear to be

$$
\gamma^{\mathcal{O}} = \frac{2\hat{g}_{12}}{k} = -\frac{2}{k} \frac{1 + \lambda^2}{1 - \lambda^2} \, .
\tag{4.34}
$$

This expression is in perfect agreement with equation (4.16) of [28], where the anomalous dimension of the perturbing operator was found by using the geometry in the space

of couplings. Notice that in the conformal point the anomalous dimension of the composite operator is twice the anomalous dimension of the holomorphic (anti-holomorphic) parafermion which equals $-\frac{1}{k}$ [27].

The reader may wonder if there are other operators with the same engineering dimension as $\mathcal{O}$ which may mix with it. The operators of such equal engineering dimension are

$$\tilde{\mathcal{O}} = \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + \mathcal{O}\left(\frac{1}{k}\right), \quad \tilde{\mathcal{O}}_\pm = \partial x_1 \bar{\partial} x_2 \pm \partial x_2 \bar{\partial} x_1 + \mathcal{O}\left(\frac{1}{k}\right), \qquad (4.35)$$

where the corrections are of same form with those for $\mathcal{O}$ in (4.28). It turns out that these operator does not mix with $\mathcal{O}$ and among themselves at the free field point and also at $\mathcal{O}(\frac{1}{k})$. Moreover, one may easily show that $\tilde{\mathcal{O}}$ has the opposite anomalous dimension as that in (4.34). The anomalous dimension of $\tilde{\mathcal{O}}_\pm$ is computed for completeness in appendix A.

4.3 Anomalous dimension of the single parafermion

The goal of this subsection is to compute, using the free field expansion, the two-point functions of the parafermions to order $\frac{1}{k}$ from which one can read the anomalous dimension of the deformed parafermion.

We first consider the correlator $\langle \Psi_+ \Psi_+ \rangle$. This vanishes at the conformal point since it is not neutral. Employing the expressions above we find that

$$\langle \Psi_+(z_1, \bar{z}_1) \Psi_+(z_2, \bar{z}_2) \rangle$$
$$= + \frac{\hat{g}_{12}}{16\pi k} \int d^2 z \langle \partial x_-(z_1) \partial x_-(z_2) \left((x_+^2 - x_-^2)(\partial x_+ \bar{\partial} x_+ - \partial x_- \bar{\partial} x_-) \right)(z, \bar{z}) \rangle$$
$$+ \frac{1}{8k(1-\lambda^2)} \left(\langle F_+(z_1, \bar{z}_1) \partial x_-(z_2) \rangle + \langle \partial x_-(z_1) F_+(z_2, \bar{z}_2) \rangle \right) \qquad (4.36)$$
$$- i \langle \partial x_-(z_1) \partial x_-(z_2) \phi_1(z_1, z_0) \rangle - i \langle \partial x_-(z_1) \partial x_-(z_2) \phi_1(z_2, z_0) \rangle = 0,$$

where for the phase ϕ_1 we will use the leading $\mathcal{O}(\frac{1}{k})$ of (4.17) with (4.23) so that the corresponding terms in (4.36) are indeed of $\mathcal{O}(\frac{1}{k})$. The second of (4.36) involves the interaction term of the action and vanishes due to the fact that the interaction term is normal ordered. The third and fourth line of (4.36) originate from the $\mathcal{O}(\frac{1}{k})$ corrections to the parafermion operators and from the phase ϕ_1, respectively. They both vanish either because in the process one necessarily encounters propagators of the form $\langle x_+ x_+ \rangle$ or $\langle x_- x_- \rangle$ which are identically zero (terms in the fourth line) or because of normal ordering (terms in the third line).

We now turn to the neutral two point function $\langle \Psi_+ \Psi_- \rangle$. To $\mathcal{O}(\frac{1}{k})$ this correlator equals to

$$\langle \Psi_+(z_1, \bar{z}_1) \Psi_-(z_2, \bar{z}_2) \rangle$$
$$= - \frac{2}{z_{12}^2} + \frac{1}{8k(1-\lambda^2)} \left(\langle F_+(z_1, \bar{z}_1) \partial x_+(z_2) \rangle - \langle \partial x_-(z_1) F_-(z_2, \bar{z}_2) \rangle \right)$$
$$+ \frac{\hat{g}_{12}}{16\pi k} \int d^2 z \langle \partial x_-(z_1) \partial x_+(z_2) \left((x_+^2 - x_-^2)(\partial x_+ \bar{\partial} x_+ - \partial x_- \bar{\partial} x_-) \right)(z, \bar{z}) \rangle \qquad (4.37)$$
$$- i \langle \partial x_-(z_1) \partial x_+(z_2) \phi_1(z_1, z_0) \rangle + i \langle \partial x_-(z_1) \partial x_+(z_2) \phi_1(z_2, z_0) \rangle.$$

In the above expression, apart from the free part, only the last two terms related to the phase factor in the parafermions are non-vanishing and equal to

$$\frac{z_{12}}{k} \frac{1+\lambda^2}{1-\lambda^2} \left(-\int_{z_0}^{z_1} dz + \int_{z_0}^{z_2} dz \right) \frac{1}{(z-z_1)^2 (z-z_2)^2}$$
$$= \frac{1}{z_{12}^2} \frac{2}{k} \frac{1+\lambda^2}{1-\lambda^2} \left(1 + \ln\frac{\varepsilon^2}{z_{12}^2} \right) + \frac{2}{k} \frac{1+\lambda^2}{1-\lambda^2} \frac{1}{\varepsilon z_{12}} \,. \tag{4.38}$$

All other terms in (4.38) vanish due to the fact that both the interaction term and $F_\pm$ are normal ordered. In evaluating the line integrals of (4.38) we have introduced a small distance cut-off ε so that the integration point never coincides with the end points of the integration. Therefore, we find that

$$\langle \Psi_+(z_1,\bar{z}_1) \Psi_-(z_2,\bar{z}_2) \rangle = -\frac{2}{z_{12}^2} \left(1 - \frac{1}{k} \frac{1+\lambda^2}{1-\lambda^2} \left(1 + \ln\frac{\varepsilon^2}{z_{12}^2} \right) \right) + \frac{2}{\varepsilon k} \frac{1+\lambda^2}{1-\lambda^2} \frac{1}{z_{12}} \,. \tag{4.39}$$

The $\frac{1}{\varepsilon}$ pole can be absorbed by a field redefinition of the $\Psi_\pm$'s, namely $\Psi_\pm \to \Psi_\pm + \delta\Psi_\pm$, where

$$\delta\Psi_\pm = -\frac{f_\pm}{\varepsilon k} x_\mp \,, \quad f_+ - f_- = \frac{1+\lambda^2}{1-\lambda^2} \,. \tag{4.40}$$

Note that we do not need to specify both parameters $f_\pm$, just their difference as above and in addition this relation is invariant under the symmetries (4.26). At $\lambda = 0$, the correlator (4.38) is in agreement with the CFT result to leading order in $\frac{1}{k}$ after one makes the following rescaling

$$\Psi = i\sqrt{2} \left(1 - \frac{1}{2k} \right) \Psi_{\rm CFT} \,. \tag{4.41}$$

Finally, from (4.39) one can read the anomalous dimension of Ψ which is given by

$$\gamma_\Psi = -\frac{1}{k} \frac{1+\lambda^2}{1-\lambda^2} \,. \tag{4.42}$$

Notice that γ_Ψ is half the anomalous dimension of the composite operator (4.34). At the conformal point it is indeed, the deviation from unity of the holomorphic dimension (equal to the anti-holomorphic one) of the parafermion which, as already noted, equals $1 - \frac{1}{k}$. For completeness, we note that the conjugate correlator $\langle \bar{\Psi}_+ \bar{\Psi}_- \rangle$ takes the form (4.39) as well.

4.4 Four-point correlation functions

In this section, we calculate four-point correlators of parafermions. To begin with, note that at the CFT point correlation functions with an odd number of parafermions vanish. In the following we will argue that the same is true for the deformed model as well. Indeed, to $\mathcal{O}(\frac{1}{k})$ the number of fields involved in the correlator is odd since the phase may contribute two and the insertion from the interacting part of the action four fields. Any correlator with an odd number of free fields vanishes. This argument should hold to all orders in the large k expansion since every extra power of $\frac{1}{k}$ contributes two more fields in the correlators.

At the CFT point correlators with odd number of holomorphic (or antiholomorphic) parafermions also vanish identically due to charge conservation. However, CFT point

correlators with an even number of holomorphic (or antiholomorphic) parafermions and neutral do not vanish. Away from the CFT point correlators with an even number of fields receive corrections. Using conformal perturbation one can easily convince oneself that the corrections come to even order in the λ expansion.

4.4.1 Correlation function $\langle\Psi_+\Psi_-\Psi_+\Psi_-\rangle$

Consider, as our first example, the four-point function

$$
\begin{aligned}
\langle\Psi_+&(z_1,\bar{z}_1)\Psi_-(z_2,\bar{z}_2)\Psi_+(z_3,\bar{z}_3)\Psi_-(z_4,\bar{z}_4)\rangle \\
&= \frac{4}{z_{12}^2 z_{34}^2} + \frac{4}{z_{14}^2 z_{23}^2} \\
&\quad + \frac{1}{8k(1-\lambda^2)}\Big(\langle F_+(z_1,\bar{z}_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\qquad\qquad\qquad + \langle F_+(z_3,\bar{z}_3)\partial x_+(z_2)\partial x_-(z_1)\partial x_+(z_4)\rangle \\
&\qquad\qquad\qquad - \langle F_-(z_2,\bar{z}_2)\partial x_-(z_1)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\qquad\qquad\qquad - \langle F_-(z_4,\bar{z}_4)\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\rangle\Big) \\
&\quad - i\langle\phi_1(z_1,z_0)\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\quad - i\langle\phi_1(z_3,z_0)\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\quad + i\langle\phi_1(z_2,z_0)\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\quad + i\langle\phi_1(z_4,z_0)\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4)\rangle \\
&\quad + \frac{\hat{g}_{12}}{16\pi k}\int d^2z\langle\partial x_-(z_1)\partial x_+(z_2)\partial x_-(z_3)\partial x_+(z_4) \\
&\qquad\qquad\qquad \times\big((x_+^2 - x_-^2)(\partial x_+\bar{\partial}x_+ - \partial x_-\bar{\partial}x_-)\big)(z,\bar{z})\rangle .
\end{aligned}
\tag{4.43}
$$

To calculate this is a rather tedious but straightforward task. To start, let us briefly describe the various contributions. The lines 3–6 of (4.43), originate from the $\frac{1}{k}$ corrections of Ψ's (4.19) and involve only free contractions, straightforwardly yielding

$$
\frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{1}{z_{14}z_{34}z_{24}^2} + \frac{1}{z_{12}z_{14}z_{13}^2} - \frac{1}{z_{12}z_{23}z_{24}^2} - \frac{1}{z_{23}z_{24}z_{12}^2}\right) .
\tag{4.44}
$$

The lines 7–10, which originate from the phase ϕ_1, give after performing ordinary integrations over z the following contribution

$$
\begin{aligned}
-\frac{8}{k}\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{14}^2 z_{23}^2}\right) &+ \frac{8}{k}\frac{1+\lambda^2}{1-\lambda^2}\frac{1}{z_{12}z_{23}z_{14}z_{34}} \\
&+ \frac{8}{k}\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{14}^2 z_{23}^2}\right)\ln\frac{z_{12}z_{23}z_{14}z_{34}}{z_{13}z_{24}} .
\end{aligned}
\tag{4.45}
$$

Finally, from the last two lines we find

$$
\frac{2\hat{g}_{12}}{k}\left(\frac{1}{z_{14}z_{34}z_{24}^2} + \frac{1}{z_{12}z_{14}z_{13}^2} - \frac{1}{z_{12}z_{23}z_{24}^2} - \frac{1}{z_{23}z_{24}z_{12}^2}\right) .
\tag{4.46}
$$

This result is easily obtained since the integration over z is immediately performed since the free contractions necessarily give contact terms of the form $\delta^{(2)}(z-z_i)$.

Adding the various contributions we finally find

$$
\begin{aligned}
\langle \Psi_+(z_1,\bar{z}_1)&\Psi_-(z_2,\bar{z}_2)\Psi_+(z_3,\bar{z}_3)\Psi_-(z_4,\bar{z}_4)\rangle \\
&= 4\frac{1-\frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2}}{z_{12}^2 z_{34}^2} + 4\frac{1-\frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2}}{z_{14}^2 z_{23}^2} \\
&\quad + \frac{8}{k}\frac{1+\lambda^2}{1-\lambda^2}\frac{1}{z_{12}z_{23}z_{14}z_{34}} + \frac{8}{k}\frac{1+\lambda^2}{1-\lambda^2}\left(\frac{1}{z_{12}^2 z_{34}^2}+\frac{1}{z_{14}^2 z_{23}^2}\right)\ln\frac{z_{12}z_{23}z_{14}z_{34}}{z_{13}z_{24}}\,.
\end{aligned}
\tag{4.47}
$$

At $\lambda=0$, this is in agreement with the CFT result [27] after the rescaling (4.41) and of course in the large k expansion up to $\mathcal{O}(\frac{1}{k})$. Furthermore, this result is in agreement with perturbation which predicts that corrections to the correlator under consideration should start at order λ^2. This is so because the contribution linear in λ will involve a single anti-holomorphic parafermion and will, thus, be vanishing. Lastly, we would like to comment on the efficiency of the method initiated in [26] and further developed here. Our method provides the exact in the deformation parameter λ correlators in contradistinction to the usual conformal perturbation theory which only gives results order by order in the λ expansion. The power of our approach resides on the fact that the exact in λ effective action (see (2.7)) is at hand.

4.4.2 Correlation function $\langle \Psi_+\Psi_+\bar{\Psi}_-\bar{\Psi}_-\rangle$

Our second four-point correlation function has the following large k expansion

$$
\begin{aligned}
\langle \Psi_+(z_1,\bar{z}_1)&\Psi_+(z_2,\bar{z}_2)\bar{\Psi}_-(z_3,\bar{z}_3)\bar{\Psi}_-(z_4,\bar{z}_4)\rangle \\
&= \frac{1}{8k(1-\lambda^2)}\Big(\langle F_+(z_1,\bar{z}_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\qquad\qquad\qquad + \langle F_+(z_2,\bar{z}_2)\partial x_-(z_1)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\qquad\qquad\qquad + \langle \bar{F}_-(z_3,\bar{z}_3)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\qquad\qquad\qquad + \langle \bar{F}_-(z_4,\bar{z}_4)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\rangle \Big) \\
&\quad -i\langle \phi_1(z_1,z_0)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\quad -i\langle \phi_1(z_2,z_0)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\quad +i\langle \phi_1(z_3,z_0)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\quad +i\langle \phi_1(z_4,z_0)\partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)\rangle \\
&\quad +\frac{\hat{g}_{12}}{16\pi k}\int \mathrm{d}^2z\,\langle \partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(z_3)\bar{\partial}x_-(\bar{z}_4) \\
&\qquad\qquad\qquad\qquad \times \big((x_+^2-x_-^2)(\partial x_+\bar{\partial}x_+ - \partial x_-\bar{\partial}x_-)\big)(z,\bar{z})\rangle\,.
\end{aligned}
\tag{4.48}
$$

A non-vanishing correlator requires equal number of x_+ and x_-'s. Clearly, only the last two lines may contribute, giving

$$
\begin{aligned}
\langle \Psi_+(z_1,\bar{z}_1)&\Psi_+(z_2,\bar{z}_2)\bar{\Psi}_-(z_3,\bar{z}_3)\bar{\Psi}_-(z_4,\bar{z}_4)\rangle \\
&= \frac{\hat{g}_{12}}{16\pi k}\int \mathrm{d}^2z\langle \partial x_-(z_1)\partial x_-(z_2)\bar{\partial}x_-(\bar{z}_3)\bar{\partial}x_-(\bar{z}_4)(x_+^2\partial x_+\bar{\partial}x_+)(z,\bar{z})\rangle
\end{aligned}
$$

$$= \frac{\hat{g}_{12}}{\pi k} \int \mathrm{d}^2 z \left(\frac{1}{(z - z_1)(z - z_2)^2(\bar{z} - \bar{z}_3)(\bar{z} - \bar{z}_4)^2} \right.$$
$$+ \frac{1}{(z - z_1)^2(z - z_2)(\bar{z} - \bar{z}_3)(\bar{z} - \bar{z}_4)^2}$$
$$+ \frac{1}{(z - z_1)(z - z_2)^2(\bar{z} - \bar{z}_3)^2(\bar{z} - \bar{z}_4)}$$
$$\left. + \frac{1}{(z - z_1)^2(z - z_2)(\bar{z} - \bar{z}_3)^2(\bar{z} - \bar{z}_4)} \right). \tag{4.49}$$

To evaluate the above integrals, we use (B.3) four times with appropriate renaming of the z_i's. Adding them up we find a vanishing result

$$\langle \Psi_+(z_1, \bar{z}_1) \Psi_+(z_2, \bar{z}_2) \bar{\Psi}_-(z_3, \bar{z}_3) \bar{\Psi}_-(z_4, \bar{z}_4) \rangle = 0 \,. \tag{4.50}$$

4.4.3 Correlation function $\langle \Psi_+ \Psi_- \bar{\Psi}_+ \bar{\Psi}_- \rangle$

The third four-point correlator that we will consider is the following

$$\langle \Psi_+(z_1, \bar{z}_1) \Psi_-(z_2, \bar{z}_2) \bar{\Psi}_+(z_3, \bar{z}_3) \bar{\Psi}_-(z_4, \bar{z}_4) \rangle$$
$$= \frac{4}{z_{12}^2 \bar{z}_{34}^2}$$
$$+ \frac{1}{8k(1 - \lambda^2)} \Big(\langle F_+(z_1, \bar{z}_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$- \langle F_-(z_2, \bar{z}_2) \partial x_-(z_1) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$- \langle \bar{F}_+(z_3, \bar{z}_3) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$+ \langle \bar{F}_-(z_4, \bar{z}_4) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \rangle \Big) \tag{4.51}$$
$$- i \langle \phi_1(z_1, z_0) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$+ i \langle \phi_1(z_2, z_0) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$- i \langle \phi_1(z_3, z_0) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$+ i \langle \phi_1(z_4, z_0) \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4) \rangle$$
$$+ \frac{\hat{g}_{12}}{16\pi k} \int \mathrm{d}^2 z \langle \partial x_-(z_1) \partial x_+(z_2) \bar{\partial} x_+(\bar{z}_3) \bar{\partial} x_-(\bar{z}_4)$$
$$\times \big((x_+^2 - x_-^2)(\partial x_+ \bar{\partial} x_+ - \partial x_- \bar{\partial} x_-) \big)(z, \bar{z}) \rangle \,.$$

To evaluate (4.51) is a rather cumbersome but straightforward computation. Let us briefly describe the various contributions. The lines 3–6, related to the $\frac{1}{k}$ corrections of the parafermions Ψ's (4.19), trivially vanish once we ignore contact terms involving only external points. The lines 7–10, that are related to the phase ϕ_1 of Ψ's, yield after performing ordinary integrations the following result

$$\frac{1}{k} \frac{1 + \lambda^2}{1 - \lambda^2} \frac{4}{z_{12}^2 \bar{z}_{34}^2} \left(-2 - \ln \frac{\varepsilon^2}{z_{12}^2} - \ln \frac{\varepsilon^2}{\bar{z}_{34}^2} + \ln \frac{|z_{23}|^2 |z_{14}|^2}{|z_{13}|^2 |z_{24}|^2} \right.$$
$$\left. + \frac{1}{2} \left(\frac{z_{12} z_{34}}{z_{24} z_{23}} + \frac{z_{12} z_{34}}{z_{13} z_{14}} + \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{23} \bar{z}_{13}} + \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{14} \bar{z}_{24}} \right) \right). \tag{4.52}$$

Finally, the last two lines of (4.51) originating from the interaction insertion equals to

$$\frac{4}{k}\frac{\hat{g}_{12}}{z_{12}^2\bar{z}_{34}^2}\left(\ln\frac{|z_{23}|^2|z_{14}|^2}{|z_{13}|^2|z_{24}|^2}+\frac{1}{2}\left(\frac{z_{12}z_{34}}{z_{24}z_{23}}+\frac{z_{12}z_{34}}{z_{13}z_{14}}+\frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{23}\bar{z}_{13}}+\frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{24}}\right)\right).\tag{4.53}$$

In order to obtain the last result, we have used twice (B.3) with an appropriate renaming of the z_i's.

Adding all the above contributions we find that

$$\langle\Psi_+(z_1,\bar{z}_1)\Psi_-(z_2,\bar{z}_2)\bar{\Psi}_+(z_3,\bar{z}_3)\bar{\Psi}_-(z_4,\bar{z}_4)\rangle$$
$$=\frac{4}{z_{12}^2\bar{z}_{34}^2}\left(1-\frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2}-\frac{1}{k}\frac{1+\lambda^2}{1-\lambda^2}\left(\ln\frac{\varepsilon^2}{z_{12}^2}+\ln\frac{\varepsilon^2}{\bar{z}_{34}^2}\right)\right).\tag{4.54}$$

At $\lambda=0$, this result is under the rescaling (4.41) in agreement with the CFT result which enforces factorization of the four-point function into two-point functions holomorphic and anti-holomorphic, respectively.

4.4.4 Correlation function $\langle\Psi_+\Psi_+\Psi_+\Psi_-\rangle$

The only contribution for this correlator comes from the free contractions involving the $\frac{1}{k}$ terms in the expression for the parafermions Ψ's (4.18) and (4.19), yielding finally

$$\langle\Psi_+(z_1,\bar{z}_1)\Psi_+(z_2,\bar{z}_2)\Psi_+(z_3,\bar{z}_3)\Psi_-(z_4,\bar{z}_4)\rangle$$
$$=\frac{4\lambda}{k(1-\lambda^2)}\left(\frac{1}{z_{14}^2z_{12}z_{13}}-\frac{1}{z_{24}^2z_{12}z_{23}}\right.$$
$$\left.+\frac{1}{z_{34}^2z_{23}z_{13}}-\frac{1}{z_{14}^2z_{24}z_{34}}-\frac{1}{z_{24}^2z_{14}z_{34}}-\frac{1}{z_{34}^2z_{14}z_{24}}\right)$$
$$=0.\tag{4.55}$$

We should mention that the result above is consistent with perturbation theory around the conformal point which predicts that it is exactly zero to all orders in the $\frac{1}{k}$ expansion since the correlator at hand is not a neutral one.

4.4.5 Correlation function $\langle\Psi_+\Psi_+\Psi_+\Psi_+\rangle$

There are two contributions for this correlator. The first one comes from the $\frac{1}{k}$ terms of the Ψ's (4.18) and (4.19), it involves only free contractions and reads

$$\frac{1+\lambda^2}{8k(1-\lambda^2)}\Big(\langle(x_+^2\partial x_+)(z_1,\bar{z}_1)\partial x_-(z_2)\partial x_-(z_3)\partial x_-(z_4)\rangle$$
$$+\langle(x_+^2\partial x_+)(z_2,\bar{z}_2)\partial x_-(z_1)\partial x_-(z_3)\partial x_-(z_4)\rangle$$
$$+\langle(x_+^2\partial x_+)(z_3,\bar{z}_3)\partial x_-(z_1)\partial x_-(z_2)\partial x_-(z_4)\rangle$$
$$+\langle(x_+^2\partial x_+)(z_4,\bar{z}_4)\partial x_-(z_1)\partial x_-(z_2)\partial x_-(z_3)\rangle\Big).\tag{4.56}$$

The second contribution comes from the interaction insertion and is given by

$$\frac{\hat{g}_{12}}{16\pi k}\int d^2z\,\langle\partial x_-(z_1)\partial x_-(z_2)\partial x_-(z_3)\partial x_-(z_4)(x_+^2\partial x_+\bar{\partial}x_+)(z,\bar{z})\rangle.\tag{4.57}$$

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To evaluate this integral we contract $\bar{\partial} x_+$ with the $(\partial x_-)_i$'s yielding contact terms proportional to $\delta^{(2)}(z - z_i)$. Evaluating then the integral we obtain an expression with free contractions among external point only, which is, after substituting $\hat{g}_{12}$ from (2.17), is precisely the opposite of that in (4.56). Combining these two contributions we find the vanishing result

$$\langle \Psi_+(z_1, \bar{z}_1) \Psi_+(z_2, \bar{z}_2) \Psi_+(z_3, \bar{z}_3) \Psi_+(z_4, \bar{z}_4) \rangle = 0. \tag{4.58}$$

As in the previous correlation function this result is in full agreement with perturbation theory around the conformal point and vanishes to all orders in the $\frac{1}{k}$ expansion.

A couple of important comments are in order. The first one concerns the $\frac{1}{\varepsilon}$ poles in the calculation of the four-point functions. These infinite contributions have been suppressed in the presentation because on can absorb them by employing precisely the same redefinition (4.40) used in the two-point function (4.37). The second comment concerns the form of the two and four-point correlators. By just inspecting the results one can see that they behave as if the theory was conformal albeit with a redefined level $\tilde{k} = k \frac{1-\lambda^2}{1+\lambda^2}$. In particular, this implies that the λ-dressed parafermions of the deformed theory will satisfy the same Poisson bracket algebra as the one satisfied by the parafermions at the conformal point but with $\tilde{k}$ in the place of k.

5 Discussion and future directions

We studied the λ-deformed SU(2)/U(1) σ-model by using a free field expansion instead of conformal perturbation and the underlying parafermionic algebras. Expanding along the lines of [26] the perturbation is organized as a series expansion for large values of k. This approach has the advantage that all deformation effects are fully encoded in the coupling constant coefficients and in the form of the operators. This prescription allowed for the computation the RG flow of the deformation parameter λ and the anomalous dimension of the parafermion bilinear perturbation, driving the theory away from the conformal point, as exact functions of λ and up to $\mathcal{O}(\frac{1}{k})$. Then, we introduced the corresponding λ-dressed parafermions containing a non-local phase in their expressions — which ensures on-shell chirality at the conformal point. Subsequently, we evaluate their anomalous dimensions and their four-point functions, odd-point correlation functions identically vanish, again to all-orders in λ and up to $\mathcal{O}(\frac{1}{k})$. The derived results are in agreement with CFT expectations at $\lambda = 0$ and are invariant under the duality-type symmetries (2.8).

The SU(2)/U(1) is the simplest possible symmetric coset space. It should be possible to study other λ-deformed coset CFT based on symmetric spaces using free fields as a basis. Furthermore, it was argued in [28] that the λ-deformed SU(2) × SU(2)/U(1) and SU(2)/U(1) CFTs are closely related since they share the same symmetries and β-function to all-orders in λ and k. It would be interesting to check and understand this relation further using the techniques of the present work.

Another interesting direction is to extend the current set for deformations of non-symmetric coset CFTs. A class of such integrable models is the λ-deformed $G_{k_1} \times G_{k_2}/G_{k_1+k_2}$ coset CFTs, which, assuming that $k_1 > k_2$, flows in the infrared to the $G_{k_2} \times G_{k_1-k_2}/G_{k_1}$ coset CFT [6]. These models have a richer parameter space and as a

result a very interesting generalization of the duality-type symmetries (2.8) (see eq. (2.17) in that work) which dictates their behavior and which reduces to the first one in (2.8) for equal levels. In addition, in these models the perturbation is driven by non-Abelian parafermion bilinears [31] which are much less understood than their Abelian counterparts [30] we utilized in the present work. We believe that our free field techniques will be instrumental in understanding these type of non-symmetric coset CFTs.

The present setup may be applied in the study of the two-parameter integrable deformations of symmetric coset spaces constructed in [7]. Such type of deformations are applicable to a restricted class of symmetric spaces provided that they satisfy a certain gauge invariance condition involving a non-trivial solution of the Yang-Baxter equation (see eq. (5.19) in that work). Examples include the $\mathrm{SU}(2)/\mathrm{U}(1)$ [7], the $\mathrm{SO}(N+1)/\mathrm{SO}(N)$ [32] and the recently studied $\mathbb{C}P^n$ models [33]. In the aforementioned examples, this second parameter can be set to zero via parameter redefinitions and/or diffeomorphisms. Such redefinitions come with caveats due to the topological nature of k and the non-trivial monodromies that the associated parafermions will presumably have. Therefore, a proper study of this class of models, in the presence context in terms of free fields, should include both deformation parameters.

A The anomalous dimension of $\tilde{\mathcal{O}}_\pm$

In this appendix we compute the anomalous dimension of the operator $\tilde{\mathcal{O}}_\pm$ defined in (4.35). The operator $\tilde{\mathcal{O}}_+$ arises, in the large k-limit, if we had we considered a perturbation of the form (4.27) but with a negative relevant sign. At the CFT point this should have anomalous dimension $-\frac{2}{k}$ since it corresponds to just the difference of two parafermion bilinears.

Consider the two-point function

$$
\begin{aligned}
\langle \tilde{\mathcal{O}}_c(z_1,\bar{z}_1)\tilde{\mathcal{O}}_c(z_2,\bar{z}_2)\rangle = \frac{1+c^2}{|z_{12}|^4} &- \frac{\hat{g}_{12}}{2\pi k}\int \mathrm{d}^2 z\, \langle \big(\partial x_1(z_1)\bar{\partial}x_2(\bar{z}_1) + c\,\partial x_2(z_1)\bar{\partial}x_1(\bar{z}_1)\big)\\
&\times (\partial x_1(z_2)\bar{\partial}x_2(\bar{z}_2) + c\,\partial x_2(z_2)\bar{\partial}x_1(\bar{z}_2))\\
&\times \big(x_1 x_2 (\partial x_1\bar{\partial}x_2 + \partial x_2\bar{\partial}x_1)\big)(z,\bar{z})\rangle
\end{aligned}
$$

$$
= \frac{1+c^2}{|z_{12}|^4} - \frac{\hat{g}_{12}}{2\pi k}\sum_{i,j,k=1}^{2} c^{i+j-2}I_{ijk}\,, \tag{A.1}
$$

where for convenience we have slightly generalize $\tilde{\mathcal{O}}_{\pm}$ to $\tilde{\mathcal{O}}_c(z, \bar{z}) = \partial x_1(z)\bar{\partial}x_2(\bar{z}) + c\,\partial x_2(z)\bar{\partial}x_1(\bar{z})$ by introducing an arbitrary relative constant c. In the various integrals I_{ijk} the index structure implies that it arises from the i-th, the j-th and the k-th term in the first, second and third factors in the integrand in (A.1). They are given by

$$I_{111} = \int \mathrm{d}^2 z \, \langle \partial x_1(z_1)\partial x_1(z_2)\partial x_1(z)x_1(z, \bar{z})\rangle \langle \bar{\partial}x_2(\bar{z}_1)\bar{\partial}x_2(\bar{z}_2)\bar{\partial}x_2(\bar{z})x_2(z, \bar{z})\rangle$$
$$= \int \mathrm{d}^2 z \left| \frac{1}{(z-z_1)^2(z-z_2)} + \frac{1}{(z-z_2)^2(z-z_1)} \right|^2 . \tag{A.2}$$

To evaluate the integral we first expand the absolute value

$$I_{111} = \int \frac{\mathrm{d}^2 z}{|z-z_1|^4 |z-z_2|^2} + \int \frac{\mathrm{d}^2 z}{|z-z_2|^4 |z-z_1|^2} \tag{A.3}$$
$$+ \int \frac{\mathrm{d}^2 z}{(z-z_1)^2(z-z_2)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)^2} + \int \frac{\mathrm{d}^2 z}{(z-z_2)^2(z-z_1)(\bar{z}-\bar{z}_2)(\bar{z}-\bar{z}_1)^2} .$$

Then we use twice (B.5) and (B.4) for the first and second line respectively, yielding

$$I_{111} = -\frac{4\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right) + \frac{4\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right) = 0 . \tag{A.4}$$

Finally, note that the integrand in (A.2) is positive definite. The explanation why the integral turns out to be zero lies in the fact that it is actually divergent and we may absorb the divergent piece by a field redefinition of the operators $\tilde{\mathcal{O}}_{\pm}$ as explained below (B.5). It is easily seen that adding to $\tilde{\mathcal{O}}_{\pm}$ in (4.35) a term proportional to $\frac{1}{\varepsilon^2 k}x_1 x_2$ does the job without giving rise to a mixing with the other operators of engineering dimension two, i.e. with $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

In addition, we have that

$$I_{112} = \int \mathrm{d}^2 z \, \langle \partial x_1(z_1)\partial x_1(z_2)\bar{\partial}x_1(\bar{z})x_1(z, \bar{z})\rangle \langle \bar{\partial}x_2(\bar{z}_1)\bar{\partial}x_2(\bar{z}_2)\partial x_2(z)x_2(z, \bar{z})\rangle , \tag{A.5}$$

has no contribution since it gives rise to contact terms on external points. In addition,

$$I_{211} = \int \mathrm{d}^2 z \, \langle \bar{\partial}x_1(\bar{z}_1)\partial x_1(z_2)\partial x_1(z)x_1(z, \bar{z})\rangle \langle \partial x_2(z_1)\bar{\partial}x_2(\bar{z}_2)\bar{\partial}x_2(\bar{z})x_2(z, \bar{z})\rangle$$
$$= \int \frac{\mathrm{d}^2 z}{|z-z_1|^2 |z-z_2|^4} , \tag{A.6}$$

and

$$I_{212} = \int \mathrm{d}^2 z \, \langle \bar{\partial}x_1(\bar{z}_1)\partial x_1(z_2)\bar{\partial}x_1(\bar{z})x_1(z, \bar{z})\rangle \langle \partial x_2(z_1)\bar{\partial}x_2(\bar{z}_2)\partial x_2(z)x_2(z, \bar{z})\rangle$$
$$= \int \frac{\mathrm{d}^2 z}{|z-z_1|^4 |z-z_2|^2} , \tag{A.7}$$

Adding the last two integrals and using twice (B.5) we find

$$I_{211} + I_{212} = -\frac{4\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right) , \tag{A.8}$$

where the divergent pieces have been dismissed (see discussion below (A.4)). Similarly,

$$
\begin{aligned}
I_{121} &= \int \mathrm{d}^2 z \, \langle \partial x_1(z_1) \bar{\partial} x_1(\bar{z}_2) \partial x_1(z) x_1(z, \bar{z}) \rangle \langle \bar{\partial} x_2(\bar{z}_1) \partial x_2(z_2) \bar{\partial} x_2(\bar{z}) x_2(z, \bar{z}) \rangle \\
&= I_{212}
\end{aligned}
\tag{A.9}
$$

and

$$
\begin{aligned}
I_{122} &= \int \mathrm{d}^2 z \, \langle \partial x_1(z_1) \bar{\partial} x_1(\bar{z}_2) \bar{\partial} x_1(\bar{z}) x_1(z, \bar{z}) \rangle \langle \bar{\partial} x_2(\bar{z}_1) \partial x_2(z_2) \partial x_2(z) x_2(z, \bar{z}) \rangle \\
&= I_{211} \,.
\end{aligned}
\tag{A.10}
$$

Finally,

$$
\begin{aligned}
I_{221} &= \int \mathrm{d}^2 z \, \langle \bar{\partial} x_1(\bar{z}_1) \bar{\partial} x_1(\bar{z}_2) \partial x_1(z) x_1(z, \bar{z}) \rangle \langle \partial x_2(z_1) \partial x_2(z_2) \bar{\partial} x_2(\bar{z}) x_2(z, \bar{z}) \rangle \,, \\
&= I_{112}
\end{aligned}
\tag{A.11}
$$

and

$$
\begin{aligned}
I_{222} &= \int \mathrm{d}^2 z \, \langle \bar{\partial} x_1(\bar{z}_1) \bar{\partial} x_1(\bar{z}_2) \bar{\partial} x_1(\bar{z}) x_1(z, \bar{z}) \rangle \langle \partial x_2(z_1) \partial x_2(z_2) \partial x_2(z) x_2(z, \bar{z}) \rangle \\
&= I_{111} \,.
\end{aligned}
\tag{A.12}
$$

Putting all together in (A.1) and using (A.8), we find:

$$
\langle \tilde{\mathcal{O}}_c(z_1, \bar{z}_1) \tilde{\mathcal{O}}_c(z_2, \bar{z}_2) \rangle = \frac{1 + c^2}{|z_{12}|^4} \left(1 + \frac{4c}{1 + c^2} \frac{\hat{g}_{12}}{k} \left(1 + \ln \frac{\varepsilon^2}{|z_{12}|^2} \right) \right) \,.
\tag{A.13}
$$

Hence, the anomalous dimension of $\tilde{\mathcal{O}}_\pm$ reads

$$
\gamma^{\tilde{\mathcal{O}}_\pm} = \pm \frac{2\hat{g}_{12}}{k} = \mp \frac{2}{k} \frac{1 + \lambda^2}{1 - \lambda^2} \,.
\tag{A.14}
$$

Hence, the operator $\tilde{\mathcal{O}}_+$ has an anomalous dimension that matches with (4.34) as expected and now has been explicitly verified. The operator $\tilde{\mathcal{O}}_-$ corresponds to a marginally irrelevant operator.

B Various integrals

In this appendix we shall provide the various (single) integrals which appear in the present work. Our regularization scheme is as follows: internal points in integrals can coincide with external ones but coincident external points are not allowed. In addition, to regularize infinities we introduce a very small disc of radius ε around external points as well.

We shall need the basic integrals

$$
\begin{aligned}
\int \frac{\mathrm{d}^2 z}{(z - z_1)(\bar{z}_2 - \bar{z})} &= \pi \ln |z_{12}|^2 \,, & \int \frac{\mathrm{d}^2 z}{(z - z_1)(\bar{z}_1 - \bar{z})} &= \pi \ln \varepsilon^2 \,, \\
\int \frac{\mathrm{d}^2 z}{(z - z_1)^2 (\bar{z}_2 - \bar{z})} &= \frac{\pi}{z_{12}} \,, & \int \frac{\mathrm{d}^2 z}{(z - z_1)(\bar{z} - \bar{z}_2)^2} &= \frac{\pi}{\bar{z}_{12}} \,,
\end{aligned}
\tag{B.1}
$$

along with the identity

$$\frac{1}{(z-z_1)(z-z_2)} = \frac{1}{z_{12}}\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right).$$ (B.2)

Using the above regularization scheme and (B.1), (B.2) we find that

$$\int \frac{\mathrm{d}^2 z}{(z-z_1)(z-z_2)^2(\bar{z}-\bar{z}_3)(\bar{z}-\bar{z}_4)^2}$$
$$= -\frac{\pi}{z_{12}^2 \bar{z}_{34}^2}\left(\ln\frac{|z_{23}|^2|z_{14}|^2}{|z_{13}|^2|z_{24}|^2} + \frac{z_{12}z_{34}}{z_{24}z_{23}} + \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{23}}\right).$$ (B.3)

$$\int \frac{\mathrm{d}^2 z}{(z-z_1)^2(z-z_2)(\bar{z}-\bar{z}_1)(\bar{z}-\bar{z}_2)^2}$$
$$= \frac{2\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right) + \frac{\pi}{|z_{12}|^2}\int \mathrm{d}^2 z\left(\frac{\bar{z}_{12}^{-1}}{(z-z_1)^2(\bar{z}-\bar{z}_1)} - \frac{z_{12}^{-1}}{(z-z_2)(\bar{z}-\bar{z}_2)^2}\right)$$
$$= \frac{2\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right),$$ (B.4)

since it can be easily seen that in our regularization scheme the integral in the last line of (B.4) is zero. For instance, $\int \frac{\mathrm{d}^2 z}{(z-z_1)(\bar{z}-\bar{z}_1)^2} = \frac{i}{2}\oint_{C_\varepsilon} \frac{\mathrm{d}z}{|z-z_1|^2}$, where C_ε is a small circle of radius ε surrounding $z = z_1$. Then, the last integral indeed vanishes. In addition, using (B.1), (B.2) we find

$$\int \frac{\mathrm{d}^2 z}{|z-z_1|^4|z-z_2|^2} = -\frac{2\pi}{|z_{12}|^4}\left(1 + \ln\frac{\varepsilon^2}{|z_{12}|^2}\right) + \frac{1}{|z_{12}|^2}\int \frac{\mathrm{d}^2 z}{|z-z_1|^4}.$$ (B.5)

Finally, the integral in the last line of (B.5) equals $\frac{\pi}{\varepsilon}2$ and is clearly divergent as $\varepsilon \to 0$. This integral arises in the computation of the anomalous dimension of the operators $\tilde{\mathcal{O}}_\pm$ in appendix A. In order to absorb this divergent piece the operators require a redefinition similar to that for the single parafermion in subsection 4.3. With this in mind we may safely ignore this term all together and set it to zero.

Finally, using (B.1), (B.2) we find

$$\int \frac{\mathrm{d}^2 z}{|z-z_1|^2|z-z_2|^2} = -\frac{2\pi}{|z_{12}|^2}\ln\frac{\varepsilon^2}{|z_{12}|^2}.$$ (B.6)

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