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# Hamilton-Jacobi approach to holographic renormalization of massive gravity

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Abstract: Recently, a practical approach to holographic renormalization has been developed based on the Hamilton-Jacobi formulation. Using a simple Einstein-scalar theory, we clarify that this approach does not conflict with the Hamiltonian constraint as it seems. Then we apply it to the holographic renormalization of massive gravity. We assume that the shift vector is falling off fast enough asymptotically. We derive the counterterms up to the boundary dimension $d = 4$. Interestingly, we find that the conformal anomaly can even occur in odd dimensions, which is different from the Einstein gravity. We check that the counterterms cancel the divergent part of the on-shell action at the background level. At the perturbation level, they are also applicable in several time-dependent cases.

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## 1 Introduction

Anti-de Sitter/conformal field theory (AdS/CFT) correspondence provides a powerful tool to study the strongly coupled field theories [1]. Among others, the Gubser-Klebanov-Polyakov-Witten dictionary that identifies the generating functional of the field theory with the on-shell gravitational action plays an essential role in the calculation [2, 3]. The most obvious technical obstacle to use the dictionary is the divergence involved on both sides of the duality [4]. According to the renormalization method to deal with the UV divergence in the field theory, the called holographic renormalization is developed to remove the IR divergence in the gravity.

There are different approaches to holographic renormalization. The first systematic one was presented in [5–7], which is usually called as the standard approach [8]. Its main procedure includes: a) solving the second-order equations of motion (EOM) in the Fefferman-Graham (FG) coordinates to obtain the asymptotic expansion of the dynamical fields [9]; b) calculating the regularized on-shell action on the boundary to separate the divergent



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terms; c) reversing the FG expansion to express the divergent terms by the local fields on the boundary. The standard approach is strict, conceptually simple, and universal for diverse situations. However, the FG expansion and its reverse are technically tedious. So it is natural to expect an alternative approach which always respects the local field expression.

Actually, such approach was put forward by de Boer, Verlinde, and Verlinde (dBVV) based on the Hamiltonian formulation of gravity [10], see ref. [11] for a nice review. To proceed in dBVV's approach, one writes down the most general ansatz for the covariant counterterms, organizes it by the derivative expansion, and specifies it by solving a series of descent equations induced from the Hamiltonian constraint, where the canonical momenta are replaced by the variations of the on-shell action with respect to boundary fields. Comparing the standard and dBVV's approaches, one can find that the latter is usually more simple than the former, mainly because the latter solves the algebraic descent equations instead of the second-order differential equations, and determines the counterterms directly on the cutoff surface without performing the FG expansion and reversion. The main drawbacks of dBVV's approach are [4, 12]: a) the solutions of some descent equations are not unique; b) the logarithmic counterterms have not been explicitly obtained; c) the ansatz may include many unnecessary terms; d) sometimes the sufficient ansatz is difficult to be figured out. In refs. [12, 13], Kalkkinen, Martelli and Muck removed the ambiguities in the descent equations by comparison with free field calculations. They also isolated the logarithmic counterterms, which are related to the breakdown of the recursion of descent equations. Subsequently, Papadimitriou and Skenderis [8, 14] developed the previous approaches where the crucial difference is that the covariant expansion is organized according to the eigenvalues of the dilatation operator. Interestingly, this approach does not rely on the ansatz and can be applied to more general backgrounds [15–17].

Besides the standard and Hamiltonian approaches, Brown and York in the early days proposed to remove the divergence of the stress tensor by subtracting the contribution from the reference spacetime [18]. This requires that a boundary with intrinsic metric is embedded in the reference spacetime, which is often not possible [19]. Moreover, instead of selecting the Dirichlet boundary conditions, the Kounterterm approach is developed where the variational principle is associated with the fixed extrinsic curvature on the boundary [20, 21]. Other attempt based the dimensional renormalization can be found in [22].

As explicitly pointed out in dBVV's work [10], the Hamiltonian constraint ensures the invariance under the diffeomorphism along the radial direction. This implies that the on-shell action does not depend on the radial coordinate explicitly and the radial Hamilton-Jacobi (HJ) equation is equivalent to the Hamiltonian constraint for any holographic theories with diffeomorphism symmetry [15]. On the contrary, by focusing on the complete HJ equation rather than the formally simpler Hamiltonian constraint, a new approach to holographic renormalization has been presented recently [23]. This approach, which we will refer as the HJ approach,[1] is partially motivated by [24], where the interesting point captured by [23] is that the HJ equation is used to isolate the infrared divergences of scalar

[1]To be clear, we have referred the previous approaches based on the Hamiltonian constraint as the Hamiltonian approaches, following [8]. However, it should be stressed that the Hamiltonian approaches also solve the HJ equation.



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fields in a fixed de Sitter background. Although the HJ approach suffers from the latter two drawbacks of dBVV's approach since the action ansatz is still required, it has been exhibited in several Einstein-scalar theories that the HJ approach is practical [23]. Here we emphasize that it is tailored to handle the systems with conformal anomalies, because the derivations of the logarithmic and power counterterms are equivalently fluent and have nothing different such as the breakdown of descent equations. However, the reason why the HJ approach does not conflict with the Hamiltonian constraint has not been clarified.[2] In this paper, one of two aims is to address this problem.

Another aim of this paper is to apply the HJ approach to the massive gravity with different dimensions. The research on massive gravity has a long history [25–29]. The two main motivations include finding a self-consistent theory with massive spin-2 graviton and modifying the Einstein gravity at long distance for self-accelerated expansion of the Universe [30]. Massive gravity has obtained revived interest since de Rham, Gabadadze, and Tolley (dRGT) proposed a covariant non-linear theory where the well-known Boulware-Deser ghost can be excluded [31–33]. Recently, massive gravity has been applied to the AdS/CFT correspondence, where the reference metric can imitate the mean-field disorder in realistic materials [34–37, 39]. The holographic renormalization of massive gravity with boundary dimension $d = 3$ has been studied previously using the standard approach [40]. However, the resultant counterterms are not general, because the Gauss normal coordinate (GNC) is adapted in the neighborhood of the boundary and some additional conditions are imposed on the characteristic tensor of massive gravity. In this paper, we will only assume that the GNC is applicable near the boundary but release the other conditions. Moreover, we will show that the conformal anomalies can occur in both odd and even dimensions, which are missed in [40]. As we have emphasized, this indicates that the HJ approach is particularly suitable for massive gravity.

The rest part of this paper is arranged as follows. In section 2, we will decompose the HJ equation and construct an equation that is actually used by holographic renormalization. In section 3, we will apply the HJ approach to the massive gravity with different dimensions. The conclusion of this paper will be given in section 4. In appendix A, we will review the HJ approach to the holographic renormalization of the Einstein gravity with massive scalars. In appendix B and C, we will provide some calculation details and basic formulas.

## 2 Decomposition of Hamilton-Jacobi equation

The bulk dynamics of a holographic theory can be formulated as a Hamiltonian system, where the Hamiltonian time is identified with the radial coordinate $r$. The Hamiltonian and on-shell action still obey the HJ equation

$$H + \frac{\partial S_{\text{on-shell}}}{\partial r} = 0, \tag{2.1}$$

[2]It was argued in [23] that the on-shell action is not diffeomorphism-invariant along the radial direction and the HJ equation cannot be reduced to the Hamiltonian constraint. Moreover, the discussion below their eq. (2.8) might suggest that the canonical momenta in the Hamiltonian constraint are not equal to the ones in the HJ equation.



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see a simple derivation in [15]. However, one should be careful that the diffeomorphism symmetry, which is respected by usual gravity theories, imposes the Hamiltonian constraint $H = 0$. It further indicates that the on-shell action does not depend on $r$ explicity. Moreover, since the Hamiltonian constraint is a part of EOM, the on-shell action cannot be well-defined before imposing the Hamiltonian constraint. Keeping these in mind, the Hamiltonian constraint is usually understood as the HJ equation in the previous Hamiltonian approaches.

In ref. [23], the Hamiltonian constraint is not imposed at the beginning as usual. Instead, the complete HJ equation is relied on. Then the coefficients in the action ansatz are allowed to depend on the radial coordinate and the HJ equation induces the one-order differential equations of the coefficients which can be solved unambiguously near the boundary. One can find that this approach to the holographic renormalization is practical indeed but its legitimacy has not been clearly stated. Here we will address this problem.

Suppose that there is a general gravity theory associated with certain terms in the action which break the diffeomorphism symmetry. Its Hamiltonian can be nonvanishing, just like the massive gravity [31–33]. But the HJ equation should still hold, if the theory is still a Hamiltonian system. Turning off the symmetry-breaking terms, one can see that $H = 0$ and $\partial S_{\text{on}-\text{shell}}/\partial r = 0$ arise. However, the HJ equation (2.1) itself is not wrong, at least formally. Thus, we can argue that the HJ equation is a more general equation than the Hamiltonian constraint and can be applicable to the theories with or without the diffeomorphism symmetry.

We proceed to separate the on-shell action into the renormalized part and the divergent part

$$S_{\text{on}-\text{shell}} = S_{\text{ren}} - S_{\text{ct}}, \tag{2.2}$$

where the divergent terms are denoted as negative counterterms. Then the HJ equation can be decomposed into

$$H_{\text{ren}} + \frac{\partial S_{\text{ren}}}{\partial r} - H_{\text{ct}} - \frac{\partial S_{\text{ct}}}{\partial r} = 0, \tag{2.3}$$

where $H_{\text{ren}}$ is defined as the part of $H$ relevant to $S_{\text{ren}}$ and $H_{\text{ct}}$ is defined as

$$H_{\text{ct}} \equiv -\left(H - H_{\text{ren}}\right). \tag{2.4}$$

We point out that what is actually used to implement the holographic renormalization of the Einstein-scalar theories in [23] is[3]

$$H_{\text{ct}} + \frac{\partial S_{\text{ct}}}{\partial r} = 0. \tag{2.5}$$

We emphasize that each term in eq. (2.3) should include the finite terms if there are conformal anomalies. This subtlety implies that eq. (2.5) is not simply the leading orders of eq. (2.1). Therefore, whether it is correct or not requires proof. In the following, we will illustrate eq. (2.5) using the Einstein gravity with massive scalars. In particular, the

[3]As an illustration, we recover the holographic renormalization of the Einstein gravity with massive scalars based on this equation in appendix A.



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Hamiltonian constraint $H = 0$ will not be involved explicitly. We argue that the extension to other theories, with or without the diffeomorphism symmetry, should be straightforward. Note that for convenience, we will refer eq. (2.5) as the counterterm part of the HJ (CPHJ) equation.

Consider that the system is described by the action

$$S = -\frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{g}\left(R[g] - g^{\mu\nu}G_{IJ}\partial_\mu\Phi^I\partial_\nu\Phi^J - V(\Phi)\right) - \frac{1}{\kappa^2}\int_{\partial M} d^d x\sqrt{\gamma}K, \tag{2.6}$$

where $G_{IJ}$ is a metric on the scalar manifold, $g_{\mu\nu}$ is the bulk metric, $\gamma_{ij}$ is the metric on the boundary, and $K$ is its extrinsic curvature. Adopting the Arnowitt-Deser-Misner (ADM) decomposition[4]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij}dx^i dx^j, \tag{2.7}$$

and selecting the usual gauge due to the diffeomorphism symmetry

$$N = 1, \qquad N_i = 0, \tag{2.8}$$

where $N$ is the lapse and $N^i$ is the shift, the Hamiltonian is given by[5]

$$H = \int_{\partial M} d^d x\left[\frac{2\kappa^2}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{4}G^{IJ}\pi_I\pi_J\right) + L_d\right], \tag{2.9}$$

where

$$L_d = \frac{\sqrt{\gamma}}{2\kappa^2}\left(R - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\right), \tag{2.10}$$

and $R$ is the scalar curvature on the boundary. The canonical momenta are defined by

$$\begin{aligned}\pi^{ij} &\equiv \frac{\partial L}{\partial\dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}(K^{ij} - K\gamma^{ij}),\\ \pi_I &\equiv \frac{\partial L}{\partial\dot{\Phi}^I} = \frac{1}{\kappa^2 N}\sqrt{\gamma}(G_{IJ}\dot{\Phi}^J - N^i G_{IJ}\partial_i\Phi^J).\end{aligned} \tag{2.11}$$

According to the standard classical mechanics [41], they should be equal to the variations of the on-shell action with respect to boundary fields

$$\pi^{ij} = \frac{\delta S_{\text{on-shell}}}{\delta\gamma_{ij}}, \qquad \pi_I = \frac{\delta S_{\text{on-shell}}}{\delta\Phi^I}. \tag{2.12}$$

Using eq. (2.12), the previous decomposition of the HJ equation indicates

$$H_{\text{ren}} = \int_{\partial M} d^d x\left[2\{-S_{\text{ct}}, S_{\text{ren}}\} + \{S_{\text{ren}}, S_{\text{ren}}\}\right], \tag{2.13}$$

$$H_{\text{ct}} = -\int_{\partial M} d^d x\left[\{S_{\text{ct}}, S_{\text{ct}}\} + L_d\right], \tag{2.14}$$

[4]We denote the bulk and boundary coordinates by Greek and Latin indices, respectively. Throughout this paper we take the Euclidean signature and set the AdS radius $l = 1$.

[5]In appendix A.1, we have reviewed briefly the Hamiltonian formalism of the Einstein-scalar theory.



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and the bracket $\{S_\mathrm{a}, S_\mathrm{b}\}$ is defined through

$$\{S_\mathrm{a}, S_\mathrm{b}\} \equiv \frac{2\kappa^2}{\sqrt{\gamma}}\left(\frac{\delta S_\mathrm{a}}{\delta\gamma_{ij}}\frac{\delta S_\mathrm{b}}{\delta\gamma_{kl}}\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\frac{\delta S_\mathrm{a}}{\delta\gamma_{ij}}\gamma_{ij}\frac{\delta S_\mathrm{b}}{\delta\gamma_{kl}}\gamma_{kl} + \frac{1}{4}G^{IJ}\frac{\delta S_\mathrm{a}}{\delta\Phi^I}\frac{\delta S_\mathrm{b}}{\delta\Phi^J}\right). \tag{2.15}$$

Keep in mind the finiteness of $S_\mathrm{ren}$ and the asymptotic behavior of the fields[6]

$$\gamma_{ij} \simeq e^{2r}\bar{\gamma}_{ij}, \qquad \Phi^I \simeq e^{-(d-\Delta_I)r}\bar{\Phi}^I, \tag{2.16}$$

where $\bar{\gamma}_{ij}$ and $\bar{\Phi}^I$ are the sources on the field theory and $\Delta_I = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m_I^2}$ is the conformal dimension. One can see immediately that $\{S_\mathrm{ren}, S_\mathrm{ren}\}$ vanishes as $r \to \infty$. Furthermore, at leading order, we have

$$-\frac{\delta S_\mathrm{ct}}{\delta\gamma_{ij}} \simeq \frac{\partial L}{\partial\dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}(K^{ij} - K\gamma^{ij}) \simeq -\frac{1}{2\kappa^2}\sqrt{\gamma}(d-1)\gamma^{ij}, \tag{2.17}$$

$$-\frac{\delta S_\mathrm{ct}}{\delta\Phi^I} \simeq \frac{\partial L}{\partial\dot{\Phi}^I} = \frac{1}{\kappa^2}\sqrt{\gamma}G_{IJ}\dot{\Phi}^J \simeq -\frac{1}{\kappa^2}\sqrt{\gamma}G_{IJ}(d-\Delta_I)\Phi^J. \tag{2.18}$$

Substituting them into eq. (2.13) gives[7]

$$\begin{aligned} H_\mathrm{ren} &\simeq \int_{\partial M} d^d x \left[2\frac{\delta S_\mathrm{ren}}{\delta\gamma_{ij}}\gamma_{ij} - (d-\Delta_I)\Phi^I\frac{\delta S_\mathrm{ren}}{\delta\Phi^I}\right] \\ &\simeq \int_{\partial M} d^d x \left[2\frac{\delta S_\mathrm{ren}}{\delta\bar{\gamma}_{kl}}\frac{\partial\left(e^{-2r}\gamma_{kl}\right)}{\partial\gamma_{ij}}\gamma_{ij} - (d-\Delta_I)\Phi^I\frac{\delta S_\mathrm{ren}}{\delta\bar{\Phi}^J}\frac{\partial\left(e^{(d-\Delta_I)r}\Phi^J\right)}{\partial\Phi^I}\right] \\ &\simeq \int_{\partial M} d^d x \left[2\frac{\delta S_\mathrm{ren}}{\delta\bar{\gamma}_{kl}}\bar{\gamma}_{ij} - (d-\Delta_I)\bar{\Phi}^I\frac{\delta S_\mathrm{ren}}{\delta\bar{\Phi}^I}\right]. \end{aligned} \tag{2.19}$$

It exactly cancels

$$\begin{aligned} \frac{\partial S_\mathrm{ren}}{\partial r} &\simeq \int_{\partial M} d^d x \left[\frac{\delta S_\mathrm{ren}}{\delta\bar{\gamma}_{ij}}\frac{\partial\left(e^{-2r}\gamma_{ij}\right)}{\partial r} + \frac{\delta S_\mathrm{ren}}{\delta\bar{\Phi}^I}\frac{\partial\left(e^{(d-\Delta_I)r}\Phi^I\right)}{\partial r}\right] \\ &\simeq \int_{\partial M} d^d x \left[-2\frac{\delta S_\mathrm{ren}}{\delta\bar{\gamma}_{ij}}\bar{\gamma}_{ij} + (d-\Delta_I)\bar{\Phi}^I\frac{\delta S_\mathrm{ren}}{\delta\bar{\Phi}^I}\right], \end{aligned} \tag{2.20}$$

that is,

$$H_\mathrm{ren} + \frac{\partial S_\mathrm{ren}}{\partial r} \simeq 0. \tag{2.21}$$

Thus, the complete HJ equation (2.1) has been reduced to the CPHJ equation (2.5).

To compare what we have done with previous references, some remarks are in order. First, the separation of the on-shell action (2.2) is different from dBVV's approach [10]. Our $-S_\mathrm{ct}$ includes all the divergent terms but $S_\mathrm{loc}$ in eq. (14) of [10] does not involve the logarithmic divergences. Second, in eq. (2.17) and eq. (2.18), we have used the well known equality between two forms of canonical momenta at leading order. Equation (2.18) is

[6]When $m_I^2 = -\frac{d^2}{4}$, the leading behaviour of $\Phi^I$ is given by [7, 12] $\Phi^I \simeq \bar{\Phi}^I r e^{-\frac{1}{2}dr}$ instead of eq. (2.16). Nevertheless, the remaining derivation of the CPHJ equation is still valid.

[7]Here and below, we have considered that $S_\mathrm{ren}$ can be taken as the functionals of $(\bar{\gamma}_{kl}, \bar{\Phi}^I)$ and $(\gamma_{kl}, \Phi^I, r)$ from the viewpoints of the field theory and its gravity dual, respectively.

nothing but the step 2 of the algorithm in [23], which is taken as a shortcut to fix some coefficients of the ansatz. Third, the CPHJ equation (2.5) is not a completely new result. In fact, a similar equation[8] has been given by eq. (27) in [4] using the Hamiltonian formulation of the renormalization group of local quantum field theories [42]. Also, eqs. (2.5) and (2.21) can be understood by the fact that both $S_{\rm ren}$ and $S_{\rm ct}$ produce a canonical transformation which can be associated with a Hamiltonian flow [43]. Moreover, it should be stressed that our derivation is similar to the part of the derivation of the dilatation operator method. In particular, the first line of (2.19) equals to the dilatation operator acting on $S_{\rm ren}$ and eq. (2.21) can be related to eq. (133) in [4]. Our contribution here is to provide a direct illustration of eq. (2.5) by holography and point out that it can be taken as a master equation to implement the holographic renormalization.

## 3 Massive gravity

We will study the massive gravity where the only dynamical field is the spacetime metric and the boundary is supposed to be the AdS at infinity. We will show that the CPHJ equation can be applied to the holographic renormalization of massive gravity. Our target boundary dimensions are the most interesting cases: $d = 2, 3, 4$. The renormalization procedure for massive gravity is only slightly different from the one for the Einstein-scalar theory, which is given in appendix A. We recommend reading it first since we will neglect some similar details here.

### 3.1 Hamilton-Jacobi formalism

Consider the massive gravity with the action [34]

$$S_{MG} = -\frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{g}\left(R[g]+d(d-1)+m^2\sum_{n=1}^{4}\beta_n e_n(\mathcal{X})\right)-\frac{1}{\kappa^2}\int_{\partial M} d^d x\sqrt{\gamma}K. \quad (3.1)$$

The mass terms are constructed subtly to avoid the Boulware-Deser ghost, where $\beta_n$ are constants and we will reparameterize them by $\alpha_n = m^2\beta_n$. The characteristic tensor $\mathcal{X}^\mu{}_\nu$ is defined as the square root of $g^{\mu\lambda}f_{\lambda\nu}$. Here $g^{\mu\lambda}$ and $f_{\lambda\nu}$ are the dynamical and reference metric, respectively. $e_n(\mathcal{X})$ are symmetric polynomials of the eigenvalues of the $(d+1)\times(d+1)$ matrix $\mathcal{X}^\mu{}_\nu$:

$$\begin{aligned} &e_1 = [\mathcal{X}], \quad e_2 = [\mathcal{X}]^2 - [\mathcal{X}^2], \quad e_3 = [\mathcal{X}]^3 - 3[\mathcal{X}][\mathcal{X}^2] + 2[\mathcal{X}^3], \\ &e_4 = [\mathcal{X}]^4 - 6[\mathcal{X}]^2[\mathcal{X}^2] + 8[\mathcal{X}^3][\mathcal{X}] + 3[\mathcal{X}^2]^2 - 6[\mathcal{X}^4], \end{aligned} \quad (3.2)$$

where we denote $[\mathcal{X}] = \mathcal{X}^\mu{}_\mu$. The reference metric can have various forms. Here we focus on

$$f_{\mu\nu} = \delta^i_\mu \delta^j_\nu f_{ij} \quad (3.3)$$

with $f_{ti} = 0$, which is popular in the application of holography [34–37, 39].

[8]To the best of our knowedge, the definition (2.13) of $H_{\rm ren}$ is new and looks very different from eq. (24) in [4]. But in terms of eq. (2.16) and the first line of eq. (2.19), they are consistent indeed.

When the Hamiltonian formulation is implemented in massive gravity, one may encounter a complication. Massive gravity explicitly breaks the diffeomorphism symmetry, which indicates that one cannot fix the gauge (2.8) in the whole bulk spacetime. These extra degrees of freedom,[9] if involved, would complicate the gravitational Hamiltonian, the relevant constraint, and the sequent holographic renormalization. For the sake of simplicity, the GNC is assumed in the neighborhood of the boundary and some additional conditions on $\mathcal{X}^{\mu}{}_{\nu}$ are imposed in ref. [40]. Here we release the conditions but still assume that the GNC can be selected in a certain region near the boundary, that is,

$$ds^2 = dr^2 + \gamma_{ij} dx^i dx^j. \tag{3.4}$$

More explicitly, we assume that the shift vector is falling off fast enough asymptotically so that it does not affect the counterterms. This assumption cannot be justified in general, but in section 3.4, we will show some interesting situations where it is true.

We would like to rewrite the mass terms by the boundary metric $\gamma_{ij}$. For this aim, let's define a tensor $X^i{}_j$ by

$$X^i{}_k X^k{}_j = \gamma^{ik} f_{kj}. \tag{3.5}$$

Due to eqs. (3.4) and (3.3), we have $[\mathcal{X}^n] = [X^n]$ and thereby $e_n(\mathcal{X}) = e_n(X)$.

We proceed to study the CPHJ equation for massive gravity. Similar to the derivation of eq. (A.5) in appendix A, one can obtain the Hamiltonian for massive gravity by a Legendre transformation of the Lagrangian

$$H \equiv \int_{\partial M} d^d x \pi^{ij} \dot{\gamma}_{ij} - L = \int_{\partial M} d^d x \mathcal{H}, \tag{3.6}$$

where

$$L = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{\gamma} \left[ R + K^2 - K_{ij} K^{ij} + d(d-1) + \sum_{n=1}^{4} \alpha_n e_n(X) \right], \tag{3.7}$$

$$\mathcal{H} = \frac{2\kappa^2}{\sqrt{\gamma}} \left( \pi_{ij} \pi^{ij} - \frac{1}{d-1} \pi^2 \right) + \frac{\sqrt{\gamma}}{2\kappa^2} \left[ R + d(d-1) + \sum_{n=1}^{4} \alpha_n e_n(X) \right]. \tag{3.8}$$

Note that we have been working in the GNC. With the Hamiltonian in hands, the CPHJ equation (2.5) for massive gravity can be built up following the same procedure in section 2. Furthermore, it can be changed into the form similar to eq. (A.21):

$$R + \mathcal{K} + d(d-1) + \sum_{n=1}^{4} \alpha_n e_n(X) + 2 \frac{\partial U}{\partial r} = 0, \tag{3.9}$$

where

$$\mathcal{K} = 4 Y_{ij} Y^{ij} - U^2 - \frac{1}{d-1} (U - 2Y)^2, \tag{3.10}$$

and the definitions of $U$ and $Y_{ij}$ can be obtained from eqs. (A.14) and (A.19) with vanishing scalar fields.

[9] We only need to worry about the effect of the shift vector. The lapse function can be gauged away since the reference metric (3.3) we choose does not break the diffeomorphism symmetry along the radial direction.

### 3.2 Action ansatz and variation

The main difference that we mentioned at the beginning of this section resides in the inverse metric expansion of $U$.[10] The definition of $X^i{}_j$ suggests that the counterterms in massive gravity may contain the terms with half-integer inverse metrics, that is,

$$U = U_{(0)} + U_{(1)} + \cdots + U_{(d)}, \qquad d = 2, 3, 4, \tag{3.11}$$

where $U_{(2k)}$ contains $k$ inverse metrics. The sufficient ansatz for each order is

$$\begin{aligned}
U_{(0)} &= A(r),\\
U_{(1)} &= B(r)[X],\\
U_{(2)} &= C_1(r)R + C_2(r)[X^2] + C_3(r)[X]^2,\\
U_{(3)} &= D_1(r)[X]R + D_2(r)R_{ij}X^{ij} + D_3(r)[X^3] + D_4(r)[X^2][X] + D_5(r)[X]^3,\\
U_{(4)} &= E_1(r)R_{ij}R^{ij} + E_2(r)R^2 + E_3(r)[X^2]R + E_4(r)[X]^2R + E_5(r)R_{ij}X^{ij}[X]\\
&\quad + E_6(r)R^i{}_jX^j{}_kX^k{}_i + E_7(r)[X^4] + E_8(r)[X^3][X] + E_9(r)[X^2][X]^2\\
&\quad + E_{10}(r)[X^2]^2 + E_{11}(r)[X]^4 + E_{12}(r)X^{ij}\nabla^k\nabla_jX_{ki} + E_{13}(r)X^{ij}\nabla^k\nabla_kX_{ij}\\
&\quad + E_{14}(r)[X]\nabla_i\nabla_jX^{ij} + E_{15}(r)[X]\nabla_i\nabla^i[X] + \cdots ,
\end{aligned} \tag{3.12}$$

where "$\cdots$" denote the terms which can be related to the existed terms by total derivatives (like the term $\sim \nabla_iX^{ij}\nabla_j[X]$) or which turns out to have the vanishing coefficients finally (like the term $\sim X^{ij}\nabla_j\nabla^kX_{ki}$). We will explain this issue later.

Taking the variation of the action ansatz with respect to the boundary metric, we can obtain each term in the expansion of $\mathcal{K}$

$$\mathcal{K}=\mathcal{K}_{(0)} + \mathcal{K}_{(1)} + \cdots + \mathcal{K}_{(d)}. \tag{3.13}$$

The detail of computation is presented in appendix B. Here we write down the results

$$\begin{aligned}
\mathcal{K}_{(0)} &= -\frac{d}{d-1}U_{(0)}^2, \\
\mathcal{K}_{(1)} &= -2U_{(0)}U_{(1)} - \frac{2}{d-1}U_{(0)}\left(U_{(1)} - 2Y_{(1)}\right),\\
\mathcal{K}_{(2)} &= 4Y_{(1)ij}Y_{(1)}^{ij} - \left(2U_{(0)}U_{(2)} + U_{(1)}^2\right) - \frac{1}{d-1}\left(2U_{(0)}(U_{(2)} - 2Y_{(2)}) + (U_{(1)} - 2Y_{(1)})^2\right),\\
\mathcal{K}_{(3)} &= 4\left(Y_{(1)ij}Y_{(2)}^{ij} + Y_{(2)ij}Y_{(1)}^{ij}\right) - 2\left(U_{(0)}U_{(3)} + U_{(1)}U_{(2)}\right)\\
&\quad - \frac{2}{d-1}\Big(U_{(0)}(U_{(3)} - 2Y_{(3)}) + (U_{(1)} - 2Y_{(1)})(U_{(2)} - 2Y_{(2)})\Big),\\
\mathcal{K}_{(4)} &= 4\left(Y_{(1)ij}Y_{(3)}^{ij} + Y_{(1)}^{ij}Y_{(3)ij} + Y_{(2)ij}Y_{(2)}^{ij}\right) - \left(2U_{(0)}U_{(4)} + 2U_{(1)}U_{(3)} + U_{(2)}^2\right)\\
&\quad - \frac{1}{d-1}\Big(2U_{(0)}(U_{(4)} - 2Y_{(4)}) + 2(U_{(1)} - 2Y_{(1)})(U_{(3)} - 2Y_{(3)}) + (U_{(2)} - 2Y_{(2)})^2\Big),
\end{aligned} \tag{3.14}$$

where the expressions of $Y_{(m)ij}Y_{(n)}^{ij}$ can be readily obtained from eq. (B.1) and $Y_{(k)}$ can be related to $U_{(k)}$ by eqs. (B.2), (B.3), (B.7) and (B.11).

[10] Usually, the derivative expansion is equivalent to the inverse metric expansion. But for massive gravity, they are different and the latter is more convenient.

### 3.3 Solution of CPHJ equation

We proceed to solve the CPHJ equation (3.9) iteratively to determine the unknown coefficients $(A, B, C_i, D_i, E_i)$.

- The order 0 descent equation is

$$\mathcal{K}_{(0)} + d(d-1) + 2\frac{\partial U_{(0)}}{\partial r} = 0, \tag{3.15}$$

which has the solution

$$A(r) = -(d-1) + \mathcal{O}(e^{-dr}). \tag{3.16}$$

We only keep the leading term. By power counting, one can see that the subleading term is not divergent.

- With the order 0 result, one is able to solve the order 1 descent equation

$$\mathcal{K}_{(1)} + \alpha_1 e_1 + 2\frac{\partial U_{(1)}}{\partial r} = 0. \tag{3.17}$$

The solution about $[X]$ is

$$B(r) = \frac{\alpha_1}{2(1-d)} + \mathcal{O}(e^{(1-d)r}). \tag{3.18}$$

So $U_{(2k)}$ does contain the term with half-integer $k$.

- It is turned to deal with the order 2 descent equation

$$R + \mathcal{K}_{(2)} + sgn(d-2)\alpha_2 e_2 + 2\frac{\partial U_{(2)}}{\partial r} = 0, \tag{3.19}$$

which is needed when $d \geq 2$. Here we have introduced the sign function

$$sgn(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}, \tag{3.20}$$

which is invoked to emphasize the polynomial $e_d(X) = 0$ under the choice $f_{t\mu} = 0$, as presented in (C.10). The independence of boundary conditions results in

$$\begin{aligned} R: &\quad 1 + 2(d-2)C_1 + 2\dot{C}_1 = 0, \\ [X^2]: &\quad \bar{B}^2 + 2(d-2)C_2 - sgn(d-2)\alpha_2 + 2\dot{C}_2 = 0, \\ [X]^2: &\quad 2(d-2)C_3 - \bar{B}^2 + sgn(d-2)\alpha_2 + 2\dot{C}_3 = 0. \end{aligned} \tag{3.21}$$

Here $\bar{B}$ is defined as a constant, denoting the solved but unfixed coefficient $B$. Later notations about $\bar{C}_i$ and $\bar{D}_i$ are similar. The above equations have the solutions:

$$C_1 = \begin{cases} \frac{1}{2(2-d)} + \mathcal{O}(e^{(2-d)r}), & d > 2 \\ -\frac{r}{2} + \mathcal{O}(1), & d = 2 \end{cases}, \qquad C_2 = \begin{cases} \frac{\bar{B}^2 - \alpha_2}{2(2-d)} + \mathcal{O}(e^{(2-d)r}), & d > 2 \\ -\frac{\bar{B}^2}{2} r + \mathcal{O}(1), & d = 2 \end{cases}$$

$$C_3 = \begin{cases} -\frac{\bar{B}^2 - \alpha_2}{2(2-d)} + \mathcal{O}(e^{(2-d)r}), & d > 2 \\ \frac{\bar{B}^2}{2} r + \mathcal{O}(1), & d = 2 \end{cases} \tag{3.22}$$

- So far we have determined all the divergent terms for $d = 2$ but not enough for $d = 3, 4$. The next is the order 3 descent equation

$$\mathcal{K}_{(3)} + sgn(d-3)\alpha_3 e_3 + 2\frac{\partial U_{(3)}}{\partial r} = 0. \tag{3.23}$$

Collecting various functional terms gives

$$\begin{aligned}
[X]R: &\quad 2(d-3)D_1 - 2\bar{B}\bar{C}_1 + 2\dot{D}_1 = 0, \\
R_{ij}X^{ij}: &\quad 4\bar{B}\bar{C}_1 + 2(d-3)D_2 + 2\dot{D}_2 = 0, \\
[X^3]: &\quad 4\bar{B}\bar{C}_2 + 2(d-3)D_3 + sgn(d-3)2\alpha_3 + 2\dot{D}_3 = 0, \\
[X^2][X]: &\quad 2(d-3)D_4 - 6\bar{B}\bar{C}_2 - sgn(d-3)3\alpha_3 + 2\dot{D}_4 = 0, \\
[X]^3: &\quad 2(d-3)D_5 + 2\bar{B}\bar{C}_2 + sgn(d-3)\alpha_3 + 2\dot{D}_5 = 0,
\end{aligned} \tag{3.24}$$

where we have used $C_3 = -C_2$. The solutions are

$$\begin{aligned}
D_1 &= \begin{cases} \frac{\bar{B}\bar{C}_1}{d-3} + \mathcal{O}(e^{(3-d)r}), & d>3 \\ \bar{B}\bar{C}_1 r + \mathcal{O}(1), & d=3 \end{cases}, \qquad
D_2 = \begin{cases} -2\frac{\bar{B}\bar{C}_1}{d-3} + \mathcal{O}(e^{(3-d)r}), & d>3 \\ -2\bar{B}\bar{C}_1 r + \mathcal{O}(1), & d=3 \end{cases} \\
D_3 &= \begin{cases} \frac{2\bar{B}\bar{C}_2+\alpha_3}{3-d} + \mathcal{O}(e^{(3-d)r}), & d>3 \\ -2\bar{B}\bar{C}_2 r + \mathcal{O}(1), & d=3 \end{cases}, \qquad
D_4 = \begin{cases} -\frac{3}{2}\frac{2\bar{B}\bar{C}_2+\alpha_3}{3-d} + \mathcal{O}(e^{(3-d)r}), & d>3 \\ 3\bar{B}\bar{C}_2 r + \mathcal{O}(1), & d=3 \end{cases} \\
D_5 &= \begin{cases} \frac{1}{2}\frac{2\bar{B}\bar{C}_2+\alpha_3}{3-d} + \mathcal{O}(e^{(3-d)r}), & d>3 \\ -\bar{B}\bar{C}_2 r + \mathcal{O}(1), & d=3 \end{cases}.
\end{aligned} \tag{3.25}$$

Specifically, one can read $D_2 = -2D_1$, $D_3 = 2D_5$, $D_4 = -3D_5$.

- Now the case $d = 3$ is completed. Let us deal with the order 4 descent equation

$$\mathcal{K}_{(4)} + sgn(d-4)\alpha_4 e_4 + 2\frac{\partial U_{(4)}}{\partial r} = 0. \tag{3.26}$$

It induces a series of equations

$$\begin{aligned}
R_{ij}R^{ij}: &\quad 4\bar{C}_1^2 + 2(d-4)E_1 + 2\dot{E}_1 = 0, \\
R^2: &\quad -\frac{d}{d-1}\bar{C}_1^2 + 2(d-4)E_2 + 2\dot{E}_2 = 0,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
[X^2]R: &\quad 2\bar{B}\bar{D}_1 - \frac{d}{d-1}2\bar{C}_1\bar{C}_2 + 2(d-4)E_3 + 2\dot{E}_3 = 0, \\
[X]^2R: &\quad -2\bar{B}\bar{D}_1 + \frac{d}{d-1}2\bar{C}_1\bar{C}_2 + 2(d-4)E_4 + 2\dot{E}_4 = 0, \\
R_{ij}X^{ij}[X]: &\quad 8\bar{B}\bar{D}_1 - 8\bar{C}_1\bar{C}_2 + 2(d-4)E_5 + 2\dot{E}_5 = 0, \\
R^i{}_j X^j{}_k X^k{}_i: &\quad -12\bar{B}\bar{D}_1 + 8\bar{C}_1\bar{C}_2 + 2(d-4)E_6 + 2\dot{E}_6 = 0,
\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
[X^4]: &\quad 12\bar{B}\bar{D}_5+4\bar{C}_2^2-sgn(d-4)6\alpha_4+2(d-4)E_7+2\dot{E}_7=0,\\
[X^3][X]: &\quad -16\bar{B}\bar{D}_5-8\bar{C}_2^2+sgn(d-4)8\alpha_4+2(d-4)E_8+2\dot{E}_8=0,\\
[X^2][X]^2: &\quad 12\bar{B}\bar{D}_5+\left(4+2\frac{d}{d-1}\right)\bar{C}_2^2-sgn(d-4)6\alpha_4+2(d-4)E_9+2\dot{E}_9=0,\\
[X^2]^2: &\quad -6\bar{B}\bar{D}_5-\frac{d}{d-1}\bar{C}_2^2+sgn(d-4)3\alpha_4+2(d-4)E_{10}+2\dot{E}_{10}=0,\\
[X]^4: &\quad -2\bar{B}\bar{D}_5-\frac{d}{d-1}\bar{C}_2^2+sgn(d-4)\alpha_4+2(d-4)E_{11}+2\dot{E}_{11}=0,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
X^{ij}\nabla^k\nabla_j X_{ki}: &\quad 8\bar{B}\bar{D}_1+2(d-4)E_{12}+2\dot{E}_{12}=0,\\
X^{ij}\nabla^k\nabla_k X_{ij}: &\quad -4\bar{B}\bar{D}_1+2(d-4)E_{13}+2\dot{E}_{13}=0,\\
[X]\nabla_i\nabla_j X^{ij}: &\quad -8\bar{B}\bar{D}_1+2(d-4)E_{14}+2\dot{E}_{14}=0,\\
[X]\nabla_i\nabla^i[X]: &\quad 4\bar{B}\bar{D}_1+2(d-4)E_{15}+2\dot{E}_{15}=0.
\end{aligned}
\tag{3.30}
$$

Although there are so many equations, their solutions are still simple. When $d = 4$, they are

$$
\begin{aligned}
&E_1 = -2\bar{C}_1^2 r + \mathcal{O}(1), && E_2 = \frac{2}{3}\bar{C}_1^2 r + \mathcal{O}(1),\\
&E_3 = \left(\frac{4}{3}\bar{C}_1\bar{C}_2 - \bar{B}\bar{D}_1\right) r + \mathcal{O}(1), && E_4 = \left(-\frac{4}{3}\bar{C}_1\bar{C}_2 + \bar{B}\bar{D}_1\right) r + \mathcal{O}(1),\\
&E_5 = (4\bar{C}_1\bar{C}_2 - 4\bar{B}\bar{D}_1)r + \mathcal{O}(1), && E_6 = (-4\bar{C}_1\bar{C}_2 + 6\bar{B}\bar{D}_1)r + \mathcal{O}(1),\\
&E_7 = (-2\bar{C}_2^2 - 6\bar{B}\bar{D}_5)r + \mathcal{O}(1), && E_8 = (4\bar{C}_2^2 + 8\bar{B}\bar{D}_5)r + \mathcal{O}(1),\\
&E_9 = \left(-\frac{4}{3}\bar{C}_2^2 - 2\bar{C}_2^2 - 6\bar{B}\bar{D}_5\right) r + \mathcal{O}(1), && E_{10} = \left(\frac{2}{3}\bar{C}_2^2 + 3\bar{B}\bar{D}_5\right) r + \mathcal{O}(1),\\
&E_{11} = \left(\frac{2}{3}\bar{C}_2^2 + \bar{B}\bar{D}_5\right) r + \mathcal{O}(1), && E_{12} = -4\bar{B}\bar{D}_1 r + \mathcal{O}(1),\\
&E_{13} = 2\bar{B}\bar{D}_1 r + \mathcal{O}(1), && E_{14} = 4\bar{B}\bar{D}_1 r + \mathcal{O}(1),\\
&E_{15} = -2\bar{B}\bar{D}_1 r + \mathcal{O}(1).
\end{aligned}
\tag{3.31}
$$

Particularly we notice the simplification

$$
\begin{aligned}
&E_7[X^4] + E_8[X^3][X] + E_9[X^2][X]^2 + E_{10}[X^2]^2 + E_{11}[X]^4\\
&= \Bigg\{\bar{B}\bar{D}_5(-6[X^4] + 8[X^3][X] - 6[X^2][X]^2 + 3[X^2]^2 + [X]^4)\\
&\quad + \frac{2}{3}\bar{C}_2^2([X^2]^2 - 2[X^2][X]^2 + [X]^4) - \frac{1}{3}\bar{C}_2^2(6[X^4] - 12[X^3][X] + 6[X^2][X]^2)\Bigg\} r\\
&= \left\{\bar{B}\bar{D}_5 e_4 + \frac{2}{3}\bar{C}_2^2 e_2^2 - \frac{1}{3}\bar{C}_2^2(-2e_1e_3 + 3e_2^2 - e_4)\right\} r\\
&= \left(\frac{2}{3}e_1e_3 - \frac{1}{3}e_2^2\right)\bar{C}_2^2 r.
\end{aligned}
\tag{3.32}
$$

Obviously, the number of divergent terms increases quickly when the spacetime dimension increases. Here we give a remark that is useful to avoid neglecting certain divergent terms. Suppose that there should be a real divergent term labeled by $a(r)\mathcal{A}_{(k)}$ in the ansatz $U_{(k)}$ and $\mathcal{A}_{(k)}$ does not appear in $\mathcal{B}$, where $\mathcal{B}$ contains every term in $H_{\rm ct}$ except $\mathcal{K}$. Then we write the order $k$ CPHJ equation as:

$$\mathcal{K}_{(k)}+\mathcal{B}_{(k)}+2\frac{\partial U_{(k)}}{\partial r}=0, \tag{3.33}$$

where

$$\mathcal{K}_{(k)}=\sum_{\substack{m+n=k,\\0<m,n<k}}4Y_{(m)ij}Y^{ij}_{(n)}-\sum_{m+n=k}U_{(m)}U_{(n)}-\frac{1}{d-1}\sum_{m+n=k}(U_{(m)}-2Y_{(m)})(U_{(n)}-2Y_{(n)}). \tag{3.34}$$

We proceed to present an assumption that will be falsified in the end: $\mathcal{A}_{(k)}$ only appears in $U_{(k)}$ in eq. (3.33). Setting $m$ (or $n$) $=0$ and using $Y_{(k)}=\frac{k}{2}U_{(k)}+$total derivatives, we have

$$\mathcal{A}_{(k)}:\quad (d-k)a+\dot{a}=0. \tag{3.35}$$

The solution is $a=\mathcal{O}(e^{(k-d)r})$, which is impossible for a real divergent term because: the scaling of $\sqrt{\gamma}a(r)\mathcal{A}_{(k)}$ is $e^{dr}\cdot e^{(k-d)r}\cdot e^{-kr}=\mathcal{O}(1)$. Thus, our previous assumption is invalid, that is, the terms other than $U_{(k)}$ and $\mathcal{B}_{(k)}$ in eq. (3.33) must contain $\mathcal{A}_{(k)}$ whose coefficient is nonvanishing. Note that all these terms can be worked out with the pre-solved $U_{(m)}$, where $m<k$.

Put it another way, suppose that one has accidentally neglected a real divergent term $a(r)\mathcal{A}_{(k)}$ in the ansatz $U_{(k)}$. When organizing the $k$th order CPHJ equation, one then will obtain an ill-defined algebraic equation about the potentially divergent term $\mathcal{A}_{(k)}$. This is implied by the above analysis. Thus, the CPHJ equation can remind one to add $a(r)\mathcal{A}_{(k)}$ which makes the ansatz sufficient.

Keeping this remark in mind, we can explain quickly why the term like $E_{16}X^{ij}\nabla_j\nabla^kX_{ki}$ in the ansatz $U_{(4)}$ is not necessary. This is because in the 4th order CPHJ equation, $X^{ij}\nabla_j\nabla^kX_{ki}$ only appears in $U_{(4)}$.

Finally, we turn back to present the counterterm action by collecting above results. It can be written as

$$S_{\rm ct}=-\frac{1}{\kappa^2}\int_\Sigma d^dx\sqrt{\gamma}\big(U_{(0)}+U_{(1)}+\ldots+U_{(d)}\big). \tag{3.36}$$

The first two terms have the uniform

$$U_{(0)}=-(d-1),\qquad U_{(1)}=\frac{\alpha_1}{2(1-d)}e_1. \tag{3.37}$$

But other terms depend on the dimensions, which will be listed as follows.

- $d=2$

$$U_{(2)}=\left(-\frac{1}{2}R+\frac{\alpha_1^2}{8}([X]^2-[X^2])\right)r=-\frac{1}{2}Rr. \tag{3.38}$$

Here we have used $e_d=0$. Then the counterterm action is

$$S_{\rm ct}=\frac{1}{\kappa^2}\int_\Sigma d^2x\sqrt{\gamma}\left(1+\frac{1}{2}\alpha_1e_1+\frac{1}{2}Rr\right). \tag{3.39}$$

- $d = 3$

$$U_{(2)} = -\frac{1}{2}R + \left(\frac{1}{2}\alpha_2 - \frac{1}{32}\alpha_1^2\right)([X^2] - [X]^2) = -\frac{1}{2}R - \left(\frac{1}{2}\alpha_2 - \frac{1}{32}\alpha_1^2\right)e_2,$$

$$U_{(3)} = \left\{\frac{\alpha_1}{8}([X]R - 2R_{ij}X^{ij}) + \left(\frac{1}{8}\alpha_1\alpha_2 - \frac{1}{128}\alpha_1^3\right)(2[X^3] - 3[X^2][X] + [X]^3)\right\} r$$
$$= \frac{\alpha_1}{8}(e_1 R - 2R_{ij}X^{ij})r. \tag{3.40}$$

The counterterm action is

$$S_{\rm ct} = \frac{1}{\kappa^2}\int_\Sigma d^3x\sqrt{\gamma}\left\{2 + \frac{1}{4}\alpha_1 e_1 + \frac{1}{2}R + \left(\frac{1}{2}\alpha_2 - \frac{1}{32}\alpha_1^2\right)e_2 + \frac{\alpha_1}{8}(2R_{ij}X^{ij} - e_1 R)r\right\}. \tag{3.41}$$

This result is the same as eq. (3.15) in [40] up to the last logarithmic terms. Note that the logarithmic terms vanish precisely if one takes the metric (B7) in [40].

- $d = 4$

$$U_{(2)} = -\frac{1}{4}R + \left(\frac{1}{4}\alpha_2 - \frac{1}{144}\alpha_1^2\right)([X^2] - [X]^2) = -\frac{1}{4}R - \left(\frac{1}{4}\alpha_2 - \frac{1}{144}\alpha_1^2\right)e_2, \tag{3.42}$$

$$U_{(3)} = \frac{\alpha_1}{24}([X]R - 2R_{ij}X^{ij}) + \left(\frac{1}{24}\alpha_1\alpha_2 - \frac{1}{864}\alpha_1^3 - \frac{\alpha_3}{2}\right)(2[X^3] - 3[X^2][X] + [X]^3)$$
$$= \frac{\alpha_1}{24}(e_1 R - 2R_{ij}X^{ij}) + \left(\frac{1}{24}\alpha_1\alpha_2 - \frac{1}{864}\alpha_1^3 - \frac{\alpha_3}{2}\right)e_3, \tag{3.43}$$

$$U_{(4)} = \left\{-\frac{1}{8}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) - \left(\frac{\alpha_1^2}{108} - \frac{\alpha_2}{12}\right)e_2 R + \left(\frac{5}{144}\alpha_1^2 - \frac{\alpha_2}{4}\right)e_1 R_{ij}X^{ij} \right. \tag{3.44}$$
$$+ \left(\frac{\alpha_2}{4} - \frac{7\alpha_1^2}{144}\right)R^i{}_j X^j{}_k X^k{}_i - \left(\frac{1}{3}e_2^2 - \frac{2}{3}e_1 e_3\right)\left(\frac{\alpha_2}{4} - \frac{\alpha_1^2}{144}\right)^2$$
$$\left. - \frac{\alpha_1^2}{72}(-2X^{ij}\nabla^k\nabla_j X_{ki} + X^{ij}\nabla^k\nabla_k X_{ij} + 2[X]\nabla_i\nabla_j X^{ij} - [X]\nabla_i\nabla^i[X])\right\} r.$$

The counterterm action is

$$S_{\rm ct} = \frac{1}{\kappa^2}\int_\Sigma d^4x\sqrt{\gamma}\left\{3 + \frac{1}{6}\alpha_1 e_1 + \frac{1}{4}R + \left(\frac{1}{4}\alpha_2 - \frac{1}{144}\alpha_1^2\right)e_2 \right. \tag{3.45}$$
$$+ \frac{1}{24}\alpha_1(2R_{ij}X^{ij} - e_1 R) - \left(\frac{1}{24}\alpha_1\alpha_2 - \frac{1}{864}\alpha_1^3 - \frac{\alpha_3}{2}\right)e_3$$
$$+ \left[\frac{1}{8}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) + \left(\frac{\alpha_1^2}{108} - \frac{\alpha_2}{12}\right)e_2 R - \left(\frac{5}{144}\alpha_1^2 - \frac{\alpha_2}{4}\right)e_1 R_{ij}X^{ij}\right.$$
$$- \left(\frac{\alpha_2}{4} - \frac{7\alpha_1^2}{144}\right)R^i{}_j X^j{}_k X^k{}_i + \left(\frac{1}{3}e_2^2 - \frac{2}{3}e_1 e_3\right)\left(\frac{\alpha_2}{4} - \frac{\alpha_1^2}{144}\right)^2$$
$$\left.\left. + \frac{\alpha_1^2}{72}(-2X^{ij}\nabla^k\nabla_j X_{ki} + X^{ij}\nabla^k\nabla_k X_{ij} + 2[X]\nabla_i\nabla_j X^{ij} - [X]\nabla_a\nabla^a[X])\right] r\right\}.$$

### 3.4 Renormalized action

Now we will show that in some situations the divergent part of the on-shell action is actually cancelled by the counterterms that we have derived. In these situations, our assumption of the shift vector is justified.

#### 3.4.1 Background level

Consider the background level at first. Select the reference metric as

$$f_{ij} = \mathrm{diag}(0, h_{ab}), \tag{3.46}$$

where $h_{ab}$ is the metric of a $(d-1)$-dimensional Einstein space with constant curvature $(d-2)\,(d-1)k$ and the parameter $k = 0, \pm 1$. There are the black-hole solutions for massive gravity in $(d+1)$-dimensional spacetimes

$$ds^2 = f(z)d\tau^2 + f^{-1}(z)dz^2 + z^2 h_{ab}dx^a dx^b, \tag{3.47}$$

where the coordinate $z$ is related to $r$ via $z = e^r$ and the blackening factor is

$$f(z) = k + z^2 - \frac{m_0}{z^{d-2}} + \frac{\alpha_1}{d-1}z + \alpha_2 + \frac{(d-2)\alpha_3}{z}, \tag{3.48}$$

with the mass parameter $m_0$. There are no cross terms in eq. (3.47), so the GNC is obviously available. In the following, we calculate the counterterms and the renormalized action $S_{\mathrm{ren}} = \lim_{z\to\infty}(S_{\mathrm{on-shell}} + S_{\mathrm{ct}})$ for different dimensions.

- $d = 2$

  Using the background metric (3.47), the counterterms (3.39) can be reduced to

$$S_{\mathrm{ct}} = \frac{V}{2\kappa^2}\left[-(a_1 + 2z)\sqrt{f} + 2f + zf'\right], \tag{3.49}$$

  where $V \equiv \int_\Sigma d^dx\sqrt{h}$ and $h$ is the determinant of the metric $h_{ab}$. Then the renormalized action can be obtained

$$S_{\mathrm{ren}} = \frac{V}{2\kappa^2}\left(z_+^2 - \frac{a_1}{4}\right), \tag{3.50}$$

  where $z_+$ denotes the location of the horizon.

- $d = 3$

  When $d = 3$, the finiteness of the renormalized action has been checked in the black-hole background [40].

- $d = 4$

  The counterterms (3.45) can be calculated as

$$S_{\mathrm{ct}} = -\frac{V}{2\kappa^2}\Bigg\{\left(6z^2 f + z^3 f'\right) - 6z^3\sqrt{f} - \alpha_1 z^2\sqrt{f} + \left(\frac{1}{12}\alpha_1^2 z - 3\alpha_2 z - 3kz\right)\sqrt{f} + \left(\frac{1}{2}\alpha_1\alpha_2 - 6\alpha_3 + \frac{1}{2}\alpha_1 k - \frac{1}{72}\alpha_1^3\right)\sqrt{f}\Bigg\}. \tag{3.51}$$

  Appending the counterterms to the on-shell action, we have

$$S_{\mathrm{ren}} = -\frac{V}{2\kappa^2}\Bigg\{\left(z_+^4 - kz_+^2 - \alpha_2 z_+^2 - 4\alpha_3 z_+\right) - \frac{3}{4}k^2 - \frac{3}{2}k\alpha_2 - \frac{3}{4}\alpha_2^2 - \alpha_1\alpha_3 + \frac{\alpha_1^2}{8}(\alpha_2 + k) - \frac{5}{1728}\alpha_1^4\Bigg\}. \tag{3.52}$$

As shown, for various dimensions, the divergent terms in the on-shell action at the background level have been canceled out. Moreover, we find that the Hawking temperature $T = f'(z_+)/(4\pi)$, the Bekenstein entropy $S = 4\pi z_+^{d-1}V/\left(2\kappa^2\right)$, and the grand potential $\Omega = -TS_{\text{ren}}$ exactly obey the thermodynamical formula $\partial\Omega/\partial T = S$. This is a self-consistent check of our results.

### 3.4.2 Perturbation level

At the perturbation level, we cannot prove in general that the shift vector is falling off fast enough. Fortunately, for the optical perturbations (finite frequency, zero wave vector) that are often studied in the holographic theories of condensed matther physics, we find that our counterterms are enough to cancel the divergence terms in some cases. To exhibit them clearly, we turn on the time-dependent linear perturbations above the black-hole background (3.47). We focus on $k = 0$ for simplicity, which denotes the flat geometry of the field theory. These perturbation modes can be separated into three groups. The shift vector appears as a vector mode but decouples with the scalar and tensor modes. Thus, our counterterms are applicable for the theories involving the scalar and tensor modes. As for the vector modes, we will show that the shift vector is actually falling fast enough in three cases below. For convenience, we write the coupled vector modes as $\delta g_{tx}\left(t,z\right) = z^2 h_{tx}(t,z)$ and $\delta g_{xz}\left(t,z\right) = z^2 h_{xz}(t,z)$. In the fourier space, they can be expressed as

$$h_{tx}(t,z) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}h_{tx}(\omega,z),\ h_{xz}(t,z) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}h_{xz}(\omega,z). \tag{3.53}$$

- $d = 2$

  Let's write down the coupled EOM of two vector modes

$$\begin{aligned} h''_{tx} + \frac{3}{z}h'_{tx} + i\omega h'_{xz} + \frac{3i}{z}h_{xz} + \frac{\alpha_1}{zf}h_{tx} &= 0, \\ h_{xz} - \frac{iz\omega h'_{tx}}{z\omega^2 + \alpha_1 f} &= 0, \end{aligned} \tag{3.54}$$

  where the prime denotes the derivative with respect to $z$. From eq. (3.54), one can see that $h_{xz}$ is completely determined by $h_{tx}$. Near the boundary, the asymptotic solutions read

$$\begin{aligned} h_{tx} &= h_{tx}^{(0)} + \frac{1}{z^1}h_{tx}^{(1)} + \frac{1}{z^2}h_{tx}^{(2)} + \cdots \\ h_{xz} &= \frac{1}{z^3}h_{xz}^{(3)} + \frac{1}{z^4}h_{xz}^{(4)} + \cdots . \end{aligned} \tag{3.55}$$

  Here the coefficient $h_{tx}^{(0)}$ is the only independent source. Two coefficients $h_{tx}^{(1)}$ and $h_{xz}^{(3)}$ are fixed by $h_{tx}^{(0)}$. The exact relations are $h_{tx}^{(1)} = \alpha_1 h_{tx}^{(0)}$ and $h_{xz}^{(3)} = -i\omega h_{tx}^{(0)}$. Other coefficients $h_{tx}^{(2)}$ and $h_{xz}^{(4)}$ rely on $h_{tx}^{(0)}$ and the incoming boundary conditions at the horizon. Note that the presence of $h_{tx}^{(1)}$ is due to the diffeomorphism breaking.

Expanding the on-shell action and the counterterm action above the background, we obtain a quadratic action

$$S^{(2)}_{\text{on-shell}} + S^{(2)}_{\text{ct}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bigg\{ -\frac{1}{4}\alpha_1 z^2 \bar{h}_{xz}h_{xz} + \frac{1}{2}iz^3\omega\bar{h}_{tx}h_{xz} + \left(z^2 - \frac{z^3}{\sqrt{f}}\right)\bar{h}_{tx}h_{tx} + \frac{1}{2}z^3\bar{h}_{tx}h'_{tx}\bigg\}, \tag{3.56}$$

where the modes with the bar have the argument $-\omega$. Substituting the asymptotic solutions (3.55) and the blackening factor (3.48) into eq. (3.56), we obtain the renormalized action:

$$S^{(2)}_{\text{ren}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{1}{8}\left(\alpha_1^2 - 4\alpha_1 z_+ - 4z_+^2 + 4\omega^2\right)\bar{h}^{(0)}_{tx}h^{(0)}_{tx} - \bar{h}^{(0)}_{tx}h^{(2)}_{tx}\right\}. \tag{3.57}$$

One can find that it is finite.

- $d = 3,\ \alpha_1 = 0,\ \alpha_2 \neq 0$

For higher dimensions, our counterterms are not enough to cancel the whole divergent part of the on-shell action in general. But when we set $\alpha_1 = 0$ for $d = 3$ or $\alpha_1 = \alpha_2 = 0$ for $d = 4$, the renormalized action is finite. Since the derivation is similar to the previous case, we will be a little abbreviated.

The EOM for $d = 3$ are

$$\begin{aligned} h''_{tx} + \frac{4}{z}h'_{tx} + i\omega h'_{xz} + \frac{4i}{z}h_{xz} + \frac{2\alpha_2}{z^2 f}h_{tx} &= 0, \\ h_{xz} - \frac{iz^2\omega h'_{tx}}{z^2\omega^2 + 2\alpha_2 f} &= 0. \end{aligned} \tag{3.58}$$

The asymptotic solutions read

$$\begin{aligned} h_{tx} &= h^{(0)}_{tx} + \frac{1}{z^2}h^{(2)}_{tx} + \frac{1}{z^3}h^{(3)}_{tx} + \cdots \\ h_{xz} &= \frac{1}{z^3}h^{(3)}_{xz} + \frac{1}{z^5}h^{(4)}_{xz} + \cdots, \end{aligned} \tag{3.59}$$

where $h^{(2)}_{tx} = (\alpha_2 + \omega^2/2)h^{(0)}_{tx}$ and $h^{(3)}_{xz} = -i\omega h^{(0)}_{tx}$. The higher order coefficients cannot be determined by the source $h^{(0)}_{tx}$ alone. The quadratic action can be obtained:

$$S^{(2)}_{\text{on-shell}} + S^{(2)}_{\text{ct}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{1}{2}iz^4\omega\bar{h}_{tx}h_{xz} + 2\left(z^3 - \frac{z^4}{\sqrt{f}}\right)\bar{h}_{tx}h_{tx} + \frac{1}{2}z^4\bar{h}_{tx}h'_{tx}\right\}. \tag{3.60}$$

It follows the renormalized action

$$S^{(2)}_{\text{ren}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\left(z_+^2 - \alpha_2 z_+\right)\bar{h}^{(0)}_{tx}h^{(0)}_{tx} + \frac{3}{2}\left(\frac{\omega^2}{2\alpha_2 + \omega^2} - 1\right)\bar{h}^{(0)}_{tx}h^{(3)}_{tx}\right\}. \tag{3.61}$$

- $d = 4,\ \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 \neq 0$

  The EOM are

$$\begin{aligned} h''_{tx} + \frac{5}{z}h'_{tx} + i\omega h'_{xz} + \frac{5i}{z}h_{xz} + \frac{6\alpha_3}{z^3 f}h_{tx} &= 0, \\ h_{xz} - \frac{iz^3\omega h'_{tx}}{z^3\omega^2 + 6\alpha_3 f} &= 0, \end{aligned} \tag{3.62}$$

  which have the asymptotic solutions

$$\begin{aligned} h_{tx} &= h^{(0)}_{tx} + \frac{1}{z^2}h^{(2)}_{tx} + \frac{1}{z^3}h^{(2)}_{tx} + \cdots \\ h_{xz} &= \frac{1}{z^3}h^{(3)}_{xz} + \frac{1}{z^5}h^{(4)}_{xz} + \cdots, \end{aligned} \tag{3.63}$$

  with $h^{(2)}_{tx} = \omega^2 h^{(0)}_{tx}/2$ and $h^{(3)}_{xz} = -i\omega h^{(0)}_{tx}$. The quadratic action is

$$S^{(2)}_{\text{on-shell}} + S^{(2)}_{\text{ct}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{1}{2}iz^5\omega\bar{h}_{tx}h_{xz} + 3\left(z^4 - \frac{z^5}{\sqrt{f}}\right)\bar{h}_{tx}h_{tx} + \frac{1}{2}z^5\bar{h}_{tx}h'_{tx}\right\}. \tag{3.64}$$

  The renormalized action is

$$S^{(2)}_{\text{ren}} = \frac{V}{2\kappa^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\left(-3\alpha_3 z_+ - \frac{3}{2}z_+^4 + \frac{18\alpha_3^3}{\omega^2}\right)\bar{h}^{(0)}_{tx}h^{(0)}_{tx} - \frac{9\alpha_3}{\omega^2}\bar{h}^{(0)}_{tx}h^{(3)}_{tx}\right\}. \tag{3.65}$$

## 4 Conclusion

As part of the foundations of AdS/CFT correspondence, holographic renormalization is a systematic procedure to remove the divergences by appending the local boundary counterterms to the on-shell action. Among several approaches to holographic renormalization, the one based on the Hamiltonian formalism has been developed recently. The new approach starts from the HJ equation and has been argued to be practical in [23]. However, it has not been clarified that whether there is a conflict with the Hamiltonian constraint, which should be respected by any theories of gravity that are invariant under the diffeomorphism. In this paper, we divide the HJ equation into two parts and point out that only one part is actually used to execute the holographic renormalization. The derivation of the CPHJ equation does not explicitly depend on the vanishing of Hamiltonian or not, hence being free of conflicts with the Hamiltonian constraint.

Then we apply the HJ approach to the massive gravity with different dimensions. Previously, by imposing the GNC and additional conditions on the characteristic tensor of massive gravity, the standard approach was used to build up the counterterms with $d = 3$ [40]. Here we only assume that the shift vector is falling off fast enough asymptotically, indicating a little more general situation than before. We have checked that our counterterms are applicable at the background level. At the perturbation level, we have shown that there are several time-dependent cases where our counterterms is enough to cancel the divergent part of the on-shell action. Thus, our results should be useful for the holographic calculation of thermodynamics and transports in the strongly coupled field theories dual to massive gravity. Moreover, we have found that the conformal anomalies appear in both odd and even dimensions. This is different from the (pure) Einstein gravity: it is well-known that there are no conformal anomalies in odd boundary dimensions [5, 6]. It would be interesting to study whether it has some profound implications on the renormalization group flow.

Our work suggests that the HJ approach is a practical approach to holographic renormalization, especially for the theories with conformal anomalies. This is because the logarithmic divergences can be identified by the same fluent procedure as the power divergences.

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## A Einstein-scalar theory

We will give a brief review on the HJ approach to the holographic renormalization of the Einstein gravity with massive scalars. More details can be found in [23]. One can find here that the master equation is the CPHJ equation and the procedure can be conveniently split into three steps, which are corresponding to three subsections.

### A.1 Hamilton-Jacobi formalism

Consider the bulk action (2.6) and the ADM metric (2.7), by which one can obtain the Lagrangian

$$\begin{aligned} L = -\frac{1}{2\kappa^2}\int_{\partial M} d^d x\sqrt{\gamma}N\Bigg[R + K^2 - K_{ij}K^{ij} - \frac{1}{N^2}G_{IJ}\dot{\Phi}^I\dot{\Phi}^J \\ + 2\frac{N^i}{N^2}G_{IJ}\dot{\Phi}^I\partial_i\Phi^J - \left(\gamma^{ij} + \frac{N^iN^j}{N^2}\right)G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\Bigg], \end{aligned} \tag{A.1}$$

where $K_{ij}$ is the extrinsic curvature and $R$ is the Ricci scalar on the boundary. Then the canonical momenta conjugate to the fields can be given by

$$\pi^{ij} \equiv \frac{\partial L}{\partial \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}(K^{ij} - K\gamma^{ij}), \tag{A.2}$$

$$\pi_I \equiv \frac{\partial L}{\partial \dot{\Phi}^I} = \frac{1}{\kappa^2 N}\sqrt{\gamma}(G_{IJ}\dot{\Phi}^J - N^iG_{IJ}\partial_i\Phi^J). \tag{A.3}$$

Since eq. (A.1) involves neither $\dot{N}$ nor $\dot{N}_i$, the shift and lapse are Lagrangian multipliers which lead to the primary constraints

$$\pi_N \equiv \frac{\partial L}{\partial \dot{N}} = 0, \qquad \pi_{N^i} \equiv \frac{\partial L}{\partial \dot{N}_i} = 0. \tag{A.4}$$

The Hamiltonian can be defined by a Legendre transformation of Lagrangian

$$H \equiv \int_{\partial M} d^d x (\pi^{ij}\dot{\gamma}_{ij} + \pi_I \dot{\Phi}^I) - L = \int_{\partial M} d^d x (N\mathcal{H} + N_i \mathcal{H}^i), \tag{A.5}$$

where

$$\begin{aligned} \mathcal{H} &= \frac{2\kappa^2}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{d-1}\pi^2 + \frac{1}{4}G^{IJ}\pi_I\pi_J\right) + \frac{\sqrt{\gamma}}{2\kappa^2}\left(R - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\right), \\ \mathcal{H}^i &= -2\nabla_j \pi^{ij} + G_{IJ}\pi^I \partial^i \Phi^J. \end{aligned} \tag{A.6}$$

An important feature of $\mathcal{H}$ and $\mathcal{H}^i$ is that they are independent with $N$ and $N_i$. Thus, the Hamilton's equations for $N$ and $N_i$ impose the secondary constraints

$$\mathcal{H} = 0, \qquad \mathcal{H}^i = 0, \tag{A.7}$$

which are called the Hamiltonian constraint and the momentum constraint, respectively.

Furthermore, due to the diffeomorphism symmetry, one can fix the gauge

$$N = 1, \qquad N_i = 0. \tag{A.8}$$

Then the bulk metric is simply

$$ds^2 = dr^2 + \gamma_{ij}dx^i dx^j, \tag{A.9}$$

and the Hamiltonian is reduced to

$$H = \int_{\partial M} d^d x \mathcal{H}. \tag{A.10}$$

Consider that the canonical momenta in the Hamiltonian formalism can be replaced by [41]

$$\pi^{ij} = \frac{\delta S_{\text{on-shell}}}{\delta \gamma_{ij}}, \qquad \pi_I = \frac{\delta S_{\text{on-shell}}}{\delta \Phi^I}. \tag{A.11}$$

One can obtain the HJ equation of Einstein-scalar theories

$$H\left(\gamma_{ij}, \Phi^I; \frac{\delta S_{\text{on-shell}}}{\delta \gamma_{ij}}, \frac{\delta S_{\text{on-shell}}}{\delta \Phi^I}\right) + \frac{\partial S_{\text{on-shell}}}{\partial r} = 0. \tag{A.12}$$

In section 2, by decomposing the HJ equation, the CPHJ equation has been built up

$$H_{\text{ct}} + \frac{\partial S_{\text{ct}}}{\partial r} = 0, \tag{A.13}$$

where $S_{\text{ct}}$ denotes the (negative) divergent part of the on-shell action and $H_{\text{ct}}$ is the part of $H$ irrelevant to the renormalized action. For later use, we rewrite $S_{\text{ct}}$ in a general form

$$S_{\text{ct}} = -\frac{1}{\kappa^2}\int_\Sigma d^d x \sqrt{\gamma} U(\gamma^{ij}, \Phi^I, r), \tag{A.14}$$



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where $\Sigma$ is the hypersurface at finite radial cutoff near the boundary. Its variation can be expressed as

$$\frac{\delta S_{\rm ct}}{\delta \gamma_{ij}} = -\frac{1}{\kappa^2}\left(\frac{1}{2}\gamma^{ij}\sqrt{\gamma}U + \int_\Sigma d^d x\sqrt{\gamma}\frac{\delta U}{\delta \gamma_{ij}}\right), \tag{A.15}$$

$$\frac{\delta S_{\rm ct}}{\delta \Phi^I} = -\frac{1}{\kappa^2}\int_\Sigma d^d x\sqrt{\gamma}\frac{\delta U}{\delta \Phi^I}. \tag{A.16}$$

Now eq. (2.14) can be written by

$$H_{\rm ct} = -\frac{1}{2\kappa^2}\int_{\partial M} d^d x\sqrt{\gamma}\left[\mathcal{K} + G^{IJ}P_I P_J + R - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\right], \tag{A.17}$$

where

$$\mathcal{K} = 4Y_{ij}Y^{ij} - \frac{1}{d-1}(U-2Y)^2 - U^2, \tag{A.18}$$

$$Y_{ij} = \frac{\tilde{\delta} U}{\tilde{\delta}\gamma^{ij}}, \qquad Y^{ij} = -\frac{\tilde{\delta} U}{\tilde{\delta}\gamma_{ij}}, \qquad Y = \gamma^{ij}Y_{ij}, \qquad P_I = \frac{\tilde{\delta} U}{\tilde{\delta}\Phi^I}, \tag{A.19}$$

and for convenience we have defined the operator:

$$\frac{\tilde{\delta}}{\tilde{\delta}\mathbb{X}} \equiv \frac{1}{\sqrt{\gamma}}\int_\Sigma d^d x\sqrt{\gamma}\frac{\delta}{\delta \mathbb{X}}. \tag{A.20}$$

Finally, the CPHJ equation takes the form

$$R + \mathcal{K} + G^{IJ}P_I P_J - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi) + 2\frac{\partial U}{\partial r} = 0, \tag{A.21}$$

which holds as an integral equation. One can find that eq. (A.21) is nothing but the master equation (2.15) in [23]. Here we have shown that it should be understood as the CPHJ equation instead of the complete HJ equation.

### A.2 Action ansatz and variation

For simplicity, we will only involve a single massive scalar below. Then the action is

$$S = -\frac{1}{2\kappa^2}\int_M d^{d+1}x\sqrt{g}(R[g] - g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m_\Phi^2\Phi^2 - 2\Lambda) - \frac{1}{\kappa^2}\int_{\partial M} d^d x\sqrt{\gamma}K. \tag{A.22}$$

Since we assume the AdS boundary, the leading asymptotic behavior of the induced metric gives

$$\sqrt{\gamma} \sim e^{dr}\sqrt{\bar{\gamma}}, \tag{A.23}$$

where $\bar{\gamma}_{ij}$ is the source of the boundary stress energy tensor. This implies that the ansatz for $U$ can be organized into the expansion

$$U = U_{(0)} + U_{(2)} + \cdots + U_{(2\lfloor\frac{d}{2}\rfloor)}, \tag{A.24}$$

where $U_{(2k)}$ contains $k$ inverse metrics (or $2k$ derivatives) and $\lfloor d/2\rfloor$ denotes the integer no more than $d/2$. For the Einstein-scalar theory, the potentially divergent terms in $U_{(2k)}$ are



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made of the scalar field $\Phi$ and boundary metric $\gamma^{ij}$. Using the leading asymptotic behavior of the scalar

$$\Phi \sim e^{-(d-\Delta_\Phi)r}\bar{\Phi}, \tag{A.25}$$

where $\Delta_\Phi = \frac{d}{2}+\sqrt{\frac{d^2}{4}+m_\Phi^2}$ is the conformal dimension of the dual operator, one can figure out the maximal number of the scalar that can be included in a potential divergent term. The ansatz for the first two order is

$$U_{(0)} = A_0(r) + A_1(r)\Phi + A_2(r)\Phi^2 + \cdots \tag{A.26}$$

$$U_{(2)} = B_0(r)R + B_1(r)R\Phi + B_2(r)R\Phi^2 + B_3(r)\Phi\Box\Phi + \cdots \tag{A.27}$$

Note that any terms are considered as equivalent if they are related by a total derivative. In addition, since action (A.22) is symmetric under $\Phi \leftrightarrow -\Phi$, the coefficients $A_1(r)$, $B_1(r)$ are simply zero.

In terms of the action ansatz, we can calculate the momenta by variations. The relevant quantities are

$$\mathcal{K} = \mathcal{K}_{(0)} + \mathcal{K}_{(2)} + \cdots \tag{A.28}$$

$$P_\Phi = P_{\Phi(0)} + P_{\Phi(2)} + \cdots \tag{A.29}$$

where

$$\mathcal{K}_{(0)} = -\frac{d}{d-1}U_{(0)}^2, \qquad \mathcal{K}_{(2)} = -\frac{2}{d-1}U_{(0)}(U_{(2)} - 2Y_{(2)}) - 2U_{(0)}U_{(2)} \tag{A.30}$$

$$P_{\Phi(0)} = 2A_2(r)\Phi, \qquad P_{\Phi(2)} = 2B_2(r)R\Phi + 2B_3(r)\Box\Phi + \cdots, \tag{A.31}$$

with

$$Y_{(2)} = \gamma^{ij}\frac{\tilde{\delta}U_{(2)}}{\tilde{\delta}\gamma^{ij}} = U_{(2)} + B_2(r)(d-1)\Box\Phi^2 + B_3(r)\left(1-\frac{1}{2}d\right)\nabla_i(\Phi\nabla^i\Phi) + \cdots . \tag{A.32}$$

### A.3 Solution of CPHJ equation

By inserting the ansatz into the CPHJ equation (A.21) and using the momentum-relevant quantities calculated above, one can solve the CPHJ equation order by order. We start with the order 0 equation

$$-\frac{d}{d-1}U_{(0)}^2 + P_{\Phi(0)}^2 - m_\Phi^2\Phi^2 + d(d-1) + 2\frac{\partial U_{(0)}}{\partial r} = 0. \tag{A.33}$$

Collecting the non-functional terms, we have

$$-\frac{d}{d-1}A_0^2 + d(d-1) + 2\dot{A}_0 = 0. \tag{A.34}$$

The solution is

$$A_0 = -(d-1) + \mathcal{O}(e^{-dr}). \tag{A.35}$$

The subleading terms give only finite contribution and can be discarded directly.[11] The coefficients of $\Phi^2$ can be organized into another differential equation

$$\Phi^2: \quad -\frac{d}{d-1}2A_0A_2 + 4A_2^2 - m_\Phi^2 + 2\dot{A}_2 = 0. \tag{A.36}$$

The mass of the scalar field is restricted by Breitenlohner-Freedman bound [46]

$$m_\Phi^2 \geq -\frac{d^2}{4}. \tag{A.37}$$

The solutions of eq. (A.36) rely on the value of mass

$$m_\Phi^2 = -\frac{d^2}{4}: \quad A_2 = -\frac{d}{4} + \frac{1}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{A.38}$$

$$m_\Phi^2 > -\frac{d^2}{4}: \quad A_2 = \frac{1}{2}(\Delta_\Phi - d) + \mathcal{O}(e^{-(2\Delta_\Phi - d)r}). \tag{A.39}$$

We will use the solution (A.39) to continue the renormalization procedure. Another branch is similar. Thus, $U_{(0)}$ has been specified

$$U_{(0)} = -(d-1) - \frac{1}{2}(d-\Delta_\Phi)\Phi^2 + \cdots. \tag{A.40}$$

We turn to the order 2 equation

$$R + \left[-\frac{2}{d-1}U_{(0)}(U_{(2)} - 2Y_{(2)}) - 2U_{(0)}U_{(2)}\right] + 2P_{(0)}P_{(2)} - \gamma^{ij}\partial_i\Phi\partial_j\Phi + 2\frac{\partial U_{(2)}}{\partial r} = 0. \tag{A.41}$$

Equation (A.41) naturally induces the following equations

$$R: \quad 1 - (4-2d)B_0 + 2\dot{B}_0 = 0, \tag{A.42}$$

$$R\Phi^2: \quad \frac{(\Delta_\Phi - d)(2-d)}{d-1}B_0 + (4\Delta_\Phi - 2d - 4)B_2 + 2\dot{B}_2 = 0, \tag{A.43}$$

$$\Phi\Box\Phi: \quad 1 + (4\Delta_\Phi - 2d - 4)B_3 + 2\dot{B}_3 = 0. \tag{A.44}$$

For $d = 2$, the solution for $B_0$ is

$$B_0 = -\frac{1}{2}r. \tag{A.45}$$

In addition, the solutions for $B_2$ and $B_3$ indicate that they are not relevant to the divergent terms. Then

$$U_{(2)} = -\frac{1}{2}rR. \tag{A.46}$$

[11] We notice that the integral constant happens to be at the subleading order. Otherwise additional boundary conditions are needed to determine the integral constant, which can complicate or even invalidate the HJ approach. This situation is interesting and can be traced back to the fact that the integral constant in the solution of the HJ equation is exactly an additive constant tacked on to the on-shell action [41, 47].

For $d>2$, the solutions are

$$B_0 = \frac{1}{4-2d} + \mathcal{O}(e^{-(d-2)r}), \tag{A.47}$$

$$B_2 = \begin{cases} \frac{d-\Delta_\Phi}{4(d-1)(2\Delta_\Phi-d-2)} + \mathcal{O}(e^{(d+2-2\Delta_\Phi)r}) & \text{for } 2\Delta_\Phi - d - 2 \neq 0 \\ \frac{d-\Delta_\Phi}{4(d-1)} r + \mathcal{O}(1) & \text{for } 2\Delta_\Phi - d - 2 = 0 \end{cases}, \tag{A.48}$$

$$B_3 = \begin{cases} -\frac{1}{2(2\Delta_\Phi-d-2)} + \mathcal{O}(e^{(d+2-2\Delta_\Phi)r}) & \text{for } 2\Delta_\Phi - d - 2 \neq 0 \\ -\frac{1}{2} r + \mathcal{O}(1) & \text{for } 2\Delta_\Phi - d - 2 = 0 \end{cases}, \tag{A.49}$$

which implies

$$U_{(2)} = \begin{cases} \frac{1}{4-2d} R - \frac{1}{2(2\Delta_\Phi-d-2)} \left[\Phi\Box\Phi - \frac{d-\Delta_\Phi}{2(d-1)} R\Phi^2\right] + \cdots & \text{for } 2\Delta_\Phi - d - 2 \neq 0 \\ \frac{1}{4-2d} R - \frac{1}{2} \left[\Phi\Box\Phi - \frac{d-\Delta_\Phi}{2(d-1)} R\Phi^2\right] r + \cdots & \text{for } 2\Delta_\Phi - d - 2 = 0 \end{cases}$$

One can further deal with higher order descent equations if needed. Finally, the counterterm action is

$$S_{\text{ct}} = -\frac{1}{\kappa^2} \int_\Sigma d^d x \sqrt{\gamma} \left[U_{(0)} + U_{(2)} + \ldots + U_{(2\lfloor \frac{d}{2} \rfloor)}\right]. \tag{A.50}$$

This result agrees with the one given by the standard approach [6, 48].

# B The details of computation

Here we present the details when dealing with $\mathcal{K}$ term in the CPHJ equation of massive gravity. The basic formulas of functional variations with respect to the boundary metric are given in appendix C.

- $Y_{(m)ij} Y^{ij}_{(n)}$

  The following equations have been used when we calculate the terms $Y_{(m)ij} Y^{ij}_{(n)}$:

$$\begin{aligned} \frac{\tilde{\delta}[X^m]}{\tilde{\delta}\gamma_{ij}} \frac{\tilde{\delta}[X^n]}{\tilde{\delta}\gamma^{ij}} &= -\frac{mn}{4}[X^{m+n}], \\ \frac{\tilde{\delta}[X]}{\tilde{\delta}\gamma_{ij}} \frac{\tilde{\delta}R}{\tilde{\delta}\gamma^{ij}} &= -\frac{1}{2} X^{ij} R_{ij}, \\ \frac{\tilde{\delta}[X]}{\tilde{\delta}\gamma_{ij}} \frac{\tilde{\delta}(X^{kl}R_{kl})}{\tilde{\delta}\gamma^{ij}} &= -\frac{3}{4} R^i{}_j X^j{}_k X^k{}_i \\ &\quad + \frac{1}{4}(2X^{ij}\nabla^k\nabla_j X_{ki} - X^{ij}\nabla^k\nabla_k X_{ij} - [X]\nabla_i\nabla_j X^{ij}), \\ \frac{\tilde{\delta}[X]}{\tilde{\delta}\gamma_{ij}} \frac{\tilde{\delta}([X]R)}{\tilde{\delta}\gamma^{ij}} &= -\frac{1}{4}[X]^2 R - \frac{1}{2}(R_{ij}X^{ij}[X] + [X]\nabla_i\nabla^i[X] - X^{ij}\nabla_i\nabla_j[X]). \end{aligned} \tag{B.1}$$

  Remind that the operator $\tilde{\delta}/\tilde{\delta}\gamma_{ij}$ has been defined in eq. (A.20).

- $Y_{(k)}$

  One can simplify the computation by utilizing the relation between $U_{(k)}$ and $Y_{(k)}$. We state it from the beginning, when $k = 1$:

$$Y_{(1)} = \gamma^{ij} \frac{\tilde{\delta} U_{(1)}}{\tilde{\delta}\gamma^{ij}} = \gamma^{ij} \left(\frac{1}{2} B X_{ij}\right) = \frac{1}{2} U_{(1)}. \tag{B.2}$$

  Similarly, when $k = 2$,

$$Y_{(2)} = \gamma^{ij} \frac{\tilde{\delta} U_{(2)}}{\tilde{\delta}\gamma^{ij}} = \gamma^{ij} (C_1 R_{ij} + C_2 [X^2]_{ij} + C_3 [X] X_{ij}) = U_{(2)}. \tag{B.3}$$

  When $k = 3$,

$$\gamma^{ij} \frac{\tilde{\delta}([X]R)}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2}[X]R + \text{total derivatives}, \tag{B.4}$$

$$\gamma^{ij} \frac{\tilde{\delta}(R_{kl} X^{kl})}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2} R_{kl} X^{kl} + \text{total derivatives}, \tag{B.5}$$

$$\gamma^{ij} \frac{\tilde{\delta}[X^3]}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2}[X^3], \quad \gamma^{ij} \frac{\tilde{\delta}([X^2][X])}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2}[X^2][X], \quad \gamma^{ij} \frac{\tilde{\delta}[X]^3}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2}[X]^3. \tag{B.6}$$

  Then one can read off

$$Y_{(3)} = \gamma^{ij} \frac{\tilde{\delta} U_{(3)}}{\tilde{\delta}\gamma^{ij}} = \frac{3}{2} U_{(3)} + \text{total derivatives}. \tag{B.7}$$

  These total derivatives can be directly dropped after considering the fact that the HJ equation is an integral equation where they are multiplied by the constant $U_{(0)}$. When $k = 4$, some results are exemplified:

$$\gamma^{ij} \frac{\tilde{\delta}(R_{kl} R^{kl})}{\tilde{\delta}\gamma^{ij}} = 2 R_{kl} R^{kl} + \text{total derivatives}, \tag{B.8}$$

$$\gamma^{ij} \frac{\tilde{\delta}(R_{kl} X^l{}_o X^{ok})}{\tilde{\delta}\gamma^{ij}} = 2 R_{kl} X^l{}_o X^{ok} + \text{total derivatives}, \tag{B.9}$$

$$\gamma^{ij} \frac{\tilde{\delta}([X]\nabla_i \nabla^i [X])}{\tilde{\delta}\gamma^{ij}} = 2[X]\nabla_i \nabla^i [X] + \text{total derivatives} \tag{B.10}$$

$$\cdots$$

  This directly gives

$$Y_{(4)} = \gamma^{ij} \frac{\tilde{\delta} U_{(4)}}{\tilde{\delta}\gamma^{ij}} = 2 U_{(4)} + \text{total derivatives}. \tag{B.11}$$

One can find that up to the total derivatives, the relation between $U_{(k)}$ and $Y_{(k)}$ looks like the Euler's homogeneous function theorem. It would be interesting to give a general proof in the future.



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## C Some basic formulas

Here we present some basic formulas that we have used. They are

$$\tilde{\mathbb{X}}\frac{\tilde{\delta}R}{\tilde{\delta}\gamma^{ij}} = R_{ij}\mathbb{X} + (\Box\mathbb{X})\gamma_{ij} - \nabla_{(i}\nabla_{j)}\mathbb{X}, \tag{C.1}$$

$$\tilde{\mathbb{X}}\frac{\tilde{\delta}(R_{kl}R^{kl})}{\tilde{\delta}\gamma^{ij}} = 2R_{k(i}R^{k}{}_{j)}\mathbb{X} + \nabla_k\nabla_l(\mathbb{X}R^{kl})\gamma_{ij} + (\Box\mathbb{X}R_{ij}) - 2\nabla^k\nabla_{(i}(\mathbb{X}R_{j)k}), \tag{C.2}$$

$$\tilde{\mathbb{X}}\frac{\tilde{\delta}(R^{k}{}_{mln})}{\tilde{\delta}\gamma^{ij}} = \frac{1}{2}\left[\nabla^k\nabla_l\mathbb{X}\gamma_{m(i}\gamma_{j)n} - \nabla_m\nabla_l\mathbb{X}\gamma_{n(i}\delta^k_{j)} - \nabla_n\nabla_l\mathbb{X}\gamma_{m(j}\delta^k_{i)}\right] - (l\leftrightarrow n), \tag{C.3}$$

$$\tilde{\mathbb{X}}\frac{\tilde{\delta}(\Box\mathbb{Y})}{\tilde{\delta}\gamma^{ij}} = \mathbb{X}\nabla_{(i}\nabla_{j)}\mathbb{Y} - \nabla_{(i}(\mathbb{X}\nabla_{j)}\mathbb{Y}) + \frac{1}{2}\nabla_k(\mathbb{X}\nabla^k\mathbb{Y})\gamma_{ij} + \Box\mathbb{X}\frac{\delta\mathbb{Y}}{\delta\gamma^{ij}}, \tag{C.4}$$

where we have defined

$$\tilde{\mathbb{X}}\frac{\tilde{\delta}}{\tilde{\delta}\gamma^{ij}} \equiv \frac{1}{\sqrt{\gamma}}\int_\Sigma d^dx\sqrt{\gamma}\mathbb{X}\frac{\delta}{\delta\gamma^{ij}}, \tag{C.5}$$

and

$$\frac{\delta[X^n]}{\delta\gamma^{ij}} = \frac{n}{2}[X^n]_{ij}, \tag{C.6}$$

$$X^{ij}\frac{\delta X^{kl}}{\delta\gamma^{ij}} = \frac{3}{2}[X^2]^{kl}, \tag{C.7}$$

$$\gamma^{ij}\frac{\delta X^{kl}}{\delta\gamma^{ij}} = \frac{3}{2}X^{kl}. \tag{C.8}$$

Note that eqs. (C.1)–(C.4) have originally been listed in [23] and eq. (C.6) was proved in [49]. The rest part of this appendix is our demonstration for eqs. (C.7) and (C.8).

### C.1 Variation of the matrix X

The variation of a square root matrix with respect to the metric was studied by Bernard, etc. The result is presented in (4.18) in [50], where $S$ is $\mathcal{X}$ in our notation. Now we multiply $\mathcal{X}_{\rho\sigma}$ on each side of that equation and then make use of Cayley-Hamilton theorem, which gives

$$\mathcal{X}_{\rho\sigma}\frac{\delta\mathcal{X}^{\lambda}{}_{\mu}}{\delta g_{\rho\sigma}} = -\frac{1}{2}[\mathcal{X}^2]^{\lambda}{}_{\mu}. \tag{C.9}$$

This equation takes the same form as (C.7). Nevertheless, (4.18) in [50] is unsuitable to our case. The main reason is following. The calculation in [50] is applicable only if the matrix $e_3I+e_1\mathcal{X}^2$ is invertible. In consideration of the gauge $f_{t\mu}=0$ that we have adopted, however, one has

$$e_{D-1}(\mathcal{X}) = e_{D-1}(X) = det(X) = 0, \tag{C.10}$$

so $e_3I+e_1\mathcal{X}^2$ is actually a singular matrix when the bulk dimension $D=4$ in our case. An applicable modification is given below. We refer [50] for more details and notation. Let us first deal with an even $d$. The Cayley-Hamilton theorem for the $d\times d$ matrix $X$ is given by

$$\sum_{n=0}^{d}(-1)^nX^{d-n}e_n(X) = 0. \tag{C.11}$$

Using (A.2) in [50] gives

$$\sum_{n=0}^{d-1}(-1)^n e_n\delta(X^{d-n}) = \sum_{n=1}^{d}\sum_{m=1}^{n}(-1)^{n+m}e_{n-m}X^{d-n}Tr[X^{m-1}\delta X]. \tag{C.12}$$

Here we have used $e_0 = 1$. By writing $X^{2n+1} = X \cdot X^{2n}$, one can convert the variation of the square root matrix $X$ to the known variation $\delta(X^{2n})$, namely

$$\begin{aligned}&\frac{(\delta X)^k{}_a}{\delta\gamma^{ij}}\,(e_1X^{d-2}+e_3X^{d-4}+\ldots+e_{d-1}I)^a{}_l \\ &=\Bigg\{\Big(\delta[X^d]^k{}_l+e_2\delta[X^{d-2}]^k{}_l+\ldots+e_{d-2}\delta[X^2]^k{}_l\Big)-X^k{}_a\Big(e_1\delta\;[X^{d-2}]^a{}_l+\ldots+e_{d-3}\delta[X^2]^a{}_l\Big) \\ &\quad-\sum_{n=1}^{d}\Big((-1)^{n+1}e_{n-1}[X^{d-n}]^k{}_l\delta[X]+\frac{1}{2}\sum_{m=2}^{n}(-1)^{n+m}e_{n-m}[X^{d-n}]^k{}_lTr[X^{m-2}\delta X^2]\Big)\Bigg\}\Big/\delta\gamma^{ij}.\end{aligned} \tag{C.13}$$

Multiplying $X^{ij}$ on both sides and working out all variation on the r.h.s. of this equation, one can obtain

$$\begin{aligned}&X^{ij}\frac{\delta X^k{}_a}{\delta\gamma^{ij}}\,(e_1X^{d-2}+e_3X^{d-4}+\ldots+e_{d-1}I)^a{}_l \\ &=\frac{1}{2}\Bigg\{\Big(d[X^{d+1}]^k{}_l+(d-2)e_2[X^{d-1}]^k{}_l+\ldots+2e_{d-2}[X^3]^k{}_l\Big) \\ &\quad-\Big((d-2)e_1[X^d]^k{}_l+(d-4)e_3[X^{d-2}]^k{}_l+\ldots+2e_{d-3}[X^4]^k{}_l\Big) \\ &\quad-\sum_{n=1}^{d}\sum_{m=1}^{n}(-1)^{n+m}e_{n-m}[X^{m+1}]\;[X^{d-n}]^k{}_l\Bigg\}.\end{aligned} \tag{C.14}$$

Here we have used (C.19) and (C.6). Using (C.22) gives the r.h.s. of (C.14) as

$$\begin{aligned}&\frac{1}{2}\Bigg\{\sum_{n=-1}^{d-3}(-1)^n(n+1-d)e_{n+1}\;[X^{d-n}]^k{}_l+\Big(e_1[X^d]^k{}_l+e_3[X^{d-2}]^k{}_l+\ldots+e_{d-3}[X^4]^k{}_l\Big) \\ &\quad-\Bigg(\sum_{n=1}^{d}(-1)^n(n+1)e_{n+1}[X^{d-n}]^k{}_l+e_1\;[X^d]^k{}_l\Bigg)\Bigg\} \\ &=\frac{1}{2}\Bigg\{-d\sum_{n=-1}^{d-1}(-1)^ne_{n+1}[X^{d-n}]^k{}_l+\Big(e_1[X^d]^k{}_l+e_3[X^{d-2}]^k{}_l+\ldots+e_{d-1}\;[X^2]^k{}_l\Big)\Bigg\} \\ &=\frac{1}{2}\Big(e_1[X^d]^k{}_l+e_3[X^{d-2}]^k{}_l+\ldots+e_{d-1}[X^2]^k{}_l\Big).\end{aligned} \tag{C.15}$$

Here we have used $e_n = 0$ for any $n > d$. That is

$$\begin{aligned}&X^{ij}\frac{\delta X^k{}_a}{\delta\gamma^{ij}}\,(e_1X^{d-2}+e_3X^{d-4}+\ldots+e_{d-1}I)^a{}_l \\ &=\frac{1}{2}[X^2]^k{}_a(e_1X^{d-2}+e_3X^{d-4}+\ldots+e_{d-1}I)^a{}_l.\end{aligned} \tag{C.16}$$

Whether one takes the gauge $f_{t\mu} = 0$ or not, $e_1 X^{d-2} + e_3 X^{d-4} + \ldots + e_{d-1} I$ is an invertible matrix commonly. Then multiplying by its inverse on each side, one has

$$X^{ij} \frac{\delta X^k{}_l}{\delta \gamma^{ij}} = \frac{1}{2} \, [X^2]^k{}_l. \tag{C.17}$$

This is just (C.7). The proof of an odd $d$ and (C.8) can be given in the same way.

### C.2 Auxiliary formulas

- For an even $n$, $X^n$ can be written as

$$[X^n]^k{}_l = [X^2]^k{}_{a_1} \cdot [X^2]^{a_1}{}_{a_2} \cdots \, [X^2]^{a_{n/2-1}}{}_l. \tag{C.18}$$

  This gives

$$X^{ij} \frac{\delta [X^n]^k{}_l}{\delta \gamma^{ij}} = \frac{1}{2} n \, [X^{n+1}]^k{}_l. \tag{C.19}$$

- Equation (2.20) in [50] gives

$$e_{n+1} = \frac{-1}{n+1} \sum_{m=1}^{n+1} (-1)^m [X^m] e_{n+1-m} = \frac{1}{n+1} \left( [X] e_n - \sum_{m=1}^{n} (-1)^{m+1} [X^{m+1}] e_{n-m} \right). \tag{C.20}$$

  That is

$$\sum_{m=1}^{n} (-1)^{m+1} [X^{m+1}] e_{n-m} = -(n+1) e_{n+1} + e_1 e_n, \tag{C.21}$$

  which induces

$$\sum_{n=1}^{d} \sum_{m=1}^{n} (-1)^{n+m} [X^{m+1}] e_{n-m} \, [X^{d-n}]^k{}_l = e_1 [X^d]^k{}_l + \sum_{n=1}^{d} (-1)^n (n+1) e_{n+1} \, [X^{d-n}]^k{}_l. \tag{C.22}$$



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