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Monster anatomy

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Abstract: We investigate the two-dimensional conformal field theories (CFTs) of $c = \frac{47}{2}$, $c = \frac{116}{5}$ and $c = 23$ 'dual' to the critical Ising model, the three state Potts model and the tensor product of two Ising models, respectively. We argue that these CFTs exhibit moonshines for the double covering of the baby Monster group, $2 \cdot \mathbb{B}$, the triple covering of the largest Fischer group, $3 \cdot \mathrm{Fi}'_{24}$ and multiple-covering of the second largest Conway group, $2 \cdot 2^{1+22} \cdot \mathrm{Co}_2$. Various twined characters are shown to satisfy generalized bilinear relations involving Mckay-Thompson series. We also rediscover that the 'self-dual' two-dimensional bosonic conformal field theory of $c = 12$ has the Conway group $\mathrm{Co}_0 \simeq 2 \cdot \mathrm{Co}_1$ as an automorphism group.

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Contents

1 Introduction

Mckay and Thompson's remarkable observation between the monster group $\mathbb{M}$ and the modular objects, especially, the j-invariant, motivated the study of the so-called 'Monstrous Moonshine' in [1]. Each Fourier coefficient of the modular invariant $j(\tau) - 744$ ($q = e^{2\pi i \tau}$), which can describe the partition function of $c = 24$ chiral CFT,

$$j(\tau) - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \cdots\,, \tag{1.1}$$

can be decomposed into the dimension of the irreducible representation of the monster group $\mathbb{M}$. Frenkel, Lepowsky and Meurman [2] provided a (heuristic) derviation of the Monstrous moonshine from an explicit construction of the chiral CFT based on the Leech lattice followed by a $\mathbb{Z}_2$ orbifold. Many examples of the generalizations of moonshine phenomena with different sporadic groups have been uncovered in the last decades [3–11].

In this article, we utilize a holomorphic bilinear relation [12] to further explore a new class of moonshine phenomena. It has been observed recently that characters $f_i(\tau)$ of a certain rational CFT with central charge c obey an intriguing bilinear relation giving a modular invariant $j(\tau)$,

$$\sum_{i=0}^{n-1} f_i(\tau)\tilde{f}_i(\tau) = j(\tau) - 744, \tag{1.2}$$

where $\tilde{f}_i(\tau)$ can be interpreted as characters of a 'dual' rational CFT with central charge $(24-c)$. For instance, the critical Ising model with $c=\frac{1}{2}$ and a rational CFT with $c=\frac{47}{2}$ satisfies the bilinear relation. Another example is a pair of rational CFTs of $c=8$ and $c=16$ having no Kac-Moody symmetry but finite group symmetry [13]. Further examples can be found in [14].

The rational CFT with $c=\frac{47}{2}$ dual to the critical Ising model exhibits Moonshine for the baby Monster group, second largest sporadic group. It is challenging to search for a dual rational CFT showing Moonshine for the sporadic groups other than the baby Monster group. The search first requires an explicit q-expansion of each character in two rational CFTs of dual pair. To do so, we make use of a modular-invariant differential equation (MDE) of the form below [15]

$$\left[\mathcal{D}_\tau^n + \sum_{k=0}^{n-1} \phi_{2(n-k)}(\tau)\mathcal{D}_\tau^k\right] f(\tau) = 0, \tag{1.3}$$

where $\mathcal{D}_\tau$ denotes the Serre derivative acting on a modular form of weight r,

$$\mathcal{D}_\tau \equiv \partial_\tau - \frac{1}{6}i\pi r E_2(\tau), \tag{1.4}$$

and $\phi_k(\tau)$ are modular forms of weight k. The MDE can be used to explore the space of rational CFTs. This is because solutions to an MDE, which furnish a finite-dimensional representation of $\mathrm{SL}(2,\mathbb{Z})$, can play a role as candidate characters $f_i(\tau)$ $(i=0,1,2,\ldots,n-1)$ in a rational CFT. One can show from (1.3) that the conformal weights h_i of primaries and the central charge c of a candidate RCFT have to satisfy the relation below

$$\sum_{i=0}^{n-1}\left[h_i - \frac{c}{24}\right] = \frac{n(n-1)}{12} - \frac{l}{6}, \tag{1.5}$$

where l is a non-negative integer other than 1. (1.5) implies that a rational CFT can be characterized by conformal weights h_i and a number n of primaries, the central charge c and an integer l. When $(\{h_i\}, n, c, l)$ of a rational CFT are related to $(\{\tilde{h}_i\}, n, \tilde{c}, \tilde{l})$ of another rational CFT as follows

$$h_0 = \tilde{h}_0 = 0, \qquad h_i + \tilde{h}_i = 2 \;\text{ for }\; i \neq 0, \tag{1.6}$$

and

$$c + \tilde{c} = 24, \qquad l + \tilde{l} = (n-3)(n-4), \tag{1.7}$$

the characters of two rational CFTs can obey the bilinear relation (1.2). Namely, one rational CFT is dual to the other.

We analyze rational CFTs dual to the three-state Potts model and to the product of two Ising models. Interestingly enough, our results can propose that the dual rational CFTs show novel connections to the triple covering of the largest Fischer group, $3\cdot\mathrm{Fi}_{24}'$ and the multiple-covering of the second largest Conway group, $2\cdot 2^{1+22}\cdot\mathrm{Co}_2$, respectively. Note

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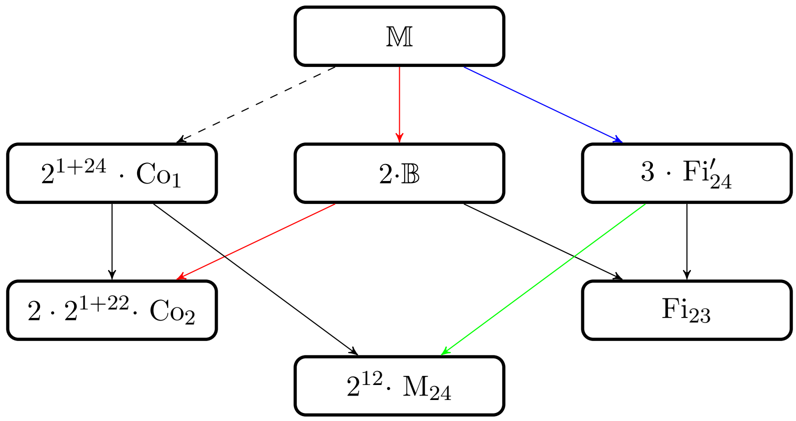

Figure 1. Partial flows of maximal subgroups from the monster group $\mathbb{M}$ [16]. Each arrow from A to B implies that B is a maximal subgroup of A.

that $3\cdot \mathrm{Fi}_{24}'$ is a maximal subgroup of $\mathbb{M}$ described in figure 1. We also observe that the self-dual theory of $c=12$ has the largest Conway group Co_0 as an automorphism group. In fact, we can show that the self-dual theory can be identified as the GSO projection of the well-known $\mathcal{N}=1$ superconformal extremal CFT of $c=12$ [2, 17]. Our observations suggest that the subgroup decomposition along the red and the blue arrows in figure 1 can be realized by the holomorphic bilinear relation. The moonshines for $2\cdot\mathbb{B}$ in baby Monster CFT and $3\cdot \mathrm{Fi}_{24}'$ in $c=\frac{116}{5}$ CFT are further supported by generalized bilinear relations involving Mckay-Thompson series. The details will be shown in section 2 and section 3.

2 Dual of the Ising model and the Baby Monster

The simplest unitary minimal model $\mathcal{M}(4,3)$ describes the critical point of the second-order phase transition of the Ising model. The critical Ising model has the identity $\mathbf{1}$, the energy density ϵ and the spin σ whose scaling dimensions are $\Delta=0,1,\frac{1}{8}$ respectively. Assuming three operators have no spin, one can relate them with allowed primary fields $\phi_{1,1},\phi_{2,1},\phi_{2,2}$ of weight $h=0,\frac{1}{2},\frac{1}{16}$ in $\mathcal{M}(4,3)$ as follows

$$\begin{aligned} \mathbf{1} &\leftrightarrow \phi_{1,1} \\ \epsilon &\leftrightarrow \phi_{2,1} \\ \sigma &\leftrightarrow \phi_{2,2} \end{aligned} \tag{2.1}$$

The $c=\frac{1}{2}$ Virasoro characters for three primary fields are

$$\begin{aligned} f_0(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}}+\sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right), \\ f_\epsilon(\tau) &= \frac{1}{2}\left(\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}}-\sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right), \\ f_\sigma(\tau) &= \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}}. \end{aligned} \tag{2.2}$$

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These characters transform into one another under $\mathrm{SL}(2,\mathbb{Z})$. In particular, the modular S-matrix is

$$\begin{pmatrix} f_0(-1/\tau) \\ f_\epsilon(-1/\tau) \\ f_\sigma(-1/\tau) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} f_0(\tau) \\ f_\epsilon(\tau) \\ f_\sigma(\tau) \end{pmatrix}. \tag{2.3}$$

It has been shown recently in [12] that the characters of the critical Ising model obey an intriguing bilinear relation,

$$j(\tau) - 744 = f_0(\tau)\cdot\tilde{f}_0(\tau) + f_\epsilon(\tau)\cdot\tilde{f}_\epsilon(\tau) + f_\sigma(\tau)\cdot\tilde{f}_\sigma(\tau), \tag{2.4}$$

where $j(\tau) - 744$ is the Monster module,

$$j(\tau) = \frac{12^3 E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots. \tag{2.5}$$

Here $\tilde{f}_0$, $\tilde{f}_\sigma$ and $\tilde{f}_\epsilon$ are three independent solutions to a third order modular differential equation

$$0 = \left[\mathcal{D}_\tau^3 + \frac{2315\pi^2}{576}E_4(\tau)\mathcal{D}_\tau - i\frac{27025\pi^3}{6912}E_6(\tau)\right]\tilde{f}, \tag{2.6}$$

and can be expanded in powers of q as follow.

$$\begin{aligned} \tilde{f}_0(\tau) &= q^{-\frac{47}{48}}\left(1 + 96256'q^2 + 9646891q^3 + 366845011q^4 + \cdots\right), \\ \tilde{f}_\epsilon(\tau) &= q^{\frac{25}{48}}\left(4371 + 1143745q + 64680601q^2 + 1829005611q^3 + \cdots\right), \\ \tilde{f}_\sigma(\tau) &= q^{\frac{23}{24}}\left(96256 + 10602496q + 420831232q^2 + 9685952512q^3 + \cdots\right). \end{aligned} \tag{2.7}$$

In fact, (2.7) can be identified as three characters of conformal weights $h = 0, \frac{31}{16}, \frac{3}{2}$ in the dual RCFT with $c = \frac{47}{2}$.

Since the $j(\tau)$ is invariant under $\mathrm{SL}(2,\mathbb{Z})$, the modular S-matrix of the dual RCFT with $c = \frac{47}{2}$, often referred to as the baby Monster CFT, obtained from that of the critical Ising model, namely

$$\begin{pmatrix} \tilde{f}_0(-1/\tau) \\ \tilde{f}_\epsilon(-1/\tau) \\ \tilde{f}_\sigma(-1/\tau) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \tilde{f}_0(\tau) \\ \tilde{f}_\epsilon(\tau) \\ \tilde{f}_\sigma(\tau) \end{pmatrix}. \tag{2.8}$$

In addition, the Verlinde formulae implies that these two RCFTs also share the same fusion rule algebra. From (2.8), the modular invariant torus partition function of the baby Monster CFT has to be

$$Z_{c=\frac{47}{2}}(\tau,\bar{\tau}) = \tilde{f}_0(\tau)\bar{\tilde{f}}_0(\bar{\tau}) + \tilde{f}_\epsilon(\tau)\bar{\tilde{f}}_\epsilon(\bar{\tau}) + \tilde{f}_\sigma(\tau)\bar{\tilde{f}}_\sigma(\bar{\tau}), \tag{2.9}$$

which agrees with numerical results in [13].

It is known that dual RCFT with $c = \frac{47}{2}$ has baby Monster group $\mathbb{B}$ as a finite group symmetry [12], because (2.7) are identical to the characters of modules $V\mathbb{B}^\natural_{(0)}, V\mathbb{B}^\natural_{(1)}$ and

$V\mathbb{B}^{\natural}_{(2)}$ in [18], respectively. In this paper, we conjecture that the finite group symmetry of $c = \frac{47}{2}$ RCFT can be promoted to the double covering of baby Monster group $2 \cdot \mathbb{B}$. As a demonstration, one can show that each coefficient in (2.7) can be expressed as a sum of dimensions of irreducible representations of $2 \cdot \mathbb{B}$, e.g.

$$\begin{aligned} 4371 &= \mathbf{4371}, \quad 96256' = \mathbf{1} \oplus \mathbf{96255}, \quad 96256 = \mathbf{96256}, \\ 9646891 &= \mathbf{1} \oplus \mathbf{96255} \oplus \mathbf{9550635}, \quad 1143745 = \mathbf{4371} \oplus \mathbf{1139374}, \\ 10602496 &= \mathbf{96256} \oplus \mathbf{10506240}, \\ 366845011 &= 2 \cdot \mathbf{1} \oplus 2 \cdot \mathbf{96255} \oplus \mathbf{9458750} \oplus \mathbf{9550635} \oplus \mathbf{347643114}, \\ 64680601 &= 2 \cdot \mathbf{4371} \oplus \mathbf{1139374} \oplus \mathbf{63532485}, \\ 420831232 &= 2 \cdot \mathbf{96256} \oplus \mathbf{10506240} \oplus \mathbf{410132480}. \end{aligned} \tag{2.10}$$

Note that $\mathbf{1}$ in the second line of (2.10) can be identified as the Virasoro descendent of the vacuum.

We apply a refined test proposed in [19, 20] to provide further supporting evidence that $c = \frac{47}{2}$ CFT exhibit moonshine for $2 \cdot \mathbb{B}$. To this end, we introduce the twined character,

$$\tilde{f}^g_i(\tau) = \mathrm{Tr}_{\mathcal{H}_i}\left[g \cdot q^{h-\frac{c}{24}}\right]. \tag{2.11}$$

where g is a group element of $2 \cdot \mathbb{B}$ and the trace is taken over all states in the Hilbert space $\mathcal{H}_i$, that consist of primary state of weight h_i and its descendants. As illustrated in [19], it is straightforward to obtain the twined characters using the character table of $2 \cdot \mathbb{B}$ in appendix B. For instance, let us consider the twined characters for $g = 2C$. Explicitly, the first term in $\tilde{f}_\sigma(\tau)$, **96256**, is replaced by **2048**. In this way, one can check that the twined characters are given by

$$\begin{aligned} \tilde{f}^{2C}_0(q) &= q^{-\frac{47}{48}}\left(1 + 2048q^2 + 37675q^3 + 470099q^4 + \cdots\right), \\ \tilde{f}^{2C}_\epsilon(q) &= q^{\frac{25}{48}}\left(275 + 9153q + 144025q^2 + \cdots\right), \\ \tilde{f}^{2C}_\sigma(q) &= q^{\frac{23}{24}}\left(2048 + 47104q + 565248q^2 + \cdots\right). \end{aligned} \tag{2.12}$$

Intriguingly, we find that (2.12) satisfy a bilinear relation of the form

$$j^{2A}(\tau) = f_0(\tau) \cdot \tilde{f}^{2C}_0(\tau) + f_\epsilon(\tau) \cdot \tilde{f}^{2C}_\epsilon(\tau) + f_\sigma(\tau) \cdot \tilde{f}^{2C}_\sigma(\tau), \tag{2.13}$$

where $j^{2A}(\tau)$ is Mckay-Thompson series of class 2A,

$$\begin{aligned} j^{2A}(\tau) &= \frac{\eta(q)^{24}}{\eta(q^2)^{24}} + \frac{2^{12}\eta(q^2)^{24}}{\eta(q)^{24}} + 24 \\ &= \frac{1}{q} + 4372q + 96256q^2 + 1240002q^3 + 10698752q^4 + \cdots. \end{aligned} \tag{2.14}$$

In appendix C, we present list of the generalized bilinear relations for various $g \in 2 \cdot \mathbb{B}$. Combined with the characters of the Ising model, all the twined characters we investigated constitutes Mckay-Thompson series of certain class. Sometimes, combination of the twined characters for different group elements and characters of the critical Ising model yields Mckay-Thompson series of identical class. More precisely, even if we replace 2C characters to 2A(or 2B) characters in the right-hand side of (2.13), the left-hand side of (2.13) still remained as $j^{2A}(\tau)$. Note that the decompositions in (2.10) are consistent to bilinear relations.

3 Dual of the three-state Potts model and $3 \cdot \mathrm{Fi}'_{24}$

We now make use of the bilinear relations but with different pairs of characters to look for new RCFTs related to the other sporadic groups. In particular, we propose in this section that a dual CFT of the three-state Potts model has the triple covering of the largest Fischer group Fi'_{24} as an automorphism group.

It is known that the three-state Potts model at the critical point can be described as a "subset" of the minimal model $\mathcal{M}(6,5)$ with $c = \frac{4}{5}$ containing ten primary fields $\phi_{(r,s)}$ of conformal weights

$$h_{r,s} = \frac{(6r-5s)^2-1}{120}, \tag{3.1}$$

where $1 \le r < 5, 1 \le s < 6$ and $(r,s) \simeq (5-r, 6-s)$. The $c = \frac{4}{5}$ Virasoro characters labelled by two integers (r,s) are given by

$$f^{(5)}_{r,s}(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} \left[q^{\frac{(60n+6r-5s)^2}{120}} - q^{\frac{(60n+6r+5s)^2}{120}} \right]. \tag{3.2}$$

Some but not all of these primary fields are present in the critical three-state Potts model. One can indeed show that a subset of the ten primary fields are closes under the fusion rules, which leads to a non-diagonal modular invariant partition function

$$Z = \sum_{r=1,2} \left| f^{(5)}_{r,1} + f^{(5)}_{r,5} \right|^2 + 2\left| f^{(5)}_{r,3} \right|^2. \tag{3.3}$$

(3.3) is the partition function of the three-state Potts model, which implies that only $\phi_{r,s}$ and two copies of $\phi_{r,3}$ with $r = 1,2$ and $s = 1,5$ are present in the theory. Notice also that the three-state Potts model has $\mathbb{Z}_3$ symmetry under which two copies of $\phi_{r,3}$ for each r transform differently. This $\mathbb{Z}_3$ symmetry plays a key role to have well-defined fusion rules of the three-states Potts model [21].

It is natural from (3.3) to define the characters of the critical three-state Potts model as follows,

$$\begin{aligned} f_0(\tau) &= f^{(5)}_{1,1}(\tau) + f^{(5)}_{1,5}(\tau), \\ f_1(\tau) &= f^{(5)}_{2,1}(\tau) + f^{(5)}_{2,5}(\tau), \\ f_2(\tau) &= f'_2(\tau) = f^{(5)}_{1,3}(\tau), \\ f_3(\tau) &= f'_3(\tau) = f^{(5)}_{2,3}(\tau). \end{aligned} \tag{3.4}$$

The modular S-matrix of the model then becomes

$$\begin{pmatrix} f_0(-1/\tau) \\ f_1(-1/\tau) \\ f_2(-1/\tau) \\ f_2'(-1/\tau) \\ f_3(-1/\tau) \\ f_3'(-1/\tau) \end{pmatrix} = \frac{2}{\sqrt{15}} \begin{pmatrix} -s_1 & s_2 & -s_1 & -s_1 & s_2 & s_2 \\ s_2 & s_1 & s_2 & s_2 & s_1 & s_1 \\ -s_1 & s_2 & -\omega s_1 & -\omega^2 s_1 & \omega s_2 & \omega^2 s_2 \\ -s_1 & s_2 & -\omega^2 s_1 & -\omega s_1 & \omega^2 s_2 & \omega s_2 \\ s_2 & s_1 & \omega s_2 & \omega^2 s_2 & \omega s_1 & \omega^2 s_1 \\ s_2 & s_1 & \omega^2 s_2 & \omega s_2 & \omega^2 s_1 & \omega s_1 \end{pmatrix} \begin{pmatrix} f_0(\tau) \\ f_1(\tau) \\ f_2(\tau) \\ f_2'(\tau) \\ f_3(\tau) \\ f_3'(\tau) \end{pmatrix}. \tag{3.5}$$

where $s_1 = \sin\left(\frac{6\pi}{5}\right), s_2 = \sin\left(\frac{12\pi}{5}\right)$ and $\omega = e^{\frac{2\pi i}{3}}$.

One can show that the characters (3.4) satisfy a bilinear relation

$$\begin{aligned} j(\tau) - 744 = {} & f_0(\tau)\tilde{f}_0(\tau) + f_1(\tau)\tilde{f}_1(\tau) + f_2(\tau)\tilde{f}_2(\tau) + f_2'(\tau)\tilde{f}_2'(\tau) \\ & + f_3(\tau)\tilde{f}_3(\tau) + f_3'(\tau)\tilde{f}_3'(\tau), \end{aligned} \tag{3.6}$$

where the four characters of a dual theory $\tilde{f}_i(\tau)$ are the solutions to a fourth order differential equation,

$$\left[\mathcal{D}_\tau^4 + \mu_1 E_4(\tau)\mathcal{D}_\tau^2 + \mu_2 E_6(\tau)\mathcal{D}_\tau + \mu_3 E_4^2(\tau)\right]\tilde{f}_i(\tau) = 0. \tag{3.7}$$

Here, we denote by $\tilde{f}_1(\tau), \tilde{f}_2(\tau), \tilde{f}_3(\tau)$ and $\tilde{f}_4(\tau)$ characters of weights $h = 0$, $\frac{8}{5}$, $\frac{4}{3}$ and $\frac{29}{15}$, respectively. We further use the q-expansion of $\tilde{f}_i(\tau)$

$$\tilde{f}_i(\tau) = q^{\left(h_i - \frac{c}{24}\right)}\left(a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots\right), \tag{3.8}$$

to fix free parameters μ_i in (3.7). The fourth order differential equation of our interest then becomes

$$0 = \left[\mathcal{D}_\tau^4 + \frac{907\pi^2}{225}E_4(\tau)\mathcal{D}_\tau^2 - i\frac{4289\pi^3}{675}E_6(\tau)\mathcal{D}_\tau - \frac{175769\pi^4}{50625}E_4^2(\tau)\right]\tilde{f}(\tau). \tag{3.9}$$

Now, It is easy to see that the solutions of (3.9) have the q-expansion

$$\begin{aligned} \tilde{f}_0(\tau) &= q^{-\frac{29}{30}}\Big(1 + 57478q^2 + 5477520q^3 + 201424111q^4 + \cdots\Big), \\ \tilde{f}_1(\tau) &= q^{\frac{19}{30}}\Big(8671 + 1675504q + 83293626q^2 + 2175548448q^3 + \cdots\Big), \\ \tilde{f}_2(\tau) &= \tilde{f}_2'(\tau) = q^{\frac{11}{30}}\Big(783 + 306936q + 19648602q^2 + \cdots\Big), \\ \tilde{f}_3(\tau) &= \tilde{f}_3'(\tau) = q^{\frac{29}{30}}\Big(64584 + 6789393q + 261202536q^2 + \cdots\Big). \end{aligned} \tag{3.10}$$

From the bilinear relation (3.6), one can read that the modular S-matrix of the dual theory is

$$\begin{pmatrix} \tilde{f}_0(-1/\tau) \\ \tilde{f}_1(-1/\tau) \\ \tilde{f}_2(-1/\tau) \\ \tilde{f}_2'(-1/\tau) \\ \tilde{f}_3(-1/\tau) \\ \tilde{f}_3'(-1/\tau) \end{pmatrix} = \frac{2}{\sqrt{15}} \begin{pmatrix} -s_1 & s_2 & -s_1 & -s_1 & s_2 & s_2 \\ s_2 & s_1 & s_2 & s_2 & s_1 & s_1 \\ -s_1 & s_2 & -\omega^2 s_1 & -\omega s_1 & \omega^2 s_2 & \omega s_2 \\ -s_1 & s_2 & -\omega s_1 & -\omega^2 s_1 & \omega s_2 & \omega^2 s_2 \\ s_2 & s_1 & \omega^2 s_2 & \omega s_2 & \omega^2 s_1 & \omega s_1 \\ s_2 & s_1 & \omega s_2 & \omega^2 s_2 & \omega s_1 & \omega^2 s_1 \end{pmatrix} \begin{pmatrix} \tilde{f}_0(\tau) \\ \tilde{f}_1(\tau) \\ \tilde{f}_2(\tau) \\ \tilde{f}_2'(\tau) \\ \tilde{f}_3(\tau) \\ \tilde{f}_3'(\tau) \end{pmatrix}. \tag{3.11}$$

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We can also verify that the dual theory with $3{\cdot}\mathrm{Fi}'_{24}$ symmetry has positive integer fusion coefficients, obtained from the above S-matrix via the Verlinde formula.

Notice that each coefficient of (3.10) can be expressed as a sum of dimensions of representations of $3{\cdot}\mathrm{Fi}'_{24}$, the triple covering of the largest Fischer group. $3{\cdot}\mathrm{Fi}'_{24}$ is one of the maximal subgroup of the Monster group, and is of order $2^{21}{\cdot}3^{17}{\cdot}5^2{\cdot}7^3{\cdot}11{\cdot}13{\cdot}17{\cdot}23{\cdot}29$. It has 256 irreducible representations including **783**, **8671** and **64584** that agree with the first coefficient of $\tilde{f}_2(\tau)$, $\tilde{f}_1(\tau)$ and $\tilde{f}_3(\tau)$. One can also show that

$$\begin{aligned}
57478 &= \mathbf{1} \oplus \mathbf{57477}, \quad 1675504 = \mathbf{8671} \oplus \mathbf{1666833}, \\
306936 &= \mathbf{783} \oplus \mathbf{306153}, \quad 6789393 = \mathbf{64584} \oplus \mathbf{6724809}, \\
5477520 &= \mathbf{1} \oplus \mathbf{57477} \oplus \mathbf{555611} \oplus \mathbf{4864431}
\end{aligned} \tag{3.12}$$

where the first term of each line in (3.12) can be understood as the Virasoro descendant of the corresponding primary field.

Now we will find the twined characters of $c = \frac{116}{5}$ CFT and examine if they form a bilinear relation analogous to (2.13). For instance, the twined characters for $g = 2A$ are given by

$$\begin{aligned}
\tilde{f}_0^{2A}(q) &= q^{-\frac{29}{30}} \left(1 + 1158q^2 + 20112q^3 + \cdots\right), \\
\tilde{f}_1^{2A}(q) &= q^{\frac{19}{30}} \left(351 + 11504q + \cdots\right), \\
\tilde{f}_2^{2A}(q) &= \tilde{f'}_2^{2A}(q) = q^{\frac{11}{30}} \left(79 + 2808q + \cdots\right), \\
\tilde{f}_3^{2A}(q) &= \tilde{f'}_3^{2A}(q) = q^{\frac{29}{30}} \left(1352 + 27729q + \cdots\right).
\end{aligned} \tag{3.13}$$

Combined with the characters of three-state Potts model, it turns out that the twined characters (3.13) merged into the Mckay-Thompson series of class 2A.

$$\begin{aligned}
j^{2A}(\tau) = {} & f_0(\tau)\cdot\tilde{f}_0^{2A}(\tau) + f_1(\tau)\cdot\tilde{f}_1^{2A}(\tau) + f_2(\tau)\cdot\tilde{f}_2^{2A}(\tau) + f'_2(\tau)\cdot\tilde{f'}_2^{2A}(\tau) \\
& + f_3(\tau)\cdot\tilde{f}_3^{2A}(\tau) + f'_3(\tau)\cdot\tilde{f'}_3^{2A}(\tau)
\end{aligned} \tag{3.14}$$

As another example, twined characters for $g = 3E$ read

$$\begin{aligned}
\tilde{f}_0^{3E}(q) &= q^{-\frac{29}{30}} \left(1 + 616q^2 + 7833q^3 + \cdots\right), \\
\tilde{f}_1^{3E}(q) &= q^{\frac{19}{30}} \left(-77 - 1925q + \cdots\right), \\
\tilde{f}_2^{3E}(q) &= q^{\frac{11}{30}} \left(54\alpha + 1485\alpha q + \cdots\right), \\
\tilde{f'}_2^{3E}(q) &= q^{\frac{11}{30}} \left(54\overline{\alpha} + 1485\overline{\alpha} q + \cdots\right), \\
\tilde{f}_3^{3E}(q) &= q^{\frac{29}{30}} \left(-297\alpha - 4158\alpha q + \cdots\right), \\
\tilde{f'}_3^{3E}(q) &= q^{\frac{29}{30}} \left(-297\overline{\alpha} - 4158\overline{\alpha} q + \cdots\right),
\end{aligned} \tag{3.15}$$

where $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $\overline{\alpha}$ is its complex conjugate. Then, we get a new type of bilinear relation of the form,

$$\begin{aligned}
j^{3A}(\tau) = {} & f_0(\tau)\cdot\tilde{f}_0^{3E}(\tau) + f_1(\tau)\cdot\tilde{f}_1^{3E}(\tau) + f_2(\tau)\cdot\tilde{f}_2^{3E}(\tau) + f'_2(\tau)\cdot\tilde{f'}_2^{3E}(\tau) \\
& + f_3(\tau)\cdot\tilde{f}_3^{3E}(\tau) + f'_3(\tau)\cdot\tilde{f'}_3^{3E}(\tau)
\end{aligned} \tag{3.16}$$

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where $j^{3A}(\tau)$ is Mckay-Thompson series of class $3A$ given by,

$$
\begin{aligned}
j^{2A}(\tau) &= \left(\left(\frac{\eta(q)}{\eta(q^3)} \right)^{12} + 27 \right)^2 \Big/ \left(\frac{\eta(q)}{\eta(q^3)} \right)^{12} - 42 \\
&= \frac{1}{q} + 783q + 8672q^2 + 65467q^3 + 371520q^4 + \cdots .
\end{aligned} \tag{3.17}
$$

In appendix C, we listed generalized bilinear relations that various twined characters satisfy. These suggest that the six-character rational CFT with $c = \frac{116}{5}$ dual to the critical three-state Potts model has $3{\cdot}\mathrm{Fi}_{24}'$ as an automorphism group.

4 Dual of the critical Ising^2 and $2 \cdot 2^{1+22} \cdot \mathrm{Co}_2$

We now in turn consider a rational CFT dual to the tensor product of the critical Ising model, the simplest example of the $c = 1$ CFTs studied in [22–24]. This theory has nine primaries of conformal weights

$$
h_1 = h_1' = \frac{1}{2},\ h_2 = h_2' = \frac{1}{16},\ h_3 = 1,\ h_4 = h_4' = \frac{9}{16},\ h_5 = \frac{1}{8}, \tag{4.1}
$$

and their characters can be expressed by characters of the critical Ising model (2.2).

$$
\begin{aligned}
&g_0(\tau) = f_0(\tau) \cdot f_0(\tau), \quad g_1(\tau) = g_1'(\tau) = f_0(\tau) \cdot f_\epsilon(\tau), \\
&g_2(\tau) = g_2'(\tau) = f_0(\tau) \cdot f_\sigma(\tau), \quad g_3(\tau) = f_\epsilon(\tau) \cdot f_\epsilon(\tau), \\
&g_4(\tau) = g_4'(\tau) = f_\sigma(\tau) \cdot f_\epsilon(\tau), \quad g_5(\tau) = f_\sigma(\tau) \cdot f_\sigma(\tau)
\end{aligned} \tag{4.2}
$$

The 9×9 extended modular matrix $\mathbb{S}$ reads,

$$
\mathbb{S} = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & \sqrt{2} & \sqrt{2} & 1 & \sqrt{2} & \sqrt{2} & 2 \\
1 & 1 & 1 & \sqrt{2} & -\sqrt{2} & 1 & \sqrt{2} & -\sqrt{2} & -2 \\
1 & 1 & 1 & -\sqrt{2} & \sqrt{2} & 1 & -\sqrt{2} & \sqrt{2} & -2 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & 2 & 0 & -\sqrt{2} & -2 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 2 & -\sqrt{2} & 0 & -2 & 0 \\
1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & 1 & -\sqrt{2} & -\sqrt{2} & 2 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -2 & 0 & -\sqrt{2} & 2 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & -2 & -\sqrt{2} & 0 & 2 & 0 \\
2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 0
\end{pmatrix}, \tag{4.3}
$$

thus the product of two critical Ising model admit diagonal modular invariants of the form

$$
\begin{aligned}
Z(\tau, \bar{\tau}) = &|g_0(\tau)|^2 + |g_1(\tau)|^2 + |g_1'(\tau)|^2 + |g_2(\tau)|^2 + |g_2'(\tau)|^2 \\
&+ |g_3(\tau)|^2 + |g_4(\tau)|^2 + |g_4'(\tau)|^2 + |g_5(\tau)|^2.
\end{aligned} \tag{4.4}
$$

We assumed that the characters (4.2) satisfy a bilinear relation

$$
\begin{aligned}
j(\tau) - 744 = &g_0(\tau)\tilde{g}_0(\tau) + g_1(\tau)\tilde{g}_1(\tau) + g_1'(\tau)\tilde{g}_1'(\tau) + g_2(\tau)\tilde{g}_2(\tau) + g_2'(\tau)\tilde{g}_2'(\tau) \\
&+ g_3(\tau)\tilde{g}_3(\tau) + g_4(\tau)\tilde{g}_4(\tau) + g_4'(\tau)\tilde{g}_4'(\tau) + g_5(\tau)\tilde{g}_5(\tau)
\end{aligned} \tag{4.5}
$$

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where the characters of a dual CFT with weights

$$\tilde{h}_1 = \tilde{h}'_1 = \frac{3}{2},\ \tilde{h}_2 = \tilde{h}'_2 = \frac{31}{16},\ \tilde{h}_3 = 1,\ \tilde{h}_4 = \tilde{h}'_4 = \frac{23}{16},\ \tilde{h}_5 = \frac{15}{8}. \tag{4.6}$$

are the solutions of a differential equation,

$$\begin{aligned} 0 = \Big[&\mathcal{D}_\tau^6 + \mu_1 E_4(\tau)\mathcal{D}_\tau^4 + \mu_2 E_6(\tau)\mathcal{D}_\tau^3 + \mu_3 E_4^2(\tau)\mathcal{D}_\tau^2 + \mu_4 E_4(\tau)E_6(\tau)\mathcal{D}_\tau \\ &+ \mu_5 E_4^3(\tau) + \mu_6 E_6^2(\tau) + \mu_7 \frac{E_4^2(\tau)}{E_6(\tau)}\mathcal{D}_\tau^5 + \mu_8 \frac{E_4^3(\tau)}{E_6(\tau)}\mathcal{D}_\tau^3 + \mu_9 \frac{E_4^4(\tau)}{E_6(\tau)}\mathcal{D}_\tau \Big] \tilde{g}(\tau). \end{aligned} \tag{4.7}$$

However, inserting q-expansion of the characters into (4.7) does not determine all μ_i in (4.7), because it give us six constraints while there are nine unfixed parameters in (4.7). To remedy it, we compared two bilinear relations (4.5) and (2.4), which eventually provide us three additional constraints

$$\begin{aligned} \tilde{f}_0(\tau) &= f_0(\tau)\tilde{g}_0(\tau) + f_\epsilon(\tau)\tilde{g}_1(\tau) + f_\sigma(\tau)\tilde{g}_2(\tau) \\ \tilde{f}_\epsilon(\tau) &= f_0(\tau)\tilde{g}'_1(\tau) + f_\epsilon(\tau)\tilde{g}_3(\tau) + f_\sigma(\tau)\tilde{g}_4(\tau) \\ \tilde{f}_\sigma(\tau) &= f_0(\tau)\tilde{g}'_2(\tau) + f_\sigma(\tau)\tilde{g}'_4(\tau) + f_\epsilon(\tau)\tilde{g}_5(\tau). \end{aligned} \tag{4.8}$$

Now one can fix nine parameters μ_i combining three equations (4.8) with six constraints from (4.7). In this way, we find that the nine parameters μ_i read,

$$\begin{aligned} &\mu_1 = \frac{2647\pi^2}{576}, && \mu_2 = -i\frac{34952636877438883\pi^3}{140037228850176}, && \mu_3 = -\frac{598979\pi^4}{82944}, \\ &\mu_4 = -i\frac{7791635802406848865\pi^5}{20165360954425344}, && \mu_5 = \frac{6348183584577513 25\pi^6}{13443573969616896}, \\ &\mu_6 = -\frac{5641332583789180993\pi^6}{120992165726552064}, && \mu_7 = i\pi, \\ &\mu_8 = i\frac{8104903469545 49\pi^3}{46679076283392}, && \mu_9 = i\frac{284686915225948007\pi^5}{6721786984808448}. \end{aligned} \tag{4.9}$$

As a result, we finally get below six different characters of dual CFT.

$$\begin{aligned} \tilde{g}_0(\tau) &= q^{-\frac{23}{24}}\left(1 + 46851q^2 + 4310154q^3 + 155027130q^4 + \cdots\right) \\ \tilde{g}_1(\tau) &= \tilde{g}'_1(\tau) = q^{\frac{13}{24}}\left(2300 + 529828q + 28051444q^2 + \cdots\right) \\ \tilde{g}_2(\tau) &= \tilde{g}'_2(\tau) = q^{\frac{47}{48}}\left(47104 + 4757504q + 178382848q^2 + \cdots\right) \\ \tilde{g}_3(\tau) &= q^{\frac{1}{24}}\left(23 + 46598q + 4311948q^2 + 155017746q^3 + \cdots\right) \\ \tilde{g}_4(\tau) &= \tilde{g}'_4(\tau) = q^{\frac{23}{48}}\left(2048 + 565248q + 31700992q^2 + \cdots\right) \\ \tilde{g}_5(\tau) &= q^{\frac{11}{12}}\left(47104 + 5230592q + 204855296q^2 + 4630417408q^3 + \cdots\right) \end{aligned} \tag{4.10}$$

One can easily show that every coefficient in (4.10) can be expressed as a sum of the dimension of the irreducible representation of $2 \cdot 2^{1+22}.\mathrm{Co}_2$. $2 \cdot 2^{1+22}.\mathrm{Co}_2$ is a maximal subgroup of the baby Monster group. Its order is $2^{15} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. This multi-covering of the second largest Conway group Co_2 has irreducible representations that includes **23**,

2048, **2300** and **47104**. These numbers are in perfect agreement with the first terms of $\tilde{g}_3(\tau)$, $\tilde{g}_4(\tau)$, $\tilde{g}_1(\tau)$, $\tilde{g}_2(\tau)$ and $\tilde{g}_5(\tau)$. Referring the table 1, the other numbers in the characters can be decomposed as follow,

$$\begin{aligned}
46851 &= \mathbf{1} \oplus \mathbf{275} \oplus \mathbf{46575}, \quad 529828 = \mathbf{2300} \oplus \mathbf{50600} \oplus \mathbf{476928}, \\
4757504 &= 2 \cdot \mathbf{47104} \oplus \mathbf{4663296}, \quad 46598 = \mathbf{23} \oplus \mathbf{46575}, \\
565248 &= \mathbf{2048} \oplus \mathbf{563200}, \\
5230592 &= \mathbf{2048} \oplus \mathbf{47104} \oplus \mathbf{518144} \oplus \mathbf{4663296} \\
4310154 &= \mathbf{1} \oplus \mathbf{253} \oplus \mathbf{275} \oplus \mathbf{46575} \oplus \mathbf{1024650} \oplus \mathbf{3238400},
\end{aligned} \tag{4.11}$$

where the first term in each decomposition can be again understood as the Virasoro descendent of the corresponding primary state. Thus we conjecture that the nine-character rational CFT of $c = 23$, dual to the product of two critical Ising model, has $2 \cdot 2^{1+22} \cdot \mathrm{Co}_2$ as an automorphism group.

The modular S-matrix of the $c = 23$ CFT is identical to the (4.3), because of the bilinear relation (4.5). Therefore, the modular invariant partition function of $c = 23$ CFT is given by

$$\begin{aligned}
Z(\tau, \bar{\tau}) = |\tilde{g}_0(\tau)|^2 + |\tilde{g}_1(\tau)|^2 + |\tilde{g}_1'(\tau)|^2 + |\tilde{g}_2(\tau)|^2 + |\tilde{g}_2'(\tau)|^2 \\
+ |\tilde{g}_3(\tau)|^2 + |\tilde{g}_4(\tau)|^2 + |\tilde{g}_4'(\tau)|^2 + |\tilde{g}_5(\tau)|^2.
\end{aligned} \tag{4.12}$$

Also, the modular S-matrix (4.3) guarantees the positive integer fusion rule algebra coefficients.

We also propose that the characters of the above RCFT (4.10) obey intriguing bilinear relations with those of the critical Ising model (2.2) to give the baby Monster modules (2.7),

$$\begin{aligned}
\tilde{f}_0(\tau) &= f_0(\tau)\tilde{g}_0(\tau) + f_\epsilon(\tau)\tilde{g}_1(\tau) + f_\sigma(\tau)\tilde{g}_2(\tau), \\
\tilde{f}_\epsilon(\tau) &= f_0(\tau)\tilde{g}_1(\tau) + f_\epsilon(\tau)\tilde{g}_3(\tau) + f_\sigma(\tau)\tilde{g}_4(\tau), \\
\tilde{f}_\sigma(\tau) &= f_0(\tau)\tilde{g}_2(\tau) + f_\epsilon(\tau)\tilde{g}_4(\tau) + f_\sigma(\tau)\tilde{g}_5(\tau).
\end{aligned} \tag{4.13}$$

5 Self-dual RCFT and $2 \cdot \mathrm{Co}_1$

We discuss in this section a "self-dual" RCFT with $c = 12$ whose torus partition function admits a natural decomposition in terms of dimensions of representation of the Conway group $2{\cdot}\mathrm{Co}_1$.

The RCFT of our interest is self-dual in a sense that its three characters, denoted by $f_0(\tau)$, $f_1(\tau)$ and $f_2(\tau)$, satisfy a bilinear relation giving,[1]

$$j(\tau) + 96 = f_0^2(\tau) + f_1^2(\tau) + \frac{1}{2} f_2^2(\tau) \tag{5.1}$$

[1] Strictly speaking, a bilinear relation (5.1) cannot be an example of Monster anatomy, because $j(\tau)+96$ is not the Monster module. Nonetheless, in this section, we discuss a self-dual RCFT of $c = 12$ because partition function of this theory also exhibit moonshine for Conway group.

with $\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$ and

$$
\begin{aligned}
f_0(\tau) &= \Big(-\frac{1}{2}\lambda^3(\tau) + \frac{3}{2}\lambda^2(\tau) - \frac{3}{2}\lambda(\tau) + 2\Big)\Big(\frac{16}{\lambda(\tau)\big(1-\lambda(\tau)\big)}\Big), \\
&= q^{-1/2}\Big(1 + 276q + 11202q^2 + 184024q^3 + \cdots\Big), \\
f_1(\tau) &= \frac{8\lambda^3(\tau)}{\lambda(\tau)\big(1-\lambda(\tau)\big)}, \\
&= 2048q\Big(1 + 24q + 300q^2 + 2624q^3 + \cdots\Big), \\
f_2(\tau) &= \frac{16\lambda^3(\tau)}{\lambda(\tau)\big(1-\lambda(\tau)\big)} + 24, \\
&= 24 + 4096q + 98304q^2 + 1228800q^3 + \cdots .
\end{aligned}
\tag{5.2}
$$

Here $f_0(\tau)$ is the vacuum character while $f_1(\tau)$ and $f_2(\tau)$ are characters for primary states of conformal weight $h = \frac{3}{2}$ and $h = \frac{1}{2}$. These characters are three independent solutions to a modular differential equation below,

$$
\Big[\partial_\tau^3 - \frac{1}{2}E_2(\tau)\partial_\tau^2 + \frac{1}{24}\Big(E_2^2(\tau) - 13E_4(\tau)\Big)\partial_\tau\Big]\, f(\tau) = 0. \tag{5.3}
$$

For later convenience, let us define $f_{--}(\tau)$, $f_{-+}(\tau)$ and $f_{+-}(\tau)$ as follows

$$
\begin{aligned}
f_{--}(\tau) &= f_0(\tau) + f_1(\tau), \\
&= q^{-1/2}\Big(1 + 276q + 2048q^{3/2} + 11202q^2 + 49152q^{5/2} + \cdots\Big), \\
f_{-+}(\tau) &= f_0(\tau) - f_1(\tau), \\
&= q^{-1/2}\Big(1 + 276q - 2048q^{3/2} + 11202q^2 - 49152q^{5/2} + \cdots\Big), \\
f_{+-}(\tau) &= f_2(\tau) \\
&= 24 + 4096q + 98304q^2 + 1228800q^3 + 10747904q^4 + \cdots .
\end{aligned}
\tag{5.4}
$$

From the modular S-matrix of the self-dual theory,

$$
\begin{pmatrix} f_{--}(1-\lambda) \\ f_{-+}(1-\lambda) \\ f_{+-}(1-\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_{--}(\lambda) \\ f_{-+}(\lambda) \\ f_{+-}(\lambda) \end{pmatrix}. \tag{5.5}
$$

one can show that the $\mathrm{SL}(2,\mathbb{Z})$ invariant partition function would be

$$
Z = \frac{1}{2}\Big(\big|f_{--}(\tau)\big|^2 + \big|f_{-+}(\tau)\big|^2 + \big|f_{+-}(\tau)\big|^2\Big) + \text{const.} \tag{5.6}
$$

Notice here that the character $f_{--}(\tau)$ is nothing but the Neveu-Schwarz (NS) partition function $K(\tau)$ of $\mathcal{N}=1$ extremal superconformal theory [17]. In fact, one can understand from (5.6) that the RCFT with $c = 12$ of our interest is the GSO projection of the $\mathcal{N}=1$

extremal SCFT where

$$
\begin{aligned}
f_{--}(\tau) &= \mathrm{tr}_{\mathrm{NS}}\left[q^{L_0-c/24}\right],\\
f_{-+}(\tau) &= \mathrm{tr}_{\mathrm{NS}}\left[(-1)^F q^{L_0-c/24}\right],\\
f_{+-}(\tau) &= \mathrm{tr}_{\mathrm{R}}\left[q^{L_0-c/24}\right],
\end{aligned}
\tag{5.7}
$$

and the constant term corresponding to the Witten index, $\mathrm{tr}_{\mathrm{R}}\left[(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right] = 24^2$. Here the each trace can be performed in either the NS Hilbert space or the Ramond Hilbert space of the $\mathcal{N}=1$ SCFT. This SCFT was first made by Frenkel, Lepowsky and Meurman [2], and revisited later by Duncan [3, 8].

The $\mathcal{N}=1$ extremal SCFT with $c=12$ is well-known to have $2{\cdot}\mathrm{Co}_1$ $=\mathrm{Co}_0$ as an automorphism group. After the GSO projection, the double covering of the largest Conway group continues to serve as an automorphism group of the self-dual theory. The Virasoro character decomposition of the partition further gives new evidence that the automorphism group of the RCFT with $c=12$ may be enhanced to a larger group $2^{1+24}\cdot\mathrm{Co}_1$, a maximal subgroup of the Monster group. To see this, let us decompose the partition function (5.6) in terms of the $c=12$ Virasoro characters,

$$
\begin{aligned}
Z(\tau,\bar{\tau}) = {}& \chi_0(q)\bar{\chi}_0(q) + 276\Big(\chi_0(\bar{q})\bar{\chi}_1(\bar{q}) + \text{c.c.}\Big) + 76176\chi_1(\bar{q})\bar{\chi}_1(\bar{q})\\
&+ 10925\Big(\chi_0(\bar{q})\bar{\chi}_2(\bar{q}) + \text{c.c.}\Big) + 3015300\Big(\chi_1(\bar{q})\bar{\chi}_2(\bar{q}) + \text{c.c.}\Big)\\
&+ 1081344\Big(\chi_{\frac{1}{2}}(q)\bar{\chi}_{\frac{5}{2}}(\bar{q}) + \text{c.c.}\Big) + 49152\chi_{\frac{1}{2}}(\bar{q})\bar{\chi}_{\frac{1}{2}}(\bar{q}) + \cdots .
\end{aligned}
\tag{5.8}
$$

It turns out that every coefficients in (5.8) is related to the irreducible representations of $2^{1+24}\cdot$ Co_1. Some details of the number decomposition are presented below.

$$
\begin{aligned}
49152 &= \mathbf{1}\oplus 2\cdot\mathbf{276}\oplus\mathbf{299}\oplus\mathbf{1771}\oplus\mathbf{8855}\oplus\mathbf{37674},\\
276 &= \mathbf{23}\oplus\mathbf{253}, \quad 10925 = \mathbf{299}\oplus\mathbf{1771}\oplus\mathbf{8855}\oplus\mathbf{10626},\\
76176 &= \mathbf{1}\oplus\mathbf{276}\oplus\mathbf{299}\oplus\mathbf{1771}\oplus\mathbf{8855}\oplus\mathbf{27300}\oplus\mathbf{37674}
\end{aligned}
\tag{5.9}
$$

6 Discussion

It is known that the CFT of $c=1$ discussed in section 4 allows a non-diagonal partition function

$$
Z(\tau,\bar{\tau}) = |g_0(\tau)+g_3(\tau)|^2 + |g_1(\tau)+g_1'(\tau)|^2 + 2|g_5(\tau)|^2,
\tag{6.1}
$$

different from (4.4). (6.1) is the partition function of another example of $c=1$ CFT studied in [23, 24]. One can show that the characters in (6.1) can obey a new bilinear relation

$$
\begin{aligned}
j(\tau) - 720 = {}& \big(g_0(\tau)+g_3(\tau)\big)\big(\tilde{g}_0(\tau)+\tilde{g}_3(\tau)\big) + \big(g_1(\tau)+g_1'(\tau)\big)\big(\tilde{g}_1(\tau)+\tilde{g}_1'(\tau)\big)\\
&+ 2g_5(\tau)\tilde{g}_5(\tau),
\end{aligned}
\tag{6.2}
$$

from which one can read the partition function of the dual theory

$$\tilde{Z}(\tau,\bar{\tau}) = |\tilde{g}_0(\tau)+\tilde{g}_3(\tau)|^2 + |\tilde{g}_1(\tau)+\tilde{g}_1'(\tau)|^2 + 2|\tilde{g}_5(\tau)|^2. \tag{6.3}$$

Notice that (6.3) can be understood as a non-diagonal partition function of the rational CFT of $c = 23$ discussed in section 4. It is obvious that the dual CFT also exhibit the moonshine for $2 \cdot 2^{1+22} \cdot \mathrm{Co}_2$. This is somewhat trivial, because it shares the characters (4.10) as building blocks. However, above example cannot be considered as a class of the Monster anatomy, because $j(\tau) - 720$ is not the Monster module.

It has been proposed that the Hecke images of the vector-valued modular form can construct a set of admissible characters for specific RCFT [25]. The Hecke image $T_p f_i(\tau)$ is characterized by the conductor N and a natural number p that is relatively prime to N. As an example, it has been known that the conductor of the critical Ising model is given by $N = 48$. Then, one can show that the Hecke images for $p = 47$ are exactly agree to the characters of the baby Monster CFT, namely (2.7). In similar way, it is easy to see that the conductor of three-state Potts model and the tensor product of the critical Ising model are given by $N = 30$ and $N = 24$, respectively. We checked that the characters (3.10) can be considered as the Hecke images of (3.4) with $p = 29$. In similar way, we found that the vector-valued modular form $(\tilde{g}_0(\tau)+\tilde{g}_3(\tau), \tilde{g}_1(\tau)+\tilde{g}_1'(\tau), \tilde{g}_5(\tau))$ in (6.3) is also realized as the Hecke images of $(g_0(\tau)+g_3(\tau), g_1(\tau)+g_1'(\tau), g_5(\tau))$ with $p = 23$. However, it turns out that the Hecke images of (4.2) cannot generate the six characters in (4.10). More precisely, the characters $\tilde{g}_2(\tau)$ and $\tilde{g}_4(\tau)$ in (4.10) are not able to realized as the Hecke images of the characters in (4.2).

One can also ask if the dual CFT of the tricritical Ising model, the next simplest unitary minimal model $\mathcal{M}(5,4)$, can exhibit the moonshine phenomena. However, it is nontrivial to obtain the characters of the candidate dual CFT with $c = \frac{233}{10}$. This is partially because one cannot determine all free parameters in the corresponding MDE completely from the known CFT data [12]. We tried to find the dual characters of $c = \frac{233}{10}$ CFT using the Hecke operator of $N = 240$ and $p = 233$, however it turns out that this Hecke operator does not produce admissible characters.

On the other hand, it is known that the tricritical Ising model is endowed with the $\mathcal{N} = 1$ supersymmetry. The NS partition function of the model can be contributed by the NS superconformal characters of the vacuum and the primary state of $h = \frac{1}{10}$. It would be interesting to search for a rational SCFT whose NS characters obey a new bilinear relation with those of the tricritical Ising model to give the $K(\tau)$-function [17]. This new bilinear relation would lead to a picture of the Conway group decomposition, instead of the Monster group decomposition. We leave them as a future project.

The bigger challenge is to find all the two-dimensional dual CFT pairs for the rest of the sporadic groups and understand the origin of bilinear relations. In particular, it would be extremely interesting to see if a CFT dual to '(three-state Potts model)$\times$(a certain CFT)' can show moonshine for the multiple-covering of the largest Mathieu group, $2^{12} \cdot \mathrm{M}_{24}$. This is analogous to the idea used to find the dual CFT with $2 \cdot 2^{1+22} \cdot \mathrm{Co}_2$ and correspond to the green arrow in figure 1.

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A Dimension of the irreducible representations

Here we give a partial list of the dimension of the irreducible representations for various sporadic groups.

	Dimension of the irreducible representations
$2\cdot\mathbb{B}$	$\{\mathbf{1}, \mathbf{4371}, \mathbf{96255}, \mathbf{96256}, \mathbf{1139374}, \mathbf{9458750}, \mathbf{9550635}, \mathbf{10506240}, \mathbf{63532485}, \mathbf{347643114}, \mathbf{356054375}, \mathbf{410132480}, \mathbf{1407126890}, \mathbf{3214743741}, \mathbf{4221380670}, \mathbf{4275362520}, \mathbf{4622913750}, \cdots\}$
$3\cdot\mathrm{Fi}'_{24}$	$\{\mathbf{1}, \mathbf{783}, \mathbf{8671}, \mathbf{57477}, \mathbf{64584}, \mathbf{249458}, \mathbf{306153}, \mathbf{555611}, \mathbf{1603525}, \mathbf{1666833}, \mathbf{4864431}, \mathbf{6724809}, \mathbf{19034730}, \mathbf{25356672}, \mathbf{32715683}, \mathbf{35873145}, \mathbf{40536925}, \mathbf{43779879}, \mathbf{48893768}, \mathbf{74837400}, \cdots,\}$
$2\cdot\mathrm{Co}_1$	$\{\mathbf{1}, \mathbf{24}, \mathbf{276}, \mathbf{299}, \mathbf{1771}, \mathbf{2024}, \mathbf{2576}, \mathbf{4576}, \mathbf{8855}, \mathbf{17250}, \mathbf{27300}, \mathbf{37674}, \mathbf{40480}, \mathbf{44275}, \mathbf{80730}, \mathbf{94875}, \mathbf{95680}, \mathbf{170016}, \mathbf{299000}, \mathbf{313950}, \mathbf{315744}, \mathbf{345345}, \mathbf{351624}, \mathbf{376740}, \mathbf{388080}, \mathbf{483000}, \mathbf{644644}, \mathbf{673750}, \mathbf{789360}, \mathbf{822250}, \mathbf{871884}, \mathbf{1434510}, \mathbf{1450449}, \mathbf{1771000}, \mathbf{1821600}, \mathbf{1841840}, \cdots\}$
$2^{1+24}\cdot\mathrm{Co}_1$	$\{\mathbf{1}, \mathbf{276}, \mathbf{299}, \mathbf{1771}, \mathbf{8855}, \mathbf{17250}, \mathbf{27300}, \mathbf{37674}, \mathbf{44275}, \mathbf{80730}, \mathbf{94875}, \mathbf{98280}, \mathbf{98304}, \mathbf{313950}, \mathbf{345345}, \mathbf{376740}, \mathbf{483000}, \mathbf{644644}, \mathbf{673750}, \mathbf{822250}, \mathbf{871884}, \mathbf{1434510}, \mathbf{1450449}, \mathbf{1771000}, \mathbf{1821600}, \mathbf{2055625}, \mathbf{2260440}, \mathbf{2417415}, \mathbf{2464749}, \mathbf{2464749}, \mathbf{2816856}, \mathbf{2877875}, \cdots\}$
$2^{1+22}\cdot\mathrm{Co}_2$	$\{\mathbf{1}, \mathbf{23}, \mathbf{253}, \mathbf{275}, \mathbf{1771}, \mathbf{2024}, \mathbf{2048}, \mathbf{2277}, \mathbf{2300}, \mathbf{4025}, \mathbf{7084}, \mathbf{9625}, \mathbf{10395}, \mathbf{12650}, \mathbf{23000}, \mathbf{31625}, \mathbf{31878}, \mathbf{37422}, \mathbf{44275}, \mathbf{46575}, \mathbf{47104}, \mathbf{50600}, \mathbf{63250}, \mathbf{91125}, \mathbf{113850}, \mathbf{129536}, \mathbf{177100}, \mathbf{184437}, \mathbf{212520}, \mathbf{221375}, \mathbf{226688}, \mathbf{239085}, \mathbf{245916}, \mathbf{253000}, \mathbf{284625}, \mathbf{312984}, \mathbf{368874}, \mathbf{398475}, \mathbf{430353}, \mathbf{442750}, \mathbf{462000}, \mathbf{467775}, \mathbf{476928}, \mathbf{518144}, \mathbf{531300}, \mathbf{558900}, \mathbf{563200}, \mathbf{579600}, \cdots\}$

Table 1. Dimension of the irreducible representation in various sporadic groups. The list of the representation read from GAP package [26].

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B Character tables of $2 \cdot \mathbb{B}$ and $3 \cdot \mathrm{Fi}_{24}'$

n	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$3A$	$6A$	$3B$	$6B$	$4B$	$4C$	$4D$	$8A$	$5A$	$10A$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4371	4371	−493	275	275	−53	78	78	−3	−3	−77	51	19	−1	21	21
3	96255	96255	4863	2047	2047	103	351	351	27	27	351	223	−1	−1	55	55
4	1139374	1139374	−26962	8878	8878	782	1000	1000	−53	−53	−978	558	−82	−26	99	99
5	9458750	9458750	118846	37950	37950	3486	2729	2729	−25	−25	2750	1214	318	−26	175	175
6	9550635	9550635	119339	35627	35627	−781	3003	3003	6	6	2827	1163	43	27	210	210
7	63532485	63532485	−468027	134597	134597	−4811	8073	8073	135	135	−7547	2437	197	1	385	385
8	347643114	347643114	1511146	392426	392426	1274	16224	16224	−219	−219	14378	4394	−5341	78	539	539
9	356054375	356054375	−1901977	419175	419175	−1377	18227	18227	−160	−160	−18425	8327	−153	27	650	650
10	1407126890	1407126890	−3199638	789866	789866	25194	25103	25103	155	155	−18326	3178	−406	26	615	615
11	3214743741	3214743741	3059133	1214653	1214653	22933	32670	32670	270	270	17501	−3875	1469	−27	616	616
12	4221380670	4221380670	−999362	1615934	1615934	−44642	37401	37401	465	465	−8130	−8130	−706	−78	595	595
13	4275362520	4275362520	10237656	1710808	1710808	48568	42471	42471	594	594	45144	20056	1240	0	770	770
14	4622913750	4622913750	11656918	2011350	2011350	−42042	58422	58422	345	345	57750	22678	−1066	−78	1275	1275
15	9287037474	9287037474	16720418	1896994	1896994	87074	19734	19734	1023	1023	24354	31522	2850	26	99	99
16	12501781215	12501781215	−15693601	2075359	2075359	65807	52404	52404	1293	1293	−47201	11935	−1057	−53	715	715
17	13508418144	13508418144	−18677152	3512928	3512928	−134368	57135	57135	1488	1488	−60576	18272	3680	0	694	694
18	27416186875	27416186875	−37543429	6369275	6369275	−44149	129349	129349	−1385	−1385	−125125	32827	2299	−1	1925	1925
19	75844139371	75844139371	−56581525	3402091	3402091	162435	−1001	−1001	2887	2887	4235	49931	875	−273	−154	−154
20	80426400000	80426400000	−87975680	10451200	10451200	104960	169170	169170	−120	−120	−172800	77056	−1792	0	1750	1750
21	90807234375	90807234375	55087175	2498375	2498375	−118625	−10725	−10725	−1815	−1815	−9625	42215	−3001	−325	0	0
22	90807234375	90807234375	55087175	2498375	2498375	−118625	−10725	−10725	−1815	−1815	−9625	42215	−3001	−325	0	0
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
185	96256	−96256	0	2048	−2048	0	352	−352	28	−28	0	0	0	0	56	−56
186	10506240	−10506240	0	45056	−45056	0	3456	−3456	−108	108	0	0	0	0	240	−240
187	410132480	−410132480	0	516096	−516096	0	23648	−23648	−4	4	0	0	0	0	880	−880
188	8844386304	−8844386304	0	3629056	−3629056	0	96096	−96096	840	−840	0	0	0	0	1904	−1904
189	36657653760	−36657653760	0	8198144	−8198144	0	146016	−146016	−432	432	0	0	0	0	1960	−1960
190	53936390144	−53936390144	0	−1835008	1835008	0	11648	−11648	1280	−1280	0	0	0	0	−56	56
191	53936390144	−53936390144	0	−1835008	1835008	0	11648	−11648	1280	−1280	0	0	0	0	−56	56
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Table 2. Partial character table of the baby Monster group $2 \cdot \mathbb{B}$.

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n	1A	3A	3B	2A	6A	6B	2B	6C	6D	3C	3D	3E	3F	3G	3H	3I	4A	12A	12B
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	8671	8671	8671	351	351	351	−33	−33	−33	247	−77	−77	−77	85	85	85	31	31	31
3	57477	57477	57477	1157	1157	1157	133	133	133	534	615	615	615	210	210	210	69	69	69
4	249458	249458	249458	2354	2354	2354	370	370	370	2705	869	869	869	167	167	167	50	50	50
5	555611	555611	555611	5083	5083	5083	91	91	91	−535	2300	2300	2300	518	518	518	155	155	155
6	1603525	1603525	1603525	1925	1925	1925	−315	−315	−315	6475	−275	−275	−275	−140	−140	−140	5	5	5
7	1603525	1603525	1603525	1925	1925	1925	−315	−315	−315	6475	−275	−275	−275	−140	−140	−140	5	5	5
8	1666833	1666833	1666833	11153	11153	11153	273	273	273	3093	−1848	−1848	−1848	987	987	987	209	209	209
9	4864431	4864431	4864431	13871	13871	13871	−1105	−1105	−1105	4836	4917	4917	4917	867	867	867	175	175	175
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
109	783	783α	$783\overline{\alpha}$	79	79α	$79\overline{\alpha}$	15	15α	$15\overline{\alpha}$	0	54	54α	$54\overline{\alpha}$	27	27α	$27\overline{\alpha}$	15	15α	$15\overline{\alpha}$
110	783	$783\overline{\alpha}$	783α	79	$79\overline{\alpha}$	79α	15	$15\overline{\alpha}$	15α	0	54	$54\overline{\alpha}$	54α	27	$27\overline{\alpha}$	27α	15	$15\overline{\alpha}$	15α
111	64584	64584α	$64584\overline{\alpha}$	1352	1352α	$1352\overline{\alpha}$	72	72α	$72\overline{\alpha}$	0	−297	-297α	$-297\overline{\alpha}$	216	216α	$216\overline{\alpha}$	72	72α	$72\overline{\alpha}$
112	64584	$64584\overline{\alpha}$	64584α	1352	$1352\overline{\alpha}$	1352α	72	$72\overline{\alpha}$	72α	0	−297	$-297\overline{\alpha}$	-297α	216	$216\overline{\alpha}$	216α	72	$72\overline{\alpha}$	72α
113	306513	306513α	$306513\overline{\alpha}$	3433	3433α	$3433\overline{\alpha}$	489	489α	$489\overline{\alpha}$	0	1431	1431α	$1431\overline{\alpha}$	351	351α	$351\overline{\alpha}$	105	105α	$105\overline{\alpha}$
114	306513	$306513\overline{\alpha}$	306513α	3433	$3433\overline{\alpha}$	3433α	489	$489\overline{\alpha}$	489α	0	1431	$1431\overline{\alpha}$	1431α	351	$351\overline{\alpha}$	351α	105	$105\overline{\alpha}$	105α
115	306513	306513α	$306513\overline{\alpha}$	2729	2729α	$2729\overline{\alpha}$	−279	-279α	$-279\overline{\alpha}$	0	1431	1431α	$1431\overline{\alpha}$	351	351α	$351\overline{\alpha}$	105	105α	$105\overline{\alpha}$
116	306513	$306513\overline{\alpha}$	306513α	2729	$2729\overline{\alpha}$	2729α	−279	$-279\overline{\alpha}$	-279α	0	1431	$1431\overline{\alpha}$	1431α	351	$351\overline{\alpha}$	351α	105	$105\overline{\alpha}$	105α
117	6724809	6724809α	$6724809\overline{\alpha}$	26377	26377α	$26377\overline{\alpha}$	−567	-567α	$-567\overline{\alpha}$	0	−3861	-3861α	$-3861\overline{\alpha}$	2079	2079α	$2079\overline{\alpha}$	393	393α	$393\overline{\alpha}$
118	6724809	$6724809\overline{\alpha}$	6724809α	26377	$26377\overline{\alpha}$	26377α	−567	$-567\overline{\alpha}$	-567α	0	−3861	$-3861\overline{\alpha}$	-3861α	2079	$2079\overline{\alpha}$	2079α	393	$393\overline{\alpha}$	393α
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

Table 3. Partial character table of the Fischer group $3 \cdot \mathrm{Fi}_{24}'$. Here $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $\overline{\alpha} = \frac{-1-i\sqrt{3}}{2}$.

g	$g_{\mathbb{M}}$	g	$g_{\mathbb{M}}$	g	$g_{\mathbb{M}}$
2A	$2A$	3B	3B	6A	6A
2B	$2A$	4A	4B	6B	6D
2C	$2A$	4B	4A	8A	8C
2D	$2B$	4D	4C	5A	5A
3A	$3A$	4C	4A	10A	10A

Table 4. Generalized bilinear relation for $2 \cdot \mathbb{B}$.

g	$g_{\mathbb{M}}$	g	$g_{\mathbb{M}}$	g	$g_{\mathbb{M}}$
3A,3B	$3A$	6C,6D	6C	3G	3A
2A	$2A$	3C	3A	3H,3I	3B
6A,6B	$6A$	3D	3B	4A	4A
2B	$2B$	3E,3F	3A	12A,12B	12A

Table 5. Generalized bilinear relation for $3 \cdot \mathrm{Fi}'_{24}$.

C Generalized bilinear relations

C.1 $2 \cdot \mathbb{B}$

We find the twined characters of baby Monster CFT for various $g \in 2 \cdot \mathbb{B}$ which are combined with the characters of Ising models and form the Mckay-Thompson series for various $g_{\mathbb{M}} \in \mathbb{M}$. More preciesly, general expression of generalized bilinear relation have a form of

$$j^{g_{\mathbb{M}}}(\tau) = f_0(\tau) \cdot \tilde{f}^g_0(\tau) + f_\epsilon(\tau) \cdot \tilde{f}^g_\epsilon(\tau) + f_\sigma(\tau) \cdot \tilde{f}^g_\sigma(\tau), \tag{C.1}$$

where $\tilde{f}^g(\tau)$ is twined character for $g \in 2 \cdot \mathbb{B}$ and $j^{g_{\mathbb{M}}}(\tau)$ denotes Mckay-Thompson series for class $g_{\mathbb{M}} \in \mathbb{M}$. Table 4 present which twined character forming Mckay-Thompson series of type $g_{\mathbb{M}}$. For instance, once we have twined characters of baby Monster CFT for $g = 2C$, they merge with the characters of Ising model to produce Mckay-Thompson series of class $2A$.

C.2 $3 \cdot \mathrm{Fi}'_{24}$

It turns out that the twined characters of $c = \frac{116}{5}$ putative CFT for various $g \in 3 \cdot \mathrm{Fi}'_{24}$ also constitute the Mckay-Thompson series of certain class $g_{\mathbb{M}} \in \mathbb{M}$ with the characters of three-states Potts model. We find that the explicit form of the generalized bilinear relation is given by

$$\begin{aligned} j^{g_{\mathbb{M}}}(\tau) = f_0(\tau) \cdot \tilde{f}^g_0(\tau) + f_1(\tau) \cdot \tilde{f}^g_1(\tau) + f_2(\tau) \cdot \tilde{f}^g_2(\tau) + f'_2(\tau) \cdot \tilde{f}'^g_2(\tau) \\ + f_3(\tau) \cdot \tilde{f}^g_3(\tau) + f'_3(\tau) \cdot \tilde{f}'^g_3(\tau), \end{aligned} \tag{C.2}$$

where $\tilde{f}^g(\tau)$ is twined character for $g \in 3 \cdot \mathrm{Fi}'_{24}$ and $j^{g_{\mathbb{M}}}(\tau)$ denotes Mckay-Thompson series for class $g_{\mathbb{M}} \in \mathbb{M}$ as before. Table 5 exhibit which twined character yields Mckay-Thompson series of certain class, like table 4 describes.

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