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Phases of $\mathcal{N} = 1$ theories in $2 + 1$ dimensions

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Abstract: We study the dynamics of 2+1 dimensional theories with $\mathcal{N} = 1$ supersymmetry. In these theories the supersymmetric ground states behave discontinuously at co-dimension one walls in the space of couplings, with new vacua coming in from infinity in field space. We show that the dynamics near these walls is calculable: the two-loop effective potential yields exact results about the ground states near the walls. Far away from the walls the ground states can be inferred by decoupling arguments. In this way, we are able to follow the ground states of $\mathcal{N} = 1$ theories in 2+1 dimensions and construct the infrared phases of these theories. We study two examples in detail: Adjoint SQCD and SQCD with one fundamental quark. In Adjoint QCD we show that for sufficiently small Chern-Simons level the theory has a non-perturbative metastable supersymmetry-breaking ground state. We also briefly discuss the critical points of this theory. For SQCD with one quark we establish an infrared duality between a U(N) gauge theory and an SU(N) gauge theory. The duality crucially involves the vacua that appear from infinity near the walls.

Keywords: Supersymmetric Gauge Theory, Chern-Simons Theories, Duality in Gauge Field Theories

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Contents

1 Introduction and summary

There has been substantial progress recently on the dynamics of 2+1 dimensional gauge theories. Many interesting phenomena such as confining phases, symmetry breaking phases, phases with topological order and dualities were uncovered. These results are supported by many nontrivial consistency checks, such as the matching of various discrete anomalies (including anomalies of space-time symmetries), consistency with renormalization group flows, various other non-perturbative constraints (such as the Vafa-Witten theorem, the study of counterterms, etc.) and the rigorous study of weakly coupled limits such as the large N limit, etc. For recent work on the phases of 2+1 dimensional gauge theories see [1–30] and references therein.

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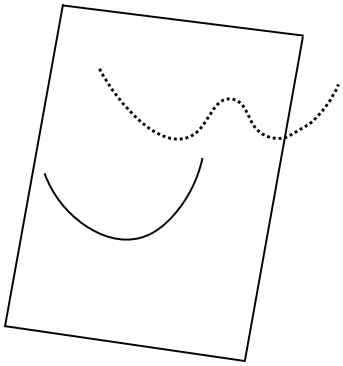

Figure 1. New vacua can emerge from infinity in field space at a wall in parameter space where the asymptotics of the superpotential changes (one such vacuum appears on the other side of the wall in the figure). The new vacua can be reliably exhibited by a perturbative two-loop computation. This combined with our proposed infrared dynamics of the vacua results in a description of the phase diagram of $\mathcal{N} = 1$ theories as a function of superpotential couplings.

Our main goal in this paper is to revisit some questions about the dynamics of supersymmetric gauge theories in 2+1 dimensions. We will investigate theories with $\mathcal{N} = 1$ supersymmetry, that is theories with two real supercharges. Theories with $\mathcal{N} = 1$ supersymmetry in 2+1 dimensions have received relatively little attention over the years. This is mostly because $\mathcal{N} = 1$ theories do not enjoy the powerful constraints on the dynamics implied by holomorphy. In addition, localization techniques essentially do not apply and indeed relatively little has hitherto been known about these theories, cf. [24, 31]. In this paper we introduce some new tools to study $\mathcal{N} = 1$ supersymmetric theories.

Since the superpotential parameters reside in real superfields, the superpotential is not protected against quantum corrections. Furthermore the number of supersymmetric ground states can jump across walls in the space of couplings where the behaviour of the potential at infinity in field space changes.[1] This can happen at co-dimension one surfaces in parameter space, which we refer to as walls. On one of the sides of the wall there can be supersymmetric ground states that "appear from infinity" in field space (see figure 1). The main point is that since these vacua appear from infinity, and since the underlying model is typically super-renormalizable, these vacua can be reliably studied by computing radiative corrections to the effective potential on the wall in parameter space. We show that a two-loop computation is necessary to establish the existence of these vacua (and sufficient for our purposes). We will see that this quantum-mechanically generated superpotential, together with our proposed infrared dynamics of each new vacuum, precisely reproduces the Witten index [33] of the theory farther away from the wall (where the index must remain constant, as the asymptotics of the potential does not change away from the wall). Together with an analysis of the physics far from these walls, one can therefore determine the phases of $\mathcal{N} = 1$ theories as a function of the superpotential parameters. In particular, one can identify the phase transitions that take place.

[1] This is in contrast to $\mathcal{N} = 2$ theories, where the number of supersymmetric ground states cannot jump as a function of real masses, cf. [32].

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We will carry out this analysis explicitly in two cases which demonstrate rather different mechanisms and principles. The methods we introduce are general and can be applied to a wide variety of $\mathcal{N}=1$ models.

We turn now to a brief summary of the theories we analyze and outline our results (the derivations and details are presented in the bulk of the paper). The gauge theories we study have a Yang-Mills term and a Chern-Simons term. We label a theory by the gauge group, the matter content and by the Chern-Simons level[2] k, which henceforth we take to be non-negative since the theory with $k<0$ can be obtained by acting with time-reversal on the theory with $k>0$. The theory with $k=0$ and with massless fermions is time-reversal invariant and typically needs a separate treatment.

$\mathcal{N}=1$ SU$(N)_k$ vector multiplet. The gauge multiplet consists of an SU(N) gauge field A, and a Majorana fermion λ. All the terms in the Lagrangian are the standard minimal couplings with a Chern-Simons term. $\mathcal{N}=1$ supersymmetry requires that the adjoint fermion mass is $-\frac{kg^2}{2\pi}$, i.e. proportional to the Chern-Simons level k. The $\mathcal{N}=1$ Lagrangian is[3]

$$\mathcal{L}=-\frac{1}{4g^2}\,\mathrm{Tr}\,F^2+i\,\mathrm{Tr}\,\lambda\not{D}\lambda+\frac{k}{4\pi}\,\mathrm{Tr}\left(AdA-\frac{2i}{3}A^3\right)-\frac{kg^2}{2\pi}\,\mathrm{Tr}\,\lambda\lambda\,. \tag{1.3}$$

This model has been studied in detail in [24, 31] and we review here those results as we build on them in this paper. This model has no adjustable continuous $\mathcal{N}=1$ preserving parameters and hence there is no wall for the Witten index to jump. For $k\gg 1$, the gauge fields and λ are both classically much heavier than the scale of interactions $kg^2\gg g^2$ and hence we can simply integrate them out. What remains after we integrate them out is a pure Chern-Simons theory. In particular, supersymmetry is unbroken. Upon integrating out the heavy particles and by virtue of (1.2) the theory flows[4] to the SU$(N)_{k-N/2}$ Chern-Simons TQFT at long distances. This analysis is reliable for $k\gg 1$. It turns out that SU$(N)_{k-N/2}$ is the correct description at long distances for $k\geq N/2$. In other words, for

[2]The level k, upon which time-reversal acts by $k\to -k$, is given in terms of the Chern-Simons level $k_{\rm bare}$ appearing in the classical Lagrangian by

$$k=k_{\rm bare}-\frac{1}{2}\sum_f T(R_f)\,, \tag{1.1}$$

where the sum is over the charged Majorana fermions in the theory and $T(R)$ is the index of the real representation R. The shift can be thought of as the contribution from the (massless) fermion determinant. Integrating out a massive Majorana fermion of mass m in a real representation R shifts the level k by [34–36]

$$k\to k+\frac{1}{2}\mathrm{sign}(m)\,T(R)\,. \tag{1.2}$$

This makes manifest the action of time-reversal, which flips the sign of the mass of a fermion. For a fermion in a complex or pseudoreal representation R the shift in (1.1)(1.2) should be multiplied by a factor of 2. We note that while the level $k\in\mathbb{Z}/2$, the ultraviolet and infrared levels $k_{\rm bare}$ and $k+\frac{1}{2}\mathrm{sign}(m)\,T(R)$ are always integrally quantized.

[3]In this paper we follow standard practice and write k in the Lagrangian, keeping in mind that it is integer for N even and half-integer for N odd.

[4]We recall that $T(\text{adjoint})=N$ and $T(\text{fundamental})=1/2$ for SU(N).

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$k \geq N/2$ supersymmetry is unbroken and the low-energy theory is the $\mathrm{SU}(N)_{k-N/2}$ TQFT. The Witten index [31] for $k \geq N/2$ agrees (possibly up to a sign) with the partition function of the $\mathrm{SU}(N)_{k-N/2}$ TQFT on the torus.

For $0 \leq k < \frac{N}{2}$ the index vanishes, supersymmetry is spontaneously broken [31] and the infrared of the theory consists of [24] a Majorana Goldstino particle G_α accompanied by a Chern-Simons TQFT[5] (the two sectors do not interact)

$$G_\alpha + U\left(\frac{N}{2} - k\right)_{\frac{N}{2}+k,N} . \tag{1.4}$$

The Goldstino operator G is the low energy limit of $\mathrm{Tr}(F\lambda)$, which is defined in the microscopic theory (1.3). This is simply because $\mathrm{Tr}(F\lambda)$ is the conserved supercurrent. It is much harder to understand the origin of the $U\left(\frac{N}{2} - k\right)_{\frac{N}{2}+k,N}$ Chern-Simons theory in terms of the microscopic degrees of freedom.[6] Nevertheless, this TQFT has many desirable properties and it passes several nontrivial checks (such as having the correct one-form symmetry and anomaly, time-reversal anomaly for $k = 0$ etc.).

$\mathcal{N} = 1$ $\mathrm{SU}(N)_k$+adjoint multiplet. The gauge multiplet consists of an $\mathrm{SU}(N)$ gauge field A, and a Majorana fermion, λ. We briefly reviewed its dynamics above. The additional adjoint multiplet includes a Majorana fermion and a real scalar (X, ψ_X), both in the adjoint representation.

This theory has an interesting family of $\mathcal{N} = 1$ supersymmetry-preserving deformations, namely, a general (real) superpotential

$$W = \sum_l \alpha_l \,\mathrm{Tr}(X^l) . \tag{1.5}$$

For one particular choice $\alpha_2 = -\frac{kg^2}{2\pi}$ (and $\alpha_{l>2} = 0$) the theory has enhanced $\mathcal{N} = 2$ supersymmetry, corresponding to a pure $\mathcal{N} = 2$ vector multiplet. We will study the dynamics of the softly mass deformed $\mathcal{N} = 2$ theory, namely, we only activate α_2 and denote $\alpha_2 \equiv m$. Therefore, we only include the superpotential mass term

$$W = m \,\mathrm{Tr}(X^2) . \tag{1.6}$$

We study the dynamics of the theory as a function of k, N, and m (in this notation, for $m = -\frac{kg^2}{2\pi}$ the theory enjoys $\mathcal{N} = 2$ supersymmetry).

We find surprisingly rich dynamics both for large k and for small k. For sufficiently large $|m|$ we can always integrate out the adjoint multiplet and the theory flows to a pure $\mathcal{N} = 1$ vector multiplet with gauge group $\mathrm{SU}(N)$ and level $k \pm N/2$ depending on whether m is large and positive or large and negative, respectively (cf. (1.2)). Then, according to what we found above, the discussion splits depending whether $k \geq N$, $0 < k < N$ or $k = 0$.

[5] We use the standard notation

$$\mathrm{U}(M)_{P,Q} = \frac{\mathrm{SU}(M)_P \times \mathrm{U}(1)_{MQ}}{\mathbb{Z}_M} .$$

Consistency requires that $Q = P \bmod M$.

[6] In [24] a duality was put forward that sheds some light on the origin of this TQFT.

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Let us now summarize the different phenomena we encounter in these various cases. A key element in our analysis is that the Witten index of $\mathcal{N}=1$ theories can jump as a function of m unlike in theories with more supersymmetry, where holomorphy forbids the index from jumping, cf. [32]. We find a new mechanism that allows the index to jump. This leads to new supersymmetric vacua (and in particular, one that supports an Abelian TQFT in the infrared) and also it leads to a mechanism for metastable supersymmetry breaking. Let us now summarize our findings in a little bit more detail:

1. $k \geq N$. Supersymmetry is unbroken for large $|m|$ and the theory flows to the TQFTs $\mathrm{SU}(N)_k$ and $\mathrm{SU}(N)_{k-N}$ for large positive m and large negative m, respectively. The Witten index therefore jumps as m is varied.[7] The $\mathcal{N}=2$ supersymmetric point is at $m=-\frac{g^2 k}{2\pi}$ and it flows to a supersymmetric vacuum with the $\mathrm{SU}(N)_{k-N}$ TQFT, a result that follows[8] for large k from (1.2) and holds all the way down to $k=N$. (This is consistent with the index [37] and the exact computation of the S^3 and $S^2\times S^1$ partition function of $\mathcal{N}=2$ theories.) The transition between the two asymptotic large mass phases happens in two steps. The wall where the index jumps is at $m=0$. At $m=0$ (where there is a classical non-compact space of supersymmetric vacua) a radiatively induced asymptotically flat (non-supersymmetric) direction with non-zero energy density opens up.[9] For infinitesimal positive m, the effective superpotential deformed by the mass term (1.6) supports $2^{N-1}-1$ new supersymmetric vacua which come in from infinity in field space and are described by new infrared Chern-Simons TQFTs. The jump in the Witten index between the two asymptotic phases is now fully accounted for by these new vacua. For infinitesimal negative m there is only one vacuum near the origin, supporting the $\mathrm{SU}(N)_{k-N}$ TQFT. As m is increased these new vacua approach the original supersymmetric vacuum with the $\mathrm{SU}(N)_{k-N}$ TQFT. Then these vacua merge in a series of second order phase transitions. When these transitions are completed, we get the $\mathrm{SU}(N)_k$ TQFT in a supersymmetric vacuum describing the asymptotic positive mass phase. See figure 2.

2. $0<k<N$. At large positive m supersymmetry is unbroken and the theory has a vacuum supporting the $\mathrm{SU}(N)_k$ TQFT. On the other hand, at large negative m supersymmetry is spontaneously broken and the supersymmetry-breaking vacuum has a Majorana Goldstino G_α accompanied by the $\mathrm{U}(N-k)_{k,N}$ TQFT (cf. (1.4)). The $\mathcal{N}=2$ theory at $m=-\frac{g^2 k}{2\pi}$ is likewise in this phase, i.e. it breaks $\mathcal{N}=2$ supersymmetry and the Majorana Goldstino fermion is joined by another Majorana fermion so that the low energy theory is a Dirac Goldstino particle with a decoupled $\mathrm{U}(N-k)_{k,N}$ TQFT. (This is consistent with the vanishing of the Witten index [37]

[7] The Witten index coincides (up to a sign) with the torus partition functions of the TQFTs.

[8] The two Majorana fermions in the $\mathcal{N}=2$ massive vector multiplet have spin 1/2 and since in 2+1d the sign of the spin of a fermion is determined by the sign of its mass, the fermions in the $\mathcal{N}=2$ vector multiplet shift the Chern-Simons level additively, cf. (1.2).

[9] The point $m=0$ does not have new symmetries. But it is still a special point — it is where the superpotential is asymptotically linear rather than quadratic.

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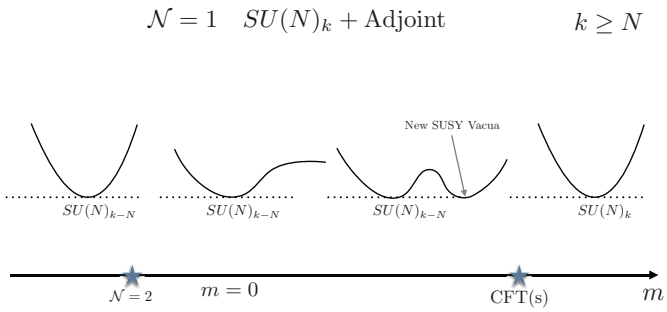

Figure 2. Proposed phase diagram for $k \geq N$. On the right hand side of the wall at $m = 0$ the theory develops new supersymmetric vacua that come in from infinity in field space. These vacua flow to specific TQFTs. As the mass is further increased vacua merge in a sequence of second order phase transitions. At the end of the sequence the physics is described by the large and positive mass asymptotic phase. On the left-hand side of the wall at $m = 0$ there is a unique supersymmetric vacuum with a TQFT that coincides with that describing the large and negative mass asymptotic phase. The new vacua at small positive mass account for the jump in the Witten between the asymptotic large mass phases.

and of the S^3 and $S^1 \times S^2$ partition function of the $\mathcal{N} = 2$ theory.[10]) The additional Majorana Goldstino becomes massive as the theory is deformed away from the $\mathcal{N} = 2$ point. The transition between the asymptotic phases at large negative and large positive m again happens in two stages. Around $m = 0$ an asymptotically flat direction opens up and new supersymmetry-preserving vacua with various Chern-Simons TQFTs come in from infinity for $m > 0$. Near the origin there is a metastable supersymmetry breaking state. As we increase m we encounter phase transitions and we eventually get to the supersymmetric vacuum with $\mathrm{SU}(N)_k$ TQFT describing the large positive mass asymptotic phase. See figure 3.

3. $k = 0$. There is a trivial supersymmetric vacuum (i.e no TQFT) at both $m \to \pm\infty$ and a non-perturbative runaway direction opens up at $m = 0$, where there is a classical non-compact space of vacua. The runaway at $m = 0$ is stabilized for both positive and negative m into a trivial supersymmetric vacuum, and there are no further phase transitions as m is varied. The $m = 0$ point for $k = 0$ is also the $\mathcal{N} = 2$ supersymmetric point and the runaway potential is due to the familiar Affleck-Harvey-Witten monopole-instanton mechanism [38]. The index remains constant everywhere where it is well-defined. In this example therefore the Witten index does not jump. The phase diagram is summarized in figure 4.

The dynamics of these theories is thus governed by the wall at $m = 0$, where the Witten index jumps. This happens in a pretty elaborate fashion, by the appearance of

[10]Both the S^3 and $S^1 \times S^2$ partition function of the $\mathcal{N} = 2$ $\mathrm{SU}(N)_k$ theory vanish for $k = 1, 2, \ldots, N-1$ (but not for $k \geq N$). This is consistent with supersymmetry breaking since an $\mathcal{N} = 2$ Goldstino on the S^3 and $S^1 \times S^2$ background has a fermionic zero mode implying that the partition function vanishes. In spite of that we will be able to provide strong evidence for our proposed infrared description of the $\mathcal{N} = 2$ $\mathrm{SU}(N)_k$ theory for $k < N$.

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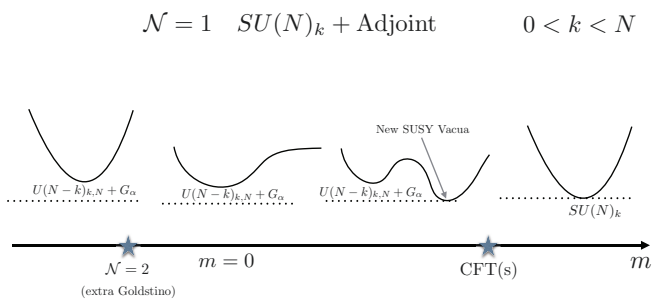

Figure 3. Proposed phase diagram for $0 < k < N$. The asymptotic large and negative mass phase breaks supersymmetry and is described by a Majorana Goldstino with a TQFT. This TQFT persists all the way down to the wall at $m = 0$, with an additional massless Majorana Goldstino appearing at the $\mathcal{N} = 2$ point. On the right-hand side of the wall at $m = 0$ the theory develops new supersymmetric vacua with novel TQFTs that come in from infinity in field space. The vacuum state at the origin in field space is a metastable supersymmetry breaking state. As the mass is further increased vacua merge in a sequence of second order phase transitions. At the end of the sequence the physics is described by the large and positive mass asymptotic phase, where supersymmetry is unbroken and described by a distinct TQFT. The new vacua at small positive mass account for the jump in the Witten between the asymptotic large mass phases.

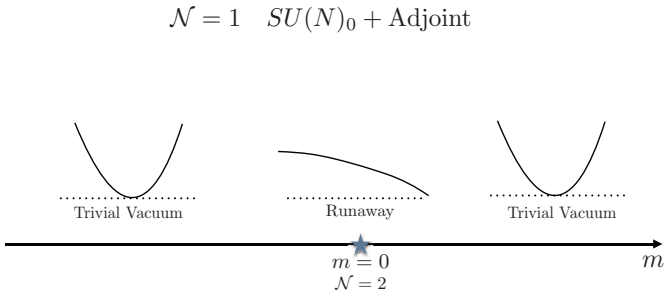

Figure 4. Phase diagram for $k = 0$. The theory flows to a supersymmetric trivial gapped vacuum everywhere, except at the $\mathcal{N} = 2$ supersymmetric point $m = 0$, where there is a runaway due to the non-perturbatively induced $\mathcal{N} = 2$ superpotential. The Witten index does not jump.

many new supersymmetric Chern-Simons vacua, which by virtue of being semiclassical match all anomalies. In particular, a vacuum with an Abelian TQFT appears. In addition, this phenomenon comes hand in hand with the appearance of a metastable supersymmetry breaking vacuum.[11] Certain conformal field theories govern the mergers of these vacua that appear from infinity. We analyse these conformal field theories in the simplest cases. We emphasize the appearance of an asymptotically flat direction with constant energy density on the wall (point) at $m = 0$, the way by which the Witten index jumps, and the implications for metastable supersymmetry breaking.

[11]Loosely speaking, one can think of a crude gravitational dual picture where the metastable supersymmetry breaking vacuum corresponds to $N - k$ anti-branes on top of each other. Then, there is a non-perturbative effect which allows this state to tunnel to a supersymmetric ground state.

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$\mathcal{N}=1$ theories with fundamental matter. The study of theories with fundamental matter will be carried out systematically elsewhere. Here we study just two examples with a single matter multiplet, demonstrating how the tools we have introduced allow to determine the phases of these theories and infer $\mathcal{N}=1$ dualities. We will see that the dynamics near the walls is necessary to understand these $\mathcal{N}=1$ dualities.

Our first example is an $\mathcal{N}=1$ SU(N) gauge theory coupled to a matter multiplet in the fundamental representation. The matter multiplet consists of a complex scalar and a Dirac fermion (Ψ, ψ_Ψ) in the fundamental representation of SU(N).

The theory has an $\mathcal{N}=1$ supersymmetry-preserving mass deformation by the superpotential

$$W = m|\Psi|^2 \,. \tag{1.7}$$

We solve for the infrared dynamics of this model as a function of k and m.

For large $|m|$ the matter multiplet can be integrated out and we are left with a pure $\mathcal{N}=1$ vector multiplet with SU(N) gauge group and level $k \pm 1/2$, depending on whether the mass is large and positive or large and negative. At $m=0$ the theory has a classical moduli space of vacua that is lifted by a radiatively induced two-loop superpotential. For infinitesimal small positive m the theory has a new supersymmetric vacuum state coming in from infinity in field space. This vacuum supports at low energies a pure $\mathcal{N}=1$ SU$(N-1)_k$ vector multiplet. For infinitesimal negative m there is only the vacuum near the origin, continuously connected to the vacuum that we see at large negative mass. Therefore, the phase diagram depends on whether $k \geq \frac{N+1}{2}$, $k = \frac{N-1}{2}$ or $k < \frac{N-1}{2}$.

1. $k \geq \frac{N+1}{2}$. There is a supersymmetric vacuum at large negative and large positive mass described by the TQFTs $\mathrm{SU}(N)_{k-\frac{N+1}{2}}$ and $\mathrm{SU}(N)_{k-\frac{N-1}{2}}$ respectively. At $m=0$ the theory has a classical moduli space of vacua that is lifted by a radiatively induced two-loop superpotential. The supersymmetric vacuum state coming in from infinity for infinitesimal small positive m flows in the infrared to a supersymmetric vacuum with the $\mathrm{SU}(N-1)_{k-\frac{N-1}{2}}$ TQFT. As the mass is increased this TQFT merges with the $\mathrm{SU}(N)_{k-\frac{N+1}{2}}$ TQFT at a second order transition, at the other side of which is the asymptotic positive mass $\mathrm{SU}(N)_{k-\frac{N-1}{2}}$ TQFT. The new supersymmetric vacuum near $m=0$ accounts for the jump in the Witten index between the asymptotic large mass phases.

2. $k = \frac{N-1}{2}$. At large positive mass there is a trivial (i.e no TQFT) supersymmetric vacuum. At large negative mass supersymmetry is spontaneously broken and the supersymmetry breaking vacuum has a Majorana Goldstino accompanied by the $\mathrm{U}(1)_N$ TQFT (cf. (1.4)). The vacuum state that appears from infinity for infinitesimal small positive m is a trivial (i.e no TQFT) supersymmetric vacuum. As the mass is increased the vacuum that has appeared from infinity may be identified with the vacuum that we have found at very large positive mass. No transition is therefore necessary in this model. The new supersymmetric vacuum near $m=0$ accounts for the jump in the Witten index between the asymptotic large mass phases.

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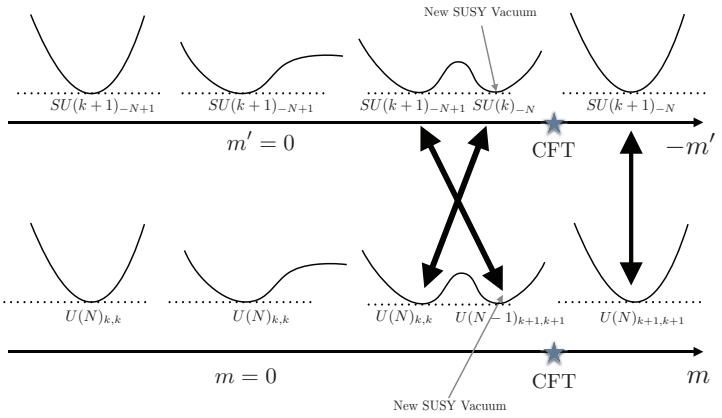

Figure 5. A duality between $\mathrm{SU}(k+1)_{-N-k/2}$ gauge theory with a fundamental matter multiplet and $\mathrm{U}(N)_{k+N/2+1/2,k+1/2}$ with a fundamental matter multiplet. The duality exchanges the vacuum that exists asymptotically in the phase diagram and the vacuum that appears from infinity near $m = m' = 0$. Note that the sign of m' is reversed. The phases match by virtue of level/rank duality.

3. $k < \frac{N-1}{2}$. Supersymmetry is broken at large $|m|$. Thus, the asymptotic phases have a Majorana Goldstino with the TQFTs $U\left(\frac{N+1}{2}-k\right)_{\frac{N-1}{2}+k,N}$ and $U\left(\frac{N-1}{2}-k\right)_{\frac{N+1}{2}+k,N}$ for large negative and large positive mass respectively. The vacuum state that appears from infinity for infinitesimal small positive m breaks supersymmetry and contains a Goldstino and the $U\left(\frac{N-1}{2}-k\right)_{\frac{N-1}{2}+k,N-1}$ TQFT.

One can similarly study the phases of the theory $\mathrm{U}(N)_{k,k'}$ coupled to a single fundamental matter multiplet Φ. This analysis leads us to propose the following duality, which is valid for $N \geq 1$ and $k \geq 0$

$$U\left(N\right)_{k+N/2+1/2,k+1/2} + \Phi \longleftrightarrow \mathrm{SU}(k+1)_{-N-k/2} + \Psi\,. \tag{1.8}$$

This duality[12] has interesting special cases,[13] for instance, $N = 1, k = 0$, which we will analyze in detail.[14] It is crucial to follow the vacua that appear from infinity at $m = 0$ in order to see that this proposal has the right phases across the transition. In fact, the vacuum that comes from infinity in one theory maps to a vacuum in an asymptotic phase of the dual theory. This is perfectly consistent with the duality, since the duality is valid in the infrared near the second order transition. See figure 5.

The 't Hooft limit of this duality can be studied explicitly [2, 41, 42] (and see references therein). In that approach, one discards the gauge kinetic term, i.e. one concentrates on

[12]The deformations map as $\Phi\Phi^\dagger \longleftrightarrow -\Psi\Psi^\dagger$. However, here we do not study the precise map between the quartic deformations. This is left for the future. We note, however, that there exist $\mathcal{N} = 2$ fixed points describing similar transitions (indeed, the SCFTs appear away from the walls, and hence the Witten index no longer jumps) and therefore in some situations there is possibly emergent $\mathcal{N} = 2$ supersymmetry in the infrared. We will say a few more words on the topic in the bulk of the paper. We thank D. Gaiotto for discussions of this topic.

[13]The case $N = 0$ is trivial, as explained in item 2 above.

[14]The duality then reduces to an $\mathcal{N} = 1$ supersymmetric version of the duality between nonsupersymmetric $\mathrm{U}(1)_{1/2}$ QED at level 1/2 with a charge 1 fermion and the XY model [7, 39, 40].

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the fixed point, which can be studied analytically in the 't Hooft limit. Our main new results concern with finding the supersymmetric vacua in the full theory (not necessarily in the 't Hooft limit or near the fixed point). It is a nontrivial consistency check that in the regime where the techniques can be both applied, they lead to the same results about the space of vacua of the theory.

Unlike in many of the non-supersymmetric dualities in the literature, where it is generally hard to establish that the transitions are second order, here it follows from the properties of the superpotential. Intuitively, since these are transitions involving (zero energy) supersymmetric vacua, a first-order transition is impossible. Hence the transitions have to be at least second order.

Similar methods can be employed to study many additional $\mathcal{N} = 1$ models. It is clearly of great interest to study models with more than one flavour multiplet, models with non-minimal charges and representations and so on. These studies are interesting in their own right, but they also naturally connect to questions about non-supersymmetric dynamics as well as to questions about theories with larger supersymmetry algebras.

The outline of the paper is as follows. In section 2 we review the pure $\mathcal{N} = 1$ vector multiplet dynamics. In section 3 we study the $\mathcal{N} = 1$ SU(N) theory with a Chern-Simons term and an additional adjoint multiplet. We consider the large k and small k dynamics, explaining how the Witten index jumps and establishing the existence of a metastable supersymmetry-breaking state for small k. In section 4 we consider SU(N) and U(N) gauge theories with a Chern-Simons term and fundamental matter representations, emphasizing duality and the important role played by the walls in establishing these dualities. Some details appear in three appendices.

2 $\mathcal{N} = 1$ vector multiplet — a review [24]

We consider Yang-Mills theory with SU(N) gauge group and a Chern-Simons term at level $k \geq 0$.[15] The vector multiplet consists of the gauge field A as well as a Majorana fermion λ in the adjoint representation. The most general renormalizable Lagrangian is

$$-\frac{1}{4g^2}\,\mathrm{Tr}\,F^2 + i\,\mathrm{Tr}\,\lambda \not{D}\lambda + \frac{k}{4\pi}\,\mathrm{Tr}\left(AdA - \frac{2i}{3}A^3\right) + m\,\mathrm{Tr}\,\lambda\lambda\,. \tag{2.1}$$

The level k is integral for N even and half-integral for N odd.

The Lagrangian (2.1) has $\mathcal{N} = 1$ supersymmetry for $m = -\frac{kg^2}{2\pi}$ (cf. (1.3)). We denote this theory as the $\mathcal{N} = 1$ SU$(N)_k$ vector multiplet. Witten found that the Witten index of this theory is [31]

$$I = \frac{\left(k + \frac{N}{2} - 1\right)!}{(N-1)!\left(k - \frac{N}{2}\right)!}\,. \tag{2.2}$$

By writing the index as

$$I = \frac{1}{(N-1)!}\left(k - \frac{N}{2} + 1\right)\left(k - \frac{N}{2} + 2\right)\cdots\left(k + \frac{N}{2} - 1\right)\,, \tag{2.3}$$

[15] Recall that $k < 0$ is obtained by acting with time-reversal on the $k > 0$ theory, which also reverses the sign of the fermion mass terms. We can therefore choose $k \geq 0$ without loss of generality.

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one finds that the index vanishes for $0 \leq k < N/2$ and it does not vanish for $k \geq N/2$. This holds for all the admissible values of k, namely integral values if N is even and half-integral values if N is odd.

Therefore supersymmetry is unbroken for $k \geq N/2$ and the number of states on $\mathbb{T}^2$ as counted by the Witten index is precisely consistent with the low energy theory in the supersymmetric vacuum being the topological $\mathrm{SU}(N)_{k-N/2}$ Chern-Simons theory. For large k the theory is weakly coupled and we can integrate out the fermion at one loop, obtaining at long distances the $\mathrm{SU}(N)_{k-N/2}$ TQFT (cf. (1.2)). It is worth noting that for $k = N/2$ the $\mathrm{SU}(N)_{k-N/2}$ TQFT trivializes, and indeed in that case the Witten index is 1, and there is a single trivial supersymmetric ground state.[16]

For $0 \leq k < N/2$ the Witten index vanishes. The standard interpretation is that supersymmety is spontaneously broken and there is a massless Majorana Goldstino particle in the vacuum. However, this cannot be the whole story because a single massless Majorana particle cannot match various discrete anomalies in the system. In particular, the system has (for generic k) an anomaly in the one-form symmetry which precludes the supersymmetry breaking vacuum from being trivial. In addition, for $k = 0$ there is a time-reversal anomaly which again would be sufficient to rule out a single Goldstino particle in the vacuum. In [24] a scenario that matches all the anomalies and passes a number of additional nontrivial tests was proposed. The proposal is that the infrared consists of the Majorana Goldstino particle, G_α, accompanied by the TQFT[17] (cf. (1.4))

$$U\left(\frac{N}{2} - k\right)_{\frac{N}{2}+k,N} . \tag{2.4}$$

In this paper we build on this recent understanding of the dynamics of the $\mathcal{N} = 1$ vector multiplet and study $\mathcal{N} = 1$ theories with matter multiplets.

3 $\mathcal{N} = 1$ vector multiplet with an adjoint matter multiplet

We consider the $\mathcal{N} = 1$ supersymmetric theory with gauge group $\mathrm{SU}(N)$ and an adjoint matter multiplet. Therefore, the theory consists of a gauge field A, two Majorana fermions λ, ψ_X and a real scalar field X, all in the adjoint representation.

The Lagrangian includes the kinetic terms

$$-\frac{1}{4g^2}\,\mathrm{Tr}\,F^2 + i\,\mathrm{Tr}\,\lambda \not{D}\lambda + i\,\mathrm{Tr}\,\psi_X \not{D}\psi_X + \mathrm{Tr}(DX)^2\,, \tag{3.1}$$

the $\mathcal{N} = 1$ Chern-Simons term

$$\frac{k}{4\pi}\,\mathrm{Tr}\left(AdA - \frac{2i}{3}A^3\right) - \frac{kg^2}{2\pi}\,\mathrm{Tr}\,\lambda\lambda\,, \tag{3.2}$$

and the Yukawa coupling

$$\sqrt{2}ig\,\mathrm{Tr}[\lambda, X]\psi_X\,. \tag{3.3}$$

[16] The $\mathcal{N} = 1$ $\mathrm{SU}(N)_k$ vector multiplet at $k = N/2$ has a holographic dual [43] (see also [44]).

[17] This TQFT is level/rank dual as spin TQFT to $U\left(\frac{N}{2} + k\right)_{-\frac{N}{2}+k,-N}$.

This theory enjoys $\mathcal{N}=1$ supersymmetry and admits an $\mathcal{N}=1$ preserving mass deformation, which can be written as a real superpotential term $W = m \operatorname{Tr} X^2$. This superpotential leads to the additional terms in the Lagrangian

$$\operatorname{Tr}\left(m^2 X^2 + m\psi_X \psi_X\right) . \tag{3.4}$$

This $\mathcal{N}=1$ theory has a global $\mathbb{Z}_2$ symmetry which acts as $(X, \psi_X) \to (-X, -\psi_X)$. In addition, for $k=0$, $m=0$ the theory is time-reversal invariant. For the value of the mass

$$m = -\frac{kg^2}{2\pi} \tag{3.5}$$

the theory has $\mathcal{N}=2$ supersymmetry and the Lagrangian is that of the pure $\mathcal{N}=2$ vector multiplet model with an $\mathcal{N}=2$ Chern-Simons term. In order to reflect the enhanced $\mathcal{N}=2$ symmetry, we sometimes use the notation m_{soft}, defined as

$$m = -\frac{kg^2}{2\pi} + m_{\text{soft}} . \tag{3.6}$$

We use m_{soft} when it is particularly natural to think about the $\mathcal{N}=1$ theory as a soft deformation of the $\mathcal{N}=2$ theory. The $\mathcal{N}=2$ theory has an $\mathrm{SO}(2)_R$ global symmetry rotating the two fermions but leaving the real adjoint boson invariant. Our goal is to determine the infrared phases of this theory as a function of k and m.[18]

3.1 Large mass asymptotic phases

The infrared dynamics in the large mass region can be solved for using semiclassical reasoning in conjunction with the results of the previous section.

We can take the mass m to be large and positive or large and negative. In this regime, the scalar X and the fermion ψ_X are very heavy classically, so that X is pinned at $X=0$ and quantum effects cannot change that. Integrating out the (X, ψ_X) multiplet we obtain a theory of an $\mathcal{N}=1$ pure vector multiplet. Using (1.2) we find that for positive m this leads to the $\mathrm{SU}(N)_{k+N/2}$ $\mathcal{N}=1$ vector multiplet while for negative m to the $\mathrm{SU}(N)_{k-N/2}$ $\mathcal{N}=1$ vector multiplet.

Using the results in the previous section, the dynamics of these theories strongly depends on whether or not the effective $\mathcal{N}=1$ level of the infrared pure vector multiplet is larger than $N/2$. Therefore, the discussion of the asymptotic phases splits into three cases depending on the value of k:

1. $k \geq N$. Since in this range $k \pm N/2 \geq N/2$ the physics is that of the "large k" phase described in the previous section. Supersymmetry is unbroken for both large positive m and large negative m. In the first case the theory flows to the $\mathrm{SU}(N)_k$ Chern-Simons TQFT and in the second case to the $\mathrm{SU}(N)_{k-N}$ TQFT. The Witten index

[18] Note that this adjoint theory appears naturally in the context of brane and domain wall dynamics [45]. More generally, $\mathcal{N}=1$ supersymmetric theories in 2+1 dimensions should arise naturally on BPS domain walls and branes of $\mathcal{N}=1$ theories in 3+1 dimensions. For some such constructions see [46–50] and therein for additional references on these matters.

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in each phase is given (up to a sign) by the partition function of the corresponding TQFT on the torus. We therefore see that the non-vanishing Witten index jumps between the two asymptotic phases.

2. $0 < k < N$. Since in this range $|k - N/2| < N/2$, supersymmetry is spontaneously broken for large negative m, but remains unbroken for large positive m. For large negative m the long distance theory consists of a Majorana Goldstino accompanied by the TQFT (cf. (2.4))

$$U\left(N-k\right)_{k,N}\,. \tag{3.7}$$

For large positive m supersymmetry is unbroken and there is a supersymmetric vacuum supporting the $\mathrm{SU}(N)_k$ TQFT. We observe once again that the Witten index jumps between the asymptotic phases, but now it is zero for large negative m and nonzero for large positive m.

3. $k = 0$. Since $|k - N/2| = N/2$ supersymmetry is unbroken for large negative m as well as for large positive m. Furthermore, the supersymmetric vacuum in both phases is trivial and gapped. Therefore, the Witten index at large positive m agrees with the Witten index at large negative m. The dynamics in this case is the simplest and does not require new tools unlike the $k \neq 0$ cases.

In summary, we have determined the asymptotic large positive and large negative mass phases for all values of k. There is a qualitative difference between $0 < k < N$ and $k \geq N$. In the former case there is spontaneous supersymmetry breaking at large negative mass and in the latter case the large $|m|$ physics always has a supersymmetric ground state. We have also noted that for $k \neq 0$ the Witten index jumps in absolute value between the two asymptotic phases.

The next subsections will tackle the infrared dynamics of the theory for small m (i.e. not parametrically large), including the $\mathcal{N} = 2$ supersymmetric point (3.5). One central question will be to understand how the Witten index jumps between asymptotic phases. Our analysis in conjunction with general considerations [33] imply that the Witten index can only jump at $m = 0$, as this is the only point where the classical asymptotics of the superpotential changes. We therefore turn next to the classical analysis of the point $m = 0$ in preparation to analyzing the full quantum theory at $m = 0$.

3.2 Classical moduli space of vacua at $m = 0$

Since for $m = 0$ the adjoint scalar field X is classically massless, the theory has a classical moduli space of $\mathcal{N} = 1$ supersymmetry preserving vacua parametrized by the eigenvalues of X

$$X = \begin{pmatrix} X_1 & 0 & 0 & \dots & 0 \\ 0 & X_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X_{N-1} & 0 \\ 0 & 0 & 0 & \dots & X_N \end{pmatrix}, \tag{3.8}$$

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where the X_i are real and obey the SU(N) invariant constraint $\sum_{i=1}^{N} X_i = 0$.[19] The residual gauge symmetry leftover after diagonalizing X implies that the classical moduli space of vacua of the theory at $m = 0$ is

$$\mathbb{R}^{N-1}/S_N\,, \tag{3.9}$$

where S_N is Weyl group of SU(N), which acts on the eigenvalues X_i by permuting them. In a generic vacuum the gauge symmetry is Higgsed down to $\mathrm{U}(1)^{N-1}$. When some eigenvalues coincide, corresponding to singular loci in the moduli space (3.9), the unbroken gauge symmetry has at least one non-Abelian factor. We now turn to a discussion of these various vacua.

3.2.1 Classical Abelian vacua

We want to determine the classical low energy theory around a generic vacuum, where the gauge symmetry is Higgsed down to $\mathrm{U}(1)^{N-1}$. The off-diagonal W-bosons and their fermionic massive partners in the $\mathcal{N} = 1$ massive vector multiplet acquire a mass from the Higgs mechanism, as well as from the Chern-Simons term. The remaining $\mathrm{U}(1)^{N-1}$ gauge bosons and fermions in the $\mathcal{N} = 1$ vector multiplet only pick up a mass from the Chern-Simons term.[20] The contribution to the mass of the off-diagonal gauge bosons W_{ij} from the Higgs mechanism depends on $g^2 X_{ij}^2$, where

$$X_{ij} \equiv X_i - X_j\,. \tag{3.10}$$

The very low energy theory therefore consists of the $N-1$ massless $\mathcal{N} = 1$ moduli multiplets (X_i, ψ_{X_i}), along with a topological theory that is associated with the unbroken $\mathrm{U}(1)^{N-1}$ gauge symmetry.[21]

The massless $\mathcal{N} = 1$ moduli multiplets (X_i, ψ_{X_i}) are trivial free fields. Let us now turn to determining the precise low energy Chern-Simons theory. The infrared Chern-Simons theory for the unbroken $\mathrm{U}(1)^{N-1}$ gauge fields is induced from the original Chern-Simons term (3.2). Hence, there is a nontrivial matrix of Chern-Simons terms for the unbroken $\mathrm{U}(1)^{N-1}$ gauge theory. This matrix can be found by simply plugging the matrix of unbroken gauge fields

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{N-1} & 0 \\ 0 & 0 & 0 & \dots & -(A_1 + \dots + A_{N-1}) \end{pmatrix} \tag{3.11}$$

[19] This constraint does not really matter since the overall U(1) part of X and its fermionic partner ψ_X are decoupled free fields.

[20] The off-diagonal components of X are eaten by the longitudinal components of the massive gauge fields.

[21] If $|X_{ij}| \gg gk$ we can write an effective field theory which includes the propagating $\mathrm{U}(1)^{N-1}$ massive photons, since their mass scales like $g^2 k$ which is much smaller than the mass of the off-diagonal W-bosons, which scales like $g|X_{ij}|$ in this limit. However, this would be unnecessary for our present purposes.

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into the Chern-Simons action (3.2). We can then immediately read off the $(N-1)\times(N-1)$ matrix $\mathbf{k}$

$$\mathbf{k} = k \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ . & . & . & . \\ 1 & 1 & \dots & 2 \end{pmatrix} \tag{3.12}$$

of Chern-Simons terms for the unbroken $\mathrm{U}(1)^{N-1}$ gauge symmetry. One can perform a change of basis with an $\mathrm{SL}(N,\mathbb{Z})$ transformation

$$\mathbf{k} \to A\mathbf{k}A^T\,, \quad A \in \mathrm{SL}(N,\mathbb{Z}) \tag{3.13}$$

in order to bring this theory to a more familiar form:[22]

$$\mathbf{k} = k \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{3.14}$$

Upon this change of basis the matrix $\mathbf{k}$ becomes the Cartan matrix of $\mathrm{SU}(N)$ times k. It is easy to see that $\det \mathbf{k} = k^{N-1}N$, which is the number of ground states of this $\mathrm{U}(1)^{N-1}$ Chern-Simons theory on the torus.

This Abelian TQFT has a dual description for $k = 1$: it is dual to $\mathrm{U}(1)_{-N}$ Chern-Simons theory [51]. This will be very useful for us below. We can think about this duality as a level/rank duality

$$\sum_{i,j=1\dots N-1} \frac{k_{ij}}{4\pi} A_i \wedge dA_j \longleftrightarrow -\frac{N}{4\pi}\tilde{A}\wedge d\tilde{A}\,, \tag{3.15}$$

with k_{ij} being the entries of the matrix $\mathbf{k}$ in (3.14) for $k = 1$.

In summary, classically, everywhere on the moduli space (3.9) except at singular loci where some $X_{ij} = 0$, the theory flows to $N-1$ free, massless $\mathcal{N} = 1$ real multiplets accompanied by the $\mathrm{U}(1)^{N-1}$ Chern-Simons theory with the $\mathbf{k}$-matrix (3.14).

We investigate later how quantum effects modify this classical analysis. We will find that quantum corrections generate a (super)-potential on the classical space of vacua (3.9). This is a new phenomenon that occurs in $\mathcal{N} = 1$ theories which is not present in theories with more supersymmetry, where the superpotential does not receive any perturbative corrections. We shall see, however, that the infrared Chern-Simons theory with the $\mathbf{k}$ matrix (3.14) will be an exact ground state for some range of parameters.

3.2.2 Classical non-Abelian vacua

We now turn to the classical analysis of the non-generic vacua where some $X_{ij} = 0$. Such vacua will play a crucial role in unraveling the phase diagram of the theory.

[22] It is easy to prove using (3.13) that one cannot bring $\mathbf{k}$ to a diagonal form. Indeed, in a diagonal form all the entries on the diagonal have to be even (to avoid a dependence on the spin structure) and nonzero, and this is impossible for $N > 2$ because $2^{N-1} > N$ for all $N > 2$.

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We divide the matrix X into L blocks such that the eigenvalue in each block is X_I and the size of the corresponding block is $S_I \times S_I$, such that

$$\sum_{I=1}^{L} S_I = N\,, \qquad \sum_{I=1}^{L} S_I X_I = 0\,.$$

These are not to be confused with partitions of N. Indeed, because of the Weyl group the eigenvalues can be without loss of generality ordered. Thus, the order in which the summands S_I appear is important. Those are called "compositions" of N. There are 2^{N-1} compositions of N.

Assuming that all the X_I ($I = 1, \ldots, L$) are distinct, all the gauge fields (and fermionic superpartners) away from these blocks acquire mass from the Higgs mechanism. The unbroken gauge symmetry in such a vacuum is thus

$$S[\mathrm{U}(S_1) \times \mathrm{U}(S_2) \cdots \mathrm{U}(S_L)]\,.$$

Let us now mention a useful way to think about the effective low energy $\mathcal{N} = 1$ gauge theory in the infrared. We first extend the $\mathrm{SU}(N)$ gauge symmetry to a $\mathrm{U}(N)$ gauge symmetry. That does not change the dynamics because the matter fields are in the adjoint representation. In that case the low energy $\mathcal{N} = 1$ Chern-Simons couplings are simply those of the product theory

$$\mathrm{U}(S_1)_{k,k} \times \mathrm{U}(S_2)_{k,k} \cdots \mathrm{U}(S_L)_{k,k}\,. \tag{3.16}$$

In order to go back to the $\mathrm{SU}(N)$ theory, we add another $\mathrm{U}(1)$ gauge field B which sets the overall trace to zero via the coupling

$$\frac{1}{2\pi} B \wedge \sum_{I=1}^{L} S_I \,\mathrm{Tr}\, A_I\,. \tag{3.17}$$

At energies below the massive W-boson multiplets, we are left with an $\mathcal{N} = 1$ vector multiplet with gauge group and Chern-Simons levels (3.16) with the additional constraint (3.17), accompanied by massless $\mathcal{N} = 1$ matter multiplets (X_I, ψ_I) in the adjoint representation of the unbroken non-Abelian gauge group.

In the vacua with Abelian gauge symmetry discussed above (i.e. $S_I = 1$), there were no light charged particles and therefore the theory at long distances (if such vacua indeed exist in the full quantum theory) could be determined right away, as it is free. For non-Abelian vacua, the low energy theory just described is still interacting, and the ultimate fate of these vacua necessitates further discussion. This will require us to understand the perturbative (and ultimately also non-perturbative) corrections to the classical analysis described here.

3.3 Semiclassical moduli space of vacua

After the discussion of the moduli space of vacua at $m = 0$ and of the low energy theories that appear in each vacuum, we now turn to the important question of how quantum effects modify the classical analysis.

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In 2+1 dimensional theories with $\mathcal{N} = 1$ supersymmetry there is no obstruction to the existence of perturbative corrections to the superpotential, in contrast with more supersymmetric theories. Therefore, there may be a nontrivial superpotential which depends on the $N-1$ coordinates parametrizing the moduli space (3.9)

$$W(X_i)\,,$$

with $\sum_{i=1}^{N} X_i = 0$. This can drastically change the classical picture (as we shall see, classical vacua get lifted).

Consider the classical vacua where the eigenvalues X_i are distinct and very well separated so that the off-diagonal degrees of freedom are very heavy. This allows to analyze what happens *far away* on the $N-1$ dimensional classical moduli space (3.9). Clearly, near the singular loci in (3.9), when some of the eigenvalues are close by, this expansion breaks down, and we postpone the discussion of the quantum behaviour of these vacua with non-Abelian gauge symmetry until later.

In order to understand how the radiatively induced superpotential ought to behave, let us first appeal to dimensional analysis arguments. We consider for simplicity expanding around large X_{ij} and let us assume that they scale uniformly $X_{ij} \sim X$. It is useful to canonically normalize the fields such that we have the following types of interactions: cubic interactions proportional to g, quartic interactions proportional to g^2 and we choose a gauge such that the Chern-Simons cubic vertex vanishes (e.g. $A_0 = 0$). The terms containing k therefore appear only in quadratic fermionic and gauge field terms. These quadratic terms are proportional to kg^2. The mass of the heavy particles that run in the loops scales like $M \sim gX$. This estimate is correct far away on the moduli space, i.e. as long as $X \gg gk$. Since we are interested in the structure of the effective superpotential far away on the moduli space, this estimate suffices.

Consider now a vacuum L loop diagram contributing to the quantum effective potential (i.e. the Coleman-Weinberg potential [52]). Such a diagram is weighted by a factor of g^{2L-2}. Then, if we do not insert the quadratic vertices depending on k, the sum over diagrams must vanish because a superpotential cannot be generated in $\mathcal{N} = 2$ theories (recall that the theory with $m = k = 0$ has enhanced $\mathcal{N} = 2$ supersymmetry). If we expand the effective scalar potential in this manner, only even powers of k can appear by virtue of parity. We therefore have at L loops a perturbative series of the form

$$V^{(L)}(X) = g^{2L-2} \sum_{n>0} d_{n;L} \frac{(kg^2)^{2n}}{(gX)^{L+2n-4}}\,, \tag{3.18}$$

with some coefficients $d_{n;L}$. This is derived by imagining insertions of k on the various edges of the diagrams. As we have explained, this representation of the effective potential is useful far away on the moduli space, and, more precisely, for $X \gg gk$. The full scalar potential is of course given by summing over all loops

$$V = \sum_{L} V^{(L)}(X)\,. \tag{3.19}$$

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It turns out that the one-loop contribution vanishes

$$V^{(1)} = 0\,.$$

This follows from the fact that the spectrum of heavy particles is supersymmetric at tree level and the one-loop potential is sensitive only to the supertrace of the classical spectrum[23]

$$\mathrm{STr}|\mathcal{M}|^3 \equiv \mathrm{Tr}|m_B|^3 - \mathrm{Tr}|m_F|^3 = 0\,. \tag{3.20}$$

As is well-known, integrating massive fermions at one loop can induce a shift in the Chern-Simons levels (cf. (1.2)). A quick inspection of the mass matrix for the off-diagonal Majorana fermions in $\mathcal{N} = 1$ vector and matter multiplets around the generic vacuum with non-degenerate eigenvalues shows that for each off-diagonal massive charged fermion there is a massive charged fermion with a mass of opposite sign.[24] Therefore, integrating the $\mathcal{N} = 1$ massive vector multiplets does not shift the levels of the $\mathbf{k}$ matrix (3.14).[25] Furthermore, the Coleman-Hill theorem [54] guarantees that the low energy Chern-Simons theory is not modified by higher loop corrections.

In summary, at one loop the $N-1$ $\mathcal{N} = 1$ matter multiplets remain massless and to all orders the infrared Chern-Simons theory is $\mathrm{U}(1)^{N-1}$ with the $\mathbf{k}$ matrix (3.14). Therefore, in order to unravel the leading quantum corrections to the classical moduli space we will need to go to two loops.

Equipped with this, the scalar potential (3.19) can be recast in terms of a superpotential

$$W(X) = kg^3 \sum_{L>1} g^L \sum_{n>0} c_{n;L} \frac{(kg^2)^{2n-2}}{g^{2n} X^{L+2n-5}}\,, \tag{3.22}$$

with some coefficients $c_{n;L}$ which can be, in principle, computed. From (3.22) we can see the utility of the semiclassical large X expansion: any given term in the $1/X$ expansion only receives contributions from *finitely* many loop orders in perturbation theory. The leading term in the large X expansion is at $L = 2, n = 1$, and it scales linearly $W(X) \sim X$. This term can only receive contributions from two-loop diagrams and determining whether this term is present or not, is absolutely crucial in order to understand the dynamics of these $\mathcal{N} = 1$ theories.

[23] The cubic and linear ultraviolet divergences vanish as they are proportional to $\mathrm{STr}(1)$ and $\mathrm{STr}|\mathcal{M}|$ respectively.

[24] Consider for simplicity the gauge group SU(2), broken to U(1) by the expectation value $X = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$. From the two adjoint Majorana fermions in the full theory we obtain two complex fermions, $\lambda^{(+2)}, \psi^{(+2)}$ of charge 2 under the unbroken U(1) gauge symmetry. Their mass matrix is proportional to

$$\mathcal{M} \sim \begin{pmatrix} -kg^2 & igx \\ -igx & 0 \end{pmatrix}. \tag{3.21}$$

Clearly, there is one positive and one negative eigenvalue.

[25] In some places in the literature it is claimed that the Chern-Simons levels can also be shifted by integrating out heavy W-bosons. See for example [53]. This stems from a confusion between the Wilsonian and 1PI effective actions.

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Fortunately, for rather different reasons, the full two-loop superpotential has been computed in [55, 56].[26] The result is[27]

$$W = -\sum_{ij} g^3 k \sqrt{g^2 k^2 + X_{ij}^2}\,, \tag{3.23}$$

expressed in terms of the eigenvalue differences $X_{ij} \equiv X_i - X_j$ defined above.

First, note that (3.23) vanishes for $k = 0$, as it should, since then the theory has $\mathcal{N} = 2$ supersymmetry and the superpotential has no perturbative corrections. Second, note that if we rewrite (3.23) in an expansion around large X_{ij} then it agrees with the general structure (3.22) derived earlier. Finally, note that the superpotential appears to be regular even when the X_{ij} go to zero, and where, classically, gauge symmetry would be enhanced. Usually, we expect the effective theory on the moduli space to indicate that it is breaking down on such loci due to the existence of new massless particles. But here the two-loop effective superpotential behaves perfectly regularly everywhere on the moduli space in spite of the fact that there are new massless particles when some of the eigenvalues coincide.

Understanding the regime of validity of (3.23) is of great importance in what follows. Very far on the moduli space, in the "far zone" $X \gg gk$, as we have shown in (3.22), there may be contributions from three loops that scale like X^0 (i.e. it could be $\log X$) and contributions from four loops that scale like $1/X$. Therefore, far out in the moduli space $X \gg gk$, the only reliable information captured by (3.23) is the linear term

$$X \gg gk: \qquad W = -g^3 k \sum_{ij} |X_{ij}|\,. \tag{3.24}$$

We will return to the "near zone" of the moduli space where X is not large compared to gk later.

In summary, we see that the classical moduli space of supersymmetric vacua is lifted starting at two loops (3.23) and that there is a radiatively induced asymptotically flat (non-supersymmetric) direction with non-zero energy density (cf. (3.24)). Next we analyze the consequences of the two-loop superpotential for the phases of the theory around $m = 0$.

3.4 Semiclassical Abelian vacuum near $m = 0$

We now proceed to study the fate of the semiclassical Abelian vacua just described when the theory is deformed by a small mass term (3.4)

$$\delta W = m \operatorname{Tr} X^2 + \lambda \operatorname{Tr} X\,, \tag{3.25}$$

[26] In fact, [55, 56] computed the scalar potential and we are inferring the superpotential from that. But there is an ambiguity in doing this, aside from the overall constant in $W(X)$ which cannot be determined and is insignificant. The sign of W is very important as soon as we add back the mass perturbation (3.4) $W = m \operatorname{Tr} X^2$ since the sign of the superpotential affects the interference between these two terms. The sign can be computed in principle by studying diagrams with external fermions. Here we simply assume that W has a certain sign so that our overall picture for the dynamics is consistent.

[27] To simplify notation, we have redefined the couplings in W in order to absorb several unimportant factors.

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where we have also added a Lagrange multiplier λ to enforce that X is a traceless matrix. Then, for infinitesimal m, by combining (3.25) and (3.23), we find that the superpotential on the moduli space takes the form[28]

$$W = -\sum_{ij} g^3 k \sqrt{g^2 k^2 + X_{ij}^2} + m \sum_i X_i^2 + \lambda \sum_i X_i \,, \tag{3.26}$$

with i, j ranging over $1, \ldots, N$. In the perturbative "far zone" regime $X_{ij} \gg gk$ we can self-consistently approximate the superpotential by

$$W = -g^3 k \sum_{ij} |X_{ij}| + m \sum_i X_i^2 + \lambda \sum_i X_i \,. \tag{3.27}$$

Supersymmetric vacua (i.e zero energy states) of the deformed theory correspond to solutions of the equations $\frac{\partial W}{\partial X_i} = 0$. The explicit equations are (we define $\mathrm{sgn}(0) = 0$)

$$-g^3 k \sum_j \mathrm{sgn}(X_{ij}) + m X_i + \frac{1}{2}\lambda = 0 \,, \tag{3.28}$$

$$\sum_i X_i = 0 \,. \tag{3.29}$$

Summing over all i in the first equation we find $\lambda = 0$. Therefore the equations can be simplified to

$$-g^3 k \sum_j \mathrm{sgn}(X_{ij}) + m X_i = 0 \,, \tag{3.30}$$

$$\sum_i X_i = 0 \,. \tag{3.31}$$

The last equation implies that at least one of the X_i has to be positive. We can now use the residual S_N group to order the eigenvalues from the most positive one, X_1, to the most negative one, X_N. Then, the first equation with $i = 1$ in (3.30) shows that all the terms in the first term are negative and the second term is also negative for $\mathrm{sgn}(m) < 0$, and hence there is no solution for $\mathrm{sgn}(m) < 0$. Therefore, we conclude that there are no supersymmetric vacua far on the moduli space for negative mass. We will discuss later the physics of the vacuum at "$X = 0$," as it depends crucially on whether k is large or small.

For small positive m a supersymmetric solution exists, and it is given up to the action of the Weyl group by[29]

$$X_i = \frac{g^3 k}{m}(N + 1 - 2i) \,. \tag{3.32}$$

The eigenvalue differences X_{ij} are all parametrically large (compared to gk) for $m \ll g^2$, and hence the existence of this supersymmetric vacuum is rigorously established in the full theory, that is, it is not an artefact of the two-loop approximation because as we

[28] This formula is valid to leading order in m since the two-loop effective potential (3.23) was computed in the massless theory.

[29] Note that this obeys (3.31).

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proved around (3.22) the higher-order corrections cannot compete far on the moduli space. Therefore, we find that for m small and positive there is an $\mathcal{N}=1$ supersymmetric massive vacuum (all $N-1$ $\mathcal{N}=1$ matter multiplets are massive) described by the $\mathrm{U}(1)^{N-1}$ Chern-Simons theory with $\mathbf{k}$ matrix (3.14).

In summary, we have established that for small positive m there is a new supersymmetric gapped vacuum that comes in from infinity in field space that supports the $\mathrm{U}(1)^{N-1}$ Chern-Simons theory with $\mathbf{k}$ matrix (3.14). Instead, for small negative m, there is no such Abelian vacuum. Intuitively, for small negative m, the potential grows everywhere at large X, at $m=0$ it becomes asymptotically flat, and as we make m slightly positive, a new supersymmetric Abelian vacuum comes in from infinity.

We will use these facts to determine the different phases of the theory at finite m, connecting them to the asymptotic phases that we found in section 3.1. We will also establish by a different semiclassical analysis at large k that the Abelian vacuum above is not the only one that appears from infinity in field space. We will find additional supersymmetric vacua with degenerate eigenvalues that approach from infinity.

We now discuss in turn the complete phase diagram of the theory for $k \geq N$ and $0 < k < N$, and $k = 0$.

3.5 Phases of the theory with $k \geq N$

Above we studied the vacuum state of the theory with an Abelian gauge symmetry that can be reliably established for arbitrary k by analyzing the theory far away in the space of vacua (i.e. large X). We now turn to the study of quantum vacua where some of the eigenvalues coincide. The low energy theory in such a vacuum has non-Abelian gauge symmetry and the theory is interacting.

Despite that the low energy theory is non-trivial around a non-Abelian vacuum, the infrared dynamics can be determined for k sufficiently large, that is for $k \geq N$.

3.5.1 Critical points of the superpotential

At sufficiently large k the physics is weakly coupled, and unlike in section 3.4, we do not exclude vacua with degenerate eigenvalues. Indeed, for $k \geq N$ we can analyze them explicitly since the theory is "semiclassical" and we can handle these interacting effective theories at these loci. The critical point equations are

$$\frac{1}{2}\lambda + mX_i = g^3 k \sum_j \frac{X_{ij}}{\sqrt{g^2k^2 + X_{ij}^2}}\,, \tag{3.33}$$

$$\sum X_i = 0\,. \tag{3.34}$$

Summing over i in the first equation we find $\lambda = 0$. Therefore the two equations can be simplified to

$$mX_i = g^3 k \sum_j \frac{X_{ij}}{\sqrt{g^2k^2 + X_{ij}^2}}\,. \tag{3.35}$$

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For the sake of analyzing this equation it is useful to rescale $gk\tilde{X} = X$ such that the equation takes the form

$$\frac{m}{g^2}\tilde{X}_i = \sum_j \frac{\tilde{X}_{ij}}{\sqrt{1+\tilde{X}_{ij}^2}}\,. \tag{3.36}$$

There is clearly the solution $X_i = 0$.

Let us now look for solutions with at least one non-vanishing X_i. First let us make a general observation. There are solutions with some nonzero X_i only for $\frac{m}{g^2} \in (0, \mathbb{N})$. To prove that, assume that at least one $\tilde{X}_i$ does not vanish, so let $\tilde{X}_1$ be the largest positive eigenvalue. Then, $\tilde{X}_{1j} \geq 0$ for all j and since

$$\sum_j \frac{\tilde{X}_{1j}}{\sqrt{1+\tilde{X}_{1j}^2}} \leq \sum_j \tilde{X}_{1j} = \sum_j \tilde{X}_1 - \sum_j \tilde{X}_j = N\tilde{X}_1$$

so that

$$\frac{m}{g^2}\tilde{X}_1 \leq N\tilde{X}_1\,,$$

and therefore we conclude that nontrivial solutions exist only if $\frac{m}{g^2} \in (0, \mathbb{N})$. The exact location where vacua with $\tilde{X}_i \neq 0$ disappear can get modified by higher loop corrections. The estimate above is reliable for asymptotically large k, i.e. $k \gg 1$.

It is rather easy to write down all the solutions explicitly when the difference of eigenvalues in each distinct block are large. These equations for small m were written in (3.30), and the solutions are labeled by choosing blocks of sizes $S_I \times S_I$ and ordered eigenvalues X_I, $I = 1, \ldots, L$

$$X_1 > X_2 > \ldots > X_L\,.$$

Using (3.30) we find that at very small positive m

$$X_I = \frac{g^3 k}{m}\left[(S_{I+1} + \cdots S_L) - (S_1 + \cdots S_{I-1})\right]\,. \tag{3.37}$$

We conclude that the vacua correspond to ordered partitions of N (a.k.a compositions of N). There are 2^{N-1} vacua in total, including the one at the origin. The only supersymmetric vacuum with small negative m is the one at the origin.

In summary, for $m < 0$ we have one supersymmetric vacuum with TQFT $\mathrm{SU}(N)_{k-N}$ and for (roughly) $m > g^2 N$ we have one supersymmetric vacuum with TQFT $\mathrm{SU}(N)_k$. The index jumps at $m = 0$ and at small $m > 0$ we have 2^{N-1} vacua with various TQFTs. As we increase m the vacua gradually merge via second order transitions but the Witten index does not jump anymore. The transitions must be second order because these vacua correspond to zeroes of W' with nontrivial Witten index. They cannot disappear without merging with other zeroes as we increase m.

How exactly these 2^{N-1} vacua merge into one vacuum is an interesting question. We find a complicated pattern where these 2^{N-1} vacua merge via a sequence of conformal field theories that appear away from the origin on the moduli space as we crank up m from zero to $g^2 N$. By the time we crank the mass up to $g^2 N$ they will have all merged into a single

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vacuum and for $m > g^2N$ we have only the $\mathrm{SU}(N)_k$ TQFT.[30] We will discuss in detail only one representative simple example of this phenomenon later.

Now we would like to analyze the vacua at small m and show that in fact these 2^{N-1} vacua precisely account for the required jump in the Witten index. To simplify the computation we now take the original gauge group to be $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$. This would simplify the combinatorics while not making any difference for the physics since the $\mathrm{U}(1)$ factor is anyway decoupled. Let us consider the superpotential (3.23) but now we omit the Lagrange multiplier term

$$W = -\sum_{ij} g^3 k \sqrt{g^2k^2 + X_{ij}^2} + m \sum X_i^2 \,. \tag{3.38}$$

Let us expand around the critical point (3.37). We take $X_i = X_i^0 + \delta X_i$ and X_i^0 is given by (3.37). Expanding the superpotential to second order in δX_i (and dropping the constant piece) we find

$$\begin{aligned} W = &-\frac{1}{2}\sum_{ij} g^3 k \sqrt{g^2k^2 + (X_{ij}^0)^2} \left(\frac{2X_{ij}^0 \delta X_{ij} + \delta X_{ij}^2}{g^2k^2 + (X_{ij}^0)^2} - \left(\frac{X_{ij}^0 \delta X_{ij}}{g^2k^2 + (X_{ij}^0)^2} \right)^2 \right) \\ &+ 2m \sum X_i^0 \delta X_i + m \sum \delta X_i^2 \,. \end{aligned} \tag{3.39}$$

Omitting the linear piece in δX_i which vanishes by virtue of the equations (3.35) we find after some simplifications

$$W = -\frac{1}{2}\sum_{ij} \frac{g^5 k^3 \delta X_{ij}^2}{\left(g^2k^2 + (X_{ij}^0)^2\right)^{3/2}} + m \sum \delta X_i^2 \,. \tag{3.40}$$

Now we restrict to infinitesimal m, where the solutions are given by (3.37). Considering the first term in (3.40) we see that there are two cases — if we consider i, j to lie in the same block then $X_{ij}^0 = 0$ and the coefficient of δX_{ij}^2 in the first term scales like g^2, which is much larger than m. If we consider i, j to lie in different blocks, the coefficient of δX_{ij}^2 scales like m^3/g^4, which is much smaller than m. The conclusion is that for i, j in the same block we should take into account the first term in (3.40) while for i, j in different blocks we can neglect it. The mass matrix for the fermions ψ_i (which are the partners of X_i) thus takes the form

$$\mathcal{M} = m \mathbb{I}_{S_I \times S_I} + g^2 \begin{pmatrix} -S_I + 1 & 1 & 1 & \dots & 1 \\ 1 & -S_I + 1 & 1 & \dots & 1 \\ 1 & 1 & -S_I + 1 & \dots & 1 \\ . & . & . & \dots & . \\ 1 & 1 & 1 & \dots & -S_I + 1 \end{pmatrix} \tag{3.41}$$

in each $S_I \times S_I$ block and is otherwise vanishing. Even though $m \ll g^2$, we have not neglected the first term on the right hand side of (3.41) since it lifts the zero mode

[30] We recall that the exact location where vacua away from origin disappear can be modified by higher loop corrections.

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$(1, 1, \ldots, 1)$. The eigenvalues of this matrix are

$$(-g^2 S_I + m, -g^2 S_I + m, \ldots, -g^2 S_I + m, m)\,. \tag{3.42}$$

Therefore, we can now complete the exercise that we have embarked on in (3.16) and compute the TQFT. For sufficiently small m, the eigenvalues are all negative other than the one that corresponds to the decoupled field $(1, 1, \ldots, 1)$. Therefore, the charged fermions under the unbroken

$$\mathrm{U}(S_1) \times \mathrm{U}(S_2) \cdots \mathrm{U}(S_L) \tag{3.43}$$

gauge symmetry all have a negative mass (we recall that the gauginos in the $\mathcal{N} = 1$ vector multiplet also have a negative mass). Since we are in the "large k" phase for each of the subgroups in (3.43), the long-distance theory can be read out by simply integrating out the fermions (matter fermions and gauginos) at one loop and supersymmetry is unbroken. Therefore the TQFT is given by

$$\mathrm{U}(S_1)_{k-S_1,k} \times \mathrm{U}(S_2)_{k-S_2,k} \cdots \mathrm{U}(S_L)_{k-S_L,k}\,. \tag{3.44}$$

The contribution of this vacuum to the Witten index is given by the number of states of this TQFT, which is simply

$$\prod_I \frac{k!}{S_I!(k-S_I)!}\,.$$

The contribution of each such vacuum (which corresponds to a composition of N) to the Witten index has to be weighted with the correct sign. The sign is simply given by counting how many fermions have a negative eigenvalue for any such given composition, and the answer is that, as we have seen above, in each block there are $S_I - 1$ fermions with a negative eigenvalue. Therefore, the weight of this vacua in the Witten index is

$$(-1)^{\sum_I (S_I - 1)} = (-1)^{N-L}\,,$$

where L is the length of the partition. Therefore the Witten index at small positive m is finally given by a sum over compositions P, $N = \sum_{I=1}^{L(P)} S_I$, where the length of each such composition is $L(P)$. We find that the Witten index is

$$I = \sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k-S_I)!}\,. \tag{3.45}$$

(Remember that this analysis is valid for $k \geq N$.) Now we observe that there is an interesting combinatorial identity (proven in the appendix C) for such compositions of N

$$\sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k-S_I)!} = \frac{(N+k-1)!}{N!(k-1)!}\,. \tag{3.46}$$

The right hand side is the torus partition function of the TQFT $\mathrm{U}(N)_{k,k}$, which is exactly the ground state of the theory at large positive m. This therefore nicely shows that the index jumps at $m = 0$ by having $2^{N-1} - 1$ vacua come from infinity, exactly reproducing

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the index of the large m phase. Therefore, the total index no longer changes as we continue to increase m. Instead, these 2^{N-1} vacua coalesce (not necessarily all at the same time, there could be multiple second order phase transitions) until eventually they combine to the form the $\mathrm{U}(N)_{k,k}$ ground state, which is visible semiclassically.

We will now study the case of SU(2) in more detail for concreteness and also because it is the simplest nontrivial case.

3.5.2 $\mathrm{SU}(2)_k$

The most general adjoint matrix X can be brought to the form

$$X = gk \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \tag{3.47}$$

and we can plug this into (3.36) to obtain

$$\frac{m}{g^2} x = \frac{2x}{\sqrt{1+4x^2}} \,. \tag{3.48}$$

The solution $x = 0$ always exists. For $m < 0$ this is the only supersymmetric vacuum, namely the vacuum at the "origin" supporting the $\mathrm{SU}(2)_{k-2}$ TQFT. For small positive m we have two vacua. One is the one we just saw at $x = 0$, supporting the $\mathrm{SU}(2)_{k-2}$ TQFT. The other supersymmetric vacuum is the Abelian TQFT described in section 3.4, i.e. the $\mathrm{U}(1)_{2k}$ pure Chern-Simons theory. Note that for small positive m we have two more solutions, related by $x \to -x$. But $x \to -x$ is the generator of the Weyl group and hence these two solutions should be deemed equivalent.

As we increase m these two vacua eventually meet at a second order phase transition (according to (3.48) this happens at $m = 2g^2$, an estimate that is reliable at asymptotically large k). Then, past this transition, there is again only one supersymmetric vacuum with an $\mathrm{SU}(2)_k$ TQFT.

Note that we can rigorously prove that the transition is second order. Indeed, for small positive m we see two vacua ($\mathrm{SU}(2)_{k-2}$ TQFT and $\mathrm{U}(1)_{2k}$ TQFT). They correspond to two zeroes of W'. At large positive m there is only one vacuum with a $\mathrm{SU}(2)_k$ TQFT. The only way that this transition can occur is by the two vacua meeting. This is because the sign of W'' in the $\mathrm{SU}(2)_{k-2}$ vacuum must change and this cannot happen without the zeroes meeting.

Therefore, the conformal field theory at (approximately) $m = 2g^2$ describes a phase transition (as we change the mass) between two isolated vacua carrying the $\mathrm{SU}(2)_k$ and $\mathrm{U}(1)_{2k}$ TQFTs and, on the other side of the transition, one isolated vacuum with $\mathrm{SU}(2)_k$ TQFT. At large k this conformal field theory can be studied systematically (and in that limit the Yang-Mills term and gaugino kinetic term can be dropped).

Strictly at $m = 0$ there is an asymptotically flat direction and one supersymmetric ground state with $\mathrm{SU}(2)_{k-2}$ TQFT near the origin. At small positive m the vacuum at the origin contributes to the Witten index $-(k-1)$ and the new Abelian vacuum contributes $2k$. Together they combine to $k+1$, which is precisely the Witten index of the vacuum at asymptotically large positive mass.

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In the SU(2) gauge theory there is therefore just one vacuum that appears from infinity for small positive mass m. And correspondingly, there is only one phase transition at positive m. The properties of this $\mathcal{N} = 1$ SCFT can be systematically computed in perturbation theory in $1/k$. Past this conformal field theory, the physics is smoothly connected to the large positive mass phase.

3.6 Phases of the theory with $0 < k < N$

We now discuss the dynamics of the $\mathcal{N} = 1$ supersymmetric model $\mathrm{SU}(N)_k$ with an adjoint multiplet and superpotential

$$W = m\,\mathrm{Tr}(X^2)$$

in the regime of "small" Chern-Simons level $0 < k < N$. In this regime non-perturbative effects dominate and the dynamics is quite rich.

First, we recall the basic facts about what happens at large $|m|$. At very large negative mass, integrating out the adjoint matter multiplet we get a pure $\mathcal{N} = 1$ vector multiplet with gauge group $\mathrm{SU}(N)$ and Chern-Simons level $k - N/2$. Since $0 < k < N$ then $|k - N/2| < N/2$ and hence this theory breaks supersymmetry spontaneously, leading to a massless Majorana Goldstino and a TQFT (according to (2.4))

$$\mathrm{U}(N-k)_{k,N} \leftrightarrow \mathrm{U}(k)_{-N+k,-N}\,. \tag{3.49}$$

We will see that it follows from our analysis that this continues to be true all the way to $m = 0$, i.e. $m_{\mathrm{soft}} = \frac{kg^2}{2\pi}$. In particular, the $\mathcal{N} = 2$ supersymmetric point has the TQFT (3.49) as well as a Dirac Goldstino particle (one Majorana fermion is lifted for nonzero m_{soft} and thus we remain with one massless Majorana fermion away from $m_{\mathrm{soft}} = 0$). At very large positive m we get a pure $\mathcal{N} = 1$ vector multiplet with gauge group $\mathrm{SU}(N)$ and Chern-Simons level $k + N/2$. Since for all non-negative k, $k + N/2 \geq N/2$, the dynamics of the vector multiplet leads to a supersymmetric vacuum with the TQFT

$$\mathrm{SU}(N)_k \simeq \mathrm{U}(k)_{-N,-N}\,. \tag{3.50}$$

We therefore see that for $0 < k < N$ at very large negative mass we have a Majorana Goldstino and TQFT (3.49) while for very large positive mass we have a supersymmetric vacuum with a (generally) different TQFT (3.50). Clearly, the Witten index jumps and we have to understand how that comes about.

In (3.37) we have found that at small positive m many new critical points of the superpotential appear. Those critical points are obtained by analyzing the two-loop+tree-level superpotential (3.38). These critical points correspond to compositions of the integer N, with unbroken gauge symmetry (3.43) (after the center-of-mass U(1) is removed). The effective field theory consists of the gauge group with bare Chern-Simons terms

$$S[\mathrm{U}(S_1)_{k,k} \times \mathrm{U}(S_2)_{k,k} \cdots \mathrm{U}(S_L)_{k,k}]\,, \tag{3.51}$$

and we also have an adjoint matter multiplet for this gauge group. The mass terms for the adjoint multiplet around this critical point have negative eigenvalues (3.42).

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For large enough k, i.e. as long as $k \geq S_I$ for all I, these supersymmetric critical points are not lifted and because the mass eigenvalues are negative we can simply integrate out these adjoint multiplets and arrive at (3.51). It is guaranteed that $k \geq S_I$ for all I for any composition $\{S_I\}$ as long as $k \geq N$.

However, when $k < N$, in some of the vacua we will have at least one $S_I > k$ and therefore these vacua would be lifted. Namely, they are no longer critical points of the full quantum superpotential due to non-perturbative effects. The vacua that remain correspond to compositions of N with all the $S_I \leq k$

$$N = \sum_I S_I \,, \qquad S_I \leq k \,. \tag{3.52}$$

In order to count the supersymmetric vacua that remain and their Witten index, it is again useful to imagine that the gauge group is $\mathrm{U}(N)$ instead of $\mathrm{SU}(N)$, which of course makes no difference for the dynamics. Therefore, those critical points that remain flow as before (since they are effectively in the "large k" phase and the mass term for the adjoint multiplet fermion and gaugino have a negative sign) to the TQFT

$$\mathrm{U}(S_1)_{k-S_1,k} \times \mathrm{U}(S_2)_{k-S_2,k} \cdots \mathrm{U}(S_L)_{k-S_L,k} \,. \tag{3.53}$$

The number of such compositions of N is obtained from the coefficient of x^N of the generating function,

$$\frac{x(1-x^k)}{1-2x+x^{k+1}} \,.$$

It is easy to see that summing over these restricted compositions the total Witten index still matches that of the single supersymmetric round state at large positive m. A simple way to see that is to consider the identity (3.46) as an identity between two polynomials in k of degree N. The terms on the left hand side that correspond to a composition P for which at least one of the S_I satisfies $S_I > k$ vanish. Therefore, the identity (3.46) remains true for $k < N$ if we restrict the left hand side to compositions that satisfy (3.52).

We can now summarize that for $m \leq 0$ there are no supersymmetric ground states, but at $m = 0$ an asymptotically flat direction with nonzero energy density opens and supersymmetric ground states appear. The supersymmetric ground states that appear at small positive m correspond to compositions of N restricted by (3.52). As we increase m these supersymmetric ground states coalesce and for sufficiently large m there is only one supersymmetric ground state, which is visible semiclassically, with TQFT $\mathrm{SU}(N)_k$. The Witten index at small positive m therefore matches the Witten index at large positive m, as it should, since the asymptotic form of the potential does not change in this domain.

An interesting special case to consider is $\mathrm{SU}(N)_1$. In that situation there is only one composition that remains of (3.52), i.e there is only one supersymmetric ground state at small positive m, corresponding to the composition

$$N = 1 + 1 + \cdots + 1 \,.$$

The TQFT in that vacuum is Abelian and given by the $\mathbf{k}$-matrix (3.14) with $k = 1$. We have shown in (3.15) that this theory has a dual description in terms of the TQFT

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$\mathrm{U}(1)_{-N}$. But now, using level/rank duality, we can also rewrite that model as $\mathrm{SU}(N)_1$. Therefore, the supersymmetric ground state at small positive m supports the $\mathrm{SU}(N)_1$ TQFT. Fortunately, this is precisely the TQFT in the supersymmetric ground state at large positive m and hence in this particular case no phase transitions at positive m are necessary at all. This is a nice consistency check since indeed there is only one gapped supersymmetric vacuum in this case and it has no other supersymmetric vacua to merge with. In other words, the deep infrared physics at small positive m and large positive m is essentially identical.

In summary, for negative m we have a supersymmetry breaking vacuum with a $\mathrm{U}(1)_{-N} \simeq \mathrm{SU}(N)_1$ TQFT accompanied by a Majorana Goldstino particle, and for positive m we have the $\mathrm{SU}(N)_1$ TQFT in a supersymmetric vacuum. This example will be important below in our discussion of metastable supersymmetry breaking. In appendix A we discuss some further consistency checks of this particular dynamics of $\mathcal{N} = 1$ $\mathrm{SU}(N)_1$ with an adjoint multiplet connecting our scenario to the "duality appetizer."

3.7 Dynamical metastable supersymmetry breaking at $0 < k < N$

Here we consider the physics of the theory with $0 < k < N$ at small $|m|$, i.e. near the wall (point) where the Witten index jumps. We have seen that for small negative m there is no supersymmetric ground state and we proposed a single supersymmetry-breaking ground state (which corresponds to the global minimum of the potential) carrying (as explained in (3.49)) the TQFT $\mathrm{U}(N-k)_{k,N}$ and a Majorana Goldstino. Let us denote the energy density in this minimum by $f(m)$, defined for negative m. We can define the limit

$$f(0) \equiv \lim_{m\to 0^-} f(m)\,.$$

At $m = 0$ we have an asymptotically flat direction with nonzero energy density (3.27) that is given by

$$V_{\text{asymp}} = 4g^6k^2\sum_i\left(\sum_j \text{sgn}(X_{ij})\right)^2 = 4g^6k^2\sum_{i=1}^{N}(N+1-2i)^2 = \frac{4}{3}g^6k^2N(N^2-1)\,. \quad (3.54)$$

(We used the fact that the kinetic terms are asymptotically approximately canonical and we evaluated the energy for a generic direction far out on the moduli space. We also relaxed the traceless-ness condition on X for simplicity.) The scaling $g^6N^3k^2$ can be understood from general large N considerations. Every insertion of k corresponds to a factor of $k\lambda/N$, with λ the usual 't Hooft coupling $\lambda = g^2N$. A generic planar two-loop contribution to the vacuum energy scales like λN^2 and hence with two insertions of k we find

$$V_{\text{asymp}} \sim k^2\lambda^3\,, \qquad \lambda = g^2N\,. \quad (3.55)$$

This agrees with (3.54).

The question now is how does the energy density of the supersymmetry-breaking ground state at small negative m compare to the energy density that opens up at infinity.

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There are a priori three options

$$\begin{aligned} f(0) &> V_{\text{asymp}} \\ f(0) &< V_{\text{asymp}} \\ f(0) &= V_{\text{asymp}} \,. \end{aligned} \tag{3.56}$$

We will now discuss these options in turn. First, let us consider the large N scaling of $f(0)$. It is especially easy to estimate $f(0)$ if $k \sim N$, i.e. when N/k is held fixed in the large N limit. Then, the action is given by $N(\cdots)$ and the only dimensionful parameter in the action is $\lambda = g^2 N$. Therefore, we should expect that $f(0) \sim \lambda^3 N^2$. This is the same scaling as would be obtained from (3.55) if k scales like N. Hence, large N considerations by themselves are not sufficient to decide among the options in (3.56).

First, let us argue that $f(0) > V_{\text{asymp}}$ is impossible. For small negative m, the superpotential has already a large region

$$g \ll X \ll \frac{g^3 k}{m} \,, \tag{3.57}$$

where it is well approximated by $W = -g^3 k \sum_{ij} |X_{ij}|$ and therefore it cannot be true that $f(0) > V_{\text{asymp}}$.

Next we consider the possibility that $f(0) < V_{\text{asymp}}$. In this case, as we increase m the supersymmetric vacua that come in from infinity must be separated by a potential barrier from the supersymmetry-breaking minimum. The distance between these vacua must scale as $\sim 1/m$ and hence the supersymmetry-breaking vacuum is arbitrarily long lived.

Finally, there is the most subtle case to consider $f(0) = V_{\text{asymp}}$. One way in which this can happen is that the supersymmetry-breaking vacuum in fact remains near the origin and the fact that the energy density coincides with what we computed asymptotically is an accident. In this case there would have to be a potential barrier between the supersymmetry breaking state and the far region and hence the state will be metastable at small positive mass. This case is morally similar to the case of $f(0) < V_{\text{asymp}}$.

However, it could also be that as we tune m to zero from the left, the supersymmetry-breaking vacuum just slides to infinity and the equality $f(0) = V_{\text{asymp}}$ simply reflects the fact that the supersymmetry breaking vacuum now resides in the far zone. In particular, the supersymmetry breaking vacuum would be in the weakly coupled region (3.57). To rule this out we need to use a new element: that the TQFT in the supersymmetry breaking vacuum, $\mathrm{U}(N-k)_{k,N}$, cannot be obtained in the semiclassical regime. Indeed, there is no weakly coupled description in the original degrees of freedom of a ground state with this TQFT.

We therefore see that to establish the existence of a long-lived supersymmetry-breaking ground state in this theory, continuity arguments near the point where the index jumped were not quite sufficient, and we had to allude also to the topological degrees of freedom in the supersymmetry-breaking ground state. Because of that, our general argument may fail in some special cases where the TQFT in the supersymmetry-breaking vacuum does accidentally coincide with the theory far out on the Coulomb branch. For example, for $k = 1$,

$$k = 1 : \qquad \mathrm{U}(N-k)_{k,N} \leftrightarrow \mathrm{U}(1)_{-N} \,,$$

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and because of the duality (3.15) this in fact coincides with the topological theory far out on the moduli space where no eigenvalues coincide. Another way to think about this special case is that if our gauge group was U(N), then for $k = 1$ the vacuum far out on the moduli space with generic eigenvalues would be trivial but also the TQFT in the supersymmetry-breaking vacuum is trivial. Therefore, in the case of $k = 1$ we essentially do not have the topology of the ground state that we used to protect the supersymmetry-breaking ground state from slipping to the weakly coupled region.

The proof that we gave here for the existence of a metastable supersymmetry-breaking state depends on the assumption that indeed our phase diagram is correct and for all negative m there is a single supersymmetry-breaking state with the properties we discussed. This assumption is motivated by the fact that the phase diagram we proposed is the simplest that is consistent with our detailed analysis of the ground states and with the 't Hooft anomalies.

In $\mathcal{N} = 1$ theories, jumps in the Witten index are generic on co-dimension one hypersurfaces. If on one side of the hypersurface the ground state breaks supersymmetry spontaneously and on the other side new supersymmetric vacua come in from infinity, it is expected that supersymmetry breaking states would be metastable at least close enough to the hypersurface. The only way in which this conclusion can be avoided is if the supersymmetry breaking ground states slip to infinity as we get near the hypersurface. But as we saw, this can be in some cases ruled out by using the properties of the supersymmetry-breaking ground state.

3.8 Phases of the theory with $k = 0$

The $\mathcal{N} = 1$ theory with an adjoint matter multiplet at $k = m = 0$ has $\mathcal{N} = 2$ supersymmetry. This implies that the superpotential on the moduli space of vacua (3.9) does not receive any perturbative corrections, and in particular, the two-loop effective potential (3.38) vanishes. As a consequence of this, the phase diagram of this theory is much simpler than for the theory with $k \neq 0$.

We recall that in section 3.1 we showed that the theory with large positive and large negative mass flows to a trivial, gapped supersymmetric vacuum (i.e. with no TQFT). Our goal here is to fill in what happens between these asymptotic phases.

Let us start at the $\mathcal{N} = 2$ supersymmetric point $m = m_{\text{soft}} = 0$. While the superpotential is not renormalized in perturbation theory, it is well known [38] that non-perturbative effects due to monopole-instantons lead to a runaway superpotential (i.e. there is no stable vacuum).

We now proceed to show that $m = 0$ is the only singular point in the phase diagram. We will demonstrate that immediately to the left and to the right of the $m = 0$ point there is a trivial vacuum with unbroken supersymmetry. Therefore, the phases infinitesimally away from $m = 0$ are identical to the asymptotic phases at large positive and negative m, and hence the phase diagram is particularly simple.

Let us first review some relevant parts of the derivation of the runaway behaviour for SU(2) for simplicity. This theory has because of the enhanced $\mathcal{N} = 2$ supersymmetry at $m = 0$ an SO$(2)_R$ R-symmetry, and a classical flat direction with unbroken U(1) gauge

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symmetry parametrized by the eigenvalues of the scalar field $X = \mathrm{diag}(x, -x)$. The Yukawa interaction (3.3) along this flat direction induces a coupling between x and the charge two fermions $\psi^{(+2)}, \lambda^{(+2)}$ and their complex conjugates, as in (3.21) (with $k = 0$)

$$igx\psi^{(+2)}\overline{\lambda^{(+2)}} + c.c.\,.$$

The $\mathrm{SO}(2)_R$ symmetry acts naturally on the linear combinations $\Psi^{(+2)} = \psi^{(+2)} + i\lambda^{(+2)}$ and $\Theta^{(+2)} = \psi^{(+2)} - i\lambda^{(+2)}$.

Integrating out $\Psi^{(+2)}, \Theta^{(+2)}$ does not generate a Chern-Simons term for the U(1) gauge field, but it does generate a mixed Chern-Simons term coupling $\mathrm{SO}(2)_R$ to the unbroken U(1) gauge symmetry. The mixed Chern-Simons term is (remembering that integrating out one fermion with charges (1,1) gives $\frac{1}{4\pi}\mathcal{B}dA$)

$$\frac{2}{2\pi}\mathcal{B}dA\,, \tag{3.58}$$

where $\mathcal{B}$ is the $\mathrm{SO}(2)_R$ background gauge field and A the unbroken U(1) gauge field. As a result, the minimal monopole operator picks up charge -2 under $\mathrm{SO}(2)_R$. Denoting the corresponding chiral superfield by

$$Y = e^{x+i\tilde{a}} = e^{u}\,,$$

we have that $\tilde{a}$, which is the scalar dual to the U(1) gauge field, is 2π periodic and transforms under $\mathrm{SO}(2)_R$ rotations as $\tilde{a} \to \tilde{a} + 2\alpha$ with α a 2π periodic parameter.

Therefore, the following superpotential gets generated non-pertubatively[31]

$$W = \frac{1}{Y} = e^{-u}\,. \tag{3.59}$$

The kinetic terms are approximately linear in terms of x and $\tilde{a}$ far out in the moduli space, where it is approximately true that

$$K \sim (\log Y + \log \overline{Y})^2\,.$$

Therefore there is a runaway potential, scaling like $V \sim \frac{1}{|Y|^2} \sim e^{-2x}$.

Let us now turn on a small mass deformation (3.4) for the $\mathcal{N} = 1$ matter multiplet and determine where the theory flows to. This can be done by writing the deformation in the ultraviolet using an $\mathcal{N} = 2$ spurion superfield M. In terms of this, the mass deformation (3.4) preserving $\mathcal{N} = 1$ takes the form (Σ is the $\mathcal{N} = 2$ chiral superfield constructed out of the $\mathcal{N} = 2$ vector multiplet, i.e. the field strength multiplet)

$$\delta\mathcal{L} = \frac{1}{2}\int d^4\theta\, M\,\mathrm{Tr}\left(\Sigma^2\right)\,,$$

with

$$M = m(\theta - \overline{\theta})^2\,.$$

[31] For SU(N) an A_{N-1} Toda superpotential is generated, see appendix B.

This choice of M preserves $\mathcal{N}=1$ supersymmetry. This choice is of course non-unique; we could have used the R-symmetry to relate any two such choices of M.

In the presence of M the standard transformation from Σ to the chiral superfield u is modified

$$\frac{1}{2}\int d^4\theta\,(-1+M)\Sigma^2+\Sigma(u+\overline{u})\,.$$

Integrating out Σ leads to the effective action in terms of u

$$\frac{1}{2}\int d^4\theta(1-M)^{-1}(u+\overline{u})^2=\int d^4\theta\left(u\overline{u}+\frac{1}{2}M(u+\overline{u})^2+\cdots\right),$$

where on the right hand side we have only kept terms to linear order in M. Expanding this action in components, and including the non-perturbative superpotential (3.59), we find (ignoring terms with derivatives)

$$e^{-u}F_u+c.c.-(e^{-u}\psi_u\psi_u+c.c.)+|F_u|^2-m(u+\overline{u})(F_u+\overline{F}_u)-\frac{1}{2}m(\psi_u+\overline{\psi}_u)^2\,.$$

Solving for the auxiliary field we find the potential

$$|e^{-u}-(u+\overline{u})m|^2\,. \tag{3.60}$$

This can be viewed as arising from the $\mathcal{N}=1$ superpotential

$$W_{\mathcal{N}=1}=e^{-\operatorname{Re}u}\cos(\operatorname{Im}u)+m(\operatorname{Re}u)^2\,.$$

For small positive m, the minimum is at $u\sim-\log 2m$ with u real, while for small negative m it is at $u\sim-\log 2|m|+i\pi$. The scalar fields are massive in these vacua, and since the vacua are $\mathcal{N}=1$ supersymmetric, so are the fermions. Therefore we have shown that the theory flows to a trivial phase for both positive and negative small m, leading to a very simple phase diagram, with a trivial massive vacuum everywhere except at $m=0$, where there is no stable vacuum.

The analysis here straightforwardly generalizes to the case of SU(N) gauge group, with identical conclusions. Some details are in appendix B.

4 $\mathcal{N}=1$ SU(N)$_k$ and U(N)$_{k,k'}$ with fundamental matter

We now discuss the phase diagram of $\mathcal{N}=1$ gauge theories with matter multiplets in the fundamental representation. We study in turn $\mathcal{N}=1$ gauge theories with U(1), SU(N) and U(N) gauge groups. Based on this we put forward infrared dualities relating different $\mathcal{N}=1$ theories.

4.1 A U(1)$_k$ warm up

We take a U(1) $\mathcal{N}=1$ gauge multiplet and couple it to matter multiplets with charges $q_i\in\mathbb{Z}$. The vector multiplet consists of A_μ,λ (λ is neutral and Majorana) and the matter multiplets consist of (Φ_i,Ψ_i), carrying charge q_i under the gauge symmetry, with $i=$

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$1, \ldots, N_f$. The Lagrangian with (classically) vanishing superpotential $W = 0$ and with a Chern-Simons term with level k is given by

$$\mathcal{L} = -\frac{1}{4g^2}F^2 + i\lambda\not{\partial}\lambda + \frac{k}{4\pi}AdA - \frac{kg^2}{2\pi}\lambda\lambda + \sum_i |D_\mu\Phi_i|^2 + i\sum_i \overline{\Psi}_i \not{D}\Psi_i + \sum_i \left(\sqrt{2}i\Phi_i\overline{\Psi}_i\lambda + c.c.\right). \tag{4.1}$$

Consistency requires that

$$k + \frac{1}{2}\sum_i q_i^2 \in \mathbb{Z}\,,$$

a condition which is equivalent to

$$k + \frac{1}{2}\sum_i q_i \in \mathbb{Z}\,.$$

Let us now consider the symmetry group of the model. If we assume that the q_i are arbitrary integers the symmetry group consists of $\mathrm{U}(1)_i$ factors acting on the multiplets (Φ_i, Ψ_i) along with $\mathrm{U}(1)_T$ which leads to a conserved magnetic charge, with conserved topological current $\frac{1}{2\pi} \star F$. One linear combination of the $\mathrm{U}(1)_i$ and $\mathrm{U}(1)_T$ is coupled to the dynamical gauge field A_μ and therefore the symmetry group is[32]

$$K = \frac{\prod_{j=1}^{N_f} \mathrm{U}(1)_j \times \mathrm{U}(1)_T}{\mathrm{U}(1)}\,. \tag{4.2}$$

If some of the charges coincide then the symmetry group K in (4.2) is enhanced to a non-Abelian group.

There is a superpotential deformation that preserves all the symmetries

$$W = \sum_i m_i \Phi_i \Phi_i^\dagger\,,$$

and again if some of the charges coincide then one can preserve the non-Abelian flavour symmetry by taking likewise some of the m_i to coincide.

If we assume that the m_i are large we can integrate out the superfields Φ_i and obtain a pure U(1) vector multiplet with an integer Chern-Simons level

$$k_{\mathrm{IR}} = k + \frac{1}{2}\sum_i q_i^2 \mathrm{sgn}(m_i)\,. \tag{4.3}$$

The long distance theory is therefore a $\mathrm{U}(1)_{k_{\mathrm{IR}}}$ TQFT. The neutral gaugino is massive and at tree level its mass is proportional to $g^2 k_{\mathrm{IR}}$.

[32] Let $(e^{is_1}, \ldots, e^{is_{N_f}}, e^{it})$ be an element of $\prod_{j=1}^{N_f} \mathrm{U}(1)_j \times \mathrm{U}(1)_T$. The identification by the U(1) in the denominator corresponds to

$$s_j \to s_j + q_j\alpha\,, \quad t \to t + \alpha k_{\mathrm{bare}}$$

for all α. Above we have defined

$$k_{\mathrm{bare}} = k + \frac{1}{2}\sum_i q_i^2\,.$$

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We can compute the Witten index as a function of these m_i, but we have to be careful about the sign of the index. We can take it to be positive (without loss of generality) when all the m_i are positive and then if some m_i crosses the origin we do not pick up a minus sign since the fermion Ψ_i has two degrees of freedom. But if the sign of the gaugino λ mass term changes its sign then we have to account for the change in fermion number. Hence, the Witten index of this theory is

$$I(m_i) = \text{sgn}(k_{\text{IR}})|k_{\text{IR}}| = k_{\text{IR}}\,. \tag{4.4}$$

Clearly, the Witten index jumps as we change the m_i.

In order to understand how this comes about it is useful to first study the theory with one charged multiplet Φ of charge q. Now the model has one mass parameter appearing in the superpotential $W = m\Phi\Phi^\dagger$. At very large positive m we get the $\text{U}(1)_{k+q^2/2}$ TQFT and at very large negative m we find the $\text{U}(1)_{k-q^2/2}$ TQFT. At $m = 0$ there is a classical flat direction. There is once again a two-loop effective potential that would lift this flat direction [57].

At small positive m we should have, due to the two-loop potential, another vacuum coming in from infinity with a condensed Φ. In this vacuum there are no massless particles. It is crucial to understand that the gauge symmetry is Higgsed down to a $\mathbb{Z}_q$ subgroup. However, the effective theory is not just a $\mathbb{Z}_q$ gauge theory as we have a Chern-Simons term for the original gauge field, which becomes a Dijkgraaf-Witten-like term [58] for the $\mathbb{Z}_q$ gauge theory. The infrared effective theory can be written as

$$\frac{q}{2\pi} A \wedge dB + \frac{k_{\text{bare}}}{4\pi} A \wedge dA \tag{4.5}$$

with A, B standard U(1) gauge fields. The contribution of (4.5) to the Witten index is

$$\Delta I = q^2\,,$$

since this is the number of lines in this Dijkgraaf-Witten $\mathbb{Z}_q$ gauge theory.

At small positive m therefore we have two supersymmetric vacua. One near the origin with the $\text{U}(1)_{k-q^2/2}$ TQFT and one very far out in the moduli space with the TQFT as in (4.5). The Witten index at small positive m is thus

$$I = k - \frac{q^2}{2} + q^2 = k + \frac{q^2}{2}\,,$$

which exactly coincides with the index at large positive m (4.4)(4.3).

For generic k the theory thus has a second order phase transition where the deformed $\mathbb{Z}_q$ gauge theory merges with a $\text{U}(1)_{k-q^2/2}$ TQFT and we get the $\text{U}(1)_{k+q^2/2}$ TQFT on the other side of the transition. A special treatment is necessary for $k = q^2/2$, in which case for negative m we have a vector multiplet with no Chern-Simons term. We dualize the vector multiplet to a real multiplet

$$(A, \lambda) \longrightarrow (\tilde{\phi}, \tilde{\psi})\,,$$

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with $\tilde{\phi}$ a pseudo-scalar and $\tilde{\psi}$ a Majorana fermion. The superpotential $W(\tilde{\phi})$ vanishes because of the $\mathrm{U}(1)_T$ symmetry. Therefore for negative m we have an S^1 of vacua parameterized by the dual photon $\tilde{\phi}$. This vacuum contributes zero to the Witten index since $\tilde{\psi}$ is massless.[33]

Let us now specialize to $q = 1$, in which case the situation is simpler as the $\mathbb{Z}_q$ gauge theories do not appear. In the case of $q = 1$, a trivial vacuum merges with the $\mathrm{U}(1)_{k-1/2}$ TQFT to become the $\mathrm{U}(1)_{k+1/2}$ TQFT. (The exceptional case of $k = \frac{1}{2}$ will be discussed in more detail soon.)

It is interesting to note that[34] the phase transition mentioned above $\mathrm{U}(1)_{k-1/2}$ TQFT+ trivial vacuum to $\mathrm{U}(1)_{k+1/2}$ TQFT for $k > 1/2$ is similar to the one that appears in $\mathcal{N} = 2$ supersymmetry [32]. In fact, for large enough k, the fixed point must have emergent $\mathcal{N} = 2$ supersymmetry in the infrared. This is not the case generically, but it is the case for U(1) gauge theories [59, 60]. For $k = 1/2$ the transition involves a circle of supersymmetric vacua that have to disappear on the other side of the transition. By contrast, in $\mathcal{N} = 2$ supersymmetric theories the supersymmetric ground states are always complex manifolds. We will discuss this example more below.

4.2 $\mathrm{SU}(N)_k$ and $\mathrm{U}(N)_{k,k'}$ with fundamental matter

We will now consider briefly a slightly different model: $\mathcal{N} = 1$ $\mathrm{SU}(N)_k$ vector multiplet with a fundamental matter multiplet ($N_f = 1$). The logic is quite similar. Imagine that we integrate out the matter multiplet with a large negative mass. Then we have a $\mathrm{SU}(N)_{k-1/2}$ pure vector multiplet. For $k - 1/2 \geq N/2$ this flows to the $\mathrm{SU}(N)_{k-1/2-N/2}$ TQFT and otherwise it breaks supersymmetry. At large positive m supersymmetry is unbroken if $k + 1/2 \geq N/2$ and we have a $\mathrm{SU}(N)_{k+1/2-N/2}$ TQFT at long distances. For infinitesimal positive m one finds [57] a new vacuum that is incoming from infinity, in which the fundamental matter field condenses. At low energies we remain with an $\mathrm{SU}(N-1)_k$ pure vector multiplet. For $k \geq N/2 - 1/2$ again supersymmetry is unbroken and we have a $\mathrm{SU}(N-1)_{k-N/2+1/2}$ TQFT at long distances.

The Witten indices match as follows for $k - 1/2 \geq N/2$:

$$\binom{N/2+k-3/2}{N-1} + \binom{N/2+k-3/2}{N-2} = \binom{N/2+k-1/2}{N-1}.$$

This is nothing but the standard Pascal identity for binomial coefficients. The extension to $0 < k - 1/2 < N/2$ is obvious: when the coefficients above vanish (written as polynomials in k) then the corresponding vacua break supersymmetry dynamically (but the formula continues to hold formally).

The $\mathrm{U}(N)_{k+N/2,k}$ vector multiplet with a fundamental matter multiplet ($N_f = 1$) behaves in a rather analogous fashion except for one additional small subtlety that needs to be taken into account. At large negative mass we get a pure vector multiplet $\mathrm{U}(N)_{k+N/2-1/2,k-1/2}$, which (for $k \geq 1/2$) flows to the TQFT $\mathrm{U}(N)_{k-1/2,k-1/2}$. At large

[33] Alternatively, since the Euler number of the circle is zero, the index vanishes.

[34] We thank D. Gaiotto for discussions on this and some related topics.

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positive mass we get the infrared TQFT $\mathrm{U}(N)_{k+1/2,k+1/2}$. The vacuum that comes from infinity at infinitesimal positive m is a little more subtle. The vacuum expectation value of the scalar field in the fundamental representation breaks the gauge symmetry to $\mathrm{U}(N-1)$. The effective theory in this Higgsed vacuum is that of an $\mathrm{U}(N-1)$ vector multiplet with a Chern-Simons term at level $\mathrm{U}(N-1)_{k+N/2,k+1/2}$. The shift of the U(1) level by $+1/2$ requires an explanation. The point is that the Lagrangian of a $\mathrm{U}(M)_{P,Q}$ Chern-Simons theory is

$$\frac{P}{4\pi}\,\mathrm{Tr}\,A\wedge dA+\frac{Q-P}{4\pi M}\,\mathrm{Tr}\,A\wedge\mathrm{Tr}\,dA\,.$$

Therefore, if the $\mathrm{U}(M)$ gauge symmetry is Higgsed to $\mathrm{U}(M-1)$, we would find a

$$\mathrm{U}(M-1)_{P,Q+(P-Q)/M}$$

Chern-Simons theory. This is why we get the theory of an $\mathcal{N}=1$ vector multiplet $\mathrm{U}(N-1)_{k+N/2,k+1/2}$ which in the infrared flows to the TQFT $\mathrm{U}(N-1)_{k+1/2,k+1/2}$.

Using level/rank duality

$$\mathrm{U}(M)_{P,P}\leftrightarrow\mathrm{SU}(P)_{-M}\,,$$

we can relate all the phases that we found above in the dynamics of the SU gauge theory to the phases of the U theory. This leads us to the following duality

$$U\,(N)_{k+N/2+1/2,k+1/2}+\Phi\longleftrightarrow\mathrm{SU}(k+1)_{-N-k/2}+\Psi\,. \tag{4.6}$$

The duality map of deformations is $\Phi\Phi^\dagger\longleftrightarrow-\Psi\Psi^\dagger$.

It is important to note that in this duality transformation the vacuum that comes from infinity in the SU theory is exchanged with a vacuum that is visible at large mass in the U theory and vice versa. This is allowed since the duality is valid near the conformal field theory (which occurs at finite distance away from the wall at $m=0$) where these vacua merge. We would like to mention that, for instance, if we take k to be very large on the left-hand side of (4.6), the conformal field theory is weakly coupled and can be analyzed explicitly [59, 60] (both sides of (4.6) can be furthermore explicitly analyzed in the 't Hooft limit [2, 42]). There are typically several close-by fixed points, and one of them has emergent $\mathcal{N}=2$ supersymmetry (and a $\mathrm{U}(1)_R$ symmetry).[35] To understand the duality (4.6) in more detail, it is thus crucial to go beyond the analysis of the phases of the theory that we have undertaken here and study the precise mapping of the quartic operators on the two sides. This can be carried out along the lines of [2, 42], where the same duality was studied in the 't Hooft limit. The duality (4.6) was summarized in figure 5 in the introduction.

The duality (4.6) is very similar to non-supersymmetric boson/fermion dualities and to $\mathcal{N}=2$ Giveon-Kutasov dualities [61]. It is not surprising that such a duality holds. We have arrived at this duality by studying in detail the walls in parameter space where the Witten index jumps. That we find the correct space of ground states and phase transitions is a nontrivial consistency check of our methods. It would be very interesting to extend

[35] If the gauge group is U(1) and the Chern-Simons level is large enough, there is only one fixed point with emergent $\mathcal{N}=2$ supersymmetry.

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the analysis to a general collection of matter multiplets. The dynamics near the walls is then more complicated, as typically more than one new ground state appears from infinity in field space.

We would like to say a few words about the special case $N = 1, k = 0$. The duality reduces then to

$$U(1)_{1/2} + \Phi \longleftrightarrow \Psi\,. \tag{4.7}$$

This appears to be a natural generalization of the non-supersymmetric duality between $\mathrm{U}(1)_{1/2}$ + fermion and the $O(2)$ model [7, 39, 40].

The phases of the model on the left hand side are a circle of vacua for negative m, a new trivial vacuum incoming from infinity at small positive m, and a trivial supersymmetric vacuum at large positive m. The Witten index for negative m therefore vanishes and the Witten index for positive m is one. On the other side of the duality we should interpret Ψ as a complex superfield, namely two real $\mathcal{N} = 1$ multiplets.

The transition of the theory on the left hand side occurs at some finite positive m, where we have a circle and a trivial vacuum on one side of the conformal field theory and a trivial vacuum on the other side of the conformal field theory. Since the Witten index of the circle vanishes, it can be that the circle disappears without merging with the trivial supersymmetric vacuum. The conformal field theory then would not involve the trivial vacuum in any essential way.[36] Let us however consider the possibility that the circle merges with the trivial vacuum. This then leads to a very natural interpretation of the duality (4.7). Indeed, let us add a quartic term in Ψ, namely

$$W = -m\Psi\Psi^\dagger + \frac{1}{2}(\Psi\Psi^\dagger)^2\,.$$

The critical point equation is $m\Psi = \Psi^2\Psi^\dagger$ and for negative m we see clearly that there is only one solution $\Psi = 0$, while for positive m we have two kinds of solutions

$$m > 0: \quad \Psi = 0, \Psi = e^{i\tilde{\phi}}\sqrt{m}\,,$$

with arbitrary $\tilde{\phi}$, parameterizing a circle. Therefore, the phase transition precisely agrees with what we have discussed for the $\mathrm{U}(1)_{1/2} + \Phi$ theory, once we flip the sign of m in the dictionary and include the term $|\Psi|^4$ in the superpotential. From the point of view of the renormalization group, the theory $W = |\Psi|^4$ can be viewed as a marginally irrelevant deformation of the free $\mathcal{N} = 1$ theory of two real multiplets. Therefore, in the very deep infrared at the critical point we have emergent $\mathcal{N} = 2$ supersymmetry as in the other U(1) gauge theories we discussed in the previous subsection. But here the quartic interaction is an important $\mathcal{N} = 1$ marginally irrelevant deformation that allows for a circle of supersymmetric vacua to exist on one side of the transition but not on the other.

[36] We thank D. Gaiotto for useful discussions of this point.

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A Relation to $\mathcal{N} = 2$ duality

We now relate our proposed small k behaviour of $\mathcal{N} = 2$ $\mathrm{SU}(N)_k$ vector multiplet in section 3.6 with consequences that stem from $\mathcal{N} = 2$ dualities that have appeared in the literature. This gives nontrivial evidence for our picture.

Consider the duality in [62][37]

$$\mathcal{N} = 2\ \mathrm{SU}(2)_1 + \text{adjoint } X \longleftrightarrow \text{free } \mathcal{N} = 2 \text{ chiral multiplet } u + \mathrm{U}(1)_2\ \mathrm{TQFT}\,, \tag{A.1}$$

and its generalization[38] to arbitrary N [63]

$$\mathcal{N} = 2\ \mathrm{SU}(N)_1 + \text{adjoint } X \longleftrightarrow \mathcal{N} = 2 \text{ free chirals } u_1, \ldots, u_N\ + \mathrm{U}(1)_{-N}\ \mathrm{TQFT}\,. \tag{A.2}$$

We like to stress that these dualities hold only after the TQFT on the right hand side is added. This will be crucial for us in what follows.

Assuming (A.1) then the following immediately follows: adding a superpotential mass for the adjoint chiral on the left hand side, we find that the left hand side flows to the $\mathcal{N} = 2$ $\mathrm{SU}(2)_1$ vector multiplet. On the right hand side, this deformation amounts to adding a linear superpotential $W \propto u$. This breaks $\mathcal{N} = 2$ supersymmetry spontaneously, leading to a Dirac goldstino. Thus the right hand side flows to $\mathrm{U}(1)_2$ TQFT + Dirac Goldstino.

Analogously, adding a superpotential mass for the adjoint chiral on the left hand side of (A.2) we get the $\mathcal{N} = 2$ $\mathrm{SU}(N)_1$ vector multiplet. This is realized on the right hand side of (A.2) by the linear superpotential $W \propto u_1$. This breaks supersymmetry,[39] and the right hand side flows to $\mathrm{U}(1)_{-N}$ TQFT + Dirac Goldstino.

[37] The S^3 partition function of both sides agree once a factor of 2 that was missing in [62] is added, and that corresponds to the contribution of the $\mathrm{U}(1)_2$ TQFT on the right hand side.

[38] We extend the duality from $\mathrm{U}(N)$ to $\mathrm{SU}(N)$ gauge group.

[39] The rest of the chiral multiplets become massive by virtue of having a nontrivial Kähler potential.

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This should be contrasted with our general proposal for $0 < k < N/2$

$$\mathcal{N} = 2\ \mathrm{SU}(N)_k \longleftrightarrow \mathrm{U}(k)_{k-N,-N} + \text{Dirac goldstino}\,.$$

Setting $k = 1$ we indeed obtain

$$\mathcal{N} = 2\ \mathrm{SU}(N)_1 \longleftrightarrow \mathrm{U}(1)_{-N} + \text{Dirac goldstino}\,, \tag{A.3}$$

and our proposal for $k = 1$ is precisely what we obtained from the dualities (A.1) and (A.2) by adding a superpotential deformation on both sides.

B The general case with $k = 0$

For SU(N) we have that the non-perturbative superpotential along the Coulomb branch is

$$W(U) \sim \sum_i e^{-(U_i - U_{i+1})}\,.$$

This can be combined with the mass deformation of the $\mathcal{N} = 1$ adjoint multiplet to the following $\mathcal{N} = 1$ superpotential

$$W \sim \sum_i\ e^{-(\Phi_i - \Phi_{i+1})} \cos(\Gamma_i - \Gamma_{i+1}) + m \sum_i \Phi_i^2,$$

where we have used that $U_i = \Phi_i + i\Gamma_i$. From the supersymmetric vacua equations we find that $\Gamma_i = 0$ and

$$e^{\Phi_{i-1} - \phi_i} - e^{\Phi_i - \phi_{i+1}} + m\Phi_i = 0.$$

It is not difficult to prove that the system above can have only one solution and there is a unique massive supersymmetric vacuum for both signs of m.

C Proof of identity (3.46).

We give an elementary proof of

$$\sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k - S_I)!} = \frac{(N + k - 1)!}{N!(k-1)!}\,. \tag{C.1}$$

We start with the generating function $F(k) = -\sum_{S_I=1}^{k} \binom{k}{S_I} x^{S_I}$. We next consider the function

$$G_k = F_k + F_k^2 + \ldots \tag{C.2}$$

Next, we expand

$$G_k = \sum_m d_k^m x^m\,.$$

It is easy to observe that

$$d_k^N = \sum_P (-1)^L \prod_{I=1}^{L} \binom{k}{S_I}\,.$$

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Indeed, considering (C.2), we see that the first term on the right hand side would correspond to a partition of N into one term, the term F_k^2 would correspond to a partition of N into two terms, etc. The coefficient $(-1)^L$ comes from the minus sign in the definition of F_k.

Now let us compute G_k explicitly. We start from the binomial formula, which leads to $F_k(x) = -(1+x)^k + 1$ and hence

$$G_k = \frac{F_k}{1-F_k} = \frac{-(1+x)^k+1}{(1+x)^k} = -1 + (1+x)^{-k}$$

We then use the standard result

$$(1+x)^{-k} = \sum \frac{1}{j!}(-k)(-k-1)\cdots(-k-j+1)x^j = \sum_j (-1)^j \binom{k+j-1}{j} x^j$$

to infer that

$$d_k^N = (-1)^N \binom{k+N-1}{N},$$

which finishes the proof.

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