BPS Kerr-AdS time machines

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ABSTRACT: It was recently observed that Kerr-AdS metrics with negative mass can describe smooth spacetimes that have a region within which naked closed time-like curves can arise, bounded by a velocity of light surface. Such spacetimes are sometimes known as time machines. In this paper we study the BPS limit of these metrics, and find that the mass and angular momenta become discretised. The completeness of the spacetime also requires that the asymptotic time coordinate be periodic, with precisely the same period as that which arises naturally for the global AdS, viewed as a hyperboliod in one extra dimension, in which the time machine spacetime is immersed. For the case of equal angular momenta in odd dimensions, we construct the Killing spinors explicitly, and show they are consistent with the global structure. Thus in examples where the solutions can be embedded in gauged supergravity, they will be supersymmetric. We also compare the global structure of the BPS AdS$_3$ time machine with the BTZ black hole, and show that the global structure allows two different supersymmetric limits.

Contents

1 Introduction

The Kerr metric [1] is arguably the most important exact vacuum solution in Einstein's theory of General Relativity. Over the years, the solution has been generalised to include a cosmological constant and also to higher dimensions [2–7]. These metrics are asymptotic to Minkowski, de Sitter (dS) or anti-de Sitter (AdS) spacetimes, depending on the cosmological constant. They carry mass (M) and angular momenta (J_i) as conserved quantities.

Black holes have played a very important role in elucidating the structure of string theory and M-theory, notably in the discussion of non-perturbative effects and duality symmetries. Among the black hole solutions the supersymmetric, or BPS, black holes are of particular interest, since they acquire some degree of protection against quantum corrections, and may thus give more trustworthy information about the non-perturbative structure of the theory. Black holes in string theory or M-theory are described, at least at leading order, as solutions in the low-energy supergravity limit of the theory in question. Typically, the solutions can carry electromagnetic charges, or higher-degree p-form charges too. The supersymmetric BPS solutions usually require these charges to be non-vanishing,

as well as having non-zero mass and possibly rotation. However, one can also consider BPS limits of rotating black hole metrics in string or M-theory that do not carry any additional electromagnetic or p-form charges.

The BPS limit of a higher-dimensional rotating Kerr-AdS black hole corresponds to the case where the mass M and angular momenta J_i satisfy

$$M = \sum_i g \, |J_i| \tag{1.1}$$

where $1/g$ is the "radius" of the asymptotic AdS spacetime in which the solution is immersed [8]. This BPS condition was studied in detail in [9] for the five-dimensional Kerr-AdS black holes, and the Killing spinors were constructed in the case where the two angular momenta were equal. The BPS limit no longer describes a black hole, however, since the singularity is no longer cloaked by a horizon. Similar conclusions arise in higher dimensions also. Interestingly, if one instead Euclideanises the spacetime and takes the cosmological constant to be positive, the Kerr-dS metrics become Einstein-Sasaki in the BPS limit. Furthermore, these can smoothly extend onto complete, compact manifolds for appropriate discretised values of the metric parameters [10, 11]. This generalises an earlier construction of smooth Einstein-Sasaki spaces in [12].

Recently, it was observed [13] that for general odd dimensions, the Kerr and Kerr-AdS metrics can extend onto smooth manifolds if the mass parameter is taken to be negative, provided that all the angular momenta are non-zero. The regularity of the spacetime manifold now requires that the asymptotic time coordinate be assigned a specific (real) periodicity. There is also a localised region within the spacetime where an azimuthal coordinate becomes timelike; such a situation, as we shall discuss in more detail below, is known as a "time machine" in the literature.

The interesting point about these Kerr-AdS time-machine metrics from our standpoint is that they continue to be smooth, non-singular, spacetimes even in the BPS limit. Thus, these solutions are of potential interest in string theory and M-theory. They will form the subject of our investigations in this paper.

Before describing these investigations in more detail, we shall first summarise some known pertinent results about the rotating black holes of supergravity and string theory.

For a given set of angular momenta, provided that the mass is sufficiently large, the metrics describe rotating black holes. Such rotating black holes contain a localised region admitting closed time-like curves (CTCs), bounded on the outside by a velocity of light surface (VLS), within which one or more periodic azimuthal angular coordinates become time-like. Such a situation is commonly referred to as a time machine.[1] In a rotating black hole, the time machine is hidden inside the black hole event horizon.

If the black hole is over-rotating, the time machine can extend outside the horizon. For example, it was demonstrated, for a supersymmetric charged black hole with equal angular momenta in five dimensions [15], that in the over-rotating situation the boundary of the time machine lies outside the horizon and so it becomes naked [16]. (See also [17–19].) An examination of geodesics showed that they could not penetrate the horizon, and

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[1]This should be distinguished from what happens in a spacetime like the Gödel universe [14], where the normal region of the spacetime is surrounded by the VLS, outside of which lie the naked CTCs.

hence the spacetime configuration is called a repulson [16]. (See also [20].) In fact the "horizon" becomes a Euclidean Killing horizon that can induce a conical singularity unless the asymptotic time coordinate itself is assigned a specific (real) period, in which case the spacetime configuration is smooth and geodesically complete [8]. One now has a situation where there are two different kinds of closed time-like curves; those associated with the local "time machine region" where an azimuthal angular coordinate has become time-like, and those associated with the global real periodicity that has been assigned to the asymptotic time coordinate.

By convention, a situation where the asymptotic time coordinate has a real periodicity is not usually referred to as a "time machine." A familiar example of this type is the strict global anti-de Sitter spacetime AdS_D, defined as a hyperboloid in $\mathbb{E}^{2,D-1}$.[2] For the sake of clarity in what follows, we shall follow this convention and reserve the term "time machine" for the situation where there is a localised region inside a VLS in which a spatial angular coordinate has become time-like. Our purpose in this paper is not to advocate the BPS Kerr-AdS metrics for time travel, but simply to investigate the intriguing global structures that can arise when the mass is taken to be negative.

In this paper, we shall remain in Lorentzian signature and with a negative cosmological constant, but now we consider the BPS Kerr-AdS metrics where the mass is taken to be negative. As mentioned above, now, unlike the example considered in [9] where the mass was assumed to be positive, this can yield a smooth time-machine spacetime. BPS time machines have been constructed previously in the literature, typically having positive mass and with additional electric charges [8, 16, 21, 22]. Our focus in this paper, however, will be on the pure gravity BPS Kerr-AdS metrics. We shall show that these metrics extend onto smooth spacetimes provided that the mass is negative, and that the asymptotic Lorenzian time coordinate is periodically identified, with a period precisely equal to that of the time coordinate in the global AdS in which the spacetime is immersed. Furthermore, in order for the various periods requried for completeness to be comensurate, the mass and angular momenta become discretised, in a manner analogous to the discretisation of the parameters in the Einstein-Sasaki spaces [10, 11], even though the spacetimes we are considering here are Lorentzian and non-compact. For Kerr-AdS metrics with equal angular momenta in odd dimensions, we construct the Killing spinors in the BPS limit explicitly, and show that they are compatible with the global structure required for the completeness of the spacetime. Thus in dimensions where the solution can be embedded within a supergravity theory, it will be superymmetric.

The paper is organised as follows. In section 2, we begin by reviewing the time machine spacetimes that were obtained in [13] from $D = (2n+1)$-dimensional Kerr-AdS spacetimes with equal angular momenta, by taking the mass to be negative, and we describe their BPS limits. We give an explicit construction of the Killing spinors in the BPS spacetimes, showing how they can be obtained by making use of the gauge-covariantly constant spinors that exist in the underlying $\mathbb{CP}^{n-1}$ spaces that form the bases of the $(2n-1)$-dimensional spher-

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[2]Throughout this paper, when we speak of "global AdS" we shall mean the one with the periodic time coordinate, arising via the hyperboloidal embedding in $E^{2,D-1}$. The covering space, where time is unwrapped to cover the entire real line, will be denoted by CAdS.

ical surfaces in the spacetimes. We also study the restrictions on the metric parameters that result from requiring completeness of the spacetimes, resulting from the compatibility conditions for periodicities at the various degenerate surfaces. These restrictions imply that the mass and angular momentum must be rational multiples of a basic unit. They also imply that the time coordinate must be periodic, with exactly the periodicity of the time coordinate in the global AdS spacetime in which the time machine is immersed.

In section 3 we consider the case of even-dimensional spacetimes, showing that Kerr-AdS metrics with equal angular momenta can give rise in the BPS limit to metrics describing foliations of the previously discussed odd-dimensional time machines. In section 4 we discuss the analogous odd and even-dimensional BPS limits of Kerr-AdS metrics with general, unequal, angular momenta. Again these give rise to time machines if the mass is taken to be negative, and we analyse the restrictions on the metric parameters to ensure global completeness of the spacetime manifolds. Again, the mass and the angular momenta are discretised, in the sense that they are constrained to be certain rational multiples of a basic unit.

In section 5 we discuss the special case of three dimensions. Here, the Kerr-AdS metric is necessarily locally isomorphic to AdS$_3$, and thus it is also locally isomorphic to the BTZ black hole [23]. We study the relation between the time machine and the BTZ spacetimes, and compare their Killing spinors in the respective BPS limits. Interestingly, the limits are different, but in each case the Killing spinors are compatible with the global structure.

Finally, after our conclusions, we include two appendices. Appendix A gives an explicit construction of the gauge-covariantly spinors in the complex projective spaces, employing an iterative construction of $\mathbb{CP}^n$ in terms of $\mathbb{CP}^{n-1}$ that was given in [26]. We use these gauge-covariantly constant spinors in the construction of Killing spinors in section 2. Appendix B contains some results relating the various vectors and tensors that can be built from Killing-spinor bilinears. These are relevant for the construction of the spinorial square roots of the time-like Killing vectors in the BPS spacetimes.

2 Equal angular momenta in $D = 2n + 1$

2.1 Kerr black holes and time machines

We begin with the Kerr-AdS metrics in $D = 2n+1$ dimensions with all n angular momenta set equal. The metric, satisfying $R_{\mu\nu} = -(D-1)g^2 g_{\mu\nu}$, contains two integration constants (m, a), and it is given by [27]

$$ds_{2n+1}^2 = -\frac{1 + g^2 r^2}{\Xi} dt^2 + \frac{U dr^2}{V - 2m} + \frac{r^2 + a^2}{\Xi}\left(\sigma^2 + d\Sigma_{n-1}^2\right) + \frac{2m}{U\Xi^2}(dt - a\sigma)^2,$$

$$\sigma = d\psi + A, \qquad U = \left(r^2 + a^2\right)^{n-1}, \qquad V = \frac{1}{r^2}\left(1 + g^2 r^2\right)\left(r^2 + a^2\right)^n, \qquad (2.1)$$

where $\Xi = 1 - a^2 g^2$, and $d\Sigma_{n-1}^2$ is the standard Fubini-Study metric on $\mathbb{CP}^{n-1}$. There is circle, parameterised by the coordinate ψ with period 2π, which is fibred over the $\mathbb{CP}^{n-1}$ base, and σ is the 1-form on the fibres, given by $\sigma = d\psi + A$ with $dA = 2J$ where J is the Kähler form on $\mathbb{CP}^{n-1}$. The terms $(\sigma^2 + d\Sigma_{n-1}^2)$ in the metric are nothing but the metric

on the unit round sphere S^{2n-1}, with $R^i_j = (n-1)\delta^i_j$. The metric (2.1) is asymptotic to anti-de Sitter spacetime with radius $\ell = 1/g$.

The mass and the (equal) angular momenta are given by

$$M = \frac{m(2n-\Xi)\mathcal{A}_{2n-1}}{8\pi\Xi^{n+1}}, \qquad J = \frac{ma\mathcal{A}_{2n-1}}{4\pi\Xi^{n+1}}, \qquad (2.2)$$

where $\mathcal{A}_k$ is the volume of a unit round S^k, given by

$$\mathcal{A}_k = \frac{2\pi^{\frac{1}{2}(k+1)}}{\Gamma\left[\frac{1}{2}(k+1)\right]}. \qquad (2.3)$$

It will be helpful to make a coordinate transformation and a redefinition of the integration constants to replace (m, a) by (μ, ν), as follows:

$$\frac{r^2 + a^2}{\Xi} \to r^2, \qquad a = \sqrt{\frac{\nu}{\mu}}, \qquad m = \frac{1}{2}\mu\left(1 - \frac{\nu}{\mu}g^2\right)^{n+1}. \qquad (2.4)$$

The metric (2.1) becomes [13]

$$ds^2_{2n+1} = \frac{dr^2}{f} - \frac{f}{W}dt^2 + r^2 W(\sigma + \omega)^2 + r^2 d\Sigma^2_{n-1},$$

$$f = \left(1 + g^2 r^2\right)W - \frac{\mu}{r^{2(n-1)}}, \qquad W = 1 + \frac{\nu}{r^{2n}}, \qquad \omega = -\frac{\sqrt{\mu\nu}}{r^{2n} + \nu}dt. \qquad (2.5)$$

The mass and angular momenta become

$$M = \frac{\mathcal{A}_{2n-1}}{16\pi}\left((2n-1)\mu + g^2\nu\right), \qquad J = -\frac{\mathcal{A}_{2n-1}}{8\pi}\sqrt{\mu\nu}. \qquad (2.6)$$

The metric (2.5) describes a rotating black hole if μ and ν are both positive, and a time machine if μ and ν are both negative [13], as we shall review later.

2.2 BPS limits

Under certain conditions the metric (2.5) will admit a Killing spinor, obeying the equation

$$\nabla_\mu\epsilon + \frac{1}{2}g\,\Gamma_\mu\epsilon = 0. \qquad (2.7)$$

A necessary condition for this to occur is that the BPS condition on the mass and angular momentum, namely

$$M = ngJ, \qquad (2.8)$$

should hold. This implies that

$$\mu = g^2\nu, \qquad \text{or} \qquad \mu = \frac{g^2\nu}{(2n-1)^2}. \qquad (2.9)$$

These two conditions correspond to $ag = 1$ (and hence $\Xi = 0$) or $ag = 2n - 1$ respectively. However, as we shall see, only the first of these cases gives a solution admitting a Killing spinor.

In AdS itself (i.e. $\mu = 0$ and $\nu = 0$), the Killing vectors

$$K_{\pm} = \frac{\partial}{\partial t} \pm g \frac{\partial}{\partial \psi} \,, \tag{2.10}$$

have the property that $g_{\mu\nu} K_{\pm}^{\mu} K_{\pm}^{\nu} = -1$, and in fact they can each be expressed in the form $K_{\pm}^{\mu} = \bar{\epsilon}_{\pm} \Gamma^{\mu} \epsilon_{\pm}$, where each of $\epsilon_{\pm}$ is one of the Killing spinors of the AdS spacetime. We expect that if the BPS spacetime where μ and ν are non-zero, obeying one or other of the conditions in (2.9), does admit a Killing spinor, then it should be such that it limits to one of the aforementioned AdS Killing spinors in the limit where μ and ν go to zero. This means that if the BPS spacetime admits a Killing spinor, the norm $K^{\mu} K_{\mu}$ should be manifestly negative (see [8] for a discussion of this). For the two cases in (2.9) we find

$$\mu = g^2 \nu : \qquad g_{\mu\nu} K_+^{\mu} K_+^{\nu} = -1 \,, \tag{2.11}$$

$$\mu = \frac{g^2 \nu}{(2n-1)^2} : \qquad g_{\mu\nu} K_+^{\mu} K_+^{\nu} = -1 + \frac{n^2 g^2 \nu}{(2n-1)^2 r^{2n-2}} \,, \tag{2.12}$$

where K_+ is defined in (2.10). This indicates that (2.11) gives rise to a true BPS limit, in the sense that the K_+ Killing vector (but not K_-) admits a spinorial square root, whereas for (2.12) it does not (nor does K_-).

For positive $\mu = g^2 \nu$, the metric has a curvature power-law naked singularity at $r = 0$. We shall thus focus on the case when $\mu = g^2 \nu$ is negative. Defining $\nu = -\alpha$, the metric becomes

$$ds^2 = -\frac{f}{W} dt^2 + \frac{dr^2}{f} + r^2 W \left(d\psi + A + \omega dt \right)^2 + r^2 d\Sigma_{n-1}^2 \,, \tag{2.13}$$

$$f = g^2 r^2 + W \,, \qquad W = 1 - \frac{\alpha}{r^{2n}} \,, \qquad \omega = \frac{\alpha g}{W r^{2n}} \,,$$

We have made the specific choice for the sign of $\sqrt{\mu\nu} \to \sqrt{\nu^2 g^2} = \nu g = -\alpha g$ when sending $\mu = \nu g$ negative, and with this choice, the Killing vector admitting the spinorial square root is again given by (2.10) with the plus sign choice, for which we now define

$$K = \frac{\partial}{\partial t} + g \frac{\partial}{\partial \psi} \,. \tag{2.14}$$

The mass and angular momentum are given by

$$M = -\frac{ng^2 \alpha}{8\pi} \mathcal{A}_{2n-1} \,, \qquad J = \frac{g\alpha}{8\pi} \mathcal{A}_{2n-1} \tag{2.15}$$

(recall that we have made the sign choice that $\sqrt{\mu\nu} \to -\alpha g$ when sending μ and ν negative). The metric has a power-law curvature singularity at $r = 0$, but there is a Euclidean Killing horizon at $r = r_0 > 0$ for which $f(r_0) = 0$. Thus we have

$$\alpha = \left(1 + g^2 r_0^2 \right) r_0^{2n} \,. \tag{2.16}$$

The absence of a conical singularity at $r = r_0$ requires that the degenerate Killing vector

$$\ell = \frac{1}{n + (n+1)g^2 r_0^2} \left(g r_0^2 \frac{\partial}{\partial t} + \left(1 + g^2 r_0^2 \right) \frac{\partial}{\partial \psi} \right) \,, \tag{2.17}$$

must generate a 2π period. As we shall discuss later, this implies that the t coordinate must be periodically identified. Note that we have scaled the Killing vector so that the corresponding Euclidean surface gravity is precisely unity.

Defining a radius $r_* \equiv \alpha^{\frac{1}{2n}}$, we see that $g_{\psi\psi} < 0$ in the region

$$r_0 < r < r_*, \qquad (2.18)$$

and thus ψ is the time coordinate in this region. (The VLS is located at $r = r_*$ where $g_{\psi\psi} = 0$.) Since ψ is periodic, with period ψ as stated earlier, it follows that there are closed timelike curves in the region defined by (2.18). This situation is commonly described as a time machine (see [8] for a more detailed discussion).

Finally, it is worth pointing out that in the case $\mu = g^2 \nu$, for which there is a Killing spinor, the corresponding metric (2.13) can be expressed, after we make a coordinate change $\psi \to \psi - g\,t$, as a time bundle over a $D = 2n$ dimensional space:

$$ds^2_{2n+1} = -\Big(dt + gr^2(d\psi + A)\Big)^2 + \frac{dr^2}{f} + r^2\Big(f\,(d\psi + A)^2 + d\Sigma^2_{n-1}\Big). \qquad (2.19)$$

The length of the time fibre is constant, and the base is a $2n$-dimensional Einstein-Kähler metric. In fact this is Lorentzian version of the situation in an Einstein-Sasaki space, which can be written, at least locally, as a constant-length circle fibration over an Einstein-Kähler base space.

2.3 Killing spinors

Here, we construct the Killing spinor η in the $(2n+1)$-dimensional BPS time machine with equal angular momenta, whose metric is given by (2.13), obeying

$$\nabla_a\eta + \frac{1}{2}g\Gamma_a\eta = 0. \qquad (2.20)$$

We shall make use of the fact that $\mathbb{CP}^{n-1}$ admits a gauge-covariantly constant spinor ξ satisfying

$$\widetilde{D}_i\,\xi + \frac{\mathrm{i}\,n}{2}\,A_i\,\xi = 0, \qquad (2.21)$$

where $\widetilde{D} = \tilde{d} + \frac{1}{4}\tilde{\omega}^{ij}\,\widetilde{\Gamma}_{ij}$ is the spinor-covariant exterior derivative and $\widetilde{D} = \tilde{e}^i\widetilde{D}_i$, with $\widetilde{\Gamma}_i$ being the Dirac matrices and $\tilde{e}^i$ denoting a vielbein basis for $\mathbb{CP}^{n-1}$.[3] With an appropriate choice of basis for the Dirac matrices one can easily establish that ξ obeys

$$J^{ij}\widetilde{\Gamma}_{ij}\,\xi = -2\mathrm{i}\,(n-1)\xi, \qquad \widetilde{\Gamma}_*\,\xi = \xi, \qquad (2.22)$$

where $\widetilde{\Gamma}_*$ denotes the chirality operator on $\mathbb{CP}^{n-1}$. (We give an iterative construction of the gauge-covariantly constant spinor ξ in appendix A.)

We introduce the vielbein basis e^a for (2.13), with

$$e^0 = u\,dt, \qquad e^1 = \frac{dr}{v}, \qquad e^2 = h(d\psi + A + \omega dt), \qquad e^i = r\tilde{e}^i, \qquad (2.23)$$

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[3]We use $\tilde{d}$ to denote the standard exterior derivative in the $(2n-2)$-dimensional $\mathbb{CP}^{n-1}$ space in order to distinguish it from d which is the exterior derivative in the full $(2n+1)$-dimensional space-time.

where

$$u = \sqrt{\frac{f}{W}}\,, \qquad v = \sqrt{f}\,, \qquad h = r\sqrt{W}\,. \tag{2.24}$$

The inverse vielbein E_a is given by

$$E_0 = \frac{1}{u}\left(\frac{\partial}{\partial t} - \omega\,\frac{\partial}{\partial \psi}\right), \quad E_1 = v\frac{\partial}{\partial r}\,, \quad E_2 = \frac{1}{h}\,\frac{\partial}{\partial \psi}\,, \quad E_i = \frac{1}{r}\left(\widetilde{E}_i - A_i\,\frac{\partial}{\partial \psi}\right), \tag{2.25}$$

where $\widetilde{E}_i$ is the inverse vielbein for $\mathbb{CP}^{n-1}$. The torsion-free spin connection ω^{ab} for the vielbein (2.23) is easily calculated, leading to the spinor-covariant exterior derivative $D = d + \frac{1}{4}\omega^{ab}\,\Gamma_{ab}$ given by

$$D = d + e^0\left(\frac{u'v}{2u}\,\Gamma_{01} - \frac{h\omega'v}{4u}\,\Gamma_{12}\right) - e^1\,\frac{h\omega'v}{4u}\,\Gamma_{02} - e^2\left(\frac{h'v}{2h}\,\Gamma_{12} + \frac{h\omega'v}{4u}\Gamma_{01} + \frac{h}{4r^2}\,J^{ij}\,\Gamma_{ij}\right)$$
$$-e^i\left(\frac{v}{2r}\,\Gamma_{1i} + \frac{h}{2r^2}\,J_i{}^j\,\Gamma_{2j}\right) + \frac{1}{4}\tilde{\omega}^{ij}\Gamma_{ij}\,. \tag{2.26}$$

Writing the $(2n+1)$-dimensional Lorentz indices as $a = (\alpha, i)$ with $\alpha = 0, 1, 2$, we may decompose the $(2n+1)$-dimensional Dirac matrices in the form

$$\Gamma_\alpha = \gamma_\alpha \otimes \widetilde{\Gamma}_*\,, \qquad \Gamma_i = \mathbb{1} \otimes \widetilde{\Gamma}_i\,, \tag{2.27}$$

where γ_α are 2×2 Dirac matrices, which we take to be

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.28}$$

It then follows that the spinor-covariant exterior derivative (2.26) is given by

$$D = \hat{d} \otimes \mathbb{1} + \mathbb{1} \otimes \widetilde{D} + e^0\left(\frac{u'v}{2u}\,\gamma_{01} - \frac{h\omega'v}{4u}\,\gamma_{12}\right) \otimes \mathbb{1} - e^1\,\frac{h\omega'v}{4u}\,\gamma_{02} \otimes \mathbb{1}$$
$$-e^2\left(\frac{h'v}{2h}\,\gamma_{12} \otimes \mathbb{1} + \frac{h\omega'v}{4u}\,\gamma_{01} \otimes \mathbb{1} + \frac{h}{4r^2}\,J^{ij}\,\mathbb{1} \otimes \widetilde{\Gamma}_{ij}\right)$$
$$-e^i\left(\frac{v}{2r}\,\gamma_1 \otimes \widetilde{\Gamma}_*\widetilde{\Gamma}_i + \frac{h}{2r^2}\,J_i{}^j\,\gamma_2 \otimes \widetilde{\Gamma}_*\widetilde{\Gamma}_j\right), \tag{2.29}$$

where $\widetilde{D}$ is the spinor-covariant exterior derivative on $\mathbb{CP}^{n-1}$ that we introduced earlier, and $\hat{d}$ denotes the standard exterior derivative in the three directions orthogonal to $\mathbb{CP}^{n-1}$, i.e. $d = \hat{d} + \tilde{d} = e^a\,E_a$ with

$$\hat{d} = e^\alpha\,E_\alpha = e^0\left(\frac{1}{u}\,\frac{\partial}{\partial t} - \frac{\omega}{u}\,\frac{\partial}{\partial \psi}\right) + e^1\,v\frac{\partial}{\partial r} + e^2\,\frac{1}{h}\,\frac{\partial}{\partial \psi}\,, \tag{2.30}$$

$$\tilde{d} = e^i\,E_i = e^i\,\frac{1}{r}\left(\widetilde{E}_i - A_i\,\frac{\partial}{\partial \psi}\right). \tag{2.31}$$

With these preliminaries, it is now straightforward to obtain the equations for the Killing spinor η in the $(2n+1)$-dimensional spacetime, satisfying (2.20). It takes the form

$$\eta = \epsilon \otimes \xi\,, \tag{2.32}$$

where ξ is the gauge-covariantly constant spinor on $\mathbb{CP}^{n-1}$ that we introduced earlier. After further straightforward computations, we find that the 2-component spinor ϵ is given by

$$\epsilon = \frac{1}{\sqrt{2}}\, W^{-\frac{1}{4}}\, \exp\left(-\frac{1}{2}\mathrm{i}\, gt - \frac{1}{2}\mathrm{i}\, n\psi\right) \begin{pmatrix} \left(gr + \mathrm{i}\,\sqrt{W}\right)^{\frac{1}{2}} \\ -\left(gr - \mathrm{i}\,\sqrt{W}\right)^{\frac{1}{2}} \end{pmatrix}. \tag{2.33}$$

We may now straightforwardly verify that the Killing vector (2.14) may be written in terms of the Killing spinor η as

$$K^a = \bar{\eta}\Gamma^a \eta\,. \tag{2.34}$$

2.4 Global considerations and discretisation of parameters

The discussion in this section is closely analogous to that in [10, 11], where the global structure of Einstein-Sasaki spaces was studied. We begin by defining the Killing vectors

$$\ell_0 = \frac{1}{g}\frac{\partial}{\partial t}\,, \qquad \ell_1 = \frac{\partial}{\partial \psi}\,, \tag{2.35}$$

where we have included a $1/g$ in the definition of ℓ_0 in order to make it dimensionless. ℓ_1 generates a 2π period. It follows from (2.17) that

$$g^2 r_0^2\, \ell_0 = \left[n + (n+1)\, g^2 r_0^2\right]\ell - \left(1 + g^2 r_0^2\right)\ell_1\,. \tag{2.36}$$

Since ℓ and ℓ_1 both generate periodic translations by 2π, the ratio of their coefficients must be rational, since otherwise one there would be identifications in the time direction, generated by ℓ_0, of arbitrarily close points in the spacetime manifold. Hence $g^2 r_0^2$ must be rational, which we shall write as $g^2 r_0^2 = p/\tilde{q}$, for coprime integers p and $\tilde{q}$. Consequently (2.36) can be written as

$$p\ell_0 = q\,\ell + q_1\,\ell_1\,, \tag{2.37}$$

where the integers q and q_1 are given by

$$q = (n+1)p + n\tilde{q}\,, \qquad q_1 = -(p + \tilde{q})\,. \tag{2.38}$$

Note that the set of integers $\{p, q, q_1\}$ are necessarily coprime, since p and $\tilde{q}$ are coprime.

It is straightforward also to see from (2.38) that since p and $\tilde{q}$ are coprime, it must also be the case that q and q_1 are coprime. It then follows from (2.37) that ℓ_0 generates a smallest translation period of 2π, and hence that gt has period 2π. Interestingly, this is precisely the same as the period of the time coordinate in a global AdS with radius g^{-1}. Thus the periodicity of t that is required in order to eliminate the conical singularity at the Euclidean Killing horizon at $r = r_0$ is exactly the same as the time periodicity of the embedding AdS spacetime itself. Consequently, the Killing spinor (2.33) is consistent with the global structure of the time machine spacetime, and hence the solution would be supersymmetric if it can be embedded in a gauged supergravity.

The fact that $g^2 r_0^2 = p/\tilde{q}$ is rational implies that the possible masses (and angular momenta) for the BPS time-machine spacetimes are discretised. From (2.15) and (2.16), we have

$$M = -nJ = -\frac{n\mathcal{A}_{n-1}}{8\pi\, g^{2n-2}}\left(1 + \frac{p}{\tilde{q}}\right)\left(\frac{p}{\tilde{q}}\right)^n. \tag{2.39}$$

3 Equal angular momenta in $D = 2n$

The Kerr-AdS metrics in even $D = 2n$ dimensions with all equal angular momenta can be expressed as [27]

$$ds^2 = -\frac{\Delta_\theta(1 + g^2 r^2)}{\Xi} dt^2 + \frac{U dr^2}{V - 2m} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{r^2 + a^2}{\Xi} \sin^2\theta[(d\psi + A)^2 + d\Sigma_{n-2}^2]$$
$$+\frac{2m}{U\Xi^2}\left[\Delta_\theta\, dt - a\sin^2\theta(d\psi + A)\right]^2, \tag{3.1}$$

where

$$U = \frac{\rho^2 \left(r^2 + a^2\right)^{n-2}}{r}, \qquad V = \frac{1}{r}\left(1 + g^2 r^2\right)\left(r^2 + a^2\right)^{n-1}$$
$$\Delta_\theta = 1 - a^2 g^2 \cos^2\theta, \qquad \rho^2 = r^2 + a^2\cos^2\theta, \qquad \Xi = 1 - a^2 g^2. \tag{3.2}$$

The mass and the (equal) angular momenta are [27]

$$M = \frac{n\, m\, \mathcal{A}_{D-2}}{4\pi\Xi^n}, \qquad J = \frac{ma\mathcal{A}_{D-2}}{4\pi\Xi^n}. \tag{3.3}$$

The BPS limit $M = ngJ$ implies that $ag = 1$ and hence $\Xi \to 0$. This requires that

$$m \sim \Xi^n \to 0, \tag{3.4}$$

so that M and J remain finite. In this limit, for the metric to be real and the coordinate θ to be spacelike, we need make the coordinate transformation

$$\theta \to \frac{1}{2}\pi - \mathrm{i}\theta \qquad r^2 + a^2 \to \Xi r^2 \to 0. \tag{3.5}$$

After some algebra we end up

$$ds_{2n}^2 = g^{-2} d\theta^2 + \cosh^2\theta\, ds_{2n-1}^2. \tag{3.6}$$

where ds_{2n-1}^2 is the time machine metric obtained earlier for odd dimensions with all equal angular momenta. In deriving this, we need to further redefine the scaled m as

$$m \to \mathrm{i}\frac{m}{g}. \tag{3.7}$$

The origin of this is that in the $(V - 2m)$ factor, there is a term of $2mr$.

4 General non-equal angular momenta

In this section, we consider the BPS limit of general Kerr-AdS black holes with general angular momenta.

4.1 $D = 5$

The Kerr-AdS metric in five dimensions was constructed in [5], given by

$$
ds_5^2 = -\frac{\Delta_r}{\rho^2} \left[dt - \frac{a \sin^2\theta}{\Xi_a} d\phi_1 - \frac{b \cos^2\theta}{\Xi_b} d\phi_2 \right]^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2} \left[a\,dt - \frac{r^2 + a^2}{\Xi_a} d\phi_1 \right]^2
$$
$$
+ \frac{\Delta_\theta \cos^2\theta}{\rho^2} \left[b\,dt - \frac{r^2 + b^2}{\Xi_b} d\phi_2 \right]^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta}
$$
$$
+ \frac{1 + g^2 r^2}{r^2 \rho^2} \left[a\,b\,dt - \frac{b\left(r^2 + a^2\right)\sin^2\theta}{\Xi_a} d\phi_1 - \frac{a\left(r^2 + b^2\right)\cos^2\theta}{\Xi_b} d\phi_2 \right]^2 , \qquad (4.1)
$$

where

$$
\Delta_r = \frac{1}{r^2}\left(r^2 + a^2\right)\left(r^2 + b^2\right)\left(1 + g^2 r^2\right) - 2m\,, \quad \Delta_\theta = 1 - a^2 g^2 \cos^2\theta - b^2 g^2 \sin^2\theta\,,
$$
$$
\rho^2 = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta\,, \qquad\qquad \Xi_a = 1 - a^2 g^2\,, \qquad \Xi_b \equiv 1 - b^2 g^2\,. \quad (4.2)
$$

The metric satisfies $R_{\mu\nu} = -4g^2 g_{\mu\nu}$. The mass and angular momenta are [27]:

$$
M = \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a \Xi_b)}{4\Xi_a^2 \Xi_b^2}\,, \qquad J_a = \frac{\pi\,m\,a}{2\Xi_a^2 \Xi_b}\,, \qquad J_b = \frac{\pi\,m\,b}{2\Xi_a \Xi_b^2}\,, \qquad (4.3)
$$

And Riemann tensor squared is

$$
\text{Riem}^2 = 40g^4 + \frac{96m^2\left(3\rho^2 - 4r^2\right)\left(\rho^2 - 4r^2\right)}{\rho^{12}}\,. \qquad (4.4)
$$

We can take the BPS limit by setting

$$
a = \frac{1}{g}\left(1 - \frac{1}{2}\alpha^2 g^2 \epsilon\right)\,, \qquad\qquad b = \frac{1}{g}\left(1 - \frac{1}{2}\beta^2 g^2 \epsilon\right)\,,
$$
$$
r^2 = -\frac{1}{g^2}\left(1 - \tilde{r}^2 g^2 \epsilon\right)\,, \qquad\qquad m = g^2 \tilde{m}\epsilon^3\,, \qquad (4.5)
$$

and sending $\epsilon \to 0$. The metric becomes

$$
ds_5^2 = -\left[dt + \frac{\left(\alpha^2 - \tilde{r}^2\right)\sin^2\theta}{\alpha^2 g} d\phi_1 + \frac{\left(\beta^2 - \tilde{r}^2\right)\cos^2\theta}{\beta^2 g} d\phi_2 \right]^2 + \frac{\tilde{\rho}^2}{\tilde{\Delta}_\theta} d\theta^2 + \frac{\tilde{\rho}^2}{\tilde{\Delta}_r} d\tilde{r}^2
$$
$$
+ \frac{\tilde{\Delta}_r \tilde{r}^2}{\tilde{\rho}^2}\left(\frac{\sin^2\theta}{\alpha^2 g^2} d\phi_1 + \frac{\cos^2\theta}{\beta^2 g^2} d\phi_2\right)^2 + \frac{\tilde{\Delta}_\theta \sin^2\theta \cos^2\theta}{\tilde{\rho}^2}\left(\frac{\alpha^2 - \tilde{r}^2}{\alpha^2 g^2} d\phi_1 - \frac{\beta^2 - \tilde{r}^2}{\beta^2 g^2} d\phi_2\right)^2 ,
$$
$$
\qquad (4.6)
$$

where

$$
\tilde{\Delta}_r = \frac{g^2 \tilde{r}^2 \left(\alpha^2 - \tilde{r}^2\right)\left(\beta^2 - \tilde{r}^2\right) + 2\tilde{m}}{\tilde{r}^2}\,,
$$
$$
\tilde{\Delta}_\theta = g^2\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right)\,,
$$
$$
\tilde{\rho}^2 = \tilde{r}^2 - \alpha^2 \cos^2\theta - \beta^2 \sin^2\theta\,, \qquad (4.7)
$$

(An analogous scaling procedure was used for five-dimensional Kerr-AdS with equal angular momenta in [9].) The metric is a constant time bundle over a four-dimensional Einstein-Kähler space. The mass and angular momenta become

$$
\tilde{M} = \frac{\pi\,\tilde{m}\left(\alpha^2 + \beta^2\right)}{2g^4 \alpha^4 \beta^4}\,, \qquad \tilde{J}_a = \frac{\pi\,\tilde{m}}{2g^5 \alpha^4 \beta^2}\,, \qquad \tilde{J}_b = \frac{\pi\,\tilde{m}}{2g^5 \alpha^2 \beta^4}\,, \qquad (4.8)
$$

satisfying the BPS condition

$$\tilde{M} = g\tilde{J}_a + g\tilde{J}_b \, . \tag{4.9}$$

The Riemann tensor squared is

$$\text{Riem}^2 = 40g^4 + \frac{1536\tilde{m}^2}{\tilde{\rho}^{12}} \, . \tag{4.10}$$

The metric has a power-law curvature singularity at $\tilde{\rho} = 0$. For positive $\tilde{m}$, the singularity is naked. However, when $\tilde{m}$ is negative, there exist a Euclidean Killing horizon at $r = r_0$ where $\tilde{\Delta}_r(r_0) = 0$. The absence of the conic singularity associated with the degenerate cycles at $\tilde{r} = r_0$, $\theta = 0$ and $\theta = \pi/2$ requires that the Killing vectors

$$
\begin{aligned}
\theta = 0: && \ell_1 &= \frac{\partial}{\partial \phi_1} \, , \\
\theta = \frac{\pi}{2}: && \ell_2 &= \frac{\partial}{\partial \phi_2} \, , \\
\tilde{r} = r_0: && \ell &= \frac{1}{\kappa}\left(\frac{\partial}{\partial t} + \frac{g\alpha^2}{r_0^2 - \alpha^2}\frac{\partial}{\partial \phi_1} + \frac{g\beta^2}{r_0^2 - \beta^2}\frac{\partial}{\partial \phi_2} \right) \, ,
\end{aligned}
\tag{4.11}
$$

must all generate 2π period. Here the Euclidean surface gravity κ on the Killing horizon is

$$\kappa = \frac{g\left(3r_0^4 - 2\left(\alpha^2 + \beta^2\right)r_0^2 + \alpha^2\beta^2\right)}{\left(\alpha^2 - r_0^2\right)\left(\beta^2 - r_0^2\right)} \, . \tag{4.12}$$

It is worth pointing out that the metric (4.7) is written in the asymptotically rotating frame. We can make a coordinate transformation $\phi_i \to \phi_i + gt$ such that the metric becomes non-rotating asymptotically. This implies that

$$\ell \to \ell = \frac{1}{\kappa}\left(\frac{\partial}{\partial t} + \frac{gr_0^2}{r_0^2 - \alpha^2}\frac{\partial}{\partial \phi_1} + \frac{gr_0^2}{r_0^2 - \beta^2}\frac{\partial}{\partial \phi_2} \right) \, . \tag{4.13}$$

Defining $\ell_0 = g^{-1}\partial_t$, we see that the Killing vectors must satisfy the linear relation

$$p\ell_0 = q\ell + q_1\ell_1 + q_2\ell_2 \, , \tag{4.14}$$

with

$$p = q + q_1 + q_2 \, . \tag{4.15}$$

Consistency requires that (p, q, q_1, q_2) are coprime integers, and consequently $\Delta t = 2\pi$. The integration constants can expressed in terms of two rational numbers $(p/q_1, p/q_2)$:

$$\alpha^2 = \left(1 + \frac{p}{q_1}\right)r_0^2 \, , \qquad \beta^2 = \left(1 + \frac{p}{q_2}\right)r_0^2 \, . \tag{4.16}$$

The mass and angular momenta are completely discretised, given by

$$
\begin{aligned}
M &= -\frac{\pi p^2 \left(pq_1 + pq_2 + 2q_2q_1\right)}{4g^2 \left(p + q_1\right)^2 \left(p + q_2\right)^2} \, , \\
J_a &= -\frac{\pi p^2 q_1}{4g^3 \left(p + q_1\right)^2 \left(p + q_2\right)} \, , && J_b = -\frac{\pi p^2 q_2}{4g^3 \left(p + q_1\right)\left(p + q_2\right)^2} \, .
\end{aligned}
\tag{4.17}
$$

4.2 $D = 2n + 1$

The Kerr-AdS metric in $D = 2n + 1$ dimensions is given by [6, 7]

$$ds_D^2 = -W\left(1 + g^2 r^2\right) dt^2 + \frac{2m}{U}\left(W dt - \sum_{i=1}^n \frac{a_i \mu_i^2}{\Xi_i} d\varphi_i\right)^2 + \sum_{i=1}^n \frac{r^2 + a_i^2}{\Xi_i}\mu_i^2 d\varphi_i^2$$
$$+ \frac{U}{V - 2m} dr^2 + \sum_{i=1}^n \frac{r^2 + a_i^2}{\Xi_i} d\mu_i^2 - \frac{g^2}{W\left(1 + g^2 r^2\right)}\left(\sum_{i=1}^n \frac{r^2 + a_i^2}{\Xi_i}\mu_i d\mu_i\right)^2, \quad (4.18)$$

where

$$W \equiv \sum_{i=1}^n \frac{\mu_i^2}{\Xi_i}, \qquad\qquad U \equiv \sum_{i=1}^n \frac{m u_i^2}{r^2 + a_i^2}\prod_{j=1}^n \left(r^2 + a_j^2\right),$$

$$V \equiv r^{-2}\left(1 + g^2 r^2\right)\prod_{j=1}^n \left(r^2 + a_j^2\right), \qquad \Xi_i \equiv 1 - a_i^2 g^2, \qquad \sum_{i=1}^n \mu_i^2 = 1. \quad (4.19)$$

They satisfy $R_{\mu\nu} = -(D-1)g^2 g_{\mu\nu}$. The mass and angular momenta are [27]

$$M = \frac{m \mathcal{A}_{D-2}}{4\pi\left(\prod_j \Xi_j\right)}\left(\sum_{i=1}^n \frac{1}{\Xi_i} - \frac{1}{2}\right), \qquad J_i = \frac{m a_i \mathcal{A}_{D-2}}{4\pi\Xi_i\left(\prod_j \Xi_j\right)}, \quad (4.20)$$

The metric is non-rotating at asymptotic infinity. We take the following transformation,

$$\psi_i = \varphi_i - a_i g^2 t, \quad (4.21)$$

so that $g_{tt} \to -1$ at asymptotic infinity. We now take the BPS limit by setting

$$a_i = \frac{1}{g}\left(1 - \frac{1}{2}b_i^2 g^2 \epsilon\right), \qquad r^2 = -\frac{1}{g^2}\left(1 - y^2 g^2 \epsilon\right), \qquad m = g^2 \tilde{m}\epsilon^{n+1}, \quad (4.22)$$

and sending $\epsilon \to 0$. The metric becomes

$$ds_d^2 = -\left(dt - \frac{1}{g}\sum_{i=1}^n \frac{y^2 - b_i^2}{b_i^2}\mu_i^2 d\psi_i\right)^2 + \frac{\Delta_\psi y^2 dy^2}{\Delta_y} + \frac{\tilde{m}}{g^4 \Delta_\psi}\left(\sum_{i=1}^n \frac{\mu_i^2}{b_i^2} d\psi_i\right)^2$$
$$+ \frac{1}{g^2}\left[\left(\sum_{i=1}^n \frac{y^2 - b_i^2}{b_i^2}\mu_i^2 d\psi_i\right)^2 + \sum_{i=1}^n \frac{y^2 - b_i^2}{b_i^2}\mu_i^2 d\psi_i^2\right]$$
$$+ \sum_{i=1}^n \frac{y^2 - b_i^2}{b_i^2 g^2} d\mu_i^2 - \frac{1}{g^2 \Delta_\mu}\left(\sum_{i=1}^n \frac{y^2 - b_i^2}{b_i^2}\mu_i d\mu_i\right)^2, \quad (4.23)$$

where

$$\Delta_\mu = y^2 \sum_{i=1}^n \frac{\mu_i^2}{b_i^2}, \qquad\qquad \Delta_\psi = \left(\sum_{i=1}^n \frac{\mu_i^2}{y^2 - b_i^2}\right)\prod_{j=1}^n \left(y^2 - b_j^2\right),$$

$$\Delta_y = \tilde{m} + g^2 y^2 \prod_{i=1}^n \left(y^2 - b_i^2\right). \quad (4.24)$$

The metric is again constant time bundle over $D = 2n$ space, indicating that the solution admits a Killing spinor. The mass and angular momenta become

$$\tilde{M} = \frac{\tilde{m}\mathcal{A}_{D-2}}{4\pi g^{2n}\left(\prod_j b_j^2\right)}\sum_{i=1}^n \frac{1}{b_i^2}\,, \qquad \tilde{J}_i = \frac{\tilde{m}\mathcal{A}_{D-2}}{4\pi g^{2n+1}\left(\prod_j b_j^2\right)}\cdot\frac{1}{b_i^2}\,.$$ (4.25)

satisfying the BPS condition

$$\tilde{M} = g\sum_{i=1}^n \tilde{J}_i\,.$$ (4.26)

The metric has a power-law curvature singularity at $\Delta_\psi = 0$. The singularity is naked for positive $\tilde{m}$, but outside the Euclidean Killing horizon y_0 with $\Delta_y = 0$. The Killing vectors associated with the degenerated null surfaces are

$$\ell = \frac{1}{\kappa}\left(\frac{\partial}{\partial t} + \sum_{i=1}^n \frac{gb_i^2}{y_0^2 - b_i^2}\frac{\partial}{\partial\psi_i}\right)\,, \qquad\qquad (y = y_0)$$

$$\ell_k = \frac{\partial}{\partial\psi_k}\,, \qquad\qquad (\mu_k = 0\,,\ k = 1\cdots n)\,.$$ (4.27)

Here the surface gravity κ on the horizon is

$$\kappa = g\left(1 + \sum_{i=1}^n \frac{y_0^2}{y_0^2 - b_i^2}\right)\,.$$ (4.28)

Making a coordinate transformation $\phi_i \to \phi_i + gt$, we find that the Killing vector ℓ becomes

$$\ell \to \ell = \frac{1}{\kappa}\left(\frac{\partial}{\partial t} + \sum_{i=1}^n \frac{gy_0^2}{y_0^2 - b_i^2}\frac{\partial}{\partial\psi_i}\right)\,.$$ (4.29)

It follows that the Killing vectors satisfy

$$p\ell_0 = q\ell + \sum_{i=1}^n q_i\ell_i\,, \qquad \text{with} \qquad p = q + \sum_{i=1}^n q_i\,.$$ (4.30)

As in the previous $D = 5$ case, consistency requires that $\Delta t = 2\pi$.

We can now expressed the n integration constant b_i as

$$b_i^2 = \left(1 + \frac{p}{q_i}\right)r_0^2\,.$$ (4.31)

The mass and angular momenta are completely discretised, given by

$$M = -\frac{\mathcal{A}_{2n-1}}{4\pi g^{2n-2}}\left(\prod_i \frac{p}{p + q_i}\right)\sum_i \frac{q_i}{p + q_i}\,,$$

$$J_i = -\frac{\mathcal{A}_{2n-1}}{4\pi g^{2n-1}}\left(\prod_j \frac{p}{p + q_j}\right)\frac{q_i}{p + q_i}\,.$$ (4.32)

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4.3 $D = 2n + 2$

The Kerr-AdS metric in $D = 2n + 2$ dimensions is given by [6, 7]

$$ds_D^2 = -W \left(1 + g^2 r^2 \right) dt^2 + \frac{2m}{U} \left(W dt - \sum_{i=1}^n \frac{a_i \mu_i^2}{\Xi_i} d\varphi_i \right)^2 + \sum_{i=1}^n \frac{r^2 + a_i^2}{\Xi_i} \mu_i^2 d\varphi_i^2$$
$$+ \frac{U}{V - 2m} dr^2 + \sum_{i=0}^n \frac{r^2 + a_i^2}{\Xi_i} d\mu_i^2 - \frac{g^2}{W \left(1 + g^2 r^2 \right)} \left(\sum_{i=0}^n \frac{r^2 + a_i^2}{\Xi_i} \mu_i d\mu_i \right)^2 , \quad (4.33)$$

where $a_0 = 0$ and

$$W \equiv \sum_{i=0}^n \frac{\mu_i^2}{\Xi_i} , \qquad U \equiv \sum_{i=0}^n \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^n \left(r^2 + a_j^2 \right) ,$$
$$V \equiv r^{-2} \left(1 + g^2 r^2 \right) \prod_{j=1}^n \left(r^2 + a_j^2 \right) , \qquad \Xi_i \equiv 1 - a_i^2 g^2 , \qquad \sum_{i=0}^n \mu_i^2 = 1 . \quad (4.34)$$

They satisfy $R_{\mu\nu} = -(D-1)g^2 g_{\mu\nu}$. The mass and angular momenta are

$$M = \frac{m \mathcal{A}_{D-2}}{4\pi \left(\prod_j \Xi_j \right)} \sum_{i=1}^n \frac{1}{\Xi_i} , \qquad J_i = \frac{m a_i \mathcal{A}_{D-2}}{4\pi \Xi_i \left(\prod_j \Xi_j \right)} . \quad (4.35)$$

As in the odd-dimensional case, we first make the coordinate transformation

$$\psi_i = \varphi_i - a_i g^2 t . \quad (4.36)$$

The BPS condition $M = g \sum_i J_i$ can be satisfied by setting

$$a_i = \frac{1}{g} \left(1 - \frac{1}{2} b_i^2 g^2 \epsilon \right) , \qquad r^2 = -\frac{1}{g^2} \left(1 - y^2 g^2 \epsilon \right) , \qquad m = g^2 \tilde{m} \epsilon^{n+1} , \quad (4.37)$$

and sending $\epsilon \to 0$. We then make the further transformations

$$\theta = i \tilde{\theta} , \qquad \mu_0 = \sin \theta , \qquad \mu_i = \cos \theta \, \tilde{\mu}_i , \, (i = 1, \cdots, n) , \quad (4.38)$$

with $\sum \tilde{\mu}_i^2 = 1$. The $(2n+2)$-dimensional metric can now be expressed as a foliation of a $(2n+1)$-dimensional BPS time machine

$$ds_{2n+2}^2 = g^{-2} d\tilde{\theta}^2 + \cosh^2 \tilde{\theta} \, ds_{2n+1}^2 . \quad (4.39)$$

So far, we have considered the general class of BPS Kerr-AdS time machines in both odd and even dimensions, with generic but non-vanishing angular momenta. When some subset of the angular momenta vanish, the BPS limits also exist. For a general Kerr-AdS black hole in D dimensions, if there are p non-vanishing angular momenta, the resulting BPS time machine metric takes the form

$$ds_D^2 = g^{-2} d\tilde{\theta}^2 + \cosh^2 \tilde{\theta} \, ds_{2p+1}^2 + \sinh^2 \tilde{\theta} \, d\Omega_{D-2p-2}^2 , \quad (4.40)$$

where ds_{2p+1}^2 is the metric for the BPS time machine in $(2p+1)$ dimensions.

5 Further comments in $D = 3$

The solutions we gave in section 2 specialise to $D = 3$ dimensions if we set $n = 1$. It is instructive to compare this with the BTZ black hole solution [23] since they are, of course, necessarily locally equivalent, both being locally just AdS$_3$.

The BTZ black hole is given by the metric [23]

$$ds^2 = -N^2 dt^2 + \frac{d\rho^2}{N^2} + \rho^2 \left(d\phi - \frac{J}{2\rho^2} dt \right)^2 ,$$
$$N^2 = -M + g^2 \rho^2 + \frac{J^2}{4\rho^2} , \tag{5.1}$$

and the mass and angular momentum are

$$M_{\text{BTZ}} = g^2 \left(\rho_+^2 + \rho_-^2 \right) , \qquad J_{\text{BTZ}} = 2g\rho_+\rho_- , \tag{5.2}$$

where ρ_+ and ρ_- are the radii of the outer and inner horizons. The BPS limit $M_{\text{BTZ}} = gJ_{\text{BTZ}}$ implies that $\rho_+ = \rho_- = \rho_0$, and then

$$ds^2 = -N^2 dt^2 + \frac{d\rho^2}{N^2} + \rho^2 \left(d\phi - \frac{g\rho_0^2}{\rho^2} dt \right)^2 ,$$
$$N^2 = \frac{g^2 \left(\rho^2 - \rho_0^2 \right)^2}{\rho^2} . \tag{5.3}$$

The rotating $D = 3$ black hole following from (2.5) by setting $n = 1$ is

$$ds_3^2 = \frac{dr^2}{f} - \frac{f}{W} dt^2 + r^2 W \left(d\phi - \frac{\sqrt{\mu\nu}}{r^2 + \nu} dt \right)^2 ,$$
$$f = \left(1 + g^2 r^2 \right) W - \mu , \qquad\qquad W = 1 + \frac{\nu}{r^2} . \tag{5.4}$$

Making the coordinate redefinition

$$r^2 = \rho^2 - \nu , \tag{5.5}$$

we see that (5.4) becomes

$$ds^2 = -h dt^2 + \frac{d\rho^2}{h} + \rho^2 \left(d\phi - \frac{\sqrt{\mu\nu}}{\rho^2} dt \right)^2 ,$$
$$h = g^2 \rho^2 + 1 - \left(g^2 \nu + \mu \right) + \frac{\mu\nu}{\rho^2} . \tag{5.6}$$

According to our general formulae (2.6), the mass and angular momentum are given by

$$M = \mu + g^2 \nu , \qquad J = 2\sqrt{\mu\nu} . \tag{5.7}$$

Comparing (5.6) with the BTZ black hole metric (5.1), we see that they match completely, with

$$M_{\text{BTZ}} = M - 1 , \qquad J_{\text{BTZ}} = J . \tag{5.8}$$

The above relations between the mass and angular momentum however give very different physical interpretations of the seemingly equivalent solution. In particular, they lead to very different BPS conditions

$$M = gJ, \qquad \text{or} \qquad M_{\text{BTZ}} = gJ_{\text{BTZ}} . \tag{5.9}$$

At the first sight, it would seem surprising if both conditions were to lead to well-defined Killing spinors.

Before solving the Killing spinor equations, we note that the vacuum for the BTZ metric with $M_{\text{BTZ}} = 0 = J_{\text{BTZ}}$ is AdS$_3$ in planar coordinates, whilst the vacuum for our metric, defined by $M = 0 = J$, yields AdS$_3$ in global coordinates:

$$M_{\text{BTZ}} = 0 = J_{\text{BTZ}} : \qquad ds^2 = -g^2\rho^2 dt^2 + \frac{d\rho^2}{g^2\rho^2} + \rho^2 d\phi^2 ,$$

$$M = 0 = J : \qquad ds^2 = -\left(g^2\rho^2 + 1\right) dt^2 + \frac{d\rho^2}{g^2\rho^2 + 1} + \rho^2 d\phi^2 . \tag{5.10}$$

To derive the Killing spinors, it is convenient to choose the vielbein basis

$$e^0 = -Ndt , \qquad e^1 = \frac{d\rho}{N} , \qquad e^2 = \rho(d\phi - \Omega dt) , \quad \text{with} \quad \Omega = \frac{J}{2\rho^2} . \tag{5.11}$$

Note that we use $(0, 1, 2)$ to denote tangent indices and (t, ρ, ψ) to denote spacetime indices. The spinor-covariant exterior derivative $D = d + \frac{1}{4}\omega^{ab}\gamma_{ab}$ is

$$D = d \otimes \mathbb{1} + e^0 \left(\frac{N'}{2}\gamma_{01} + \frac{\rho\Omega'}{4}\gamma_{12} \right) + e^1 \frac{\rho\Omega'}{4}\gamma_{02} - e^2 \left(\frac{N}{2\rho}\gamma_{12} - \frac{\rho\Omega'}{4}\gamma_{01} \right) ,$$

$$d = e^0 \left(\frac{1}{N}\frac{\partial}{\partial t} + \frac{\Omega}{N}\frac{\partial}{\partial \phi} \right) + e^1 N\frac{\partial}{\partial \rho} + e^2 \frac{1}{\rho}\frac{\partial}{\partial \phi} , \tag{5.12}$$

where the Dirac matrices are defined in (2.28). We find that the two-component Killing spinor is given by

$$\zeta = e^{\frac{1}{2}\Delta(gt+\phi)} \begin{pmatrix} \zeta_+(\rho) \\ \zeta_-(\rho) \end{pmatrix} . \tag{5.13}$$

where (ζ_+, ζ_-) satisfy the constraints

$$0 = 2\rho \left(J \pm 2\rho\Delta - g\rho \right) \zeta'_\pm + \left(J + 2g\rho^2 \right) \zeta_\pm ,$$

$$\frac{\zeta_+}{\zeta_-} = -\frac{J^2 - 4M\rho^2 + 4g^2\rho^4}{2\rho \left(2\rho\Delta + 2g\rho^2 - J \right)} , \tag{5.14}$$

and the exponent Δ is given by

$$\Delta = \sqrt{M_{\text{BTZ}} - gJ_{\text{BTZ}}} , \qquad \text{or equivalently,} \qquad \Delta = \sqrt{M - gJ - 1} . \tag{5.15}$$

The situation becomes clear now with the explicit Killing spinor solutions. Owing to the fact that the three-dimensional metric is locally AdS$_3$, the Killing spinors exist locally for all mass and charge, regardless whether they satisfy the BPS conditions or not. For the BTZ black holes $M_{\text{BTZ}} > gJ_{\text{BTZ}}$, the local Killing spinor has real exponential dependence

on the ϕ coordinate. However, since ϕ must be periodic in order for the solution to describe a black hole, as opposed to AdS$_3$, the Killing spinor can only be well defined when $M_{\mathrm{BTZ}} = gJ_{\mathrm{BTZ}}$, implying that Δ becomes zero and so the Killing spinor no longer depends on ϕ. Note that for the Killing vector $K = \partial_t + g\partial_\phi$, we have

$$g(K, K) = \Delta^2 \geq 0. \tag{5.16}$$

Thus, the Killing vector associated with the Killing spinor is null for the supersymmetric BTZ black hole, corresponding to $\Delta = 0$.

This is not the only way to achieve the supersymmetry, however. We can instead impose $M = gJ$, corresponding to $M_{\mathrm{BTZ}} - gJ_{\mathrm{BTZ}} = -1$, in which case, we have

$$\Delta = \sqrt{-1} = \mathrm{i}, \qquad g(K, K) = \Delta^2 = -1. \tag{5.17}$$

In this case, the Killing vector is time-like, and the Killing spinor now has periodic dependence on ϕ, with the same period as that in the global AdS$_3$. The resulting metric with negative mass then leads to the BPS time machine.

Killing spinors of BTZ black holes were also studied in [24, 25].

6 Conclusions

In this paper, we studied the global structure of the Kerr-AdS metrics in general dimensions, when the mass and angular momenta satisfy the BPS condition (1.1). In odd dimensions with equal angular momenta, we constructed explicitly the Killing spinors.

For positive mass, the solutions have naked power-law curvature singularities with no horizon to cloak them. For negative mass, the BPS solutions can describe smooth spacetime configurations that are called time machines. These smooth spacetime configurations are purely gravitational and there is no matter energy-momentum tensor source at all. The completeness of the spacetime requires that the asymptotic Lorentzian time coordinate be periodically identified, with precisely the same time period as that of the global AdS spacetime in which the solutions are immersed. Furthermore, the mass and angular momenta become discretised. The Killing spinors are periodic in time, with a period that is consistent with the global structure of the time machines. Thus in cases where they solutions can be embedded in gauged supergravities, they are supersymmetric.

In the AdS/CFT correspondence, the time coordinate in both the global or the planar AdS spacetime is taken to lie on the real line, describing the infinite covering CAdS of AdS in the global case. In this case, the BPS time machines constructed in this paper would all have a conical singularity at the Euclidean Killing horizon. However, if we consider the asymptotic AdS$_D$ as being the strict hyperboloid in $\mathbb{E}^{2,D-1}$, then the time machines described in this paper are precisely consistent with the boundary conditions. The breaking of the time translational $\mathbb{R}$ symmetry in our BPS and the general non-BPS [13] Kerr-AdS time machines is reminiscent of the time crystals proposed by Wilczek [28]. Although it lies beyond the scope of the present paper, it would be interesting to investigate the implications of a periodic global AdS time coordinate within the framework of the AdS/CFT correspondence, and also to see what consequences result from the closed time-like curves associated with the time machine region of the bulk spacetime.

Acknowledgments

A $\mathbb{CP}^n$ and gauge-covariantly constant spinor

Here we make use of the iterative construction of $\mathbb{CP}^n$ in terms of $\mathbb{CP}^{n-1}$ that was obtained in [26], in order to give an explicit iterative construction of the gauge-covariantly constant spinor that we employed in the construction of the Killng spinor in the previous section. As was shown in [26], the Fubini-Study metric $d\Sigma_n^2$ on $\mathbb{CP}^n$ can be written in terms of the Fubini-Study metric $d\Sigma_{n-1}^2$ on $\mathbb{CP}^{n-1}$ as follows:

$$d\Sigma_n^2 = d\chi^2 + \sin^2\chi\,\cos^2\chi\left(d\psi + \widetilde{A}\right)^2 + \sin^2\chi\, d\Sigma_{n-1}^2\,, \tag{A.1}$$

where $\widetilde{J} = \frac{1}{2}d\widetilde{A}$ is the Kähler form of $\mathbb{CP}^{n-1}$. The Kähler form of $\mathbb{CP}^n$ is given by $J = \frac{1}{2}dA$, where

$$A = \sin^2\chi\left(d\psi + \widetilde{A}\right)\,. \tag{A.2}$$

We define the vielbein e^a for $\mathbb{CP}^n$, with[4]

$$e^0 = d\chi\,, \qquad e^1 = \sin\chi\,\cos\chi\left(d\psi + \widetilde{A}\right)\,, \qquad e^i = \sin\chi\,\tilde{e}^i\,, \tag{A.3}$$

where $\tilde{e}^i$ is a vielbein for $\mathbb{CP}^{n-1}$. The inverse vielbein is then given by

$$E_0 = \frac{\partial}{\partial\chi}\,, \qquad E_1 = \frac{1}{\sin\chi\,\cos\chi}\frac{\partial}{\partial\psi}\,, \qquad E_i = \frac{1}{\sin\chi}\left(\widetilde{E}_i - \widetilde{A}_i\frac{\partial}{\partial\psi}\right)\,. \tag{A.4}$$

A straightforward calculation shows that the spinor-covariant exterior derivative $D = d + \frac{1}{4}\omega^{ab}\,\Gamma_{ab}$ on $\mathbb{CP}^n$ is given by

$$D = d + +\frac{1}{4}\tilde{\omega}^{ij}\,\Gamma_{ij} - e^1\left(\cot 2\chi\,\Gamma_{01} + \frac{1}{4}\cot\chi\,\widetilde{J}^{ij}\,\Gamma_{ij}\right) - \frac{1}{2}e^i\cot\chi\left(\Gamma_{0i} + \widetilde{J}_i{}^j\,\Gamma_{1j}\right)\,. \tag{A.5}$$

Decomposing the $2n$-dimensional Dirac matrices Γ_a for $\mathbb{CP}^n$ as

$$\Gamma_0 = \sigma_2 \otimes \widetilde{\Gamma}_*\,, \qquad \Gamma_1 = \sigma_1 \otimes \widetilde{\Gamma}_*\,, \qquad \Gamma_i = \mathbb{1} \otimes \widetilde{\Gamma}_i\,, \tag{A.6}$$

---

[4]Note that although we are using a 0 index here, in this section it refers to a Euclidean direction not a time direction. This section is intended to be self-contained, and not all notation or symbols used here are the same as in the previous section.

where $\widetilde{\Gamma}_i$ are the $(2n-2)$-dimensional Dirac matrices for $\mathbb{CP}^{n-1}$, it can be seen that the spinor-covariant exterior derivative (A.5) can be written as

$$D = \mathbb{1} \otimes \widetilde{D} + e^0 \frac{\partial}{\partial \chi} + e^1 \frac{1}{\sin \chi \cos \chi} \frac{\partial}{\partial \psi} - e^i \frac{1}{\sin \chi} \widetilde{A}_i \frac{\partial}{\partial \psi} + \mathrm{i}\, e^1 \cot 2\chi\, \sigma_3 \otimes \mathbb{1}$$
$$- \frac{1}{4} e^1 \cot \chi\, \widetilde{J}^{ij}\, \mathbb{1} \otimes \Gamma_{ij} - \frac{1}{2} e^i \cot \chi \left(\sigma_2 \otimes \widetilde{\Gamma}_* \widetilde{\Gamma}_i + \widetilde{J}_i{}^j \sigma_1 \otimes \widetilde{\Gamma}_* \widetilde{\Gamma}_j \right), \quad \text{(A.7)}$$

where $\widetilde{D} = \tilde{d} + \frac{1}{4} \tilde{\omega}^{ij} \widetilde{\Gamma}_{ij}$ is the spinor-covariant exterior derivative on $\mathbb{CP}^{n-1}$.

Assuming that the $\mathbb{CP}^{n-1}$ admits a gauge-covariantly constant spinor $\tilde{\xi}$ satisfying

$$\widetilde{D} + \frac{\mathrm{i}}{2}\, n\, \widetilde{A}\, \tilde{\xi} = 0\,, \qquad \widetilde{J}_i j\, \widetilde{\Gamma}_j\, \tilde{\xi} = -\mathrm{i}\, \widetilde{\Gamma}_j\, \tilde{\xi}\,, \qquad \widetilde{\Gamma}_*\, \tilde{\xi} = \tilde{\xi} \quad \text{(A.8)}$$

(the middle equation also implies $\widetilde{J}^{ij}\, \widetilde{\Gamma}_{ij}\, \tilde{\xi} = -2\mathrm{i}\,(n-1)\, \tilde{\xi}$), it then follows that $\mathbb{CP}^n$ admits a gauge-covariantly constant spinor $\xi = \nu \otimes \tilde{\xi}$ satisfying

$$D\xi + \frac{\mathrm{i}}{2}\,(n+1)\, A\, \xi = 0\,, \qquad J_{ab}\, \Gamma_{ab}\, \xi = -\mathrm{i}\, \Gamma_a\, \xi\,, \qquad \Gamma_*\, \xi = \xi\,, \quad \text{(A.9)}$$

where $\Gamma_* = \sigma_3 \otimes \widetilde{\Gamma}_*$ is the chirality operator on $\mathbb{CP}^n$, and where the 2-component spinor ν has ψ dependence $e^{-\frac{\mathrm{i}}{2} n \psi}$, it depends on no other coordinates, and it obeys $\sigma_3 \nu = \nu$. In other words, the gauge-covariantly constant spinor on $\mathbb{CP}^n$ can be taken to be

$$\xi = e^{-\frac{\mathrm{i}}{2} n \psi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \tilde{\xi}. \quad \text{(A.10)}$$

It also follows that ξ obeys $J^{ab}\, \Gamma_{ab}\, \xi = -2\mathrm{i}\, n\, \xi$.

If we denote the fibre coordinate ψ in the construction (A.1) of $\mathbb{CP}^n$ from $\mathbb{CP}^{n-1}$ by ψ_n we therefore have an iterative construction of the gauge-covariantly constant spinor:

$$\xi(\mathbb{CP}^n) = e^{-\frac{\mathrm{i}}{2} n \psi_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \xi\left(\mathbb{CP}^{n-1}\right). \quad \text{(A.11)}$$

An almost trivial calculation confirms that for $n=1$ the spinor

$$\xi\left(\mathbb{CP}^1\right) = e^{-\frac{\mathrm{i}}{2} \psi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(A.12)}$$

indeed satisfies all the properties assumed above, and so by induction we arrive at the expression

$$\xi(\mathbb{CP}^n) = \exp\left[-\frac{\mathrm{i}}{2} \sum_{p=1}^n p\, \psi_p \right] \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{n \text{ factors}} \quad \text{(A.13)}$$

for the gauge-covariantly constant spinor on $\mathbb{CP}^n$. (Note that for $n=1$, writing $\chi = \frac{1}{2}\theta$ and $\psi_1 = \phi$ puts the metric (A.1) in the standard form $d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$.)

B Identities for spinorial square roots

In this appendix, we record some basic results for spinors in odd dimensions, which are related to our discussion about the Killing vector (2.14) in the time-machine spacetimes.

In the odd dimension $D = 2n + 1$, the Fierz identity can be written in the form

$$\chi\bar{\chi} = \sum_{p=0}^{n} \frac{(-1)^{\frac{1}{2}p(p-1)}}{2^n \, p!} \, \bar{\chi}\Gamma_{\mu_1\cdots\mu_p}\chi \, \Gamma^{\mu_1\cdots\mu_p} \, , \tag{B.1}$$

where χ is any commuting spinor. A useful identity is

$$\Gamma_{\nu_1\cdots\nu_q} \, \Gamma_{\mu_1\cdots\mu_p} \, \Gamma^{\nu_1\cdots\nu_q} = c(q,p) \, \Gamma_{\mu_1\cdots\mu_p} \, , \tag{B.2}$$

where [29]

$$c(q,p) = (-1)^{\frac{1}{2}q(q-1)} \, (-1)^{pq} \, q! \sum_{i=0}^{\min(q,p)} \binom{p}{i} \binom{2n+1-q}{q-i} (-1)^i \, , \, . \tag{B.3}$$

If we define the tensors

$$T_{\mu_1\cdots\mu_p} = \bar{\chi}\Gamma_{\mu_1\cdots\mu_p}\chi \, , \tag{B.4}$$

and their norms

$$N_{(p)} = T^{\mu_1\cdots\mu_p} \, T_{\mu_1\cdots\mu_p} \, , \tag{B.5}$$

then it is straightforward to see from (B.1) and (B.2) that these satisfy the set of linear relations

$$N_{(q)} = \sum_{p=0}^{n} \frac{(-1)^{\frac{1}{2}p(p-1)}}{2^n \, p!} \, c(q,p) \, N_{(p)} \, , \qquad 0 \le q \le n \, . \tag{B.6}$$

(One does not need to consider $q > n$, since $\Gamma_{\mu_1\cdots\mu_p}$ is proportional to $\Gamma_{\mu_1\cdots\mu_{2n+1-p}}$.) The equations (B.6) are not, in fact, all linearly independent. For example, for the first few cases we find the relations imply:

$$
\begin{aligned}
D = 3: & \quad N_{(1)} = N_{(0)} \, , \\
D = 5: & \quad N_{(1)} = N_{(0)} \, , & & N_{(2)} = -4N_{(0)} \, , \\
D = 7: & \quad N_{(2)} = -6N_{(1)} \, , & & N_{(3)} = -42N_{(0)} + 24N_{(1)} \, , \\
D = 9: & \quad N_{(3)} = -36N_{(0)} + 36N_{(1)} + 3N_{(2)} \, , & & N_{(4)} = 216N_{(0)} + 120N_{(1)} + 24N_{(2)} \, , \\
D = 11: & \quad N_{(3)} = -30N_{(0)} + 30N_{(1)} + 3N_{(2)} \, , & & N_{(4)} = 240N_{(0)} + 120N_{(1)} + 12N_{(2)} \, , \\
& \quad N_{(5)} = 1920N_{(0)} - 120N_{(1)} + 60N_{(2)} \, .
\end{aligned}
\tag{B.7}
$$

Only in the first two cases, in $D = 3$ and $D = 5$ dimensions, we see that $N_{(1)}$ is simply equal to $N_{(0)}$. This means that in these two cases, and only in these cases, one has the relation

$$(\bar{\chi}\Gamma^\mu\chi)\,(\bar{\chi}\Gamma_\mu\chi) = (\bar{\chi}\chi)^2 \, , \tag{B.8}$$

where χ is *any* commuting spinor.[5]

---

[5]We emphasise that the spinor χ here is completely arbitrary, and need not be Majorana. If one does require χ to be Majorana, then (B.8) will hold in $D = 9$ also, since $C\Gamma_{\mu\nu}$ and $C\Gamma_{\mu\nu\rho}$ are antisymmetric in $D = 9$, so then $N_{(2)} = 0$ and $N_{(3)} = 0$.

The fact that (B.8) holds for any commuting spinor in $D = 3$ or $D = 5$ implies in particular that in these dimensions, any Killing vector K^μ that has a spinorial square root, meaning that it can be written as in terms of a Killing spinor η as $K^\mu = \bar{\eta}\Gamma^\mu\eta$, will necessarily have constant (negative) norm.

The Killing vector (2.14) in the BPS time-machine spacetime has constant and negative norm $K^\mu K_\mu = -1$ in any odd dimension, and we saw in section 2.3 that it always has a spinorial square root, as in (2.34). In odd dimensions $D \geq 7$, the fact that the norm is constant therefore depends upon special additional properties of the Killing spinor η that would, a priori, not necessarily hold for an arbitrary Killing spinor.

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