Consistent deformations of free massive field theories in the Stueckelberg formulation

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ABSTRACT: Cohomological techniques within the Batalin-Vilkovisky (BV) extension of the Becchi-Rouet-Stora-Tyutin (BRST) formalism have proved invaluable for classifying consistent deformations of gauge theories. In this work we investigate the application of this idea to massive field theories in the Stueckelberg formulation. Starting with a collection of free massive vectors, we show that the cohomological method reproduces the cubic and quartic vertices of massive Yang-Mills theory. In the same way, taking a Fierz-Pauli graviton on a maximally symmetric space as the starting point, we are able to recover the consistent cubic vertices of nonlinear massive gravity. The formalism further sheds light on the characterization of Stueckelberg gauge theories, by demonstrating for instance that the gauge algebra of such models is necessarily Abelian and that they admit a Born-Infeld-like formulation in which the action is simply a combination of the gauge-invariant structures of the free theory.

Contents

1 Introduction

Given the free theory of some set of fields, what are the possible interactions that one can add in a consistent manner? This question was first addressed decades ago in the context of Einstein gravity, with several works — see e.g. [1–3] and the references given by Preskill and Thorne in their foreword to the Feynman lectures on Gravitation [4] — showing via different methods that, under certain minimal assumptions, general relativity (GR) can be derived as the unique nonlinear extension of the Fierz-Pauli action for a massless spin-2 particle. A systematic analysis of the problem of introducing consistent interactions in a gauge theory was given in [5]. Perhaps the most important landmark in this program was the cohomological reformulation [6] of the analysis [5] within the Batalin-Vilkovisky (BV) antifield formalism [7, 8]. The latter approach extends the Becchi-Rouet-Stora-Tyutin (BRST) formalism [9, 10] by the adjunction of extra structures and antifields. In particular, the cohomological approach [6] generalizes the approach of Wald, who studied interactions for massless spin-1 and spin-2 fields by demanding consistency of the gauge algebra [11, 12].

The cohomological antifield method goes one step further as it unifies into a purely algebraic framework the problems of deforming consistently both the gauge symmetry and the action functional of a theory.

Originally introduced as an instrument to examine quantum aspects of gauge theories with an open gauge algebra [7, 8, 13] such as their renormalizability and anomalies, the BRST-BV formalism has proved extremely powerful also at the classical level, see [14] for a review. In this setting, full classifications of consistent interaction vertices have been achieved in a number of theories including Yang-Mills [15, 16], massless vector-scalar models [17, 18], Einstein and Weyl multi-gravity [19, 20] as well as pure supergravity [21]. See [22–25] for other references where the cohomological approach for consistent deformations of classical actions was used. A common property of all these examples is the presence of gauge symmetries. This would seem to preclude its use in the study of *massive* theories, which of course do not possess any gauge symmetry in their minimal covariant formulation. It is, however, well understood that gauge invariance does not have any true physical meaning, but rather encodes a redundancy in the phase space of a theory. For massless and partially massless field theories, such a redundancy is useful because it allows for a description that is manifestly local and Lorentz invariant — or invariant under possibly other spacetime isometries. This is not a problem in massive theories, but even in this case one may wish to work with a gauge invariant formulation as a tool for studying some of their physical properties. This method is known as the Stueckelberg procedure. It simply amounts to introducing an additional set of fields and corresponding gauge symmetries in such a way that the propagating degrees of freedom remain unchanged. Many successful instances where this approach was followed to clarify and discover properties of massive systems can be found in, e.g., [26–42]. There is therefore no fundamental limitation to the application of the cohomological method of [6] to examine consistent deformations of massive field theories, and it is the purpose of the present paper to initiate a systematic implementation of this idea. For completeness, we mention that another algebraic method for the study of consistent deformations of massive and massless field theories was provided in [43] and used, e.g., in [44, 45]. We note that the method exposed in [43] is more general, in the sense that it does not require any Lagrangian field equations as a starting point.

Concretely, our main goals will be to understand the general properties that characterize Stueckelberg gauge theories in the BRST-BV language and to see whether the cohomological approach of [6] can indeed be successfully applied in classifying interaction vertices. We address the first point in section 2, where in addition to an elementary review of the formalism we draw some general conclusions that are valid for any theory with Stueckelberg fields. We then proceed to illustrate our techniques by examining two relatively simple yet interesting models: massive Yang-Mills theory (section 3) and massive gravity (section 4).

An important conclusion of our analysis is that the cohomological formalism of [6] allows for a precise characterization of the gauge structure of massive theories. We will show in particular that Stueckelberg models possess the following properties:

- They retain the Abelian nature of the initial gauge algebra;

- Their actions are of the Abelian Born-Infeld type for an appropriate choice of field variables, i.e., are expressible solely in terms of gauge invariant building blocks of the Abelian, free gauge theory;

- The gauge transformation laws of the Stueckelberg fields can always be reduced to a pure shift, with no field-dependent corrections;

- After an appropriate decoupling limit, our procedure gives a sum of two Lagrangians, one of which gives a nonlinear sigma model, i.e., a non-linear realisation of some non-Abelian group.

It is worth emphasizing that the previous properties are often not manifest, and for this reason they are perhaps rather unexpected. Indeed, Stueckelberg theories are usually formulated in a way that makes the study of their degrees of freedom more transparent, but with the fields transforming non-trivially under the gauge symmetries. Our results show, however, that it is always possible to perform a redefinition of the fields and gauge parameters so that the transformation laws of the fields and the invariances of the action greatly simplify. We are thus uncovering here what may be thought of as a dual picture in which the gauge structure of the theory becomes very clear — in fact Abelian — but with the catch being that the vertices often involve more derivatives than in the standard parametrization, thereby making power counting and the identification of the relevant interaction scales more subtle.

As advertised, in order to make these notions more tangible and to test the usefulness of the BRST-BV deformation procedure in the context of massive field theories, we consider massive Yang-Mills theory and massive gravity as our first case studies. For the first model the starting point is the free action for a collection of vector fields A_μ^a and as many Stueckelberg scalars π^a. We push the calculation up to second order in the deformation analysis, allowing us to recover the known cubic and quartic vertices of the gauge invariant formulation of massive Yang-Mills theory, derived for instance in [28], which we also confirm by comparing with the full result obtained by starting with the interacting theory and performing a non-linear Stueckelberg replacement. Lastly we make explicit the field redefinition that takes the "standard" action into its Born-Infeld form, where the vertices are written as contraction of tensors that are invariant under the gauge symmetries of the *free* theory.

We next repeat the analysis for the case of a massive spin-2 field $h_{\mu\nu}$ on a maximally symmetric space. The calculations here are more involved and we only study the first order deformations that lead to the 3-point vertices of the theory. Here too we are able to rederive all the known structures (with up to four derivatives) previously classified for instance in [46], and again make explicit the Born-Infeld formulation of the action and the field redefinition connecting it to the standard parametrization. We also briefly examine the special case where the graviton mass is tuned relative to the cosmological constant as $m^2 = \Lambda/(D-1)$ (with D the spacetime dimension), for which the spin-2 particle becomes *partially massless* [47–50]. We highlight the peculiarities of the partially massless theory

relative to the generic massive set-up, and as a by-product we recover the unique cubic vertex that is known to exist only in four dimensions (see e.g. [51]).

2 Massive theories in the BRST-BV formalism

2.1 BRST-BV deformation method

The cohomological reformulation of the deformation procedure exposed in [5] was proposed in [6]. It exploits the antifield formalism [7, 8] for gauge theories. A detailed and accessible introduction to the BRST-BV deformation procedure can be found in [52]; see also the introduction of [19]. Here we reproduce the essential aspects in order to fix the necessary notation and describe the main steps of our approach in a self-contained manner as possible.

Consider a theory for a set of gauge fields φ^i defined by an action $S[\varphi^i]$ that is invariant under the infinitesimal gauge symmetries

$$\delta_\epsilon \varphi^i = R^i{}_\alpha \, \epsilon^\alpha \,, \tag{2.1}$$

where, using De Witt's notation, a summation over repeated indices also means that an integral over an omitted dummy coordinate is implied, so that (2.1) gives a local relation depending on the gauge parameters ϵ^α and their derivatives up to some finite order. The operator $R^i{}_\alpha$ may depend on the gauge fields and some of their successive derivatives.

In the BRST formalism [9, 10], one proceeds by associating a ghost field C^α to every gauge parameter ϵ^α, with a shift in the Grassmann parity: $|C^\alpha| = |\epsilon^\alpha| + 1 \pmod 2$. If the theory is reducible, such as for p-form gauge theories with $p > 1$, one also introduces a hierarchy of higher-degree ghosts (ghosts of ghosts). We will not discuss these cases here and stick to irreducible gauge theories. In the antifield extension of the BRST formalism due to Batalin and Vilkovisky, to the gauge fields φ^i and the BRST ghosts C^α one associates the antifields φ_i^* and C_α^*, respectively.[1] For concreteness we will take both the fields φ^i and the gauge parameters ϵ^α to be bosonic, thereby excluding the discussion of supergravity theories, although the general case can be treated in much the same way modulo some obvious changes, see for example the pedagogical review [55]. In this situation the ghost antifields are Grassmann-even or commuting variables as well, while the ghosts C^α and antifields φ_i^* are Grassmann-odd or anticommuting. Next we introduce two gradings, the ghost number "gh" and the antifield number "antifld", according to table 2.1. It is also useful to keep track of the number of differentiated or undifferentiated ghosts (not counting the associated antifields), via the pureghost number "puregh". We collectively denote the fields and ghosts by the variables Φ^A, while their associated antifields are denoted by Φ_A^*.

To the initial theory with gauge-invariant action $S[\varphi^i]$, one associates a BV functional $W[\Phi^A, \Phi_A^*]$ in the following way:

$$W[\Phi^A, \Phi_A^*] = S[\varphi^i] + \varphi_i^* \, R^i{}_\alpha \, C^\alpha + \frac{1}{2} \, C_\gamma^* \, f^\gamma{}_{\alpha\beta} \, C^\alpha C^\beta + \frac{1}{4} \, \varphi_i^* \varphi_j^* \, M^{ij}_{\alpha\beta} \, C^\alpha C^\beta + \cdots \,. \tag{2.2}$$

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[1]The antifields C_α^* are often called *antighosts*, in this context, although they should not be confused with the antighosts that appear in the Faddeev-Popov procedure. The latter form trivial pairs with the Lautrup-Nakanishi fields and can be eliminated from the BRST cohomology, see e.g. [53, 54].

	gh	antifld	puregh
Fields φ^i	0	0	0
Ghosts C^α	1	0	1
Antifields φ_i^*	-1	1	0
Ghost antifields C_α^*	-2	2	0

Table 1. Ghost, antifield and pureghost quantum numbers.

By construction, as it is required to start with the classical action and to have definite ghost number and Grassmann parity, the BV functional W is defined to be bosonic (Grassmann-even) with ghost number zero. More importantly, it is required to satisfy what is called the *classical master equation*:

$$(W, W) = 0 \,, \tag{2.3}$$

where the antibracket, also called BV bracket, is defined by[2]

$$(A, B) = \frac{\delta^R A}{\delta \Phi^A} \frac{\delta^L B}{\delta \Phi_A^*} - \frac{\delta^R A}{\delta \Phi_A^*} \frac{\delta^L B}{\delta \Phi^A} \,. \tag{2.4}$$

At zero antifield number, the master equation yields the Noether identities associated with the original gauge symmetries of the action,

$$\frac{\delta S}{\delta \varphi^i} R^i{}_\alpha = 0 \,. \tag{2.5}$$

Next, the terms with antifield number one in (2.3) produce

$$R^j{}_\alpha \frac{\delta R^i{}_\beta}{\delta \varphi^j} - R^j{}_\beta \frac{\delta R^i{}_\alpha}{\delta \varphi^j} = R^i{}_\gamma f^\gamma{}_{\alpha\beta} + \frac{\delta S}{\delta \varphi^j} M^{ji}_{\alpha\beta} \,, \tag{2.6}$$

which is nothing but the gauge algebra of the transformations (2.1), which gives the interpretation of $f^\gamma{}_{\alpha\beta}$ as "structure functionals", since in general they are operators that may depend on the fields. The presence of the term proportional to $M^{ij}_{\alpha\beta}$ implies that the algebra will in general be "open", meaning that two gauge transformations will only close upon use of the equations of motion. Similarly, at the following order one finds the Jacobi identity satisfied by the structure functionals, and continuing in this manner generates a tower of consistency relations involving the higher order tensors — the ellipsis in (2.2) — that characterize the gauge group of the theory. We refer to [55] for detailed discussions and review.

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[2]Right and left functional derivatives are defined via

$$\delta A = \int \frac{\delta^R A}{\delta \Phi^A} \delta \Phi^A + \int \frac{\delta^R A}{\delta \Phi_A^*} \delta \Phi_A^* = \int \delta \Phi^A \frac{\delta^L A}{\delta \Phi^A} + \int \delta \Phi_A^* \frac{\delta^L A}{\delta \Phi_A^*} \,.$$

The distinction between left and right derivatives of course only makes a difference when the derivative is with respect to a fermionic (Grassmann-odd) variable.

The main idea of the deformation analysis is to revert this story. The full action S and its gauge symmetries are unknown, and we seek to determine them by perturbatively solving the master equation, knowing an initial action S_0 invariant under the gauge transformations

$$\delta_0 \varphi^i = R_0{}^i{}_\alpha \, \epsilon^\alpha \, . \tag{2.7}$$

The BV functional, as we recalled, encodes all the information about the gauge structure of the theory. Hence the BV formalism is equivalent to other more direct approaches that aim at determining a theory based on the action and gauge transformation, although we will see that it is in many ways more powerful.

In order to solve the classical master equation, we consider the functional W as a perturbation series in some overall deformation parameter g, i.e.,

$$W = W_0 + g\, W_1 + g^2\, W_2 + \cdots \, . \tag{2.8}$$

Here W_0 corresponds to the BV functional associated with the theory that is already known and that one wishes to deform perturbatively. In the scenarios we focus on in this paper, the known theories will be free, so that the deformation procedure amounts to studying the consistent interaction vertices one can add to it. However, the formalism is not restricted to this case, since for instance one can also apply it to known models which are themselves already interacting, see for example [17, 18] for recent analyses.

2.2 Local BRST cohomology

The master equation approach becomes particularly powerful when rephrased as a cohomological problem [6]. The BV functional W_0 for the free theory S_0 is viewed as the generator of BRST transformations, in the sense that

$$sA \coloneqq (W_0, A) \, , \tag{2.9}$$

for any local functional A, with s denoting the BRST differential of the free theory. By a local functional A, one means the integral $\int a$ of a D-form a that depends on the fields (including the ghosts), their associated antifields and their derivatives up to some arbitrary but finite order, which we indicate by the notation $a = a([\Phi], [\Phi^*], dx)$. Moreover, in this work we assume that all the fields and their derivative vanish at infinity, or alternatively that they have compact support, which enables us to discard all boundary terms. With this assumption, any local D-form $a([\Phi], [\Phi^*], dx)$ is equivalent to $a([\Phi], [\Phi^*], dx) + db([\Phi], [\Phi^*], dx)$ where $b([\Phi], [\Phi^*], dx)$ is a local $(D-1)$-form and where d is the total exterior differential $d = dx^\mu \partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x^\mu} + \partial_\mu z^M \frac{\partial}{\partial z^M} + \dots$ is the total derivative that takes into account the spacetime dependence of the fields and antifields that we have collectively denoted by $z^M \coloneqq (\Phi^A, \Phi^*_A)$. We use conventions whereby s anticommutes with d, or equivalently, for the type of theories we will deal with in this paper, $dx^\mu C^\alpha + C^\alpha dx^\mu = 0$ and $dx^\mu \varphi^*_i + \varphi^*_i dx^\mu = 0$. One has the following isomorphism of cohomological classes, $H^g(s) \cong H^{g,D}(s|d)$, where $H^g(s)$ denotes the cohomology of s in the class of ghost number g local functionals and $H^{g,D}(s|d)$ is the cohomology of the differential s, at ghost number

g, in the space of local D-forms. In other words, the cohomology class in $H^{g,D}(s|d)$ is defined up to the $\sim$ equivalence relation by a representative solution $a^{g,D}$ of

$$sa^{g,D} + da^{g+1,D-1} = 0, \quad a^{g,D} \sim a^{g,D} + sb^{g-1,D} + db^{g,D-1}. \tag{2.10}$$

The first superscript refers to the ghost number and the second one to the form degree.

The basic properties of the BRST differential s then follow from the properties of the antibracket; in particular the nilpotency, $s^2 = 0$, is a consequence of the (graded) Jacobi identity and of the master equation $(W_0, W_0) = 0$.

Given a free theory with action $S_0[\varphi^i]$ invariant under irreducible gauge symmetries as in (2.7), we can write the effect of a BRST transformation on the fields by decomposing $s = \gamma + \delta$, such that

$$\gamma\varphi^i = R_0{}^i{}_\alpha C^\alpha, \qquad \delta\varphi_i^* = \frac{\delta S_0}{\delta\varphi^i}, \qquad \delta C_\alpha^* = \varphi_i^* R_0{}^i{}_\alpha, \tag{2.11}$$

while the action of γ and δ on the variable not shown is by definition zero. It follows immediately that

$$\gamma^2 = 0, \qquad \delta^2 = 0, \qquad \gamma\delta + \delta\gamma = 0, \tag{2.12}$$

which together again amount to the nilpotency of s. We recall that, with our conventions, $\{\gamma, d\} = 0 = \{\delta, d\}$. Observe that δ decreases the antifield number by one unit, while γ does not change it. From eqs. (2.3) and (2.8), one has that the master equation, up to order g^2, yields

$$sW_0 = 0, \qquad sW_1 = 0, \qquad sW_2 = -\frac{1}{2}(W_1, W_1). \tag{2.13}$$

The first of this holds trivially of course since the free action S_0 and its gauge invariance is known. The second equation then states that the first order deformation of the BV functional is BRST-closed. Any BRST-exact expression $W_1 = sB$ for a local functional B at ghost number -1 is of course a solution, but it is trivial in that it can be obtained from W_0 by means of a generalized non-linear field redefinition, where by "generalized" we mean one that may involve the ghosts and antifields as well. At the level of the original action such transformations translate into redefinitions of the field variables and/or gauge parameters. This leads to the conclusion that non-trivial first order (in the deformation parameter g) solutions of the master equation belong to the local cohomological group of s at ghost number zero, denoted by $H^{0,D}(s|d)$. The same considerations apply to all higher order deformations. For instance W_2 now satisfies an inhomogeneous equation, but once a particular solution is found the homogeneous part will admit trivial terms arising from field redefinitions of W_2.

2.3 Stueckelberg procedure

Generalities. The Stueckelberg procedure has proved to be a convenient technique to understand a number of important aspects relevant to the consistency of massive field theories, for example in counting the degrees of freedom, in analyzing their stability, and

in obtaining certain high-energy regimes (the so-called decoupling limits). More relevant to us is its application to the analysis of deformations of free theories [28], although an implementation of this idea in the context of the BRST-BV formalism has not been worked out yet, a gap that we want to fill with the present work.

A pedagogical review of the Stueckelberg method through several examples is given in [56]; see also the references given in the introduction for other applications. Here we restrict ourselves with a brief abstract explanation, which will be helpful in proving certain general attributes that characterize Stueckelberg gauge theories. Take again an action $S[\varphi]$ of the generic form

$$S = \int d^D x \, (\mathcal{L}_k + \mathcal{L}_m) \,, \tag{2.14}$$

where $\mathcal{L}_k$ is invariant under the gauge transformation

$$\varphi^i \to \varphi^{i\,\prime} = F^i([\varphi], [\epsilon]) \,, \tag{2.15}$$

with local parameters ϵ^α and such that $F^i([\varphi], 0) = \varphi^i$. The second term, $\mathcal{L}_m$, however breaks this symmetry explicitly. (The subscripts stand for "kinetic" and "mass", as this is the typical situation, although here we do not make any assumptions regarding the dimensions of the operators entering in $\mathcal{L}_k$ and $\mathcal{L}_m$.) We can restore the invariance of the action by introducing a set of fields σ^α, the "Stueckelberg fields", by performing the replacement

$$\varphi^i \to F^i([\varphi], [\sigma]) \,, \tag{2.16}$$

everywhere in $S[\varphi]$. The kinetic term is of course unaffected, while the mass term changes as

$$\mathcal{L}_m(\varphi) \to \mathcal{L}'_m([\varphi], [\sigma]) \coloneqq \mathcal{L}_m \left(F([\varphi], [\sigma]) \right) \,. \tag{2.17}$$

The claim is that the new action has a gauge symmetry, written infinitesimally as

$$\delta_\epsilon \varphi^i = R^i{}_\alpha \epsilon^\alpha \,, \qquad \delta_\epsilon \sigma^\alpha = S^\alpha_\beta([\varphi], [\sigma]) \epsilon^\beta \,, \tag{2.18}$$

with $R^i{}_\alpha \coloneqq \delta F^i / \delta \sigma^\alpha \big|_{\sigma=0}$. Without further knowledge of the gauge group we cannot say much about the form of S^α_β, but a crucial property is the fact that

$$S^\alpha_\beta([\varphi], 0) = -\delta^\alpha_\beta \,. \tag{2.19}$$

To see this, observe that the condition of invariance of the Lagrangian $\mathcal{L}'_m$ implies the Noether identity

$$\frac{\delta F^i}{\delta \varphi^j} \, R^j{}_\alpha + \frac{\delta F^i}{\delta \sigma^\beta} \, S^\beta_\alpha = 0 \,, \tag{2.20}$$

since $\mathcal{L}_k$ is invariant by itself. Eq. (2.19) then follows by evaluating at $\sigma^\alpha = 0$. Thus, perturbatively we have that

$$\delta_\epsilon \sigma^\alpha = -\epsilon^\alpha + \cdots \,, \tag{2.21}$$

for any Stueckelberg gauge theory. This simple result already gives us quite a lot of information.[3] It tells us that there exists a gauge condition, the so-called unitary gauge,

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[3]Eq. (2.21) can also be understood from the fact that Stueckelberg fields can be interpreted as the Goldstone modes associated with the breaking of the gauge group down to its global subgroup [57, 58].

in which $\sigma^\alpha = 0$, which establishes that the theory is dynamically equivalent to the one we started with. Moreover, unitary gauge can be applied directly at the level of the action (it is a "good" gauge choice in the terminology of [59]), since the equations of motion of the Stueckelberg fields are always implied by those of the dynamical fields. Indeed, using the invariance of the kinetic Lagrangian and the Noether identity (2.20) we find

$$\frac{\delta \mathcal{L}'_m}{\delta \sigma^\alpha} = -\frac{\delta \mathcal{L}'_m}{\delta \varphi^i} R^i{}_\beta (S^{-1})^\beta_\alpha \,, \tag{2.22}$$

which is only possible of course because S^β_α is (perturbatively) invertible.

Applications to the BRST-BV formalism. In the context of the BV deformation analysis, eq. (2.21) implies that the cohomology of the differential γ is trivial in strictly positive pureghost number g: $H^{g>0}(\gamma) \cong 0$. Indeed, for the free theory we infer that

$$C^\alpha = \gamma(-\sigma^\alpha) \,, \tag{2.23}$$

and hence the Stueckelberg ghosts (as well as all their derivatives) are always γ-exact. This leads to some interesting consequences that will become more explicit in the examples we analyze below. We will see, in particular, that any deformation of the gauge algebra of Stueckelberg transformations is necessarily trivial.

A simple but important theorem that follows from the property $H^{g>0}(\gamma) \cong 0$ is that Stueckelberg gauge theories do not admit Chern-Simons-like vertices. Denote by $a_0^{0,D}$ the antifield-zero part of (the integrand of) the BV functional, at first order in deformation around the free theory. In $a_0^{0,D}$ will appear the possible cubic vertices to be added to the quadratic Lagrangian. Then, if $a_0^{0,D}$ solves the master equation, we can always add to it a homogeneous solution $\bar{a}_0^{0,D}$ that satisfies

$$\gamma \bar{a}_0^{0,D} + db^{1,D-1} = 0 \,, \tag{2.24}$$

where $b^{1,D-1}$ has ghost number 1 and form degree $D-1$. Such $\bar{a}_0^{0,D}$ terms correspond to vertices that are invariant, modulo a total derivative, under the gauge symmetries of the free theory. We can then identify two types of such vertices: Born-Infeld terms, which are exactly invariant, i.e., $\gamma \bar{a}_0^{0,D} = 0$; and Chern-Simons terms, for which $b^{1,D-1} \neq 0$. Of course, starting from a Born-Infeld term one can always integrate by parts, but the $b^{1,D-1}$ generated in this way will be γ-exact modulo d, due to the fact that γ and d anticommute. Thus, a true Chern-Simons vertex is defined by a $b^{1,D-1}$ that is nontrivial in the cohomology $H^{1,D-1}(\gamma|d)$ and such that $db^{1,D-1}$ is trivial in $H^1(\gamma)$.

Now, operate in equation (2.24) with γ to get $0 = d(\gamma b^{1,D-1})$, which implies, by using the algebraic Poincaré Lemma recalled in [14], that there exists $b^{2,D-2}$ such that $\gamma b^{1,D-1} + db^{2,D-2} = 0$. Acting repeatedly with γ, one produces the descent of equations

$$\gamma \bar{a}_0^{0,D} + db^{1,D-1} = 0 \,, \tag{2.25}$$

$$\gamma b^{1,D-1} + db^{2,D-2} = 0 \,, \tag{2.26}$$

$$\vdots$$

$$\gamma b^{k,D-k} = 0 \,, \qquad k \leqslant D \,, \tag{2.27}$$

where the last equation is reached either because $k = D$ and a zero-form cannot be d-exact, or for $k < D$ in case one happens to reach an element $b^{k,D-k}$ in the cohomology of γ. But the last equation of the descent yields that $b^{k,D-k} = 0$ in the cohomology because of the fact that $H^{g>0}(\gamma) \cong 0$ in a Stueckelberg field theory, which then implies, working backwards along the ladder, that $b^{p,D-p} = 0$ for all p. Therefore $\gamma \bar{a}_0^{0,D} = 0$, which shows that any vertex that does not deform the gauge symmetry of a free Stueckelberg theory is necessarily of the Born-Infeld type. This result will prove very convenient, since in general Born-Infeld vertices are much easier to classify than Chern-Simons ones, and indeed we will be able to perform an exhaustive analysis for the example theories we study.

3 Massive Yang-Mills theory

Our first example is the massive version of Yang-Mills theory in the Stueckelberg formulation. The starting point is the free theory of a collection[4] of massive spin-1 fields A_μ^a,

$$S_0 = \int d^D x \left[-\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - \frac{1}{2} \mathcal{D}_\mu \pi_a \mathcal{D}^\mu \pi^a \right], \tag{3.1}$$

where the Stueckelberg scalars are π^a, and $F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ will always denote the field strength of the free theory. We have also defined

$$\mathcal{D}_\mu \pi^a := \partial_\mu \pi^a - m A_\mu^a, \tag{3.2}$$

and note that all the fields are assumed to have the same mass m. The above action enjoys the gauge symmetry

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a, \qquad \delta_\epsilon \pi^a = m \epsilon^a. \tag{3.3}$$

The choice of unitary gauge $\pi^a = 0$ then returns the usual free Proca action for the spin-1 fields A_μ^a; see section 2.3.

To construct the BV functional we introduce ghosts C^a, equal in number to the gauge symmetries, and antifields $A_a^{*\mu}$ and C_a^*. From (2.2) we then have, at quadratic order,

$$W_0 = S_0 + \int d^D x \left[A_a^{*\mu} \partial_\mu C^a + m \pi_a^* C^a \right], \tag{3.4}$$

which can be checked to satisfy the master equation $(W_0, W_0) = 0$.

The BRST differential (of the free theory) is decomposed as $s = \gamma + \delta$, where

$$\gamma A_\mu^a = \partial_\mu C^a, \qquad\qquad\qquad \gamma \pi^a = m C^a, \tag{3.5}$$

$$\delta A_a^{*\mu} = \frac{\delta S_0}{\delta A_\mu^a} = \partial_\nu F_a^{\nu\mu} - m^2 A_a^\mu + m \partial^\mu \pi_a, \qquad \delta \pi_a^* = \frac{\delta S_0}{\delta \pi^a} = \Box \pi_a - m \partial_\mu A_a^\mu, \tag{3.6}$$

$$\delta C_a^* = -\partial_\mu A_a^{*\mu} + m \pi_a^*, \tag{3.7}$$

and the action of γ and δ on the variables not shown is by definition zero.

---

[4]The color indices $a, b, \ldots$ run over a finite number n_v of values. In our conventions, they are raised and lowered using an Euclidean metric δ_{ab} in the internal space of vector fields.

3.1 Cubic deformations

We write W_1 as

$$W_1 = \int d^D x \, (a_0 + a_1 + a_2) \,, \tag{3.8}$$

where $\text{antifld}(a_k) = k$; the reason why the expansion stops at antifield number 2 is explained in [19] and results from the general theorems of [15] that can be applied to the local, irreducible theory (3.4). Referring to eq. (2.2), the interpretation of the various terms a_i in antifield numbers 0, 1 and 2 is as follows. The local scalar density a_0 encodes the infinitesimal deformations of the Lagrangian, while the local scalar density a_1 encodes the first-order deformation of the gauge transformation laws. Finally, a_2 gives the information about the first-order deformations of the Abelian gauge algebra. The equation $sW_1 = 0$ can now be decomposed with respect to the antifield number, with the results [15]

$$\begin{cases} \gamma a_0 + \delta a_1 + \text{t.d.} = 0 \,, \\ \gamma a_1 + \delta a_2 + \text{t.d.} = 0 \,, \\ \qquad\qquad \gamma a_2 = 0 \,. \end{cases} \tag{3.9}$$

We assume that the deformation starts at cubic order and modifies the gauge algebra of the free theory. Therefore, we write

$$a_2 = \frac{g}{2} \, f^a{}_{bc} C_a^* C^b C^c \,, \tag{3.10}$$

where $f^a{}_{bc} = f^a{}_{[bc]}$ is an otherwise arbitrary (at this stage) constant tensor. This is the most general expression at lowest order in derivatives. Obviously $\gamma a_2 = 0$, in agreement with the last equation in (3.9), but in fact a_2 is moreover γ-exact,

$$a_2 = \gamma \left(\frac{g}{2m} \, f^a{}_{bc} C_a^* \pi^b C^c \right) \,, \tag{3.11}$$

which follows because $C^a = \gamma(\pi^a/m)$. This is the first instance of the phenomenon we anticipated in section 2.3: the fact that the ghosts C^a do not belong to the cohomology $H(\gamma)$ implies a trivial a_2, and therefore we conclude that there is no nontrivial deformation of the gauge algebra. Note that this remains true if one considers higher derivative contractions, and will also hold at all orders in perturbations. This is in contrast to what occurs in the massless case [16, 60], where the term in (3.10) is precisely the starting point that leads to Yang-Mills theory.

It should be emphasized that the triviality of the above a_2 does not mean we cannot use it. The γ-exactness of a_2 implies that there exists a choice of field-dependent gauge parameters for which $a_2 = 0$, but such choice is not necessarily the smartest one. Indeed, here we are interested in seeing explicitly how the deformation analysis is modified in the presence of mass terms, and hence it is useful to keep the same point of departure as in the massless case. We will have more to say about this aspect below, in section 3.5, where we make explicit the field redefinition that maps a solution constructed starting from (3.10) to the one where both a_2 and a_1 are taken to be zero. The approach we follow in this section

is dictated by the wish to reproduce the Yang-Mills theory in the unitary gauge followed by the massless limit. Notice that a similar guiding principle was followed in [61–63].

With this discussion in mind, we continue the calculation to get a_1 from the second equation in (3.9). First we need

$$\delta a_2 = g f^a{}_{bc} A_a^{*\mu} \partial_\mu C^b C^c + \frac{g}{2} f^a{}_{bc} \pi_a^* (mC^b) C^c + \text{t.d.}$$
$$= -\gamma \left(g f^a{}_{bc} A_a^{*\mu} A_\mu^b C^c + \frac{g}{2} f^a{}_{bc} \pi_a^* \pi^b C^c \right) + \text{t.d.} \,, \tag{3.12}$$

and to obtain the second line we note that, given any two Grassmann-odd variables α and β, one has $\gamma(\alpha\beta) = (\gamma\alpha)\beta - \alpha(\gamma\beta)$. Here we encounter another subtle point that is characteristic of the Stueckelberg formulation: the expression $\partial_\mu C^a$ can be written both as γA_μ^a or as $\gamma(\partial_\mu \pi^a/m)$, meaning that our a_2 will admit two independent "liftings". We have checked however that the second option leads to a cubic vertex identically null, $a_0 \equiv 0$, hence we discard it. Using the result of (3.12) in the second line of (3.9), we obtain the particular solution

$$a_1 = g f^a{}_{bc} A_a^{*\mu} A_\mu^b C^c + \frac{g}{2} f^a{}_{bc} \pi_a^* \pi^b C^c \,. \tag{3.13}$$

From this we can read off the order-g deformation of the gauge symmetry:

$$\delta_\epsilon^{(1)} A_\mu^a = g f^a{}_{bc} A_\mu^b \epsilon^c \,, \qquad \delta_\epsilon^{(1)} \pi^a = \frac{g}{2} f^a{}_{bc} \pi^b \epsilon^c \,. \tag{3.14}$$

Note that to the above a_1 we may add a solution to the homogeneous equation $\gamma a_1 + \text{t.d.} = 0$, which we denote by $\bar{a}_1$. In order to stay as close as possible to the massless spin-1 case for which there exists no such $\bar{a}_1$'s, we choose not to take any such representatives, which anyway are trivial in the cohomology (recall that $H^g(\gamma) \cong 0$ for $g > 0$) and hence can be eliminated by a field redefinition.[5]

Finally, to find a_0 we compute

$$\delta a_1 = -g f^a{}_{bc} \left[\frac{1}{2} F_a^{\mu\nu} F_{\mu\nu}^b C^c + m^2 A_a^\mu A_\mu^b C^c + F_a^{\mu\nu} A_\nu^b \partial_\mu C^c - \partial^\mu \pi_a A_\mu^b (mC^c) \right.$$
$$\left. + \frac{1}{2} \partial^\mu \pi_a \partial_\mu \pi^b C^c + \frac{1}{2} \partial^\mu \pi_a \pi^b \partial_\mu C^c + \frac{1}{2} \partial_\mu A_a^\mu \pi^b (mC^c) \right] + \text{t.d.} \,. \tag{3.15}$$

This is γ-exact without a priori imposing $f_{abc} := \delta_{ad} f^d{}_{bc} = f_{[abc]}$. Indeed we obtain

$$\delta a_1 + \text{t.d.} = -\gamma \left(-\frac{g}{2} f^a{}_{bc} F_a^{\mu\nu} A_\mu^b A_\nu^c - \frac{g}{2} f^a{}_{bc} A_a^\mu \partial_\mu \pi^b \pi^c - g f^a{}_{bc} \partial^\mu \pi_a A_\mu^b \pi^c \right.$$
$$\left. + \frac{g}{2m} f^a{}_{bc} F_a^{\mu\nu} F_{\mu\nu}^b \pi^c + gm f^a{}_{bc} A_a^\mu A_\mu^b \pi^c + \frac{g}{2m} f^a{}_{bc} \partial_\mu \pi_a \partial^\mu \pi^b \pi^c \right) \,, \tag{3.16}$$

implying the consistency at first order in g of the "exotic" vertex

$$a_0^{\text{exotic}} = -\frac{g}{2} f^a{}_{bc} F_a^{\mu\nu} A_\mu^b A_\nu^c - \frac{g}{2} f^a{}_{bc} A_a^\mu \partial_\mu \pi^b \pi^c - g f^a{}_{bc} \partial^\mu \pi_a A_\mu^b \pi^c$$
$$+ \frac{g}{2m} f^a{}_{bc} F_a^{\mu\nu} F_{\mu\nu}^b \pi^c + gm f^a{}_{bc} A_a^\mu A_\mu^b \pi^c + \frac{g}{2m} f^a{}_{bc} \partial_\mu \pi_a \partial^\mu \pi^b \pi^c \,. \tag{3.17}$$

---

[5]For example, $\bar{a}_1 = A_a^{*\mu} \mathcal{D}_\mu \pi^b C^c f^a{}_{bc}$ is a candidate that is readily seen to be γ-exact.

However, since our guiding principle is that, in the unitary gauge followed by massless limit, one should recover massless Yang-Mills theory, one must finally impose the relation $f_{abc} = f_{[abc]}$. As a matter of fact, the first term of a_0^{exotic} is the right cubic part of the Yang-Mills Lagrangian, only when the structure constants satisfy $f_{abc} = f_{[abc]}$. We thus arrive at the result

$$a_0 = \frac{g}{2} f^a{}_{bc} \left(-F_a^{\mu\nu} A_\mu^b A_\nu^c + A_a^\mu \partial_\mu \pi^b \pi^c \right), \quad f_{abc} = f_{[abc]}, \tag{3.18}$$

which corresponds to the cubic vertices of the deformed theory. We can again envisage adding homogeneous solutions satisfying $\gamma \bar{a}_0 + \text{t.d.} = 0$. From the theorem given in section 2.3, we know that all such solutions must be of the Born-Infeld type, i.e. solution of $\gamma \bar{a}_0 = 0$. The cohomology of γ in the space of fields and their derivatives is generated by[6]

$$H^0(\gamma) \cong \left\{ f([F_{\mu\nu}^a], [\mathcal{D}_\mu \pi^a]) \right\}. \tag{3.19}$$

It is easy to see that there exists no two-derivative cubic invariant contractions built out of these elements. The simplest ones are the three-derivative vertices

$$\begin{aligned}
\bar{a}_0^{(1)} &= f^a{}_{bc} F_a^{\mu\nu} \mathcal{D}_\mu \pi^b \mathcal{D}_\nu \pi^c, \\
\bar{a}_0^{(2)} &= f^a{}_{bc} F_a^{\mu\nu} F^b_{\nu\rho} F^{c\rho}{}_\mu.
\end{aligned} \tag{3.20}$$

Note that such Born-Infeld terms, when taken alone, automatically solve the master equation and hence cannot generate an obstruction at higher orders in perturbations — although they will get Yang-Mills covariantized, meaning that at the end of the deformation procedure they become Born-Infeld terms under the non-linear completion of the gauge symmetries. For this reason it is consistent to ignore them for now, although we will consider such deformations later.

3.2 Quartic deformations

It is easy to see that the antibracket of W_1 with itself takes the form

$$(W_1, W_1) = \int d^D x \, (\alpha_0 + \alpha_1 + \alpha_2), \tag{3.21}$$

where again antifld$(\alpha_k) = k$. We find

$$\begin{aligned}
\alpha_2 &= -g^2 f^a{}_{b[c} f_{|a|de]} C^{*b} C^c C^d C^e, \\
\alpha_1 &= 3g^2 f^a{}_{b[c} f_{|a|de]} A^{*b\mu} A_\mu^c C^d C^e + g^2 f^a{}_{b[c} f_{|a|d]e} \pi^{*b} \pi^c C^d C^e, \\
\alpha_0 &= -\frac{3g^2}{2} f^a{}_{b[c} f_{|a|de]} F_{\mu\nu}^b A^{c\mu} A^{d\nu} C^e + g^2 f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d \partial_\nu C^e \\
&\quad + \frac{g^2}{2} f^a{}_{bc} f_{ade} \left(2\partial^\mu \pi^b \pi^c A_\mu^d + \partial_\mu A^{b\mu} \pi^c \pi^d + 2A^{b\mu} \partial_\mu \pi^c \pi^d \right) C^e + \text{t.d.}.
\end{aligned} \tag{3.22}$$

---

[6]Note that the variables $[F_{\mu\nu}^a]$ and $[\mathcal{D}_\mu \pi^a]$ are not linearly independent due to the identity $2\partial_{[\mu} \mathcal{D}_{\nu]} \pi^a + mF_{\mu\nu}^a \equiv 0$. A basis can be obtained by taking the set $\{\partial_{(\mu_1} \ldots \partial_{\mu_{n-1}} F^a_{\mu_n)\nu}, \partial_{(\mu_1} \ldots \partial_{\mu_{n-1}} \mathcal{D}_{\mu_n)} \pi^a\}$, $n = 1, 2, \ldots$.

As before we decompose the BV functional W_2 with respect to the antifield number,

$$W_2 = \int d^D x \, (b_0 + b_1 + b_2) \,, \tag{3.23}$$

so that the order g^2 master equation $sW_2 + \frac{1}{2}(W_1, W_1) = 0$ yields

$$
\begin{cases}
\gamma b_0 + \delta b_1 + \dfrac{1}{2}\,\alpha_0 + \text{t.d.} = 0 \,, \\[2mm]
\gamma b_1 + \delta b_2 + \dfrac{1}{2}\,\alpha_1 + \text{t.d.} = 0 \,, \\[2mm]
\qquad \gamma b_2 + \dfrac{1}{2}\,\alpha_2 + \text{t.d.} = 0 \,.
\end{cases}
\tag{3.24}
$$

A priori the structure constants do not have to satisfy the Jacobi identity since the last equation can be solved for b_2 for general structure constants f_{abc} (which would not be possible in the massless case because $C^{*b}C^c C^d C^e$, being now in $H(\gamma)$, would yield an obstruction). Explicitly we get

$$b_2 = \frac{g^2}{2m} f^a{}_{b[c} f^b{}_{de]} C^*_a \pi^c C^d C^e \,, \tag{3.25}$$

up to a solution $\bar{b}_2$ of the homogeneous equation $\gamma b_2 = 0$. Again, because $H^g(\gamma)$ is trivial in strictly positive pureghost number, we have the choice to discard such possibilities. Moreover, the $\bar{b}_2$'s that exist for the massive theory would bring too many derivatives in the Lagrangian.

Moving on to the next term, from (3.22) and (3.25) one can calculate

$$- \delta b_2 - \frac{1}{2}\alpha_1 + \text{t.d.} = \cdots - \frac{g^2}{8} f^a{}_{b[c} f^b{}_{de]} \pi^*_a \pi^c C^d C^e \,. \tag{3.26}$$

The presence of the last term makes it impossible to solve the second equation in (3.24) for b_1, since it cannot be produced by γb_1 due to the antisymmetrization of color indices. It is therefore at this stage that we finally have to impose the Jacobi identity,

$$f^a{}_{b[c} f^b{}_{de]} = 0 \,, \tag{3.27}$$

so that b_2 vanishes, as in the Yang-Mills case. A useful equivalent form of (3.27) is

$$f^a{}_{b[c} f_{|a|d]e} = -\frac{1}{2} f^a{}_{cd} f_{abe} \,. \tag{3.28}$$

Coming back to the part with antifield number one in (3.22) and using (3.27) and (3.28) we can calculate

$$
\begin{aligned}
\alpha_1 &= -\frac{g^2}{2} f^a{}_{be} f_{acd} \pi^{*b} \pi^c C^d C^e \\
&= \gamma \left(-\frac{g^2}{6m} f^a{}_{bc} f_{ade} \pi^{*b} \pi^c \pi^d C^e \right) \,.
\end{aligned}
\tag{3.29}
$$

Since now $b_2 = 0$ by virtue of the Jacobi identity, the second equation in (3.24) implies

$$b_1 = \frac{g^2}{12m} f^a{}_{bc} f_{ade} \pi^{*b} \pi^c \pi^d C^e \,, \tag{3.30}$$

which results in the following second-order deformation of the gauge symmetry:

$$\delta_\epsilon^{(2)} A_\mu^a = 0\,, \qquad \delta_\epsilon^{(2)} \pi^a = \frac{g^2}{12m}\, f^a{}_{bc} f^c{}_{de} \pi^b \pi^d \epsilon^e\,. \tag{3.31}$$

The final step is to find b_0 from the first equation in (3.24). After some manipulations α_0 simplifies to

$$\begin{aligned}
\alpha_0 = {}& 2g^2 f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d \partial_\nu C^e - \frac{g^2}{2}\, f^a{}_{bc} f_{ade} A^{b\mu} \pi^c \pi^d \partial_\mu C^e \\
& + \frac{g^2}{2}\, f^a{}_{be} f_{acd} A^{b\mu} \partial_\mu \pi^c \pi^d C^e + \text{t.d.}\,.
\end{aligned} \tag{3.32}$$

We observe that the first term gives the quartic Yang-Mills vertex,

$$2g^2 f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d (\gamma A_\nu^e) = \gamma \left(\frac{g^2}{2}\, f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d A_\nu^e \right)\,. \tag{3.33}$$

Expanding δb_1 and collecting terms we get

$$\begin{aligned}
\frac{1}{2}\, \alpha_0 + \delta b_1 = {}& \gamma \left(\frac{g^2}{4}\, f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d A_\mu^e \right) + \frac{g^2}{3}\, f^a{}_{b(d} f_{|ac|e)} A^{b\mu} \partial_\mu \pi^c \pi^d C^e \\
& - \frac{g^2}{6}\, f^a{}_{bc} f_{ade} A^{b\mu} \pi^c \pi^d \partial_\mu C^e - \frac{g^2}{12m}\, f^a{}_{bc} f_{ade} \partial^\mu \pi^b \pi^c \partial_\mu \pi^d C^e \\
& - \frac{g^2}{12m}\, f^a{}_{bc} f_{ade} \partial^\mu \pi^b \pi^c \pi^d \partial_\mu C^e + \text{t.d.}\,.
\end{aligned} \tag{3.34}$$

By using a general Ansatz for b_0,

$$\begin{aligned}
b_0 = {}& -\frac{g^2}{4}\, f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d A_\nu^e + x_1 g^2 f^a{}_{bc} f_{ade} A^{b\mu} \pi^c \pi^d A_\mu^e + \frac{x_2 g^2}{m}\, f^a{}_{bc} f_{ade} A^{b\mu} \pi^c \partial_\mu \pi^d \pi^e \\
& + \frac{x_3 g^2}{m^2}\, f^a{}_{bc} f_{ade} \partial^\mu \pi^b \pi^c \partial_\mu \pi^d \pi^e\,,
\end{aligned} \tag{3.35}$$

and substituting in (3.24), we find that $x_1 = 0$, $x_2 = -1/6$ and $x_3 = 1/24$, so that the final result for the quartic vertices of the theory is given by

$$\begin{aligned}
b_0 = {}& -\frac{g^2}{4}\, f^a{}_{bc} f_{ade} A^{b\mu} A^{c\nu} A_\mu^d A_\nu^e - \frac{g^2}{6m}\, f^a{}_{bc} f_{ade} A^{b\mu} \pi^c \partial_\mu \pi^d \pi^e \\
& + \frac{g^2}{24m^2}\, f^a{}_{bc} f_{ade} \partial^\mu \pi^b \pi^c \partial_\mu \pi^d \pi^e\,.
\end{aligned} \tag{3.36}$$

We could also consider adding the quartic solutions of the homogeneous equation $\gamma \bar{b}_0 = 0$, which are of the schematic form F^4, $F^2(\mathcal{D}\pi)^2$ and $(\mathcal{D}\pi)^4$, with the color indices being contracted with either a pair of structure constants or with δ_{ab}. We omit the list of all such contractions as we will have no occasion to use them, but the task is of course straightforward.

3.3 Comparison with the full nonlinear theory

It is instructive to check our results by expanding the full non-linear model obtained by performing a Stueckelberg replacement of massive Yang-Mills theory, see e.g. [56, 57]. To this end it is helpful to switch to matrix notation and write $\mathbf{A}_\mu = A_\mu^a T_a$, with T_a the generators of the gauge group normalized such that

$$[T_a, T_b] = if^c{}_{ab}T_c. \tag{3.37}$$

The action is

$$S = \int d^D x \left[-\frac{1}{2}\operatorname{tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) - \frac{m^2}{2\mathcal{T}}\operatorname{tr}(\mathbf{A}^\mu\mathbf{A}_\mu) \right], \tag{3.38}$$

where $\mathbf{F}_{\mu\nu}$ is the Yang-Mills field strength and $\mathcal{T}$ is the Dynkin index of the fundamental representation of the group, defined via $\operatorname{tr}(T_a T_b) = \mathcal{T}\delta_{ab}$. We introduce the Stueckelberg fields π^a mimicking the gauge transformation under which the kinetic term remains invariant,

$$\mathbf{A}_\mu \to \mathbf{A}'_\mu = U\mathbf{A}_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1}, \qquad U := e^{ig\pi^a T_a/m}. \tag{3.39}$$

Let us ignore the pure vector part of the action and focus on the new operators involving the Stueckelberg modes,

$$\mathcal{L}' = -\frac{m^2}{2\mathcal{T}}\left[\frac{2i}{g}\operatorname{tr}(\mathbf{A}^\mu U^{-1}\partial_\mu U) + \frac{1}{g^2}\operatorname{tr}(\partial^\mu U\partial_\mu U^{-1}) \right]. \tag{3.40}$$

Both terms in $\mathcal{L}'$ are functions of the matrix

$$U^{-1}\partial_\mu U = \frac{ig}{m}\partial_\mu \pi^a \ell_a{}^b(\pi)T_b, \tag{3.41}$$

where we defined

$$\ell_a{}^b(\pi) := \delta_a^b + \sum_{k=1}^\infty \frac{(g/m)^k}{(k+1)!}(\pi^{b_1}\cdots\pi^{b_k})(f_{b_1 a}{}^{c_1} f_{b_2 c_1}{}^{c_2} \cdots f_{b_k c_{k-1}}{}^b). \tag{3.42}$$

It is straightforward to check that $\ell_a{}^b$ satisfies the differential equation

$$\frac{\partial \ell_a{}^c}{\partial \pi^b} - \frac{\partial \ell_b{}^c}{\partial \pi^a} = -\frac{g}{m}f_{de}{}^c \ell_a{}^d \ell_b{}^e, \tag{3.43}$$

which can be used as an alternative definition as was done in [28]. Expressing $\ell_a{}^b$ in closed form is not possible for generic gauge groups, but for the particular case of SU(2) we can perform the sum to find

$$\ell_{ab} = \delta_{ab}\frac{\sin\left(\frac{g}{m}\sqrt{\pi^2}\right)}{\frac{g}{m}\sqrt{\pi^2}} + \frac{\pi_a\pi_b}{\pi^2}\left(1 - \frac{\sin\left(\frac{g}{m}\sqrt{\pi^2}\right)}{\frac{g}{m}\sqrt{\pi^2}}\right) - \frac{\epsilon_{abc}\pi^c}{\sqrt{\pi^2}}\left(\frac{1 - \cos\left(\frac{g}{m}\sqrt{\pi^2}\right)}{\frac{g}{m}\sqrt{\pi^2}}\right), \tag{3.44}$$

with $\pi^2 := \delta_{ab}\pi^a\pi^b$.

Returning to the general case, we substitute (3.41) into the Lagrangian to obtain the Stueckelberg part of the action,

$$\mathcal{L}' = m\partial^\mu\pi^a \ell_a{}^b(\pi)A_\mu^b - \frac{1}{2}\gamma_{ab}(\pi)\partial^\mu\pi^a\partial_\mu\pi^b\,, \tag{3.45}$$

where $\gamma_{ab} := \ell_a{}^c\ell_{bc}$, which gives the interpretation of $\ell_a{}^b$ as a field space vielbein [64]. It is now direct to expand this expression using the explicit result (3.42) to recover the cubic and quartic vertices we obtained through the deformation procedure.

3.4 Decoupling limit

The quartic vertices we have derived, eq. (3.36), are clearly singular in the massless limit, i.e. we cannot simply take $m \to 0$. A more sensible thing to do with an interacting theory is to study the decoupling limit, that is to consider the limiting value of the coupling constants in such a way that (1) the number of degrees of freedom is preserved, (2) the Stueckelberg fields become gauge invariant and hence physical, carrying propagating degrees of freedom rather than pure gauge ones, and (3) the resulting theory retains some nonlinearities and is therefore nontrivial. In practice we can achieve these requirements by identifying the smallest energy scale — let's call it M — that plays a role in the theory, and to take the limit of the parameters of the model such that M is kept fixed but all other scales that are greater than M go to infinity.

By inspection of the vertices and modified gauge transformations, it is clear that the decoupling limit of massive YM theory is achieved by letting $g, m \to 0$ with the scale $M := (m/g)^{2/(D-2)}$ kept finite.[7] The resulting action is

$$S = \int d^D x \left[-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi^a + \frac{1}{24M^{D-2}}f^a{}_{bc}f_{ade}\partial_\mu\pi^b\pi^c\partial^\mu\pi^d\pi^e + \cdots\right]\,, \tag{3.46}$$

and the omitted terms correspond to higher order scalar vertices suppressed by increasing powers of M. The appearance of a dimensionful parameter (even though the fields are massless) means that the theory of the scalar sector is non-renormalizable and must be regarded as an EFT with UV cutoff M [56]. In the decoupling limit the scalar and vector sectors are truly decoupled from one another (unlike what happens for instance in massive gravity); the vector part of the action is nothing but a sum of Maxwell terms, meaning that we must necessarily lose the Yang-Mills interactions when taking this limit, but the scalar self-interaction on the other hand are nontrivial and correspond to a non-linear sigma model. Indeed, in this decoupling limit the gauge symmetry reduces to

$$\delta_\epsilon A_\mu^a = \partial_\mu\epsilon^a\,, \qquad \delta_\epsilon\pi^a = 0\,, \tag{3.47}$$

consistent with the expectation that in this limit the vector fields become massless, with the longitudinal modes now being physically propagated by the scalars π^a.

---

[7]Note that in D dimensions the coupling constant g has mass dimension $-(D-4)/2$ (since the canonically normalized fields have dimension $(D-2)/2$). The scale M thus carries mass dimension one.

3.5 Triviality of the gauge algebra

Determination of the field redefinition. We have seen that, in Stueckelberg gauge theories, any deformation of the gauge transformation laws and of the gauge algebra — encoded respectively in the terms a_1 and a_2 for the first order deformation of the BV functional — are ultimately trivial by virtue of the fact that the ghosts are γ-exact. In this subsection we make explicit the field redefinition and field-dependent gauge parameter redefinition that provide the mapping from the solution of the first-order deformation (3.8), which we solved for in section 3.1, to the trivial solution, modulo Born-Infeld type vertices, defined by

$$a_2^t = \gamma \left(\frac{g}{2m} f^a{}_{bc} C_a^* \pi^b C^c \right) , \tag{3.48}$$

$$a_1^t = \delta \left(\frac{g}{2m} f^a{}_{bc} C_a^* \pi^b C^c \right) , \tag{3.49}$$

$$a_0^t = 0 . \tag{3.50}$$

Since

$$a_2 = \frac{g}{2} f^a{}_{bc} C_a^* C^b C^c = \gamma \left(\frac{g}{2m} f^a{}_{bc} C_a^* \pi^b C^c \right) = a_2^t , \tag{3.51}$$

the resolution of $\delta a_2 + \gamma a_1 = $ t.d. has to give the same solution for a_1 and a_1^t modulo γ-closed terms and modulo total derivatives, i.e. $\gamma \left(a_1^t - a_1 \right) = \partial_\mu j^\mu$. Because of the triviality of the cohomology of γ modulo d in strictly positive pureghost number, $a_1^t - a_1$ has to be γ-exact modulo a total derivative. Thus the goal is to solve for c_1 in

$$\delta \left(\frac{g}{2m} f^a{}_{bc} C_a^* \pi^b C^c \right) - g f^a{}_{bc} A_a^{*\mu} A_\mu^b C^c - \frac{g}{2} f^a{}_{bc} \pi_a^* \pi^b C^c = \gamma c_1 + \text{t.d.} . \tag{3.52}$$

The knowledge of c_1 will give us the field redefinition connecting the two solutions. Noting that the left-hand side of (3.52) can be written as

$$\text{t.d.} + \frac{g}{2m} f^a{}_{bc} A_a^{*\mu} \left(\partial_\mu \pi^b C^c + \pi^b \partial_\mu C^c \right) - g f^a{}_{bc} A_a^{*\mu} A_\mu^b C^c , \tag{3.53}$$

one finds that the solution for c_1 is provided by

$$c_1 = \frac{g}{2m^2} f^a{}_{bc} A_a^{*\mu} \pi^b \left(\partial_\mu \pi^c - 2m A_\mu^c \right) . \tag{3.54}$$

In conclusion, the field redefinition at cubic order that enables us to go from the cubic vertex a_0 to the trivial cubic vertex $a_0^t = 0$, up to strictly gauge-invariant vertices, is given by

$$A_\mu^a \longrightarrow A_\mu^a + \frac{g}{2m^2} f^a{}_{bc} \pi^b \left(\partial_\mu \pi^c - 2m A_\mu^c \right) , \tag{3.55}$$

and the redefinition of the gauge parameters that trivializes the cubic deformation of the gauge algebra can be read in (3.51) as

$$\epsilon^a \longrightarrow \epsilon^a + \frac{g}{2m} f^a{}_{bc} \pi^b \epsilon^c . \tag{3.56}$$

Determination of the non-trivial cubic deformation. From the resolution of $\delta a_1 + \gamma a_0 = \text{t.d.}$ and (3.52) we have that the difference between the two solutions a_0 and $a_0^t = 0$ summed with δc_1 is γ-closed modulo a total derivative. Indeed,

$$\gamma\left(a_0 - a_0^t + \delta c_1\right) = \gamma a_0 + \gamma \delta c_1 = -\delta\left(a_1 + \gamma c_1\right) + \text{t.d.}$$
$$= -\delta\left(\delta\left(\frac{g}{2m}f^a{}_{bc}C_a^*\pi^b C^c\right)\right) + \text{t.d.} = \text{t.d.} \ . \tag{3.57}$$

Since a pureghost-zero solution of $\gamma a_0 + \text{t.d.} = 0$ is equivalent to $\gamma\bar{a}_0 = 0$, as we have seen in section 2.3, one can add total derivatives to the γ modulo d cocycle $a_0 - a_0^t + \delta c_1$ so that it becomes strictly annihilated by γ. Therefore we have

$$\bar{a}_0 = a_0 + \delta c_1 \ . \tag{3.58}$$

Explicitly, the cocycle $\bar{a}_0$ reads

$$\bar{a}_0 = -\frac{g}{2}f^a{}_{bc}F_a^{\mu\nu}A_\mu^b A_\nu^c + \frac{g}{m}f^a{}_{bc}F_a^{\mu\nu}\partial_\mu\pi^b A_\nu^c - \frac{g}{2m^2}f^a{}_{bc}F_a^{\mu\nu}\partial_\mu\pi^b\partial_\nu\pi^c \ . \tag{3.59}$$

To make manifest that it belongs to $H^0(\gamma)$, we rewrite it as follows:

$$\bar{a}_0 = -\frac{g}{2m^2}f^a{}_{bc}F_a^{\mu\nu}\mathcal{D}_\mu\pi^b\mathcal{D}_\nu\pi^c \ . \tag{3.60}$$

This is actually the most general deformation which mixes A_μ^a and π^a; see (3.20). Indeed, this $\bar{a}_0$ is the most general cubic term in $H^0(\gamma)$, at ghost number zero, that involves both A_μ^a and π^a.

In conclusion, the cubic deformation found in section 3.1 is obtained from the deformation of action

$$S_0 = \int d^D x \left(-\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{2}\mathcal{D}_\mu\pi^a\mathcal{D}^\mu\pi_a\right) \longrightarrow S = S_0 - \frac{g}{2m^2}\int d^D x f^a{}_{bc}F_a^{\mu\nu}\mathcal{D}_\mu\pi^b\mathcal{D}_\nu\pi^c \ ,$$

followed by the field redefinition (3.55), keeping only cubic terms.

Then one can formally follow the same procedure at each order in perturbation to find:

(1) The total field redefinition[8] which maps from the full solution (3.45) to the solution with trivial gauge algebra;

(2) All the vertices at all orders which are obtained after doing the total field redefinition and which are by construction in the cohomology $H^0(\gamma)$, at ghost number zero.

4 Massive gravity

We now carry out the deformation analysis for a single massive spin-2 field on a maximally symmetric D-dimensional background. The components of the background metric are written as $g_{\mu\nu}$, while g denotes its determinant and ∇_μ the associated Levi-Civita covariant derivative. We introduce the parameter σ that takes the value 1 in the anti-de Sitter (AdS)

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[8]This field redefinition is thus order-by-order invertible.

background and the value -1 in the de Sitter (dS) geometry. The (A)dS radius will be denoted by L and is related to the cosmological constant Λ via $\Lambda = -\frac{(D-1)(D-2)}{2\,\sigma L^2}$. One can reach the Minkowski background by taking the limit $\Lambda \longrightarrow 0$, or equivalently, $L \longrightarrow \infty$. In our conventions, the commutator of background covariant derivatives acting on a (co)vector is $[\nabla_\mu, \nabla_\nu]V_\sigma = -\frac{2}{\sigma L^2} g_{\sigma[\mu} V_{\nu]}$.

The free theory is given by the massive Fierz-Pauli action in its Stueckelberg form,[9]

$$
S_0 = \int d^D x \bigg[-\frac{1}{2}\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} + \nabla_\rho h^{\mu\nu}\nabla_\mu h^\rho{}_\nu - \nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{2}\nabla^\mu h \nabla_\mu h - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}
$$
$$
- \frac{1}{2}(\nabla\varphi)^2 + 2m(h\nabla_\mu B^\mu - h^{\mu\nu}\nabla_\mu B_\nu) + \mu B^\mu \nabla_\mu \varphi - \left(\frac{D-1}{\sigma L^2} + m^2\right) h^{\mu\nu}h_{\mu\nu}
$$
$$
+ \frac{1}{2}\left(\frac{D-1}{\sigma L^2} + 2m^2\right)h^2 - \mu m\, h\varphi + \frac{Dm^2}{D-2}\varphi^2 - \frac{D-1}{\sigma L^2}B^\mu B_\mu \bigg]\sqrt{-g}\,,
$$

$$(4.1)$$

with Stueckelberg fields B_μ and φ. We use the notation $F_{\mu\nu} := 2\nabla_{[\mu}B_{\nu]}$ and introduce the following parameter μ with dimension of mass:

$$
\mu := \sqrt{\frac{2(D-1)}{D-2}}\sqrt{\frac{D-2}{\sigma L^2} + 2m^2}\,. \tag{4.2}
$$

In these conventions[10] the fields are all canonically normalized and the action is manifestly unitary as far as the real mass parameter m has a value such that the parameter μ is real. There are two gauge symmetries given by

$$
\delta_\xi h_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,, \qquad \delta_\xi B_\mu = -2m\,\xi_\mu\,, \qquad \delta_\xi \varphi = 0\,, \tag{4.3}
$$
$$
\delta_\epsilon h_{\mu\nu} = \frac{2\,m}{D-2}g_{\mu\nu}\epsilon\,, \qquad \delta_\epsilon B_\mu = \nabla_\mu \epsilon\,, \qquad \delta_\epsilon \varphi = \mu\,\epsilon\,. \tag{4.4}
$$

The ghosts corresponding to these symmetries will be denoted by C_μ and C, respectively. The free part of the BV functional is then given by

$$
W_0 = S_0 + \int \sqrt{-g}\bigg[h^{*\mu\nu}\left(2\nabla_{(\mu}C_{\nu)} + \frac{2\,m}{D-2}g_{\mu\nu}C\right) + B^{*\mu}\left(\nabla_\mu C - 2\,mC_\mu\right) + \mu\,\varphi^* C\bigg]\,. \tag{4.5}
$$

The action of the differential γ on the fields is

$$
\gamma h_{\mu\nu} = 2\nabla_{(\mu}C_{\nu)} + \frac{2m}{D-2}g_{\mu\nu}C\,, \quad \gamma B_\mu = \nabla_\mu C - 2mC_\mu\,, \quad \gamma\varphi = \mu\,C\,. \tag{4.6}
$$

---

[9]Note that the physical graviton mass is related to the scale m by $m_{\text{graviton}}^2 = 2m^2$. Our choice of parametrization simplifies many of the equations that follow.

[10]As a result the partially massless point can only be obtained in the dS background. For conventions where the partially massless limit is possible with AdS as background but manifestly non-unitary (albeit real) Lagrangian, see [65].

For the action of δ on the antifields, we have

$$\delta h^{*\mu\nu} = \Box h^{\mu\nu} - 2\nabla_\rho \nabla^{(\mu} h^{\nu)\rho} + \nabla^\mu \nabla^\nu h + g^{\mu\nu} \left(\nabla_\rho \nabla_\sigma h^{\rho\sigma} - \Box h\right)$$
$$- 2m \left(\nabla^{(\mu} B^{\nu)} - g^{\mu\nu} \nabla_\rho B^\rho\right) - 2\left(\frac{D-1}{\sigma L^2} + m^2\right) h^{\mu\nu}$$
$$+ \left(\frac{D-1}{\sigma L^2} + 2m^2\right) g^{\mu\nu} h - \mu m \, g^{\mu\nu} \varphi \,, \tag{4.7}$$

$$\delta B^{*\mu} = \nabla_\nu F^{\nu\mu} + 2m\left(\nabla_\nu h^{\mu\nu} - \nabla^\mu h\right) + \mu \nabla^\mu \varphi - \frac{2(D-1)}{\sigma L^2} B^\mu \,, \tag{4.8}$$

$$\delta \varphi^* = \Box \varphi - \mu \left(\nabla_\mu B^\mu + mh\right) + \frac{2Dm^2}{D-2} \varphi \,, \tag{4.9}$$

while on the antighosts it is

$$\delta C^{*\mu} = -2\nabla_\nu h^{*\mu\nu} - 2m B^{*\mu} \,, \quad \delta C^* = \frac{2m}{D-2} h^* - \nabla_\mu B^{*\mu} + \mu \, \varphi^* \,. \tag{4.10}$$

4.1 Cubic deformations

We follow the same steps as in the case of massive Yang-Mills theory. Our guiding principle is to reproduce the cubic vertex of the Einstein-Hilbert Lagrangian with cosmological constant, plus other terms with no more than two derivatives. This principle was also followed in the analyses of Zinoviev, see e.g. [28, 61–63]. It turns out that, for this to be the case, one must take the following expression for the gauge algebra deforming term a_2,

$$a_2 = -\kappa \, C^{*\mu} C^\nu \nabla_\nu C_\mu + g \, C^{*\mu} C_\mu C \,, \tag{4.11}$$

with deformation parameters κ and g. The first term is what leads to the Einstein-Hilbert cubic vertex [19], with the constant κ that coincides with $1/M_P^{(D-2)/2}$, M_P being the Planck mass. The second term is needed in order to lift a_2 to a cubic vertex a_0 with at most two derivatives.

The two contractions in (4.11) can be lifted independently, that is they both satisfy the master equation at antifield number one, $\gamma a_1 + \delta a_2 + \text{t.d.} = 0$. Defining

$$a_2^{(\text{GR})} = C^{*\mu} C^\nu \nabla_\nu C_\mu \,, \qquad a_2^{(\text{extra})} = C^{*\mu} C_\mu C \,, \tag{4.12}$$

we eventually obtain

$$a_1^{(\text{GR})} = - h^{*\mu\nu} \left(C^\rho \nabla_\rho h_{\mu\nu} + 2\nabla_{(\mu} C^\rho h_{\nu)\rho}\right) - \frac{2m}{D-2} h^{*\mu\nu} C h_{\mu\nu} + \frac{2m}{D-2} h^* C^\mu B_\mu$$
$$+ m \, B^{*\mu} C^\nu h_{\mu\nu} + \frac{1}{2} B^{*\mu} C^\nu F_{\mu\nu} - \frac{2m^2}{\mu(D-2)} B^{*\mu} C_\mu \varphi \,, \tag{4.13}$$

$$a_1^{(\text{extra})} = h^{*\mu\nu} C h_{\mu\nu} - 2h^{*\mu\nu} C_\mu B_\nu + \frac{2m}{\mu} B^{*\mu} C_\mu \varphi \,. \tag{4.14}$$

Next we must also consider the homogeneous terms satisfying $\gamma \bar{a}_1 + \text{t.d.} = 0$. There are several candidates with at most one derivative in the gauge transformation laws. A priori, we could say that none of them are interesting since anyway they lead to trivial vertices,

obtained from the free theory by a redefinition of the fields. Among them, there is a single $\bar{a}_1$ that we will consider for the reason that, when added to $a_1 := -\kappa a_1^{(\text{GR})} + g a_1^{(\text{extra})}$ with a well-chosen coefficient, it gives a cubic vertex containing at most two derivatives in total. The other candidates do not have the same effect. Explicitly, the $\bar{a}_1$ that enables one to obtain a two-derivative cubic vertex is

$$\bar{a}_1 = \varphi^* C^\mu \left(\nabla_\mu \varphi - \mu B_\mu \right) . \tag{4.15}$$

Thus, at this stage we have the antifield-1 solution

$$a_1 = -\kappa a_1^{(\text{GR})} + g a_1^{(\text{extra})} + \beta \bar{a}_1 , \tag{4.16}$$

with three free parameters κ, g and β.

At the last step of the resolution of the master equation at first order in deformation, when solving $\gamma a_0 + \delta a_1 + \text{t.d.} = 0$ for the vertex a_0, we find a unique solution for a_0 that requires the constants in (4.16) to take the following values:

$$g = -\frac{m\kappa}{2} , \qquad \beta = \left[\frac{D+2}{4(D-1)} + \frac{D-4}{2\mu^2 \sigma L^2} \right] \kappa . \tag{4.17}$$

The resulting cubic vertex has a lengthy expression (A.1) that we give in appendix A. In the unitary gauge, it reduces to

$$a_0 \Big|_{B_\mu = 0 = \varphi} = \kappa \mathcal{L}_{\text{EH}}^{(3)} + \kappa m^2 \left(h^{\mu\nu} h_\mu{}^\rho h_{\nu\rho} - \frac{5}{4} h h^{\mu\nu} h_{\mu\nu} + \frac{1}{4} h^3 \right) , \tag{4.18}$$

with $\mathcal{L}_{\text{EH}}^{(3)}$ the cubic part of the Einstein-Hilbert Lagrangian. The potential vertex proportional to m^2 coincides with the one found in [28, 66], and is a particular member of the dRGT class of massive gravity theories (see [56, 67] for reviews). In fact, the complete solution (4.18) happens to be special, in that it is the unique massive graviton cubic interaction consistent with positivity constraints of eikonal scattering amplitudes and absence of superluminality [46] (see also [68–74] for other studies of the S-matrix in massive gravity). It is also interesting that, in $D = 4$, this vertex corresponds to the unique nonlinear action of a partially massless spin-2 field [28, 51, 75, 76], although as is well known the theory happens to be obstructed at higher orders [77–79] (more on this below).

This is not the end of the story, as we still have the option of adding homogeneous solutions $\bar{a}_0$, such that $\gamma \bar{a}_0 + \text{t.d.} = 0$. Recall from the theorem of section 2.3 that all such solutions are in fact of the Born-Infeld type, i.e. they satisfy $\gamma \bar{a}_0 = 0$. The goal is therefore to determine the cohomology of the differential γ in the fields. Interestingly, the answer turns out to be very simple:

$$H^0(\gamma) \cong \{ f[H_{\mu\nu}] \} , \tag{4.19}$$

where

$$H_{\mu\nu} := h_{\mu\nu} + \frac{1}{m} \nabla_{(\mu} B_{\nu)} - \frac{1}{\mu m} \nabla_\mu \nabla_\nu \varphi - \frac{2m}{\mu(D-2)} g_{\mu\nu} \varphi , \tag{4.20}$$

is the unique Stueckelberg gauge invariant combination of the fields, as all other invariants can be built out of $H_{\mu\nu}$ and its derivatives. In particular the linearized Weyl tensor (which

must clearly be invariant since the transformation of $h_{\mu\nu}$ is nothing but a linear diffeomorphism plus a Weyl rescaling) can be expressed in terms of appropriately projected second derivatives of $H_{\mu\nu}$.

The $\bar{a}_0$ vertices we seek are thus given by all the possible cubic combinations of the tensor $H_{\mu\nu}$ and derivatives thereof. At the lowest order in derivatives we can simply take contractions of powers of $H_{\mu\nu}$ with no extra derivatives. But in fact there are more possibilities, since the tensors

$$H^{\boxminus}_{\mu\nu\rho} := \nabla_\mu H_{\nu\rho} - \nabla_\nu H_{\mu\rho} , \tag{4.21}$$

$$H^{\boxplus}_{\mu\nu\rho\sigma} := \nabla_\mu \nabla_{[\rho} H_{\sigma]\nu} - \nabla_\nu \nabla_{[\rho} H_{\sigma]\mu} + \nabla_\rho \nabla_{[\mu} H_{\nu]\sigma} - \nabla_\sigma \nabla_{[\mu} H_{\nu]\rho} , \tag{4.22}$$

also contain no more than two derivatives and can be used to construct gauge invariant vertices. As the notation suggests, these correspond respectively to the "hook" and "window" Young projections of the first and second derivatives of $H_{\mu\nu}$. Now, generic cubic contractions of $H_{\mu\nu}$, $H^{\boxminus}_{\mu\nu\rho}$, and $H^{\boxplus}_{\mu\nu\rho\sigma}$ will contain six derivatives, but there are three special combinations, namely

$$\bar{a}_0^{(\text{dRGT})} = \epsilon^{\mu_1\cdots\mu_3\rho_1\cdots\rho_{D-3}} \epsilon^{\nu_1\cdots\nu_3}{}_{\rho_1\cdots\rho_{D-3}} H_{\mu_1\nu_1} H_{\mu_2\nu_2} H_{\mu_3\nu_3} , \tag{4.23}$$

$$\bar{a}_0^{(\text{PL}_1)} = \epsilon^{\mu_1\cdots\mu_4\rho_1\cdots\rho_{D-4}} \epsilon^{\nu_1\cdots\nu_4}{}_{\rho_1\cdots\rho_{D-4}} H^{\boxplus}_{\mu_1\nu_1\mu_2\nu_2} H_{\mu_3\nu_3} H_{\mu_4\nu_4} , \tag{4.24}$$

$$\bar{a}_0^{(\text{PL}_2)} = \epsilon^{\mu_1\cdots\mu_5\rho_1\cdots\rho_{D-5}} \epsilon^{\nu_1\cdots\nu_5}{}_{\rho_1\cdots\rho_{D-5}} H^{\boxplus}_{\mu_1\nu_1\mu_2\nu_2} H^{\boxplus}_{\mu_3\nu_3\mu_4\nu_4} H_{\mu_5\nu_5} , \tag{4.25}$$

which actually contain only four derivatives (upon integration by parts).[11] When $\bar{a}_0^{(\text{dRGT})}$ is added to the solution of the deformation procedure, eq. (4.18), it yields the full one-parameter cubic interactions of dRGT massive gravity. The terms $\bar{a}_0^{(\text{PL})}$ on the other hand correspond to the so-called pseudo-linear vertices analyzed in [80, 81]. Note however that $\bar{a}_0^{(\text{PL}_2)}$ is only nontrivial in $D \geq 5$ dimensions.

4.2 Partially massless theory

As mentioned before, there exists a unique cubic vertex for a partially massless (PM) spin-2 field in four dimensions, which moreover happens to belong to the dRGT class of massive gravity theories. Since we have recovered all the cubic interactions of dRGT theory via the deformation analysis, it is clear that the PM vertex should appear as a by-product. Indeed, we observed above that the solution in unitary gauge (4.18) of the descent equations precisely matches the cubic PM Lagrangian previously found via other methods [51].

But this conclusion is a bit too quick since the calculations that led to eq. (4.18) involved the use of the relation $C = \frac{1}{\mu}\gamma\varphi$, and now $\mu := \sqrt{\frac{2(D-1)}{D-2}}\sqrt{\frac{D-2}{\sigma L^2} + 2m^2} = 0$ because $\sigma = -1$ (dS background) and $2L^2 m^2 = (D-2)$ at the PM point. Thus, for a PM spin-2, the ghost C is now in the cohomology of γ, which turns out to significantly differ from the one associated with the massive set-up, and generates a standard (as opposed to Stueckelberg) gauge symmetry, as we of course expected. In practical terms this means

---

[11]Note that $\bar{a}_0^{(\text{PL}_1)}$ and $\bar{a}_0^{(\text{PL}_2)}$ may be alternatively written in terms of $H^{\boxminus}_{\mu\nu\rho}$ after integrating by parts.

that any step where we divided by μ is not allowed. A first implication is that the antifield-2 terms $a_2^{(\text{GR})}$ and $a_2^{(\text{extra})}$ in eq. (4.12) can no longer be lifted independently, but instead we have

$$a_1^{(\text{PM})} = -\kappa \left(a_1^{(\text{GR})} + \frac{m}{D-2} a_1^{(\text{extra})} \right) . \tag{4.26}$$

The combination in parenthesis is finite at the PM point and moreover does not depend on φ, which is decoupled from the beginning. For that reason no $\bar{a}_1$ should be considered since the only one used in the massive case, eq. (4.15), contains φ.

Continuing with the last descent equation we find that $a_1^{(\text{PM})}$ can be lifted to a cubic vertex only if one sets

$$D = 4 . \tag{4.27}$$

The expression for the vertex is given in (A.5). Actually this solution cannot directly be obtained from our results concerning the massive case and taking the PM limit (A.4) in $D = 4$ as some terms diverge and the scalar mode does not decouple from the other modes. Moreover, (4.17) would have given $\beta = \frac{\kappa}{2}$ while it should have been zero since (4.15) involves the scalar field. In order to see the PM case as a limit of the massive case, one should therefore treat the generic massive theory without (4.15), which is in reality nothing else than doing a field redefinition of φ, since (4.15) is γ-exact, but the resulting vertices will have more than two derivatives. In fact all these higher-derivative terms contain the scalar field which decouples in the PM limit, and this is why one can find a vertex with no more than two derivatives in the PM case without adding (4.15). We also expect other important features peculiar to a PM field to show up at the next order in the deformation analysis, since it is known that PM theory is obstructed at quartic order whereas massive gravity is not [77–79].

So far we have ignored the homogeneous solutions satisfying $\gamma \bar{a}_0 + \text{t.d.} = 0$. Here the analysis is quite different in PM relative to the generic case, since obviously the tensor $H_{\mu\nu}$ defined in (4.20) no longer makes sense. If we define instead

$$\mathcal{H}_{\mu\nu} := h_{\mu\nu} + \frac{1}{m} \nabla_{(\mu} B_{\nu)} , \tag{4.28}$$

we find that

$$\mathcal{F}_{\mu\nu\rho} := \nabla_\mu \mathcal{H}_{\nu\rho} - \nabla_\nu \mathcal{H}_{\mu\rho} , \tag{4.29}$$

is γ-closed and generates the cohomology of γ in the fields; the tensor $\mathcal{F}_{\mu\nu\rho}$ is in fact nothing but the field strength of a PM graviton after one fixes unitary gauge. The possible Born-Infeld type vertices, $\gamma \bar{a}_0^{(\text{BI})} = 0$, are therefore given by all the possible contractions of $\mathcal{F}_{\mu\nu\rho}$ and its derivatives. Importantly, the dRGT and pseudo-linear terms in eqs. (4.23)–(4.25) are not allowed in the PM case. Moreover, because the ghost C does not correspond to a Stueckelberg symmetry anymore, the theorem of section 2 does not apply and the possibility of constructing Chern-Simons type vertices for a PM spin-2 field is open. A complete analysis of the PM set-up in the BRST-BV formalism, including multiple fields (for which cubic vertices were already studied in [82]) as well as the coupling to massless spin-2 particles, will be presented elsewhere.

4.3 Triviality of the gauge algebra

The procedure described in subsection 3.5 in the context of massive Yang-Mills can be applied in a very analogous way to massive gravity. The result will make it explicit that the cubic vertex (4.18) (see (A.1) for the full expression) can be written as a Born-Infeld type vertex followed by a field redefinition.

Writing a_2 as

$$
\begin{aligned}
a_2 &= -\kappa\, C^{*\mu} C^{\nu} \nabla_\nu C_\mu - \frac{m\kappa}{2}\, C^{*\mu} C_\mu C \\
&= \gamma \left(\frac{\kappa}{2}\, C^{*\mu} C^{\nu} \left(h_{\mu\nu} + \frac{1}{2m} F_{\mu\nu} \right) + \frac{m\kappa}{\sqrt{2}\,\mu} \left(\frac{D-4}{D-2} \right) C^{*\mu} C_\mu \varphi \right) = a_2^t\,,
\end{aligned}
\tag{4.30}
$$

allows us to read off the redefinition of the gauge parameter

$$
\xi_\mu \longrightarrow \xi_\mu + \frac{\kappa}{2} \left(h_{\mu\nu} + \frac{1}{2m} F_{\mu\nu} \right) \xi^\nu + \frac{m\kappa}{\sqrt{2}\,\mu} \left(\frac{D-4}{D-2} \right) \varphi\, \xi_\mu
\tag{4.31}
$$

that trivializes the cubic deformation of the gauge algebra (4.11).

The next step in the procedure is the resolution of the analogue of eq. (3.52), which gives the information about the field redefinition. Here we have to solve for c_1 in

$$
\delta \left(\frac{\kappa}{2}\, C^{*\mu} C^{\nu} \left(h_{\mu\nu} + \frac{1}{2m} F_{\mu\nu} \right) + \frac{m\kappa}{\sqrt{2}\,\mu} \left(\frac{D-4}{D-2} \right) C^{*\mu} C_\mu \varphi \right) - a_1 = \gamma c_1 + \text{t.d.}\,,
\tag{4.32}
$$

where a_1 is the one given in (4.16) with the deformation parameters related through (4.17). The simplest solution of this equation is rather lengthy and will be given in (A.6) in appendix A. From the resulting c_1 we obtain the field redefinition (A.7) that makes the gauge transformation trivial, that is equal to that of the free theory we started with.

Lastly the Born-Infeld vertex is given by

$$
\bar{a}_0 = a_0 + \delta c_1\,,
\tag{4.33}
$$

where the explicit expressions for a_0 and c_1 can be found in the appendix A, respectively in (A.1) and (A.6). We emphasize again that the fact that this vertex is of the Born-Infeld type follows by construction, see eq. (3.57). It is actually easy to construct $\bar{a}_0$ in its manifestly gauge invariant form, i.e., in terms of $H_{\mu\nu}$. Indeed, any analytic function $f(h_{\mu\nu}, B_\mu, \varphi)$ which is gauge-invariant, $\gamma f = 0$, can be written as

$$
f(h_{\mu\nu}, B_\mu, \varphi) = f(h_{\mu\nu}, B_\mu, \varphi)\Big|_{B_\mu = 0 = \varphi\,,\ h_{\mu\nu} \to H_{\mu\nu}}\,.
\tag{4.34}
$$

This can be proven by observing the following:
(1) Any strictly gauge invariant function can be written as a function of $H_{\mu\nu}$ (and its derivatives):

$$
f(h_{\mu\nu}, B_\mu, \varphi) = \tilde{f}(H_{\mu\nu})\,;
\tag{4.35}
$$

(2) Any analytic function of $h_{\mu\nu}$, B_μ and φ can be split as a function of solely $h_{\mu\nu}$ plus a function of all the fields which vanishes in unitary gauge:

$$
f(h_{\mu\nu}, B_\mu, \varphi) = f_1(h_{\mu\nu}) + f_2(h_{\mu\nu}, B_\mu, \varphi)\,, \qquad f_2(h_{\mu\nu}, B_\mu, \varphi)\Big|_{B_\mu = 0 = \varphi} = 0\,.
\tag{4.36}
$$

Combining these two equations one obtains $\tilde{f}(H_{\mu\nu}) = f_1(h_{\mu\nu}) + f_2(h_{\mu\nu}, B_\mu, \varphi)$, which implies, upon going to unitary gauge, $\tilde{f}(h_{\mu\nu}) = f_1(h_{\mu\nu})$. Using this in (4.35) proves the result (4.34).

Thus $\bar{a}_0$ can be easily written in compact form as

$$
\begin{aligned}
\bar{a}_0 &= \left(a_0 + \delta c_1\right)\Big|_{B_\mu = 0 = \varphi\,,\; h_{\mu\nu} \to H_{\mu\nu}} \\
&= \kappa\, \mathcal{L}_{\mathrm{EH}}^{(3)}(H_{\mu\nu}) + \kappa\, m^2 \left(H^{\mu\nu} H_\mu{}^\rho H_{\nu\rho} - \frac{5}{4}\, H H^{\mu\nu} H_{\mu\nu} + \frac{1}{4}\, H^3 \right) \\
&\quad + \frac{\kappa}{4} \Bigg[\Box H^{\mu\nu} - 2\nabla_\rho \nabla^{(\mu} H^{\nu)\rho} + \nabla^\mu \nabla^\nu H + g^{\mu\nu} \left(\nabla_\rho \nabla_\sigma H^{\rho\sigma} - \Box H \right) \\
&\qquad - 2\left(\frac{D-1}{\sigma L^2} + m^2 \right) H^{\mu\nu} + \left(\frac{D-1}{\sigma L^2} + 2m^2 \right) g^{\mu\nu} H \Bigg] H_{\mu\lambda} H_\nu{}^\lambda\,.
\end{aligned}
\tag{4.37}
$$

In conclusion, at the cubic level, the deformation obtained by deforming solely the action — and not the gauge transformations — by the addition of this $\bar{a}_0$ is equivalent to the deformation performed in section 4.1, as they differ by a redefinition of the fields and gauge parameters of the theory.

5 Discussion

In this paper we have initiated the study of the consistent deformations of massive field theories using the BRST-BV formalism as a tool. Although such theories of course do not possess any gauge invariance in their usual parametrizations, it is well known that gauge symmetries can be straightforwardly introduced via the Stueckelberg procedure. Our goal in this work was two-fold: to understand the peculiarities of Stueckelberg gauge theories in the context of the BRST-BV formulation, and to see whether the method can be successfully applied to obtain consistent interaction vertices for massive fields.

Regarding the first question, we have unveiled an interesting picture of Stueckelberg models by showing that they always admit a choice of field variables in which the gauge algebra is manifestly Abelian, and with the action being constructed out of the gauge invariant building blocks of the initial free theory, what is called a Born-Infeld type model.

We illustrated these properties by studying the possible interaction vertices of massive Yang-Mills theory and of massive gravity on a maximally symmetric space. We showed that the BRST-BV formalism allows one to derive all the vertex structures, modulo some assumptions on the number of derivatives, that have been previously classified using other approaches. We also analyzed the special case in which the graviton is partially massless, and showed that the method allows us to recover, as a by-product, the known cubic vertex of PM gravity in four dimensions. We expect that the BRST-BV method will display its full potential with more complicated theories; we plan to tackle some of them in dedicated investigations, in particular the case of multiple massive and massless gravitons. It would also be interesting to gain further insight regarding the physically special spin-2 vertex, eq. (4.18), that one gets if one were to ignore γ-closed solutions.

It is worth emphasizing that the interactions given by the BRST-BV deformation procedure are consistent in the sense that they preserve the number of gauge symmetries.

On the other hand, in order for a deformation to maintain the number of degrees of freedom, one also ought to pay attention to Ostrogradski modes associated to equations of motion of order higher than two. The BRST-BV method is oblivious of this aspect, which therefore calls for a separate analysis, for instance a full Hamiltonian counting or a study of the decoupling limit theory. In this respect, however, massive theories in the Stueckelberg language are not at all different from standard gauge theories. The restriction on the number of derivatives appearing in the deformations is thus important for recovering "healthy" theories such as dRGT massive gravity.

Acknowledgments

A   Full cubic vertex of massive gravity

The full cubic vertex arising in the deformation of linearized massive gravity in the Stueckelberg formulation, and which reduces to (4.18) in the unitary gauge, is given by

$$
\begin{aligned}
a_0 =\ & \kappa \mathcal{L}_{\text{EH}}^{(3)} + \kappa\, m^2 \left(h^{\mu\nu} h_\mu{}^\rho h_{\nu\rho} - \frac{5}{4}\, h h^{\mu\nu} h_{\mu\nu} + \frac{1}{4}\, h^3 \right) + \frac{3m\kappa}{2} A^\sigma h^{\mu\nu} \nabla_\sigma h_{\mu\nu} \\
& - m\kappa A^\mu h^{\nu\sigma} \nabla_\sigma h_{\mu\nu} - 2m\kappa A^\mu h_{\mu\nu} \nabla_\sigma h^{\nu\sigma} + \frac{m\kappa}{2} A_\mu h \nabla_\nu h^{\mu\nu} + \frac{3m\kappa}{2} A_\mu h^{\mu\nu} \nabla_\nu h \\
& - \frac{m\kappa}{2} A_\mu h \nabla^\mu h - \frac{\kappa}{16} h F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{4} h_{\mu\nu} F^{\mu\sigma} F_\sigma{}^\mu + \frac{\kappa\tilde{\mu}^2}{2\mu^2} h^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - \frac{\kappa\tilde{\mu}^2}{4\mu^2} h \nabla_\mu \varphi \nabla^\mu \varphi \\
& + \frac{m\kappa}{8\mu} \left(\frac{D-4}{D-2} \right) \varphi F_{\mu\nu} F^{\mu\nu} - \frac{\kappa\tilde{\mu}^2}{\mu} h^{\mu\nu} A_\mu \nabla_\nu \varphi + \frac{\kappa\tilde{\mu}^2}{2\mu} h A^\mu \nabla_\mu \varphi - \frac{m\kappa\tilde{\mu}^2}{\mu^2} A^\mu \varphi \nabla_\mu \varphi \\
& + \frac{m\kappa\tilde{\mu}^2}{2\mu^3} \varphi \nabla_\mu \varphi \nabla^\mu \varphi + \frac{m\kappa}{2\mu} \left(\frac{2(D-1)}{\sigma L^2} + \frac{3Dm^2}{D-2} \right) \varphi h_{\mu\nu} h^{\mu\nu} - \frac{\kappa m\tilde{\mu}^2}{2\mu} \varphi h^2 \\
& + \frac{\kappa}{2} \left(\frac{D-1}{\sigma L^2} - m^2 \right) h_{\mu\nu} A^\mu A^\nu - \frac{\kappa}{4} \left(\frac{D-1}{\sigma L^2} + m^2 \right) h A_\mu A^\mu + \frac{m\kappa\tilde{\mu}^2}{2\mu} \varphi A_\mu A^\mu \\
& + \frac{m^2\kappa}{2\mu^2} \left(\frac{D-1}{D-2} \right) \left(\frac{D+2}{\sigma L^2} + \frac{12m^2}{D-2} \right) h \varphi^2 - \frac{m^3\kappa}{3\mu^3} \frac{D(D-1)}{(D-2)^2} \left(\frac{D+2}{\sigma L^2} + \frac{12m^2}{D-2} \right) \varphi^3 \,,
\end{aligned}
\tag{A.1}
$$

where the Einstein-Hilbert cubic piece is written as

$$
\begin{aligned}
\mathcal{L}_{\text{EH}}^{(3)} =\ & \frac{1}{2} h^{\mu\nu} \nabla_\mu h^{\rho\sigma} \nabla_\nu h_{\rho\sigma} - \frac{1}{2} h^{\mu\nu} \nabla_\mu h \nabla_\nu h + 2 h^{\mu\nu} \nabla_\rho h_\mu{}^\rho \nabla_\nu h - h^{\mu\nu} \nabla_\rho h \nabla^\rho h_{\mu\nu} \\
& + \frac{1}{4} h \nabla^\rho h \nabla_\rho h - 2 h^{\mu\nu} \nabla_\rho h_\mu{}^\rho \nabla_\sigma h_\nu{}^\sigma - 2 h^{\mu\nu} \nabla_\sigma h_\rho{}^\sigma \nabla_\nu h_\mu{}^\rho + h \nabla_\nu h^{\nu\rho} \nabla_\sigma h_\rho{}^\sigma
\end{aligned}
$$

$$+ h^{\mu\nu}\nabla_\sigma h_\nu{}^\sigma\nabla^\sigma h_{\mu\nu} - \frac{1}{2}h\nabla^\sigma h\nabla_\sigma h_\rho{}^\sigma + h^{\mu\nu}\nabla^\sigma h_\mu{}^\rho\nabla_\rho h_{\nu\sigma} + h^{\mu\nu}\nabla^\sigma h_\mu{}^\rho\nabla_\sigma h_{\nu\rho}$$

$$- \frac{1}{2}h\nabla_\sigma h_{\nu\sigma}\nabla^\sigma h^{\nu\rho} - \frac{1}{4}h\nabla_\sigma h_{\nu\rho}\nabla^\sigma h^{\nu\rho} - \frac{2(D+2)}{3\sigma L^2}h^{\mu\nu}h_\mu{}^\rho h_{\nu\rho} + \frac{3}{\sigma L^2}hh_{\mu\nu}h^{\mu\nu}$$

$$+ \frac{(D-7)}{6\sigma L^2}h^3 \tag{A.2}$$

and with the definition

$$\tilde{\mu} := \sqrt{\frac{D-1}{\sigma L^2} + \left(\frac{D+2}{D-2}\right)m^2} \tag{A.3}$$

for the parameter $\tilde{\mu}$ which has dimension of mass.

In the dS_4 background, the PM limit of this vertex is not finite but can be written as

$$a_0^{(\text{PM limit})} = \kappa\,\mathcal{L}_{\text{EH}}^{(3)} + \lim_{m\to\frac{1}{L}}\Bigg[\kappa\,m^2\left(h^{\mu\nu}h_\mu{}^\rho h_{\nu\rho} - \frac{5}{4}\,hh^{\mu\nu}h_{\mu\nu} + \frac{1}{4}\,h^3\right) + \frac{3m\kappa}{2}A^\sigma h^{\mu\nu}\nabla_\sigma h_{\mu\nu}$$

$$- m\kappa A^\mu h^{\nu\sigma}\nabla_\sigma h_{\mu\nu} - 2m\kappa A^\mu h_{\mu\nu}\nabla_\sigma h^{\nu\sigma} + \frac{m\kappa}{2}A_\mu h\nabla_\nu h^{\mu\nu} + \frac{3m\kappa}{2}A_\mu h^{\mu\nu}\nabla_\nu h$$

$$- \frac{m\kappa}{2}A_\mu h\nabla^\mu h - \frac{\kappa}{16}hF_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{4}h_{\mu\nu}F^{\mu\sigma}F_\sigma{}^\mu + \frac{\kappa}{4}h^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi$$

$$- \frac{\kappa}{8}h\nabla_\mu\varphi\nabla^\mu\varphi - \frac{m\kappa}{2}A^\mu\varphi\nabla_\mu\varphi - 2m^2\kappa\,h_{\mu\nu}A^\mu A^\nu + \frac{m^2\kappa}{2}hA_\mu A^\mu + \frac{3m^2\kappa}{4}h\varphi^2$$

$$+ \frac{m\kappa}{\sqrt{6}\sqrt{m^2 - \frac{1}{L^2}}}\left(\frac{1}{4}\,\varphi\nabla_\mu\varphi\nabla^\mu\varphi - m^2\,\varphi^3\right)\Bigg]. \tag{A.4}$$

Instead of doing the PM limit of the result (A.1), one can follow the deformation procedure as in section 4.2 to eventually obtain

$$a_0^{(\text{PM})} = \kappa\,\mathcal{L}_{\text{EH}}^{(3)} + \kappa\,m^2\left(h^{\mu\nu}h_\mu{}^\rho h_{\nu\rho} - \frac{5}{4}\,hh^{\mu\nu}h_{\mu\nu} + \frac{1}{4}\,h^3\right) + \frac{3m\kappa}{2}A^\sigma h^{\mu\nu}\nabla_\sigma h_{\mu\nu}$$

$$- m\kappa A^\mu h^{\nu\sigma}\nabla_\sigma h_{\mu\nu} - 2m\kappa A^\mu h_{\mu\nu}\nabla_\sigma h^{\nu\sigma} + \frac{m\kappa}{2}A_\mu h\nabla_\nu h^{\mu\nu} + \frac{3m\kappa}{2}A_\mu h^{\mu\nu}\nabla_\nu h$$

$$- \frac{m\kappa}{2}A_\mu h\nabla^\mu h - \frac{\kappa}{16}hF_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{4}h_{\mu\nu}F^{\mu\sigma}F_\sigma{}^\mu - 2m^2\kappa\,h_{\mu\nu}A^\mu A^\nu + \frac{m^2\kappa}{2}hA_\mu A^\mu, \tag{A.5}$$

where $m = \frac{1}{L}$. The above expression exactly is (A.4) with the scalar field φ put to zero before taking the limit.

In section 4.3 we calculated the field redefinition mapping the algebra-deforming vertices to the Born-Infeld-like deformation. This field redefinition is encoded in the object c_1 (see eq. (4.32)) which is written in full as

$$c_1 = \kappa h^{*\mu\nu}\Bigg[\frac{1}{4}h_{\mu\sigma}h_\nu{}^\sigma - \frac{1}{4m}h_\mu{}^\rho F_{\nu\rho} + \frac{1}{2m}A^\rho\nabla_\nu h_{\mu\rho} - \frac{1}{2m}A^\rho\nabla_\rho h_{\mu\nu} - \frac{1}{2\mu m}\nabla^\rho\varphi\nabla_\nu h_{\mu\rho}$$

$$+ \frac{1}{2\mu m}\nabla^\rho\varphi\nabla_\rho h_{\mu\nu} - \frac{1}{4\mu m^2}\nabla^\rho\varphi\nabla_\nu F_{\mu\rho} + \frac{1}{2\mu m^2\,\sigma L^2}g_{\mu\nu}A^\rho\nabla_\rho\varphi$$

$$- \frac{1}{2\mu m^2}\left(\frac{1}{\sigma L^2} - \left(\frac{D-4}{D-2}\right)m^2\right)A_\mu\nabla_\nu\varphi + \frac{1}{16m^2}F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4m^2}A^\rho\nabla_\mu F_{\nu\rho}$$

$$
+ \frac{1}{4\mu^2 m^2} \left(\frac{1}{\sigma L^2} - \left(\frac{D-6}{D-2} \right) m^2 \right) \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{8(D-1)m^2} g_{\mu\nu} \nabla_\rho \varphi \nabla^\rho \varphi
$$
$$
- \frac{(D-4)m}{2(D-2)\mu} \varphi h_{\mu\nu} + \frac{1}{4m^2} \left(\frac{1}{\sigma L^2} - m^2 \right) A_\mu A_\nu
$$
$$
- \frac{1}{4m^2} \left(\frac{1}{\sigma L^2} - \frac{2m^2}{D-2} \right) g_{\mu\nu} A_\rho A^\rho + \frac{(D-4)m^2}{2(D-2)^2 \mu^2} g_{\mu\nu} \varphi^2 \Bigg] . \tag{A.6}
$$

This translates into the field redefinition

$$
h_{\mu\nu} \longrightarrow h_{\mu\nu} + \frac{\kappa}{4} h_{\mu\sigma} h_\nu{}^\sigma + \dots \tag{A.7}
$$

where the dots refer to terms proportional to the spin-1 and spin-0 modes and vanish in the unitary gauge.

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