

Stringy AdS_3 from the worldsheet

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ABSTRACT: We investigate the behaviour of the bosonic string on AdS_3 with H-flux at stringy scales, looking in particular for a ‘tensionless’ limit in which there are massless higher spin gauge fields. We do this by revisiting the physical spectrum of the $\mathfrak{sl}(2, \mathbb{R})_k$ WZW model and considering the limit in which k becomes small. At $k = 3$ we find that there is an infinite stringy tower of massless higher spin fields which are part of a continuum of light states. This can be viewed as a novel tensionless limit, which appears to be distinct from that inferred from the symmetric orbifold description of superstring AdS_3 vacua.

KEYWORDS: Higher Spin Symmetry, AdS-CFT Correspondence, Bosonic Strings, Long strings

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1 Introduction

Bosonic string theory on three-dimensional anti-de Sitter space AdS_3 with H-flux or the superstring on AdS_3 with pure NS-NS background flux has a description in terms of a WZW model based on $\mathfrak{sl}(2, \mathbb{R})$ [1–3]. The theory has two dimensionful parameters, the string tension T and the AdS cosmological constant Λ . The dimensionless ratio T/Λ is proportional to the WZW level k , which is a continuous real parameter for $\mathfrak{sl}(2, \mathbb{R})$. The spectrum contains massive higher spin states with mass scale set by the string tension, so it is natural to ask whether an interacting massless higher spin theory could arise from a tensionless limit of the string theory in AdS [4–6]. The naive tensionless string limit would be to take $k \rightarrow 0$, but it is also possible interesting behaviour could emerge as k approaches a (small) critical value. In [7], the limit $k \rightarrow 2$ was considered, corresponding to the critical level of the $\mathfrak{sl}(2, \mathbb{R})$ affine algebra, and the possibility of massless higher spins arising in this limit was discussed.

Our purpose in this note is to examine the space-time spectrum of strings on AdS_3 (tensored with an internal CFT so as to give a critical string theory) as a function of k . In particular, we analyse whether the spectrum contains massless higher spin states at some critical value of k , corresponding to a tensionless limit of string theory on this background. We find that, while the $k \rightarrow 2$ limit appears problematic, $k = 3$ gives a spectrum with an infinite number of massless higher spin fields.

These massless higher spin fields are the lowest members of a continuum of physical states. This continuum arises from the continuous representations of $\mathfrak{sl}(2, \mathbb{R})$ in the so-called spectrally flowed sector (with ‘winding number’ $w = 1$). Physically, for large k these correspond to modes of a ‘long string’ near the boundary of AdS [1, 8, 9]. The tension of these strings tends to zero in the limit $k \rightarrow 3$, so the limit we are considering can be thought of as that in which long fundamental strings become effectively tensionless due to a cancellation between the contributions of the H-flux and their actual tension. Since these are strings stretched near the boundary of AdS, they have radial momentum excitations

which give rise to the continuum. Roughly speaking, as k becomes small, the curvature of AdS approaches the string scale and massive states become effectively localised toward the centre of AdS and only massless states can approach the boundary, so the only way states associated with the long string can stay near the boundary is for the string to become tensionless.

This is somewhat different from the tensionless limit that one sees in other AdS vacua, in which the actual tension of the fundamental string becomes small (in comparison to the AdS scale). In AdS_d with $d > 3$ the tensionless limit corresponds to a free gauge theory, which has additional conserved currents, thus implying an enhanced gauge symmetry in the bulk. It corresponds to the leading Regge trajectory of stringy states becoming massless. This is the subsector which is believed to be described by a Vasiliev theory [10] of higher spin gauge fields [4–6].

For the AdS_3 superstring (with R-R three-form flux), the tensionless limit is that in which the dual CFT_2 is the free symmetric orbifold theory. There, in contrast to the higher dimensional cases, we have a stringy tower of massless higher spin fields rather than a single Regge trajectory. This tensionless limit has been studied recently in [11–13], and a large underlying unbroken symmetry, dubbed the “higher spin square”, has been uncovered. The tensionless limit discussed here appears to be a more singular one in that the continuum of states would lead to a similar continuum in the dual CFT signalling, perhaps, a noncompact direction. However, it shares with the regular tensionless limit of the AdS_3 superstring the feature of having a stringy tower of massless states. We comment more on these similarities and differences in the discussion section. Before that, in the next section, we briefly review the bosonic string spectrum of Maldacena-Ooguri [1–3], and then go on to examine the special features at small k , including the new tensionless limit.

2 The worldsheet description of the spectrum

The $\text{SL}(2, \mathbb{R})$ group manifold is a three-dimensional hyperboloid with periodic time, giving rise to a discrete energy spectrum. We will work with AdS_3 , which is the universal covering space of this hyperboloid, and which has non-compact time so that the energy spectrum is not forced to be discrete. We will investigate the critical bosonic string theory formulated in terms of the WZW model of level k on AdS_3 , combined with an ‘internal’ CFT to give a CFT with total central charge $c = 26$. This theory has been studied extensively in [1–3, 14–18], and we will make extensive use of the results of [1].

2.1 Conventions and setup

Let us begin by fixing our notation and conventions, mostly following [1]. For general level k , the commutation relations of the $\mathfrak{sl}(2, \mathbb{R})$ algebra are

$$[J_m^3, J_n^\pm] = \pm J_{m+n}^\pm \tag{2.1}$$

$$[J_m^+, J_n^-] = -2 J_{m+n}^3 + k m \delta_{m,-n} \tag{2.2}$$

$$[J_m^3, J_n^3] = -\frac{k}{2} m \delta_{m,-n} . \tag{2.3}$$

For $k \neq 2$, the Sugawara construction is well defined,

$$\begin{aligned}
 L_n &= \frac{1}{k-2} \sum_{m=0}^{\infty} (J_{n-m}^+ J_m^- + J_{n-m}^- J_m^+ - 2J_{n-m}^3 J_m^3) \quad (n \neq 0) \\
 L_0 &= \frac{1}{k-2} \left[\frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3 + \sum_{m=1}^{\infty} (J_{-m}^+ J_m^- + J_{-m}^- J_m^+ - 2J_{-m}^3 J_m^3) \right],
 \end{aligned}
 \tag{2.4}$$

and the resulting Virasoro generators satisfy the commutation relations

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}
 \tag{2.5}$$

$$[L_m, J_n^a] = -n J_{m+n}^a,
 \tag{2.6}$$

where the central charge takes the value

$$c = \frac{3k}{k-2}.
 \tag{2.7}$$

For this to satisfy $c \leq 26$ requires $k \geq 52/23$, which in particular excludes $k = 2$, so that the $k = 2$ WZW model cannot be part of a conventional critical string theory.

The critical level is $k_{\text{crit}} = 2$ since then the Virasoro generators (2.4) are ill-defined. For this reason, it will be useful to define also the generators $l_n \equiv (k-2) L_n$, i.e.

$$l_n = \sum_{m=0}^{\infty} (J_{n-m}^+ J_m^- + J_{n-m}^- J_m^+ - 2J_{n-m}^3 J_m^3) \quad (n \neq 0)
 \tag{2.8}$$

$$l_0 = \left[\frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3 + \sum_{m=1}^{\infty} (J_{-m}^+ J_m^- + J_{-m}^- J_m^+ - 2J_{-m}^3 J_m^3) \right].
 \tag{2.9}$$

They satisfy the commutation relations

$$[l_m, l_n] = (k-2)(m-n) l_{m+n} + (3k(k-2)) m(m^2-1) \delta_{m,-n}
 \tag{2.10}$$

$$[l_m, J_n^a] = -(k-2)n J_{m+n}^a.
 \tag{2.11}$$

In particular, for $k = 2$, the l_m generators are central, i.e. they commute both with the current modes J_n^a , and with themselves. The existence of these commuting charges suggests that the theory at $k = 2$ could be integrable.

At $k = 2$, the conventional Virasoro constraints involving L_n cannot be imposed, giving the possibility of evading the exclusion of $k = 2$ [7]. Moreover, at the critical level of $k = 2$, affine representations have a large number of null states of the form [19]

$$l_{n_1} l_{n_2} \dots l_{n_p} |\phi\rangle,
 \tag{2.12}$$

where $n_i > 0$ are positive integers and $|\phi\rangle$ is any state satisfying $J_n^a |\phi\rangle = 0$ for $n > 0$. One may take this as evidence that also the space-time theory will develop a large gauge symmetry at $k = 2$, and a natural idea could be that this gauge symmetry is related to a higher spin theory or a tensionless string theory [7].¹ However, this expectation does not seem to be born out by a careful analysis of the spectrum. Instead, it seems that something special happens for $k = 3$.

¹Similar ideas have been pursued in [20, 21], see also [22].

2.2 Affine representations

We shall be interested in representations of the affine $\mathfrak{sl}(2, \mathbb{R})$ algebra based on Virasoro highest weight representations (which, however, need not be affine highest weight). In section 2.4 we will look also at more general classes of representations that arise as the spectrally flowed images of these representations and which are not even Virasoro highest weight [1]; however, let us first consider this simpler setting.

We will denote the Virasoro highest weight states by $|\hat{c}, m\rangle$, where \hat{c} is the eigenvalue of the quadratic Casimir C of $\mathfrak{sl}(2, \mathbb{R})$

$$C = \frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3, \tag{2.13}$$

while m is the eigenvalue of J_0^3 , i.e.

$$C|\hat{c}, m\rangle = \hat{c}|\hat{c}, m\rangle, \quad J_0^3|\hat{c}, m\rangle = m|\hat{c}, m\rangle. \tag{2.14}$$

Here ‘Virasoro highest weight’ means that

$$J_n^a|\hat{c}, m\rangle = 0, \quad \text{for } n > 0 \tag{2.15}$$

since this implies, in particular, that $L_n|\hat{c}, m\rangle = 0$ for $n > 0$; however, we do not assume that any of the states is annihilated by J_0^+ (or indeed J_0^-), i.e. we do not assume that it is ‘affine highest (or lowest) weight’.

There are five classes of unitary affine representations, and of these only two play a role in the spectrum of [1]. The two classes of (Virasoro) highest weight representations that appear in the spectrum of [1] are: (i) the discrete lowest weight representations \mathcal{D}_j^+ for which $m = j, j + 1, j + 2, \dots$ and $\hat{c} = \hat{c}(j) \equiv -j(j - 1)$ (with j positive), and (ii) the continuous representations $\mathcal{C}(p, \alpha)$ labelled by $p, \alpha \in \mathbb{R}$ for which the J_0^3 spectrum is unbounded both from below and above — m takes the values $m \in \alpha + \mathbb{Z}$ — and $\hat{c} = \frac{1}{4} + p^2$. Without loss of generality, the real parameter α can be taken to be in the range $0 \leq \alpha < 1$. For $k \neq 2$, we can immediately read off the conformal dimension of the ground states; indeed it follows from (2.4) that

$$L_0|\hat{c}, m\rangle = \frac{\hat{c}}{k-2}|\hat{c}, m\rangle \implies h_{\text{SL}(2, \mathbb{R})} = \frac{\hat{c}}{k-2}. \tag{2.16}$$

The full affine representation is then obtained from these Virasoro highest weight states by the action of the negative current modes, modulo null vectors. The physical states are characterised by the condition that they are annihilated by L_n with $n > 0$. In addition, they have to satisfy the mass-shell condition

$$h_{\text{SL}(2, \mathbb{R})} + h_{\text{int}} + N = \frac{\hat{c}}{k-2} + h_{\text{int}} + N = 1, \tag{2.17}$$

where \hat{c} is the value of the Casimir operator C on the ground states, while h_{int} is the conformal dimension of the internal CFT (making up the additional directions of the full string theory, in addition to AdS_3), and N is the excitation number of the state. In the

following we shall concentrate on the case $h_{\text{int}} = 0$, and take N to arise entirely from $\mathfrak{sl}(2, \mathbb{R})$ excitations, i.e. we take the ‘internal’ CFT to be in its trivial ground state. (It will become obvious from the following discussion that our conclusions will not be changed if we included excitations in the internal CFT.)

We can now relate these worldsheet quantities to spacetime variables. Physical states in AdS_3 fit into unitary representations of the anti-de Sitter group $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$, and for lowest-weight representations the energy is bounded below. The spacetime energy E and spin s of a state with J_0^3 eigenvalue m and \bar{J}_0^3 eigenvalue \bar{m} are

$$E = m + \bar{m}, \quad s = m - \bar{m}. \quad (2.18)$$

The AdS_3 mass is given by the quadratic Casimir and is defined by

$$m_{\text{AdS}_3}^2 = (E - |s|)(E + |s| - 2). \quad (2.19)$$

There is a Breitenlohner-Freedman (BF) bound $m_{\text{AdS}_3}^2 \geq -(|s| - 1)^2$.

Massless higher spin fields correspond to states for which $E = |s|$. For a lowest weight representation of $\text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R$ with lowest weights given by $j_L = m_L = (E + s)/2$ and $j_R = m_R = (E - s)/2$, the representation becomes reducible if $E = |s|$; if this is the case the representation contains a sub-representation of null states that describe longitudinal (pure gauge) modes which decouple. The corresponding spacetime fields are then massless higher spin gauge fields.

The conformal dimension of the spacetime CFT, h_{CFT} , is to be identified with the J_0^3 eigenvalue m , and likewise for the right-movers, i.e. $\bar{h}_{\text{CFT}} = \bar{m}$, the eigenvalue of \bar{J}_0^3 . The Breitenlohner-Freedman (BF) bound follows from the requirement of unitarity of the dual CFT, i.e. $h_{\text{CFT}}, \bar{h}_{\text{CFT}} \geq 0$.

2.3 Massless higher spin states: the spectrally unflowed case

Returning to the worldsheet, let us now see whether there exist massless higher spin states in the spectrum for some (small) value of k . We will first discuss the so-called spectrally unflowed sector corresponding to the two classes of Virasoro highest weight representations of the previous subsection. We will discuss the spectrally flowed versions of these representations in the following subsection.

For the continuous representations, the first term in (2.17) is positive for $k > 2$, and the only possible solution is $N = 0$ (since $h_{\text{int}} \geq 0$). These representations have spacetime energies unbounded from below — they violate the BF bound. The corresponding spacetime state is a tachyon; such a state was to be expected for bosonic string theory.

For the discrete (lowest weight) representations labelled by j , the Casimir $\hat{c} = \hat{c}(j) \equiv -j(j - 1)$ is negative, and there are more interesting solutions. First we solve (2.17) for j (as a function of N), to conclude that

$$j = \frac{1}{2} + \sqrt{\frac{1}{4} + (k - 2)(h_{\text{int}} + N - 1)}. \quad (2.20)$$

The no-ghost theorem [14, 15] implies that j has to be bounded by

$$0 \leq j \leq \frac{k}{2}. \quad (2.21)$$

Actually, in the analysis of [1] a somewhat stronger bound is imposed, namely

$$\frac{1}{2} < j < \frac{k-1}{2} . \tag{2.22}$$

Here the reason for the lower bound $\frac{1}{2} < j$ comes from the requirement that the corresponding ground state $\mathfrak{sl}(2, \mathbb{R})$ representation should lead to square-integrable wave-functions,² and this, via spectral flow, then also leads to the stronger upper bound $j < \frac{k-1}{2}$. Furthermore, it seems from the analysis of [2] that the MO-bound (2.22) is required for modular invariance.

As we have seen, massless higher spin fields correspond to states for which $E = |s|$. In terms of the conformal dimensions of the spacetime CFT, this means that either $h_{\text{CFT}} = 0$ or $\bar{h}_{\text{CFT}} = 0$, so that the operators describe chiral fields. Thus we need to investigate whether the physical string spectrum contains states with $h_{\text{CFT}} = m = 0$, say.

Given the ground state spin j together with the excitation number N , the possible $\mathfrak{sl}(2, \mathbb{R})$ lowest weights \hat{j} lie in the range $j - N \leq \hat{j} \leq j + N$ — this just follows from the fact that the oscillators sit in the adjoint representation of $\mathfrak{sl}(2, \mathbb{R})$, and tensoring with the adjoint representation can change the spin at most by ± 1 . Thus the minimal value of m that can appear is $m_{\text{min}} = j - N$. The question is then whether $j - N = 0$ is possible, provided that j is given by (2.20) and j obeys the unitarity bound (2.21) or the Maldacena-Ooguri bound (2.22). [Since $h_{\text{CFT}} = \hat{j}$ is the conformal dimension of the dual CFT, the no-ghost theorem (i.e. unitarity) guarantees that $h_{\text{CFT}} \geq 0$ — so the only way $h_{\text{CFT}} = 0$ can arise is for $h = j - N = 0$.]

One solution is obvious: if we take $N = 1$ (with $h_{\text{int}} = 0$), then $j = 1$ and $j - N = 0$. Tensoring this state together with the corresponding right-mover with $\bar{j} = j + N = 2$ gives then the helicity two component of the graviton. This state is allowed by unitarity for $k \geq 2$, whereas the MO-bound (2.22) only allows the massless graviton for $k \geq 3$.

In the following we want to study whether there are other solutions for $N \geq 2$, corresponding to higher spin. Going back to the original equation (2.17) for $h_{\text{int}} = 0$ and with $\hat{c}(j) = -j(j-1)$, we have to solve³

$$-\frac{j(j-1)}{k-2} + N = 1 , \tag{2.23}$$

which we can rewrite as

$$j^2 - j - (k-2)(N-1) = 0 . \tag{2.24}$$

In order for $h = j - N = 0$, this has to be solved for $j = N$. Provided that $N \geq 2$, plugging $j = N$ into (2.24) leads to

$$N^2 - N - (k-2)(N-1) = 0 \quad \implies \quad N = k - 2 . \tag{2.25}$$

²This conclusion may be relaxed by considering an alternate quantisation of modes with $0 < j < \frac{1}{2}$, see page 18 of [1].

³It is clear that there can be no solutions with non-zero h_{int} : if $h_{\text{int}} > 0$, the relevant j in (2.20) is larger than the value for $h_{\text{int}} = 0$, and hence $j - N$ cannot be zero — unitarity of the dual CFT (which follows from the no-ghost theorem) implies that already the original $j - N$ was non-negative, so increasing j cannot make it zero.

For $N = 2, 3, \dots$, corresponding to $k = 4, 5, \dots$, it then follows that

$$j = N = k - 2 \geq \frac{k - 1}{2}, \tag{2.26}$$

in violation of the upper bound on j from (2.22). We note in passing that if we were to relax the upper bound on j to the unitary bound (2.21), then we have the solution $k = 4$ with $N = 2 = j$. However, the corresponding state

$$J_{-1}^- J_{-1}^- |j = 2\rangle \tag{2.27}$$

is null since it is a descendant of the null-state $J_{-1}^- |j = 2\rangle$ at level $k = 4$.

2.4 The spectrally flowed representations

It was seen in [2] that, as well as the positive energy representations discussed above, the formulation of string theory in AdS₃ also requires spectrally flowed representations that are not Virasoro highest weight.⁴ One way to describe the spectrally flowed representations is as follows. We start with the vector space corresponding to a Virasoro highest weight representation of $\mathfrak{sl}(2, \mathbb{R})_k$ — in our case, this will either be a lowest weight representation \mathcal{D}_j^+ or a continuous representation $\mathcal{C}(p, \alpha)$. On this vector space we then define the action of $\mathfrak{sl}(2, \mathbb{R})_k$ and the Virasoro algebra by the action of the hatted operators defined via

$$\begin{aligned} \hat{J}_n^\pm &= J_{n \mp w}^\pm \\ \hat{J}_n^3 &= J_n^3 + \frac{k}{2} w \delta_{n,0} \\ \hat{L}_n &= L_n - w J_n^3 - \frac{k}{4} w^2 \delta_{n,0}. \end{aligned} \tag{2.28}$$

With respect to the hatted operators the vector space is then not a conventional (Virasoro) highest weight representation. However, provided that $w > 0$, the representation (with respect to the hatted modes) consists of lowest weight representations of the global $\mathfrak{sl}(2, \mathbb{R})$ algebra — this is the case of physical interest, since then the energy spectrum in AdS₃ is bounded from below (and so the spectrum of dimensions in the dual CFT will be bounded from below).

With respect to this hatted action we impose the usual mass-shell condition, which now takes the form (from now on we are setting $h_{\text{int}} = 0$)

$$\frac{\hat{c}}{k - 2} - w m - \frac{k}{4} w^2 + N = 1, \tag{2.29}$$

where m is the J_0^3 eigenvalue of the state in question. Furthermore, the conformal dimension of the state in the dual CFT is now $h_{\text{CFT}} = m + \frac{k}{2} w$. As before the presence of a massless higher spin state requires $h_{\text{CFT}} = m + \frac{k}{2} w = 0$, i.e. that we take $m = -\frac{k}{2} w$. Plugging this into (2.29) leads to

$$\frac{\hat{c}}{k - 2} + \frac{k}{4} w^2 + N = 1. \tag{2.30}$$

⁴The relevance of these representations for string theory on AdS₃ was already observed in [23].

Let us now consider the two classes of representations in turn. For the discrete representations $\hat{c} = -j(j-1)$ and, since j still has to satisfy the unitarity bound (2.21), we conclude that

$$-\frac{j(j-1)}{k-2} \geq -\frac{k}{4}, \tag{2.31}$$

and hence that

$$1 - N - \frac{k}{4}(w^2 - 1) \geq 0. \tag{2.32}$$

Thus the only cases to consider are $w = 1$ with $N = 0, 1$. Furthermore, since $m = -\frac{k}{2}w = -\frac{k}{2}$, but $j \geq 0$ and the minimal value of m is $j - N = -1$ (for $j = 0, N = 1$), the only possible solution is $k = 2$, which is not allowed — at $k = 2$ the denominator in the stress energy tensor blows up.⁵ Thus there are no massless higher spin fields from the spectrally flowed discrete representations.

For the continuous representations $\hat{c} = p^2 + \frac{1}{4}$. Since all three terms are non-negative, the only possible solution arises for $N = 0$ and $w = 1$, for which we obtain the equation

$$p^2 + \frac{1}{4} = -\frac{k^2}{4} + \frac{3}{2}k - 2. \tag{2.33}$$

It is not difficult to show that the only solution arises for $k = 3$ and $p = 0$. For these values, i.e. $k = 3, p = 0$, and $w = 1$, the solution to the mass-shell condition (2.29) is then

$$-m - \frac{1}{2} + N = 1, \tag{2.34}$$

which leads to $m = N - \frac{3}{2}$. Thus, in addition to the massless states, we get a whole tower of states (each at the bottom of a continuum labelled by p) with $h_{\text{CFT}} = N$ and $\bar{h}_{\text{CFT}} = \bar{N}$, where we recall that $h_{\text{CFT}} = m + \frac{k}{2}w = m + \frac{3}{2}$, and similarly for the right-movers. To summarise, the infinite tower of massless higher spin fields are given by considering the continuous representation with $p = 0$ in the $w = 1$ sector at $k = 3$, where we take

$$m = -\frac{3}{2} + N, \quad \bar{m} = -\frac{3}{2} + \bar{N}, \tag{2.35}$$

with $N, \bar{N} \in \mathbb{N}_0$ and either $N = 0$ or $\bar{N} = 0$. Note that these states (for any N, \bar{N}), satisfy the level matching condition [1], for non-zero spectral flow, which requires that $w(m - \bar{m})$ be an integer.

We should also note that the graviton which was part of the $w = 0$ discrete spectrum merges at this point with the continuum since it has $j = 1 = \frac{(k-1)}{2}$ and thus can be identified with the state in the $w = 1$ continuous representation with $N = 0, \bar{N} = 2$ (as well as the state where N and \bar{N} are interchanged), cf. the comments around page 25 of [1].

3 Discussion

We have found, from a close examination of the bosonic string spectrum on AdS₃ with H-flux, that there is a novel kind of tensionless limit at small k . This happens at $k = 3$,

⁵In addition, $j = 0$ is not allowed by the Maldacena-Ooguri bound, eq. (2.22).

rather than the critical value $k = 2$ which one might have expected. In fact, as we have seen, according to the Maldacena-Ooguri bound (2.22), the massless graviton (with $j = 1$) is only part of the physical spectrum for $k \geq 3$. Therefore, the meaning of the theories with $2 \leq k < 3$ is somewhat unclear. (For $k = 2$, there is a massless spin-two state consistent with the unitarity bound (2.21), but it is not normalisable. It is possible that an alternative approach to the quantisation of the $k = 2$ case, perhaps with a non-standard norm on the space of states, could be useful here.)

It will be interesting to study the physics of this tensionless limit further, and understand the similarities and differences with the limit corresponding to the symmetric orbifold point. Seiberg and Witten [9], in work preceding that of Maldacena and Ooguri, studied long fundamental strings in AdS_3 and argued that a long string is effectively described, for $k > 3$, by a Liouville theory with background charge Q and central charge $c = 1 + 3Q^2$, where

$$Q = (k - 3)\sqrt{\frac{2}{k - 2}}. \tag{3.1}$$

The Liouville field is associated with the distance of the long string from the boundary of AdS_3 . The Liouville theory has a spectrum that is discrete up to a threshold energy

$$\Delta_0 = \frac{Q^2}{8} = \frac{(k - 3)^2}{4(k - 2)}, \tag{3.2}$$

and continuous above this, see eq. (4.15) of [9]. This is exactly the behaviour that we find from the microscopic analysis of Maldacena and Ooguri — if we put $p = 0$, $w = 1$, and $N = 0$, and solve for m from the on-shell condition (2.29), we find that $h_{\text{CFT}} = m + \frac{k}{2} = \frac{(k-3)^2}{4(k-2)}$. In the limit $k \rightarrow 3$, the threshold tends to zero and we obtain a continuous spectrum. The discussion of Seiberg and Witten [9] suggests that this continuous spectrum could be associated with a limit of Liouville theory in which the background charge $Q \rightarrow 0$ giving a $c = 1$ CFT. A natural hypothesis is that this could be a free scalar field, corresponding to an extra non-compact dimension emerging in this limit.⁶

Seiberg and Witten [9] also considered the supersymmetric case, resulting from the near horizon limit of a system of Q_1 fundamental strings inside Q_5 NS 5-branes, finding a similar behaviour with a threshold energy of

$$\Delta_0 = \frac{Q^2}{8} = \frac{(Q_5 - 1)^2}{4Q_5}, \tag{3.3}$$

so that $Q_5 \rightarrow 1$ is the limit in which $\Delta_0 \rightarrow 0$ and there are extra massless states.⁷ Further, it is conjectured in [9] that the dual CFT has a $(\mathbb{R}^4)^{Q_1}/S_{Q_1}$ factor. The spectrum of the superstring case is treated in more detail in [26].

⁶Note, however, that taking this limit directly in Liouville theory is tricky since the theory is only conformally invariant if the charge Q is related to the parameter γ in the exponential Liouville potential $\mu^2 e^{\gamma\phi}$ by $Q = \gamma + 2/\gamma$ (see [24] and references therein). Therefore, taking $Q < 2\sqrt{2}$ is not possible for real γ . An accurate comparison with a candidate dual non-compact CFT will therefore need to go beyond the effective Liouville description (which breaks down for $k \approx 8$) and be based on the microscopic analysis of [2].

⁷See also [25], where it was argued that $Q_5 = 1$ separates two phases of quantum gravity. This was based on examining the behaviour of the high energy density of states.

The appearance of a continuum in the dual CFT is also somewhat reminiscent of the light states in the vicinity of the (Vasiliev) tower of higher spin currents in the \mathcal{W}_N minimal models in the large N limit, see [27, 28]. It would be useful to see whether this can be made more precise.

We have seen that there is a stringy tower of massless higher spin states that become massless in the limit $k \rightarrow 3$. One may ask about the unbroken gauge symmetry that they generate. Naively, it seems to be similar in structure to that of the Higher Spin Square [12, 13] that is seen at the symmetric orbifold point. This would indeed be so if we can identify these states in the dual CFT with the single particle states in a symmetric product of free non-compact bosons of the kind mentioned above.

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