

Bootstrapping mixed correlators in 4D $\mathcal{N} = 1$ SCFTs

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ABSTRACT: The numerical conformal bootstrap is used to study mixed correlators in $\mathcal{N} = 1$ superconformal field theories (SCFTs) in $d = 4$ spacetime dimensions. Systems of four-point functions involving scalar chiral and real operators are analyzed, including the case where the scalar real operator is the zero component of a global conserved current multiplet. New results on superconformal blocks as well as universal constraints on the space of 4D $\mathcal{N} = 1$ SCFTs with chiral operators are presented. At the level of precision used, the conditions under which the putative “minimal” 4D $\mathcal{N} = 1$ SCFT may be isolated into a disconnected allowed region remain elusive. Nevertheless, new features of the bounds are found that provide further evidence for the presence of a special solution to crossing symmetry corresponding to the “minimal” 4D $\mathcal{N} = 1$ SCFT.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory, Supersymmetry and Duality

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1 Introduction

The modern revival of the conformal bootstrap program [1] has led to remarkable progress in our understanding of conformal field theories (CFTs) in $d > 2$ spacetime dimensions. By studying the constraints of crossing symmetry and unitarity, it is possible to derive rigorous bounds on the scaling dimensions and operator product expansion (OPE) coefficients of any CFT. This approach relies on very few assumptions and can thus be used to study and discover theories without a known Lagrangian description.

A striking result of the numerical conformal bootstrap is that the bounds can develop kinks, or singularities, corresponding to known theories. This was observed in the 3D Ising [2] and $O(N)$ vector models [3] and was correlated with the decoupling of certain operators. This intuition was further developed in [4]. With the introduction of multiple correlators and additional assumptions on the number of relevant scalars, small regions surrounding the known theories can be isolated from other solutions of the bootstrap equations, i.e. the kinks become islands [5, 6]. Consequently, the known theory is essentially the unique consistent solution of the crossing equations in a certain region in parameter space, given certain mild assumptions.

In $d = 4$ a kink was observed for $\mathcal{N} = 1$ superconformal theories (SCFTs) with a chiral scalar operator ϕ [7–9]. More specifically, the scaling dimension bound for the first real scalar in the $\bar{\phi} \times \phi$ OPE develops a kink as a function of Δ_ϕ at the same point where the lower bound for the three-point function coefficient $c_{\phi\phi\phi^2}$ disappears. Similar behavior was also observed for theories in $2 \leq d \leq 4$ with four supercharges [10]. In [9] it was conjectured that there is a 4D superconformal field theory (SCFT) that saturates the bootstrap bounds at the kink, referred to as the minimal 4D $\mathcal{N} = 1$ SCFT. Based on the position of the kink and a corresponding local minimum in the lower bound on the central charge, this minimal theory was predicted to have $c_{\text{minimal}} = \frac{1}{9}$ and a chiral multiplet with scaling dimension $\Delta_\phi = \frac{10}{7}$, which also satisfies the chiral ring condition $\phi^2 = 0$. Various speculations about this minimal theory have appeared [11, 12]. In these proposals $\phi^2 = 0$ is explicitly satisfied, but the central charge and the critical Δ_ϕ have not been successfully reproduced. As a result, the identity of this minimal theory remains elusive.

Motivated by this open problem, we study here the mixed correlator bootstrap for 4D $\mathcal{N} = 1$ theories for the system of correlators $\{\langle \bar{\phi}\phi\bar{\phi}\phi \rangle, \langle \bar{\phi}R\phi R \rangle, \langle RRRR \rangle\}$, where R is a generic real scalar and ϕ is a chiral scalar. We consider both the case where R is the first real scalar in the $\bar{\phi} \times \phi$ OPE (beyond the identity operator of course), and that where R saturates the unitarity bound. In the latter case it sits in a linear multiplet, which we will label by J . The bootstrap equations for the $\langle \bar{\phi}\phi\bar{\phi}\phi \rangle$ correlator were first considered in [7] and for $\langle JJJJ \rangle$ in [13], and for $\langle RRRR \rangle$ in [14]. Here we present new results for the superconformal blocks of $\langle \bar{\phi}R\phi R \rangle$ and $\langle \bar{\phi}J\phi J \rangle$. To be precise, we find superconformal blocks when the superconformal primary of the exchanged multiplet appears in a (j, \bar{j}) representation of $SO(3,1)$, with $j \neq \bar{j}$. In this case the corresponding superconformal primary does not appear in the OPE of the external operators, but some of its superconformal descendants do. We also compute superconformal blocks of superconformal primaries in integer-spin representations; our results agree with the literature [14–16].

Our main results are new numerical constraints on 4D $\mathcal{N} = 1$ theories. Studying the single correlator $\langle JJJJ \rangle$, where J corresponds to a $U(1)$ linear multiplet, we improve upper bounds on the OPE coefficients for $\langle JJJ \rangle$ and $\langle JJV \rangle$ where V is the spin-one multiplet containing the stress-energy tensor $T^{\mu\nu}$. We also study these bounds as a function of the first unprotected scalar in the $J \times J$ OPE, deriving an upper bound on this operators scaling dimension and the $\langle JJO \rangle$ OPE coefficient. With the mixed correlator system for ϕ and R , with R the first real scalar in the $\bar{\phi} \times \phi$ OPE, we will derive stronger lower bounds on the central charge c and upper and lower bounds on $c_{\phi\bar{\phi}R}$. In both cases we find interesting

features near the minimal $\mathcal{N} = 1$ point. Finally, studying the mixed correlator system for ϕ and J we will derive new bounds on $c_{\phi\bar{\phi}J}$ and $c_{\bar{\phi}J(\phi J)}$ where (ϕJ) is the second scalar appearing in the $\phi \times J$ OPE.

In sections 2 and 3 we give the complete set of conformal blocks for the mixed correlator system involving a generic real scalar multiplet R and the linear multiplet J respectively. In sections 4 and 5 we give the corresponding crossing relations for R and J . In section 6 we present results for the ϕ and R system. In section 7 we present results for the ϕ and J system. In appendix A we will go over the approximations used in the numerical implementation of the crossing equations and in appendix B we give some details on the derivation of the superconformal blocks.

2 Four-point functions, conformal and superconformal blocks

In this section we present our results for the superconformal block decomposition of the various four-point functions used in our bootstrap analysis. In particular we include results for the four-point function $\langle \bar{\phi}(x_1)\phi(x_2)\bar{\phi}(x_3)\phi(x_4) \rangle$, first obtained in [7, 17], and new results for the four-point function $\langle \bar{\phi}(x_1)R(x_2)\phi(x_3)R(x_4) \rangle$, with R a real operator, in the $\bar{\phi} \times R$ channel. In our numerical analysis we also use the four-point function $\langle \bar{\phi}(x_1)R(x_2)\phi(x_3)R(x_4) \rangle$ in the $\bar{\phi} \times \phi$ channel, results for which were first obtained in [14] (see also [16]). This forces us to also consider $\langle R(x_1)R(x_2)R(x_3)R(x_4) \rangle$, where again we use results of [14].

Four-point functions can be reduced and computed via the OPE. Consider the four-point function $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4) \rangle$ where all operators are conformal primary. We can use the OPEs $\mathcal{O}_i(x_1) \times \mathcal{O}_j(x_2)$ and $\mathcal{O}_k(x_3) \times \mathcal{O}_l(x_4)$ to obtain

$$\begin{aligned} \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4) \rangle &= \frac{1}{r_{12}^{\Delta_i+\Delta_j} r_{34}^{\Delta_k+\Delta_l}} \left(\frac{r_{24}}{r_{14}}\right)^{\Delta_{ij}} \left(\frac{r_{14}}{r_{13}}\right)^{\Delta_{kl}} \\ &\times \sum_{\substack{\text{conformal} \\ \text{primaries} \\ \mathcal{O}_m}} \delta_{mn} c_{ij}^m c_{kl}^n g_{\Delta_m, \ell_m}^{\Delta_{ij}, \Delta_{kl}}(u, v), \end{aligned} \quad (2.1)$$

where $r_{ij} = (x_{ij}^2)^{\frac{1}{2}}$, $x_{ij} = x_i - x_j$, $\Delta_{ij} = \Delta_i - \Delta_j$ and similarly for Δ_{kl} , Δ_m, ℓ_m is the scaling dimension and spin of the exchanged operator, and

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}) \quad (2.2)$$

are the two independent conformally-invariant cross ratios constructed out of four points in space. The conformal blocks $g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}$ are functions that account for the sum over conformal descendants. They are given by [18, 19]¹

$$\begin{aligned} g_{\alpha, \beta}^{\gamma, \delta}(z, \bar{z}) &= (-1)^\beta \frac{z\bar{z}}{z-\bar{z}} \left(k_{\alpha+\beta}^{\gamma, \delta}(z) k_{\alpha-\beta-2}^{\gamma, \delta}(\bar{z}) - (z \leftrightarrow \bar{z}) \right), \\ k_{\alpha}^{\beta, \gamma}(x) &= x^{\alpha/2} {}_2F_1\left(\frac{1}{2}(\alpha-\beta), \frac{1}{2}(\alpha+\gamma); \alpha; x\right). \end{aligned} \quad (2.3)$$

¹Compared to their original definition we drop a factor of $2^{-\beta}$ in $g_{\alpha, \beta}^{\gamma, \delta}$, i.e. $(g_{\alpha, \beta}^{\gamma, \delta})^{\text{here}} = 2^\beta (g_{\alpha, \beta}^{\gamma, \delta})^{\text{D\&O}}$, by rescaling appropriately the OPE coefficients in (2.1).

In $\mathcal{N} = 1$ superconformal theories some of the conformal primaries in the sum in (2.1) are superconformal descendants, and so their contributions to the four-point function can also be accounted for by computing “superconformal blocks”. The dimensions of the exchanged operators are constrained by unitarity to be [20, 21]

$$\Delta \geq \left| q - \bar{q} - \frac{1}{2}(j - \bar{j}) \right| + \frac{1}{2}(j + \bar{j}) + 2, \quad (2.4)$$

where $(\frac{1}{2}j, \frac{1}{2}\bar{j})$ is the representation of \mathcal{O} under the Lorentz group, viewed here as $SU(2) \times SU(2)$, and q and \bar{q} give the scaling dimension and R-charge of an operator via

$$\Delta = q + \bar{q}, \quad R = \frac{2}{3}(q - \bar{q}). \quad (2.5)$$

2.1 Four-point function $\langle \bar{\phi}(x_1) \phi(x_2) \bar{\phi}(x_3) \phi(x_4) \rangle$

The four-point function $\langle \bar{\phi}(x_1) \phi(x_2) \bar{\phi}(x_3) \phi(x_4) \rangle$ involving the chiral operator ϕ and its complex conjugate can be expressed in terms of $12 \rightarrow 34$ contributions as [7]

$$\langle \bar{\phi}(x_1) \phi(x_2) \bar{\phi}(x_3) \phi(x_4) \rangle = \frac{1}{r_{12}^{2\Delta_\phi} r_{34}^{2\Delta_\phi}} \sum_{\substack{\text{superconformal} \\ \text{primaries} \\ \mathcal{O}_\ell \in \bar{\phi} \times \phi}} |c_{\bar{\phi}\phi\mathcal{O}_\ell}|^2 (-1)^\ell \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi; \bar{\phi}\phi}(u, v), \quad (2.6)$$

where we used $c_{\bar{\phi}\phi\mathcal{O}_\ell} = (-1)^\ell c_{\bar{\phi}\phi\mathcal{O}_\ell}^*$ and

$$\mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi; \bar{\phi}\phi} = g_{\Delta,\ell} - c_1 g_{\Delta+1,\ell+1} - c_2 g_{\Delta+1,\ell-1} + c_1 c_2 g_{\Delta+2,\ell}, \quad g_{\alpha,\beta} \equiv g_{\alpha,\beta}^{0,0}, \quad (2.7)$$

with

$$c_1 = \frac{\Delta + \ell}{4(\Delta + \ell + 1)}, \quad c_2 = \frac{\Delta - \ell - 2}{4(\Delta - \ell - 1)}. \quad (2.8)$$

The unitarity bound here is $\Delta \geq \ell + 2$ and, when it is saturated, c_2 becomes zero.

If we flip the last two operators and consider $\langle \bar{\phi}(x_1) \phi(x_2) \phi(x_3) \bar{\phi}(x_4) \rangle$, then we can write, in the $12 \rightarrow 34$ channel,

$$\langle \bar{\phi}(x_1) \phi(x_2) \phi(x_3) \bar{\phi}(x_4) \rangle = \frac{1}{r_{12}^{2\Delta_\phi} r_{34}^{2\Delta_\phi}} \sum_{\substack{\text{superconformal} \\ \text{primaries} \\ \mathcal{O}_\ell \in \bar{\phi} \times \phi}} |c_{\bar{\phi}\phi\mathcal{O}_\ell}|^2 \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi; \phi\bar{\phi}}(u, v), \quad (2.9)$$

where we used $c_{\phi\bar{\phi}\mathcal{O}_\ell} = c_{\bar{\phi}\phi\mathcal{O}_\ell}^*$ and

$$\mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi; \phi\bar{\phi}} = g_{\Delta,\ell} + c_1 g_{\Delta+1,\ell+1} + c_2 g_{\Delta+1,\ell-1} + c_1 c_2 g_{\Delta+2,\ell}. \quad (2.10)$$

The difference between (2.7) and (2.10) is just in the sign of the $g_{\Delta+1,\ell\pm 1}$ contributions.

In this work we will also decompose $\langle \bar{\phi}(x_1) \phi(x_2) \phi(x_3) \bar{\phi}(x_4) \rangle$ in the $14 \rightarrow 32$ channel [17],

$$\langle \bar{\phi}(x_1) \phi(x_2) \phi(x_3) \bar{\phi}(x_4) \rangle = \frac{1}{r_{14}^{2\Delta_\phi} r_{23}^{2\Delta_\phi}} \sum_{\substack{\text{conformal} \\ \text{primaries} \\ \bar{\mathcal{O}}_\ell \in \bar{\phi} \times \bar{\phi}}} |c_{\bar{\phi}\bar{\phi}\bar{\mathcal{O}}_\ell}|^2 \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\bar{\phi}; \phi\phi}(v, u), \quad (2.11)$$

where we used $c_{\phi\phi\mathcal{O}_\ell} = c_{\bar{\phi}\bar{\phi}\bar{\mathcal{O}}_\ell}^*$ and

$$\mathcal{G}_{\Delta,\ell}^{\bar{\phi}\bar{\phi};\phi\phi} = g_{\Delta,\ell}. \quad (2.12)$$

In this case no superconformal block needs to be computed, but we need to include all classes of conformal primaries that can appear in the $\phi \times \phi$ OPE. This has been done in [17] and uses the fact that the product $\phi \times \phi$ is chiral and that the three-point function $\langle \Phi(z_1) \Phi(z_2) \mathcal{O}_I(z_3) \rangle$ is symmetric under $z_1 \leftrightarrow z_2$. Here $z = (x, \theta, \bar{\theta})$ is a point in superspace, and the index I denotes Lorentz indices. The contributions we need to include turn out to be the superconformal primary ϕ^2 , protected even-spin operators of the form $\bar{Q}\mathcal{O}_\ell$ with dimension $\Delta = 2\Delta_\phi + \ell$, and unprotected even-spin operators of the form $\bar{Q}^2\mathcal{O}_\ell$ with dimension satisfying $\Delta \geq |2\Delta_\phi - 3| + 3 + \ell$. When $\Delta_\phi < \frac{3}{2}$ there is a gap in the dimensions of the unprotected and protected operators.

2.2 Four-point function $\langle \bar{\phi}(x_1) R(x_2) \phi(x_3) R(x_4) \rangle$

The four-point function $\langle \bar{\phi}(x_1) R(x_2) \phi(x_3) R(x_4) \rangle$, involving the chiral operator ϕ and the real operator R , can be expanded in the $12 \rightarrow 34$ channel as

$$\langle \bar{\phi}(x_1) R(x_2) \phi(x_3) R(x_4) \rangle = \frac{1}{r_{12}^{\Delta_\phi + \Delta_R} r_{34}^{\Delta_\phi + \Delta_R}} \left(\frac{r_{24}}{r_{13}} \right)^{\Delta_\phi - \Delta_R} \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |c_{\bar{\phi}R\bar{\mathcal{O}}_\ell}|^2 \mathcal{G}_{\Delta,\ell,\Delta_\phi - \Delta_R}^{\bar{\phi}R;\phi R}(u, v), \quad (2.13)$$

where Δ_ϕ, Δ_R are the scaling dimensions of ϕ, R respectively, Δ, ℓ are the scaling dimension and spin of \mathcal{O} , $\bar{c}_{\bar{\phi}R\bar{\mathcal{O}}_\ell}$ is the coefficient of the three-point function $\langle \bar{\phi}(x_1) R(x_2) \mathcal{O}_I(x_3) \rangle$, and we use $\bar{c}_{\phi R\bar{\mathcal{O}}_\ell} = \bar{c}_{\bar{\phi}R\bar{\mathcal{O}}_\ell}^*$. As we will see below the sum in the right-hand side of (2.13) contains contributions from multiple classes of operators.

In order to compute $\mathcal{G}_{\Delta,\ell,\Delta_\phi - \Delta_R}^{\bar{\phi}R;\phi R}$ we need the general form of the three-point function $\langle \bar{\Phi}(z_1) \mathcal{R}(z_2) \mathcal{O}_I(z_3) \rangle$, where \mathcal{O}_I is a superconformal primary operator. To obtain this we use the results of [22, 23]. To start, we note that $\bar{\Phi}$ has superconformal weights $q_{\bar{\Phi}} = 0$ and $\bar{q}_{\bar{\Phi}} = \Delta_\phi$, while \mathcal{R} has $q_{\mathcal{R}} = \bar{q}_{\mathcal{R}} = \frac{1}{2}\Delta_{\mathcal{R}}$. General superconformal constraints imply that the three-point function is proportional to a function of X_3, Θ_3 and $\bar{\Theta}_3$ [23],

$$\langle \bar{\Phi}(z_1) \mathcal{R}(z_2) \mathcal{O}_I(z_3) \rangle = \frac{1}{x_{13}^{2\Delta_\phi} x_{23}^{\Delta_R} x_{32}^{\Delta_R}} t_I(X_3, \Theta_3, \bar{\Theta}_3), \quad (2.14)$$

with the homogeneity property

$$\begin{aligned} t_I(\lambda\bar{\lambda}X, \lambda\Theta, \bar{\lambda}\bar{\Theta}) &= \lambda^{2a} \bar{\lambda}^{2\bar{a}} t_I(X, \Theta, \bar{\Theta}), \\ a - 2\bar{a} &= \bar{q}_{\bar{\Phi}} + \bar{q}_{\mathcal{R}} - q_{\mathcal{O}}, \quad \bar{a} - 2a = q_{\bar{\Phi}} + q_{\mathcal{R}} - \bar{q}_{\mathcal{O}}. \end{aligned} \quad (2.15)$$

Quantities appearing in (2.14) are defined as

$$\begin{aligned} X_3 &= \frac{x_{31}\bar{x}_{12}x_{23}}{x_{13}^2 x_{32}^2}, \quad x_{\alpha\dot{\alpha}} = \sigma_{\mu\alpha\dot{\alpha}} x^\mu, \quad \bar{x}^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} x_{\beta\dot{\beta}}, \\ \Theta_3 &= i \left(\frac{1}{x_{13}^2} x_{31} \bar{\theta}_{31} - \frac{1}{x_{23}^2} x_{32} \bar{\theta}_{32} \right), \quad \bar{\Theta}_3 = \Theta_3^*, \end{aligned} \quad (2.16)$$

with $\bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j$ and the supersymmetric interval between x_i and x_j defined by

$$x_{\bar{i}j} = -x_{j\bar{i}} \equiv x_{ij} - i\theta_i\sigma\bar{\theta}_i - i\theta_j\sigma\bar{\theta}_j + 2i\theta_j\sigma\bar{\theta}_i. \quad (2.17)$$

Let us first assume that \mathcal{O}_I has $q = \frac{1}{2}(\Delta + \Delta_\phi)$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi)$, as would be the case if the zero component of $\bar{\mathcal{O}}_I$ appeared in the $\bar{\phi} \times R$ OPE. Then, $a = \bar{a}$, which implies that t_I in (2.14) can only be a function of the product $\Theta_3\bar{\Theta}_3$. Furthermore, the Ward identity following from the antichirality property of $\bar{\Phi}$ implies that t_I cannot be a function of $\bar{\Theta}_3$. Therefore, t_I can only be a function of X_3 in this case.

With the constraints we just described the operator \mathcal{O}_I in (2.14) is an integer-spin traceless-symmetric superconformal primary $\mathcal{O}_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}$, with the dotted and undotted indices symmetrized independently of each other, for which we can write

$$t_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}(X_3) = \bar{c}_{\bar{\phi}R\mathcal{O}_\ell} X_3^{(\alpha_1(\dot{\alpha}_1 \cdots X_3^{\alpha_\ell)\dot{\alpha}_\ell)} X_3^{\Delta-\ell-\Delta_\phi-\Delta_R}, \quad (2.18)$$

where the dotted indices are symmetrized independently of the undotted ones. With (2.18) the θ expansion of both sides of (2.14) can be performed with *Mathematica* by extending the code developed for the purposes of [24]. We need the superconformal primary zero-components of $\bar{\Phi}$ and \mathcal{R} , but then the possible contributions to the three-point function come not only from the zero component of $\mathcal{O}_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}$, but also from the conformal primaries in its $\theta\bar{\theta}$ and $\theta^2\bar{\theta}^2$ components. Taking into account all these contributions and using results of [24] leads to the superconformal block

$$\bar{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R} = g_{\Delta,\ell}^{\Delta_\phi-\Delta_R} + \bar{c}_1 g_{\Delta+1,\ell+1}^{\Delta_\phi-\Delta_R} + \bar{c}_2 g_{\Delta+1,\ell-1}^{\Delta_\phi-\Delta_R} + \bar{c}_1\bar{c}_2 g_{\Delta+2,\ell}^{\Delta_\phi-\Delta_R}, \quad g_{\alpha,\beta}^\gamma \equiv g_{\alpha,\beta}^{\gamma,\gamma}, \quad (2.19)$$

with

$$\begin{aligned} \bar{c}_1 &= \frac{(\Delta + \ell - \Delta_\phi)(\Delta + \ell + \Delta_\phi - \Delta_R)^2}{4(\Delta + \ell)(\Delta + \ell + 1)(\Delta + \ell + \Delta_\phi)}, \\ \bar{c}_2 &= \frac{(\Delta - \ell - \Delta_\phi - 2)(\Delta - \ell + \Delta_\phi - \Delta_R - 2)^2}{4(\Delta - \ell - 1)(\Delta - \ell - 2)(\Delta - \ell + \Delta_\phi - 2)}. \end{aligned} \quad (2.20)$$

The unitarity bound on \mathcal{O}_ℓ that follows from (2.4) is

$$\Delta \geq \Delta_\phi + \ell + 2. \quad (2.21)$$

When the unitarity bound (2.21) is saturated, we see from (2.20) that $\bar{c}_2 = 0$ as expected.²

The block $\bar{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}$ we just computed constitutes merely one of the possible contributions to the right-hand side of (2.13). Further, we note that, in general, R is an operator exchanged in the $\bar{\phi} \times \phi$ OPE, and so we also need to consider the three-point function

$$\langle \bar{\Phi}(z_1) \mathcal{R}(z_2) \Phi(z_3) \rangle = \frac{\bar{c}_{\bar{\phi}R\phi}}{x_{13}^{2\Delta_\phi} x_{23}^{\Delta_R} x_{32}^{\Delta_R}} X_3^{-\Delta_R}. \quad (2.23)$$

²As an aside we note here that, for a general scalar operator \mathcal{S} with superconformal weights q_S and \bar{q}_S , we get an expression similar to (2.19) for the corresponding block $\bar{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_S}^{\bar{\phi}S;\phi S}$, with the coefficients

$$\begin{aligned} \bar{c}_1 &= \frac{(\Delta + \ell - \Delta_\phi + q_S - \bar{q}_S)(\Delta + \ell + \Delta_\phi - q_S - \bar{q}_S)^2}{4(\Delta + \ell)(\Delta + \ell + 1)(\Delta + \ell + \Delta_\phi - q_S + \bar{q}_S)}, \\ \bar{c}_2 &= \frac{(\Delta - \ell - \Delta_\phi + q_S - \bar{q}_S - 2)(\Delta - \ell + \Delta_\phi - q_S - \bar{q}_S - 2)^2}{4(\Delta - \ell - 1)(\Delta - \ell - 2)(\Delta - \ell + \Delta_\phi - q_S + \bar{q}_S - 2)}. \end{aligned} \quad (2.22)$$

Since Φ has $\bar{q} = \bar{j} = 0$, the unitarity bound (2.4) is modified to $q \geq j + 1$. This implies that Φ has $\Delta \geq 1$. In this case we only need to consider a conformal block $g_{\Delta_\phi, 0}^{\Delta_\phi - \Delta_R}$. Note that due to this contribution there is always a gap in the scalar spectrum of the $\bar{\phi} \times R$ OPE.

We should also consider the case where the zero component of $\bar{\mathcal{O}}$ does not contribute to the $\bar{\phi} \times R$ OPE. Due to the antichirality property of $\bar{\Phi}$ it is still true that there cannot be a $\bar{\Theta}_3$ in t_I , but now both Θ_3 and Θ_3^2 are allowed.

In the first case, relevant operators are of the form $\mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ for some ℓ and with $q = \frac{1}{2}(\Delta + \Delta_\phi - \frac{3}{2})$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi + \frac{3}{2})$, so that $Q^\alpha \bar{\mathcal{O}}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ is a spin- ℓ conformal primary that can appear in the $\bar{\phi} \times R$ OPE.³ In this case

$$t_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}(X_3) = \hat{c}_{\bar{\phi}R\mathcal{O}_\ell} \Theta_3^\alpha X_{3\alpha} X_{3(\dot{\alpha}_1} \dots X_{3\alpha_\ell)\dot{\alpha}_\ell} X_3^{\Delta - \ell - \Delta_\phi - \Delta_R - \frac{3}{2}}, \quad (2.24)$$

and a superconformal block computation gives

$$\hat{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; \phi R} = \hat{c}_1 g_{\Delta + \frac{1}{2}, \ell}^{\Delta_\phi - \Delta_R} + \hat{c}_2 g_{\Delta + \frac{3}{2}, \ell + 1}^{\Delta_\phi - \Delta_R}, \quad (2.25)$$

where

$$\begin{aligned} \hat{c}_1 &= \frac{\ell + 2}{(\ell + 1)(2(\Delta - \ell - \Delta_\phi) - 3)}, \\ \hat{c}_2 &= \frac{(2\Delta - 3)(2(\Delta + \ell - \Delta_\phi) + 5)(2(\Delta + \ell + \Delta_\phi - \Delta_R) + 1)^2}{4(2\Delta - 1)(2(\Delta + \ell) + 1)(2(\Delta + \ell) + 3)(2(\Delta - \ell - \Delta_\phi) - 3)(2(\Delta + \ell + \Delta_\phi) - 3)}. \end{aligned} \quad (2.26)$$

The block $\hat{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; \phi R}$ is another contribution to (2.13). We should note here that if the shortening condition $Q_{(\beta} \bar{\mathcal{O}}_{\alpha_1 \dots \alpha_\ell); \dot{\alpha}_1 \dots \dot{\alpha}_\ell} = 0$ is satisfied, then \mathcal{O} is forced to have $\bar{q} = -\frac{1}{2}(\ell + 1)$ [23]. As a result, the dimension of such \mathcal{O} is fixed to be $\Delta = \Delta_\phi - \ell - \frac{5}{2}$. This is below the unitarity bound $\Delta \geq \Delta_\phi + \ell + \frac{3}{2}$ for this class of operators, but it nevertheless provides a check on \hat{c}_2 of (2.26).⁴

There is another case to consider with a Θ_3 , i.e. when we have a superconformal primary of the form $\mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_2 \dots \dot{\alpha}_\ell}$ for some $\ell \geq 1$, again with $q = \frac{1}{2}(\Delta + \Delta_\phi - \frac{3}{2})$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi + \frac{3}{2})$. Unitarity requires $\Delta \geq |\Delta_\phi - 2| + \ell + \frac{3}{2}$. Then, the conformal primary $Q_{(\alpha_1} \bar{\mathcal{O}}_{\alpha_2 \dots \alpha_\ell); \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ has spin ℓ and can contribute to the $\bar{\phi} \times R$ OPE. Corresponding to (2.14) we here have

$$t_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_2 \dots \dot{\alpha}_\ell}(X_3) = \check{c}_{\bar{\phi}R\mathcal{O}_\ell} \Theta_3(\alpha_1 X_{3\alpha_2}(\dot{\alpha}_2 \dots X_{3\alpha_\ell)\dot{\alpha}_\ell} X_3^{\Delta - \ell - \Delta_\phi - \Delta_R + \frac{1}{2}}, \quad \ell \geq 1, \quad (2.28)$$

and the associated superconformal block is

$$\check{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; \phi R} = \check{c}_1 g_{\Delta + \frac{1}{2}, \ell}^{\Delta_\phi - \Delta_R} + \check{c}_2 g_{\Delta + \frac{3}{2}, \ell - 1}^{\Delta_\phi - \Delta_R}, \quad \ell \geq 1, \quad (2.29)$$

³The three-point function $\langle \bar{\Phi}(z_1) R(z_2) \mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}(z_3) \rangle$ is proportional to Θ_3 , for (2.15) gives $2(a - \bar{a}) = 1$.

⁴For a general scalar operator \mathcal{S} we get a block $\hat{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_S}^{\bar{\phi}S; \phi \bar{S}}$ similar to (2.25) but with

$$\begin{aligned} \hat{c}_1 &= \frac{\ell + 2}{(\ell + 1)(2(\Delta - \ell - \Delta_\phi + q_S - \bar{q}_S) - 3)}, \\ \hat{c}_2 &= \frac{(2\Delta - 3)(2(\Delta + \ell - \Delta_\phi + q_S - \bar{q}_S) + 5)(2(\Delta + \ell + \Delta_\phi - q_S - \bar{q}_S) + 1)^2}{4(2\Delta - 1)(2(\Delta + \ell) + 1)(2(\Delta + \ell) + 3)(2(\Delta - \ell - \Delta_\phi + q_S - \bar{q}_S) - 3)(2(\Delta + \ell + \Delta_\phi - q_S + \bar{q}_S) - 3)}. \end{aligned} \quad (2.27)$$

with

$$\begin{aligned}\check{c}_1 &= \frac{1}{2(\Delta + \ell - \Delta_\phi) + 1}, \\ \check{c}_2 &= \frac{(\ell+1)(2\Delta-3)(2(\Delta-\ell-\Delta_\phi)+1)(2(\Delta-\ell+\Delta_\phi-\Delta_R)-3)^2}{4\ell(2\Delta-1)(2(\Delta-\ell)-1)(2(\Delta-\ell)-3)(2(\Delta+\ell-\Delta_\phi)+1)(2(\Delta-\ell+\Delta_\phi)-7)}.\end{aligned}\tag{2.30}$$

For operators \mathcal{O} of this class such that $Q^\alpha \bar{\mathcal{O}}_{\alpha\alpha_3\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell} = 0$, it follows that \mathcal{O} has $\bar{q} = \frac{1}{2}(\ell+1)$ [23]. This implies that the dimension of such \mathcal{O} is $\Delta = \Delta_\phi + \ell - \frac{1}{2}$, providing a check on \check{c}_2 of (2.30).⁵ Note that this dimension of \mathcal{O} is consistent with the unitarity bound for this class of operators only if $\Delta_\phi \geq 2$.

If Θ_3^2 appears in t_I only the superconformal descendant $Q^2\mathcal{O}_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}$ of a superconformal primary $\mathcal{O}_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}$ with $q = \frac{1}{2}(\Delta + \Delta_\phi - 3)$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi + 3)$ needs to be considered. The associated conformal block we have to include is $g_{\Delta+1,\ell}^{\Delta_\phi-\Delta_R}$. The unitarity bound here is $\Delta \geq |\Delta_\phi - 3| + \ell + 2$.

To summarize we may write, in (2.13),

$$\begin{aligned}\sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |c_{\bar{\phi}R\mathcal{O}_\ell}|^2 \mathcal{G}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(u,v) &= \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |\bar{c}_{\bar{\phi}R\mathcal{O}_\ell}|^2 \bar{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(u,v) \\ &+ \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\hat{c}_{\bar{\phi}R(Q\bar{\mathcal{O}})_\ell}|^2 \hat{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(u,v) \\ &+ \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\check{c}_{\bar{\phi}R(Q\bar{\mathcal{O}})_\ell}|^2 \check{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(u,v) \\ &+ \sum_{(Q^2\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |c_{\bar{\phi}R(Q^2\mathcal{O})_\ell}|^2 g_{\Delta+1,\ell}^{\Delta_\phi-\Delta_R}(u,v),\end{aligned}\tag{2.32}$$

with the appropriate unitarity bounds, and with the contribution associated to (2.23) implicitly included in the first sum on the right-hand side.

Let us finally consider $\langle \bar{\phi}(x_1) R(x_2) R(x_3) \phi(x_4) \rangle$ both in the $12 \rightarrow 34$ and the $14 \rightarrow 32$ channel. For the former we have

$$\begin{aligned}\langle \bar{\phi}(x_1) R(x_2) R(x_3) \phi(x_4) \rangle &= \frac{1}{r_{12}^{\Delta_\phi+\Delta_R} r_{34}^{\Delta_\phi+\Delta_R}} \left(\frac{r_{13} r_{24}}{r_{14}^2} \right)^{\Delta_\phi-\Delta_R} \\ &\times \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |c_{\bar{\phi}R\mathcal{O}_\ell}|^2 (-1)^\ell \mathcal{G}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;R\phi}(u,v),\end{aligned}\tag{2.33}$$

where one contribution comes from

$$\bar{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;R\phi} = \tilde{g}_{\Delta,\ell}^{\Delta_\phi-\Delta_R} - \bar{c}_1 \tilde{g}_{\Delta+1,\ell+1}^{\Delta_\phi-\Delta_R} - \bar{c}_2 \tilde{g}_{\Delta+1,\ell-1}^{\Delta_\phi-\Delta_R} + \bar{c}_1 \bar{c}_2 \tilde{g}_{\Delta+2,\ell}^{\Delta_\phi-\Delta_R}, \quad \tilde{g}_{\alpha,\beta}^\gamma \equiv g_{\alpha,\beta}^{\gamma,-\gamma}.\tag{2.34}$$

⁵For a general scalar operator \mathcal{S} we get a block $\check{\mathcal{G}}_{\Delta,\ell,\Delta_\phi-\Delta_S}^{\bar{\phi}S;\phi\bar{S}}$ similar to (2.29) but with

$$\begin{aligned}\check{c}_1 &= \frac{1}{2(\Delta + \ell - \Delta_\phi + q_S - \bar{q}_S) + 1}, \\ \check{c}_2 &= \frac{(\ell+1)(2\Delta-3)(2(\Delta-\ell-\Delta_\phi+q_S-\bar{q}_S)+1)(2(\Delta-\ell+\Delta_\phi-q_S-\bar{q}_S)-3)^2}{4\ell(2\Delta-1)(2(\Delta-\ell)-1)(2(\Delta-\ell)-3)(2(\Delta+\ell-\Delta_\phi+q_S-\bar{q}_S)+1)(2(\Delta-\ell+\Delta_\phi-q_S+\bar{q}_S)-7)}.\end{aligned}\tag{2.31}$$

As before, there are also contributions corresponding to superconformal descendants whose primary does not appear in the $\bar{\phi} \times R$ OPE. In particular, corresponding to (2.25) and (2.29) we have

$$\hat{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; R\phi} = \hat{c}_1 \tilde{g}_{\Delta + \frac{1}{2}, \ell}^{\Delta_\phi - \Delta_R} - \hat{c}_2 \tilde{g}_{\Delta + \frac{3}{2}, \ell + 1}^{\Delta_\phi - \Delta_R}, \quad (2.35)$$

and

$$\check{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; R\phi} = \check{c}_1 \tilde{g}_{\Delta + \frac{1}{2}, \ell}^{\Delta_\phi - \Delta_R} - \check{c}_2 \tilde{g}_{\Delta + \frac{3}{2}, \ell - 1}^{\Delta_\phi - \Delta_R}, \quad \ell \geq 1, \quad (2.36)$$

while we also have the $\tilde{g}_{\Delta + 1, \ell}^{\Delta_\phi - \Delta_R}$ conformal block contribution. The unitarity bounds are explained above.

In the $14 \rightarrow 32$ channel we can use results of [14] to obtain

$$\langle \bar{\phi}(x_1) R(x_2) R(x_3) \phi(x_4) \rangle = \frac{1}{r_{14}^{2\Delta_\phi} r_{23}^{2\Delta_R}} \sum_{\substack{\mathcal{O}_\ell \in \bar{\phi} \times \phi \\ \mathcal{O}_\ell \in R \times R}} (-1)^\ell \mathcal{G}_{\Delta, \ell}^{\bar{\phi}\phi; RR}(v, u), \quad (2.37)$$

where

$$\mathcal{G}_{\Delta, \ell \text{ even}}^{\bar{\phi}\phi; RR} = c_{\bar{\phi}\phi\mathcal{O}_\ell}^* c_{RR\mathcal{O}_\ell}^{(0)} g_{\Delta, \ell} - \frac{c_{\bar{\phi}\phi\mathcal{O}_\ell}^* \left((\Delta + \ell)^2 c_{RR\mathcal{O}_\ell}^{(0)} - 8(\Delta - 1) c_{RR\mathcal{O}_\ell}^{(2)} \right)}{16\Delta(\Delta - \ell - 1)(\Delta + \ell + 1)} g_{\Delta + 2, \ell}, \quad (2.38)$$

and

$$\mathcal{G}_{\Delta, \ell \text{ odd}}^{\bar{\phi}\phi; RR} = -\frac{c_{\bar{\phi}\phi\mathcal{O}_\ell}^* c_{RR\mathcal{O}_\ell}^{(1)}}{2(\Delta + \ell + 1)} g_{\Delta + 1, \ell + 1} - \frac{c_{\bar{\phi}\phi\mathcal{O}_\ell}^* \left(c_{RR\mathcal{O}_\ell}^{(1)} + \frac{\ell + 1}{\ell} c_{RR\mathcal{O}_\ell}^{(3)} \right)}{2(\Delta - \ell - 1)} g_{\Delta + 1, \ell - 1}. \quad (2.39)$$

2.3 Four-point function $\langle R(x_1) R(x_2) R(x_3) R(x_4) \rangle$

In the $12 \rightarrow 34$ channel we can write

$$\langle R(x_1) R(x_2) R(x_3) R(x_4) \rangle = \frac{1}{r_{12}^{2\Delta_R} r_{34}^{2\Delta_R}} \sum_{\mathcal{O}_\ell \in R \times R} \mathcal{G}_{\Delta, \ell}^{RR; RR}(u, v). \quad (2.40)$$

Here the sum runs over superconformal primaries, but also over just conformal primaries if a superconformal primary does not contribute but one of its descendants does. Only even-spin operators can be exchanged in the $R \times R$ OPE. These can come from even- or odd-spin superconformal primaries, so that the sum in (2.40) runs over \mathcal{O}_ℓ 's with both even and odd spin. The block $G_{\Delta, \ell}^{RR; RR}$, then, receives separate contributions from even- and odd-spin superconformal primaries. There are no constraints on R , except that it is a real operator of dimension $\Delta \geq \ell + 2$ by unitarity, and so from results of [14] we see that we cannot fix the coefficients of the conformal block contributions to the superconformal blocks. The best we can do is write

$$\mathcal{G}_{\Delta, \ell \text{ even}}^{RR; RR} = |c_{RR\mathcal{O}_\ell}^{(0)}|^2 g_{\Delta, \ell} + \frac{\left| (\Delta + \ell)^2 c_{RR\mathcal{O}_\ell}^{(0)} - 8(\Delta - 1) c_{RR\mathcal{O}_\ell}^{(2)} \right|^2}{16\Delta^2(\Delta - \ell - 1)(\Delta - \ell - 2)(\Delta + \ell)(\Delta + \ell + 1)} g_{\Delta + 2, \ell}, \quad (2.41)$$

and

$$\mathcal{G}_{\Delta, \ell \text{ odd}}^{RR; RR} = \frac{|c_{RR\mathcal{O}_\ell}^{(1)}|^2}{(\Delta + \ell)(\Delta + \ell + 1)} g_{\Delta + 1, \ell + 1} + \frac{\left| c_{RR\mathcal{O}_\ell}^{(1)} + \frac{\ell + 1}{\ell} c_{RR\mathcal{O}_\ell}^{(3)} \right|^2}{(\Delta - \ell - 1)(\Delta + \ell + 1)} g_{\Delta + 1, \ell - 1}. \quad (2.42)$$

A superconformal primary that is not an integer-spin Lorentz representation can have superconformal descendant conformal primary components that contribute to (2.40). It turns out that we only need to consider superconformal primaries of the form $\mathcal{O}_{\alpha\alpha_1\dots\alpha_\ell;\dot{\alpha}_2\dots\dot{\alpha}_\ell}$ with even $\ell \geq 2$ and $q = \bar{q} = \frac{1}{2}\Delta$.⁶ The relevant operator is then the conformal primary contained in the superconformal descendant $\bar{Q}_{(\dot{\alpha}_1} Q^\alpha \mathcal{O}_{\alpha\alpha_1\dots\alpha_\ell;\dot{\alpha}_2\dots\dot{\alpha}_\ell)}$, where the undotted indices are the only ones that are symmetrized with $\dot{\alpha}_1$. The conformal block we need to include is $g_{\Delta+1,\ell}$ with even $\ell \geq 2$ and $\Delta \geq \ell + 3$ by unitarity.

3 Four-point functions with linear multiplets

So far we have analyzed four-point functions including a chiral operator ϕ , its conjugate $\bar{\phi}$, and a real field R . The results we have obtained can be easily adapted to the case where the corresponding real superfield \mathcal{R} is a linear multiplet \mathcal{J} , containing a U(1) vector current j^μ . Linear multiplets have $q_{\mathcal{J}} = \bar{q}_{\mathcal{J}} = 1$, and appear in theories with global symmetries. The superspace three-point function $\langle \mathcal{J}(z_1) \mathcal{J}(z_2) \mathcal{O}(z_3) \rangle$ was considered in [25], where the superconformal blocks for $\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle$ were computed. Bootstrap constraints from $\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle$ were obtained in [13]. Our aim here is to obtain bounds using the system of correlators $\langle \bar{\phi}(x_1) \phi(x_2) \bar{\phi}(x_3) \phi(x_4) \rangle$, $\langle \bar{\phi}(x_1) J(x_2) \phi(x_3) J(x_4) \rangle$, and $\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle$.

The associated superconformal-block decomposition of these four-point functions can be obtained from the results of section 2, given that \mathcal{J} is a particular case of a real superfield with $q_{\mathcal{J}} = \bar{q}_{\mathcal{J}} = 1$. Since $Q^2(J) = \bar{Q}^2(J) = 0$ and $Q_\alpha(\bar{\phi}) = 0$, we also need to make sure that the operators in the right hand side of the $\bar{\phi} \times J$ OPE are annihilated by Q^2 . This last requirement implies that a superconformal primary of the form $\mathcal{O}_{\alpha_1\dots\alpha_\ell;\dot{\alpha}_1\dots\dot{\alpha}_\ell}$, as considered around (2.18) above, can only have $\bar{q} = 1$ and $\ell = 0$ [23], i.e. it can be a scalar with $\Delta = \Delta_\phi + 2$. This implies that, analogously to the blocks defined in (2.19) and (2.34), we only need

$$\bar{\mathcal{G}}_{\Delta_\phi+2,0,\Delta_\phi}^{\bar{\phi}J;\phi J} = g_{\Delta_\phi+2,0}^{\Delta_\phi-2}, \quad \bar{\mathcal{G}}_{\Delta_\phi+2,0,\Delta_\phi}^{\bar{\phi}J;J\phi} = \tilde{g}_{\Delta_\phi+2,0}^{\Delta_\phi-2}. \quad (3.1)$$

Without any changes other than $\Delta_R \rightarrow \Delta_J = 2$ we can define $\hat{\mathcal{G}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;\phi J}$, $\hat{\mathcal{G}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi}$, $\check{\mathcal{G}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;\phi J}$, and $\check{\mathcal{G}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi}$ using (2.25), (2.35), (2.29), and (2.36), respectively, as well as $g_{\Delta+1,\ell}^{\Delta_\phi-2}$ with $\Delta \geq |\Delta_\phi - 3| + \ell + 2$.

For the blocks defined in (2.38), (2.39), (2.41), and (2.42) we need to use relations that exist between $c_{JJ\mathcal{O}_\ell}^{(2)}$ and $c_{JJ\mathcal{O}_\ell}^{(0)}$, as well as between $c_{JJ\mathcal{O}_\ell}^{(3)}$ and $c_{JJ\mathcal{O}_\ell}^{(1)}$, namely [14]

$$c_{JJ\mathcal{O}_\ell}^{(2)} = -\frac{1}{8}(\Delta + \ell)(\Delta - \ell - 4)c_{JJ\mathcal{O}_\ell}^{(0)}, \quad c_{JJ\mathcal{O}_\ell}^{(3)} = -\frac{2(\Delta - 2)}{\Delta + \ell}c_{JJ\mathcal{O}_\ell}^{(1)}. \quad (3.2)$$

Using this we can define, in the $14 \rightarrow 32$ channel,

$$\langle \bar{\phi}(x_1) J(x_2) J(x_3) \phi(x_4) \rangle = \frac{1}{r_{14}^{2\Delta_\phi} r_{23}^4} \sum_{\substack{\mathcal{O}_\ell \in \bar{\phi} \times \phi \\ \mathcal{O}_\ell \in J \times J}} c_{\phi\mathcal{O}_\ell}^* c_{JJ\mathcal{O}_\ell} (-1)^\ell \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi;JJ}(v,u), \quad (3.3)$$

⁶The three-point function $\langle \mathcal{R}(z_1) \mathcal{R}(z_2) \mathcal{O}_I(z_3) \rangle$ is symmetric under $z_1 \leftrightarrow z_2$, something that restricts the possible non-integer-spin superconformal primary operators we can consider. We thank Ran Yacoby for discussions on this point.

where

$$\mathcal{G}_{\Delta,\ell \text{ even}}^{\bar{\phi}\phi;JJ} = g_{\Delta,\ell} - \frac{(\Delta-2)(\Delta+\ell)(\Delta-\ell-2)}{16\Delta(\Delta-\ell-1)(\Delta+\ell+1)} g_{\Delta+2,\ell}, \quad (3.4)$$

and

$$\mathcal{G}_{\Delta,\ell \text{ odd}}^{\bar{\phi}\phi;JJ} = -\frac{1}{2(\Delta+\ell+1)} g_{\Delta+1,\ell+1} + \frac{(\ell+2)(\Delta-\ell-2)}{2\ell(\Delta+\ell)(\Delta-\ell-1)} g_{\Delta+1,\ell-1}. \quad (3.5)$$

Finally, in the $12 \rightarrow 34$ channel we can write

$$\langle J(x_1)J(x_2)J(x_3)J(x_4) \rangle = \frac{1}{r_{12}^4 r_{34}^4} \sum_{\mathcal{O}_\ell \in J \times J} |c_{RR\mathcal{O}_\ell}|^2 \mathcal{G}_{\Delta,\ell}^{JJ;JJ}(u,v), \quad (3.6)$$

with

$$\mathcal{G}_{\Delta,\ell \text{ even}}^{JJ;JJ} = g_{\Delta,\ell} + \frac{(\Delta-2)^2(\Delta+\ell)(\Delta-\ell-2)}{16\Delta^2(\Delta-\ell-1)(\Delta+\ell+1)} g_{\Delta+2,\ell}, \quad (3.7)$$

and

$$\mathcal{G}_{\Delta,\ell \text{ odd}}^{JJ;JJ} = \frac{1}{(\Delta+\ell)(\Delta+\ell+1)} g_{\Delta+1,\ell+1} + \frac{(\ell+2)^2(\Delta-\ell-2)}{\ell^2(\Delta+\ell)^2(\Delta-\ell-1)} g_{\Delta+1,\ell-1}. \quad (3.8)$$

We should also mention here that there are conformal primary superconformal descendant operators that contribute to the four-point functions involving J , but whose corresponding superconformal primaries do not. This type of operators has been analyzed in detail in [13]. The result is that in order to account for these operators we need to include $g_{\Delta+1,\ell}$ with even $\ell \geq 2$ and $\Delta \geq \ell + 3$ by unitarity.

4 Crossing relations

Using the results of section 2 we can now write down the crossing equations that we use in our numerical analysis. It is well-known that from $\langle \bar{\phi}(x_1)\phi(x_2)\bar{\phi}(x_3)\phi(x_4) \rangle$ we obtain three crossing relations [8]. We get another three from $\langle \bar{\phi}(x_1)R(x_2)\phi(x_3)R(x_4) \rangle$ (for these we will assume that $1 \leq \Delta_\phi < 2$), and a final crossing relation from $\langle R(x_1)R(x_2)R(x_3)R(x_4) \rangle$. In total we have seven crossing relations.

4.1 Chiral-chiral and chiral-antichiral

From $\langle \bar{\phi}(x_1)\phi(x_2)\bar{\phi}(x_3)\phi(x_4) \rangle$ we find the crossing relations [8]

$$\sum_{\mathcal{O}_\ell \in \bar{\phi} \times \phi} |c_{\bar{\phi}\phi\mathcal{O}_\ell}|^2 \begin{pmatrix} \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) \\ \mathcal{H}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) \\ (-1)^\ell \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) \end{pmatrix} + \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times \bar{\phi}} |c_{\bar{\phi}\bar{\phi}\bar{\mathcal{O}}_\ell}|^2 \begin{pmatrix} F_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\phi}(u,v) \\ -H_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\phi}(u,v) \\ 0 \end{pmatrix} = 0, \quad (4.1)$$

where

$$\begin{aligned} \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) &= u^{-\Delta_\phi} \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) - (u \leftrightarrow v), \\ \mathcal{H}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) &= u^{-\Delta_\phi} \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi;\bar{\phi}\phi}(u,v) + (u \leftrightarrow v), \\ \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\phi}(u,v) &= u^{-\Delta_\phi} \mathcal{G}_{\Delta,\ell}^{\bar{\phi}\phi;\phi\phi}(u,v) - (u \leftrightarrow v), \\ F_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\phi}(u,v) &= u^{-\Delta_\phi} g_{\Delta,\ell}(u,v) - (u \leftrightarrow v), \\ H_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\phi}(u,v) &= u^{-\Delta_\phi} g_{\Delta,\ell}(u,v) + (u \leftrightarrow v). \end{aligned} \quad (4.2)$$

4.2 Chiral-real

From $\langle \bar{\phi}(x_1) R(x_2) R(x_3) \phi(x_4) \rangle$ we find

$$\begin{aligned}
 & \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |\bar{c}_{\bar{\phi} R \mathcal{O}_\ell}|^2 (-1)^\ell \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\hat{c}_{\bar{\phi} R (Q\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell \hat{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} \\
 & + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\check{c}_{\bar{\phi} R (Q\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell \check{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} + \sum_{(Q^2\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |c_{\bar{\phi} R (Q^2\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} \quad (4.3) \\
 & + \sum_{\mathcal{O}_\ell \in \bar{\phi} \times \phi} c_{\bar{\phi} \phi \mathcal{O}_\ell}^* c_{RR \mathcal{O}_\ell} (-1)^\ell \mathcal{F}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR} = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |\bar{c}_{\bar{\phi} R \mathcal{O}_\ell}|^2 (-1)^\ell \bar{\mathcal{H}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\hat{c}_{\bar{\phi} R (Q\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell \hat{\mathcal{H}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} \\
 & + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\check{c}_{\bar{\phi} R (Q\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell \check{\mathcal{H}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} + \sum_{(Q^2\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |c_{\bar{\phi} R (Q^2\bar{\mathcal{O}})_\ell}|^2 (-1)^\ell H_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi} \quad (4.4) \\
 & - \sum_{\mathcal{O}_\ell \in \bar{\phi} \times \phi} c_{\bar{\phi} \phi \mathcal{O}_\ell}^* c_{RR \mathcal{O}_\ell} (-1)^\ell \mathcal{H}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR} = 0,
 \end{aligned}$$

where

$$\begin{aligned}
 \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} \bar{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi} R; R\phi}(u, v) - (u \leftrightarrow v), \\
 \bar{\mathcal{H}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} \bar{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi} R; R\phi}(u, v) + (u \leftrightarrow v),
 \end{aligned} \quad (4.5)$$

and similarly for $\hat{\mathcal{F}}, \hat{\mathcal{H}}, \check{\mathcal{F}}, \check{\mathcal{H}}$, using $\hat{\mathcal{G}}, \check{\mathcal{G}}$,

$$\begin{aligned}
 F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} \tilde{\mathcal{g}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi} R; R\phi}(u, v) - (u \leftrightarrow v), \\
 H_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; R\phi}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} \tilde{\mathcal{g}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi} R; R\phi}(u, v) + (u \leftrightarrow v),
 \end{aligned} \quad (4.6)$$

and, if ℓ is even, $c_{RR \mathcal{O}_\ell} = c_{RR \mathcal{O}_\ell}^{(0)}$ and

$$\begin{aligned}
 \mathcal{F}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR}(u, v) &= u^{-\Delta_R} \mathcal{G}_{\Delta, \ell \text{ even}}^{\bar{\phi} \phi; RR}(u, v) - (u \leftrightarrow v), \\
 \mathcal{H}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR}(u, v) &= u^{-\Delta_R} \mathcal{G}_{\Delta, \ell \text{ even}}^{\bar{\phi} \phi; RR}(u, v) + (u \leftrightarrow v),
 \end{aligned} \quad (4.7)$$

while, if ℓ is odd, $c_{RR \mathcal{O}_\ell} = c_{RR \mathcal{O}_\ell}^{(1)}$ and

$$\begin{aligned}
 \mathcal{F}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR}(u, v) &= u^{-\Delta_R} \mathcal{G}_{\Delta, \ell \text{ odd}}^{\bar{\phi} \phi; RR}(u, v) - (u \leftrightarrow v), \\
 \mathcal{H}_{\Delta, \ell, \Delta_R}^{\bar{\phi} \phi; RR}(u, v) &= u^{-\Delta_R} \mathcal{G}_{\Delta, \ell \text{ odd}}^{\bar{\phi} \phi; RR}(u, v) + (u \leftrightarrow v).
 \end{aligned} \quad (4.8)$$

Note that in (4.7) and (4.8) the superconformal blocks of (2.38) and (2.39) have been rescaled by $c_{\bar{\phi} \phi \mathcal{O}_\ell}^* c_{RR \mathcal{O}_\ell}^{(0)}$ and $c_{\bar{\phi} \phi \mathcal{O}_\ell}^* c_{RR \mathcal{O}_\ell}^{(1)}$, respectively.

The crossing relation arising from $\langle \bar{\phi}(x_1) R(x_2) \phi(x_3) R(x_4) \rangle$ is

$$\begin{aligned} & \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |\bar{c}_{\bar{\phi} R \mathcal{O}_\ell}|^2 \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\hat{c}_{\bar{\phi} R(Q\bar{\mathcal{O}})_\ell}|^2 \hat{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R} \\ & + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\check{c}_{\bar{\phi} R(Q\bar{\mathcal{O}})_\ell}|^2 \check{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R} + \sum_{(Q^2\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |c_{\bar{\phi} R(Q^2\bar{\mathcal{O}})_\ell}|^2 F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R} = 0, \end{aligned} \quad (4.9)$$

where

$$\begin{aligned} \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} \bar{\mathcal{G}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi} R; \phi R}(u, v) - (u \leftrightarrow v), \\ F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi} R; \phi R}(u, v) &= u^{-\frac{1}{2}(\Delta_\phi + \Delta_R)} g_{\Delta, \ell}^{\Delta_\phi - \Delta_R}(u, v) - (u \leftrightarrow v), \end{aligned} \quad (4.10)$$

and similarly for $\hat{\mathcal{F}}, \check{\mathcal{F}}$.

4.3 Real-real

From $\langle R(x_1) R(x_2) R(x_3) R(x_4) \rangle$ we find the crossing relation

$$\sum_{\mathcal{O}_\ell \in R \times R} |c_{RR\mathcal{O}_\ell}|^2 \mathcal{F}_{\Delta, \ell, \Delta_R}^{RR; RR} + \sum_{(Q\mathcal{O})_\ell \in R \times R} |c_{RR(Q\mathcal{O})_\ell}|^2 F_{\Delta, \ell, \Delta_R}^{RR; RR} = 0, \quad (4.11)$$

with

$$\begin{aligned} \mathcal{F}_{\Delta, \ell, \Delta_R}^{RR; RR}(u, v) &= u^{-\Delta_R} \mathcal{G}_{\Delta, \ell}^{RR; RR}(u, v) - (u \leftrightarrow v), \\ F_{\Delta, \ell, \Delta_R}^{RR; RR}(u, v) &= u^{-\Delta_R} g_{\Delta, \ell}(u, v) - (u \leftrightarrow v), \end{aligned} \quad (4.12)$$

and for ℓ even we define $c_{RR\mathcal{O}_\ell} = c_{RR\mathcal{O}_\ell}^{(0)}$ and use (2.41) rescaled by $|c_{RR\mathcal{O}_\ell}^{(0)}|^2$, while for ℓ odd we define $c_{RR\mathcal{O}_\ell} = c_{RR\mathcal{O}_\ell}^{(1)}$ and use (2.42) rescaled by $|c_{RR\mathcal{O}_\ell}^{(1)}|^2$.

4.4 System of crossing relations

The crossing relations (4.1), (4.3), (4.4), (4.9) and (4.11) can now be written in the form

$$\begin{aligned} & \sum_{\substack{\mathcal{O}_\ell \in \bar{\phi} \times \phi \\ \bar{\mathcal{O}}_\ell \in R \times R}} \left(c_{\bar{\phi}\phi\mathcal{O}_\ell}^* c_{RR\mathcal{O}_\ell}^* c_{RR\bar{\mathcal{O}}_\ell}^* \right) \vec{V}_{\Delta, \ell, \Delta_\phi, \Delta_R} \begin{pmatrix} c_{\bar{\phi}\phi\mathcal{O}_\ell} \\ c_{RR\mathcal{O}_\ell} \\ c_{RR\bar{\mathcal{O}}_\ell}^* \end{pmatrix} + \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times \bar{\phi}} |c_{\bar{\phi}\bar{\phi}\bar{\mathcal{O}}_\ell}|^2 \vec{W}_{\Delta, \ell, \Delta_\phi} \\ & + \sum_{\bar{\mathcal{O}}_\ell \in \bar{\phi} \times R} |\bar{c}_{\bar{\phi} R \bar{\mathcal{O}}_\ell}|^2 \vec{X}_{\Delta, \ell, \Delta_\phi, \Delta_R} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\hat{c}_{\bar{\phi} R(Q\bar{\mathcal{O}})_\ell}|^2 \vec{X}_{\Delta, \ell, \Delta_\phi, \Delta_R} \\ & + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |\check{c}_{\bar{\phi} R(Q\bar{\mathcal{O}})_\ell}|^2 \vec{X}_{\Delta, \ell, \Delta_\phi, \Delta_R} + \sum_{(Q^2\bar{\mathcal{O}})_\ell \in \bar{\phi} \times R} |c_{\bar{\phi} R(Q^2\bar{\mathcal{O}})_\ell}|^2 \vec{Y}_{\Delta, \ell, \Delta_\phi, \Delta_R} \\ & + \sum_{(Q\mathcal{O})_\ell \in R \times R} |c_{RR(Q\mathcal{O})_\ell}|^2 \vec{Z}_{\Delta, \ell, \Delta_R} = 0, \end{aligned} \quad (4.13)$$

where the seven-vector $\vec{V}_{\Delta, \ell, \Delta_\phi, \Delta_R}$ contains the 3×3 matrices

$$\begin{aligned}
 V_{\Delta, \ell, \Delta_\phi}^1 &= \begin{pmatrix} \mathcal{F}_{\Delta, \ell, \Delta_\phi}^{\bar{\phi}\phi; \phi\bar{\phi}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & V_{\Delta, \ell, \Delta_\phi}^2 &= \begin{pmatrix} \mathcal{H}_{\Delta, \ell, \Delta_\phi}^{\bar{\phi}\phi; \phi\bar{\phi}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 V_{\Delta, \ell, \Delta_\phi}^3 &= \begin{pmatrix} (-1)^\ell \mathcal{F}_{\Delta, \ell, \Delta_\phi}^{\bar{\phi}\phi; \phi\bar{\phi}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 V_{\Delta, \ell, \Delta_R}^4 &= \begin{pmatrix} 0 & \frac{1}{2}(-1)^\ell F_{1, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & \frac{1}{2}(-1)^\ell F_{2, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} \\ \frac{1}{2}(-1)^\ell F_{1, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & 0 & 0 \\ \frac{1}{2}(-1)^\ell F_{2, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & 0 & 0 \end{pmatrix}, \\
 V_{\Delta, \ell, \Delta_R}^5 &= \begin{pmatrix} 0 & \frac{1}{2}(-1)^{\ell+1} H_{1, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & \frac{1}{2}(-1)^{\ell+1} H_{2, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} \\ \frac{1}{2}(-1)^{\ell+1} H_{1, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & 0 & 0 \\ \frac{1}{2}(-1)^{\ell+1} H_{2, \Delta, \ell, \Delta_R}^{\bar{\phi}\phi; RR} & 0 & 0 \end{pmatrix}, \\
 V^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & V_{\Delta, \ell, \Delta_R}^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & F_{1, \Delta, \ell, \Delta_R}^{RR; RR} & 0 \\ 0 & 0 & F_{2, \Delta, \ell, \Delta_R}^{RR; RR} \end{pmatrix}, \tag{4.14}
 \end{aligned}$$

and the remaining vectors are given by

$$\vec{W}_{\Delta, \ell, \Delta_\phi} = \begin{pmatrix} F_{\Delta, \ell, \Delta_\phi}^{\bar{\phi}\phi; \phi\phi} \\ -H_{\Delta, \ell, \Delta_\phi}^{\bar{\phi}\phi; \phi\phi} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{X}_{\Delta, \ell, \Delta_\phi, \Delta_R} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (-1)^\ell \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; R\phi} \\ (-1)^\ell \bar{\mathcal{H}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; R\phi} \\ \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; \phi R} \\ 0 \end{pmatrix}, \tag{4.15}$$

with definitions for \vec{X} and \vec{Y} similar to that for \vec{X} but involving $\hat{\mathcal{F}}, \hat{\mathcal{H}}, \check{\mathcal{F}}, \check{\mathcal{H}}$, and

$$\vec{Y}_{\Delta, \ell, \Delta_\phi, \Delta_R} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (-1)^\ell F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; R\phi} \\ (-1)^\ell H_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; R\phi} \\ F_{\Delta, \ell, \Delta_\phi, \Delta_R}^{\bar{\phi}R; \phi R} \\ 0 \end{pmatrix}, \quad \vec{Z}_{\Delta, \ell, \Delta_R} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F_{\Delta, \ell, \Delta_R}^{RR; RR} \end{pmatrix}. \tag{4.16}$$

We should note here that the entries of $\vec{V}_{\Delta,\ell,\Delta_\phi,\Delta_R}$ are 3×3 matrices because (2.38), (2.39), (2.41), and (2.42) do not contain their conformal block contributions with fixed relative coefficients. The subscripts 1 and 2 in the functions F and H of $V_{\Delta,\ell,\Delta_R}^4, V_{\Delta,\ell,\Delta_R}^5$ and $V_{\Delta,\ell,\Delta_R}^7$ denote the first and second part of the corresponding \mathcal{F} and \mathcal{H} functions defined in (4.7), (4.8) and (4.12), as obtained when the blocks (2.38), (2.39), (2.41) and (2.42) are used and the coefficient $c'_{RR\mathcal{O}_\ell}$ is appropriately defined. For example, for even ℓ we have $F_{2,\Delta,\ell,\Delta_R}^{\bar{\phi}\phi;RR} = -\frac{1}{16\Delta(\Delta-\ell-1)(\Delta+\ell+1)} g_{\Delta+2,\ell}$ and $c'_{RR\mathcal{O}_\ell} = (\Delta + \ell)^2 c_{RR\mathcal{O}_\ell}^{(0)} - 8(\Delta - 1)c_{RR\mathcal{O}_\ell}^{(2)}$ as follows from (2.38). Note that we can neglect $\vec{Z}_{\Delta,\ell,\Delta_R}$ for its contributions are already contained in $V_{\Delta,\ell,\Delta_R}^7$.

The crossing relation (4.13) can be used with the usual numerical methods. This requires polynomial approximations for derivatives of the various functions that participate. We describe the required results in appendix A. For numerical optimization we use SDPB [26]. The functional search space is governed by the parameter Λ , where each component α_i of a seven-functional $\vec{\alpha}$ is a linear combination of $\frac{1}{2} \lfloor \frac{\Lambda+2}{2} \rfloor (\lfloor \frac{\Lambda+2}{2} \rfloor + 1)$ independent nonvanishing derivatives, $\alpha_i \propto \sum_{m,n} a_{mn}^i \partial_z^m \partial_{\bar{z}}^n |_{1/2,1/2}$ with $m+n \leq \Lambda$. For example, for $\Lambda = 17$, a common choice in the plots below, the search space is 315-dimensional.

5 Crossing relations with linear multiplets

The crossing relations obtained in this case can be brought to the form

$$\begin{aligned}
 & \sum_{\substack{\mathcal{O}_\ell \in \bar{\phi} \times \phi \\ \mathcal{O}_\ell \in J \times J}} \left(c_{\bar{\phi}\phi\mathcal{O}_\ell}^* c_{JJ\mathcal{O}_\ell}^* \right) \vec{V}_{\Delta,\ell,\Delta_\phi} \begin{pmatrix} c_{\bar{\phi}\phi\mathcal{O}_\ell} \\ c_{JJ\mathcal{O}_\ell} \end{pmatrix} + \sum_{\mathcal{O}_\ell \in \bar{\phi} \times \bar{\phi}} |c_{\bar{\phi}\bar{\phi}\mathcal{O}_\ell}|^2 \vec{W}_{\Delta,\ell,\Delta_\phi} \\
 & + \sum_{\bar{\mathcal{O}} \in \bar{\phi} \times J} |\bar{c}_{\bar{\phi}J\bar{\mathcal{O}}}|^2 \vec{X}_{\Delta,0,\Delta_\phi} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times J} |\bar{c}_{\bar{\phi}J(Q\bar{\mathcal{O}})_\ell}|^2 \vec{X}_{\Delta,\ell,\Delta_\phi} + \sum_{(Q\bar{\mathcal{O}})_\ell \in \bar{\phi} \times J} |\bar{c}_{\bar{\phi}J(Q\bar{\mathcal{O}})_\ell}|^2 \vec{X}_{\Delta,\ell,\Delta_\phi} \\
 & + \sum_{(Q^2\bar{\mathcal{O}}) \in \bar{\phi} \times J} |c_{\bar{\phi}J(Q^2\bar{\mathcal{O}})}|^2 \vec{Y}_{\Delta,\ell,\Delta_\phi} + \sum_{(Q\mathcal{O})_\ell \in J \times J} |c_{JJ(Q\mathcal{O})_\ell}|^2 \vec{Z}_{\Delta,\ell} = 0, \tag{5.1}
 \end{aligned}$$

where $\vec{X}_{\Delta,0,\Delta_\phi}$ goes over just two scalar operators with dimension Δ_ϕ and $\Delta_\phi + 2$. Due to the determined coefficients in the superconformal blocks (3.4), (3.5), (3.7), and (3.8), the seven-vector $\vec{V}_{\Delta,\ell,\Delta_\phi}$ contains 2×2 matrices now, contrary to the case in (4.13) where $\vec{V}_{\Delta,\ell,\Delta_\phi,\Delta_R}$ contained 3×3 matrices. Here, $\vec{V}_{\Delta,\ell,\Delta_\phi}$ contains the matrices

$$\begin{aligned}
 V_{\Delta,\ell,\Delta_\phi}^1 &= \begin{pmatrix} \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\bar{\phi}} & 0 \\ 0 & 0 \end{pmatrix}, & V_{\Delta,\ell,\Delta_\phi}^2 &= \begin{pmatrix} \mathcal{H}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\phi\bar{\phi}} & 0 \\ 0 & 0 \end{pmatrix}, \\
 V_{\Delta,\ell,\Delta_\phi}^3 &= \begin{pmatrix} (-1)^\ell \mathcal{F}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\phi;\bar{\phi}\phi} & 0 \\ 0 & 0 \end{pmatrix}, & V_{\Delta,\ell}^4 &= \begin{pmatrix} 0 & \frac{1}{2}(-1)^\ell \mathcal{F}_{\Delta,\ell}^{\bar{\phi}\phi;JJ} \\ \frac{1}{2}(-1)^\ell \mathcal{F}_{\Delta,\ell}^{\bar{\phi}\phi;JJ} & 0 \end{pmatrix}, \\
 V_{\Delta,\ell}^5 &= \begin{pmatrix} 0 & \frac{1}{2}(-1)^{\ell+1} \mathcal{H}_{\Delta,\ell}^{\bar{\phi}\phi;JJ} \\ \frac{1}{2}(-1)^{\ell+1} \mathcal{H}_{\Delta,\ell}^{\bar{\phi}\phi;JJ} & 0 \end{pmatrix}, & V_{\Delta,\ell}^6 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & V_{\Delta,\ell}^7 &= \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{F}_{\Delta,\ell}^{JJ;JJ} \end{pmatrix}, \tag{5.2}
 \end{aligned}$$

and the remaining vectors are given by

$$\vec{W}_{\Delta,\ell,\Delta_\phi} = \begin{pmatrix} F_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\bar{\phi};\phi\phi} \\ -H_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}\bar{\phi};\phi\phi} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{X}_{\Delta,\ell,\Delta_\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{F}_{\Delta,0,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ \bar{H}_{\Delta,0,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ \bar{F}_{\Delta,0,\Delta_\phi}^{\bar{\phi}J;\phi J} \\ 0 \end{pmatrix}, \quad \vec{\tilde{X}}_{\Delta,\ell,\Delta_\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (-1)^\ell \hat{\mathcal{F}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ (-1)^\ell \hat{\mathcal{H}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ \hat{\mathcal{F}}_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;\phi J} \\ 0 \end{pmatrix}, \quad (5.3)$$

with a similar definition for \vec{X} , and

$$\vec{Y}_{\Delta,\ell,\Delta_\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (-1)^\ell F_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ (-1)^\ell H_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;J\phi} \\ F_{\Delta,\ell,\Delta_\phi}^{\bar{\phi}J;\phi J} \\ 0 \end{pmatrix}, \quad \vec{Z}_{\Delta,\ell} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F_{\Delta,\ell}^{JJ;JJ} \end{pmatrix}. \quad (5.4)$$

The various functions F, \mathcal{F} and H, \mathcal{H} here are defined similarly to the analogous functions defined in section 4, using the superconformal blocks of section 3. We note that contrary to the case in section 4, the contributions of $\vec{Z}_{\Delta,\ell}$ are not identical to those in $V_{\Delta,\ell}^7$, and so $\vec{Z}_{\Delta,\ell}$ needs to be included in our numerical analysis.

6 Bounds in theories with ϕ and R

6.1 Using only the chiral-chiral and chiral-antichiral crossing relations

A bound on the dimension of the first unprotected scalar operator R in the $\bar{\phi} \times \phi$ OPE using just (4.1) was first obtained in [8] and recently reproduced in [9]. This bound, for $\Lambda = 21$ and $\Lambda = 29$, is shown in figure 1, and displays a mild kink at $\Delta_\phi \approx 1.4$. The bound for $\Lambda = 21$ was first obtained in [8]. Here we provide a slightly stronger bound at $\Lambda = 29$. If we assume that $\phi^2 = 0$, then the allowed region on the left of the kink disappears [9, 10], turning the kink into a sharp corner. The precision analysis of [9] suggests that the kink is at $\Delta_\phi = \frac{10}{7}$, although this relies on extrapolation.

Using (4.1) we can also obtain a lower bound on the central charge. This is shown in figure 2 for $\Lambda = 25$. The corresponding bound for $\Lambda = 21$ first appeared in [8], and was later improved in [9]. The bound contains a feature slightly to the right of the kink of figure 1. Close to the origin the bound sharply falls just below the free chiral multiplet value of $c = \frac{1}{24}$ in our normalization [7]. We may further assume that Δ_R lies on the

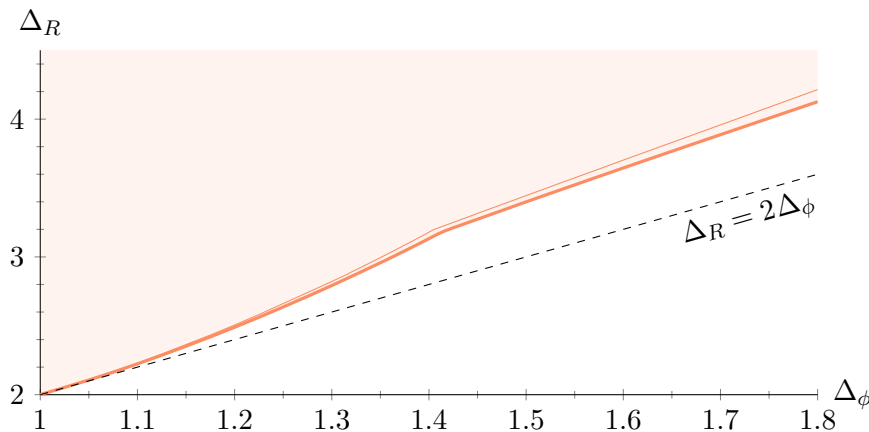


Figure 1. Upper bound on the dimension of the operator R as a function of Δ_ϕ using only (4.1). The generalized free theory dashed line $\Delta_R = 2\Delta_\phi$ is also shown. The shaded area is excluded. In this plot we use $\Lambda = 21$ for the thin and $\Lambda = 29$ for the thick line.

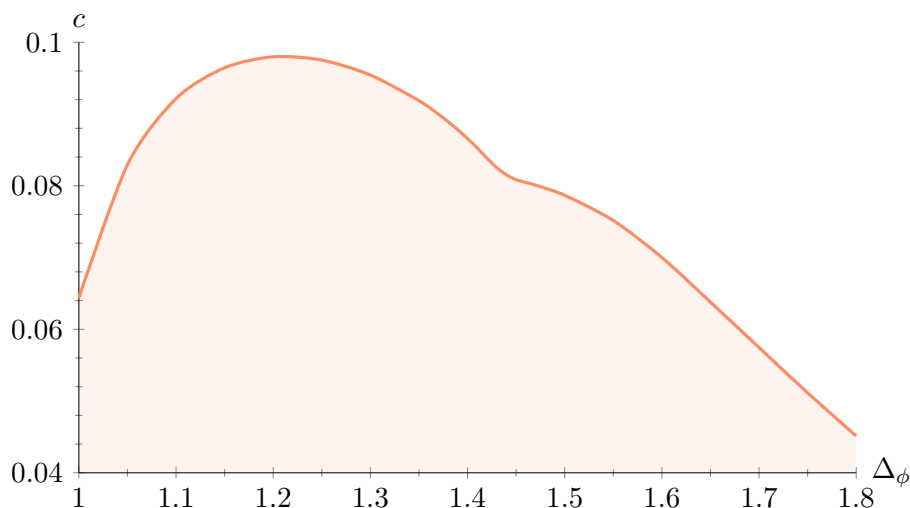


Figure 2. Lower bound on the central charge as a function of Δ_ϕ . The shaded area is excluded. In this plot we use $\Lambda = 25$.

bound of figure 1, and that R is the first scalar after the identity operator in the $\bar{\phi} \times \phi$ OPE. The lower bound on the central charge obtained in this case is shown in figure 3. As we see, these extra assumptions strengthen the bound globally, but have the weakest effect around the free theory and $\Delta_\phi \approx 1.4$. At that Δ_ϕ , which coincides with the position of the kink, we observe a local minimum of the lower bound on c . This result has also been discussed in [10], and is similar to the corresponding bound obtained in $d = 3$ in [2], although the free theory of a single chiral operator in our case has a lower c than the minimum in figure 3. The assumption $\phi^2 = 0$ excludes the region to the left of $\Delta_\phi \approx 1.4$. Therefore, we may conjecture that the putative theory that lives on the kink minimizes c among $\mathcal{N} = 1$ superconformal theories that have a chiral operator ϕ that satisfies $\phi^2 = 0$.

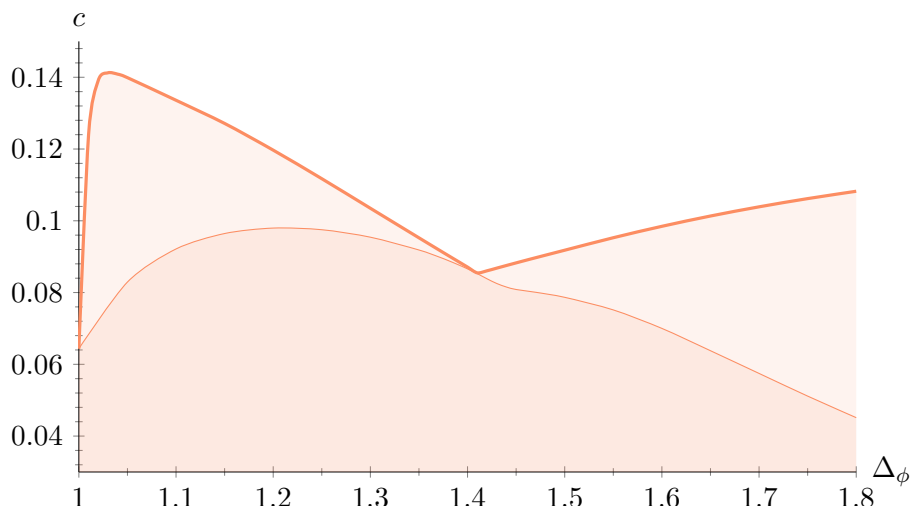


Figure 3. The thick line is the lower bound on the central charge as a function of Δ_ϕ , assuming that Δ_R lies on the bound of figure 1. The thin line is the bound of figure 2. The shaded area is excluded. In this plot we use $\Lambda = 25$.

Such theories were obtained recently [11, 12] from deformations of $\mathcal{N} = 2$ Argyres-Douglas theories [27–29], but they appear to have larger c than the one obtained for the minimal theory in [9], namely $c_{\text{minimal}} = \frac{1}{9}$ after extrapolating to $\Lambda \rightarrow \infty$.

6.2 Using the full set of crossing relations involving ϕ and R

We will now explore bootstrap constraints using the full system of crossing relations (4.13). The virtue of considering mixed correlators is that they allow us to probe a larger part of the operator spectrum, e.g. we can obtain bounds on operator dimensions and OPE coefficients of operators in the $\bar{\phi} \times R$ OPE. In this subsection we assume that Δ_R lies on the (stronger) bound of figure 1. We also impose $c_{\bar{\phi}R\phi} = c_{\bar{\phi}\phi R}$ — the implementation of this follows [6], i.e. we add a single constraint for $\vec{V}_{\Delta_R, 0, \Delta_\phi, \Delta_R} + \vec{X}_{\Delta_\phi, 0, \Delta_\phi, \Delta_R} \otimes \text{diag}(1, 0, 0)$ to our optimization problem. Finally, we introduce a gap of one between the dimension of R and that of the next unprotected real scalar in the spectrum, R' . We have found that for low values of this gap the bounds below are not sensitive to the choice of the gap.

First we would like to obtain a bound on the OPE coefficient of the operator $\bar{\phi}$ in the $\bar{\phi} \times R$ OPE. We can obtain both an upper and a lower bound; they are both shown in figure 4. As we see there is a minimum of the upper bound slightly to the right of $\Delta_\phi \approx 1.4$. Note that the bound of $c_{\bar{\phi}R\phi}$ at the minimum is lower than the free theory value which is equal to one.

Using mixed correlators we can also obtain a bound on the central charge similar to that of figure 3, i.e. assuming that Δ_R saturates its bound. The bound is shown in figure 5. As we see, even though we use the mixed correlator crossing relations the bound obtained is very similar to the corresponding bound in figure 3. The bound of figure 5 is weaker than that of figure 3 due to the lower Λ used in the former.

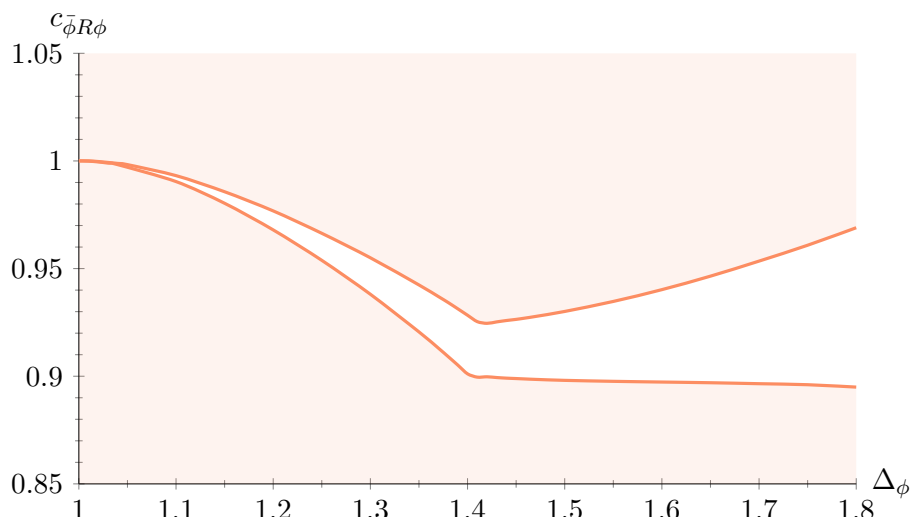


Figure 4. Upper and lower bounds on the OPE coefficient of the operator $\bar{\phi}$ in the $\bar{\phi} \times R$ OPE as a function of Δ_ϕ , assuming Δ_R lies on the bound of figure 1 and demanding $c_{\bar{\phi}R\phi} = c_{\bar{\phi}\phi R}$. We also impose a gap equal to one between Δ_R and $\Delta_{R'}$. The shaded area is excluded. In this plot we use $\Lambda = 17$.

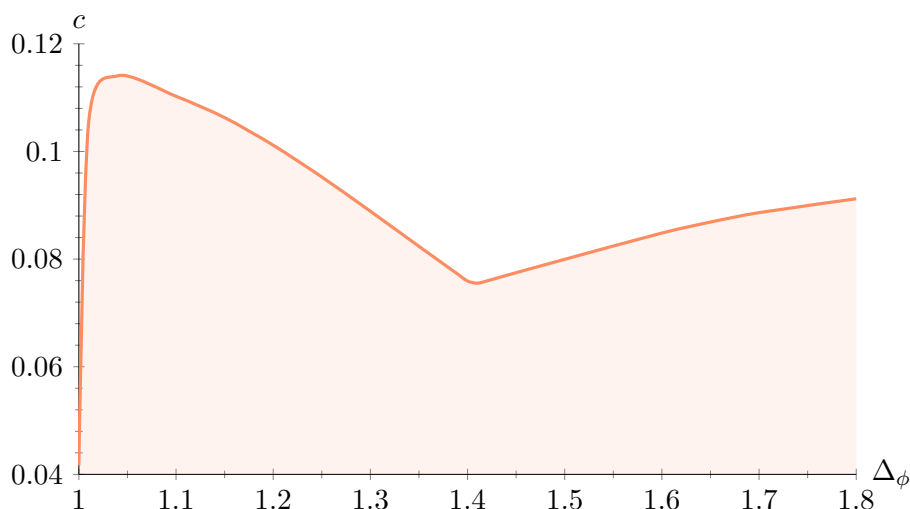


Figure 5. Lower bound on the central charge as a function of Δ_ϕ , assuming that Δ_R lies on the bound of figure 1 and demanding $c_{\bar{\phi}R\phi} = c_{\bar{\phi}\phi R}$. We also impose a gap equal to one between Δ_R and $\Delta_{R'}$. The shaded area is excluded. In this plot we use $\Lambda = 17$.

With the inclusion of the crossing relations (4.3), (4.4) and (4.9) we can attempt to constrain scaling dimensions of operators with R-charge equal to that of $\bar{\phi}$. In particular, we can attempt to find a bound on the dimension of the first scalar superconformal primary after $\bar{\phi}$ in the $\bar{\phi} \times R$ OPE, called $\bar{\phi}'$, assuming that Δ_R lies on the (stronger) bound of figure 1.

Numerically, this turned out to be a hard problem. For $\Lambda = 11$ a bound on $\Delta_{\phi'}$ did not arise for any value of Δ_ϕ . With the assumption that there are no Q -exact scalar operators in the $\bar{\phi} \times R$ OPE, i.e. neglecting the \vec{X} and \vec{Y} scalar contributions in (4.13), we

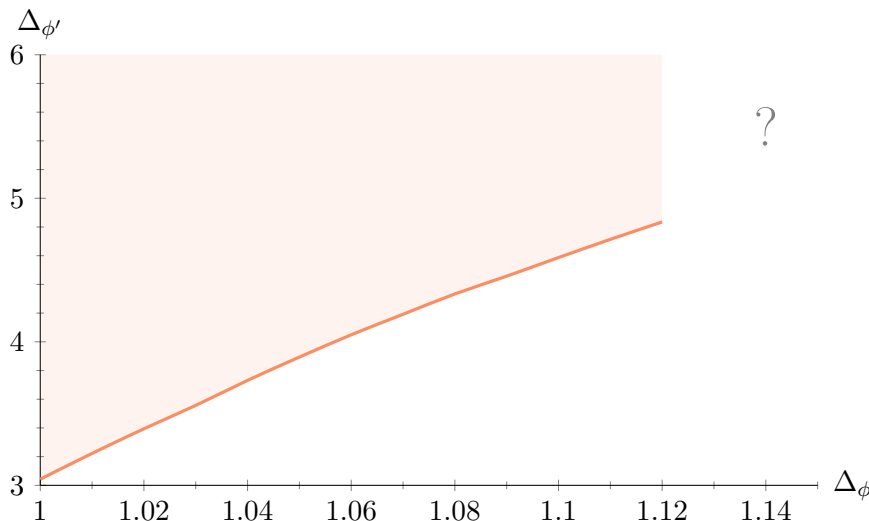


Figure 6. Upper bound on $\Delta_{\phi'}$ as a function of Δ_ϕ , assuming that Δ_R lies on the bound of figure 1 and imposing $c_{\bar{\phi}R\phi} = c_{\bar{\phi}\phi R}$. Here we neglect \vec{X} and \vec{Y} scalar contributions in (4.13), and impose a gap equal to one between Δ_R and $\Delta_{R'}$. The shaded area is excluded. In this plot we use $\Lambda = 11$.

managed to obtain a bound on $\Delta_{\phi'}$ but only for $\Delta_\phi \lesssim 1.12$, after which point the bound was abruptly lost. This bound is shown in figure 6. Increasing our functional search space by taking $\Lambda = 13$, $\Lambda = 17$ and $\Lambda = 19$ we find a bound on $\Delta_{\phi'}$ up to $\Delta_\phi \approx 1.27$, $\Delta_\phi \approx 1.32$ and $\Delta_\phi \approx 1.34$, respectively. At the corresponding Δ_ϕ the bound is again abruptly lost. Note that for these results we do not actually obtain the bound, but rather we ask if the spectrum with $\bar{\phi}$ as the only scalar in the $\bar{\phi} \times R$ OPE is allowed or not. We believe that numerical analysis for higher Λ will yield bounds on $\Delta_{\phi'}$ for higher Δ_ϕ , but it is puzzling that in going from $\Lambda = 17$ to $\Lambda = 19$ we have a very small gain in the Δ_ϕ up to which a bound on $\Delta_{\phi'}$ can be obtained.

The various features we have seen in plots of this section indicate the existence of a CFT with a chiral operator of dimension $\Delta_\phi \approx 1.4$, or $\Delta_\phi = \frac{10}{7}$ based on the analysis of [9]. Unfortunately the mixed correlator analysis has not allowed us to isolate this putative CFT from the allowed region around it, particularly from the allowed region for higher Δ_ϕ . We remind the reader that the region for $\Delta_\phi < \frac{10}{7}$ can be excluded by imposing that $\phi^2 = 0$ as a primary [9, 10]. The set of conditions that isolate this putative CFT from solutions to crossing symmetry with higher Δ_ϕ have not been found in this paper. We hope that future work will be able to identify these conditions, or uncover a physical reason for their absence.

7 Bounds in theories with global symmetries

7.1 Using the crossing relation from $\langle JJJJ \rangle$

Bootstrap bounds arising from the four-point function $\langle J(x_1)J(x_2)J(x_3)J(x_4) \rangle$ were obtained recently in [13]. In fact, [13] considered the more complicated nonabelian case. Here we will consider just the Abelian case, where J carries no adjoint index, and obtain some further bounds that have not appeared before.

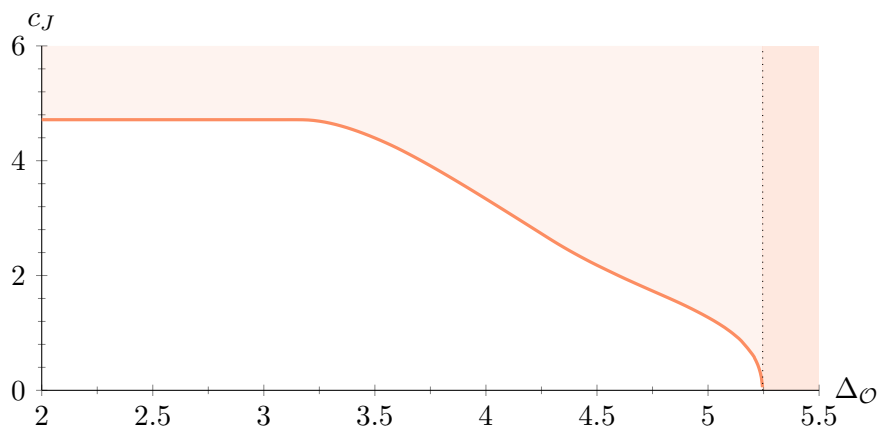


Figure 7. Upper bound on the OPE coefficient of J in the $J \times J$ OPE as a function of the dimension of the first unprotected scalar in the $J \times J$ OPE. The region to the right of the dotted vertical line at $\Delta_{\mathcal{O}} = 5.246$ is not allowed. In this plot we use $\Lambda = 29$.

Since the dimension of J is fixed by symmetry, no external operator dimension can be used as a free parameter. For the plots in this section we will instead use the dimension of the first unprotected operator \mathcal{O} in the $J \times J$ OPE as the parameter in the horizontal axis. Note that there is an upper bound to how large that dimension can get, and so our plots will not extend past that bound. This bound is found here by looking at the value for which the square of the plotted OPE coefficient turns negative.

First, we obtain an upper bound on the OPE coefficient of J in the $J \times J$ OPE. The bound is shown in figure 7. It contains a plateau that eventually breaks down, leading to a violation of unitarity past $\Delta_{\mathcal{O}} = 5.246$. This is a reflection of the fact that the dimension of the first unprotected scalar in the $J \times J$ OPE cannot be larger than $\Delta_{\mathcal{O}} = 5.246$ consistently with unitarity. The $J \times J$ OPE also contains contributions arising from the dimension-three vector multiplet that contains the stress-energy tensor. We can obtain a bound on the OPE coefficient c_V of these contributions; see figure 8. A lower bound on the central charge c can then be derived from these results, since $c_V^2 = \frac{1}{90c}$ in our conventions. Close to the origin we get $c \gtrsim 0.00064$, a bound much weaker than that in figure 2.

The bounds in figures 7 and 8 were obtained using $\Lambda = 29$.⁷ We can also obtain bounds for other values of Λ . We do this here letting \mathcal{O} saturate its unitarity bound, i.e. choosing $\Delta_{\mathcal{O}} = 2$. The plots are shown in figure 9. As Λ gets larger we see observe an approximately linear distribution of the bounds, which we then fit and extrapolate to the origin. The fits are given by

$$c_J^{(\text{fit})} = 3.311 + \frac{39.412}{\Lambda}, \quad c_V^{(\text{fit})} = 2.256 + \frac{56.279}{\Lambda}. \quad (7.1)$$

The limit $\Lambda \rightarrow \infty$ gives us an estimate of the converged optimal bound that can be obtained.

⁷For lower values of Λ , e.g. $\Lambda = 21$, we do not find an upper bound on $\Delta_{\mathcal{O}}$, i.e. c_J^2 and c_V^2 never turn negative. The upper bounds for c_J and c_V in those cases converge to values that do not change with $\Delta_{\mathcal{O}}$ no matter how large $\Delta_{\mathcal{O}}$ becomes.

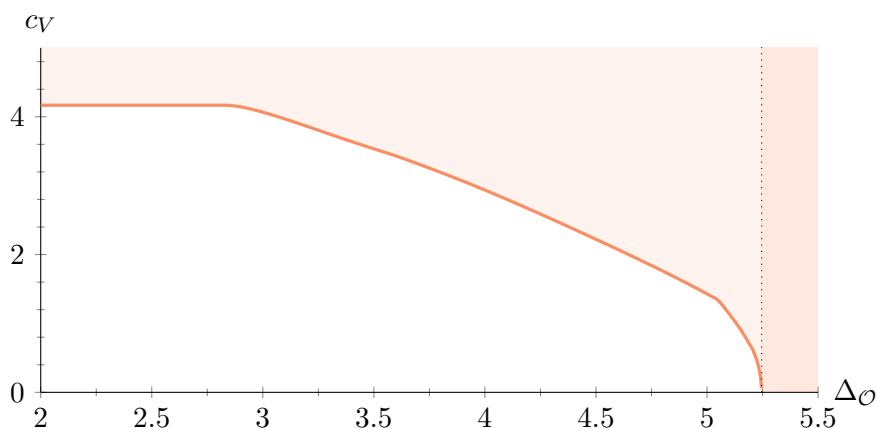


Figure 8. Upper bound on the OPE coefficient of the contributions to the $J \times J$ OPE arising from the leading vector superconformal primary V as a function of the dimension of the first unprotected scalar in the $J \times J$ OPE. The region to the right of the dotted vertical line at $\Delta_{\mathcal{O}} = 5.246$ is not allowed. In this plot we use $\Lambda = 29$.

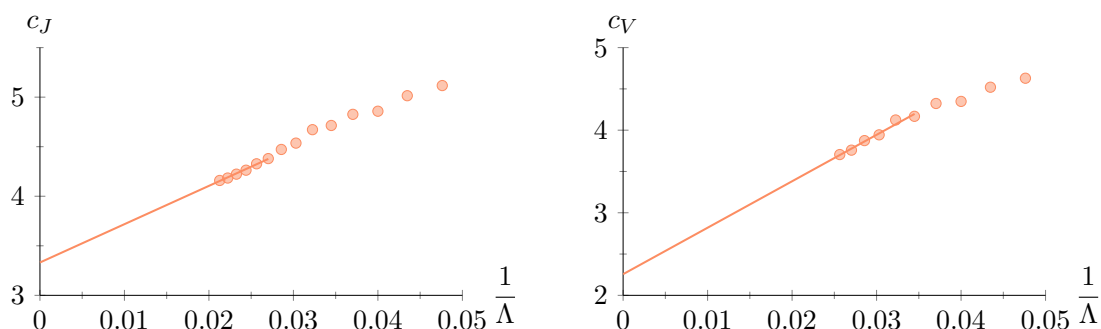


Figure 9. The upper bounds on c_J and c_V with $\Delta_{\mathcal{O}} = 2$ as functions of the inverse cutoff $1/\Lambda$, and linear extrapolations of the six points closest to the origin.

Finally, we also find an upper bound on the OPE coefficient of \mathcal{O} as a function of the dimension of \mathcal{O} ; see figure 10.

7.2 Using the full set of crossing relations involving ϕ and J

Similarly to subsection 6.2 we can here obtain constraints on operators that appear in the $\bar{\phi} \times J$ OPE. One such operator is $\bar{\phi}$ itself, and we can obtain a bound on its OPE coefficient. This OPE coefficient is equal to that of J in the $\bar{\phi} \times \phi$ OPE, and its meaning has been analyzed in [7], where it was denoted by $\tau_{IJ} T_{11}^I T_{11}^J$. The bound is shown in figure 11. One application of this bound is in $SU(N_c)$ SQCD with N_f flavors Q^i and \tilde{Q}_i . Mesons in this theory have scaling dimension $\Delta_M = 3(1 - N_c/N_f)$, which can be close to one at the lower end of the conformal window, $N_f \sim \frac{3}{2}N_c$. This was considered first in [7], where the meson M_1^1 was taken as the chiral operator and the relation

$$\tau_{IJ} T_{11}^I T_{11}^J = 2 \frac{N_f - 1}{3N_c^2} \tag{7.2}$$

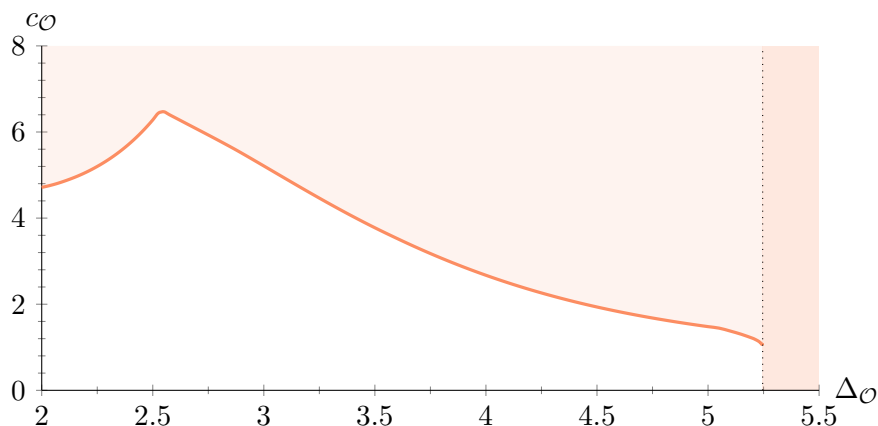


Figure 10. Upper bound on the OPE coefficient of the first unprotected scalar operator in the $J \times J$ OPE as a function of its dimension. The region to the right of the dotted vertical line at $\Delta_{\mathcal{O}} = 5.246$ is not allowed. In this plot we use $\Lambda = 29$.

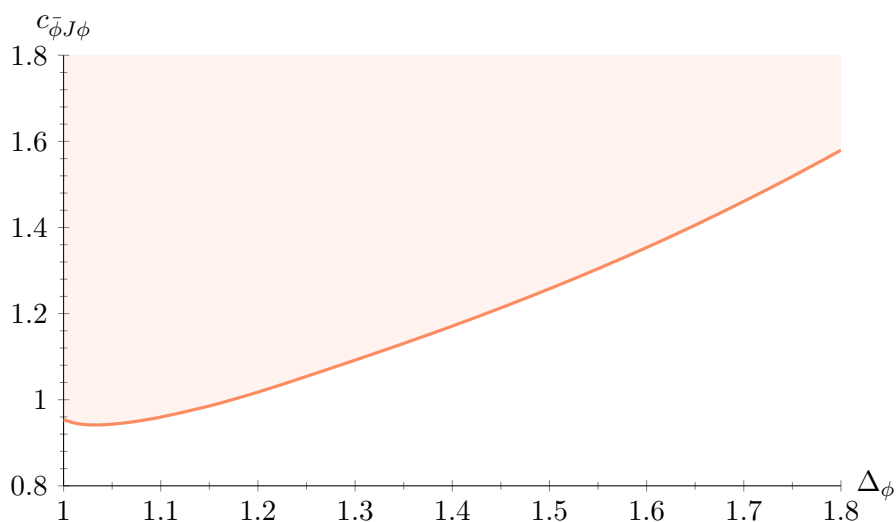


Figure 11. Upper bound on the OPE coefficient of the operator $\bar{\phi}$ in the $\bar{\phi} \times J$ OPE as a function of Δ_{ϕ} , demanding $c_{\bar{\phi}J\phi} = c_{\bar{\phi}\phi J}$. In this plot we use $\Lambda = 17$.

was obtained for the contributions of the flavor currents of the symmetry group $SU(N_f)_L \times SU(N_f)_R$ of SQCD. This satisfies our bound in figure 11 comfortably. For example, for $N_c = 3$ and $N_f = 5$, in which case $\Delta_M = 1.2$, we have $\tau_{IJ} T_{1\bar{1}}^I T_{1\bar{1}}^J \approx 0.3$ with the bound constraining this to be lower than approximately one. Even with these numerical results we are far away from saturating the bound with SQCD, although we can hope that by pushing the numerics further we will get much closer in the near future.

We should also note here that very close to $\Delta_{\phi} = 1$ our bound appears to be converging to a value for $c_{\bar{\phi}J\phi}$ below one, thus excluding the free theory of a free chiral operator charged under a $U(1)$. While we have not been able to obtain a bound very close to one, i.e. 10^{-15} or so away from it, we believe that the bound abruptly jumps right above one as $\Delta_{\phi} \rightarrow 1$ in order to allow the free theory solution. This behavior of the bound has also been seen in [8].

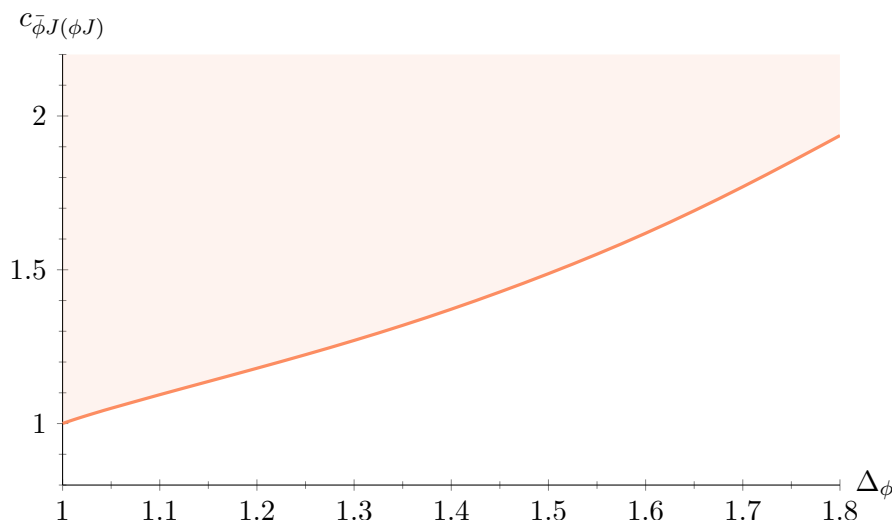


Figure 12. Upper bound on the OPE coefficient of the operator $\bar{\phi}J$ in the $\bar{\phi} \times J$ OPE as a function of Δ_ϕ , demanding $c_{\bar{\phi}J\phi} = c_{\bar{\phi}\phi J}$. In this plot we use $\Lambda = 17$.

As we have already seen the second scalar in the $\bar{\phi} \times J$ OPE has dimension $\Delta_\phi + 2$. We will call it $\bar{\phi}J$. We can obtain a bound on its OPE coefficient, again imposing $c_{\bar{\phi}J\phi} = c_{\bar{\phi}\phi J}$. The bound is seen in figure 12, and is strongest close to $\Delta_\phi = 1$ where it approaches the expected value of $c_{\bar{\phi}J(\phi J)} = 1$.

8 Discussion

This work is the first numerical bootstrap study of mixed correlator systems in SCFTs with four supercharges. In this paper we focused on 4D $\mathcal{N} = 1$ SCFTs and used the crossing symmetry and positivity in the $\{\langle \bar{\phi}\phi\bar{\phi}\phi \rangle, \langle \bar{\phi}R\phi R \rangle, \langle RRRR \rangle\}$ system, where R is a generic real scalar and ϕ is a chiral scalar. We also studied the special case with $R \rightarrow J$, where J is the superconformal primary in a linear multiplet that contains a conserved global symmetry current. In all these cases we computed all necessary superconformal blocks, obtaining some new results.

We found new rigorous bounds on 4D $\mathcal{N} = 1$ SCFTs that are stronger than those previously obtained. The features of our results strongly suggest the existence of a minimal 4D $\mathcal{N} = 1$ SCFT with a chiral operator of dimension $\Delta_\phi \approx 1.4$. Nevertheless, further studies are needed in this system of crossing relations. In particular, we did not find an isolated island of viable solutions to the crossing equations similar to that obtained in [5, 6]. We believe that in order to address this more definitively we need to overcome the current practical limits on the dimension of the functional search space we can use with the available computational resources. When that becomes possible, we expect certain dimension bounds to become much more constraining. However, this will likely require a new level of both algorithmic efficiency and computational power. We expect to return to this system when such resource becomes available.

Acknowledgments

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A Polynomial approximations

In this work we consider crossing relations for four-point functions involving operators with different scaling dimensions Δ_1 and Δ_2 , e.g.

$$\sum_{\mathcal{O}} |c|^2 \mathcal{F}_{\Delta, \ell, \Delta_1, \Delta_2}(u, v) = 0, \tag{A.1}$$

where $\mathcal{F}_{\Delta, \ell, \Delta_1, \Delta_2}(u, v) = u^{-(\Delta_1 + \Delta_2)/2} \mathcal{G}_{\Delta, \ell, \Delta_1, \Delta_2}(u, v) - (u \leftrightarrow v)$, with \mathcal{G} a superconformal block. The superconformal block contains ordinary conformal blocks defined in (2.3). In order to use semidefinite programming techniques we have to approximate derivatives on \mathcal{F} and \mathcal{F}' as positive functions times polynomials [8]. Here we explain how we do this for expressions like (A.1), assuming first that \mathcal{F} contains a single conformal block. To signify this we will use F instead of \mathcal{F} .⁸

From (2.3) and using $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ we have

$$(z - \bar{z})F_{\Delta, \ell, \Delta_1, \Delta_2}(z, \bar{z}) = (-1)^\ell (u_{\Delta+\ell}^{\beta, \gamma, \delta}(z) u_{\Delta-\ell-2}^{\beta, \gamma, \delta}(\bar{z}) + u_{\Delta+\ell}^{\beta, \gamma, \delta}(1-z) u_{\Delta-\ell-2}^{\beta, \gamma, \delta}(1-\bar{z})) - (z \leftrightarrow \bar{z}), \tag{A.2}$$

where β and γ can here be either $\Delta_1 - \Delta_2$ or $\Delta_2 - \Delta_1$ depending on the four-point function we are considering, $\delta = \frac{1}{2}(\Delta_1 + \Delta_2)$, and

$$u_{\alpha}^{\beta, \gamma, \delta}(z) = z^{1-\delta} k_{\alpha}^{\beta, \gamma}(z). \tag{A.3}$$

The constants $\alpha, \beta, \gamma, \delta$ have specific relations to $\Delta, \ell, \Delta_1, \Delta_2$ when appearing in (A.2), but below we will keep them general. As we see the crossing relation (A.1) takes a convenient form in terms of the function $u_{\alpha}^{\beta, \gamma, \delta}(z)$. For our bootstrap analysis we now need to compute derivatives of $u_{\alpha}^{\beta, \gamma, \delta}$ with respect to z or \bar{z} , and evaluate them at $z = \bar{z} = \frac{1}{2}$. An easy way to do this is to use a power series expansion. Indeed, the function $u_{\alpha}^{\beta, \gamma, \delta}(z)$ can be expanded as

$$u_{\alpha}^{\beta, \gamma, \delta}(z) = \sum_{n=0}^{\infty} C_{\alpha, \beta, \gamma, \delta}^n \frac{1}{n!} \left(z - \frac{1}{2}\right)^n, \tag{A.4}$$

⁸Polynomial approximations of conformal blocks corresponding to four-point functions involving operators with different scaling dimensions were recently considered in [30].

with

$$C_{\alpha,\beta,\gamma,\delta}^n = 2^{n-\frac{1}{2}\alpha+\delta-1} \frac{\Gamma(\frac{1}{2}(\alpha-2\delta+4))}{\Gamma(\frac{1}{2}(\alpha-2\delta+4-2n))} \times {}_3F_2\left(\frac{1}{2}(\alpha-\beta), \frac{1}{2}(\alpha+\gamma), \frac{1}{2}\alpha-\delta+2; \alpha, \frac{1}{2}\alpha-\delta+2-n; \frac{1}{2}\right). \quad (\text{A.5})$$

$C_{\alpha,\beta,\gamma,\delta}^n$ as given in (A.5) is nonpolynomial and thus not appropriate for our analysis. Hence, we take an alternate route here, based on that suggested in [7]. Using the hypergeometric differential equation it is easy to verify that $u_{\alpha}^{\beta,\gamma,\delta}$ satisfies the differential equation

$$\left(z^2(1-z) \frac{d^2}{dz^2} + \frac{1}{2}z((\beta-\gamma-4\delta+2)z+4(\delta-1)) \frac{d}{dz} + \frac{1}{4}((\beta-2\delta+2)(\gamma+2\delta-2)z - (\alpha-2\delta+2)(\alpha+2\delta-4)) \right) u_{\alpha}^{\beta,\gamma,\delta}(z) = 0. \quad (\text{A.6})$$

If we use (A.4), then taking $n-2$ derivatives on (A.6) and evaluating at $z = \frac{1}{2}$ we find the recursion relation

$$C_{\alpha,\beta,\gamma,\delta}^n = -(2n+\beta-\gamma+4\delta-10)C_{\alpha,\beta,\gamma,\delta}^{n-1} + (4n(n-\beta+\gamma-3) + 2\alpha(\alpha-2) - \beta(\gamma+2\delta-10) + \gamma(2\delta-10) - 4\delta(\delta-4) - 4)C_{\alpha,\beta,\gamma,\delta}^{n-2} + 2(n-2)(2n-\beta+2\delta-8)(2n+\gamma+2\delta-8)C_{\alpha,\beta,\gamma,\delta}^{n-3}. \quad (\text{A.7})$$

This allows us to write

$$C_{\alpha,\beta,\gamma,\delta}^n = P_n(\alpha, \beta, \gamma, \delta) 2^{\delta-1} k_{\alpha}^{\beta,\gamma} \left(\frac{1}{2}\right) + Q_n(\alpha, \beta, \gamma, \delta) 2^{\delta-1} (k_{\alpha}^{\beta,\gamma})' \left(\frac{1}{2}\right), \quad (\text{A.8})$$

where $(k_{\alpha}^{\beta,\gamma})'$ is the z -derivative of $k_{\alpha}^{\beta,\gamma}$ and the polynomials P and Q can be determined from (A.7).

In order to be able to use semidefinite programming we need to further express appropriately the right-hand side of (A.8), for it still involves the nonpolynomial quantities $k_{\alpha}^{\beta,\gamma}$ and $(k_{\alpha}^{\beta,\gamma})'$ evaluated at $\frac{1}{2}$. To proceed, we perform a series expansion around $z = 0$ of $k_{\alpha}^{\beta,\gamma}(\rho)$ and $(k_{\alpha}^{\beta,\gamma})'(\rho)$, where we use the coordinate $\rho = z/(1 + \sqrt{1-z})^2$ [31]. The expansion in ρ converges faster than that in z . We perform this expansion to a fixed order w for $k_{\alpha}^{\beta,\gamma}$ and $w-1$ for $(k_{\alpha}^{\beta,\gamma})'$, so that both expressions have the same poles in α , and then we substitute $\rho = \rho(\frac{1}{2}) = 3 - 2\sqrt{2}$. Then, in the right-hand side of (A.8) we can pull out a positive factor equal to $(2^{-\frac{1}{2}\alpha} \alpha D_w(\alpha))^{-1}$,⁹ where $D_w(\alpha)$ is the denominator of the power series expansion of $k_{\alpha}^{\beta,\gamma}$ evaluated at $\rho(\frac{1}{2})$. Doing so we can bring (A.8) to the form

$$C_{\alpha,\beta,\gamma,\delta}^n \rightarrow C_{\alpha,\beta,\gamma,\delta,w}^n \approx 2^{\delta+\frac{1}{2}\alpha-1} \frac{1}{\alpha D_w(\alpha)} R_{n,w}(\alpha, \beta, \gamma, \delta), \quad \alpha D_w(\alpha) > 0 \text{ for } \alpha > -1, \quad (\text{A.9})$$

⁹Since α is here $\Delta + \ell$ or $\Delta - \ell - 2$ we may have $\alpha = -1$, in which case $\alpha D(\alpha) = 0$. This corresponds to the case where the exchanged operator is a free scalar.

where $R_{n,w}(\alpha, \beta, \gamma, \delta)$ is polynomial in its arguments, given by

$$R_{n,w}(\alpha, \beta, \gamma, \delta) = N_{1,w}(\alpha, \beta, \gamma) P_n(\alpha, \beta, \gamma, \delta) + N_{2,w}(\alpha, \beta, \gamma) Q_n(\alpha, \beta, \gamma, \delta), \quad (\text{A.10})$$

where $N_{1,w}$ is $2^{-\frac{1}{2}\alpha}$ times the numerator of the power series expansion of $k_\alpha^{\beta,\gamma}$ evaluated at $\rho(\frac{1}{2})$, and $N_{2,w}$ is the power series expansion of $(k_\alpha^{\beta,\gamma})'$ multiplied with $2^{-\frac{1}{2}\alpha} \alpha D_w(\alpha)$. The approximation to $C_{\alpha,\beta,\gamma,\delta}^n$ in (A.9) becomes better as we increase the order w of the power series expansion of (A.8).¹⁰ For the remainder of this appendix we will ignore the label w .

Using (A.2), (A.4) and (A.9), derivatives of $(z - \bar{z})F_{\Delta,\ell,\Delta_1,\Delta_2}(z, \bar{z})$ evaluated at $z = \bar{z} = \frac{1}{2}$ can now be written as

$$\partial_z^m \partial_{\bar{z}}^n \left((z - \bar{z}) F_{\Delta,\ell,\Delta_1,\Delta_2}(z, \bar{z}) \right) \Big|_{z=\bar{z}=\frac{1}{2}} \approx \chi(\Delta, \ell, \delta) U_{m,n}(\Delta, \ell, \beta, \gamma, \delta), \quad (\text{A.11})$$

where

$$\chi(\Delta, \ell, \delta) = \frac{2^{2(\delta-1)+\Delta}}{(\Delta + \ell)(\Delta - \ell - 2)D(\Delta + \ell)D(\Delta - \ell - 2)} \quad (\text{A.12})$$

is positive in unitary theories, and

$$U_{m,n}(\Delta, \ell, \beta, \gamma, \delta) = \frac{1}{2} (1 + (-1)^{m+n}) (-1)^\ell (R_m(\Delta + \ell, \beta, \gamma, \delta) R_n(\Delta - \ell - 2, \beta, \gamma, \delta) - (m \leftrightarrow n)) \quad (\text{A.13})$$

is a polynomial in $\Delta, \ell, \beta, \gamma, \delta$. In the case of H instead of F we find an expression similar to (A.11) but instead of the overall factor of $1 + (-1)^{m+n}$ in (A.13) we have the factor $1 - (-1)^{m+n}$.

Finally, let us consider derivatives of the function $\mathcal{F}_{\Delta,\ell,\Delta_1,\Delta_2}(z, \bar{z})$ at $z = \bar{z} = \frac{1}{2}$. Here we will focus on $\bar{\mathcal{F}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(z, \bar{z})$ of (4.10), but other \mathcal{F} 's can be treated similarly. We can again multiply with $z - \bar{z}$ as in (A.2), and then it is straightforward to obtain

$$\begin{aligned} & \partial_z^m \partial_{\bar{z}}^n \left((z - \bar{z}) \bar{\mathcal{F}}_{\Delta,\ell,\Delta_\phi-\Delta_R}^{\bar{\phi}R;\phi R}(z, \bar{z}) \right) \Big|_{z=\bar{z}=\frac{1}{2}} \approx \chi(\Delta, \ell, \delta) \\ & \times \left[U_{m,n}(\Delta, \ell, \beta, \gamma, \delta) \right. \\ & + 4\rho \left(\frac{1}{2} \right) \bar{c}_1 \frac{(\Delta + \ell) \tilde{D}(\Delta + \ell)}{(\Delta + \ell + 2) \tilde{D}(\Delta + \ell + 2)} U_{m,n}(\Delta + 1, \ell + 1, \beta, \gamma, \delta) \\ & + 4\rho \left(\frac{1}{2} \right) \bar{c}_2 \frac{(\Delta - \ell - 2) \tilde{D}(\Delta - \ell - 2)}{(\Delta - \ell) \tilde{D}(\Delta - \ell)} U_{m,n}(\Delta + 1, \ell - 1, \beta, \gamma, \delta) \\ & \left. + 16\rho^2 \left(\frac{1}{2} \right) \bar{c}_1 \bar{c}_2 \frac{(\Delta + \ell)(\Delta - \ell - 2) \tilde{D}(\Delta + \ell) \tilde{D}(\Delta - \ell - 2)}{(\Delta + \ell + 2)(\Delta - \ell) \tilde{D}(\Delta + \ell + 2) \tilde{D}(\Delta - \ell)} U_{m,n}(\Delta + 2, \ell, \beta, \gamma, \delta) \right], \quad (\text{A.14}) \end{aligned}$$

where $\beta = \gamma = \Delta_\phi - \Delta_R$, $\delta = \frac{1}{2}(\Delta_\phi + \Delta_R)$, \bar{c}_1 and \bar{c}_2 are given by (2.20), and $\tilde{D}(\alpha) = (2\rho(\frac{1}{2}))^{\frac{1}{2}\alpha} D(\alpha)$ is polynomial in α . Now, since $\tilde{D}(\alpha)$ is a polynomial of degree w of the

¹⁰In this work we have typically used w around 20.

form $\alpha(\alpha + 1) \cdots (\alpha + w - 1)$, it is

$$\frac{\alpha \tilde{D}(\alpha)}{(\alpha + 2) \tilde{D}(\alpha + 2)} = \frac{\alpha^2(\alpha + 1)}{(\alpha + 2)(\alpha + w)(\alpha + w + 1)}. \quad (\text{A.15})$$

As a result, (A.14) can be written as

$$\begin{aligned} \partial_z^m \partial_{\bar{z}}^n \left((z - \bar{z}) \bar{\mathcal{F}}_{\Delta, \ell, \Delta_\phi - \Delta_R}^{\bar{\phi}R; \phi R}(z, \bar{z}) \right) \Big|_{z=\bar{z}=\frac{1}{2}} &\approx \frac{\chi(\Delta, \ell, \delta)}{f(\Delta + \ell) f(\Delta - \ell - 2)} \\ &\times \left[f(\Delta + \ell) f(\Delta - \ell - 2) U_{m,n}(\Delta, \ell, \beta, \gamma, \delta) \right. \\ &\quad + \bar{c}_1 g(\Delta + \ell) f(\Delta - \ell - 2) U_{m,n}(\Delta + 1, \ell + 1, \beta, \gamma, \delta) \\ &\quad + \bar{c}_2 f(\Delta + \ell) g(\Delta - \ell - 2) U_{m,n}(\Delta + 1, \ell - 1, \beta, \gamma, \delta) \\ &\quad \left. + \bar{c}_1 \bar{c}_2 g(\Delta + \ell) g(\Delta - \ell - 2) U_{m,n}(\Delta + 2, \ell, \beta, \gamma, \delta) \right], \end{aligned} \quad (\text{A.16})$$

where

$$f(\alpha) = (\alpha + 2)(\alpha + w)(\alpha + w + 1)(\alpha + \Delta_\phi), \quad g(\alpha) = 4\rho \left(\frac{1}{2} \right) \alpha^2(\alpha + 1)(\alpha + \Delta_\phi). \quad (\text{A.17})$$

The quantity $\chi(\Delta, \ell, \delta)/f(\Delta + \ell) f(\Delta - \ell - 2)$ is positive in unitary theories since $w > 1$. Furthermore, the factors in the denominators of \bar{c}_1 and \bar{c}_2 are also contained in the corresponding g that multiplies them in (A.16). Therefore, the right-hand side of (A.16) is of the form of a positive quantity times a polynomial and so it can be used in our bootstrap analysis.

B On the derivation of superconformal blocks

In this appendix we briefly describe the method we used to compute the superconformal blocks of section 2. Despite significant developments on $\mathcal{N} = 1$ superconformal blocks [7, 13–16, 25], blocks that arise from superdescendants whose corresponding primaries do not contribute have not been treated systematically. An example has been worked out in [25], while, in the case of interest for this paper, namely regarding the $\bar{\phi} \times R$ OPE, an example is the superconformal primary $\mathcal{O}_{\dot{\alpha}}$, which cannot appear because it does not have integer spin, but whose descendants $\bar{Q}^{\dot{\alpha}} \mathcal{O}_{\dot{\alpha}}$ and (the primary component of) $\bar{Q}^2 Q_{\dot{\alpha}} \mathcal{O}_{\dot{\alpha}}$ may both appear and form a superconformal block.

As mentioned in section 2, there are two types of such operators for the four-point function we are interested in. The first has $\bar{j} = j + 1$, that is, it has one more dotted than undotted index. The superconformal primary $\mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ has zero three-point function with two scalars because it does not have integer spin. The superdescendant $Q^\alpha \bar{\mathcal{O}}_{\alpha \alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ has spin ℓ and the primary component of the superdescendant $Q^2 \bar{Q}_{(\dot{\alpha}} \bar{\mathcal{O}}_{\alpha \alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell)}$ has spin $\ell + 1$. These two superdescendants have nonzero three-point function with $\bar{\phi}$ and R if the weights of the associated superconformal primary $\mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ satisfy $q = \frac{1}{2}(\Delta + \Delta_\phi - \frac{3}{2})$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi + \frac{3}{2})$.

There is a second class of operators $\mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_2 \dots \dot{\alpha}_\ell}$, $\ell \geq 1$, that has one more undotted index. When $q = \frac{1}{2}(\Delta + \Delta_\phi - \frac{3}{2})$ and $\bar{q} = \frac{1}{2}(\Delta - \Delta_\phi + \frac{3}{2})$, the superdescendant $Q_{(\alpha_1} \bar{\mathcal{O}}_{\alpha_2 \dots \alpha_\ell); \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ and the primary component of $Q^2 \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_2 \dots \alpha_\ell; \dot{\alpha}_2 \dots \dot{\alpha}_\ell}$ have nontrivial three-point functions with $\bar{\phi}$ and R .

In this appendix we summarize the calculation of such superconformal blocks in four-dimensional $\mathcal{N} = 1$ SCFTs. We focus on the contribution of an exchanged superconformal multiplet in the $\bar{\phi} \times R$ channel of the four-point function $\langle \bar{\phi} R \phi R \rangle$. In $d \geq 3$ dimensions, a superconformal multiplet includes a finite number of conformal multiplets. Therefore, the superconformal block is a linear combination of conformal blocks with coefficients fixed by supersymmetry. For each conformal primary component \mathcal{O} of the supermultiplet, this coefficient is given by $c_{\bar{\phi}R\mathcal{O}}c_{\phi R\bar{\mathcal{O}}}/c_{\bar{\mathcal{O}}\mathcal{O}}$, where $c_{\bar{\phi}R\mathcal{O}}$ and $c_{\phi R\bar{\mathcal{O}}}$ are the three-point function coefficients and $c_{\bar{\mathcal{O}}\mathcal{O}}$ is the two-point function coefficient. The construction of primary components and their two-point function coefficients $c_{\bar{\mathcal{O}}\mathcal{O}}$ for any 4D $\mathcal{N} = 1$ superconformal multiplet has been worked out in [24]. The form of the superfield three-point function was originally worked out in [22, 23], and reproduced for the cases of interest here in (2.14), (2.24) and (2.28). Using the *Mathematica* package developed in [24], we expand these three-point functions in θ and $\bar{\theta}$. Using the explicit construction of the superfield at each $\theta, \bar{\theta}$ order worked out in [24], we match the result of the expansion of the superfield three-point functions to the expected form of conformal three-point functions and solve for the three-point function coefficients $c_{\bar{\phi}R\mathcal{O}}$.

As an illustration, we elaborate more on this calculation for the first class of operators mentioned above. Expanding (2.14) with (2.24) to first order in $\bar{\theta}_3$, we have

$$\langle \bar{\Phi}(z_1) \mathcal{R}(z_2) \mathcal{O}(z_3, \eta, \bar{\eta}) \rangle_{\bar{\theta}_3} = -i \frac{1}{r_{13}^{2\Delta_\phi} r_{23}^{2\Delta_R}} (Z_3^2)^{\frac{1}{2}(\Delta - \ell + \frac{1}{2} - \Delta_\phi - \Delta_R)} (\eta Z_3 \bar{\eta})^\ell \bar{\eta} \bar{\theta}_3, \quad (\text{B.1})$$

where $r_{ij} = (x_{ij}^2)^{\frac{1}{2}}$, $Z_3^\mu = -x_{13}^\mu/x_{13}^2 + x_{23}^\mu/x_{23}^2$, $Z_{3\alpha\dot{\alpha}} = Z_{3\mu} \sigma_{\alpha\dot{\alpha}}^\mu$, $Z_3^2 = x_{12}^2/x_{13}^2 x_{23}^2$, and we have used bosonic auxiliary spinors η and $\bar{\eta}$ to saturate all free spinor indices on \mathcal{O} :

$$\mathcal{O}(z, \eta, \bar{\eta}) \equiv \frac{1}{(\ell!)^2} \eta^{\alpha_1} \dots \eta^{\alpha_\ell} \bar{\eta}^{\dot{\alpha}_1} \dots \bar{\eta}^{\dot{\alpha}_\ell} \mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}. \quad (\text{B.2})$$

Note that the x -dependence on the right-hand side of (B.1) has exactly the form of a three-point function of conformal primaries. It corresponds to the contribution from $\bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$ in the three-point function. Using the superfield structure worked out in [24],

$$e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\ell, \ell+1} |_{\bar{\theta}} = i \bar{\theta} \bar{Q} \mathcal{O}_{\ell, \ell+1} = i \bar{\theta} \partial_{\bar{\eta}} (\bar{Q} \mathcal{O})_{\ell, \ell+2} + i \frac{\ell+1}{\ell+2} \bar{\theta} \bar{\eta} (\bar{Q} \mathcal{O})_{\ell, \ell}. \quad (\text{B.3})$$

Here $(\bar{Q} \mathcal{O})_{\ell, \ell+2}$ and $(\bar{Q} \mathcal{O})_{\ell, \ell}$ are the two conformal primaries obtained from symmetrizing or antisymmetrizing the index of \bar{Q} with the dotted indices of the superconformal primary \mathcal{O} . Only the later can appear in the three-point function with scalars because it has integer spin. Plugging (B.3) into the left-hand side of (B.1) we find that the three-point function coefficient of $\langle \bar{\phi} R (\bar{Q} \mathcal{O})_{\ell, \ell} \rangle$ is

$$c_{\bar{\phi}R(\bar{Q}\mathcal{O})} = \frac{\ell+2}{\ell+1}. \quad (\text{B.4})$$

To get the three-point function coefficient for the $\bar{Q}^2 Q \mathcal{O}$ descendant, we first work out the θ -expansion of the superfield three-point function. The result is

$$\begin{aligned} \langle \bar{\Phi}(z_1) \mathcal{R}(z_2) \mathcal{O}_I(z_3, \eta, \bar{\eta}) \rangle_{\theta_3 \bar{\theta}_3^2} &= \frac{1}{r_{13}^{2\Delta_\phi} r_{23}^{2\Delta_R}} (Z_3^2)^{\frac{1}{2}(\Delta - \ell + \frac{1}{2} - \Delta_\phi - \Delta_R)} (\eta Z_3 \bar{\eta})^\ell \\ &\bar{\theta}_3^2 \left((\Delta_\phi - \ell - 2) \frac{1}{x_{13}^2} \theta_3 x_{13} \bar{\eta} \right. \\ &\quad \left. - \frac{1}{4} (2(\Delta + 3\ell - \Delta_\phi - \Delta_R) + 9) \theta_3 Z_3 \bar{\eta} \right). \end{aligned} \quad (\text{B.5})$$

This does not take the form of a three-point function involving conformal primaries. This is expected since at this order in θ and $\bar{\theta}$ the three-point function also contains contributions from conformal descendants. In particular, following notation of [24], we have

$$\begin{aligned} e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\ell, \ell+1} |_{\theta \bar{\theta}^2} &= -\frac{i}{4} \bar{\theta}^2 \theta \partial_\eta \left((\bar{Q}^2 Q \mathcal{O})_{\ell+1, \ell+1; p}^\eta + 2i\bar{c}_5 \partial_{\bar{\eta}} \partial_x \eta (\bar{Q} \mathcal{O})_{\ell, \ell+2; p} \right. \\ &\quad \left. - 2i\bar{c}_6 \frac{\ell+1}{\ell+2} \eta \partial_x \bar{\eta} (\bar{Q} \mathcal{O})_{\ell, \ell; p} \right) \\ &+ \frac{i}{4} \frac{\ell+1}{\ell+2} \bar{\theta}^2 \theta \eta \left((\bar{Q}^2 Q \mathcal{O})_{\ell-1, \ell+1; p} + 2i\bar{c}_7 \partial_\eta \partial_x \partial_{\bar{\eta}} (\bar{Q} \mathcal{O})_{\ell, \ell+2; p} \right. \\ &\quad \left. - 2i\bar{c}_8 \frac{\ell+1}{\ell+2} \bar{\eta} \partial_x \partial_\eta (\bar{Q} \mathcal{O})_{\ell, \ell; p} \right), \end{aligned} \quad (\text{B.6})$$

and we see that two different descendants have integer spins and can contribute to the three-point function with $\bar{\phi}$ and R . The relevant coefficients can be obtained from [24]:

$$\bar{c}_6 = -2 \frac{\Delta_\phi - 2}{(\ell+1)(2(\Delta+\ell)+1)}, \quad \bar{c}_8 = -2 \frac{\Delta_\phi - \ell - 3}{\ell(2\Delta-1)}. \quad (\text{B.7})$$

Removing these contributions from the superfield correlator we indeed get a conformal primary three-point function with coefficient

$$c_{\bar{\phi} R (\bar{Q}^2 Q \mathcal{O})_p} = i \frac{(2(\Delta - \Delta_\phi + \ell) + 5)(2(\Delta + \Delta_\phi - \Delta_R + \ell) + 1)}{(\ell+1)(2(\Delta+\ell)+1)}. \quad (\text{B.8})$$

Finally, using the two-point function coefficient derived in [24], we get the results (2.25) and (2.26). For the second class of operators we carried out a similar procedure and obtained (2.29) and (2.30).

Although we will not present the details here, this calculation is easily generalized to cases where the operator R is not real and carries an R-charge. The relevant results can be found in (2.27) and (2.31). More generally, for other scalar $\mathcal{N} = 1$ superconformal four-point functions, there may be intermediate operators of this type that do not correspond to (2.24) or (2.28). We have not calculated such superconformal blocks, but our method should apply straightforwardly to such cases. Indeed, this method is a feasible way of computing any $\mathcal{N} = 1$ scalar superconformal block in a case by case basis.

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References

- [1] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [[arXiv:0807.0004](https://arxiv.org/abs/0807.0004)] [[INSPIRE](#)].
- [2] S. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys. Rev. D* **86** (2012) 025022 [[arXiv:1203.6064](https://arxiv.org/abs/1203.6064)] [[INSPIRE](#)].
- [3] F. Kos, D. Poland and D. Simmons-Duffin, *Bootstrapping the $O(N)$ vector models*, *JHEP* **06** (2014) 091 [[arXiv:1307.6856](https://arxiv.org/abs/1307.6856)] [[INSPIRE](#)].
- [4] S. El-Showk and M.F. Paulos, *Extremal bootstrapping: go with the flow*, [arXiv:1605.08087](https://arxiv.org/abs/1605.08087) [[INSPIRE](#)].
- [5] F. Kos, D. Poland and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, *JHEP* **11** (2014) 109 [[arXiv:1406.4858](https://arxiv.org/abs/1406.4858)] [[INSPIRE](#)].
- [6] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *Bootstrapping the $O(N)$ Archipelago*, *JHEP* **11** (2015) 106 [[arXiv:1504.07997](https://arxiv.org/abs/1504.07997)] [[INSPIRE](#)].
- [7] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, *JHEP* **05** (2011) 017 [[arXiv:1009.2087](https://arxiv.org/abs/1009.2087)] [[INSPIRE](#)].
- [8] D. Poland, D. Simmons-Duffin and A. Vichi, *Carving Out the Space of 4D CFTs*, *JHEP* **05** (2012) 110 [[arXiv:1109.5176](https://arxiv.org/abs/1109.5176)] [[INSPIRE](#)].
- [9] D. Poland and A. Stergiou, *Exploring the Minimal 4D $\mathcal{N} = 1$ SCFT*, *JHEP* **12** (2015) 121 [[arXiv:1509.06368](https://arxiv.org/abs/1509.06368)] [[INSPIRE](#)].
- [10] N. Bobev, S. El-Showk, D. Mazac and M.F. Paulos, *Bootstrapping SCFTs with Four Supercharges*, *JHEP* **08** (2015) 142 [[arXiv:1503.02081](https://arxiv.org/abs/1503.02081)] [[INSPIRE](#)].
- [11] D. Xie and K. Yonekura, *Search for a Minimal $\mathcal{N} = 1$ Superconformal Field Theory in 4D*, *Phys. Rev. Lett.* **117** (2016) 011604 [[arXiv:1602.04817](https://arxiv.org/abs/1602.04817)] [[INSPIRE](#)].
- [12] M. Buican and T. Nishinaka, *Small deformation of a simple $\mathcal{N} = 2$ superconformal theory*, *Phys. Rev. D* **94** (2016) 125002 [[arXiv:1602.05545](https://arxiv.org/abs/1602.05545)] [[INSPIRE](#)].
- [13] M. Berkooz, R. Yacoby and A. Zait, *Bounds on $\mathcal{N} = 1$ superconformal theories with global symmetries*, *JHEP* **08** (2014) 008 [*Erratum ibid.* **01** (2015) 132] [[arXiv:1402.6068](https://arxiv.org/abs/1402.6068)] [[INSPIRE](#)].
- [14] Z.U. Khandker, D. Li, D. Poland and D. Simmons-Duffin, *$\mathcal{N} = 1$ superconformal blocks for general scalar operators*, *JHEP* **08** (2014) 049 [[arXiv:1404.5300](https://arxiv.org/abs/1404.5300)] [[INSPIRE](#)].
- [15] A.L. Fitzpatrick, J. Kaplan, Z.U. Khandker, D. Li, D. Poland and D. Simmons-Duffin, *Covariant Approaches to Superconformal Blocks*, *JHEP* **08** (2014) 129 [[arXiv:1402.1167](https://arxiv.org/abs/1402.1167)] [[INSPIRE](#)].
- [16] Z. Li and N. Su, *The Most General 4D $\mathcal{N} = 1$ Superconformal Blocks for Scalar Operators*, *JHEP* **05** (2016) 163 [[arXiv:1602.07097](https://arxiv.org/abs/1602.07097)] [[INSPIRE](#)].

- [17] A. Vichi, *Improved bounds for CFT's with global symmetries*, *JHEP* **01** (2012) 162 [[arXiv:1106.4037](#)] [[INSPIRE](#)].
- [18] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys. B* **599** (2001) 459 [[hep-th/0011040](#)] [[INSPIRE](#)].
- [19] F.A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, *Nucl. Phys. B* **678** (2004) 491 [[hep-th/0309180](#)] [[INSPIRE](#)].
- [20] M. Flato and C. Fronsdal, *Representations of Conformal Supersymmetry*, *Lett. Math. Phys.* **8** (1984) 159 [[INSPIRE](#)].
- [21] V.K. Dobrev and V.B. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys. Lett. B* **162** (1985) 127 [[INSPIRE](#)].
- [22] J.-H. Park, *$N = 1$ superconformal symmetry in four-dimensions*, *Int. J. Mod. Phys. A* **13** (1998) 1743 [[hep-th/9703191](#)] [[INSPIRE](#)].
- [23] H. Osborn, *$\mathcal{N} = 1$ superconformal symmetry in four-dimensional quantum field theory*, *Annals Phys.* **272** (1999) 243 [[hep-th/9808041](#)] [[INSPIRE](#)].
- [24] D. Li and A. Stergiou, *Two-point functions of conformal primary operators in $\mathcal{N} = 1$ superconformal theories*, *JHEP* **10** (2014) 37 [[arXiv:1407.6354](#)] [[INSPIRE](#)].
- [25] J.-F. Fortin, K. Intriligator and A. Stergiou, *Current OPEs in Superconformal Theories*, *JHEP* **09** (2011) 071 [[arXiv:1107.1721](#)] [[INSPIRE](#)].
- [26] D. Simmons-Duffin, *A Semidefinite Program Solver for the Conformal Bootstrap*, *JHEP* **06** (2015) 174 [[arXiv:1502.02033](#)] [[INSPIRE](#)].
- [27] P.C. Argyres and M.R. Douglas, *New phenomena in $SU(3)$ supersymmetric gauge theory*, *Nucl. Phys. B* **448** (1995) 93 [[hep-th/9505062](#)] [[INSPIRE](#)].
- [28] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **461** (1996) 71 [[hep-th/9511154](#)] [[INSPIRE](#)].
- [29] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Study of $N = 2$ superconformal field theories in four-dimensions*, *Nucl. Phys. B* **471** (1996) 430 [[hep-th/9603002](#)] [[INSPIRE](#)].
- [30] M.S. Costa, T. Hansen, J. Penedones and E. Trevisani, *Radial expansion for spinning conformal blocks*, *JHEP* **07** (2016) 057 [[arXiv:1603.05552](#)] [[INSPIRE](#)].
- [31] M. Hogervorst and S. Rychkov, *Radial Coordinates for Conformal Blocks*, *Phys. Rev. D* **87** (2013) 106004 [[arXiv:1303.1111](#)] [[INSPIRE](#)].