

Exact WKB analysis of $\mathcal{N} = 2$ gauge theories

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ABSTRACT: We study $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(2)$ coupled to fundamental flavours, covering all asymptotically free and conformal cases. We re-derive, from the conformal field theory perspective, the differential equations satisfied by ϵ_1 - and ϵ_2 -deformed instanton partition functions. We confirm their validity at leading order in ϵ_2 via a saddle-point analysis of the partition function. In the semi-classical limit we show that these differential equations take a form amenable to exact WKB analysis. We compute the monodromy group associated to the differential equations in terms of ϵ_1 -deformed and Borel resummed Seiberg-Witten data. For each case, we study pairs of Stokes graphs that are related by flips and pops, and show that the monodromy groups allow one to confirm the Stokes automorphisms that arise as the phase of ϵ_1 is varied. Finally, we relate the Borel resummed monodromies with the traditional Seiberg-Witten variables in the semi-classical limit.

KEYWORDS: Extended Supersymmetry, Solitons Monopoles and Instantons, Supersymmetric gauge theory

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Contents

1	Introduction	1
2	The conformal field theory perspective	3
2.1	The five-point block	3
2.2	The null vector decoupling equations for irregular blocks	5
2.3	The semi-classical limit	6
2.4	The semi-classical irregular blocks	7
3	The exact WKB analysis of differential equations	8
3.1	The exact WKB method	8
3.2	The applicability of the exact WKB analysis	15
3.3	Theorems on Stokes automorphisms	15
4	The monodromy group	16
4.1	Pure super Yang-Mills	17
4.2	One flavour	19
4.3	Two flavours	21
4.4	Three flavours	23
4.5	The conformal theory	25
5	The gauge theory perspective	32
5.1	The Seiberg-Witten variables	32
5.2	The non-perturbative prepotential	34
6	Conclusions	35
A	The null vector decoupling with irregular blocks	37
A.1	One irregular puncture	37
A.1.1	$N_f = 3$	37
A.1.2	$N_f = 2$: asymmetric realization	38
A.2	Two irregular punctures	38
A.2.1	$N_f = 2$: symmetric realization	39
A.2.2	$N_f = 1$	39
A.2.3	$N_f = 0$	39
B	Stokes graphs	40
B.1	The double flip	40
B.2	The pop	40

C	The saddle point analysis	40
C.1	$N_f = 4$	43
C.2	$N_f = 3$	43
C.3	$N_f = 2$: asymmetric realization	44
C.4	$N_f = 2$: symmetric realization	45
C.5	$N_f = 1$	46
C.6	$N_f = 0$	46

1 Introduction

For some time now, we have been able to compute the low-energy effective action of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. In [1, 2], the solution for the low-energy theory was given in terms of an algebraic curve and an associated differential. Subsequent works have simplified and clarified many aspects of the Seiberg-Witten solution. The Seiberg-Witten curves may be intuitively pictured in terms of M-theory five-branes [3], and this geometric picture has inspired a description of class \mathcal{S} theories in terms of punctured Riemann surfaces [4]. In a parallel development, it has also become possible to compute instanton contributions by invoking the powerful machinery of equivariant localization [5]. Of particular note, the calculation of the gauge theory partition function on S^4 via localization naturally incorporates these instanton sums [6]. All these developments were key to writing a dictionary between observables in four-dimensional gauge theories and those in two-dimensional conformal field theories: the 2d/4d correspondence [7].

The 2d/4d correspondence makes it possible to use the technology of conformal field theory to gain deeper insights into the behavior of $\mathcal{N} = 2$ gauge theories. For instance, the Ω -deformed gauge theory partition function with a surface operator insertion maps to the meromorphic solution of a null vector decoupling equation [8–11]. Thus, an analysis of conformal blocks in two-dimensional conformal field theory yields information about surface operators in gauge theories. These conformal blocks can be viewed as solutions to Riemann-Hilbert problems specified by a differential equation with singularities and associated monodromies [12]. We expect this exact picture to be valid in gauge theory (and the field theory limit of topological string theory) [13].

In this paper, we study quantum chromodynamics with $\mathcal{N} = 2$ supersymmetry and gauge group $SU(2)$, and the corresponding Virasoro conformal blocks. In particular, we study the differential equation that the instanton partition function with surface operator insertion satisfies. This corresponds to an analysis of null vector decoupling equations in the presence of irregular blocks. The differential equations satisfied by correlators involving irregular blocks were described in [10, 11, 13–15]. The equations are exact in the Ω -deformation parameters (ϵ_1, ϵ_2) , and provide for a map to standard gauge theory expressions for the Seiberg-Witten curve, including ϵ_i corrections.

We then concentrate on the limit $\epsilon_2/\epsilon_1 \rightarrow 0$ [16], which is a large central charge limit in the conformal field theory. It has been shown in e.g. [17–19] that a WKB analysis of the null vector decoupling equations in this semi-classical limit reproduces the non-convergent

ϵ_1 -expansion of the instanton partition function of the gauge theory.¹ There is a rich literature [22–31] on methods which may be used to enhance these results non-perturbatively. Using the exact WKB analysis, we study the resulting differential equations satisfied by the (ir)regular conformal blocks (equivalently, the ϵ_1 -deformed surface operator partition function). This allows us to compute the monodromy group of each of the differential equations as a function of (i) the parameters of the differential equations, and (ii) the Borel resummed monodromies that are properties of individual solutions. The monodromy group contains information about the instanton partition function with surface operator insertion, which is non-perturbative in ϵ_1 . In doing so, we provide the underlying exact picture [12] with a detailed description of how these beautiful and abstract mathematical constructs reduce to the more hands-on limiting analysis of $\mathcal{N} = 2$ gauge theories to which we have become accustomed.

In this physical set-up, we apply the theorems of [31], thereby drawing on intuition from both gauge theory and the mathematical study of singular perturbation theory [29]. As a by-product, we add details to the WKB analysis and provide a calculation of the monodromy group of the differential equation in terms of deformed gauge theory data. For instance, we analyze the occurrence of a double flip, consisting of simultaneous single flips. Two different ways of splitting the double flip into two single flips give the same monodromy group and Stokes automorphism. Although we demonstrate this result in the context of $N_f = 4$ theory, this is a new result in the exact WKB method and we believe it is valid in a more general context.

In [32], a WKB analysis of the Hitchin systems corresponding to circle compactifications of undeformed $SU(2)$ gauge theories was undertaken. Our work may be viewed as an alternative route to the WKB analysis, which is closely related to [32] at zeroth order in ϵ_1 .

Our broader goal is to communicate the extreme generality of the correspondence between ϵ_1 -deformed $\mathcal{N} = 2$ gauge theories — specifically, their instanton partition functions with surface operator insertions — and certain Schrödinger equations amenable to exact WKB analysis. As a first step, we show the extent to which the program applied to pure $\mathcal{N} = 2$ super Yang-Mills in [33] generalizes to theories with matter.

We will now briefly present the structure of our paper. In section 2, we present a derivation of the null vector decoupling equation satisfied by the five-point conformal block with a light degenerate insertion, which has a null vector at level two. We apply the collision procedure of [34] to produce irregular conformal blocks and derive the null vector decoupling equations satisfied by the limit blocks. We then consider the semi-classical limit (of infinite central charge) of these differential equations. These equations will be the starting point for the exact WKB analysis of section 3. In this section, we briefly review the exact WKB approach, and in section 4 apply it to the calculation of the monodromy groups of our differential equations. We make contact with the standard undeformed Seiberg-Witten perspective in section 5 and end with comments and future directions for work in section 6. The appendices collect details regarding the derivation of the ϵ_2 -exact differential equations for the asymptotically free theories, and an independent

¹For non-perturbative results in the context of topological strings we refer to [20, 21].

check of the semi-classical differential equations via the saddle-point analyses of Nekrasov partition functions [35].

2 The conformal field theory perspective

In this section, we present the null vector decoupling equation satisfied by the five-point conformal block with one degenerate operator insertion. We then list the corresponding equations satisfied by irregular blocks that arise when punctures collide [34]. We study these equations within the framework of conformal field theory, and finally, exploit the fact that these conformal blocks also capture the ϵ_1 -deformed instanton partition function of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions with $SU(2)$ gauge group and a varying number of flavours [5]. We thus lay the groundwork for further analysis of these partition functions, which will be non-perturbative in the deformation parameter ϵ_1 . For completeness, we provide the details of the derivation of all these equations in appendix A.

We start our analysis by considering regular conformal blocks with four ordinary primary operator insertions on the sphere and one degenerate operator insertion with a null vector at level two, which remains light in the limit of large central charge. On the gauge theory side of the 2d/4d correspondence, this set-up corresponds to the conformal $N_f = 4$ case. To get asymptotically free (lower N_f) theories, we sequentially collide primary operators on the sphere in such a way that they generate irregular conformal blocks [34]. The case of three flavours will correspond to one irregular block, the case of two flavours can correspond to either one or two irregular blocks, while a lower number of flavours corresponds to two irregular blocks in the conformal field theory. For all these collision limits, we give the corresponding null vector decoupling equations.

2.1 The five-point block

We study a conformal field theory with central charge

$$c = 1 + 6Q^2, \quad \text{where} \quad Q = b + b^{-1} \quad \text{and} \quad b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \quad (2.1)$$

We consider a five-point chiral conformal block Ψ with four primary operator insertions V_{α_i} and an insertion of a degenerate field $\Phi_{2,1}(z)$ of the Virasoro algebra [9]:

$$\Psi(z_i, z) = \left\langle \Phi_{2,1}(z) : \prod_{i=1}^4 V_{\alpha_i}(z_i) : \right\rangle. \quad (2.2)$$

The degenerate field $\Phi_{2,1}$ has conformal dimension $\Delta_{2,1}$

$$\Delta_{2,1} = -\frac{1}{2} - \frac{3}{4} \frac{\epsilon_2}{\epsilon_1}, \quad (2.3)$$

while the conformal dimensions of the generic primaries are denoted Δ_{α_i} . We have chosen the degenerate insertion such that it remains light in the limit of large central charge

$\epsilon_2/\epsilon_1 \rightarrow 0$. The degenerate field $\Phi_{2,1}$ has a null vector at level two, and consequently satisfies the null vector condition

$$\frac{\epsilon_1}{\epsilon_2} \partial^2 \Phi_{2,1}(z) + :T(z)\Phi_{2,1}(z): = 0, \quad (2.4)$$

where the operator $T(z)$ is the holomorphic stress tensor of the conformal field theory. Using the operator product expansion between the stress tensor and the primary fields, the second term can be written as:

$$\left\langle :T(z)\Phi_{2,1}(z): \prod_{i=1}^4 V_{\alpha_i}(z_i) \right\rangle = \sum_{i=1}^4 \left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) \left\langle \Phi_{2,1}(z) \prod_{i=1}^4 V_{\alpha_i}(z_i) \right\rangle. \quad (2.5)$$

Imposing global conformal invariance allows us to express the derivatives with respect to z_1, z_3 and z_4 in terms of the derivatives at z_2 and z . Then, setting the insertions to be at $(z, 0, q, 1, \infty)$, the null vector decoupling equation takes the form

$$\left[\frac{\epsilon_1}{\epsilon_2} \frac{\partial^2}{\partial z^2} + \left(\frac{\Delta_{\alpha_2}}{(z-q)^2} + \frac{q(q-1)}{z(z-1)(z-q)} \frac{\partial}{\partial q} \right) - \frac{2z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{\Delta_{\alpha_1}}{z^2} + \frac{\Delta_{\alpha_3}}{(z-1)^2} - \frac{\Delta_{2,1} + \Delta_{\alpha_1} + \Delta_{\alpha_2} + \Delta_{\alpha_3} - \Delta_{\alpha_4}}{z(z-1)} \right] \Psi(z, q) = 0 \quad (2.6)$$

The null vector decoupling on the five point conformal block was also studied in [9–11, 36, 37]. The conformal dimensions Δ_i of the primary fields V_{α_i} can be written in terms of the momenta α_i as

$$\Delta_{\alpha_i} = \alpha_i(Q - \alpha_i). \quad (2.7)$$

We further parameterize the momenta α_i in terms of the four masses m_i :

$$\begin{aligned} \alpha_1 &= \frac{Q}{2} + \frac{m_1 - m_2}{2\sqrt{\epsilon_1\epsilon_2}}, & \alpha_2 &= \frac{Q}{2} + \frac{m_1 + m_2}{2\sqrt{\epsilon_1\epsilon_2}}, \\ \alpha_3 &= \frac{Q}{2} - \frac{m_3 + m_4}{2\sqrt{\epsilon_1\epsilon_2}}, & \alpha_4 &= \frac{Q}{2} - \frac{m_3 - m_4}{2\sqrt{\epsilon_1\epsilon_2}}. \end{aligned} \quad (2.8)$$

As a function of the masses, the conformal dimensions are

$$\begin{aligned} \Delta_{\alpha_1} &= \frac{(\epsilon_1 + \epsilon_2)^2 - (m_1 - m_2)^2}{4\epsilon_1\epsilon_2}, & \Delta_{\alpha_2} &= \frac{(\epsilon_1 + \epsilon_2)^2 - (m_1 + m_2)^2}{4\epsilon_1\epsilon_2}, \\ \Delta_{\alpha_3} &= \frac{(\epsilon_1 + \epsilon_2)^2 - (m_3 + m_4)^2}{4\epsilon_1\epsilon_2}, & \Delta_{\alpha_4} &= \frac{(\epsilon_1 + \epsilon_2)^2 - (m_3 - m_4)^2}{4\epsilon_1\epsilon_2}. \end{aligned} \quad (2.9)$$

In terms of these variables that are appropriate for comparison to the four dimensional gauge theory, the null vector decoupling equation for the $N_f = 4$ theory takes the following

form:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_1 - m_2)^2}{4z^2} + \frac{(m_1 + m_2)^2}{4(z - q)^2} + \frac{(m_3 + m_4)^2}{4(z - 1)^2} + \frac{m_1^2 + m_2^2 + 2m_3 m_4}{2z(1 - z)} \right. \\ \left. - \epsilon_1^2 \left(\frac{q^2 - 2qz + z^2 (z^2 - 2z + 2)}{4(z - 1)^2 z^2 (q - z)^2} \right) + \epsilon_1 \epsilon_2 \left(\frac{q(1 - q)}{z(z - 1)(z - q)} \frac{\partial}{\partial q} + \frac{2z - 1}{z(z - 1)} \frac{\partial}{\partial z} \right) \right. \\ \left. + \frac{q^2 (-z^2 + z - 1) + 2qz (z^2 - z + 1) + z^2 (-2z^2 + 3z - 2)}{2(z - 1)^2 z^2 (q - z)^2} \right) \\ \left. + \epsilon_2^2 \left(\frac{q^2 (-3z^2 + 3z - 1) + 2qz (3z^2 - 3z + 1) + z^2 (-4z^2 + 5z - 2)}{4(z - 1)^2 z^2 (q - z)^2} \right) \right] \Psi(z, q) = 0. \quad (2.10)$$

2.2 The null vector decoupling equations for irregular blocks

We now take limits of the five-point null vector decoupling equation (2.6) in which various primary operators V_{α_i} collide to form irregular conformal blocks of order one [34]. These limiting configurations are in direct correspondence with the ϵ_i -deformed SU(2) gauge theories with $N_f < 4$. We list below the null vector decoupling equations for each of these cases and refer to appendix A for a detailed derivation. A summary of these equations can also be found in [10, 13].

$N_f = 3$. In this case, we have one irregular block of order one with a fourth order pole at $z = 0$. In the gauge theory variables, we take $q \rightarrow 0$ and $m_2 \rightarrow \infty$, keeping the dynamical scale $\Lambda_3 = q m_2$ finite. The resulting differential equation is:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3 + m_4)^2}{4(z - 1)^2} + \frac{m_3 m_4}{z(1 - z)} + \frac{m_1 \Lambda_3}{z^3} + \frac{\Lambda_3^2}{4z^4} + \epsilon_1 \epsilon_2 \left(\frac{1 - 2z}{z(1 - z)} \frac{\partial}{\partial z} + \frac{1 - 2z}{2z(z - 1)^2} \right) \right. \\ \left. + \frac{1}{z^2(1 - z)} \left(-\epsilon_1 \epsilon_2 \Lambda_3 \frac{\partial}{\partial \Lambda_3} + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) - \frac{\epsilon_1^2}{4(z - 1)^2} + \epsilon_2^2 \frac{(3 - 4z)}{4z(z - 1)^2} \right] \Psi_3(z, \Lambda_3) = 0.$$

$N_f = 2$. There are two ways to reach the case with two flavours from the case with three flavours. One could decouple either the flavour with mass m_1 or one of those with masses $m_{3,4}$. As shown in [32], these lead to inequivalent Hitchin systems and give rise to distinct differential equations.

Let us first consider the irregular block of order one with a third order pole at $z = 0$. This corresponds to decoupling m_1 . We refer to this as the asymmetric configuration and the associated null vector decoupling equation becomes:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3 + m_4)^2}{4(z - 1)^2} + \frac{m_3 m_4}{z(1 - z)} + \frac{\Lambda_2^2}{z^3} - \frac{\epsilon_1 \epsilon_2}{2z^2(1 - z)} \Lambda_2 \frac{\partial}{\partial \Lambda_2} \right. \\ \left. + \epsilon_1 \epsilon_2 \left(\frac{1 - 2z}{z(1 - z)} \frac{\partial}{\partial z} + \frac{1 - 2z}{2z(z - 1)^2} \right) - \frac{\epsilon_1^2}{4(z - 1)^2} + \epsilon_2^2 \frac{(3 - 4z)}{4z(z - 1)^2} \right] \Psi_{2,A}(z, \Lambda_2) = 0 \quad (2.11)$$

Alternatively, one can consider two irregular blocks of order one, with equal fourth order poles. This corresponds to decoupling m_3 while keeping m_1 and m_4 finite. We

refer to this as the symmetric configuration and the associated null vector decoupling equation reads:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_2^2}{4z^4} + \frac{\Lambda_2 m_1}{z^3(z-1)^2} - \frac{\Lambda_2 m_4}{z(z-1)^3} + \frac{\Lambda_2^2}{4(z-1)^4} + \frac{2-3z}{4z(z-1)^2} (2\epsilon_1 \epsilon_2 + 3\epsilon_2^2) \right. \\ \left. + \frac{1}{z^2(z-1)^2} \left(-\epsilon_1 \epsilon_2 \Lambda_2 \frac{\partial}{\partial \Lambda_2} - 2\Lambda_2 m_1 + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} \right] \Psi_{2,S}(z, \Lambda_2) = 0.$$

$\mathbf{N}_f = \mathbf{1}$. We consider two irregular blocks of order one with one fourth order pole and one third order pole. This corresponds to decoupling m_4 and the null vector decoupling equation takes the form

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_1^2}{4z^4} + \frac{\Lambda_1 m_1}{z^3(z-1)^2} - \frac{\Lambda_1^2}{4z(z-1)^3} + \frac{2-3z}{4z(z-1)^2} (2\epsilon_1 \epsilon_2 + 3\epsilon_2^2) + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} \right. \\ \left. + \frac{1}{z^2(z-1)^2} \left(-\epsilon_1 \epsilon_2 \Lambda_1 \frac{\partial}{\partial \Lambda_1} - 2\Lambda_1 m_1 + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) \right] \Psi_1(z, \Lambda_1) = 0. \quad (2.12)$$

$\mathbf{N}_f = \mathbf{0}$. Finally, we consider the case with two irregular blocks of order one with equal third order poles. All masses have been decoupled and the null vector decoupling equation becomes

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_0^2}{z^3(z-1)^2} + \frac{1}{z^2(z-1)^2} \left(-\frac{1}{2} \epsilon_1 \epsilon_2 \Lambda_0 \frac{\partial}{\partial \Lambda_0} - 2\Lambda_0^2 \right) + \frac{\Lambda_0^2}{z(z-1)^3} \right. \\ \left. + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{2-3z}{4z(z-1)^2} (2\epsilon_1 \epsilon_2 + 3\epsilon_2^2) \right] \Psi_0(z, \Lambda_0) = 0. \quad (2.13)$$

This completes the list of six differential equations that we refer to throughout.

2.3 The semi-classical limit

In the rest of our paper, we will concentrate on the limit $\epsilon_2/\epsilon_1 \rightarrow 0$, which is a large central charge limit. We keep the ratio of the mass parameters m_i and the deformation parameter ϵ_1 fixed. In this limit, the primary insertions V_{α_i} are heavy, while the degenerate insertion $\Phi_{2,1}$ is light. Thus, in this limit, the differential equation (2.6) simplifies, and we can drop the term proportional to ∂_z , while the terms proportional to the conformal dimensions Δ_{α_i} grow large. To simplify the equation further, we must specify the leading dependence of the q -derivative of the five-point block on ϵ_2 . To that end, we make the semi-classical $\epsilon_2 \rightarrow 0$ ansatz

$$\Psi(z, q) = \exp \left(-\frac{\tilde{F}(q, m_i, \epsilon_i)}{\epsilon_1 \epsilon_2} \right) \psi(z, q). \quad (2.14)$$

We suppose that the q -derivative of the remaining function $\psi(z, q)$ is sub-dominant in the small ϵ_2/ϵ_1 limit, and observe that the leading dependence in ϵ_2 is only on the cross-ratio q of the heavy operators. We then define the quantity

$$\tilde{u} = q(1-q) \partial_q \tilde{F}. \quad (2.15)$$

The parameter \tilde{u} is identified with the Coulomb modulus of the gauge theory up to shifts that depend on the masses. Substituting this parameterization into the null vector decoupling equation and taking the semi-classical limit $\epsilon_2 \rightarrow 0$ leads to the Schrödinger equation

$$\left(-\epsilon_1^2 \frac{d^2}{dz^2} + Q(z, \epsilon_1)\right) \psi(z, q) = 0, \tag{2.16}$$

where the potential function Q has an ϵ_1 expansion which terminates at second order

$$Q(z) = Q_0(z) + \epsilon_1 Q_1(z) + \epsilon_1^2 Q_2(z). \tag{2.17}$$

The coefficient functions are

$$\begin{aligned} Q_0(z) &= -\frac{\tilde{u}}{z(z-1)(z-q)} + \frac{(m_1-m_2)^2}{4z^2} + \frac{(m_1+m_2)^2}{4(z-q)^2} + \frac{(m_3+m_4)^2}{4(z-1)^2} + \frac{m_1^2+m_2^2+2m_3m_4}{2z(1-z)}, \\ Q_1(z) &= 0, \\ Q_2(z) &= -\frac{1}{4z^2} - \frac{1}{4(z-1)^2} - \frac{1}{4(z-q)^2} + \frac{1}{2z(z-1)}. \end{aligned} \tag{2.18}$$

2.4 The semi-classical irregular blocks

The same type of ansatz (2.14) can be used in order to obtain the differential equations for the irregular blocks in the semi-classical $\epsilon_2 \rightarrow 0$ limit. The variable parameterizing the Coulomb modulus is now defined as

$$\tilde{u} = \Lambda_{N_f} \frac{\partial \tilde{F}}{\partial \Lambda_{N_f}}, \tag{2.19}$$

where Λ_{N_f} is the corresponding strong coupling scale of the $N_f < 4$ gauge theory. As in the conformal case, the prepotential of the gauge theory will differ mildly from \tilde{F} . However, what is of importance to us is the pole structure of the functions $Q_k(z)$, and we choose a parameterization that descends naturally from the conformal theory and that allows for a simple presentation of the differential equations. In the following, we present all the asymptotically free cases:

- $N_f = 3$: the Schrödinger equation which governs the ϵ_1 -deformed gauge theory is given by

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3+m_4)^2}{4(z-1)^2} + \frac{m_3m_4}{z(1-z)} + \frac{m_1\Lambda_3}{z^3} + \frac{\Lambda_3^2}{4z^4} + \frac{\tilde{u}}{z^2(1-z)} - \frac{\epsilon_1^2}{4(z-1)^2}\right] \psi_3(z, \Lambda_3) = 0. \tag{2.20}$$

- $N_f = 2$ (asymmetric realization): the differential equation in the semi-classical limit takes the form

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3+m_4)^2}{4(z-1)^2} + \frac{m_3m_4}{z(1-z)} + \frac{\Lambda_2^2}{z^3} + \frac{\tilde{u}}{z^2(1-z)} - \frac{\epsilon_1^2}{4(z-1)^2}\right] \psi_{2,A}(z, \Lambda_2) = 0 \tag{2.21}$$

- $N_f = 2$ (symmetric realization):

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_2^2}{4z^4} + \frac{\Lambda_2 m_1}{z^3(z-1)^2} - \frac{\Lambda_2 m_4}{z(z-1)^3} + \frac{\Lambda_2^2}{4(z-1)^4} + \frac{\tilde{u}}{z^2(z-1)^2} \right] \psi_{2,S}(z, \Lambda_2) = 0. \quad (2.22)$$

- $N_f = 1$:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_1^2}{4z^4} + \frac{\Lambda_1 m_1}{z^3(z-1)^2} - \frac{\Lambda_1^2}{4z(z-1)^3} + \frac{\tilde{u}}{z^2(z-1)^2} \right] \psi_1(z, \Lambda_1) = 0. \quad (2.23)$$

- $N_f = 0$: finally, for the pure super Yang-Mills theory, the equation reads

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_0^2}{z^3(z-1)^2} + \frac{\tilde{u}}{z^2(z-1)^2} + \frac{\Lambda_0^2}{z(z-1)^3} \right] \psi_0(z, \Lambda_0) = 0. \quad (2.24)$$

We have thus obtained the differential equations which we analyze in detail in section 4.

3 The exact WKB analysis of differential equations

In this section, we review the exact WKB approach to the analysis of differential equations and apply it to the null vector decoupling equations in the semi-classical limit. We will carry out the exact WKB analysis with respect to the small parameter ϵ_1 . Our analysis will therefore be valid to zeroth order in ϵ_2 and non-perturbatively in ϵ_1 . Below, we briefly review the salient features of the exact WKB analysis and refer the reader to [29, 31] for a more comprehensive treatment of the same.

3.1 The exact WKB method

The differential equations that we study can be written in the form of a Schrödinger equation:

$$\left(-\epsilon_1^2 \frac{d^2}{dx^2} + Q(x) \right) \psi(x, \epsilon_1) = 0. \quad (3.1)$$

We allow the function Q to have an expansion of the form

$$Q(x) = Q_0(x) + \epsilon_1 Q_1(x) + \epsilon_1^2 Q_2(x) + \dots. \quad (3.2)$$

For the null vector decoupling equations that we study, the only non-zero coefficient functions are Q_0, Q_1 and Q_2 . We choose a WKB ansatz for the solution to this differential equation, which takes the form

$$\psi(x, \epsilon_1) = \exp \left(\int_{x_0}^x dx' S(x', \epsilon_1) \right), \quad (3.3)$$

with $S(x, \epsilon_1)$ expanded as a formal power series in ϵ_1 as

$$S(x, \epsilon_1) = \frac{1}{\epsilon_1} S_{-1}(x) + S_0(x) + \epsilon_1 S_1(x) + \dots. \quad (3.4)$$

Substituting this ansatz into the differential equation, we get recursion relations governing the coefficients S_k

$$S_{-1}^2 = Q_0, \quad (3.5)$$

$$2S_{-1}S_{n+1} + \sum_{k+l=n} S_k S_l + \frac{dS_n}{dx} = Q_{n+2} \quad \text{for } n \geq -1. \quad (3.6)$$

We see that the initial conditions governing the system of recursion relations allow for two possible sets of solutions to these recursion relations, as $S_{-1} = \pm\sqrt{Q_0}$. We also note the crucial feature that the zeroes of Q_0 , which we call turning points, introduce branch cuts on the Riemann surface Σ on which our differential equation and its exact solutions live. Thus, in our exact WKB treatment, we introduce a new manifold $\hat{\Sigma}$, which is a double cover of the Riemann surface, and we move between sheets as we pass branch cuts that emanate from turning points, or odd order poles. From hereon, we will distinguish the choice of WKB solution by attaching to it the subscript (\pm) . We also observe that in the ϵ_1 -expansion of $S(x, \epsilon_1)$, the sets of odd and even coefficients are dependent. If we define

$$S_{\text{odd}} = \sum_{j \geq 0} S_{2j-1} \epsilon_1^{2j-1} \quad \text{and} \quad S_{\text{even}} = \sum_{j \geq 0} S_{2j} \epsilon_1^{2j}, \quad (3.7)$$

we have the relation

$$S_{\text{even}} = -\frac{1}{2} \frac{d}{dx} \log S_{\text{odd}}. \quad (3.8)$$

Putting all this together, we can write down a formal expression for the two linearly independent solutions to our differential equation:

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left\{ \pm \int_{x_0}^x dx' S_{\text{odd}} \right\}. \quad (3.9)$$

This formal expression should be understood as an analytic function of x multiplying an asymptotic series in ϵ_1 :

$$\psi_{\pm} = \exp \left\{ \pm \frac{1}{\epsilon_1} \int_{x_0}^x dx' \sqrt{Q_0(x')} \right\} \epsilon_1^{1/2} \sum_{k=0}^{\infty} \epsilon_1^k \psi_{\pm,k}(x). \quad (3.10)$$

Borel resummation. In the exact WKB approach, it is convenient to normalize wavefunctions at distinguished points of the differential equation. As mentioned earlier, in addition to the singularities of the coefficient functions of the differential equations, their zeros (turning points) also play an important role. We will normalize our solutions with respect to the turning points, i.e. choose the starting point x_0 of the integration path to be a turning point t ,

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left\{ \pm \int_t^x dx' S_{\text{odd}} \right\}. \quad (3.11)$$

Formal WKB solutions are generically divergent. To remedy this, we invoke Borel resummation: a technique that constructs an analytic function whose asymptotic expansion matches the formal WKB series. The Borel transformed series is defined as

$$\psi(\epsilon_1) = \sum_{k=0}^{\infty} \psi_k \epsilon_1^k \xrightarrow{\text{Borel transform}} \tilde{\psi}(y) = \sum_{k=1}^{\infty} \psi_k \frac{y^{k-1}}{(k-1)!}. \quad (3.12)$$

Next, define the function [31]

$$\Psi(\epsilon_1) = \psi_0 + \int_{\ell_\theta} dy e^{-y/\epsilon_1} \tilde{\psi}(y), \quad (3.13)$$

where ℓ_θ is the line connecting a point at which the series $\tilde{\psi}(y)$ converges² — typically, a turning point — to the point at infinity at an angle θ . If this integral exists, $\Psi(\epsilon_1)$ is the requisite analytic function, called the Borel sum.

Notice that the Borel sum contains an angular dependence. In order to understand this better, one must appreciate that Borel sums are typically defined only in regions of the complex ϵ_1 -plane, and not throughout. These regions are bounded by Stokes lines, defined by the condition

$$\text{Im} \left[\int_{x_0}^x dx' \sqrt{Q_0(x')} \right] = 0, \quad (3.14)$$

and different Stokes regions are assigned different linear combinations of a given basis of analytic solutions to the differential equation, arrived at via Borel resummation. One of the key components of the exact WKB analysis is understanding how solutions in different Stokes regions are related by analytic continuation; these often go by the name of “connection formulae”. However, before we address this transition behaviour, we will find it necessary to endow Stokes lines with an orientation. To this end, we adopt the convention that Stokes lines are oriented *away* (i.e. the arrow on the Stokes line is pointing away from a turning point) if

$$\text{Re} \left[\int_{x_0}^x dx' \sqrt{Q_0(x')} \right] > 0 \quad (3.15)$$

along the Stokes line. Else, the arrow points towards the turning point.

Three Stokes lines emanate from a first order zero of $Q_0(x)$, which is also referred to as a simple turning point. Thus one end of any Stokes line is at a turning point. The other end can either be at a singularity or at a turning point. When both the end points of a Stokes line in a given Stokes graph terminate at turning points then the corresponding Stokes graph is called “critical”.³

Connection formulae. We are now in a position to state the connection formulae. For a Stokes graph which is not critical, consider two regions U_1 and U_2 separated by a Stokes curve Γ , and consider Ψ_\pm^j to be the Borel sums of WKB solutions in each of the regions U_j . The connection formulae for the Borel sums in different Stokes domains are given by:

$$\text{if } \text{Re} \left[\int_{x_0}^x dx' \sqrt{Q_0(x')} \right] < 0 \text{ on } \Gamma \quad : \quad \begin{cases} \Psi_+^1 &= \Psi_+^2, \\ \Psi_-^1 &= \Psi_-^2 \pm i\Psi_+^2, \end{cases} \quad (3.16)$$

$$\text{if } \text{Re} \left[\int_{x_0}^x dx' \sqrt{Q_0(x')} \right] > 0 \text{ on } \Gamma \quad : \quad \begin{cases} \Psi_+^1 &= \Psi_+^2 \pm i\Psi_-^2, \\ \Psi_-^1 &= \Psi_-^2. \end{cases} \quad (3.17)$$

²To be precise, this is true for Gevrey-1 series, which in our context corresponds to the following statement. If ψ_k is the k th coefficient of the asymptotic series then the series is Gevrey-1 type if growth of ψ_k is bounded by $\psi_k \leq AB^k k!$ for some constants A and B . If ψ_k is a function of a continuous variable, say, $x \in \mathbb{C}$ then this condition applies to the supremum of $\psi_k(x)$ in a compact subset of \mathbb{C} .

³The general behaviour of Stokes lines is discussed in [31]. We restrict ourselves to situations that are relevant in this work.

In the above connection formulae, there is an ambiguity (\pm) that is fixed by noting that the turning point that Γ originates from serves as a point of reference. If the path of analytic continuation crosses Γ counter-clockwise as seen from the turning point, we pick the (+) sign, and if this path crosses Γ clockwise, we pick the (−) sign. Later in this section, we will write down the Stokes matrices that multiply wave-functions; these are equivalent to the above result.

The global properties of solutions to the differential equations we consider are governed by the monodromy group and the Stokes phenomena around singular points. The monodromy group of these differential equations can be expressed entirely in terms of two sets of quantities: (a) the characteristic exponents at each singular point s_k , and (b) the contour integrals of S_{odd} around branch cuts. We now parameterize the characteristic exponents conveniently.

As a system of solutions to our differential equation, we consider the WKB solutions (3.9), and define the characteristic exponents as residues of the differential:

$$M_k = \text{Res} \sqrt{Q_0(x)} \Big|_{x=s_k} . \tag{3.18}$$

From the null vector decoupling equations we derived in the previous section, one can check that the residues M_k are linear combinations of the mass parameters of the gauge theory. As the monodromy group computations will use WKB wave-functions (3.9), we relate the residues of S_{odd} to our characteristic exponents as⁴

$$\text{Res} S_{\text{odd}}(x, \epsilon) \Big|_{x=s_k} = \frac{M_k}{\epsilon_1} \sqrt{1 + \frac{\epsilon_1^2}{4M_k^2}} . \tag{3.19}$$

Finally, upon exponentiating this contribution, we get the multiplier that affects WKB wave-functions:

$$\nu_k^\pm = \exp \left[i\pi \left(1 \pm \sqrt{\frac{4M_k^2}{\epsilon_1^2} + 1} \right) \right] . \tag{3.20}$$

Notice that $\nu_k^+ = 1/\nu_k^-$, a fact that we will use repeatedly. Since the base point x_0 will not always be a turning point, the modified connection formulae can be obtained by a composition of the contour integrals. We find it convenient to use a matrix notation to exhibit the connection formulae. As an example, let us consider analytically continuing the Borel resummed wave-functions from Stokes region U_1 to Stokes region U_2 . As shown in figure 1, there are two distinct possibilities. If the contour crosses a Stokes line that is directed inwards to a turning point as in figure 1 (A), we find the connection formula:

$$\left(\Psi_+^1, \Psi_-^1 \right) \implies \left(\Psi_+^2, \Psi_-^2 \right) \begin{pmatrix} 1 \pm i u_i^{-1} & \\ & 1 \end{pmatrix} . \tag{3.21}$$

In the above equation, we use the notation,

$$u_j = \exp \left(2 \int_{\gamma_j} dx S_{\text{odd}} \right) , \tag{3.22}$$

⁴This is true under the assumptions that $\text{Re} M_k \neq 0$.

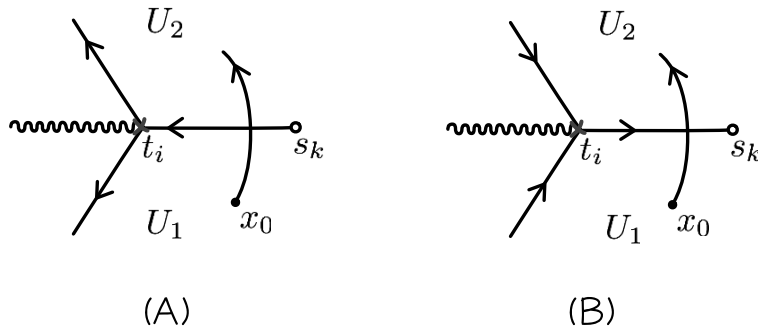


Figure 1. Analytic continuation of wave-functions from U_1 to U_2 .

where γ_j is an oriented curve from the base point to the turning point t_j . Along a contour that crosses a Stokes line which is directed outwards from a turning point as in figure 1 (B), we have the connection formula:

$$\left(\Psi_+^1, \Psi_-^1\right) \implies \left(\Psi_+^2, \Psi_-^2\right) \begin{pmatrix} 1 & 0 \\ \pm i u_i & 1 \end{pmatrix}. \quad (3.23)$$

In the above, the $+(-)$ sign is chosen for counter-clockwise (clockwise) crossing of the contour from one Stokes region to the other, with respect to the turning point. For more complicated contours, it is important to take into account contributions from any branch cuts and/or singularities enclosed along the closed contour from the base point to the intersection point, the turning point and then back to the base point. As a simple example of this phenomenon, let us suppose the contour chosen happens to encircle a branch cut — say between t_j and t_i as in figure 2 — counter-clockwise. Here, the curves γ_i are those that define the parameter u_i . The closed contour γ_{ji} that encircles the branch cut has a contribution of the form

$$u_{ji} = \exp\left(\int_{\gamma_{ji}} dx' S_{\text{odd}}\right), \quad (3.24)$$

where from the figure it is clear that

$$u_{ji} = u_j^{-1} u_i. \quad (3.25)$$

One can see that although the u_i by itself is dependent on the base point, the contour integral is independent of this choice.

Contour encircling a turning point. Let us make another important preliminary point regarding the choice of cycles. In order to define the monodromy group, we first choose a base point and define a basis of closed loops that encircle just the singularities. In some of the cases we encounter, there are branch cuts between turning points and singularities. In such cases, we choose the contours to also include these turning points.

In order to prove that this is consistent with the usual definition of the monodromy group, let us consider a contour that only encircles the turning point, as shown in figure 3.

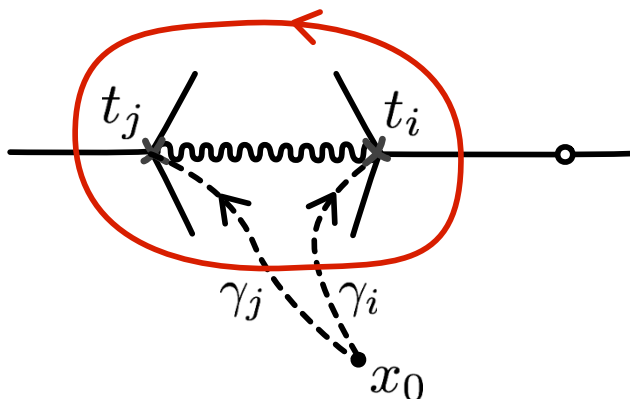


Figure 2. Encircling branch cuts.

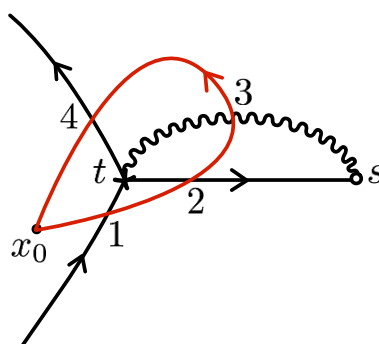


Figure 3. Contour with base point x_0 encircling a turning point t .

If we choose to normalize the wave-functions at x_0 , the wave-functions undergo the following transformation as we travel along the path:

$$M_{x_0, \text{path}} = \begin{pmatrix} 1 & 0 \\ \frac{i}{u_1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_1} \\ 0 & 1 \end{pmatrix} \quad (3.26)$$

$$= \begin{pmatrix} u_1 & 0 \\ 0 & \frac{1}{u_1} \end{pmatrix} \quad (3.27)$$

Here we have associated the matrix $-i\sigma_1$ to the branch-cut crossing, which ensures that we remain on the same sheet of the Riemann surface. It can be easily shown that for any base point that one may choose, the answer is trivial as above. If we chose the turning point itself to be the base-point, $u_1 = 1$ and the matrix reduces to the identity matrix. Since the net result is simply the identity matrix, in order to calculate the monodromy matrix for the contour that encircles the singularity s , one may just as well compute the monodromy of the wave-functions around the cycle that encircles both the turning point t and the singularity s . We will make use of this repeatedly in those cases in which the branch cut extends between a turning point and a singularity.

It is instructive to square this situation with the solution of a differential equation near an ordinary point. It is known that any solution of a differential equation can be written as

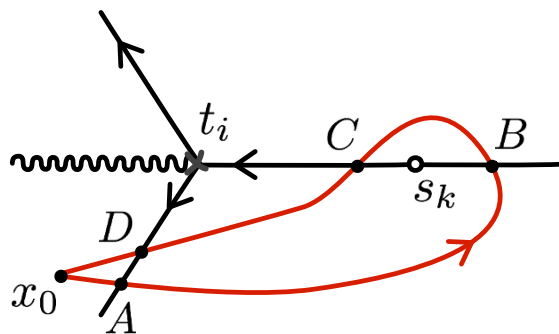


Figure 4. Evaluation of Stokes matrices: effect of singularities.

a Taylor series in the neighbourhood of an ordinary point. The radius of convergence of this solution is at least as much as the distance from the chosen point to the nearest singularity. The Taylor series solution will clearly have trivial monodromy property. Although the WKB analysis assigns a special status to turning points, from the differential equation point of view the turning point is an ordinary point. Clearly, the branch cut and the Stokes lines emanating from a turning point are artefacts of the WKB approximation and the insertion of the matrix $-i\sigma_1$ restores the fact that the turning point is an ordinary point of the differential equation.

Contours encircling a singular point. Let us now consider the toy example, as shown in figure 4, where the contour encloses a singularity.⁵

In figure 4, at the first intersection point A , the contour crosses counter-clockwise a Stokes line emanating from t_i . Thus, the Stokes matrix is

$$\begin{pmatrix} 1 & 0 \\ +iu_i & 0 \end{pmatrix}. \tag{3.28}$$

In order to determine the Stokes matrix at B , we need to know to which turning point the Stokes line is connected. Since this is irrelevant to the present discussion, we move on to consider the third intersection point C . This time the contour crosses a Stokes line going into t_i , and the crossing is clockwise as seen from t_i . Further, when this contour is completed using γ_j , we see that a singularity is encircled counter-clockwise. Taking this into account, the Stokes matrix is

$$\begin{pmatrix} 1 & -iu_i^{-1}v_k^{-2} \\ 0 & 1 \end{pmatrix}. \tag{3.29}$$

Finally, at the fourth intersection point D , the contour crosses the Stokes line clockwise. In fact, it is very similar to the first intersection, except that now there is a singularity

⁵This example will illustrate the manner in which the Stokes matrices at each intersection are written down. The Stokes lines here don't end at turning points or singularities; the reader is encouraged to think of the figure as a part of a complete Stokes graph that has been zoomed into.

encircled. Consequently, the Stokes matrix is

$$\begin{pmatrix} 1 & 0 \\ -iu_j \nu_k^2 & 0 \end{pmatrix}. \tag{3.30}$$

This concludes our brief review of the exact WKB analysis. We refer the reader to [29, 31] for a more detailed discussion and further references.

3.2 The applicability of the exact WKB analysis

The application of the exact WKB techniques depends on the precise differential equation under consideration. Before we apply the exact WKB method to the equations derived in the previous section, it is important to point out the subtleties in the applicability of this analysis. In the Schrödinger type differential equations listed in sections 2.3 and 2.4, the parameter ϵ_1 functions as the Planck's constant \hbar in the WKB approximation scheme. For our null vector decoupling equations, the potential has zeroth, first and second order terms in ϵ_1 . In order to apply the exact WKB techniques to the solution of the differential equation, the ϵ_1 -deformed potential must satisfy certain conditions. These consistency conditions not only ensure normalizability of the wave-functions at singularities but also are useful in proving Borel summability of the WKB wave-functions.

The necessary conditions (eq. (2.8) and (2.9) in [31]) are:

- If the leading coefficient Q_0 has a pole of order $m \geq 3$, then the order of $Q_{n \geq 1}$ at that pole should be smaller than $1 + m/2$.
- If the pole of Q_0 (at, say $z = z_0$) is of order $m = 2$, then $Q_{n \neq 2}$ may have at most a simple pole there and Q_2 should have a double pole:

$$Q_2 = -\frac{1}{4(z - z_0)^2}(1 + O(z - z_0)) \quad \text{as } z \rightarrow z_0. \tag{3.31}$$

It is easily checked that the potentials that appear in the various Schrödinger type differential equations in sections 2.3 and 2.4 satisfy these conditions.

3.3 Theorems on Stokes automorphisms

Since all the equations listed in sections 2.3 and 2.4 satisfy the necessary conditions, the theorems proved in [31] using these conditions can be directly applied to our equations. There is however, an interesting exception and we will comment on it momentarily. In particular, the results of [31] include theorems on the Stokes automorphisms that relate WKB resummed monodromies with a given Borel resummation angle, to monodromies with another Borel resummation angle.

We will now list the relevant results from these theorems. Consider a closed curve γ on the double cover $\hat{\Sigma}$ of the Riemann surface Σ encircling either a singularity or a turning point. We then define the Voros symbol e^{V_γ} as a formal power series using the integral

$$V_\gamma(\epsilon_1) = \oint_\gamma dz S_{\text{odd}}(z, \epsilon_1). \tag{3.32}$$

The Borel sums of the Voros symbol are then defined as $S_{\pm}[e^{V\gamma}]$. They satisfy the Stokes automorphism formula

$$S_-[e^{V\gamma}] = S_+[e^{V\gamma}](1 + S_+[e^{V\gamma_0}])^{-(\gamma_0, \gamma)} \tag{3.33}$$

whereby we suppose a simple flip, with the critical Stokes cycle being denoted by γ_0 , and (γ_0, γ) is the intersection number of the critical cycle with the cycle γ defining the Voros symbol. The resummations S_{\pm} are the Borel resummations of the Voros symbol on either side of (and close enough to) the critical graph. The intersection numbers are defined using the convention that, if the cycle γ_1 has the arrow pointing outwards in the positive x direction and the cycle γ_2 , which crosses γ_1 , with the arrow pointing towards the upper half-plane, then $(\gamma_1, \gamma_2) = +1$. When we have Borel sums on either side of a pop rather than a flip, the Voros symbols (importantly, associated to closed cycles) are trivially related

$$S_-[e^{V\gamma}] = S_+[e^{V\gamma}]. \tag{3.34}$$

These two theorems govern the transformation of Voros symbols associated to closed cycles. In the next section we will frequently use results of these theorems to study global properties of our differential equations.

In the case of the conformal $SU(2)$ gauge theory (with $N_f = 4$ flavours) however, the extra assumptions of [31] are not always fully satisfied. In particular, in this case we find that pairs of Stokes graphs that are related by a simultaneous or double flip, excluded in [31]. When such a double flip occurs, we show that the formulae for the Stokes automorphisms derived for single flips compose without change to give the Stokes automorphism for the double flip. This is an extension of the results of [31]. We will discuss this case in detail in the next section.

4 The monodromy group

In this section, we study global properties of the differential equations derived in section 2. The differential equations are second order and hence have two linearly independent global solutions. The solutions undergo a monodromy as we analytically continue them around a singular point. The monodromies, defined up to a change of basis, form a group called the monodromy group. The monodromy group of the differential equations we consider can be expressed entirely in terms of two sets of quantities: (i) the characteristic exponents ν_k at the singular points s_k and (ii) the Borel resummed contour integrals of the WKB differential S_{odd} around branch cuts, which we denote by u_{ij} .

The connection formulae which relate the Borel resummed wave functions in the various Stokes regions are sufficient to completely determine the monodromy group associated to the relevant null vector decoupling equation. The Borel resummed exact WKB contour integrals depend on the Borel resummation angle, (equivalently, on the phase of ϵ_1) and undergo Stokes automorphisms as a function of these parameters. Thus, the expression of the monodromy group in terms of the resummed integrals varies, and we determine the explicit transformation rules as we pass through a critical graph. In this section, we

calculate the monodromy groups, starting with the simplest case of zero flavours, with no regular singular points in the differential equation, and we end with the conformal case ($N_f = 4$) which has four regular singular points.

We stress the fact that there is a dictionary between the Borel resummation angle θ , and the phase of the zeroth order differential which is determined by the phase of ϵ_1 in our set-up. (See e.g. [31] for the details, which follow from the definition of the Borel sum.) We see that this dictionary is given a natural home in ϵ_1 -deformed $\mathcal{N} = 2$ gauge theories. The formal dependence on the Borel resummation angle that induces the Stokes automorphism, has a physical counterpart in the dependence of all non-perturbatively resummed monodromies on the phase of the deformation parameter ϵ_1 .

A brief summary of our analysis. Throughout this section, we perform the calculation of the monodromy group in a strong coupling regime. In all the examples, we will plot the Stokes graphs emphasizing the connectivity of the graphs and the choice of branch cuts; we refer to [32] for various possible sequences of Stokes graphs. To illustrate the detailed coding of the monodromy group in terms of the characteristic exponents and the resummed monodromies, as well as the ambiguity of their formal expression in terms of the monodromies, we calculate the monodromy groups associated to two distinct Stokes graphs. Equating the invariants constructed from the monodromy groups of the two graphs gives us the Stokes automorphism relating the variables in each description. We will thus find concrete descriptions of the monodromy group, as well as the Stokes automorphisms that the exact WKB parameters undergo. The Stokes automorphisms must satisfy the theorems of [31] and this fact serves as a consistency check of our analysis.

4.1 Pure super Yang-Mills

The semi-classical null vector decoupling equation corresponding to the case of pure super Yang-Mills theory has been discussed in detail in [33]. The description was mostly in terms of variables that resulted after mapping the sphere onto a cylinder, such that the differential equation became the Mathieu equation, and the monodromy group was coded in the Floquet exponent. Below, we perform an equivalent analysis on the sphere, which will prepare us to include flavours. A WKB analysis of the Mathieu equation can be found in [38, 39] and further in [40] in the context of exact WKB and the 2d/4d correspondence.

The Stokes graph only depends on the leading potential term $Q_0(z)$. For the pure $\mathcal{N} = 2$ super Yang-Mills theory, the zeroth order term is given by (2.24)

$$Q_0(z) = \frac{\Lambda_0^2}{z^3(z-1)^2} + \frac{\tilde{u}}{z^2(z-1)^2} + \frac{\Lambda_0^2}{z(z-1)^3}. \tag{4.1}$$

In figure 5, we exhibit various Stokes graphs in the strong coupling region of the pure super Yang-Mills theory.⁶ We first draw the critical graph 5 that has a finite WKB line

⁶Using the form of the $N_f = 0$ differential as in [32], we are within the strong coupling region if we make the choice $\Lambda = 1$ and $u = 1/2$. A series of conformal transformations and rescalings relate the differential presented here and the one presented in [32]. At the end of this series of transformations, we are led to the choice of parameters presented in the caption of figure 5.

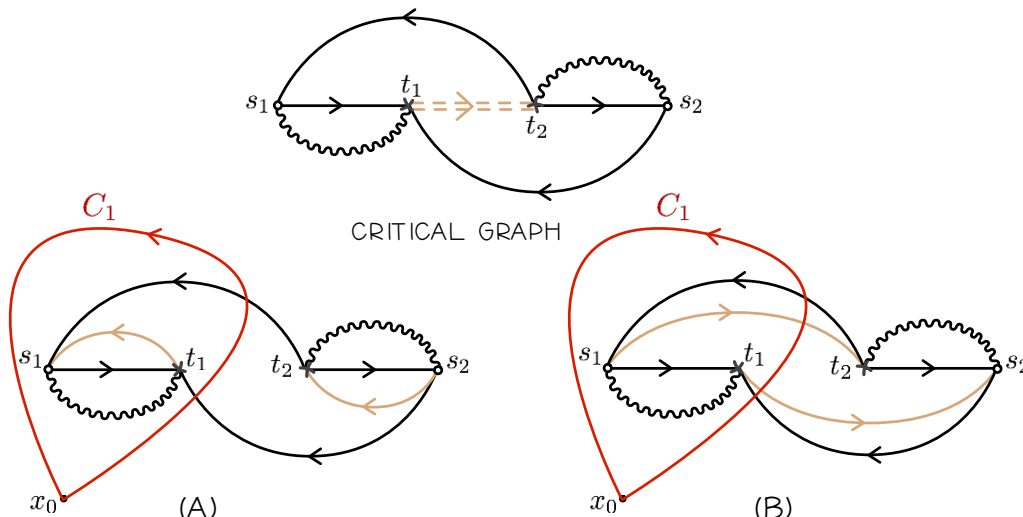


Figure 5. The two Stokes graphs of the $N_f = 0$ case that are related by a simple flip. We also exhibit the contour used to calculate the monodromy matrix. These graphs were obtained with the parameters $\Lambda_0 = e^{-i\frac{\pi}{4}}$ and $\tilde{u} = -1 + i$, with the critical graph observed at $\theta = \pi$.

connecting the turning points t_1 and t_2 . The Stokes graphs we work with are related by a flip [32] about this finite WKB line.

The monodromy group. In this case, there is a single independent generator of the monodromy group and we choose the contour enclosing the singularity s_1 and the turning point t_1 to be it. Consider first the Stokes graph 5(A). The contour intersects two Stokes lines; the monodromy matrix is given by

$$M_{A,s_1} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & +iu_1^{-1} \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

Note that the matrices are written from right to left as we go around the branch cut. The final matrix encodes the overall normalization factor as we return to the base point x_0 . The variable x which appears there is identified with the overall monodromy around the branch cut connecting s_1 and t_1 .

We now turn to the second Stokes graph 5(B). We see that the contour intersects four Stokes lines, including two lines arising from the flip. The monodromy matrix is given by

$$\begin{aligned} M_{B,s_1} &= \begin{pmatrix} \tilde{x} & 0 \\ 0 & \tilde{x}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i\tilde{u}_2^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ +i\tilde{u}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & +i\tilde{u}_1^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{x}(\tilde{u}_1 + \tilde{u}_2) & i\tilde{x} \\ \tilde{u}_2 & \tilde{u}_1 \\ -i\frac{\tilde{u}_2}{\tilde{x}} & \frac{\tilde{u}_2}{\tilde{x}\tilde{u}_1} \end{pmatrix}. \end{aligned} \tag{4.3}$$

We have denoted the variables in Stokes graph 5(B) by variables with tildes since they correspond to a different Borel resummation. The monodromy matrix M_{A,s_1} must be equivalent to the monodromy matrix calculated on the basis of graph (B), since the monodromy (equivalence class) is a property of the exact solutions on the Riemann surface Σ .

The Stokes automorphism. Above, we have the explicit expressions for the monodromy matrices for the two Stokes graphs. The independent Stokes variables are given by x and u_{21} in graph 5(A) and the tilde-variables in graph 5(B). Using this notation, we calculate the conjugation invariant traces of the two monodromy matrices:

$$\text{Tr} (M_{A,s_1}) = x + \frac{1}{x} + \frac{1}{u_{21}x} \tag{4.4}$$

$$\text{Tr} (M_{B,s_1}) = \tilde{x} + \tilde{u}_{21}\tilde{x} + \frac{1}{\tilde{u}_{21}\tilde{x}}. \tag{4.5}$$

Requiring that the traces of the two monodromy matrices match leads to the map between the parameters appearing in the two graphs:

$$\begin{aligned} u_{21} &= \tilde{u}_{21} \\ x &= \tilde{x}(1 + \tilde{u}_{21}). \end{aligned} \tag{4.6}$$

This agrees with the Stokes automorphisms derived in [33]. This is also consistent with the general analysis in [31]. Let us expand on this briefly: the two Stokes graphs lie on either side of the $t_1 - t_2$ flip in the critical graph 5. Since the $t_1 - t_2$ cycle corresponding to u_{12} has zero intersection number with itself, the variable u_{12} is unaffected by the flip. However, the x variable changes because the contour around the branch cut has intersection number 1 with the $t_1 - t_2$ cycle.

4.2 One flavour

The Stokes graphs corresponding to the differential in the case of one flavour are determined by the corresponding zeroth order differential (2.23):

$$Q_0(z) = \frac{\Lambda_1^2}{4z^4} + \frac{\Lambda_1 m_1}{z^3(z-1)^2} - \frac{\Lambda_1^2}{4z(z-1)^3} + \frac{\tilde{u}}{z^2(z-1)^2}. \tag{4.7}$$

From the form of the differential, one can see that in the z -plane, there are three turning points and two irregular singularities (at $z = 0$ and $z = 1$). The WKB triangulations are given in figure 61 of [32]. We consider a particular pair that are separated by a flip⁷ and draw only the corresponding Stokes graphs. The two Stokes graphs correspond to a flip about the $t_2 - t_3$ finite Stokes line in the critical graph (see figure 6).

The monodromy group. We proceed to calculate the monodromy group for both the Stokes graphs. There are two irregular singularities in the graphs and one expects two independent generators of the monodromy group. We choose the two corresponding generators of the monodromy group as shown in figure 6. The contour around the singularity s_2 is treated in much the same way as the irregular singular point in the $N_f = 0$ case, while the singularity s_1 behaves slightly differently.

⁷These are the first and the third out of the six triangulations given in figure 61 of [32].

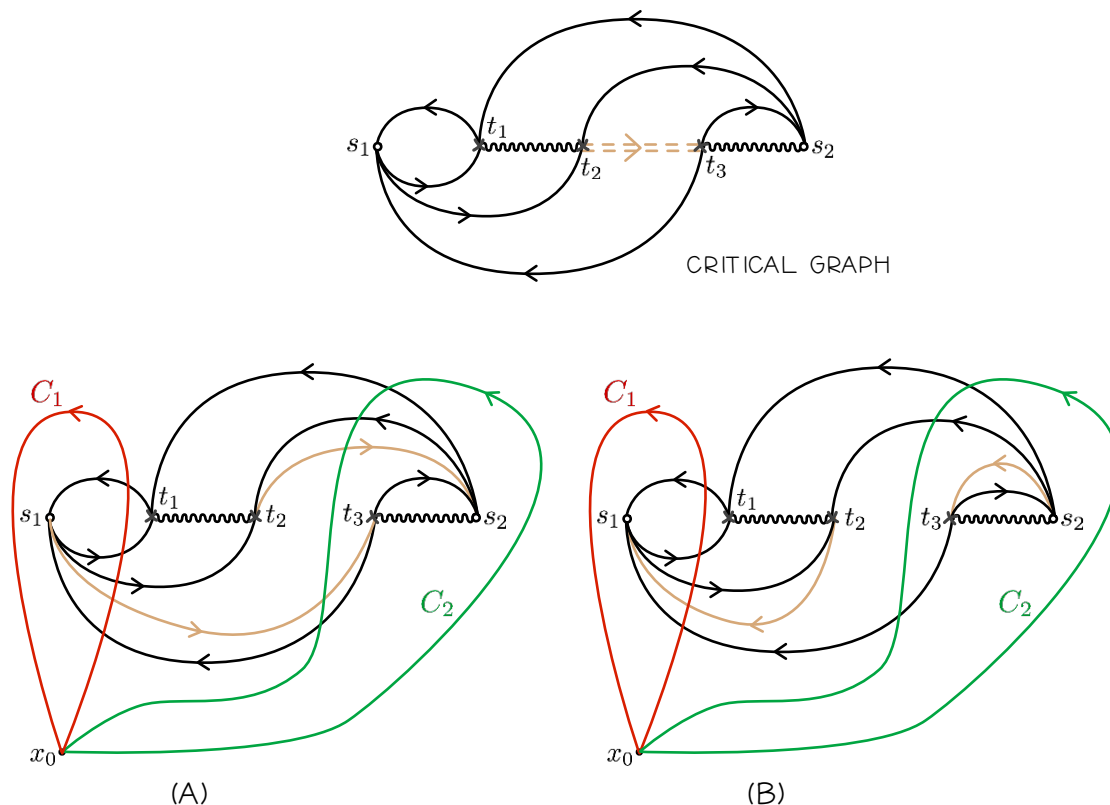


Figure 6. The critical graph, the pair of Stokes graphs related by a flip and the contours that define the monodromy group for the $N_f = 1$ case. The parameters chosen were $\Lambda_1 = 2$, $\tilde{u} = -1/2$, and $m_1 = 1$, and the critical graph was observed at $\theta = \pi$.

Let us first consider Stokes graph 6(A) and calculate the monodromy matrices; we find

$$\begin{aligned}
 M_{A,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_1} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_3} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_3 & 1 \end{pmatrix}, \\
 M_{A,s_2} &= \begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i u_3 x^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_3 x^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_2 x^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_2 x^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_1 u_{12}^2 x^2} \\ 0 & 1 \end{pmatrix}.
 \end{aligned}
 \tag{4.8}$$

The matrix element on the extreme left in the second monodromy matrix is the naive WKB monodromy around the branch cut connecting t_3 and s_2 . This contribution x satisfies the relation,

$$x u_{12} \nu_1 = 1. \tag{4.9}$$

Let us now turn to the Stokes graph 6(B). The monodromy matrices are given by

$$\begin{aligned}
 M_{B,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i \tilde{u}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{\tilde{u}_1} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{\tilde{u}_2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i \tilde{u}_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i \tilde{u}_3 & 1 \end{pmatrix}, \\
 M_{B,s_2} &= \begin{pmatrix} \tilde{x} & 0 \\ 0 & \frac{1}{\tilde{x}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i \tilde{u}_3 \tilde{x}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_2 \tilde{x}^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_1 \tilde{u}_{12}^2 \tilde{x}^2} \\ 0 & 1 \end{pmatrix}.
 \end{aligned}
 \tag{4.10}$$

As before, we define $\tilde{x} = \frac{1}{\tilde{u}_{12} \nu_1}$.

The Stokes automorphism. Now that we have the two sets of monodromy matrices, we calculate the traces of the two sets and obtain

$$\begin{aligned}
 \text{Tr}(M_{A,s_1}) &= -\nu_1 \left(\frac{u_3}{u_1} + \frac{u_3}{u_2} \right) - \frac{1}{\nu_1} \left(\frac{u_1}{u_2} + \frac{u_1}{u_3} \right), \\
 \text{Tr}(M_{A,s_2}) &= \frac{1}{\nu_1} (u_{31} + u_{21}) + (u_{13} + u_{23})\nu_1, \\
 \text{Tr}(M_{B,s_1}) &= -\nu_1 \left(\frac{\tilde{u}_2}{\tilde{u}_1} + \frac{\tilde{u}_3}{\tilde{u}_2} + \frac{\tilde{u}_3}{\tilde{u}_1} \right) - \frac{1}{\nu_1} \frac{u_1}{u_2}, \\
 \text{Tr}(M_{B,s_2}) &= \frac{1}{\nu_1} (\tilde{u}_{21}) + (\tilde{u}_{12} + \tilde{u}_{13} + \tilde{u}_{23})\nu_1.
 \end{aligned} \tag{4.11}$$

Substituting $u_{13} = u_{12}u_{23}$, and equating the expressions for the traces in powers of ν_1 (where we use the fact that the characteristic exponents are true invariants of the differential equation), we can extract the Stokes automorphism formulae for the independent contour integrals u_{21} and u_{23} , namely:

$$\begin{aligned}
 \tilde{u}_{23} &= u_{23}, \\
 \tilde{u}_{21} &= u_{21}(1 + u_{32}).
 \end{aligned} \tag{4.12}$$

Since there is more than one generator of the monodromy group, one can calculate higher-order invariants by calculating traces of products of the matrices. Using the Stokes automorphism, one can check that the trace of the products also coincide, thus confirming the identification of the monodromy group.

4.3 Two flavours

In this section, we consider the $SU(2)$ gauge theory with two flavours. We concentrate on the asymmetric configuration. The zeroth order potential function is given by (2.21)

$$Q_0(z) = \frac{(m_3 + m_4)^2}{4(z-1)^2} + \frac{m_3 m_4}{z(1-z)} + \frac{\Lambda_2^2}{z^3} + \frac{\tilde{u}}{z^2 - z^3}. \tag{4.13}$$

In the z -plane, the quadratic differential has three singularities, and three turning points. One of these is an irregular singularity at $z = 0$. As before, we work in a strong coupling limit, where $\tilde{u} \ll \Lambda_2^2$. We consider the critical graph (see figure 7), and by a flip about the $t_1 - t_3$ finite WKB line, obtain the two Stokes graphs, as shown in the figure. An important difference from the earlier cases is that we have regular singularities at s_1 and s_2 .

The monodromy group. In order to calculate the monodromy group, we first consider the Stokes graph 7(A) and determine the generators

$$\begin{aligned}
 M_{A,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_2 \nu_1^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_2 \nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_3 \nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_1 \nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_2} \\ 0 & 1 \end{pmatrix}, \\
 M_{A,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_3 u_{32}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i u_2 & 1 \end{pmatrix}, \\
 M_{A,s_3} &= \begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_2 x^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i u_1 \nu_1^2 x^2 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_3 \nu_1^2 x^2} & \\ 0 & 1 \end{pmatrix}.
 \end{aligned} \tag{4.14}$$

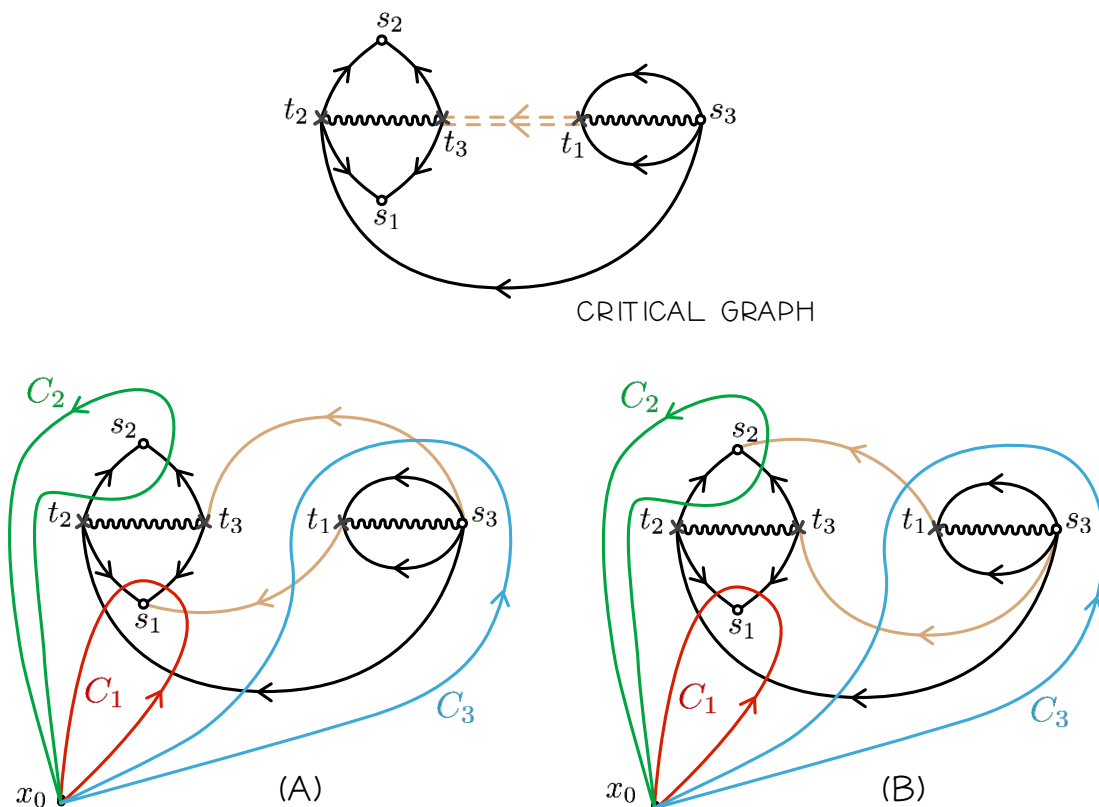


Figure 7. The critical graph and the Stokes graphs for the $N_f = 2$ case. While plotting the figures, we used a potential that is conformally equivalent to (4.13). We set $\Lambda_2 \rightarrow i, \tilde{u} \rightarrow \frac{1}{2}, m_3 \rightarrow 0, m_4 \rightarrow -2$. The two Stokes graphs presented were observed at $\theta = \frac{2\pi}{3}$ and $\theta = \frac{3\pi}{4}$.

In the above, x is the naive WKB monodromy around the branch cut connecting t_1 and s_3 . This contribution satisfies the relation,

$$u_{23} \nu_1 \nu_2 x = 1. \tag{4.15}$$

A similar calculation for Stokes graph 7(B), gives us the following monodromy matrices for circling the singularities

$$\begin{aligned} M_{B,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{\tilde{u}_2 \nu_1^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 \nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 \nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{\tilde{u}_2} \\ 0 & 1 \end{pmatrix}, \\ M_{B,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_1 \tilde{u}_{32}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 \tilde{u}_{32}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 & 1 \end{pmatrix}, \\ M_{B,s_3} &= \begin{pmatrix} \tilde{x} & 0 \\ 0 & \frac{1}{\tilde{x}} \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{\tilde{u}_2 \tilde{x}^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{\tilde{u}_3 \nu_1^2 \tilde{x}^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_1 \nu_1^2 \tilde{x}^2 & 1 \end{pmatrix}. \end{aligned} \tag{4.16}$$

Again we have the relation,

$$u_{23} \nu_1 \nu_2 \tilde{x} = 1. \tag{4.17}$$

Stokes automorphisms. We now compare the traces of the generators of the monodromy group:

$$\begin{aligned}
 \text{Tr } M_{A,s_1} &= \text{Tr } M_{B,s_1} = \nu_1 + \frac{1}{\nu_1}, \\
 \text{Tr } M_{A,s_2} &= \text{Tr } M_{B,s_2} = \nu_2 + \frac{1}{\nu_2}, \\
 \text{Tr } M_{A,s_3} &= \frac{1}{\nu_1\nu_2}u_{32} + \frac{\nu_1}{\nu_2}u_{21}u_{32} + \nu_1\nu_2 \left(u_{21} + \frac{1}{u_{32}} \right), \\
 \text{Tr } M_{B,s_3} &= \frac{1}{\nu_1\nu_2}\tilde{u}_{32}(\tilde{u}_{21}\tilde{u}_{32} + 1) + \frac{\nu_1}{\nu_2}\tilde{u}_{21}\tilde{u}_{32} + \nu_1\nu_2\frac{1}{\tilde{u}_{32}}.
 \end{aligned}
 \tag{4.18}$$

These equations illustrate a recurring feature: the traces of the monodromy matrices around regular singular points will always be given by the critical exponents, with no u_{ij} monodromy factors entering the expression. This is because the Stokes lines are either all going in or coming out at such regular singular points. As a result, the relevant Stokes matrices are all either upper triangular or lower triangular, respectively. This leads to the trivial nature of the trace. The irregular singularity, on the other hand, has non-trivial structure even at the level of the simple traces.

Matching the traces between the graphs 7(A) and 7(B) leads to the Stokes automorphism relations,

$$\begin{aligned}
 u_{31} &= \tilde{u}_{31}, \\
 u_{32} &= \tilde{u}_{32}(1 + \tilde{u}_{31}), \\
 u_{21} &= \tilde{u}_{21}(1 + \tilde{u}_{31})^{-1}.
 \end{aligned}
 \tag{4.19}$$

This is once again as expected from the general results of [31] and the intersection numbers between the various cycles. As a consistency check on the monodromy matrices, we have also computed the traces of products of matrices, and a similar analysis as above confirms the Stokes automorphisms (4.19).

4.4 Three flavours

We move on to the SU(2) theory with three flavours. The Seiberg-Witten differential is

$$Q_0(z) = \frac{(m_3 + m_4)^2}{4(z-1)^2} + \frac{m_3m_4}{z(1-z)} + \frac{m_1\Lambda_3}{z^3} + \frac{\Lambda_3^2}{4z^4} + \frac{\tilde{u}}{z^2 - z^3}.
 \tag{4.20}$$

There are four turning points and three singularities on the z -plane. The two Stokes graphs in figure 8 are related by a flip about the $t_2 - t_3$ finite line in the critical graph.

The monodromy group. For Stokes graph 8(A), we find the generators of the monodromy group:

$$\begin{aligned}
 M_{A,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_3\nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_4\nu_1^2\nu_2^2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_1} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_2} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{i}{u_3} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix}, \\
 M_{A,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix},
 \end{aligned}$$

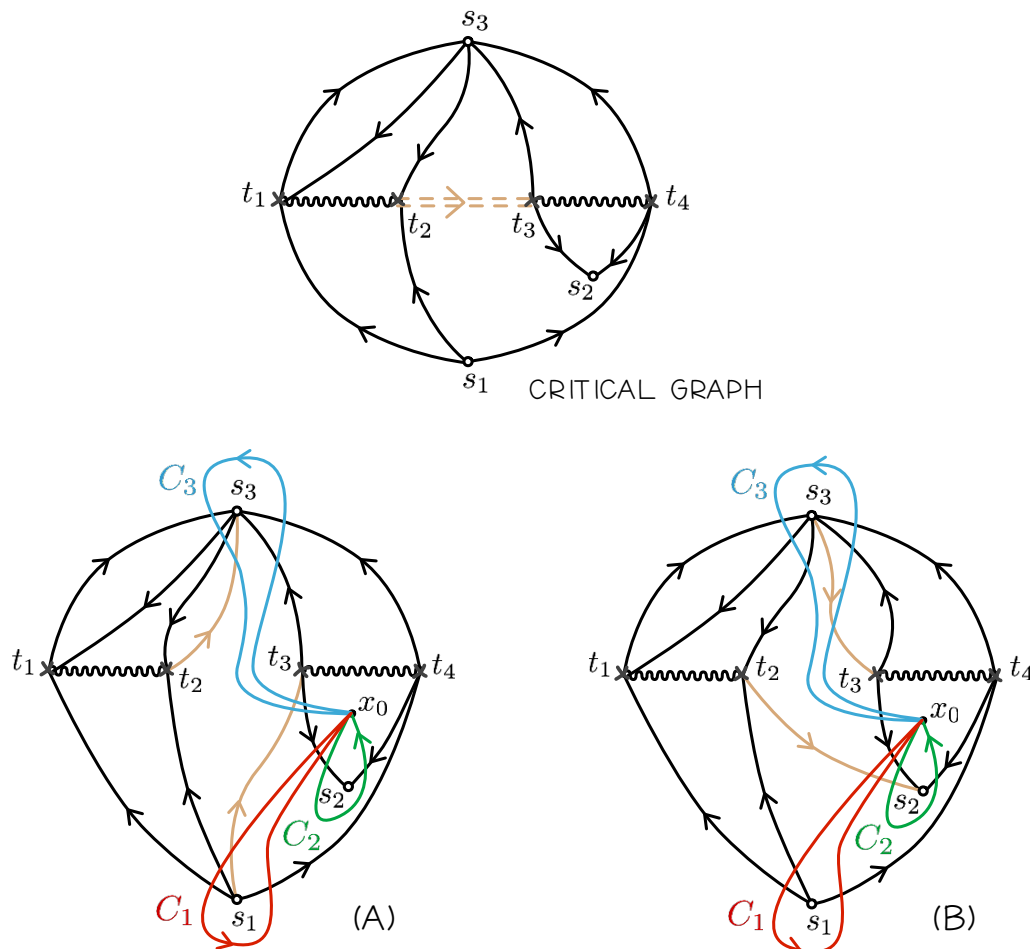


Figure 8. The critical graph and the Stokes graphs for the $N_f = 3$ case. While plotting the figures, we used a potential that is conformally equivalent to (4.20). We set $\Lambda_3 \rightarrow 1, \tilde{u} \rightarrow 2, m_1 \rightarrow -1, m_3 \rightarrow 0, m_4 \rightarrow -2$. The two Stokes graphs presented were observed at $\theta = \frac{\pi}{2}$ and $\theta = \frac{7\pi}{12}$.

$$\begin{aligned}
 M_{A,s_3} = & \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_3\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_3\nu_3^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_2\nu_3^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_1\nu_3^2 u_{12}^2} \\ 0 & 1 \end{pmatrix} \\
 & \times \begin{pmatrix} 1 & 0 \\ -iu_1\nu_3^2 u_{12}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 u_{43}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix}.
 \end{aligned} \tag{4.21}$$

For the Stokes graph 8(B), a similar calculation yields:

$$\begin{aligned}
 M_{B,s_1} = & \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_3\nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_2\nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_4\nu_1^2\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_2} \\ 0 & 1 \end{pmatrix} \\
 & \times \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 M_{B,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix}, \\
 M_{B,s_3} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_3\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_2\nu_3^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_1\nu_3^2\tilde{u}_{12}^2} \\ 0 & 1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_1\nu_3^2\tilde{u}_{12}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4\tilde{u}_{43}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix}.
 \end{aligned} \tag{4.22}$$

The Stokes automorphism. As in the previous examples, the Stokes automorphism can be obtained by comparing the invariants built out of the monodromy matrices. The trace of the monodromy around the irregular singular point s_3 is non-trivial and it is important that we express it in terms of independent Stokes variables. The variables are constrained by the relation

$$u_{12}u_{34}\nu_1\nu_2\nu_3 = 1, \tag{4.23}$$

and similarly for the \tilde{u} variables. We solve for u_{34} using this relation and choose the independent variables to be u_{12} and u_{23} . In terms of these variables, we find that

$$\text{Tr } M_{A,s_3} = -\nu_3 u_{12} - \nu_1 \nu_2 u_{23} (1 + u_{12}) - \frac{1}{\nu_3 u_{12}} (1 + u_{23} + u_{12} u_{23}). \tag{4.24}$$

Similarly, from the monodromy around s_3 in Stokes graph 8(B) we find

$$\text{Tr } M_{B,s_3} = -\nu_3 \tilde{u}_{12} \left(\frac{1}{\tilde{u}_{23}} + 1 \right) - \nu_1 \nu_2 (\tilde{u}_{12} + \tilde{u}_{23} + \tilde{u}_{12} \tilde{u}_{23}) - \frac{\tilde{u}_{23}}{\nu_3 \tilde{u}_{12}} (1 + \tilde{u}_{12}). \tag{4.25}$$

Matching the traces leads to the Stokes automorphisms:

$$\begin{aligned}
 u_{23} &= \tilde{u}_{23}, \\
 u_{12} &= \tilde{u}_{12} \left(1 + \frac{1}{\tilde{u}_{23}} \right).
 \end{aligned} \tag{4.26}$$

As a check of our monodromy matrices, we computed the traces of the products of the monodromy matrices. These imply the same Stokes automorphism as above.

4.5 The conformal theory

We consider Stokes graphs in a strong coupling region of the conformal $SU(2)$ theory. The zeroth order potential has four regular singular points and four turning points.

In particular, we consider the Stokes graphs corresponding to two out of the six triangulations in figure 74 of [32] (see figure 9(A) and (C)).⁸ It can be seen that the two Stokes graphs are related by a double flip, a simultaneous flip about the $t_1 - t_3$ and $t_2 - t_4$ finite WKB lines in the critical graph. We realize the double flip as two alternative sequences of two single flips. We provide the corresponding intermediate graphs after the single flips and perform the calculation we have familiarized ourselves with by now.

Let us first consider the flip from Stokes graph 9(A) to 9(B') via the $t_1 - t_3$ flip. The relevant Stokes graphs and contours that generate the monodromy group are given in figure 10.

⁸Our graphs are topologically equivalent to those appearing in [32].

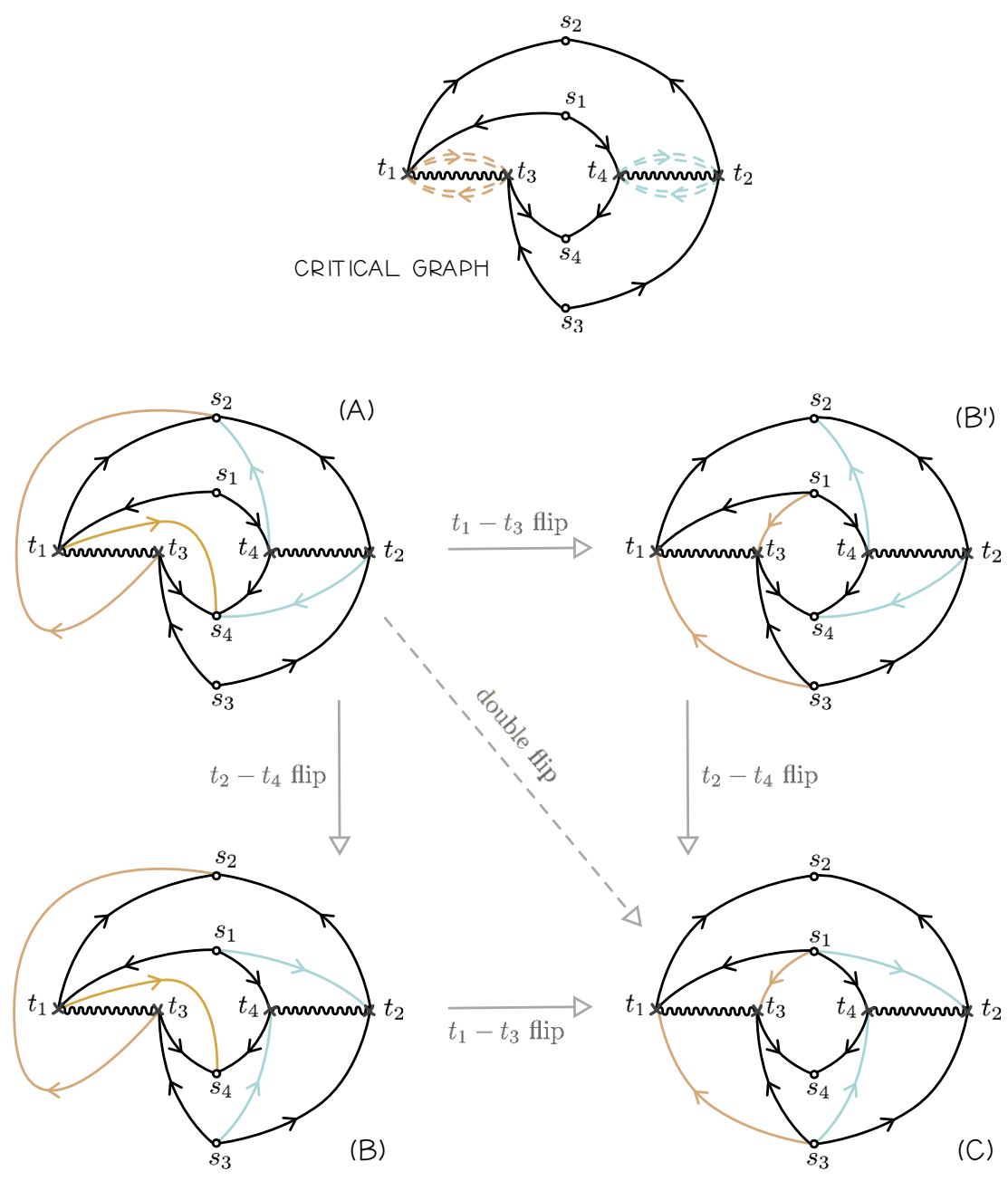


Figure 9. The critical graph and a pair of Stokes graphs for the conformal $SU(2)$ theory. A double flip relates one graph to the other. We refer the reader to [B.1](#) for details.

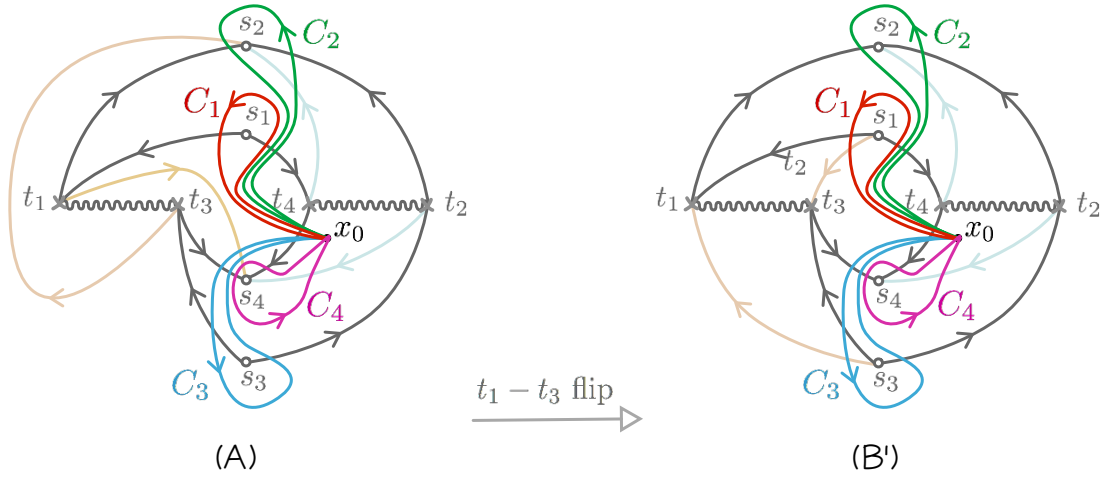


Figure 10. The two Stokes graphs related by the $t_1 - t_3$ flip.

The monodromy group. For Stokes graph 9(A), using the contours as shown in figure 10(A), the monodromy matrices are given by

$$M_{A,s_1} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_4\nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_1 u_{13}^2 \nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_4} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}, \quad (4.27)$$

$$M_{A,s_2} = \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_4\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_4\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1 u_{13}^2 \nu_1^2 \nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 u_{13}^2 \nu_1^2 \nu_2^2 & 1 \end{pmatrix} \\ \times \begin{pmatrix} 1 & 0 \\ -iu_2 u_{24}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_4} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}, \quad (4.28)$$

$$M_{A,s_3} = \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_4\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_1 u_{13}^2 \nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_3 \nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_2 \nu_3^2 \nu_4^2} \\ 0 & 1 \end{pmatrix} \\ \times \begin{pmatrix} 1 & -\frac{i}{u_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1 u_{13}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}, \quad (4.29)$$

$$M_{A,s_4} = \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2 \nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1 u_{13}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}. \quad (4.30)$$

Next we consider the Stokes graph 9(B') and move along the contours C_k around the singularities as shown in figure 10(B'). We compute the monodromy matrices:

$$M_{B',s_1} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_4\nu_1^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_3 \nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_1 \tilde{u}_{13}^2 \nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_4} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4 & 1 \end{pmatrix}, \\ M_{B',s_2} = \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_4\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{\tilde{u}_4 \nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_1 \tilde{u}_{13}^2 \nu_1^2 \nu_2^2 & 1 \end{pmatrix} \\ \times \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2 \tilde{u}_{24}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_4} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4 & 1 \end{pmatrix},$$

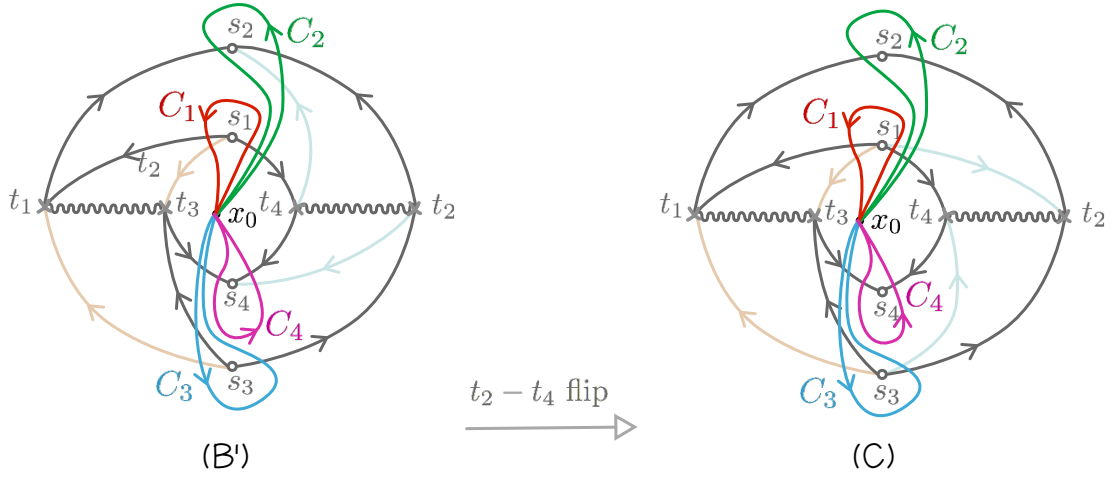


Figure 11. The $t_2 - t_4$ flip and the contours around the singularities.

$$\begin{aligned}
 M_{B',s_3} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_4\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_3\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_2\nu_3^2\nu_4^2} \\ 0 & 1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{\tilde{u}_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4 & 1 \end{pmatrix}, \\
 M_{B',s_4} &= \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2\nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4 & 1 \end{pmatrix}. \tag{4.31}
 \end{aligned}$$

We now implement the $t_2 - t_4$ flip to go from Stokes graph 9(B') to graph 9(C). The relevant Stokes graphs and contours are given in figure 11. Notice that the base point of the contours in figure 11 is different from that used in figure 10, however, this is irrelevant because the monodromy group is independent of the choice of a base point. The monodromy matrices for the (B') graph are given by

$$\begin{aligned}
 M_{B',s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_3\nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_1u_{13}^2\nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_4} \\ 0 & 1 \end{pmatrix}, \\
 M_{B',s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_4\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_1u_{13}^2\nu_1^2\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2u_{24}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_4} \\ 0 & 1 \end{pmatrix}, \\
 M_{B',s_3} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_3\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_2\nu_3^2\nu_4^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix}, \\
 M_{B',s_4} &= \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4\nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix}. \tag{4.32}
 \end{aligned}$$

Finally, we consider Stokes graph 9(C) and calculate the generators of the monodromy group

$$\begin{aligned}
 M_{C,s_1} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_3\nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_1\tilde{u}_{13}\nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_2\tilde{u}_{24}^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_4} \\ 0 & 1 \end{pmatrix}, \\
 M_{C,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{\tilde{u}_4\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{\tilde{u}_2\tilde{u}_{24}\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_1\tilde{u}_{13}^2\nu_1^2\nu_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_2\tilde{u}_{24}^2 & 1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_2\tilde{u}_{24}^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_4} \\ 0 & 1 \end{pmatrix}, \\
 M_{C,s_3} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tilde{u}_3\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_4\nu_3^2\nu_4^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_2\nu_3^2\nu_4^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{\tilde{u}_3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix}, \\
 M_{C,s_4} &= \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_4\nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tilde{u}_3 & 1 \end{pmatrix}. \tag{4.33}
 \end{aligned}$$

The Stokes automorphism for the double flip. The monodromy matrices obtained above by encircling the singularities in both the pairs of graphs in the sequence of flips have standard traces, since all the singularities are regular. Hence, in order to compute the transformation of Voros symbols, we compute the traces of products of monodromy matrices. We express the traces in terms of the variables u_{13} and u_{34} in the t_1-t_3 flip and in terms of the variables u_{42} and u_{21} in the t_2-t_4 flip. Since the entries of the matrices are a bit cumbersome, we merely present the results; here the variables in a given Stokes graph are denoted with the appropriate superscript: this gives us the Stokes automorphism relations.

$$u_{13}^A = u_{13}^{B'}, \quad u_{34}^A = u_{34}^{B'}(1 + u_{13}^{B'}) \tag{4.34}$$

$$u_{42}^{B'} = u_{42}^C, \quad u_{21}^{B'} = u_{21}^C(1 + u_{42}^C) \tag{4.35}$$

Upon composing the Stokes relations from the two single flips, we obtain the desired Stokes relation for the double flip from Stokes graph (A) to (C). If we consider a counter-clockwise loop encircling both the branch cuts and the four singularities in graphs (A), (B') and (C), we get the relation,

$$u_{13}u_{42}\nu_1\nu_2\nu_3\nu_4 = 1 \tag{4.36}$$

There is an analogous condition that is given by

$$u_{43}u_{12}\nu_1\nu_2\nu_3\nu_4 = 1. \tag{4.37}$$

Using these and the Stokes automorphisms for the sequence of flips, we obtain the following relations for the independent variables of the Stokes graphs (A) and (C):

$$\begin{aligned}
 u_{13}^A &= u_{13}^C \\
 u_{34}^A &= u_{34}^C \frac{1 + u_{13}^C}{1 + u_{42}^C}. \tag{4.38}
 \end{aligned}$$

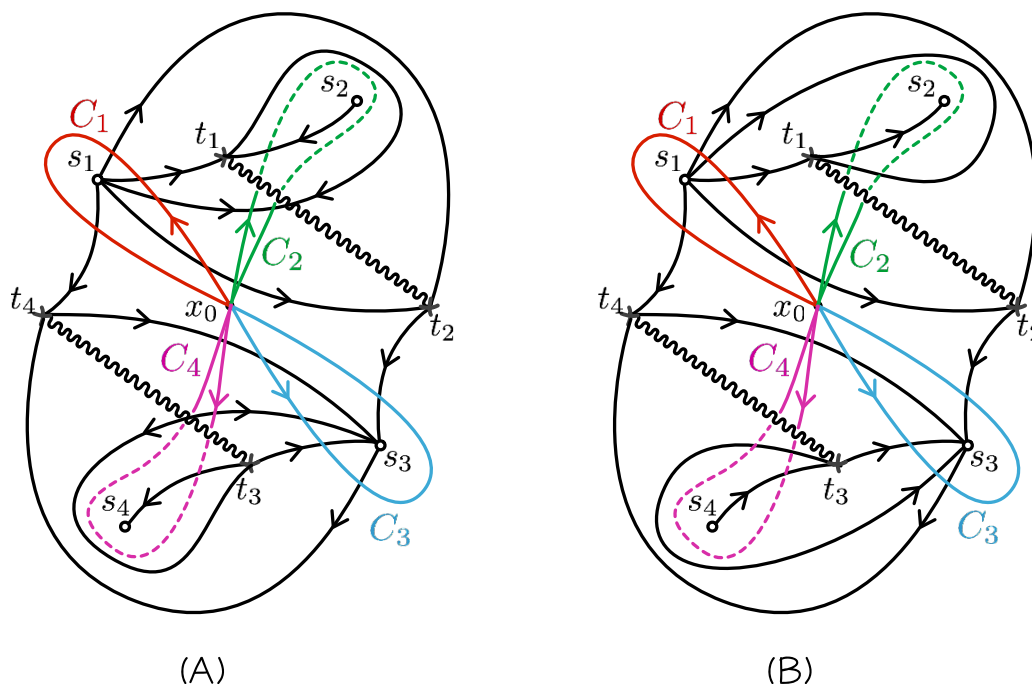


Figure 12. The two Stokes graphs related by a pop. We refer the reader to B.2 for details.

Because the double flip is composed of single flips, each taking place within their own arena, the final result for the double flip is a composition of the result for single flips [31].

So far, we have implemented the double flip via a sequence of two single flips, (A)→(B')→(C) as in figure 9. The double flip can equivalently be implemented by a different sequence of two single flips, (A)→(B)→(C) as shown in figure 9. We have checked that this results in the same Stokes automorphism relations as were arrived at earlier. This is further confirmation of the rules for computing the monodromy groups and of our resolution of the double flip into two single flips.

Pops. Finally, we consider two Stokes graphs that are related by a pop rather than a flip, in the conformal gauge theory. We concentrate on the situation depicted in figure 12 (see appendix B for more details). This corresponds to the degenerate triangulations in figure 74 of [32]. The pop is expected to give rise to a trivial Stokes automorphism for our closed loop Voros symbols. We first consider graph 12(A) and calculate the generators of the monodromy group. As the closed contour goes around s_1 counter-clockwise,

$$M_{A,s_1} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_4 \nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i \nu_2^2}{u_2 u_{21}^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_1} \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{i \nu_2^2}{u_1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_2} \\ 0 & 1 \end{pmatrix}}. \quad (4.39)$$

Next we compute the monodromy matrix as we traverse a closed contour that goes around the singularity s_2

$$\begin{aligned}
 M_{A,s_2} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_2\nu_2^2} \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{i}{u_1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & iu_1 \\ 0 & 1 \end{pmatrix}}_{(4.40)} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
 &\times \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{i\nu_2^2}{u_1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{u_2} \\ 0 & 1 \end{pmatrix}}_{(4.40)}.
 \end{aligned}$$

Around the singularity s_3 and s_4 , we find

$$\begin{aligned}
 M_{A,s_3} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{iu_4u_{34}^2}{\nu_4^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \\
 &\quad \times \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{iu_3}{\nu_4^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}}_{(4.41)}, \\
 M_{A,s_4} &= \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_4\nu_4^2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & iu_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{i}{u_3} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}_{(4.41)} \\
 &\quad \times \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{iu_3}{\nu_4^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}}_{(4.41)}.
 \end{aligned}$$

In the above, the set of matrices clubbed together by an underbrace gives the rule for crossing a Stokes line that runs through a branch cut into the other sheet and approaches the turning point. We repeat the above exercise for Stokes graph 12(B) and the monodromy matrices around the singularities in this case are given by

$$\begin{aligned}
 M_{B_{s_1}} &= \begin{pmatrix} \nu_1 & 0 \\ 0 & \frac{1}{\nu_1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_4\nu_1^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i\nu_2^2}{u_2u_{21}^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i\nu_2^2}{u_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_2} \\ 0 & 1 \end{pmatrix}, \\
 M_{B_{s_2}} &= \begin{pmatrix} \nu_2 & 0 \\ 0 & \frac{1}{\nu_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{u_2\nu_2^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & -iu_1\nu_2^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{i}{u_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{-i}{u_2} \\ 0 & 1 \end{pmatrix}, \\
 M_{B_{s_3}} &= \begin{pmatrix} \nu_3 & 0 \\ 0 & \frac{1}{\nu_3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_2\nu_3^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{iu_4u_{34}^2}{\nu_4^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-iu_3}{\nu_4^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}, \\
 M_{B_{s_4}} &= \begin{pmatrix} \nu_4 & 0 \\ 0 & \frac{1}{\nu_4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ iu_4\nu_4^2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-i}{u_3\nu_4^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & iu_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{i}{u_3} & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -iu_4 & 1 \end{pmatrix}.
 \end{aligned}
 \tag{4.42}$$

Again, it can be checked that the monodromy matrices have standard traces in both the graphs. Similarly it can be checked that the product of matrices from both the graphs have the same traces. Thus, we have a consistency check on our calculation, which is the triviality of the Stokes automorphism acting on the Voros symbols of graphs related by a pop (3.34).

5 The gauge theory perspective

We have computed the monodromy groups associated to the differential equations governing the instanton partition function with surface operator insertion in terms of the exponents ν_i characterizing the singularities, and the Borel resummed monodromies u_{ij} . The characteristic exponents are readily calculated in terms of the masses of the gauge theory using the explicit expression of the differential equation. In this section, we further relate the exact WKB parameters u_{ij} with the Seiberg-Witten periods a and a_D , which are leading order approximations. As a result, we obtain the monodromy groups in terms of the (deformed, resummed) gauge theory data. Next, we present ideas on how to exploit this information to obtain non-perturbative corrections in ϵ_1 to the prepotential.

5.1 The Seiberg-Witten variables

We start by relating the monodromies u_{ij} to more standard Seiberg-Witten data. In the following figures, we mark only the turning points and the singularities on the Riemann surface, and identify the $\hat{\alpha}$ and $\hat{\beta}$ cycles of the genus one Seiberg-Witten curve. For the conformal case [37], the cycles are identified as in figure 13. The cycles have a smooth limit when the masses are set to zero, i.e. when the turning points coincide with the singularities. Further, in [37], it was explicitly checked that the prepotential of the conformal theory, obtained by calculating the period integrals with this choice of cycles, matches the results from equivariant localization methods. Once this identification is made, it is possible to go down in the number of flavors sequentially, each time identifying the $\hat{\alpha}$ and $\hat{\beta}$ cycles, until we finally reach the $N_f = 0$ theory, where we obtain agreement with the results of [33].

The conformal theory. From figure 13, we read off the identification

$$u_{13}\nu_1\nu_3 = e^a, \quad u_{34}\nu_3\nu_4 = e^{a_D}. \quad (5.1)$$

The Stokes automorphisms we have derived for the conformal theory then imply the following relations for the gauge theory variables between the Stokes graphs 9(A) and 9(C):

$$\begin{aligned} (e^a)_A &= (e^a)_C, \\ (e^{a_D})_A &= (e^{a_D})_C \left[\frac{1 + \nu_1^{-1}\nu_3^{-1}(e^a)_C}{1 + \nu_2^{-1}\nu_4^{-1}(e^{-a})_C} \right]. \end{aligned} \quad (5.2)$$

We will comment on the meaning of these relations after we list the cycles and appropriate gauge theory variables for the asymptotically free cases.

Three flavours. From figure 14, for the three flavours case, we find the relation

$$u_{12}\nu_1\nu_2 = e^a, \quad u_{23}\nu_2\nu_3 = e^{a_D}. \quad (5.3)$$

As before we can write the Stokes automorphisms in terms of the gauge theory variables but we suppress these details and only give the choice of cycles in our subsequent examples.

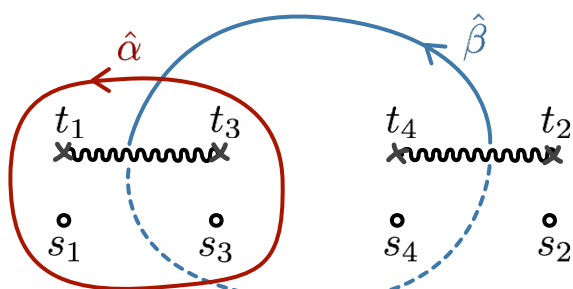


Figure 13. $N_f = 4$ cycles.

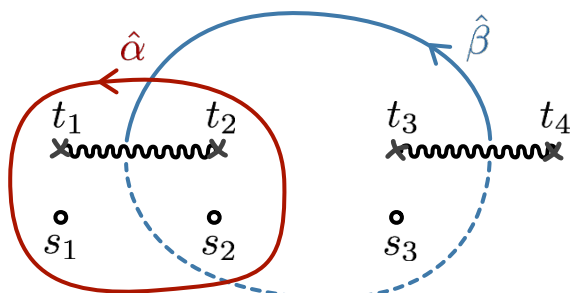


Figure 14. $N_f = 3$ cycles.

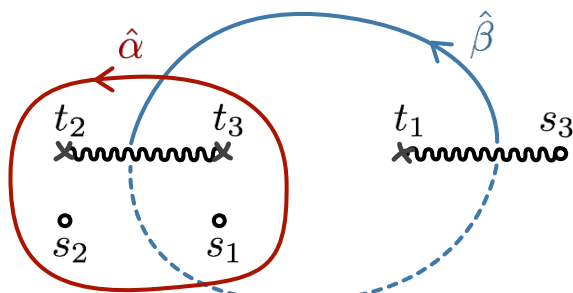


Figure 15. $N_f = 2$ cycles.

Two flavours. For two flavours, from figure 15 we have,

$$u_{23}\nu_1\nu_2 = e^a, \quad u_{31}\nu_1 = e^{aD}. \tag{5.4}$$

One flavour. When we are left with a single flavour, we find from figure 16

$$u_{12}\nu_1 = e^a, \quad u_{23} = e^{aD}. \tag{5.5}$$

Pure super Yang-Mills. Finally, for pure super Yang-Mills, we have from figure 17

$$u_{21} = e^{aD}, \quad x = e^a. \tag{5.6}$$

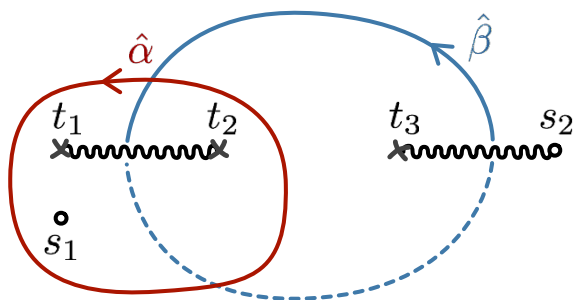


Figure 16. $N_f = 1$ cycles.

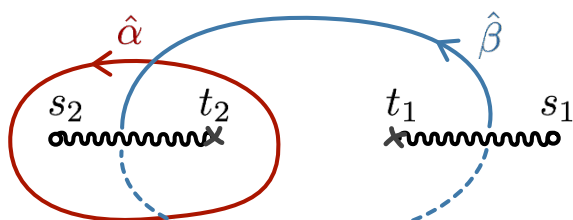


Figure 17. $N_f = 0$ cycles.

The Stokes automorphisms derived in section 4.1 can then be reinterpreted in terms of the gauge theory variables as a relation between a and a_D ,

$$\begin{aligned} e^{a_D} &= e^{\tilde{a}_D}, \\ e^a &= e^{\tilde{a}}(1 + e^{\tilde{a}_D}). \end{aligned} \tag{5.7}$$

One can check that these precisely coincide with the Stokes automorphisms obtained for the pure super Yang-Mills case in [33].

5.2 The non-perturbative prepotential

In this subsection, we restrict ourselves to some preliminary remarks on how to exploit the results we have obtained on the monodromy group to find non-perturbative corrections in ϵ_1 to the prepotential (parameterized in terms of the given exact WKB monodromy data). We note that the expressions $e^{a(\epsilon_1)}$ and $e^{a_D(\epsilon_1)}$ used above (where we have now rendered explicit the ϵ_1 dependence) refer to the exact WKB, Borel resummed ϵ_1 perturbation series. At leading order in ϵ_1 , the first term in the expansion matches the Seiberg-Witten periods $a(0)$ and $a_D(0)$. In the ϵ_1 -deformed theory, these are corrected as a perturbation series in ϵ_1 . In the exact WKB approach, these perturbation series are Borel resummed, but the result depends on the region of Borel resummation, and the resummed expressions $e^{a(\epsilon_1)}$ depend on the phase of Borel resummation, or equivalently, on the phase of ϵ_1 . Thus, the above identifications undergo the Stokes automorphisms of which we have discussed many an example. Note that when we refer to such expressions, we have in mind that they are valid with a given resummation angle.

The monodromy group itself, and in particular the gauge invariant traces of products of monodromy matrices, are exact invariants of the solution to the differential equations.

In [33], it was proposed in the context of the theory with zero flavours to solve the equation for the Borel resummed period $a(\epsilon_1)$ in terms of the invariant quantity a^{exact} , determined by the exact periodicity (or Floquet exponent, or monodromy) of the exact solution to the (Mathieu) differential equation. This solution is valid at a particular resummation angle, and a is identified with a^{exact} , up to non-perturbative corrections. Near a point in moduli space where a perturbative expansion is possible, like the weak, magnetic or dyonic point, we can then solve the relation in terms of a transseries [30]. On the other hand, since in this case $a_D(\epsilon_1)$ is independent of the Borel resummation angle, we can posit $a_D^{\text{exact}} = a_D(\epsilon_1)$ (independently of the specification of the resummation angle). The variable u determining the derivative of the prepotential is known as a function of a , and therefore as a function of a^{exact} . Inverting this relation and calculating the dual period integrals allows one to calculate non-perturbative corrections in ϵ_1 to the prepotential \mathcal{F} .

Thus, one application of the identifications above is to attempt to integrate up the non-perturbative relation between the period a and the parameters coding the exact monodromy group, towards non-perturbative corrections to the prepotential \mathcal{F} . The encompassing case in which to execute this program is the conformal $N_f = 4$ theory.

6 Conclusions

In our work, we filled in many details of the connection between conformal field theory and four-dimensional gauge theory which was made handily available starting from the matching of partition functions and correlators in [9]. We applied the technology developed in [10, 13–15, 34] to study in greater detail SU(2) super Yang-Mills theories with a varying number of flavours. We thus provided a complete list of ϵ_i -deformed differential equations satisfied by the five-point conformal block with surface operator insertion, in the irregular conformal block limit. Subsequently, we took a semi-classical limit and analyzed the resulting differential equations with exact WKB methods. We used this technology to give a detailed parameterization of the monodromy groups in terms of Voros symbols and external conformal dimensions or, in the language of gauge theory, in terms of Borel resummed ϵ_1 -deformed Seiberg-Witten periods and masses. The Borel resummed variables depend intrinsically on the resummation angle (and the phase of ϵ_1), and we illustrated how to bridge this ambiguity using Stokes automorphisms, with at least one illustration for each number of flavours.

The Stokes automorphisms we obtained are consistent with the general theorems proved in [31]. The one subtlety arose in the case of the conformal gauge theory: in this case, we analyzed a pair of Stokes graphs that were related by a simultaneous flip along two independent finite WKB lines. In such a case, we showed that the resulting Stokes automorphisms could be obtained by treating the double-flip as a sequence of single flips. We demonstrated consistency of this approach by checking that the final Stokes automorphisms between the Stokes graphs were independent of the order in which the single flips were taken.

We believe it is instructive to develop these most elementary of $\mathcal{N} = 2$ models in still further detail. There are many avenues to explore. As was mentioned in the previous

section, it should be possible to calculate non-perturbative corrections in ϵ_1 to the prepotential, by suitably generalizing the analysis that was done for the case of pure super Yang-Mills [33]. Similarly, it should be informative to match the cluster algebra description of Stokes automorphisms of [31] to the cluster algebra description of the spectrum of BPS states [41]. (See also e.g. [31, 42, 43].)

In the present work, we studied general properties of the Borel resummed wavefunctions of the null vector decoupling equations in the semi-classical limit, without focusing on the specific form of the wave-function. An interesting direction would be to study in greater detail the dependence of the five-point conformal block on the insertion point of the degenerate operator. This should yield non-trivial information about the gauge theory in the presence of a surface operator. Most interestingly, perturbative and non-perturbative corrections in ϵ_2 can now be studied from the differential equation point of view.

All these projects have straightforward extensions, both to the $\mathcal{N} = 2^*$ theory, where the Riemann surface is of genus one [7], as well as to higher rank conformal gauge theories, the simplest of which is super Yang-Mills theory with $SU(N)$ gauge group and $2N$ fundamental flavours. As explained in section 5, one not only needs the description of the monodromy group, but also requires a local expansion of the period integrals in terms of the Coulomb moduli in order to carry out the goal of calculating corrections to the prepotential that are non-perturbative in ϵ_1 . For the $\mathcal{N} = 2^*$ theory with gauge group $SU(2)$, the null vector decoupling equation for the toroidal block has been well studied in the semi-classical limit (see e.g. [18, 44, 45]) and the instanton series for the prepotential has been resummed in terms of modular functions [46]. One can therefore attempt to study the monodromy group along the lines of the present paper and hope to carry out the proposal put forward in section 5. For higher rank $\mathcal{N} = 2^*$ theories, there has been much progress on the gauge theory side in resumming the instanton expansion for the prepotential using modular anomaly equations in deformed gauge theories with arbitrary gauge group [47, 48]. It remains an open problem to reproduce these successes using conformal field theory methods and to do the corresponding WKB analysis.

For the higher rank (and undeformed) SQCD theories, the instanton series has been resummed in a special locus with \mathbb{Z}_N symmetry [49, 50]. From the CFT approach to the problem, one has to work with Toda theory [51] and the corresponding null vector decoupling equations in Toda have been analyzed recently in [52]. In the semi-classical limit, such differential equations can also be derived using a saddle point analysis of the Nekrasov integrand [35] and the resulting deformed Seiberg-Witten curve. It would be interesting to analyze these higher order differential equations using exact WKB methods and make contact with the general approach to these systems using spectral networks [53, 54].

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A The null vector decoupling with irregular blocks

In this section, we derive in detail the null vector decoupling equations that involve irregular conformal blocks. These equations were summarized in different forms in [10, 13]. Our final results are listed in section 2.

A.1 One irregular puncture

The case of $N_f = 4$ is standard and is described in section 2 in sufficient detail. We turn to the conformal block involving an irregular puncture. This involves rendering one flavour very massive.

A.1.1 $N_f = 3$

From the gauge theory point of view, the flavour decoupling is carried out by taking the limit:

$$m_2 \rightarrow \infty \quad q \rightarrow 0 \quad \text{with} \quad \Lambda_3 = q m_2 \quad \text{finite.} \quad (\text{A.1})$$

The parameter Λ_3 has mass-dimension one and is the strong coupling scale of the SU(2) gauge theory with $N_f = 3$. As shown in [15], in the conformal field theory this involves a collision of the regular conformal block at $z = q$ and $z = 0$ and leads to an irregular conformal block at $z = 0$. We now consider the four point conformal block with the insertion of the degenerate field $\Phi_{2,1}(z)$:

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) \mathcal{I}^{(4)}(0) \Phi_{2,1}(z) \rangle. \quad (\text{A.2})$$

The Ward identity for this chiral conformal block can be obtained by considering the five point block relevant for the conformal $N_f = 4$ and scaling the wave-function in the following way [13, 15]:

$$\Psi(z, q) = q^{-2\alpha_1\alpha_2} \psi_3(z, \Lambda_3). \quad (\text{A.3})$$

On the rescaled wave-function $\psi_3(z)$, the q -derivative is traded for a Λ_3 -derivative. Combining these, and taking the decoupling limit, we obtain the null vector decoupling equation for the $N_f = 3$ theory:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3+m_4)^2}{4(z-1)^2} + \frac{m_3 m_4}{z(1-z)} + \frac{m_1 \Lambda_3}{z^3} + \frac{\Lambda_3^2}{4z^4} - \frac{\epsilon_1^2}{4(z-1)^2} + \epsilon_2^2 \frac{(3-4z)}{4z(z-1)^2} \right. \\ \left. + \frac{1}{z^2-z^3} \left(-\epsilon_1 \epsilon_2 \Lambda_3 \frac{\partial}{\partial \Lambda_3} + m_1^2 + m_1 (\epsilon_1 + \epsilon_2) \right) + \epsilon_1 \epsilon_2 \left(\frac{1-2z}{z-z^2} \frac{\partial}{\partial z} + \frac{1-2z}{2z(z-1)^2} \right) \right] \psi_3(z, \Lambda_3) = 0. \quad (\text{A.4})$$

The quartic pole at $z = 0$ in the OPE between the stress tensor and the irregular block explains our notation: we denote such an irregular block by $\mathcal{I}^{(4)}(0)$.

A.1.2 $N_f = 2$: asymmetric realization

One can take a further limit in which we decouple the mass m_1 :

$$m_1 \rightarrow \infty \quad \Lambda_3 \rightarrow 0 \quad \text{with} \quad \Lambda_2^2 = m_1 \Lambda_3 \quad \text{finite.} \quad (\text{A.5})$$

Simultaneously, we rescale the wave-function by

$$\psi_3(z, \Lambda_3) = \Lambda_3^{\frac{1}{\epsilon_1 \epsilon_2} (m_1^2 + m_1 (\epsilon_1 + \epsilon_2))} \psi_2(z, \Lambda_2). \quad (\text{A.6})$$

From the conformal field theory perspective, this amounts to tuning the coefficient of the quartic pole to zero, leaving behind only a cubic pole. This corresponds to the four point conformal block

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) \mathcal{I}^{(3)}(0) \Phi_{2,1}(z) \rangle. \quad (\text{A.7})$$

The null vector decoupling equation takes the form:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{(m_3 + m_4)^2}{4(z-1)^2} + \frac{m_3 m_4}{z(1-z)} + \frac{\Lambda_2^2}{z^3} - \frac{\epsilon_1 \epsilon_2}{2(z^2 - z^3)} \Lambda_2 \frac{\partial}{\partial \Lambda_2} \right. \\ \left. + \epsilon_1 \epsilon_2 \left(\frac{1-2z}{z-z^2} \frac{\partial}{\partial z} + \frac{1-2z}{2z(z-1)^2} \right) - \frac{\epsilon_1^2}{4(z-1)^2} + \epsilon_2^2 \frac{(3-4z)}{4z(z-1)^2} \right] \psi_2(z, \Lambda_2) = 0. \quad (\text{A.8})$$

The equation exhibits a cubic pole at the irregular singularity $z = 0$.

A.2 Two irregular punctures

Let us begin with the five point conformal block in which we have regular conformal primaries at z_i , with $i \in \{1, 2, 3, 4\}$ and the degenerate field at z . In equation (2.5), we have obtained the null vector decoupling equation for this case. We now set $z_1 = 0$, $z_2 = q$ and $z_4 = 1$ and consider the simultaneous collision of punctures such that $q \rightarrow 0$ and $z_3 \rightarrow 1$. We rescale the wave-function as in [15]

$$\Psi(z_i) = q^{-2\alpha_1 \alpha_2} (z_3 - 1)^{-2\alpha_3 \alpha_4} \psi(z, \Lambda, \tilde{\Lambda}). \quad (\text{A.9})$$

In order to get finite results, we take the limit

$$m_2 \rightarrow \infty, \quad q \rightarrow 0 \quad \text{with} \quad \Lambda = q m_2 \quad \text{finite,} \\ m_3 \rightarrow \infty, \quad (z_3 - 1) \rightarrow 0 \quad \text{with} \quad \tilde{\Lambda} = (z_3 - 1) m_3 \quad \text{finite.} \quad (\text{A.10})$$

Note that this introduces two independent scales in the problem. The conformal block we are considering is a three point function, with two irregular punctures of quartic order and a degenerate field $\Phi_{2,1}(z)$:

$$\langle \mathcal{I}^{(4)}(1) \mathcal{I}^{(4)}(0) \Phi_{2,1}(z) \rangle. \quad (\text{A.11})$$

The parameters Λ and $\tilde{\Lambda}$ respectively represent the quartic pole coefficient at $z = 0$ and $z = 1$. After the wave-function rescaling and the decoupling limits, we obtain the null

vector decoupling equation for such a conformal block with two such irregular singularities and the degenerate field:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda^2}{4z^4} + \frac{\Lambda m_1}{z^3(z-1)^2} + \frac{\tilde{\Lambda} m_4}{z(z-1)^3} + \frac{\tilde{\Lambda}^2}{4(z-1)^4} + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{(2\epsilon_1 \epsilon_2 + 3\epsilon_2^2)(2-3z)}{4z(z-1)^2} + \frac{1}{z^2(z-1)^2} \left(-\epsilon_1 \epsilon_2 \Lambda \frac{\partial}{\partial \Lambda} - 2\Lambda m_1 + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) \right] \psi_2(z, \Lambda, \tilde{\Lambda}) = 0. \quad (\text{A.12})$$

We can tune the two scales suitably in order to obtain the null vector decoupling equations for the remaining gauge theories of our focus.

A.2.1 $N_f = 2$: symmetric realization

We set $\Lambda = -\tilde{\Lambda} = \Lambda_2$, in which case we obtain the null vector decoupling equation:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_2^2}{4z^4} + \frac{\Lambda_2 m_1}{z^3(z-1)^2} - \frac{\Lambda_2 m_4}{z(z-1)^3} + \frac{\Lambda_2^2}{4(z-1)^4} + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{(2\epsilon_1 \epsilon_2 + 3\epsilon_2^2)(2-3z)}{4z(z-1)^2} + \frac{1}{z^2(z-1)^2} \left(-\epsilon_1 \epsilon_2 \Lambda_2 \frac{\partial}{\partial \Lambda_2} - 2\Lambda_2 m_1 + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) \right] \psi_2(z, \Lambda_2) = 0. \quad (\text{A.13})$$

A.2.2 $N_f = 1$

We set $\Lambda = \Lambda_1$ and $\tilde{\Lambda} \rightarrow 0$ such that $\tilde{\Lambda} m_4 = -\frac{\Lambda_1^2}{4}$ is a finite combination. In other words, the relevant conformal block of interest is

$$\langle \mathcal{I}^{(3)}(1) \mathcal{I}^{(4)}(0) \Phi_{2,1}(z) \rangle. \quad (\text{A.14})$$

The coefficient of the cubic pole at $z = 1$ is tuned and related to that of the quartic pole at $z = 0$. Such a conformal block satisfies the null vector decoupling equation:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_1^2}{4z^4} + \frac{\Lambda_1 m_1}{z^3(z-1)^2} - \frac{\Lambda_1^2}{4z(z-1)^3} + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{2-3z}{4z(z-1)^2} (2\epsilon_1 \epsilon_2 + 3\epsilon_2^2) + \frac{1}{z^2(z-1)^2} \left(-\epsilon_1 \epsilon_2 \Lambda_1 \frac{\partial}{\partial \Lambda_1} - 2\Lambda_1 m_1 + m_1^2 + m_1(\epsilon_1 + \epsilon_2) \right) \right] \psi_1(z, \Lambda_1) = 0. \quad (\text{A.15})$$

A.2.3 $N_f = 0$

As for the earlier case with $N_f = 2$ in the asymmetric realization, in order to obtain finite results, one has to rescale the wave-function

$$\psi(z, \Lambda, \tilde{\Lambda}) = \Lambda^{\frac{1}{\epsilon_1 \epsilon_2} (m_1^2 + m_1(\epsilon_1 + \epsilon_2))} \psi_0(z, \Lambda_0). \quad (\text{A.16})$$

We set $\Lambda \rightarrow 0$ and $\tilde{\Lambda} \rightarrow 0$ such that the combinations $m_1 \Lambda = \Lambda_0^2$ and $m_4 \tilde{\Lambda} = \Lambda_0^2$ are equal and finite. The relevant conformal block is given by

$$\langle \mathcal{I}^{(3)}(1) \mathcal{I}^{(3)}(0) \Phi_{2,1}(z) \rangle. \quad (\text{A.17})$$

This satisfies the null vector decoupling equation:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \frac{\Lambda_0^2}{z^3(z-1)^2} + \frac{\Lambda_0^2}{z(z-1)^3} + \epsilon_1 \epsilon_2 \frac{3z-1}{z(z-1)} \frac{\partial}{\partial z} + \frac{2-3z}{4z(z-1)^2} (2\epsilon_1 \epsilon_2 + 3\epsilon_2^2) + \frac{1}{z^2(z-1)^2} \left(-\frac{1}{2} \epsilon_1 \epsilon_2 \Lambda_0 \frac{\partial}{\partial \Lambda_0} - 2\Lambda_0^2 \right) \right] \psi_0(z, \Lambda_0) = 0. \tag{A.18}$$

B Stokes graphs

In this section, we plot actual machine-generated Stokes graphs for the $N_f = 4$ theory as representative examples. The Stokes lines are defined by the condition

$$\text{Im} \left[\int_{x_0}^x dx' \sqrt{Q_0(x')} \right] = 0. \tag{B.1}$$

The red dots indicate singularities, and the blue dots indicate turning points. It is important to remember that what is relevant for the monodromy calculations is the topology of the Stokes graph. The cartoons in the body of the paper abstract away from graphs given in this appendix.

B.1 The double flip

The potential we use to plot these Stokes graphs has a convenient \mathbb{Z}_4 symmetric form [32]

$$Q_0(z) = \frac{z^4 - u(z^4 - 1)}{(z^4 - 1)^4}, \tag{B.2}$$

where the singularities are at $z_1 = 1$, $z_2 = i$, $z_3 = -1$, and $z_4 = -i$. This potential is arrived at via an $\text{SL}(2, \mathbb{C})$ transformation of (2.18) that maps all singularities to finite points for ease of plotting. The masses of the fundamental hypermultiplets are $m_a = \frac{1}{4}z_a$. We have made the choice $u = \frac{1}{2}$ for plotting the double flip Stokes graphs in figure 18.

B.2 The pop

We continue to work with the potential (B.2) and, in order visualize the pops in figure 19, we have made the choice $u = \frac{1}{2} \exp\left(\frac{3}{10}i\pi\right)$.

C The saddle point analysis

We discussed regular (irregular) conformal blocks associated to two-dimensional conformal field theories that, via the 2d/4d correspondence, are dual to four-dimensional Ω -deformed conformal (respectively, asymptotically free) gauge theories. The null vector decoupling equations together with the conformal Ward identities allow us to arrive at Schrödinger equations that govern an integrable system related to the Ω -deformed gauge theory. While these analyses are exact in ϵ_2 , it is enlightening to see how the $\epsilon_2 \rightarrow 0$ limit of these

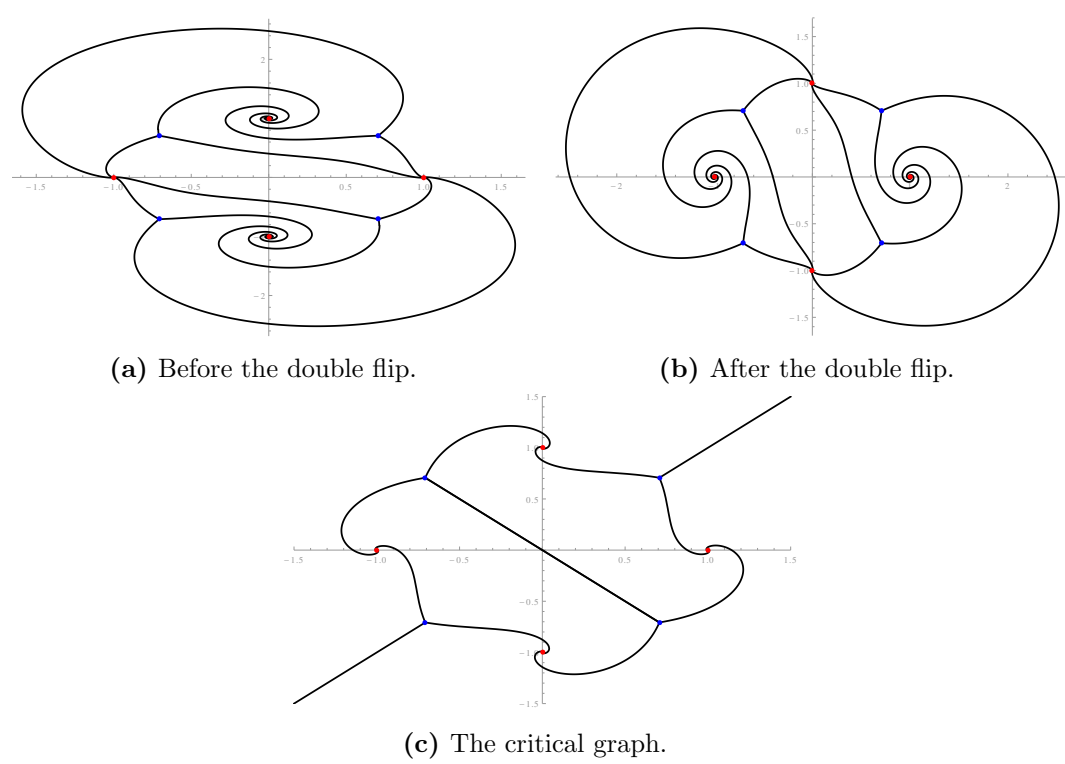


Figure 18. Sequence of Stokes graphs related by a double flip about the critical graph.

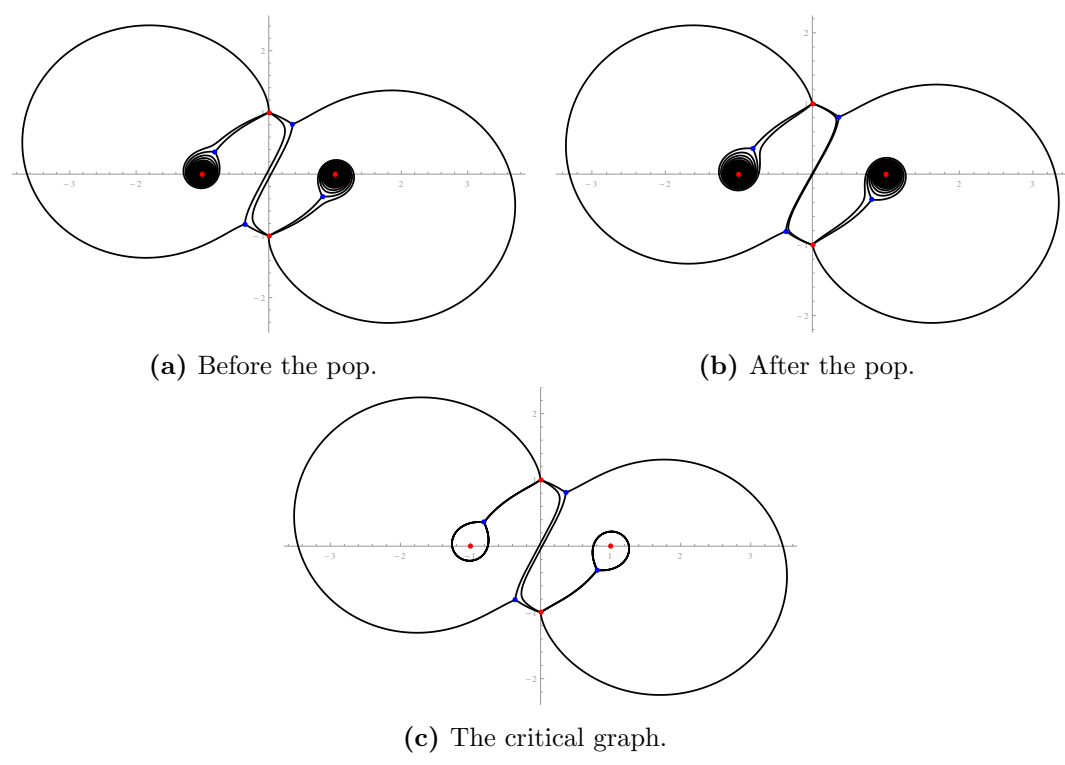


Figure 19. Sequence of Stokes graphs related by a pop about the critical graph.

differential equations are derived from purely gauge-theoretic considerations. This will serve as a consistency check of the 2d/4d correspondence, and our calculations.

In this section, we explain how to derive differential equations starting from saddle-point difference equations, valid for any $SU(N)$ theory with $N_f = 2N$ fundamental hypermultiplets. We specialize to the case of the conformal $SU(2)$ theory ($N_f = 4$) and consider various decoupling limits that give asymptotically free theories with fewer fundamental hypermultiplets.

Our analysis begins with the Nekrasov partition function, and considers its saddle-points in the limit $\epsilon_2 \rightarrow 0$, with ϵ_1 held constant and finite [35, 55–57]. The result of this analysis is the ϵ_1 -deformed Seiberg-Witten equation:

$$y(x) + q \frac{M(x - \epsilon_1)}{y(x - \epsilon_1)} = (1 + q) P(x). \tag{C.1}$$

Here, q is the instanton counting parameter. The gauge polynomial $P(x)$ is of degree N and encodes the Coulomb moduli of the gauge theory. The flavour polynomial $M(x)$ is of degree $2N$, and we choose to factorize it into two pieces:

$$M(x) = A(x)D(x), \tag{C.2}$$

where $A(x)$ and $D(x)$ are degree N polynomials. As should be evident, this decomposition is far from being unique. Different decompositions can be mapped to the different ways in which the flavour symmetry is realized in the type II construction using two NS5 branes and a stack of D4 branes. One can associate the number of semi-infinite D4 branes on each side of the NS5 branes with the degree of the polynomials $A(x)$ and $D(x)$. We now peel off a factor of $A(x)$ from the function $y(x)$ and express the remainder as the ratio of some rational function $\mathbf{Q}(x)$ as

$$y(x) = A(x) \frac{\mathbf{Q}(x)}{\mathbf{Q}(x - \epsilon_1)}. \tag{C.3}$$

There are other ways to perform the split but this one will reduce, in the $\epsilon_1 \rightarrow 0$ limit, to the correct M-theory curve. The result of these decompositions gives us the Baxter $T\mathbf{Q}$ -relation

$$A(x)\mathbf{Q}(x) + qD(x - \epsilon_1)\mathbf{Q}(x - 2\epsilon_1) - (1 + q)P(x)\mathbf{Q}(x - \epsilon_1) = 0. \tag{C.4}$$

We now trade $\mathbf{Q}(x)$ for its Fourier transform [35]

$$\Psi(t) = \sum_{x \in \Gamma} \mathbf{Q}(x) e^{-x/\epsilon_1}, \tag{C.5}$$

and arrive at the differential equation of order N :

$$[A(x) + qD(x - \epsilon_1) t^{-2} - (1 + q)P(x) t^{-1}] \Psi(t) = 0, \tag{C.6}$$

where in the above equation, the Fourier transform effectively sends $x \mapsto -\epsilon_1 t \partial_t$.

C.1 $N_f = 4$

We specialize to the case of the conformal SU(2) gauge theory, with $N_f = 4$. The matter polynomials we start with are

$$D(x) = \left(x - m_1 + \frac{\epsilon_1}{2}\right) \left(x - m_2 + \frac{\epsilon_1}{2}\right), \tag{C.7}$$

$$A(x) = \left(x - m_3 + \frac{\epsilon_1}{2}\right) \left(x - m_4 + \frac{\epsilon_1}{2}\right). \tag{C.8}$$

The gauge polynomial is

$$P(x) = x^2 - q \left(\frac{m_1 + m_2 + m_3 + m_4}{1 + q}\right) x - \frac{u_S}{1 + q}. \tag{C.9}$$

Substituting this into the difference equation and taking the Fourier transform as described earlier leads to a second order differential equation. We would like this differential equation to be of the Schrödinger form, i.e. with no linear derivative terms. This may be achieved by peeling off an appropriate factor.⁹ The resulting differential equation is

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial t^2} + Q(t, \epsilon_1)\right] \Psi(t) = 0, \tag{C.10}$$

where the potential term $Q(t, \epsilon_1)$ has an expansion in powers of ϵ_1 . In addition, we further shift of the Coulomb modulus u_S ,

$$u_S = -\tilde{u} + \frac{q-1}{2} (m_1^2 + m_2^2) + q \left[m_1 m_2 + m_3 m_4 + \frac{1}{2} (m_1 + m_2) (m_3 + m_4) \right] + \frac{\epsilon_1^2}{4} (1 + q) \tag{C.11}$$

Denoting the order ϵ_1^m coefficient in the potential by $Q_m(t)$, we obtain the non-zero potential terms:

$$Q_0(t) = -\frac{\tilde{u}}{t(t-1)(t-q)} + \frac{(m_1 - m_2)^2}{4t^2} + \frac{(m_1 + m_2)^2}{4(t-q)^2} + \frac{(m_3 + m_4)^2}{4(t-1)^2} + \frac{m_1^2 + m_2^2 + 2m_3 m_4}{2t(1-t)}, \tag{C.12}$$

$$Q_2(t) = -\frac{1}{4t^2} - \frac{1}{4(t-1)^2} - \frac{1}{4(t-q)^2} + \frac{1}{2(t-1)(t-q)}. \tag{C.13}$$

This matches the potentials in the text obtained from the null vector decoupling equations (2.18).

C.2 $N_f = 3$

The $N_f = 3$ case is obtained by decoupling one of the masses; we choose here to send $m_2 \rightarrow \infty$, while simultaneously sending $q \rightarrow 0$ such that the combination $\Lambda_3 = q m_2$ remains finite: we identify Λ_3 to be the strong coupling scale in the SU(2) gauge theory with $N_f = 3$. The difference equation takes the form

$$A(x)\mathbf{Q}(x) - \Lambda_3 D(x - \epsilon_1)\mathbf{Q}(x - 2\epsilon_1) - P(x)\mathbf{Q}(x - \epsilon_1) = 0. \tag{C.14}$$

⁹The terms we peel off are proportional to the products of square roots of eigenfunctions of the monodromy at 0, 1 and q .

The polynomial functions are given by

$$D(x) = \left(x - m_1 + \frac{\epsilon_1}{2}\right), \tag{C.15}$$

$$A(x) = \left(x - m_3 + \frac{\epsilon_1}{2}\right) \left(x - m_4 + \frac{\epsilon_1}{2}\right), \tag{C.16}$$

while the gauge polynomial is

$$P(x) = x^2 - \Lambda_3 x - u_S. \tag{C.17}$$

The differential equation can be obtained in a similar fashion as in the conformal case by taking the Fourier transform as in (C.5). The analysis leading to the Schrödinger type equation can be repeated as before and we obtain the differential equation:

$$\left[-\epsilon_1^2 \frac{\partial^2}{\partial t^2} + \sum_{m=0}^2 Q_m(t) \epsilon_1^m\right] \Psi(t) = 0. \tag{C.18}$$

The u_S we use in the saddle-point analysis must be shifted in order to make contact with the form of the potential in the text (2.20) and the shift is given by

$$u_S = \tilde{u} - \frac{\Lambda_3}{2} (2m_1 + m_3 + m_4) + \frac{\epsilon_1^2}{4}. \tag{C.19}$$

The shift of the Coulomb modulus is accompanied by a global rescaling $m_1 \rightarrow \frac{m_1}{2}$ and $\Lambda_3 \rightarrow \frac{\Lambda_3}{2}$. After these shifts and rescalings, the non-zero potential functions $Q_m(t)$ are given by

$$Q_0(t) = \frac{(m_3 + m_4)^2}{4(t-1)^2} + \frac{m_3 m_4}{t(1-t)} + \frac{m_1 \Lambda_3}{t^3} + \frac{\Lambda_3^2}{4t^4} + \frac{\tilde{u}}{t^2(1-t)}, \tag{C.20}$$

$$Q_2(t) = -\frac{1}{4(t-1)^2}. \tag{C.21}$$

which match the potential in equation (2.20).

C.3 $N_f = 2$: asymmetric realization

There are two distinct cases to be considered when $N_f = 2$. These correspond to the fashion in which we decouple fundamental matter. We first consider the case when

$$m_1 \rightarrow \infty \quad \text{and} \quad \Lambda_3 \rightarrow 0 \quad \text{with} \quad \Lambda_2^2 = m_1 \Lambda_3 \quad \text{finite}. \tag{C.22}$$

The difference equation takes the form

$$A(x)\mathbf{Q}(x) + \Lambda_2^2 \mathbf{Q}(x - 2\epsilon_1) - P(x)\mathbf{Q}(x - \epsilon_1) = 0. \tag{C.23}$$

The polynomials are given by

$$A(x) = \left(x - m_3 + \frac{\epsilon_1}{2}\right) \left(x - m_4 + \frac{\epsilon_1}{2}\right), \tag{C.24}$$

while the gauge polynomial is

$$P(x) = x^2 - u_S. \tag{C.25}$$

The differential equation is once again given as in (C.18), and after the shift

$$u_S = \tilde{u} - \Lambda^2 + \frac{\epsilon_1^2}{4}, \tag{C.26}$$

We find that the non-zero potential functions $Q_m(t)$ are

$$Q_0(t) = \frac{(m_3 + m_4)^2}{4(t-1)^2} + \frac{m_3 m_4}{t(1-t)} + \frac{\Lambda_2^2}{t^3} + \frac{\tilde{u}}{t^2(1-t)}, \tag{C.27}$$

$$Q_2(t) = -\frac{1}{4(t-1)^2}. \tag{C.28}$$

which matches the potential in (2.21).

C.4 $N_f = 2$: symmetric realization

An inequivalent way to realize the $N_f = 2$ theory is to start from the $N_f = 3$ differential equation and consider the limit

$$m_3 \rightarrow \infty \quad \text{and} \quad \Lambda_3 \rightarrow 0 \quad \text{with} \quad \Lambda_2^2 = m_3 \Lambda_3 \quad \text{finite}. \tag{C.29}$$

The difference equation takes the form

$$\Lambda_2 A(x) \mathbf{Q}(x) + \Lambda_2 D(x - \epsilon_1) \mathbf{Q}(x - 2\epsilon_1) - P(x) \mathbf{Q}(x - \epsilon_1) = 0. \tag{C.30}$$

The polynomial $A(x)$ is given by

$$D(x) = \left(x - m_1 + \frac{\epsilon_1}{2}\right), \tag{C.31}$$

$$A(x) = \left(x - m_4 + \frac{\epsilon_1}{2}\right), \tag{C.32}$$

while the gauge polynomial is given by (C.25). The differential equation still takes the Schrödinger form (C.18).

Finally, in order to match with the form of the differential in the text (2.22), we need to perform a conformal transformation ($t \rightarrow \frac{z-1}{z}$). We choose to shift away the $O(\epsilon_1^2)$ term by redefining of the Coulomb modulus:

$$u_S = \tilde{u} + m_1 \Lambda_2 - \frac{\Lambda_2^2}{2} + \frac{\epsilon_1^2}{4}. \tag{C.33}$$

After this, the differential matches the form in (2.22), with

$$Q_0(z) = \frac{\Lambda_2^2}{4z^4} + \frac{\Lambda_2 m_1}{z^3(z-1)^2} - \frac{\Lambda_2 m_4}{z(z-1)^3} + \frac{\Lambda_2^2}{4(z-1)^4} + \frac{\tilde{u}}{z^2(z-1)^2}. \tag{C.34}$$

C.5 $N_f = 1$

We start with the symmetric realization of the $N_f = 2$ case and take the limit

$$m_4 \rightarrow \infty \quad \text{and} \quad \Lambda_2 \rightarrow 0 \quad \text{with} \quad \Lambda_1^3 = m_4 \Lambda_2^2 \quad \text{finite.} \quad (\text{C.35})$$

The difference equation in this case takes the form

$$\Lambda_1^2 \mathbf{Q}(x) + \Lambda_1 D(x - \epsilon_1) \mathbf{Q}(x - 2\epsilon_1) - P(x) \mathbf{Q}(x - \epsilon_1) = 0. \quad (\text{C.36})$$

The gauge polynomial is once again given by (C.25), and the flavour polynomial $D(x)$ is given by

$$D(x) = \left(x - m_1 + \frac{\epsilon_1}{2} \right). \quad (\text{C.37})$$

While the differential equation in this case also takes the form of a Schrödinger equation, with the potential (after shifting u_S so as to set $Q_2(t)$ to zero for convenience)

$$Q_0(t) = \frac{\Lambda_1^2}{4t^4} - \frac{m_1 \Lambda_1}{t^3} + \frac{u_S}{t^2} + \frac{\Lambda_1^2}{t}, \quad (\text{C.38})$$

its form requires a series of conformal transformations and rescalings before it can be easily compared with the form in (2.23); for convenience, we reproduce the transformations below:

$$\begin{aligned} t &\longrightarrow \frac{1}{z} \\ u_S &\longrightarrow 2^{4/3} (\tilde{u} + m_1 \Lambda_1) \\ m_1 &\longrightarrow -2^{2/3} m_1 \\ z &\longrightarrow 2^{2/3} w \\ w &\longrightarrow \frac{1-z}{z}. \end{aligned} \quad (\text{C.39})$$

After this sequence of transformations, we get the form of the potential as in (2.23):

$$Q_0(z) = \frac{\Lambda_1^2}{4z^4} + \frac{\Lambda_1 m_1}{z^3(z-1)^2} - \frac{\Lambda_1^2}{4z(z-1)^3} + \frac{\tilde{u}}{z^2(z-1)^2}. \quad (\text{C.40})$$

C.6 $N_f = 0$

The pure super Yang-Mills case is obtained by starting with the $N_f = 1$ case and taking the limit,

$$m_1 \rightarrow \infty \quad \text{and} \quad \Lambda_1 \rightarrow 0 \quad \text{with} \quad \Lambda_0^4 = m_1 \Lambda_1^3 \quad \text{finite.} \quad (\text{C.41})$$

The difference equation takes the form:

$$\Lambda_0^2 \mathbf{Q}(x) + \Lambda_0^2 \mathbf{Q}(x - 2\epsilon_1) - P(x) \mathbf{Q}(x - \epsilon_1) = 0. \quad (\text{C.42})$$

While the differential equation in this case also takes the form of a Schrödinger equation, with the potential (after shifting u_S so as to set $Q_2(t)$ to zero for convenience)

$$\phi_2(t) = \frac{\Lambda_0^2}{t^3} + \frac{u_S}{t^2} + \frac{\Lambda_0^2}{t}, \quad (\text{C.43})$$

its form requires a series of conformal transformations and rescalings before it can be literally compared with the form in (2.24); for convenience, we reproduce these transformations below:

$$\begin{aligned}
 t &\longrightarrow iw \\
 u_S &\longrightarrow -(\tilde{u} + \Lambda_0^2) \\
 \Lambda_0 &\longrightarrow e^{i\frac{\pi}{4}} \Lambda_0 \\
 w &\longrightarrow \frac{1-z}{z}.
 \end{aligned}
 \tag{C.44}$$

After this sequence of transformations, we get the potential as in the bulk of the paper (2.24):

$$Q_0(z) = \frac{\Lambda_0^2}{z^3(z-1)^2} + \frac{\tilde{u}}{z^2(z-1)^2} + \frac{\Lambda_0^2}{z(z-1)^3}.
 \tag{C.45}$$

Thus, we completed the derivation of the null vector decoupling equations (at $\epsilon_2 = 0$) from the purely gauge theoretic instanton partition function.

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