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Classical Virasoro irregular conformal block

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Abstract: Virasoro irregular conformal block with arbitrary rank is obtained for the classical limit or equivalently Nekrasov-Shatashvili limit using the beta-deformed irregular matrix model (Penner-type matrix model for the irregular conformal block). The same result is derived using the generalized Mathieu equation which is equivalent to the loop equation of the irregular matrix model.

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1 Introduction

Virasoro irrregular module, so called Gaiotto state or Whittaker state [1] turns out to be in connection with the four dimensional $\mathcal{N}$=2 super Yang-Mills theory [2]. The irregular module is different from the regular one in that it is the simultaneous eigenstate of positive Virasoro generators L_n with $n > 0$ instead of L_0. According to the AGT conjecture [3] there is a duality between 4D gauge theory and 2D conformal field theory: the instanton partition sector of the Nekrasov partition of class $T_{g,n}$ in SU(2) quiver gauge theory is given as the Liouville conformal block on a Riemann surface with genus g and n-regular punctures. However, there can arise irregular punctures whose degree is higher than 2 when the number of flavors is less than the regular one. In this case, the partition is constructed in terms of the inner product of Gaiotto states and provides the information on the Argyres-Douglas theory [4, 5].

On the other hand, it is noted that the irregular conformal block (ICB) is obtained from the colliding limit of the regular conformal block [6, 7]. The colliding limit is the fusion of primary vertex operators when the Liouville charge is very big so that their moment constructed as the product of the charge with powers of its position is finite. The ICB can be easily studied in terms of Penner-type matrix model which consists of logarithmic potential and finite Laurent series (finite number of positive powers and negative powers of matrix) [8]. We will call this matrix model hereafter, the irregular matrix model (IMM).

This model can describe the inner product between Gaiotto states of arbitrary rank and also provide the ICB.

The Liouville conformal block can be also studied using the classical limit. The 3-point conformal block was studied in [9] and the 4-point one is given in terms of Painlevé VI in [10]. From the AGT-conjecture point of view, the classical limit is equivalent to the Nekrasov and Shatashvili (NS) limit: the ϵ parameter of the Nekrasov partition function with the Ω background is identified with the Planck constant. Nekrasov and Shatashvili obtained the lowest energy using the first quantization of integrable systems [11]. In addition, the classical limit was equivalently investigated in terms of β-deformed Penner-type matrix model in [12].

Considering the current trend of research one may wonder if there is the classical limit of ICB. The simplest case is obtained using the Bohr-Sommerfeld periods of one-dimensional sine-Gordon model [13]. Similarly, degenerate conformal block is used to obtain the information on the classical limit of the ICB in [10, 14] In this paper, we will provide NS limit of the inner product of irregular modules for arbitrary rank using the IMM and evaluate the partition function in a systematic way.

This paper is organized as follows. In section 2, the NS limit of the IMM is provided. We review the property of the beta-deformed regular Penner-type matrix model (regular matrix model) and obtain the NS limit. The NS limit of the IMM is obtained from the colliding limit of the NS limit of the regular matrix model. It is noted that the order of the NS limit and the colliding limit is immaterial. One may equally take the colliding limit first and the NS limit later. We present the NS limit first in this section since NS limit of the regular conformal block appeared in [12] already. In section 3, the NS limit of the IMM is identified with the NS limit of the irregular module with arbitrary rank and the explicit representation is provided. In addition, the exponentiated form is provided for the NS limit of ICB. Furthermore, explicit form of the dominant contribution is provided for the arbitrary rank case. In section 4, the same result is obtained with a slightly different method. We use the degenerated primary operator method to find a second order differential equation for the rank 1. As a result, the Schrödinger equation is obtained which has two cosine potentials. We find a similar result for the arbitrary rank whose derivation is given using the IMM. A differential equation is obtained with the combination of exponential (cosine) terms which will be called the generalized Mathieu equation. We provide a check of the result obtained in section 3 with that from the second order differential equation. Section 5 is the summary and discussion.

2 Irregular matrix model and NS limit

2.1 Liouville conformal block and regular matrix model

Liouville conformal block is given as the holomorphic part of the correlation with N number of screening operators

$$\mathcal{F}^{(n+2)} = \left\langle\!\!\left\langle \left(\int d\lambda_I e^{2b\phi(\lambda_I)} \right)^N \quad \left(\prod_{a=0}^{n+1} e^{2\alpha_a \phi(z_a)} \right) \right\rangle\!\!\right\rangle , \tag{2.1}$$

where the double bracket stands for the expectation value with respect to free field action. Using the free field correlation $\langle\!\langle\phi(z)\phi(\omega)\rangle\!\rangle \sim -\frac{1}{2}\log(z-\omega)$ one has the conformal block in Selberg integrals:

$$\mathcal{F}^{(n+2)} = \prod_{0\le k<I\le n+1}(z_k-z_l)^{-2\alpha_k\alpha_l} \int\left(\prod_{I=1}^{N} d\lambda_I\right)\Delta(\{\lambda_I\})^{-2b^2}\prod_a(\lambda_I-z_a)^{-2b\alpha_a}\,, \tag{2.2}$$

where $\Delta(\{\lambda_I\}) = \prod_{I<J}(\lambda_I-\lambda_J)$ is the Vandermonde determinant.

We will fix the position $z_0 = 0$ as the reference point and put $z_{n+1}\to\infty$. The β-deformed Penner-type matrix model [15, 16] is defined from the conformal block by factoring out the prefactor $\prod_{0\le k<I\le n+1}(z_k-z_l)^{-2\alpha_k\alpha_l}$ as well as the terms containing z_{n+1}:

$$Z_\beta = \int\left(\prod_{I=1}^{N} d\lambda_I\right)\prod_{I<J}(\lambda_I-\lambda_J)^{2\beta}e^{\frac{\sqrt{\beta}}{g}\sum_I V(\lambda_I)}\,, \tag{2.3}$$

with $b = i\sqrt{\beta}$ and the Penner potential $V(\lambda)$ is defined as

$$V(\lambda) = \sum_{a=0}^{n}\hat{\alpha}_a\log(\lambda-z_a)\,. \tag{2.4}$$

Here we introduce the scaling parameter g_s so that α is scaled: $\alpha_k \equiv \hat{\alpha}_k/g_s$ for later convenience. The expansion parameter g is related with g_s as $g = ig_s/2$ so that $\sqrt{\beta}/g = -2b/g_s$. The β-deformed matrix model reduces to the hermitian matrix model when $\beta = 1$. We call this beta-deformed Penner-type matrix model the regular matrix model to distinguish from the IMM.

The Liouville charge $\hat{\alpha}$ satisfies the neutrality condition

$$\sum_{a=0}^{n+1}\hat{\alpha}_a + bg_sN = g_sQ\,, \tag{2.5}$$

where Q is the Liouville background charge $Q = b+1/b$ and $bg_s = 2g\sqrt{\beta}$. Be aware that the Liouville charge $\hat{\alpha}_{n+1}$ located at ∞ is included in the neutrality condition even though the matrix model does not contain the term at the spacial infinity. If one takes the conformal transformation $\lambda\to 1/\lambda$, then the Liouville charge at infinity appears naturally through the neutrality condition.

It is well-known that the symmetric property of the matrix model is given in terms of the loop equation which corresponds to the Ward-identity [17]:

$$4W(z)^2 + 4V'(z)W(z) + 2g_sQW'(z) - g_s^2W(z,z) = f(z)\,, \tag{2.6}$$

where $W(z_1,\cdots,z_s)$ is the s-point resolvent and is defined as

$$W(z_1,\ldots,z_s) = \beta\left(\frac{g}{\sqrt{\beta}}\right)^{2-s}\left\langle\sum_{I_1,\cdots,I_s}\frac{1}{(z_1-\lambda_{I_1})\cdots(z_s-\lambda_{I_s})}\right\rangle_{\text{conn}}. \tag{2.7}$$

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Explicitly one and two point resolvents are given as $W(z) = g\sqrt{\beta}\left\langle \sum_I \frac{1}{z-\lambda_I}\right\rangle_{\text{conn}}$ and $W(z,w) = \beta\left\langle \sum_I \frac{1}{(z-\lambda_I)(w-\lambda_I)}\right\rangle_{\text{conn}}$ respectively. The bracket $\langle O\cdots\rangle_{\text{conn}}$denotes the connected part of the expectation value of $O\cdots$ with respect to the matrix model (2.3).

$f(z)$ is the expectation value determined by the potential $V(z)$

$$f(z) = 4g\sqrt{\beta}\left\langle \sum_I \frac{V'(z)-V'(\lambda_I)}{z-\lambda_I}\right\rangle_{\text{conn}} . \tag{2.8}$$

With the Penner type potential in (2.4), $f(z) = \sum_{a=0}^{n} \frac{d_a}{z-z_a}$ and d_a is determined by the derivatives of the partition function

$$d_a = -2bg_s\left\langle \sum_I \frac{\hat{\alpha}_a}{\lambda_I - z_a}\right\rangle = -g_s^2\frac{\partial \log Z_\beta}{\partial z_a} . \tag{2.9}$$

One may rewrite the loop equation (2.6) as

$$x(z)^2 + g_sQx'(z) - g_s^2W(z,z) = -g_s^2\varphi(z)\,, \tag{2.10}$$

where $x(z) = 2W(z)+V'(z)$ and $\varphi(z)$ turns out to be the expectation value of the energy-momentum tensor whose explicit form is given as [17]

$$\begin{aligned}\varphi(z) &\equiv -\frac{1}{g_s^2}\left\{(V'(z))^2 + g_sQV''(z) + f(z)\right\}\\ &= \sum_a\left(\frac{\Delta_a}{(z-z_a)^2} + \frac{1}{z-z_a}\frac{\partial\log Z_{\text{eff}}}{\partial z_a}\right) ,\end{aligned} \tag{2.11}$$

with $Z_{\text{eff}} = Z_\beta \times \prod_{0\le a<b\le n}(z_a-z_b)^{-2\alpha_a\alpha_b}$.

The Liouville conformal block has a direct relation [12] with gauge theory through the AGT conjecture. One may put the position of the vertex operators $z_0 = 0$, $z_1 = 1$ and $w_a = q_1q_2\ldots q_{a-1}$ for $a = 2,\cdots,n$ and $q_a = e^{2\pi i\tau_a}$ with τ_a the gauge coupling constants. Then one has $d_a = (u_{a-1}-u_a)/(2\pi i z_a)$ where $u_a = -g_s^2\partial\log Z/\partial\tau_a$. Here the relation $2\pi i z_a\frac{\partial}{\partial w_a} = \frac{\partial}{\partial\tau_{a-1}} - \frac{\partial}{\partial\tau_a}$ is used. In this way, d_a's relate with the Higg's field expectation value $\langle\text{Tr}\,\Phi^2\rangle$.

2.2 Regular matrix model and NS limit

The Ω deformation parameters ϵ_1 and ϵ_2 of the gauge theory are related with the Liouville parameter b. One may identify $\epsilon_1 = b$ and $\epsilon_2 = 1/b$ so that $\epsilon_1\epsilon_2 = 1$. On the other hand, one may define the gauge theory in the NS limit where $\epsilon_2 = 0$ but ϵ_1 is finite. To get the corresponding limit for the matrix model one may define the parameter relation in a different way so that the NS limit is obtained easily. To achieve this, one may rescale Ω deformation parameters so that one has

$$\epsilon_1 = g_sb, \quad \epsilon_2 = \frac{g_s}{b}\,, \tag{2.12}$$

which provides the overall scale $\epsilon_1\epsilon_2 = g_s^2$. In this new convention, NS limit ($\epsilon_2\to 0$ and ϵ_1 finite) corresponds to the limit $g_s\to 0$ and $b\to\infty$. Note that the Liouville theory has

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$b \to 1/b$ duality and this duality for the gauge theory is $\epsilon_1 \to \epsilon_2$. Therefore, NS limit is equivalent to the classical limit $b \to 0$. Or one may equally put $\epsilon_1 \to 0$ instead of ϵ_2. In this way, we will not distinguish the NS limit from the classical limit.

Under the NS limit, physical quantities are to be rescaled properly. The background charge scales as $Q \to \epsilon_1/g_s$ and the central charge $c = 1 + 6Q^2 \to 6\epsilon_1^2/g_s^2$. The conformal dimension $\Delta(\alpha) = \alpha(Q - \alpha)$ of primary operator will scale as $\Delta(\alpha) = \delta_\alpha/g_s^2$ where $\delta_\alpha = \hat{\alpha}(\epsilon_1 - \hat{\alpha})$ with $\hat{\alpha}$ finite. In addition, the neutrality condition (2.5) maintained with no scaling of N.

Note that the potential in (2.4) dose not change but the power β of the Vandermonde determinant should scale as $\beta \to -\epsilon_1 b/g_s$ and $g_s Q \to \epsilon_1$. Besides, in NS limit the 2-point resolvent $g_s^2 W(z,z)$ vanishes since $W(z,z)$ is finite [21]. As a result, the loop equation (2.10) has the form

$$x(z)^2 + \epsilon_1 x'(z) + U(z) = 0\,, \tag{2.13}$$

and $U(z)$ is the NS limit of $g_s^2\varphi(z)$

$$U(z) = \sum_{a=0}^{n} \frac{\delta_{\alpha_a}}{(z - z_a)^2} + \sum_{a=0}^{n} \frac{\chi_a}{z - z_a}\,. \tag{2.14}$$

The coefficients are defined as $\delta_{\alpha_a} = \hat{\alpha}_a(\epsilon_1 - \hat{\alpha}_a)$ and $\chi_a = \sum_{b(\neq a)} 2\hat{\alpha}_a\hat{\alpha}_b/(z_a - z_b) - d_a$ where d_a is defined in (2.9).

It is noted that the loop equation turns into the second order differential equation with n regular singularities presented in $U(z)$:

$$\left(\epsilon_1^2 \frac{\partial^2}{\partial z^2} + U(z)\right)\Psi(z) = 0\,, \tag{2.15}$$

where $\Psi(z) = \exp\left(\frac{1}{\epsilon_1}\int^z x(z')dz'\right)$. Therefore, one may view the loop equation (2.13) as the Hamilton-Jacobi like equation. In [10], the 4-point classical block ($n = 4$) with the position identified as $(0, 1, t, \infty)$ is converted into the conventional Hamilton-Jacobi equation and the function $z(t)$ is noted to satisfy the Painlevé VI.

2.3 Colliding limit and irregular matrix model

The colliding limit is used to find the ICB, where many primary vertex operators are put at the reference point ($z_k \to z_0 = 0$) but with the Liouville charge infinite ($\hat{\alpha}_k \to \infty$) so that their products have finite results, $\hat{c}_k = \sum_{a=0}^{n} \hat{\alpha}_a (z_a)^k, (k = 0, 1, \cdots, n)$. The colliding limit of $n + 1$ vertex operators provides the maximum number $n + 1$ of the non-vanishing moments c_k and the potential of the form

$$V(z; \{\hat{c}_k\}) = \hat{c}_0 \log z - \sum_{k=1}^{n} \frac{\hat{c}_k}{k z^k}\,. \tag{2.16}$$

$f(z)$ is defined in (2.8) and has the form [17],

$$f(z) = -g_s^2 \sum_{k=0}^{n-1} \frac{v_k(\log Z)}{z^{2+k}}\,, \tag{2.17}$$

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where $v_k = \sum_\ell \ell\ \hat{c}_{\ell+k} \frac{\partial}{\partial \hat{c}_\ell}$. Here the notation $\hat{c}_\ell = 0$ is used when $\ell \geq n+1$. One has no term proportional to $1/z$ due to the identity $\langle \sum_I \hat{V}'(\lambda_I) \rangle = 0$. The loop equation (2.13) maintains the same form and $U(z)$ is given explicitly as

$$U(z) = \sum_{k=0}^{2n} \frac{\Lambda_k}{z^{k+2}} + \sum_{k=0}^{n-1} \frac{v_k(g_s^2 \log \hat{Z})}{z^{2+k}}\,, \tag{2.18}$$

where $\Lambda_k = (k+1)\epsilon_1 \hat{c}_k - \sum_{\ell=0}^{n} \hat{c}_\ell \hat{c}_{k-\ell}$. Noting that v_k is the representation of Virasoro operators in $\{c_k\}$ space [7], one realizes that $U(z)$ is the expectation value of the energy momentum tensor $T(z)$ where non-negative moment is non-vanishing. Therefore, one can define the non-negative moment of Virasoro generators as $\mathcal{L}_k = \Lambda_k + v_k$ with $k = 0, 1, \cdots, 2n$ with the notation $v_k = 0$ if $k = n, \cdots, 2n$. This identification realizes the Virasoro commutation relation on the irregular module

$$[\mathcal{L}_k, \mathcal{L}_\ell] = (\ell - k)\mathcal{L}_{k+\ell}\,. \tag{2.19}$$

The new feature is that the Virasoro generator has non-vanishing expectation values Λ_k when $k = n, n+1, \cdots, 2n$. This demonstrate that IMM is based on the rank n irregular module which is defines as

$$\mathcal{L}_k |I_n\rangle = \Lambda_k |I_n\rangle\,, \tag{2.20}$$

when $k = n, n+1, \cdots, 2n$. The partition function is identified as the NS limit of the inner product between a regular module located at infinity and a rank n irregular module located at origin.

3 Classical irregular conformal block

3.1 Irregular conformal block and NS limit

The partition function given in section 2.3 is the NS limit of the inner product between a regular module located at infinity and a rank n irregular module located at origin. On the other hand, ICB is given as the inner product between two different irregular modules, one at the origin and one at infinity. It is noted that NS limit of IMM is equivalent to the colliding limit of NS limit of regular matrix model: the order of the two limiting procedure is commutative. One can equally take the colliding limit first and NS limit next. In this section we present the classical conformal block by taking the NS limit to the IMM.

The partition function corresponding to the inner product between irregular modules has the potential of the form

$$\frac{1}{g_s} V_{(m:n)}(z; c_0, \{c_k\}, \{c_{-\ell}\}) = c_0 \log z - \sum_{k=1}^{n} \left(\frac{c_k}{k z^k} \right) + \sum_{\ell=1}^{m} \left(\frac{c_{-\ell}\ z^\ell}{\ell} \right)\,, \tag{3.1}$$

where $c_0 = \sum_{r=0}^{n} \alpha_r$ and $c_k = \sum_{r=1}^{n} \alpha_r (z_r)^k$ are the moments. Positive k corresponds to the contribution at the origin and negative k at infinity. The ICB $\mathcal{F}_\Delta^{(m:n)}(\{c_{-\ell} : c_k\})$ is obtained from the normalized inner product between irregular module so that $\mathcal{F}_\Delta^{(m:n)}(\{c_{-\ell} : c_k\})$

$= \langle I_m | I_n\rangle / (\langle I_m|\Delta\rangle\langle\Delta|I_n\rangle)$ where $|\Delta\rangle$ is the regular module with conformal dimension Δ. The explicit form of the ICB is given using the IMM [8]:

$$\mathcal{F}_\Delta^{(m:n)}(\{c_{-\ell}:c_k\}) = \frac{e^{\zeta_{(m:n)}} Z_{(m:n)}(c_0;\{c_k\};\{c_{-\ell}\})}{Z_{(0:n)}(c_0;\{c_k\}) Z_{(0:m)}(c_\infty;\{c_{-\ell}\})}\,, \tag{3.2}$$

where extra factor $e^{\zeta_{(m:n)}}$ is needed due to the limiting procedure $z_a \to \infty$ and $z_b \to 0$. Note that original conformal block has the factors $\prod_{a,b}(z_a - z_b)^{-2\alpha_a\alpha_b}$ which we factored out but the limiting procedure results in a finite contribution, so called U(1) contribution $e^{\zeta_{(m:n)}}$, where $\zeta_{(m:n)} = \sum_k^{\min(m,n)} 2c_k c_{-k}/k$. Therefore, to have the right conformal block we need to include this extra factor in the definition. In addition, $\langle I_m|\Delta\rangle$ is expressed as $Z_{(0:m)}(c_\infty;\{c_{-\ell}\})$ with the change of variable $\lambda_i \to 1/\lambda_i$.

The evaluation of ICB is done in [18]. Note that the information of the irregular module at the origin is obtained if one regards the potential $V_0 = V_{(0:n)}(\{\lambda_i\}; c_0, \{c_k\})$ as the reference one and ΔV_0 as its perturbation:

$$\frac{1}{g_s} V_0 = \sum_{I=1}^{N_0}\left(c_0 \log\lambda_I - \sum_{k=1}^{n}\frac{c_k}{k}\lambda_I^{-k}\right); \quad \frac{1}{g_s}\Delta V_0 = \sum_{I=1}^{N_0}\left(\sum_{\ell=1}^{n}\frac{c_{-\ell}}{\ell}\lambda_I^{\ell}\right). \tag{3.3}$$

That is, V_0 is the potential for the partition function $Z_{(0:n)}$ with $N_0(\le N)$ number of screening operators. At infinity one has the reference potential $\sum_{J=1}^{N_\infty}\left(c_0\log\lambda_J - \sum_{\ell=1}^n c_{-\ell}\lambda_i^{\ell}/\ell\right)$ and its perturbation $-\sum_{J=1}^{N_\infty}\left(\sum_{k=1}^n c_k\lambda_J^{-k}/k\right)$. We introduce the number N_∞ of screening operators at infinity so that $N_\infty + N_0 = N$.

The ICB is obtained using the perturbation theory. For example, ICB for rank 1 is given in power of $\eta_0 \equiv c_1 c_{-1}$ up to order $\mathcal{O}(\eta_0^2)$ as

$$\mathcal{F}_\Delta^{(1:1)} = 1 + \eta_0\frac{2\bar c_0 \bar c_\infty}{\Delta} + \eta_0^2\frac{4\bar c_0{}^2 \bar c_\infty{}^2 c/\Delta + 4\Delta + 2 + 12(\bar c_0{}^2 + \bar c_\infty{}^2) + 32\bar c_0{}^2\bar c_\infty{}^2}{c + 2c\Delta + 2\Delta(8\Delta-5)}\,, \tag{3.4}$$

where $\bar c_k = Q - c_k$. This is compared with the Gaiotto notation

$$\langle\widetilde{G_2}|\widetilde{G_2}\rangle = 1 + \Lambda\Lambda'\frac{mm'}{2\Delta} + (\Lambda\Lambda')^2\frac{m^2m'^2c/4\Delta + 4\Delta + 2 - 3(m^2+m'^2) + 2m^2m'^2}{c + 2c\Delta + 2\Delta(8\Delta-5)}\,, \tag{3.5}$$

and reveals the parameter relation $\Lambda^2 = -c_1^2$ and $m\Lambda = 2c_1\bar c_0$,

To have the NS limit, one has the scaling $c_k = \hat c_k/g_s$ and $\eta_0 = \hat\eta_0/g_s^2$. Following the conjecture in [9], one may put the classical ICB in an exponentiated form

$$\mathcal{F}_\Delta^{(1:1)}(\eta_0) \overset{g_s\to 0}{\sim} \exp\left\{\frac{1}{g_s^2} f_\delta(\hat\eta_0)\right\}. \tag{3.6}$$

If one expresses the exponentiated term $f_\delta(\hat\eta_0)$ in the power series in $\hat\eta_0$, one has

$$f_\delta(\hat\eta_0) = \lim_{g_s\to 0} g_s^2 \log\mathcal{F}_\Delta^{(1:1)}(\eta_0) = \sum_{n=1}(\hat\eta_0)^n f_\delta^{(n)}\,. \tag{3.7}$$

Explicit result is obtained from (3.4)

$$f_\delta^{(1)} = \frac{2\hat{\bar c}_0\hat{\bar c}_\infty}{\delta}, \qquad f_\delta^{(2)} = \frac{(10-\hat c)\hat{\bar c}_0{}^2\hat{\bar c}_\infty{}^2 + 2\delta^3 + 6\delta^2(\hat{\bar c}_0{}^2 + \hat{\bar c}_\infty{}^2)}{(c+8\delta)\delta^3}\,, \tag{3.8}$$

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or in Gaiotto variables

$$f_\delta^{(1)} = \frac{\hat{m}\hat{m}'}{2\delta}, \qquad f_\delta^{(2)} = \frac{(5\delta - 3\epsilon_1^2)\hat{m}^2\hat{m}'^2 - 12(\hat{m}^2 + \hat{m}'^2)\delta^2 + 16\delta^3}{16\delta^3(4\delta + 3\epsilon_1^2)}. \tag{3.9}$$

3.2 Dominant behavior of the exponentiated term

The conjecture (3.6) equally applies to higher ranks. We explicitly check this conjecture for the dominant part at the NS limit. Suppose ICB at the NS limit is exponentiated. Then one expands the ICB in powers of expansion parameters. For example, rank 1 has the expansion parameter $\hat{\eta}_0$ and has the form

$$\begin{aligned}
\mathcal{F}_\Delta^{(1:1)}(\eta_0) &\overset{g_s \to 0}{\sim} \exp\left\{ \frac{1}{g_s^2} \sum_{n=1} (\hat{\eta}_0)^n f_\delta^{(n)} \right\} \\
&= 1 + \hat{\eta}_0 \frac{1}{g_s^2} f_\delta^{(1)} + (\hat{\eta}_0)^2 \left\{ \frac{1}{g_s^2} f_\delta^{(2)} + \frac{1}{2} \frac{1}{g_s^4} \left(f_\delta^{(1)}\right)^2 \right\} \\
&\quad + (\hat{\eta}_0)^3 \left\{ \frac{1}{g_s^2} f_\delta^{(3)} + \frac{1}{g_s^4} f_\delta^1 f_\delta^{(2)} + \frac{1}{3!} \frac{1}{g_s^6} \left(f_\delta^{(1)}\right)^3 \right\} + \ldots
\end{aligned} \tag{3.10}$$

and can be compared with the ICB and finds the NS limit by putting

$$\mathcal{F}_\Delta^{(1:1)}(\eta_0) = \sum_{n=0} (\eta_0)^n F_\Delta^{(n)}(g_s^2) = 1 + \hat{\eta}_0 \frac{1}{g_s^2} F_\Delta^{(1)} + (\hat{\eta}_0)^2 \frac{1}{g_s^4} F_\Delta^{(2)} + (\hat{\eta}_0)^3 \frac{1}{g_s^6} F_\Delta^{(3)} + \ldots \tag{3.11}$$

The dominant contribution is given as $\lim_{g_s \to 0} F_\Delta^{(n)} = \frac{1}{n!}\left(f_\delta^{(1)}\right)^n$. Indeed, the first few terms show the exponentiated behavior: $\lim_{g_s \to 0} F_\Delta^{(1)} = 2\hat{c}_0 \hat{c}_\infty / \delta = f_\delta^{(1)}$ and $\lim_{g_s \to 0} F_\Delta^{(2)} = 2(\hat{c}_0 \hat{c}_\infty)^2/\delta^2 = (f_\delta^{(1)})^2/2$.

One may find more rigorous proof for this dominant behavior. Let us first consider the simplest case (the rank 1/2) given in [14]. (This corresponds to $N_f = 0$, SU(2) and is obtained using some appropriate limit from the rank 1 case.)

$$\mathcal{F}_{c,\Delta}(\Lambda) = \sum_{n=0} \Lambda^{4n} \left[G_{c,\Delta}^n\right]^{(1^n)(1^n)} = \sum_{n=0} \left(\frac{\hat{\Lambda}^4}{g_s^2}\right)^n \frac{1}{g_s^{2n}} \left[G_{c,\Delta}^n\right]^{(1^n)(1^n)}. \tag{3.12}$$

Setting the scaling $\Lambda = \hat{\Lambda}/g_s$ one has to demonstrate the dominant behavior

$$\lim_{g_s \to 0} \frac{1}{g_s^{2n}} \left[G_{c,\Delta}^n\right]^{(1^n)(1^n)} = \frac{1}{n!} \left(\lim_{g_s \to 0} \frac{1}{g_s^2} \left[G_{c,\Delta}^n\right]^{(1)(1)} \right)^n. \tag{3.13}$$

We use the property of the Gram matrix. Gram matrix is defined at each level as follows: At level 1 : $\{L_{-1}|\, \nu_\Delta\,\rangle\}$,

$$G_{c,\Delta}^{n=1} = \langle L_{-1}\nu_\Delta \,|\, L_{-1}\nu_\Delta\rangle = \langle \nu_\Delta \,|\, L_1 L_{-1}\nu_\Delta\rangle = 2\Delta.$$

At level 2: $\{L_{-2}|\, \nu_\Delta\,\rangle, L_{-1}L_{-1}|\, \nu_\Delta\,\rangle\}$,

$$G_{c,\Delta}^{n=2} = \begin{pmatrix} \langle L_{-2}\nu_\Delta \,|\, L_{-2}\nu_\Delta\rangle & \langle L_{-1}^2\nu_\Delta \,|\, L_{-2}\nu_\Delta\rangle \\ \langle L_{-2}\nu_\Delta \,|\, L_{-1}^2\nu_\Delta\rangle & \langle L_{-1}^2\nu_\Delta \,|\, L_{-1}^2\nu_\Delta\rangle \end{pmatrix} = \begin{pmatrix} \frac{c}{2} + 4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix}.$$

At level 3: $\{L_{-3}|\,\nu_\Delta\,\rangle, L_{-2}L_{-1}|\,\nu_\Delta\,\rangle, L_{-1}L_{-1}L_{-1}|\,\nu_\Delta\,\rangle\}$,

$$G^{n=3}_{c,\Delta} = \begin{pmatrix} 2c+6\Delta & 10\Delta & 24\Delta \\ 10\Delta & \Delta(c+8\Delta+8) & 12\Delta(3\Delta+1) \\ 24\Delta & 12\Delta(3\Delta+1) & 24\Delta(\Delta+1)(2\Delta+1) \end{pmatrix},$$

and so on.

At the NS limit, the scaling behavior shows that $c = \hat{c}/g_s^2$ and $\Delta = \delta/g_s^2$ are of the same order. In addition, every commutator $[L_m, L_{-m}]$ leads to the order of $\mathcal{O}(\Delta)$. Therefore, the term with $\langle \nu_\Delta \,|\, L_1^n L_{-1}^n \nu_\Delta \rangle$ at level n will result in the order of $\mathcal{O}(\Delta)^n$, the highest order in the inverse powers of g_s and provides the dominant behavior in the same row or column.

Note that $N \times N$ square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1N} \\ a_{i1} & a_{i2} & \dots a_{iN} \\ a_{N1} & a_{n2} & \dots a_{NN} \end{pmatrix},$$

has the cofactor matrix

$$A^\star = \begin{pmatrix} A_{11} & A_{21} & \dots A_{N1} \\ A_{12} & A_{22} & \dots A_{N2} \\ A_{1n} & A_{2n} & \dots A_{NN} \end{pmatrix},$$

where A_{ij} is the cofactor of a_{ij} and the determinant of the matrix A is given as $|A| = AA^\star$: $a_{N1}A_{N1} + a_{N2}A_{N2} + \cdots + a_{NN}A_{NN} = |A|$. Let's turn back to $G^n_{c,\Delta}$. In this case,

$$\left[G^n_{c,\Delta}\right]^{(1^n)(1^n)} = \frac{A_{NN}}{|A|}. \tag{3.14}$$

As discussed in the above the NS limit picks up the $a_{NN}A_{NN}$ as the dominant term and the Gram matrix reduces to

$$\lim_{g_s\to 0} \left[G^n_{c,\Delta}\right]^{(1^n)(1^n)} = \lim_{g_s\to 0} \frac{1}{a_{nn}} = \lim_{g_s\to 0} \frac{1}{\langle L_{-1}^n \nu_\Delta \,|\, L_{-1}^n \nu_\Delta \rangle}. \tag{3.15}$$

Note that $\langle L_{-1}^n \nu_\Delta \,|\, L_{-1}^n \nu_\Delta \rangle = \langle \nu_\Delta \,|\, L_1^n L_{-1}^n \nu_\Delta \rangle$ is not simple to evaluate because one needs to apply the commutator repeatedly. For example, to calculate $\langle \nu_\Delta \,|\, L_1^n L_{-1}^n \nu_\Delta \rangle$, we need $[L_1, L_{-1}] = 2L_0$, and $[L_0, L_{-1}] = L_{-1}$ an d so on. However at the NS limit the dominant term simplifies since $[L_0, L_{-1}] = L_{-1}$ gives the lower contribution in $\mathcal{O}(\Delta)$. Therefore, at the NS limit, one can regard technically $[L_0, L_{-1}] = 0$. In this case, one uses the relation $[A, B^n] = n[A,B]B^{n-1}$ where $A = L_1$ and $B = L_{-1}$. Then since $[A, B]$ commutes with B, one has $[L_1, L_{-1}^n]\,|\,\nu_\Delta\rangle = 2n\Delta L_{-1}^{n-1}\,|\,\nu_\Delta\rangle$. Repeatedly using this method, one concludes that

$$\lim_{g_s\to 0} \langle L_{-1}^n \nu_\Delta \,|\, L_{-1}^n \nu_\Delta \rangle = \lim_{g_s\to 0} (2^n n!)\Delta^n\,. \tag{3.16}$$

As a result,

$$\lim_{g_s\to 0} \frac{1}{g_s^{2n}} \left[G^n_{c,\Delta}\right]^{(1^n)(1^n)} = \frac{1}{(2^n n!)\delta^n} = \frac{1}{n!}\left(\lim_{g_s\to 0} \frac{1}{g_s^2} \left[G^n_{c,\Delta}\right]^{(1)(1)} \right)^n. \tag{3.17}$$

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To consider rank 1 case, remind that the irregular module for rank n is conjectured as [8, 19, 20]

$$|\widetilde{G_{2n}}\rangle = \sum_{\ell,Y,\ell_p} \Lambda^{\ell/n} \left\{\prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i}\right\} m^{\ell_n} Q_\Delta^{-1}\Big(1^{\ell_1}2^{\ell_2}\cdots(2n-1)^{\ell_{2n-1}}(2n)^{\ell_{2n}};Y\Big) L_{-Y}|\Delta\rangle\,. \tag{3.18}$$

One has the ICB using this notation

$$\mathcal{F}_\Delta^{(1:1)} = \langle\widetilde{G_2}|\widetilde{G_2}\rangle = \sum_{k=0}(\Lambda\Lambda')^k \sum_{\ell_i,\ell_i'} m'^{\ell_1'} m^{\ell_1} Q_\Delta^{-1}\Big(1^{\ell_1}2^{\ell_2};1^{\ell_1'}2^{\ell_2'}\Big)\,. \tag{3.19}$$

Using the scaling $\Lambda = \hat\Lambda/g_s$, $m=\hat m/g_s$, $\Delta=\delta/g_s^2$, one has $\mathcal{F}_\Delta^{(1:1)} = \sum\limits_{k=0}\left(\hat\Lambda\hat\Lambda'/g_s^2\right)^k F_2^{(k)}$ where

$$F_2^{(k)} = \sum_{\ell_i,\ell_i'}(\hat m'/g_s)^{\ell_1'}(\hat m/g_s)^{\ell_1} Q_\Delta^{-1}\Big(1^{\ell_1}2^{\ell_2};1^{\ell_1'}2^{\ell_2'}\Big)\,. \tag{3.20}$$

The scaling of the Gram matrix results in the scaling of $F_2^{(k)}$ as $(\hat m\hat m')^k/(2^k k!\delta^k)$. This concludes $\lim\limits_{g_s\to 0} F_2^{(k)} = \frac{1}{k!}\left(\lim\limits_{g_s\to 0} F_2^{(1)}\right)^k$ which demonstrates that the dominant part of $\mathcal{F}_\Delta^{(1:1)}(\eta_0)$ is exponentiated.

One may demonstrate the same behavior for arbitrary rank. However, for the rank $n\ge 2$, there appears one subtle behavior of the irregular module related with the coefficient b_k in (3.18). In fact, b_k is not a simple constant but is to be fixed as [8] $\Lambda^{k/n}b_k \to \Lambda_k + v_k(\log Z_{(0:n)})$. If one understands b_k as the replacement, one can identify $|\widetilde{G_{2n}}\rangle$ with $|I_n\rangle$ in (2.18). For notational convenience, we will use either $|I_n\rangle$ or $|\widetilde{G_{2n}}\rangle$ without distinction but its revised form is tacitly assumed.

The eigenvalues Λ_k for $|I_n\rangle$ in are identified with the coefficients given in (3.18) by the comparison between the two expression of irregular states $|\widetilde{G_{2n}}\rangle$ and $|I_n\rangle$ [20]:

$$\begin{aligned}\Lambda_{2n-s} &= \frac{\langle\Delta|L_W L_{2n-s}|\widetilde{G_{2n}}\rangle}{\langle\Delta|L_W|\widetilde{G_{2n}}\rangle} = \Lambda^{2n-s/n}a_s \qquad \text{for } 0\le s<n,\\ \Lambda_n &= \frac{\langle\Delta|L_W L_n|\widetilde{G_{2n}}\rangle}{\langle\Delta|L_W|\widetilde{G_{2n}}\rangle} = \Lambda m\,,\end{aligned} \tag{3.21}$$

and $\langle\Delta|L_W|\widetilde{G_{2n}}\rangle = \Lambda^{\ell/n}\left\{\prod_{i=1}^{n-1}a_i^{\ell_{2n-i}}b_i^{\ell_i}\right\}m^{\ell_n}$ when $W=1^{\ell_1}2^{\ell_2}\cdots(2n)^{\ell_{2n}}$.

Putting $\langle\widetilde{G_{2n}}|\widetilde{G_{2n}}\rangle \equiv \sum\limits_{k=0}\left((\Lambda\Lambda')^{1/n}/g_s^2\right)^k F_n^{(k)}$ one has

$$F_n^{(k)} = g_s^{2k(n-1)/n}\sum_{\ell_i,\ell_i'}\left\{\prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} a_i'^{\ell_{2n-i}'} b_i^{\ell_i} b_i'^{\ell_i'}\right\} m^{\ell_n} m'^{\ell_n'} Q_\Delta^{-1}\Big(1^{\ell_1}\cdots(2n)^{\ell_{2n}};1^{\ell_1'}\cdots(2n)^{\ell_{2n}'}\Big). \tag{3.22}$$

Since the parameters scale as $\Lambda = \hat{\Lambda}/g_s$, $m = \hat{m}/g_s$, $\Delta = \delta/g_s^2$ and $c_k = \hat{c}_k/g_s$ one has $a_i = \hat{a}_i/(g_s^{i/n})$, $b_i = \hat{b}_i/(g_s^{2-i/n})$. From this scaling one immediately finds the dominant contribution of the Gram matrix is $Q_\Delta^{-1}\left(1^n; 1^n\right)$. Therefore, one has $F_n^{(k)} \to (\hat{b}_1\hat{b}_1')^k/(k!2^k\delta^k)$ which is the exponentiated form with $\lim_{g_s\to 0} F_n^{(k)} = \left(\lim_{g_s\to 0} F_n^{(1)}\right)^k/k!$.

4 Classical irregular conformal block and second order differential equation

4.1 Null vector approach

In section 3 the NS limit of ICB is shown using IMM. In this section, we present a different approach to find the same quantity. Conformal block with addition of a degenerate primary operator (degenerate conformal block) satisfies the null condition, which is written as a differential equation. This method is used for the NS limit of the rank 1/2 in [10, 14] to obtain the Mathieu equation.

The degenerate primary operator $V_+(z) \equiv V_{\Delta_+}(z)$ with the Liouville charge $\alpha = -1/(2b)$ has the conformal dimension $\Delta_+ = -\frac{1}{2} - \frac{3}{4b^2}$. At level 2, the null vector arises:

$$\chi_+(z) = \left[\widehat{L}_{-2}(z) - \frac{3}{2(2\Delta_+ + 1)}\,\widehat{L}_{-1}^2(z)\right] V_+(z)\,.$$

The null vector needs to vanish when evaluated between any states, i.e., $\langle\, I_\ell \,|\, \chi_+(z) \,|\, I_k \,\rangle = 0$. This provides a non-trivial constraint

$$\langle\, I_\ell \,|\, \widehat{L}_{-2}(z) V_+(z) \,|\, I_k \,\rangle + b^2 \,\langle\, I_\ell \,|\, \widehat{L}_{-1}^2(z) V_+(z) \,|\, I_k \,\rangle = 0. \tag{4.1}$$

Let us consider the case of rank 1. Let us denote degenerate irregular 3-point block as $\Phi(\Lambda, z) = \langle\, I_1; \Delta_L, m_L, \Lambda_L | V_+(z) | I_1; \Delta_R, m_R, \Lambda_R \,\rangle$ where each irregular module is assumed to be constructed with the highest state with conformal dimension Δ_L and Δ_R, respectively. In addition, $m_{L,R}$ and $\Lambda_{L,R}$ are the eigenvalues characterizing the irregular module. However, we will restrict ourselves to the case when all the L parameters are the same with the R parameters: $\Delta_L = \Delta_R = \Delta$, $m_L = m_R = m$ and $\Lambda_L = \Lambda_R = \Lambda$. In this case, the constraint (4.1) reduces to the second order differential equation,

$$\left[b^2\, z^2 \frac{\partial^2}{\partial z^2} - \frac{3z}{2}\frac{\partial}{\partial z} + \Lambda^2\left(z^2 + \frac{1}{z^2}\right) + m\Lambda\left(z + \frac{1}{z}\right) + \frac{\Lambda}{2}\frac{\partial}{\partial \Lambda} + \kappa\right]\Phi(\Lambda, z) = 0\,, \tag{4.2}$$

with $\kappa = \Delta - \Delta_+/2$. One may normalize the 3-point block considering the conformal dimension of the degenerate operator and inner product of the irregular model:

$$\Phi(\Lambda, z) \equiv z^{-\Delta_+} \langle\, I_1 | I_1 \rangle \psi(\Lambda, z)\,. \tag{4.3}$$

Now we put the inner product as the exponential form $\langle\, I_1 | I_1 \rangle \overset{g_s\to 0}{\sim} \exp\left\{\frac{1}{{g_s}^2} f_\delta(\hat{\Lambda})\right\}$ and use the scaled quantities $\Delta_+ \to -1/2$ and $\Delta \to \delta/g_s^2$ with $\delta = \epsilon_1^2(\frac{1}{4} - \xi^2)$ and $m \to \hat{m}/g_s$.

Multiplying (4.2) by g_s^2 so that $\epsilon_1 = bg_s$ finite, one has

$$\left[\epsilon_1^2\left(z^2\frac{\partial^2}{\partial z^2}+z\frac{\partial}{\partial z}\right)+\hat{\Lambda}^2\left(z^2+\frac{1}{z^2}\right)+\hat{m}\hat{\Lambda}\left(z+\frac{1}{z}\right)+\frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}f_\delta\left(\hat{\Lambda}\right)-\epsilon_1^2\xi^2\right]\psi(z)=0, \tag{4.4}$$

where we use the limit $\lim_{g_s\to 0} g_s^2\frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}\psi(\Lambda,z)=0$.

This equation can be considered on the unit circle $z = \mathrm{e}^{2ix}$ with real x:

$$\left[-\epsilon_1^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+(8\hat{\Lambda}^2\cos 4x+8\hat{m}\hat{\Lambda}\,\cos 2x)\right]\psi(x)=E\,\psi(x)\,, \tag{4.5}$$

where $E = 4\epsilon_1^2\xi^2 - 2\hat{\Lambda}\,\partial_{\hat{\Lambda}}f_\delta\left(\hat{\Lambda}\right)$. This is the Schrödinger equation for $\psi(\Lambda,x)$ with the potential real. It is noted that we have the real potential since we put all the parameters of L and R the same: the ψ corresponds to the expectation value of the irregular module.

4.2 Differential equation and loop equation

The same differential equation can also be derived if one uses the loop equation. In fact, the derivation using the loop equation is simpler and can be easily generalized into higher rank cases. Let us define the conformal block with the degenerate operator $V_{-1/(2b)}(z)$ [12],

$$\mathcal{F}^{(n+m+2)}_{-1/(2b)}(z)=\left\langle\!\!\left\langle V_{-1/(2b)}(z)\left(\int d\lambda e^{2b\phi(\lambda)}\right)^{N_+}\prod_{0\le k\le n+m+1}V_{\alpha_k}(w_k)\right\rangle\!\!\right\rangle, \tag{4.6}$$

where $w_0 = 0$ and $w_{n+m+1}\to\infty$. In addition, n number of operators $w_k, 1\le k\le n$ are assumed to lie around 0 and m number of operators $w_k, n+1\le k\le n+m$ around ∞ so that rank n (and m) colliding limit is obtained. Explicitly,

$$\begin{aligned}\mathcal{F}^{(n+m+2)}_{-1/(2b)}(z)=&\prod_{0\le k<\ell\le n+m+1}(w_k-w_\ell)^{-\frac{2\hat{\alpha}_k\hat{\alpha}_\ell}{g_s^2}}\prod_{k=0}^{n+m+1}(z-w_k)^{\frac{\hat{\alpha}_k}{bg_s}}\\&\times\int\prod_{I=1}^{N_+}d\lambda_I\prod_{I<J}(\lambda_I-\lambda_J)^{-2b^2}\prod_I\prod_{k=0}^{n-2}(\lambda_I-w_k)^{\frac{-2b\hat{\alpha}_k}{g_s}}(z-\lambda_I).\end{aligned} \tag{4.7}$$

One may normalize the above with $\mathcal{F}^{(n+m+2)} = \left\langle\!\!\left\langle\left(\int d\lambda e^{2b\phi(\lambda)}\right)^N\prod_{0\le k\le n+m+1}V_{\alpha_k}(w_k)\right\rangle\!\!\right\rangle$. However, one needs to care about the neutrality condition. For the expectation value one has the neutrality condition $-1/(2b)+\sum_k\alpha_k+N_+b=Q$ where as the partition function has the neutrality condition $\sum_k\alpha_k+Nb=Q$. This requires that $N_+-N=1/(2b^2)$, which shows that one needs different number of screening operators for the evaluation of the partition function and for the expectation value. However, this unpleasant feature disappears when NS limit is achieved: $N_+-N=1/(2b^2)\to 0$ and one may identity N_+ with N and find the normalized degenerate conformal block as

$$\frac{\mathcal{F}^{(n+m+2)}_{-1/(2b)}(z)}{\mathcal{F}^{(n+m+2)}}=\prod_{k=0}^{n+m+1}(z-w_k)^{\hat{\alpha}_k/\epsilon_1}\left\langle\left(\prod_I(z-\lambda_I)\right)\right\rangle, \tag{4.8}$$

so that

$$\log\left(\frac{\mathcal{F}^{(n+m+2)}_{-1/(2b)}(z)}{\mathcal{F}^{(n+m+2)}}\right)=\frac{V_{(m:n)}(z)}{\epsilon_1}+\log\left\langle\prod_I(z-\lambda_I)\right\rangle. \tag{4.9}$$

We introduce the bracket to denote the normalized expectation value of λ_I's.
Define $\eta(z)\equiv\langle(\prod_I(z-\lambda_I))\rangle$, noting that

$$z-\lambda_I=C(\lambda_I;z_0)\;e^{\int_{z_0}^{z}\frac{dz'}{z'-\lambda_I}}\,, \tag{4.10}$$

where $C(\lambda_I;z_0)$ is a z-independent normalization one may put $\eta(z)/\eta(z_0)$ in terms of exponential form i.e., the irreducible effective action whose explicit form is given as

$$\log\left(\frac{\eta(z)}{\eta(z_0)}\right)=\sum_{k=1}^{\infty}\frac{1}{k!}\left\langle\left(\sum_I\int_{z_0}^{z}\frac{dz'}{z'-\lambda_I}\right)^k\right\rangle_{\mathrm{conn}}, \tag{4.11}$$

where the bracket with the subscript c denotes the connected part of the expectation value. This quantity is given in terms of the multi-point of the resolvent

$$\log\left(\frac{\eta(z)}{\eta(z_0)}\right)=\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{g}{\sqrt{\beta}}\right)^{k-2}\int_{z_0}^{z}\prod_{\ell=1}^{k}dz'_\ell\;W(z'_1,z'_2,\cdots,z'_k)\,, \tag{4.12}$$

where the multi-point resolvent is defined in (2.7). At the NS limit, all the multi-point resolvent vanishes except the one-point resolvent [21]. Therefore the expectation value at the NS limit is given as

$$\log\left(\frac{\eta(z)}{\eta(z_0)}\right)=\frac{1}{g\sqrt{\beta}}\int_{z_0}^{z}dz'\;W(z')\,. \tag{4.13}$$

Recall that $x(z)=2W(z)+V'(z)$, we find

$$\frac{\mathcal{F}^{(n+m+2)}_{-1/(2b)}(z)}{\mathcal{F}^{(n+m+2)}}=\Psi(z),\quad \Psi(z)=\exp\left(\frac{1}{\epsilon_1}\int^{z}x(z')dz'\right), \tag{4.14}$$

exactly the one we defined before.

Noting that the resolvent satisfies the loop equation we have the second order differential equation for $\Psi(z)$ similar to the one in (2.15)

$$\left(\epsilon_1^2\frac{\partial^2}{\partial z^2}+U_{(m:n)}(z)\right)\Psi(z)=0\,, \tag{4.15}$$

with the potential $U_{(m:n)}(z)$, NS limit of the potential $V_{m:n}(z)$

$$\begin{aligned}U_{(m:n)}(z)&=-\left(V'_{(m:n)}(z)\right)^2-\epsilon_1V''_{(m:n)}(z)-f(z)\\&=\sum_{k=-2m}^{2n}\frac{\tilde\Lambda_k}{z^{k+2}}-\sum_{k=-m}^{n-1}\frac{\tilde d_k}{z^{2+k}}\,,\end{aligned} \tag{4.16}$$

where $\tilde{\Lambda}_k = (k+1)\epsilon_1 \hat{c}_k - \sum_{\ell=-m}^{n} \hat{c}_\ell \hat{c}_{k-\ell}$ and according to [18],

$$\begin{aligned} -g_s^2\, v_k(\log Z_{(m:n)}) &= \tilde{d}_k && \text{for } 0 \le k \le n-1\,, \\ -g_s^2\, u_k(\log Z_{(m:n)}) &= \tilde{d}_{-k} - 2\epsilon_1 N \hat{c}_{-k} && \text{for } 1 \le k < m-1\,, \\ 2\epsilon_1 N \hat{c}_{-m} &= \tilde{d}_{-m}\,. \end{aligned} \tag{4.17}$$

Here u_k is a differential operator corresponding to $\hat{c}_{-\ell}$, $u_{k>0} = \sum_{\ell>0} \ell\, \hat{c}_{-\ell-k} \frac{\partial}{\partial \hat{c}_{-\ell}}$. The potential has degree of poles higher than 2 and non-vanishing zeros. Notice that the partition function $Z_{(m:n)}$ should also have the classical behavior $Z_{(m:n)} \overset{g_s \to 0}{\sim} \exp\left\{\frac{1}{g_s^2}\varsigma_{(m:n)}\right\}$. we have Generalized Mathieu equation on the circle $z = \mathrm{e}^{2ix}$ by putting $\Psi(z) = z^{-\Delta_+}\psi(z)$,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k=-2m}^{2n} \frac{4\tau_k}{\epsilon_1^2}\,\mathrm{e}^{-i2kx} + \sum_{k=0}^{n-1} \frac{4v_k(\varsigma_{(m:n)})}{\epsilon_1^2}\mathrm{e}^{-i2kx} + \sum_{k=1}^{m-1} \frac{4u_k(\varsigma_{(m:n)})}{\epsilon_1^2}\mathrm{e}^{i2kx} - 1\right]\psi(x) = 0, \tag{4.18}$$

where

$$\tau_k = \begin{cases} \tilde{\Lambda}_k & 0 \le k \le 2n \\ \tilde{\Lambda}_k - 2\epsilon_1 N \hat{c}_k & -m \le k \le -1 \\ \tilde{\Lambda}_k & -2m \le k < -m\,. \end{cases} \tag{4.19}$$

We see when $m = n$, with proper choice of τ_k, $\mathrm{e}^{-i2kx} + \mathrm{e}^{i2kx}$ will reproduce $2\cos 2kx$.

4.3 Example of the classical irregular conformal block

We present here an explicit calculation of the classical conformal block for rank 1. Introducing new parameters which rescales the original quantities such that $E = \epsilon_1^2\lambda$, $h = 2\hat{\Lambda}/\epsilon_1$ and $M = 2\hat{m}/\epsilon_1$ we have (4.5) as

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \left(\lambda - 2h^2\cos 4x - 2hM\cos 2x\right)\psi(x) = 0, \tag{4.20}$$

We are looking for a quasi-periodic solution $\psi(x)$ with a Floquet exponent ν

$$\psi(x+\pi) = \mathrm{e}^{-i\pi\nu}\psi(x)\,, \tag{4.21}$$

where $\mathrm{e}^{-i\pi\nu}$ is called the Bloch factor. We provide a brief procedure to solve the equation perturbatively for small h and M, whose method can be found in [22]. The solution may have value $\lambda = \nu^2 - 2h^2\zeta$, where ζ is a small quantity. In this case we may rearrange the equation (4.20) into the following form

$$D_\nu\psi = \left(2h^2\zeta + 2h^2\cos 4x + 2hM\cos 2x\right)\psi, \tag{4.22}$$

and use the perturbation in powers of h and M. Here $D_\nu \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \nu^2$ is the ordinary differential operator independent of h and M.

The lowest order the solution $\psi^{(0)} \equiv \psi_\nu$ has either of the form $\cos\nu x$, $\sin\nu x$, $e^{\pm i\nu x}$ or their combinations. Using the product-to-sum formula (similar tricks also works for the exponential terms in (4.18)), we know ψ_ν always satisfies that

$$2\cos(tx)\,\psi_\nu = \psi_{\nu+t} + \psi_{\nu-t}. \tag{4.23}$$

Inserting $\psi^{(0)} \equiv \psi_\nu$ into the right hand side of (4.22) we have

$$R_\nu^{(0)} = 2h^2\zeta\ \psi_\nu + h^2(\psi_{\nu+4} + \psi_{\nu-4}) + hM(\psi_{\nu+2} + \psi_{\nu-2})\,, \tag{4.24}$$

which should be of higher order than h. Thus we add the perturbation so that $\psi = \psi^{(0)} + \psi^{(1)}$. Since $D_\nu\psi_\nu = 0$ and $D_{\nu+t}\psi_{\nu+t} = 0$ we have $D_\nu\psi_{\nu+t} = -t(2\nu+t)\psi_{\nu+t}$ where we use the relation $D_{\nu+t} = D_\nu + t(2\nu+t)$. Therefore, we can cancel the term $\psi_{\nu+t}$ in $R_\nu^{(0)}$ by adding $\frac{\psi_{\nu+t}}{-t(2\nu+t)}$. We may choose

$$\psi^{(1)} = h^2\left(\frac{\psi_{\nu+4}}{-4(2\nu+4)} + \frac{\psi_{\nu-4}}{4(2\nu-4)}\right) + hM\left(\frac{\psi_{\nu+2}}{-2(2\nu+2)} + \frac{\psi_{\nu-2}}{2(2\nu-2)}\right), \tag{4.25}$$

which will cancel the last four terms in $R_\nu^{(0)}$.

Now we have new contribution $R_\nu^{(1)}$ in the right hand side of (4.22)

$$R_\nu^{(1)} = h^2\left(\frac{R_{\nu+4}^{(0)}}{-4(2\nu+4)} + \frac{R_{\nu-4}^{(0)}}{4(2\nu-4)}\right) + hM\left(\frac{R_{\nu+2}^{(0)}}{-2(2\nu+2)} + \frac{R_{\nu-2}^{(0)}}{2(2\nu-2)}\right), \tag{4.26}$$

which results in the combination (up to the terms proportional to ψ_ν):

$$R_\nu^{(0)} + R_\nu^{(1)} = \psi_\nu\left\{2h^2\zeta + h^2M^2\left(\frac{1}{-2(2\nu+2)} + \frac{1}{2(2\nu-2)}\right) + \mathcal{O}(h)^4\right\} + \dots. \tag{4.27}$$

This sum should be forced to vanish up to $\mathcal{O}(h^4)$, so we have $\zeta = -\frac{M^2}{4(\nu^2-1)}$.

Repeating the similar but lengthy calculation, we find the eigenvalue λ as:

$$\lambda = \nu^2 + h^2\frac{M^2}{2\left(\nu^2-1\right)} + h^4\frac{\left(5\nu^2+7\right)M^4 + 24\left(\nu^2-1\right)^2M^2 + 16\left(\nu^2-1\right)^3}{32\left(\nu^2-4\right)\left(\nu^2-1\right)^3} + \dots. \tag{4.28}$$

One may compare the definition of λ with the eigenvalues obtained above order by order, using $f_\delta(\hat\Lambda) = \sum\limits_{n=1}\left(\hat\Lambda\right)^{2n} f_\delta^{(n)}$,

$$\begin{aligned}
\lambda &= \nu^2 + \frac{4\hat\Lambda^2}{\epsilon_1^4}\frac{2\hat m^2}{\left(\nu^2-1\right)} + \frac{16\hat\Lambda^4}{\epsilon_1^8}\frac{\left(5\nu^2+7\right)\hat m^4 + 6\epsilon_1^2\left(\nu^2-1\right)^2\hat m^2 + \epsilon_1^4\left(\nu^2-1\right)^3}{2\left(\nu^2-4\right)\left(\nu^2-1\right)^3} + \dots. \\
&= 4\xi^2 - \frac{2\hat\Lambda}{\epsilon_1^2}\,\partial_{\hat\Lambda}\left[\sum_{n=1}\left(\hat\Lambda\right)^{2n} f_\delta^{(n)}\right] \\
&= 4\xi^2 - \frac{4\hat\Lambda^2}{\epsilon_1^2}\,f_\delta^{(1)} - \frac{8\hat\Lambda^4}{\epsilon_1^2}\,f_\delta^{(2)} - \dots
\end{aligned} \tag{4.29}$$

To the lowest order one has $\xi = \frac{\nu}{2}$. Hence $\delta \equiv \epsilon_1^2\left(\frac14 - \xi^2\right) = \epsilon_1^2\left(\frac14 - \frac{\nu^2}{4}\right)$. We can read off the value of $f_\delta^{(n)}$ from the above, and they are indeed consistent with those found in (3.9).

It is observed that if we take the limit $\hat m \to \infty$ and $h \to 0$ with $\hat m h = \tilde h^2$ constant, the above reduces to the rank 1/2 ($N_f = 0$ case) given in [14].

5 Summary and discussion

In this paper we provide two ways to evaluate the classical limit of the irregular (2-point) conformal block with arbitrary rank. One is to take the direct limit from the irregular conformal block which is obtained using the irregular matrix model as presented in section 3. The classical irregular conformal block is given in an exponential form whose dominant contribution is checked by taking the classical limit of the irregular conformal block in section 3.2.

The other way is to solve the second order differential equation as given in section 4, which is obtained by the null condition of degenerate primary operator. The differential equation is derived for arbitrary rank. If one consider the expectation value of the degenerate primary field on the unit circle, then the equation turns out to be the generalized Mathieu equation whose potential is given as the superposition of exponential (cosine) terms. We provide an explicit solution for the rank 1. The method is easily generalized for higher rank. It is noted that for rank $n \geq 1$, there are n number of coefficients which are given in terms of the differential form of the classical conformal block with respect to the eigenvalues of L_k's. This will provide the classical analogue of the flow equation obtained in [18].

It is known that the classical limit of the irregular conformal block is not simple to evaluate. However, the generalized Mathieu equation provides an alternative approach to evaluate the classical conformal block in a systematic way. One may have 3-point conformal block with one degenerate primary field still in terms of the generalized Mathieu equation whose potential is not real but complex. In this case, it is more convenient to solve the differential equation on the complex plane rather than on a circle. The solution is given in Laurent series expansion of z with a fractional power term attached. The series expansion can be done where potential terms are given as perturbation. It will be interesting to find the complete solution and to investigate its analytical structure. In this paper we only consider the case on the sphere but it is not hard to extend to higher genus case. For the genus 1, classical regular conformal block is discussed in [12].

Note added. According to the referee report we add comments on a special limit of the irregular matrix model $Z_{(0;n)}$ which may reduce to $|G_m\rangle$ which appears in [19]. Suppose one scales the parameters $\hat{c}_k \to q\hat{c}_k$ $(1 \leq k \leq n)$ of the potential in (2.16) and considers the limit $q \to 0$. As a result all $\Lambda_k \to 0$. However, the ratio $\hat{c}_\ell/\hat{c}_n = c_\ell/c_n$ $(1 \leq \ell \leq n-1)$ is finite and one may still find the partition function in terms of the ratios. Note that v_1 and v_{n-1} commute with each other $[v_1, v_{n-1}] = 0$, and one may regard $\tilde{d}_1$ and $\tilde{d}_{n-1}$ in (4.17) as the eigenvalues of v_1 and v_{n-1}, respectively. In this case, $\tilde{d}_1$ and $\tilde{d}_{n-1}$ are independent of coefficients $\hat{c}_k$ and considered as input parameters where the filling fractions N_i are given as functions of $\tilde{d}_1$ and $\tilde{d}_{n-1}$.

In this framework, one may solve the equation

$$v_k(-g_s^2 \ln Z_{0,n}) = \tilde{d}_k \quad \text{for } 0 \leq k \leq n-1\,, \tag{5.1}$$

and $\tilde{d}_k$ (with $k \neq 1, n-1$) is to be found using the loop equation (2.13). The solution will have the form

$$-\,g_s^2 \ln Z_{(0;n)} = \tilde{d}_{n-1} h_{n-1} + \tilde{d}_1 \left(\frac{c_{n-1}}{c_n (n-1)} \right) + g(\{c_k/c_n\})\,, \tag{5.2}$$

where $g(\{c_k/c_n\})$ has to be determined by the condition $v_1(g) = v_{n-1}(g) = 0$ and $v_k(g) = \tilde{d}_k$ with $k \neq 1, n-1$ whose explicit solution is beyond the scope of this added note. On the other hand, h_{n-1} has explicit form which is the function of c_k/c_n's and depends on n: for example, when $n = 2$, h_1 reduces to c_1/c_2 and when $n = 4$, $h_3 = \big(c_1/c_4 - c_2 c_3/(3c_4^2) + 2c_3^3/(27c_4^3)\big)$. In general when $n \geq 3$, one has

$$h_{n-1} = \sum_{k=1}^{n-2} \frac{c_k}{c_n} \left(\frac{-c_{n-1}}{(n-1)c_n} \right)^{k-1} - (n-2) \left(\frac{-c_{n-1}}{(n-1)c_n} \right)^{n-1}. \tag{5.3}$$

It is also noted that the explicit form of irregular state $|G_m\rangle$ first constructed in [19] is different from $|\widetilde{G}_{2n}\rangle$ in (3.18):

$$|G_m\rangle = \sum_{\ell, Y, \ell_p} \Lambda^{2\ell/m} \left\{ \prod_{i=1}^{[\frac{m}{2}]} a_i^{\ell_{2m-i}} \right\} \left\{ \prod_{j=1}^{[\frac{m-1}{2}]} b_j^{\ell_j} \right\} Q_\Delta^{-1} \Big((m)^{\ell_m} (m-1)^{\ell_{m-1}} \cdots 2^{\ell_2} 1^{\ell_1}; Y \Big) L_{-Y} |\Delta\rangle\,. \tag{5.4}$$

This $|G_m\rangle$ does not reduce to $|\widetilde{G}_{2n}\rangle$ when $m = 2n$. The big difference is that the order of Young diagram in Q_Δ^{-1} is opposite to that of $|\widetilde{G}_{2n}\rangle$. In fact $|\widetilde{G}_{2n}\rangle$ is the simultaneous eigenstate of $\mathcal{L}_k$ with $n \leq k \leq 2n$. However, the opposite ordering makes $|G_m\rangle$ simultaneous eigenstate of $\mathcal{L}_1$ and $\mathcal{L}_m$: $\mathcal{L}_1 |G_m\rangle = \Lambda^{\frac{2}{m}} b_1 |G_m\rangle$ and $\mathcal{L}_m |G_m\rangle = \Lambda^2 |G_m\rangle$. This demonstrates that most of parameters except Λ and b_1 given in (5.4) are not fixed by the eigenvalue condition. The other parameters should be determined by solving the differential equation (5.1) as done in [8] for the case $|\widetilde{G_{2n}}\rangle$.

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