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Constraints on the quantum state of pairs produced by semiclassical black holes

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Abstract: The pair-production process for a black hole (BH) is discussed within the framework of a recently proposed semiclassical model of BH evaporation. Our emphasis is on how the requirements of unitary evolution and strong subadditivity act to constrain the state of the produced pairs and their entanglement with the already emitted BH radiation. We find that the state of the produced pairs is indeed strongly constrained but that the semiclassical model is consistent with all requirements. We are led to the following picture: Initially, the pairs are produced in a state of nearly maximal entanglement amongst the partners, with a parametrically small entanglement between each positive-energy partner and the outgoing radiation, similar to Hawking's model. But, as the BH evaporation progresses past the Page time, each positive-energy partner has a stronger entanglement with the outgoing radiation and, consequently, is less strongly entangled with its negative-energy partner. We present some evidence that this pattern of entanglement does not require non-local interactions, only EPR-like non-local correlations.

Keywords: Models of Quantum Gravity, Black Holes

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Contents

1 Introduction

In a recent series of papers [1–7], we have developed a new theory of semiclassical black hole (BH) evaporation, extending Hawking's seminal works [8–10] to finite-mass BHs. As motivated in [11–14], we have included the fluctuations of the background geometry by way of a wavefunction for the horizon of the incipient BH. Our guiding principle was to closely follow Hawking's original calculations — but, instead of assuming a classically fixed (Schwarzschild) metric, we evaluated background-dependent quantities as quantum expectation values in the state of the BH. This prescription guaranties that, on average, the emission rate from a BH is the standard thermal rate and that the fluctuations about the average rate are small. We have also included the classical time dependence of the particle emissions and their classical back-reaction on the BH.

The quantum corrections that arise from the background fluctuations are proportional to the inverse of the BH entropy, $1/S_{BH}$. This is contrary to the standard methods of effective field theory in a fixed curved-space background, for which one only expects exponentially small corrections from the quantum back-reaction of the matter fields on the classical geometry. But our framework goes beyond that of an effective field-theory description of BH evaporation. From the standard point of view, the corrections arising from our model would correspond to non-perturbative contributions to expectation values that vanish to all orders in perturbation theory.

Let us emphasize that BHs do not "burn" like normal black bodies do; rather, their burning (evaporation) is of a highly quantum nature. For example, the Sun and a BH at the

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same temperature will emit differently. The Sun emits radiation in a highly classical state with high occupation numbers for the emitted modes, whereas the BH, on the other hand, will emit a highly quantum state with low occupation numbers. (See, for instance, [6].) As Hawking showed in [10], these quantum emissions can (but do not have to) be described in terms of pair production. Indeed, by adapting Schwinger's pair-production formula to the case of a BH, one obtains the correct rate of emission (see below).

In [3], we have described BH evaporation in our semiclassical framework from the perspective of pair production. Just as in Hawking's model, the negative-energy pair partners are subsumed into the BH interior at a rate that is determined by the thermal rate of emission. But, for a BH of finite mass — and contrary to the situation in Hawking's model of an effectively eternal BH — this process acts to bound the number of entangled pairs in the near-horizon zone at any given time, such that their number is parametrically smaller than the BH entropy. This is the key fact that allows information to be released from the BH without over-exciting the state of the near-horizon region. However, in [3], we assumed that the pairs are produced in a pure state or, equivalently, that the produced pairs and the outgoing radiation are not entangled. Here, this assumption is relaxed and we in fact show that the pairs must be entangled to some degree with the outgoing radiation.

Let us consider a single pair, which is produced in a process that is akin to a gravitational version of the Schwinger effect. Schwinger's famous equation [15] predicts the rate per unit volume $R^{\rm E}_{PP}$ of electron-positron pair production in an electric field $\mathcal{E}$, where $R^{\rm E}_{PP} = \frac{\alpha^2}{\pi^2}\mathcal{E}^2 e^{-\frac{\pi m^2}{e\mathcal{E}}}$. Now suppose that one substitutes the gravitational force $F_{\rm G}$ for the electric force $F_{\rm E} = \mathcal{E}e$ in Schwinger's equation. If m denotes the relativistic mass of the positive-energy partner, the Newtonian gravitational force is given by $F_{\rm G} = \frac{G_N M_{BH} m}{r^2}$.[1] The resulting expression for the rate of gravitational pair production per unit volume $\mathcal{R}^{\rm G}_{PP}$ in the near-horizon limit is then

$$\mathcal{R}^{\rm G}_{PP} \sim \frac{\hbar}{R_S^4}\left(\frac{R_S m}{2\pi\hbar}\right)^2 e^{-\frac{2\pi R_S m}{\hbar}} . \tag{1.1}$$

This rate is maximized when $m = \frac{\hbar}{\pi R_S}$, which is of the order of the Hawking temperature $T_H = \frac{\hbar}{4\pi R_S}$; in which case, $\mathcal{R}^{\rm G}_{PP} \sim \frac{\hbar}{R_S^4}$. Meaning that, as expected, one Hawking pair is produced per light-crossing time R_S from a volume R_S^3. Away from the horizon ($r > R_S$), the rate of pair production becomes exponentially small, and so the idea that the quantum emission process originates from pair production near the horizon is indeed supported.

Schwinger's formula and its gravitational analogue provide the rate for the pair-production mechanism but contain no information about the state of the produced pairs. In Hawking's model, the entanglement is maximal, corresponding to the original vacuum state in the vicinity of the horizon. In the absence of additional physics introducing new scales, this follows from the adiabaticity of the collapse process [8, 9, 16]. However, in our model, the shell (incipient BH) is fluctuating and so adiabatic considerations need to be

[1]Here, G_N is Newton's constant, $M_{BH} = \frac{R_S}{2G_N}$ is the BH mass, R_S is the Schwarzschild radius, and the speed of light and Boltzmann's constant are set to unity.

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revised. This issue was discussed for harmonic oscillators in [14], but we would still like to understand in detail how the situation is changed for our semiclassical model. Another example in which adiabaticity is modified is given by the emission from fuzzballs [17, 18].

The overall goal of the current paper is to better understand the nature of the quantum state of the pairs that are produced in this gravitational Schwinger process in the context of our semiclassical model for BH evaporation.

The dependence of the degree of entanglement of pairs on the state of the producing field was addressed for electron-positron pair production by an electromagnetic field in, for example, [19, 20]. The state of a pair can vary between maximally entangled and a product state, depending on the properties of the background field. In our case, we do not have such a detailed knowledge of the dependence of the state of the pairs on the production conditions. We will, instead, choose to parametrize the possible states and look for general constraints from the conditions of unitarity evolution and strong subadditivity (SSA) of entropy.

In the context of BH pair production, entanglement is expressed in terms of quantum correlations of the times, frequencies and (possibly) the polarizations of the emissions. Because we consider a Schwarzschild BH, the angular momenta of the emitted particles has to sum up to zero and each particle has to be emitted within the thermal-frequency window. Consequently, the main variable whose quantum correlations determine the amount of entanglement is the emission time of the particles. These correlations determine the off-diagonal elements of both the single-particle density matrix for the emitted radiation and the density matrix for the produced pairs. In many discussions on the state of the emitted radiation, the entanglement is modeled in terms of spin degrees of freedom (qubits). One can also map the emission times formally into qubit states; however, it is important to remember that the true physical variables are different.

But an outside observer can only determine the state of the outgoing radiation. She might want to use her knowledge of this state to determine the state of the produced pairs. However, it is clear that such a determination can only be achieved by supplying additional ingredients about the physics of the pair-production process. Here, we will not attempt to construct such a physical model but, rather, use quantum-information concepts to constrain the possible models and to show that our semi-classical framework is indeed consistent with all the requirements.

The plan of the paper is as follows: The next section recalls some of the basic elements of our semiclassical model, as needed for the rest of the discussion. Then, in section 3, we consider the constraints that quantum theory imposes on the process of BH evaporation, on both general grounds and in the context of our semiclassical picture. In particular, we address Mathur's argument about the conflict between unitarity and SSA [21–24]. These ideas are put on a quantitative level in section 4, where the conditions of unitarity and SSA are used to constrain the state of the pairs in our framework. We are able to demonstrate that our model is consistent with both of these principles. The paper ends in section 5 with a brief summary.

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2 Review of semiclassical finite-mass black holes

Before proceeding, we will review some basic elements of our semiclassical framework. A four-dimensional Schwarzschild BH with radius R_S and mass M_{BH} is assumed for concreteness. All fundamental constants are set to unity except for $\hbar$ and Newton's constant G, or equivalently, the Planck length $l_P = \sqrt{\hbar G}$.

2.1 The basics

- The classicality parameter C_{BH}

  We define C_{BH} as the ratio of the Compton wavelength of the BH $\frac{\hbar}{2\pi M_{BH}}$ to its Schwarzschild radius R_S, then $C_{BH} = \frac{l_P^2}{\pi R_S^2} = S_{BH}^{-1}$. This parameter characterizes the deviation from a classically fixed, curved spacetime (i.e., $G \to 0$, $M_{BH} \to \infty$, $R_S \gg 1$ but finite), so that the semiclassical regime is when $C_{BH} \ll 1$ but finite. The semiclassical corrections to physical quantities typically come as a power series in C_{BH}. These corrections are non-perturbative from the point of view of an effective field theory in a fixed, curved background; hence, their inclusion is the essential difference between our analysis and the standard discussions in the literature.

  The parameter $C_{BH} = C_{BH}(R_S(t))$ can be thought of as a dimensionless, time-dependent $\hbar$. It is formally introduced into our theory as the dimensionless width (squared) of the BH wavefunction. The same parameter has appeared in a different guise in [25–29] and also corresponds to the (small) parameter $1/N^2$ in the AdS/CFT correspondence [30] when N is large but finite.

- The emission rate of Hawking radiation

  The Stefan-Boltzmann law determines the approximate classical time dependence of the Schwarzschild radius, $R_S(t) = R_S(0)\left[1 - \left(\frac{t}{\tau_{BH}}\right)^{1/3}\right]$, where $\tau_{BH} \simeq S_{BH}(0) R_S(0)$ is the BH lifetime. We may now use $S_{BH} \sim R_S^2$ along with the relation between the number of particle emissions[2] N and the entropy, $\Delta N = -\frac{\Delta M_{BH}}{E} \simeq -\frac{\hbar \Delta M_{BH}}{T_H} = -\Delta S_{BH}$, to obtain $N(t) = S_{BH}(0)\left(\frac{t}{\tau_{BH}}\right)^{2/3}$. This allows us to replace t by N and use the latter as our (dimensionless) time coordinate. For instance, the Page time [33, 34] is simply the "time" N when $S_{BH}(N) = N$. The average emission rate of the semiclassical model is equal to the Hawking emission rate with small fluctuations, of order C_{BH}, about the average rate.

- The semiclassical single-particle density matrix

  The single-particle density matrix of the outgoing Hawking radiation ρ_{SC}[3] is the semiclassical two-point function or number operator, but with an appropriate normalization. For a free theory, the full density matrix is completely determined by the

[2] This use of N should not be confused with the rank of the field theory in AdS/CFT.
[3] We previously called this matrix the "density matrix" or "radiation matrix".

corresponding single-particle density matrix. This is already evident in Hawking's work [10] and is further clarified in [6].

Each entry in the single-particle density matrix depends in principle on the frequency, polarization and emission time of the radiated particles. We ignore the polarization dependence. In the semiclassical model, the matrix ρ_{SC} no longer has Hawking's diagonal form and, as a result, the evaporation process becomes unitary even though the thermal-like emission spectrum is kept [5, 6]. The matrix ρ_{SC} picks up off-diagonal contributions that are uniform in terms of frequency but suppressed relative to the diagonal elements by $C_{BH}^{1/2}(N)$ [1].

The elements of ρ_{SC} do have a non-uniform suppression in terms of emission time; modes emitted at different times tend to decohere [2]. Nonetheless, if the radiation is being regularly monitored at intervals of $\Delta N \sim \sqrt{S_{BH}}$ or less, then this suppression can be compensated [5] (and see below). Thus, off-diagonal elements of the matrix ρ_{SC} can be regarded as uniform in magnitude with respect to both frequency and emission time. Knowing this form allows us to calculate the Rényi entropy of the full density matrix [6].

As implicit in our earlier works and explained in [6], ρ_{SC} can be viewed as an $N \times N$ matrix, with the indices running over the wave-packet modes with non-vanishing occupation number and with the diagonal elements given by the average occupation number for each mode. The elements can then be expressed to good approximation as [1, 2]

$$\begin{aligned} (\rho_{SC})_{ii} &= 1 , \\ (\rho_{SC})_{i\neq j} &= \sqrt{C_{\rm BH}(N)} e^{i\theta_{ij}} , \end{aligned} \tag{2.1}$$

where the phases θ_{ij} can be treated as random for most purposes.

- Tracking

  An observer is said to be tracking the radiation when she monitors and records the amplitudes and phases of all the non-vanishing elements of the single-particle density matrix ρ_{SC} at regular time intervals. These time intervals should be short enough (as described above) to allow the observer eventually to record all the entries of ρ_{SC} and, thus, reconstruct the full density matrix. Then, via the off-diagonal elements, she will possess knowledge about all the correlations of the emitted particles, even those that have since decohered. She will also have to compensate for the classical time-dependence of C_{BH} as explained in [5]. It will always be assumed that the radiation is being tracked.

- Applicability

  It should be noted that our formal methods employed Schwarzschild coordinates. This is appropriate because our framework, just like Hawking's, is formulated from the perspective of an external, stationary observer. It is possible that a free-falling observer — for whom Kruskal coordinates would be a better choice — would disagree on frame-dependent quantities such as the size of the horizon fluctuations. See [31, 32] for a case in point.

3 Constraints from unitarity and strong subadditivity

We will now address the apparent conflict between unitary evolution and SSA, as has often been emphasized by Mathur, and then, in the next section, explicitly verify that our model can satisfy both of these conditions. Our discussion focuses on parametric dependence; hence, numerical factors are sometimes omitted for clarity.

3.1 Unitarity

It has long been accepted — at least since the advent of the gauge-gravity duality — that an evaporating BH respects unitarity, as would be the case for any other process of quantum-mechanical evolution. The benchmark model for describing BH evaporation as a unitary process is the Page model [33, 34]. (Also see [35].) Page assumes that the BH starts off in a pure state and, at later times, views the remaining BH as the "purifier" of the emitted radiation in some random basis. Meaning that a random unitary transformation relates this basis to that in which the density matrix of the composite BH-radiation system has a single entry. With these assumptions, Page is describing the minimal requirement for a BH to release all of its information before the end of evaporation.

3.2 Strong subadditivity

Here, we recall Mathur's arguments [21–24] about the SSA inequality and explain their implication to our framework. Mathur's perspective is closely related to but distinct from that of the "firewall" proponents [36] (also, [37–41]). In brief, Mathur assumes a unitary model of BH evaporation which allows for only small corrections to Hawking's picture and contends that, for the system consisting of entangled pairs plus escaped Hawking particles, the entanglement entropy will grow monotonically throughout the pair-production process.

Mathur starts by assuming that the pairs are produced in a state which is approximately pure and that the pair partners are approximately in a state of maximal entanglement. This implies that the pairs are produced in a state which is approximately equal to that of the Hawking model. The deviations of the pair state from maximal entanglement is parametrized by a small parameter ϵ. Specifically, each of the produced pairs is assumed to have an associated entanglement entropy in bits of $S_{pair} = 1 - \epsilon$, where ϵ is meant as a small number. Mathur then applies the SSA inequality to conclude that, after the production of N pairs, the total entropy of the outgoing radiation is bounded from below by NS_{pair}. And so, as long as deviations from Hawking's model are small (i.e., $\epsilon \ll 1$), the entanglement entropy is necessarily large. Mathur correctly observes that a monotonically growing entanglement entropy is contrary to the behavior of a "normal" burning body.

The key assumption which is made by Mathur is that ϵ is roughly constant and small for each of the produced pairs. In particular, ϵ is assumed to have at most a weak dependence on the history of the BH; meaning that it is essentially an N-independent number. The constancy of ϵ is attributed to locality. It will be shown that this critical assumption is

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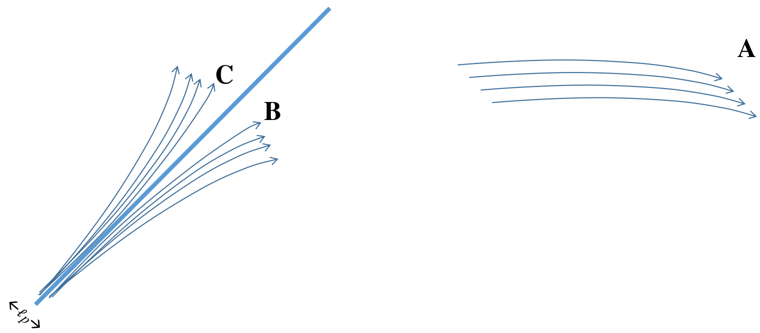

Figure 1. The ABC system shown on a spacetime diagram. The outgoing Hawking particle A is far from the BH horizon while the B and C modes are near the horizon.

modified in our framework and indications will be given that non-local interactions are *not* needed, only non-local correlations.

Let us next recall the precise meaning of SSA; first in general and then in the current context. The SSA inequality relies on the unitarity of quantum mechanics and is a statement about a tripartite quantum system $|A\rangle \otimes |B\rangle \otimes |C\rangle$. It asserts that the associated von Neumann entropies must satisfy the bound [42] $S_{AB} + S_{BC} \geq S_{ABC} + S_B$ or, equivalently,

$$S_{AB} + S_{BC} \;\geq\; S_A + S_C \; . \tag{3.1}$$

Here, $S_X = -\mathrm{Tr}_X[\widehat{\rho}_X \ln \widehat{\rho}_X]$ such that $\widehat{\rho}_X$ is the reduced density matrix for subsystem X. For instance, $\widehat{\rho}_{AB} = \mathrm{Tr}_C[\widehat{\rho}_{ABC}]$, $\widehat{\rho}_A = \mathrm{Tr}_{BC}[\widehat{\rho}_{ABC}]$ and so forth. Equality is obtained if and only if ABC is in a pure state, $S_{ABC} = 0$.

Now, in the context of BH pair production, Mathur takes subsystem A to be the positive-energy modes that have already moved far from the BH and B and C to be, respectively, the positive- and negative-energy modes of a newly formed entangled pair (see figure 1). If ϵ is indeed approximately constant and small, it is clear that $S_{BC} = 0$ up to irrelevant corrections and the reduced density matrix is then "thermal", $S_B = S_C = 1$. Hence, the bound reduces to

$$S_{AB} \geq S_A + 1 \; . \tag{3.2}$$

Now suppose that this newly formed pair is the N^{th} such pair produced. It would follow, according to Mathur, that $S_A = N - 1$ (as each escaped Hawking mode is still presumed to share nearly maximal entanglement with its partner) and thus

$$S_{AB} \geq N \; . \tag{3.3}$$

Extrapolating this process to the Page time, when the BH entropy is reduced to half its original size [33, 34], one obtains

$$S_{rad} \geq \frac{1}{2} S_{BH}(0) \; , \tag{3.4}$$

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where S_{rad} means the entropy of the external Hawking radiation. Mathur then contends that the process continues in this fashion until the BH can no longer be regarded as semiclassical. But, by then, it is too late to make S_{rad} small as the remaining BH lacks the entropy storage capacity to purify the radiation.

Mathur has argued[4] that, if one restricts the discussion to the context of effective field theory in a fixed background, then the small parameter ϵ cannot depend on the number of emitted particles N without an accompanying violation of locality. This is because the effects of a strictly local interaction can not depend on the characteristics nor the number of the previously emitted Hawking modes. We believe that this argument is indeed correct within the realm of effective field theory on a fixed, curved background. However, we will argue — pending further investigations — that, when the background is treated as a quantum state and its fluctuations are taken into account, such N-dependence is the result of EPR-like, non-local correlations rather than non-local interactions.

The essence of our argument is that the degree of entanglement could depend on the state of the BH (and possibly on the state of the pairs near the horizon), as typically happens in normal quantum systems. And, given unitary evolution, the state of the BH must depend on the state of the outgoing radiation. In fact, an example of such a phenomenon was given by Chowdhury and Mathur in the context of a fuzzball model [17, 18]. There, the early emissions of the Hawking particles indeed change the state of both the BH and the near-horizon matter such that later emissions are correlated with the early ones. In contrast, our semiclassical model, as of now, does not include the physics that allows us to estimate the degree of entanglement of the pairs. We then have to resort to general arguments and constraints.

It is standard, classically or semiclassically, to regard the entangled pairs as being created in a pure state. Our claim, however, is that the pairs are produced in an entangled state with the outgoing radiation and so cannot, by themselves, form a pure state. Initially, though, the entanglement of the pairs with the outgoing radiation is small and the deviation of the pair partners from maximal entanglement can be estimated. What we find is that the value of ϵ is not necessarily constant. Indeed, our model suggests that, at early times much before the Page time,

$$\epsilon(N) \lesssim \frac{2NC_{BH}}{(1+NC_{BH})} = \frac{2N}{S_{BH}(0)} \tag{3.5}$$

as follows from Bound (4.12) below. Besides the explicit dependence on N, one can also notice that ϵ can become large (order one) when approaching the Page time.

As will be shown later, the pairs are indeed significantly entangled with the outgoing radiation after the Page time; in which case, the parameter ϵ will have lost its meaning. This result suggests that, at times later than the Page time, the pairs are produced in a state that is becoming approximately a product state. This is really the essence of how Mathur's "small-corrections" theorem is evaded: The large quantum fluctuations as parametrized by ϵ are causing the state of the pairs to deviate from the Hawking state. A semiclassical background is, in itself, insufficient for this purpose [24].

[4]See, especially, the discussions about locality in [22].

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Now, to understand the origin of $\epsilon = \epsilon(N)$, let us consider the perspective of a tracking observer. Such an observer does not have direct knowledge about the state of the pairs. She may believe that it is Hawking's state of maximal entanglement plus some small corrections, but this is only a guess — just like, in the standard EPR situation, if Alice were to guess the spin of Bob's particle before even performing a measurement on her own. But, because of the entanglement between the external particles and the pairs, a tracking observer will naturally acquire knowledge about the near-horizon state. The more measurements that she performs, the more precise her knowledge about the state will become. At the same time, each such measurement will project the state of the pairs further away from a maximally entangled state. If the observer is continuously monitoring the radiation, then the number of measurements must be of order N^2.

The fact that the number of emissions at the Page time is of order of the BH entropy has another important consequence. So many emissions can change the state of the BH itself, even though each one on its own makes only a small change. This is because the emissions are coherent and, therefore, their effects can accumulate. Consequently, the deviation of the state of the pairs near the Page time from the early-time maximally entangled state could also scale with N, without requiring non-local interactions. From the perspective of an effective field theory in a fixed background, it would be impossible to see this effect at any order in perturbation theory. As such is the case, Mathur's argument about the necessity of non-local interactions would actually be correct.

One might still be tempted to attribute the N dependence of ϵ to non-local interactions rather than entanglement. However, if this were so, then the modifications of ϵ would be expected to depend on N times some coupling of the fields to the BH rather than be a function of NC_{BH}.

4 Constraining the semiclassical model

Of course, the previous discussion is a moot point if the evaporation process fails to be unitary or if the SSA inequality is invalidated. It is, therefore, worth checking that our semiclassical model is able to fulfill these minimal requirements.

A first test for any candidate model of unitary BH evaporation would be a demonstration that the rate of information release from the BH is at least fast as that predicted by the Page model [43]. Remarkably, our semiclassical model with tracking passes this first hurdle, as can be observed in [5, 6]. We now want to understand this information-transfer process from the pair-production perspective.

Given that unitary evolution is indeed viable, we will use the SSA inequality to constrain the state of the pairs, as well as the entanglement pattern between the pairs and the outgoing Hawking particles. Our model obeys the resulting constraints, but we still hope to be able to derive the state of the pairs from a physical model of pair production in future research.

For the purposes of this presentation, we will follow Mathur's argument and take subsystem A to be the escaped (early) Hawking particles, B to be a positive-energy mode in the near-horizon zone (or late Hawking radiation) and C to be its negative-energy partner

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in the zone. Although only one of the pairs need be considered, we could just as well work with a fraction of the pairs and arrive at the same conclusions.

4.1 Unitarity

An essential point that we would like to reemphasize is that the pair-production picture is inherently ambiguous as far as an external observer is concerned [10]. Such an observer can only know for sure about what she learns by measuring the emitted particles; namely, the state of the outgoing radiation.

This state was determined in [6], where the density matrix of the external radiation $\widehat{\rho}$ was evaluated. This led to a calculation of the Rényi entropy $H_2(\widehat{\rho})$, which is the main quantity that will be used to constrain the state of the pairs. This entropy is, up to subdominant corrections,

$$H_2(N) \;=\; \frac{N}{1+NC_{BH}} \;=\; \frac{N(S_{BH}(0)-N)}{S_{BH}(0)}\;. \tag{4.1}$$

It is worth mentioning that these expressions for the Rényi entropy have higher-order corrections in C_{BH} *but not* in NC_{BH}. Consequently, any of our findings are accurate as long as $C_{BH}=1/S_{BH}\ll 1$, which is assured until the final stages of evaporation.

The Rényi entropy stops growing when $NC_{BH}=1$ (i.e., at the Page time) and then decreases monotonically for the remainder of the BH's lifetime. That the entropy depends on N in just this way is indicative of a unitary process of evaporation, but it is also the main aspect of Mathur's argument that unitarity is in contradiction with the SSA constraint.

4.2 Strong subadditivity

To check the status of the SSA condition, we consider the bound

$$H_2(A)+H_2(C) \;\leq\; H_2(AB)+H_2(BC)\;, \tag{4.2}$$

which also implies the bounds

$$|H_2(A)-H_2(B)| \;\leq\; H_2(AB) \;\leq\; H_2(A)+H_2(B)\;. \tag{4.3}$$

Bound (4.2) is saturated when the ABC system is in a pure state, and a sharp inequality is expected when ABC is mixed. We may use the Rényi entropy, rather than the von Neumann entropy, because the Hawking modes are in a Gaussian state [6, 10] and the SSA inequality is respected by the Rényi entropy for such a state [44] as a consequence of the Hadamard-Fisher inequality [45].

As a measure of entanglement, we will use $E(X|Y)$, the (negative of the) conditional entropy [46]. This is actually a lower bound on the true entanglement between X and Y. The conditional Rényi entropy of the AB system is given by

$$E(B|A) \;=\; H_2(A)-H_2(AB) \tag{4.4}$$

and, similarly, for the BC system,

$$E(B|C) \;=\; H_2(C)-H_2(BC)\;. \tag{4.5}$$

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The SSA inequality (4.2) tells us that

$$E(B|A) + E(B|C) \;\leq\; 0\;, \tag{4.6}$$

which is essentially a statement of "monogamy of entanglement". For instance, if B is strongly entangled with C, then $E(B|C) \sim 1$ and so $E(B|A) \lesssim -1$, implying that B is weakly entangled with A (and *vice versa*).

We will first be considering the constraints at times later than the Page time when $NC_{BH} = 1$. A discussion about earlier times then follows.

We begin here with a relation that follows from eqs. (4.1) and (4.4),

$$\begin{aligned} E(B|A) \;&=\; H_2(N) - H_2(N+1) \\ &=\; \frac{2N+1}{S_{BH}(0)} - 1 \;\simeq\; \frac{2N}{S_{BH}(0)} - 1 \\ &=\; \frac{NC_{BH}-1}{1+NC_{BH}}\;. \end{aligned} \tag{4.7}$$

Since $H_2(N)$ starts to decrease only after the Page time, it is only after this time that $E(B|A)$ becomes positive and the SSA inequality becomes useful. The measure of entanglement $E(B|A)$ then grows monotonically from zero at $NC_{BH} = 1$ to unity when $NC_{BH} \gg 1$. This tells us that the positive-energy partner B is becoming more strongly entangled with the outgoing radiation as the evaporation proceeds.

From Bound (4.6), we also find that

$$E(B|C) \;\leq\; -E(B|A) \;=\; \frac{1-NC_{BH}}{1+NC_{BH}}\;. \tag{4.8}$$

After the Page time, the right-hand side of this bound is smaller than zero and, therefore, the conditional entropy $E(B|C)$ is negative. The entanglement between B and C is then getting weaker after the Page time; in particular, the BC system has to deviate significantly from a pure and maximally entangled state. Eventually, for $NC_{BH} \gg 1$, the difference $H_2(BC) - H_2(C)$ has to be of order 1, which means that the BC "pair" has to be substantially mixed — essentially, a product state as then $H_2(BC) \sim H_2(B) + H_2(C) \sim 2$. This stands to reason because, as seen above, the positive energy-partner already has a strong entanglement with A.

We can also use this formalism to place limits on the "small" parameter ϵ from the previous section. Since $H_2(A) \geq H_2(B) \sim 1$, at least until the late stages of the evaporation, it follows from the left-most relation in Bound (4.3) that

$$H_2(B) \;\geq\; H_2(A) - H_2(AB) \;=\; \frac{NC_{BH}-1}{1+NC_{BH}}\;. \tag{4.9}$$

Then, parametrizing $H_2(B) = 1 - \epsilon(N)$, we find that

$$\epsilon(N) \;\leq\; \frac{2}{1+NC_{BH}} \;=\; \frac{2(S_{BH}(0)-N)}{S_{BH}(0)}\;. \tag{4.10}$$

As one can see, $\epsilon(N)$ is large (order unity) close to the Page time, when B can be expected to have significant entanglement with both A and C. On the other hand, if

$NC_{BH} \gg 1$, then $\epsilon(N) \ll 1$. This is another indication that the AB system is becoming approximately pure and maximally entangled at times well past the Page time. Then, by monogamy of entanglement, the amount of entanglement between the pair partners has to decrease accordingly.

Let us now discuss the situation before the Page time. At these times, the SSA inequality is not particularly useful unless additional information about the pairs is provided. This is because of the minimal amount of entanglement between the pairs and the external radiation. This is all that an external observer can truly know about the state of the pairs at early times — as far as she is concerned, the BC system could just as well be in a product state, a maximally entangled state or somewhere in between.

However, we do know that, at early enough times, the semiclassical model can be viewed as Hawking's plus perturbatively small corrections. Hence, it can still be expected that the BC system is approximately pure and maximally entangled up to corrections of order $NC_{BH} \ll 1$.

And, in spite of the built-in ambiguity, we can still address the size of ϵ at early times. The right-most relation in Bound (4.3) leads to

$$H_2(B) \ \geq \ \frac{1 - NC_{BH}}{1 + NC_{BH}} \ . \tag{4.11}$$

Again parametrizing $H_2(B) = 1 - \epsilon(N)$, we then obtain

$$\epsilon(N) \ \leq \ \frac{2NC_{BH}}{1 + NC_{BH}} \ = \ \frac{2N}{S_{BH}(0)} \ . \tag{4.12}$$

This fits very well with the expected behavior of the BC system, being approximately pure and maximally entangled at early times and deviating from this picture as the Page time is approached.

4.3 A proposal for the entanglement

From the previous discussion, one can observe the pivotal role that is played by the Page time for which $NC_{BH} = 1$. We expect that the amount of entanglement between the outgoing radiation and the produced pair is very small at early times and large (order unity) at the later stages of evaporation. This suggests that, initially, the positive-energy horizon mode is almost in a product state with the outgoing radiation and almost maximally entangled with its negative-energy partner, $H_2(BC) \ll 1$. But, well after the Page time, when the horizon mode is significantly entangled with the outgoing radiation, it is almost in a product state with its negative-energy partner, $H_2(BC) \sim 1$. We further expect that, at intermediate stages close to the Page time, the horizon mode is substantially entangled with both its partner and the outgoing radiation, but with the sum of these two entanglements bounded by unity from above.

We can make the above expectations more quantitative by adding the assumptions that the amount of entanglement is symmetric under the exchange $N \leftrightarrow S_{BH}(0) - N$ or, equivalently, $NC_{BH} \leftrightarrow \frac{1}{NC_{BH}}$ (see section 5.3 of [6]) and that the total entanglement is fixed on the grounds of unitary evolution. On this basis, we can propose that the following equalities are valid at *all times* (*cf*, eq. (4.7)):

$$E(B|C) \;=\; -E(B|A) \;=\; \frac{NC_{BH}-1}{1+NC_{BH}}\,. \tag{4.13}$$

It is straightforward to check that all of the above expectations are realized by this proposal.

5 Summary and conclusion

In brief, we have revisited the pair-production picture of our semiclassical model of BH evaporation and have shown that it can be constrained in a way that is consistent with both unitary evolution and the SSA constraint on entropy (i.e., the monogamy of entanglement). The SSA inequality has proven to be a useful tool for constraining the state of the near-horizon modes, but a physical description of the process is still lacking in our model. We hope to address this matter at a future time.

Our findings are in general agreement with the arguments of Mathur, who argues that, within the realm of an effective theory of fields on a fixed background geometry, the corrections to Hawking's model are small and cannot depend on the number of emitted particles without violating locality. However, when additional non-perturbative effects due to quantum fluctuations of the BH itself are taken into account, these assumptions are no longer in effect. Indeed, the corrections from our model fail to satisfy both of Mathur's conditions. However, we argue that this violation can be attributed to non-local entanglement rather than non-local interactions. Further analysis will be required to settle this matter conclusively.

The SSA inequality necessitated an entanglement between the positive-energy pair partners and the external Hawking radiation; the degree of which becomes substantial after the Page time. It would then seem that a (so-called) firewall would be an inevitable feature in our model. Nonetheless, another consequence of this framework is an upper bound on the number of Hawking pairs in the near-horizon zone that is parametrically smaller than the BH entropy. This bound also limits the degree of excitement of the near-horizon state. The details of this argument are clarified elsewhere [4, 7].

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