Spin(7) duality for $\mathcal{N} = 1$ CS-matter theories

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ABSTRACT: In this paper we propose a duality for non-holomorphic $\mathcal{N} = 1$ CS-matter theories living on M2 branes probing Spin(7) cones. We call this duality Spin(7) duality. Two theories are named Spin(7) dual if they have the same moduli space: a real Spin(7) cone with base a weak G_2 manifold, and they are hence holographic dual to the same $AdS_4 \times G_2$ M theory solution. We provide a systematic way to generate these dualities, derived by combining toric duality for $\mathcal{N} = 2$ CS-matter theories and generalized non-holomorphic orientifold projections to $\mathcal{N} = 1$. Brane construction, AdS/CFT correspondence, and the computation of the moduli space support our proposal at the classical level and provide some arguments at the quantum strong coupling regime. The relation with Seiberg-like duality is also analyzed.

Contents

1 Introduction

Strongly coupled systems are interesting both from phenomenological and theoretical perspectives. However, understanding their dynamics is usually quite difficult. An interesting strategy to explore such phenomena consists of looking for an alternative, weakly coupled, description of the same system, the so called "dual description". Dualities have been discovered and studied in many contexts and they provided a deep insight in strongly coupled physics. Supersymmetric field theories are an useful laboratory to explore duality maps. Seiberg duality for SQCD [1] in four dimensions and its generalizations to $\mathcal{N} = 2$ three dimensional field theories [2–8] are examples of this map. Another well known duality for three dimensional field theories is the AdS$_4$/CFT$_3$ correspondence [9–11] that relates Chern-Simons (CS) matter theories to M theory AdS$_4$ solutions. In this case it has been shown that there are different UV field theory descriptions of the IR theory living on M2 branes probing the same toric Calabi-Yau four dimensional cone CY$_4$ [6, 12–15]. This phenomenon has been named toric duality and it is the three dimensional extension of

the previously discovered toric duality for the four dimensional field theories living on D3-branes probing Calabi-Yau three dimensional cones CY_3 [16–19]. For four dimensional field theories toric duality coincides with Seiberg duality [20, 21].

It was shown in [6] that, for some classes of theories, toric duality for M2 branes is a generalization of the $\mathcal{N} = 2$ Seiberg-like duality of [5]. $\mathcal{N} = 2$ CS-matter theories can be further reduced to $\mathcal{N} = 1$ CS-matter theories[1] living on stacks of M2 branes at certain conical singularities [26]: the so called Spin(7) cones. These theories can be obtained with a generalized orientifold projection from parents $\mathcal{N} = 2$ holomorphic theories describing stacks of N M2 branes probing the tip of toric CY_4 cones. In the geometric language this projection corresponds to the quotient done by an anti-holomorphic involution on the CY_4, that breaks the SU(4) holonomy to Spin(7)[2] [26, 29].

Inspired by the $\mathcal{N} = 2$ case one may ask if there are extensions of toric (and of Seiberg-like) duality to the $\mathcal{N} = 1$ case.[3] In this paper we use a geometrical approach and define a Spin(7) duality in analogy with the toric duality of the toric CY_4 case. Namely we say that two $\mathcal{N} = 1$ CS-matter theories are Spin(7) dual if they have the same classical moduli space for one regular M2 brane and if it coincides with the Spin(7) cone of the dual geometry. We provide a general picture to generate $\mathcal{N} = 1$ Spin(7) dual pairs obtained from parent toric dual $\mathcal{N} = 2$ theories. Some control on these dualities beyond the classical level is provided by the existence of the same AdS_4 dual geometry for both the dual CFTs and by planar equivalence.

In some cases the orientifold projects the $\mathcal{N} = 2$ theory to $\mathcal{N} = 1$ theories with only unitary groups. In these cases we argue that the Spin(7) duality is also an $\mathcal{N} = 1$ three dimensional Seiberg-like duality. Indeed it corresponds to move $\mathcal{N} = 1$ branes in the Hanany-Witten [31] projected setup.

The paper is organized as follows. In section 2 we review the main aspects of the projection of the CY_4 to Spin(7) and its interpretation in terms of an orientifold. In section 3 we state the main claim of the paper about the $\mathcal{N} = 1$ Spin(7) duality and explain the general idea behind it. In section 4 we provide some examples of dual pairs and give some checks about the validity of the duality. In section 5 we show examples where the Spin(7) duality can be regarded as a Seiberg-like duality. In section 6 we discuss the extension of $\mathcal{N} = 1$ Seiberg like duality to more general models. In section 7 we conclude. To complete the paper we provide also two appendices. In appendix A we explain the projection of the $\mathcal{N} = 2$ superspace to $\mathcal{N} = 1$ while in appendix B we present the $\mathcal{N} = 1$ superconformal algebra.

2 From $\mathcal{N} = 2$ CY_4 to $\mathcal{N} = 1$ Spin(7)

In this section we briefly review the non-holomorphic orbifold of the CY_4 geometry that we will use in the rest of the paper [29], and we will provide a short discussion of the associated

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[1]See [22–25] for some recent analysis of $\mathcal{N} = 1$ theories in three dimensions

[2]We refer the reader to [27, 28] where the reduction of M-theory on Spin(7) manifolds constructed by this method has been considered too.

[3]We refer the reader to [30] for another proposal of Seiberg-like duality in $\mathcal{N} = 1$ theories.

orientifold projection in field theory.[4] An interesting class of $\mathcal{N} = 2$ SCFTs [10, 11] describes the low energy dynamics of a stack of N M2 branes at the tip of a non compact eight-dimensional CY$_4$ real cone: C(H_7), where H_7 is a seven dimensional compact Sasaki-Einstein manifold at the base of the cone.

The field theory is a quiver gauge theory. A quiver is a graph with nodes connected by arrows. Each node represents a gauge factor $U(N_i)$. There are also matter fields, represented by oriented arrows. Arrows with the tip and the tail on the same node are fields in the adjoint representation of the gauge group, arrows connecting the i-th with the j-th node are associated to fields in the bifundamental representation. In the Lagrangian each $U(N_i)$ factor has CS action with integer level k_i, and no Yang-Mills (YM) action. From now on we will keep track of the CS level and the rank of the gauge group factor by using the notation: $U(N_i)_{k_i}$.

These field theories are dual, in the gauge/gravity correspondence, to M-theory on the AdS$_4 \times H_7$ background. In this paper we consider a particular projection of this theory that breaks the four real supercharges down to two real supercharges.[5] The resulting theory is still a superconformal CS-matter theory, like before, but with only $\mathcal{N} = 1$ supersymmetry in three dimension. Moreover it does not have holomorphic properties: fields and superpotential are real. It describes the low energy dynamics of N M2 branes living at the tip of a Spin(7) cone: $C(G_2)$, where G$_2$ is a seven dimensional compact weak G$_2$ manifold. These theories are dual, in the gauge/gravity correspondence to M-theory on the AdS$_4 \times G_2$ background.

On the geometry the projection is obtained by modding the original CY$_4$ by the action of an anti-holomorphic involution Θ [29]. This geometric procedure is implemented in field theory by projecting the lagrangian using an orientifold projection [26] as we will review in the rest of this section and in the following section.

A CY$_4$ has a Kahler $(1,1)$ form J and a holomorphic $(4,0)$ form ω, that are left invariant by the holonomy group of the manifold: SU(4). Following [29] we use the action of an anti-holomorphic involution Θ to define a Spin(7) manifold. Θ acts on J and ω as $\Theta : \omega \to \overline{\omega}$, and $\Theta : J \to -J$, and it breaks the SU(4) holonomy to Spin(7). Using the defining forms of the CY$_4$ it is indeed possible to construct a closed self dual four form

$$\Omega_4 = \frac{1}{2} J \wedge J + Re(\omega) \tag{2.1}$$

that is left invariant under the action of Θ and hence defines a Spin(7) manifold [29]. In the field theory it is possible to interpret a class of these quotients as an orientifold [26, 34–37]. Because there are no open strings in M-theory it is easier to define its action by looking at the type IIA limit. Indeed the CY$_4$ cone Y that we consider can be written as a double fibration of a CY$_3$ Z, over a real line, parameterized by the real coordinate σ, and a circle, parameterized by an angle ψ [10, 11, 38–40]. The angle ψ parameterizes the M-theory

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[4]In the next section we will report some more details on the field theory.

[5]In M-theory the background $\mathbb{R}^{1,2} \times C$ preserves four real supercharges ($\mathcal{N} = 2$ susy in three dimension), if C is a CY$_4$ manifold, or two real supercharges ($\mathcal{N} = 1$ susy in three dimension), if C is a Spin(7) manifold [32, 33].

circle while σ is the expectation value of a particular combination of the D terms in field theory. In the type IIA limit one describes the worldvolume theory of D2 branes probing a seven dimensional manifold given by Z fibered over a line. The four form ω is locally

$$\omega \sim f(z_i)dz_1 \wedge dz_2 \wedge dz_3 \wedge (d\sigma + id\psi) \tag{2.2}$$

where z_i are the holomorphic coordinates of Z and $f(z_i)$ is a holomorphic function. We choose an anti-holomorphic involution Θ that acts on the M-theory circle as $\Theta : \psi \to -\psi$, and that leaves invariant the coordinate σ. This class of quotients in M-theory can then be interpreted as an orientifold projection [26, 34–37]. One then concludes that the field theory living on the M2 branes at the tip of Y/Θ geometry is the IR strong coupling limit in M-theory of the $\mathcal{N} = 1$ orientifold theory living on a stack of N D2 branes in type IIA [26]. From now on we refer to the $\mathcal{N} = 2$ theories as the "parent theories", while we refer to the $\mathcal{N} = 1$ theories as the "projected theories".

3 Spin(7) duality: our strategy

In this section we discuss our approach to generate and check Spin(7) dualities between $\mathcal{N} = 1$ three dimensional CS-matter theories with gravity duals.

First we give some general remarks of the Spin(7) duality that we are proposing. Two UV $\mathcal{N} = 1$ field theories are Spin(7) dual if their moduli spaces coincide and they are equivalent to the Spin(7) cone probed by one M2 brane. This duality is the analogous of the toric duality for $\mathcal{N} = 2$ theories living at the tip of toric CY_4 cones. There are some important differences between the two dualities. First in the $\mathcal{N} = 2$ case the gauge theory that lives on an M2 brane is abelian, while in the $\mathcal{N} = 1$ case the theory for a single M2 is usually non-abelian. Second, both toric and Spin(7) duality are classical dualities. In the $\mathcal{N} = 2$ case the duality is valid also at quantum level. The $\mathcal{N} = 1$ theories are not holomorphic and one may expect quantum corrections. Anyway the underlining AdS/CFT duality provides some arguments supporting the duality also in the quantum strongly coupled regime. Further studies are however required to understand the quantum properties of the proposed Spin(7) duality, and we leave them for future works. A last important remark concerns the relation between Spin(7) duality and Seiberg-like duality. For $\mathcal{N} = 2$ three dimensional CS-matter theories it has been shown that, for a particular class of theories, the so called $\widetilde{L_k^{aba}}$ models, some toric dualities are actually Seiberg-like dualities [6]. In this paper we will discuss some cases in which also the Spin(7) duality is a Seiberg-like duality.

In the following we provide a step by step illustration of our strategy to obtain $\mathcal{N} = 1$ pairs, and to check the validity of the Spin(7) duality. We start by introducing in some details the $\mathcal{N} = 2$ parent theories living on N M2 branes at the tip of a CY_4 cone, and discuss their moduli space. A discussion on the orientifold projection to $\mathcal{N} = 1$ in field theory follows. Then we explain our general strategy to obtain the moduli space of $\mathcal{N} = 1$ field theories and to match the moduli space and the geometry of $\mathcal{N} = 1$ field theory dual pairs. We conclude with a discussion on the relation between Spin(7) duality and Seiberg-like duality. More details could be found in [6, 10, 11, 26]. From now on we will refer to

the $\mathcal{N} = 2$ field theory as the "parent theory", while we will call the $\mathcal{N} = 1$ theory the "projected theory".

The $\widetilde{L^{aba}}_{k_i}$ $\mathcal{N} = 2$ CS-matter theories. The $\mathcal{N} = 2$ parents theories we consider are three dimensional extensions of L^{aba} four dimensional quiver gauge theories [41–43], introduced in [10, 11]. They are CS-matter theories with a product of $U(N_i)$ gauge groups and CS levels k_i, with $i = 1, \ldots a + b$, with pairs of bifundamental-antibifundamental connecting each pair of consecutive $U(N_i)$ and possibly adjoint fields. Every field appears twice in the superpotential with opposite sign, such that every F-term is an equality between two monomials with the same sign. From now on we will refer to these theories as $\widetilde{L^{aba}}_{k_i}$. Examples of the $\widetilde{L^{aba}}_{k_i}$ quivers are given in figure 1 and 2.

These are the low energy theories living on M2 branes at particular CY_4 singularities that are the double fibration of the L^{aba} CY_3 singularity over a segment parameterized by σ and a circle parameterized by ψ. In the UV they have a simple type IIB description in terms of branes [9, 31, 44, 45].

Brane setup and dualities. The $\widetilde{L^{aba}}_{k_i}$ theories can be engineered as a stack of D3 branes on a circle ending on a set of $(1, p_i)$ five-branes, where $i = 1, \ldots, a + b$: N_i D3s for every interval between a $(1, p_i)$ and a $(1, p_{i+1})$ five-brane. This construction corresponds to a circular quiver with $a + b$ gauge groups and a pair bifundamental-antibifundamental connecting each pair of consecutive nodes that are actually the type IIB strings stretching through the i-th five-branes. The N D3 branes are extended along the directions (x_0, x_1, x_2) and the direction x_6 compactified on a circle. The NS5 and the D5 branes, that recombine into the five-branes, are divided in two sets. In the first case one NS is extended along $(x_0, x_1, x_2, x_3, x_4, x_5)$ and the corresponding p_i D5 are extended along $(x_0, x_1, x_2, x_4, x_5, x_7)$. In the second case one NS is extend-end along $(x_0, x_1, x_2, x_3, x_8, x_9)$ and the corresponding p_i D5 are extended along $(x_0, x_1, x_2, x_7, x_8, x_9)$. There are a $(1, p_i)$ five-branes of the first type and b five-branes of the second type. The SCFT lives in the (x_0, x_1, x_2) directions common to all the branes. The NS branes and the corresponding D5 branes get deformed in $(1, p_i)$ five-branes at angles $\tan \theta_i \simeq p_i$. The Chern-Simons levels are associated with the relative angle of the branes in the $(3, 7)$ directions, they are $k_i = p_i - p_{i+1}$, such that $\sum_i k_i = 0$. When the $(1, p_i)$ and the $(1, p_{i+1})$ five-branes are parallel there is a massless adjoint field associated to the i-th gauge group. In the minimal phase there are $b - a$ nodes with an adjoint fields and $2a$ nodes without the adjoint.

By exchanging two consecutive (non parallel) five-branes one has a local transformation on the quiver, that corresponds to a Seiberg-like duality in field theory. If this action is performed on the i-th gauge group we have the transformation [6]

$$
\begin{aligned}
U(N)_{k_{i-1}} &\to U(N)_{k_i + k_{i-1}} \\
U(N)_{k_i} &\to U(N + |k_i|)_{-k_i} \\
U(N)_{k_{i+1}} &\to U(N)_{k_i + k_{i+1}}
\end{aligned}
\tag{3.1}
$$

It is possible to demonstrate in full generality that this local transformation preserves the moduli space [6]: CY_4 moduli space associated to the same dual supergravity background.

Moduli space. The moduli space of these theories is the set of values of the scalar fields that solve the zero condition for the bosonic potential. This boils down to solve the following set of equations.

$$\partial_{X_{ab}} W = 0$$
$$D_a(X) = \frac{k_a \sigma_a}{2\pi}$$
$$\sigma_a X_{ab} - X_{ab}\sigma_b = 0$$

(3.2)

where W is the superpotential, X_{ab} are scalar components of the bifundamental fields between the $U(N_a)$ and the $U(N_b)$ factor of the gauge group,[6] $D_a(X)$ is a real function of the bifundamental fields that corresponds to the usual D-terms, and σ_a are the real scalar components of the vector multiplet for the $U(N_a)$ factors.

For N M2 branes at the tip of the cone, without fractional branes, we have: $N_a = N_b = N$. The moduli space is then simply the N-times symmetric product of the moduli space for one brane. For one regular M2 brane, the gauge group is simply $U(1)^G$. The moduli space is found by imposing the set of three equations in (3.2) and by quotienting by the appropriate gauge group factors. It is important to notice that in the abelian case the third equation in (3.2) simply imposes: $\sigma_a = \sigma$, while one of the D-term equations is redundant, because $\sum_a k_a = \sum_a D_a(X) = 0$. Then we are left with $G - 1$ linearly independent equations. One of these equations can be written along the direction of the CS levels and it fixes the value of $\sigma_a = \sigma$, while the remaining $G - 2$ are orthogonal to this direction and equate the $G - 2$ linear combinations of D terms to zero. We should then quotient by the associated $G - 2$ $U(1)$ factors, while the $U(1)$ corresponding to the D-term orthogonal to the CS is broken to $\mathbb{Z}_{gcd\{k_a\}} = \mathbb{Z}_k$ and only imposes an additional discrete quotient. The moduli space of an $\mathcal{N} = 2$ CS-matter theory is then in general a $\mathbb{Z}_k$ quotient of a CY_4 cone Y, where k is the maximum common divisor of the CS levels [10, 11].

The analysis of the moduli space of the dual pairs generated using the transformation (3.1) for the $\widetilde{L^{aba}}_{k_i}$ theories was done in [6] and it was shown that these models have the same moduli space and are toric dual. The main claim of this paper is that similar dualities exist in the $\mathcal{N} = 1$ case, when the dual geometry is described by a Spin(7) manifold obtained as explained in section 2. To support this claim we provide a coherent geometrical and brane-orientifold construction. The CY_4 moduli space of two toric dual $\mathcal{N} = 2$ theories is projected on the same Spin(7).

We first show how to compute in the $\mathcal{N} = 1$ case the moduli space for a single M2 brane probing a Spin(7) cone. This is the non-holomorphic quotient of the original CY_4. We then check that the two $\mathcal{N} = 1$ theories, claimed to be Spin (7) dual, obtained by projecting the parent $\mathcal{N} = 2$ theories, have the same moduli space.

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[6]With some abuse of notation we will often use the same symbol: X_{ab} to refer both to the superfield or to its lowest scalar component. We hope that the reader will not get confused. What we meant should be clear from the context.

Field theory projection to $\mathcal{N} = 1$. As explained in section 2 the Spin(7) cone is obtained by quotienting the CY$_4$ Y by the anti-holomorphic involution Θ. This corresponds to a real orbifold of Y in M-theory and it acts as an orientifold on the dual field theory. In this subsection we briefly discuss the action of the projection on the field theory lagrangian, while in the next subsection we show that the moduli space of the projected theory is actually the Spin(7) geometry [26].

There are two interesting classes of orientifold projections. In the first class the orientifold action identifies the gauge groups with themselves, projecting the unitary $U(N)$ groups of the $\mathcal{N} = 2$ parent theory to orthogonal $O(2N)$ and/or symplectic $SP(2N)$ groups in the $\mathcal{N} = 1$ projected theory.[7] In the second class the orientifold action instead identifies pairs of $U(N)$ gauge group factors of the $\mathcal{N} = 2$ parent theory projecting them to a single $U(2N)$ group in the $\mathcal{N} = 1$ projected theory. It is important to underline that the orientifold acts in general as an anti-holomorphic involution on the matter fields in the lagrangian and it breaks the holomorphic structure of the $\mathcal{N} = 2$ theory, preserving only $\mathcal{N} = 1$ supersymmetry.

In the first class, where the projection identifies the a-th group with itself, the orientifold acts on the gauge and matter fields as:

$$
\begin{aligned}
A_\mu^a &\to -\Omega_a (A_\mu^a)^T \Omega_a^{-1} \\
X_{ab} &\to \Omega_a X_{ab}^* \Omega_a^{-1} \\
\sigma_a &\to \Omega_a \sigma_a^T \Omega_a^{-1} \\
D_a &\to \Omega_a D_a^T \Omega_a^{-1}
\end{aligned}
\tag{3.3}
$$

where Ω_a could be either the identity or the symplectic matrix. When $\Omega_a = I_{2N}$ it projects the unitary to an orthogonal group, if instead $\Omega_a = J_{2N}$ it projects the unitary to a symplectic group.

In the second class, where instead the projection identifies pairs of groups, $a_i \leftrightarrow b_i$, the orientifold acts on the gauge and matter fields as:

$$
\begin{aligned}
A_\mu^a &\to -\Omega_{ab} (A_\mu^b)^T \Omega_{ab}^{-1} \\
X_{a_1 a_2} &\to \Omega_{a_1 b_1} X_{b_1 b_2}^* \Omega_{a_2 b_2}^{-1} \\
\sigma_a &\to \Omega_{ab} \sigma_a^T \Omega_{ab}^{-1} \\
D_a &\to \Omega_{ab} D_b^T \Omega_{ab}^{-1}
\end{aligned}
\tag{3.4}
$$

where, as before, the Ω_{ab} matrix could be either the identity or the symplectic matrix.

In both cases, because A and σ have different transformation rules, the $\mathcal{N} = 2$ vector multiplet is broken to the sum of the $\mathcal{N} = 1$ vector multiplet and the real $\mathcal{N} = 1$ matter multiplet. Moreover it is manifest in (3.3) and (3.4) that the involution breaks the holomorphic structure of the superpotential. The details on the $\mathcal{N} = 1$ lagrangian are reported in appendix A.

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[7]Please observe that the standard orientifold procedure implies that we should double the ranks of the gauge groups and the CS-levels before quotienting the theory. Here we use the convention $SP(2)_k = \mathrm{SU}(2)_{2k}$ for the symplectic cases.

It is maybe important to remind that the action for which we quotient the $\mathcal{N} = 2$ theory is a symmetry of the theory itself.

The moduli space of $\mathcal{N} = 1$ theories and Spin(7) duality. As discussed above we have a Spin(7) duality if the proposed pair of field theories have the same moduli space for one M2 brane: the Spin(7) cone obtained as the non-holomorphic quotient of the CY$_4$ cone moduli space of the parent theories. Here we sketch our strategy to compute the moduli space and verify the Spin(7) duality.

The moduli space for one M2 brane is obtained by setting $N = 1$ in all the gauge group factors. It is important to underline that finding the $\mathcal{N} = 1$ moduli space for one M2 brane is in general a difficult task. Indeed, first of all, even for one brane the gauge group is in general non abelian: namely it is the product of SU(2), $U(2)$ and $O(2)$ gauge groups, and hence the equation defining the moduli space are two by two matrix equations. Moreover the moduli space of an $\mathcal{N} = 1$ field theory in three dimension is real and non-holomorphic and hence one cannot use the powerful tools of the algebraic complex geometry. Following [26] we proceed as follows. We provide an ansatz for the two by two matrices describing the matter fields of the $\mathcal{N} = 1$ theory in terms of the complex scalar fields of the $\mathcal{N} = 2$ parent theory for one M2 brane. It follows that the zero potential condition for the $\mathcal{N} = 1$ theory reproduces exactly the same equations of the parent theory (3.2) in terms of the ansatz fields. We then verify that the ansatz exhausts the vacuum space of the $\mathcal{N} = 1$ theory, i.e. that there are no other connected flat directions.

The moduli space is obtained by quotienting by the action of the gauge group. The ansatz we use is perfectly suited for this scope. Indeed, as we will explicitly see in the following examples, our ansatz breaks the gauge group down to its abelian subgroup: a bunch of SO(2)s plus the discrete non-holomorphic Θ. The SO(2)s that leave the ansatz invariant act as the $U(1)$s of the parent theory on the ansatz fields. Hence the quotient by the SO(2)s exactly reproduces the CY$_4$ Y cone quotiented by the additional discrete action $\mathbb{Z}_k$ associated to the CS levels. The remaining discrete action Θ is generated by the parity inversion $\sigma_3 \in O(2)$ and the element $i\sigma_3$ of SU(2) and $U(2)$. This last action exactly generates the needed anti-holomorphic involution to obtain the Spin(7) cone as explained in section 2.

By following this procedure we systematically check that the moduli spaces for one M2 brane for pairs of theories, claimed to be dual, are the same and that they coincide with the Spin(7) cone obtained by the anti-holomorphic involution on the CY$_4$ cone of the associated parent theories.

In the near horizon limit the AdS/CFT correspondence provides some arguments to support the fact that the dual pairs of theories previously constructed are actually two equivalent UV descriptions of the same IR strong coupling fixed point, dual to M-theory on AdS$_4 \times G_2$ background.

Relation with Seiberg-like duality. When the orientifold action leaves unitary groups we can sometimes argue that the Spin(7) duality is a Seiberg-like duality. In this case we can think to a type IIB brane setup that is locally $\mathcal{N} = 2$, but globally $\mathcal{N} = 1$. Supersymmetry is broken to $\mathcal{N} = 1$ because of the orientifold on some gauge group or on

	$O(2N)_{-2k}$	$U(2N)_{2k}$	$SP(2N)_{-k}$
$\mathcal{Q}_1$	□	□	□
$\widetilde{\mathcal{Q}}_1$	□	□	□
$\mathcal{Q}_2$	□	□	□
$\widetilde{\mathcal{Q}}_2$	□	□	□

Table 1. Representations of the bifundamental fields under the gauge groups.

some bifundamental fields not involved in the duality. In this case we can move consecutive $(1, p_i)$ branes and locally reproduce the same transformation as in (3.1). We claim that the resulting theory is Seiberg-like dual to the first theory. Indeed it has been obtained by applying the usual rules for brane exchange and brane creation.

A first check of the duality is that the $\mathcal{N} = 1$ theory obtained by moving the branes is indeed exactly the theory that we would have obtained instead projecting the Seiberg-like dual theory of the parent $\mathcal{N} = 2$ theory, closing in this way the circle of dualities.

4 Examples

In this section we study examples of Spin(7) dualities between pairs of three dimensional gauge theories along the lines explained in the previous section. We adopt the following strategy. First we introduce the $\mathcal{N} = 1$ conjectured dual pairs and then we show that these models describe the same IR physics.

We show that two conjectured $\mathcal{N} = 1$ dual theories can be obtained by projecting two $\widetilde{L^{aba}}_{k_i}$ $\mathcal{N} = 2$ toric dual models. These $\mathcal{N} = 2$ models are toric quiver gauge theories associated to CY$_4$ singularities. By projecting these dual pairs with the anti-holomorphic involution introduced in section 2 we obtain $\mathcal{N} = 1$ dual pairs that reproduce the same Spin(7) geometry. These models are Spin(7) dual.

First we present a very simple example. It is a toy model, where the Spin(7) duality actually coincides with a parity transformation, that should however help the comprehension of our strategy. In the second example we increase the complexity studying a more intricate example of Spin(7) duality.

4.1 First example

The first Seiberg-like dual pair that we consider consists of $\mathcal{N} = 1$ CS matter theories with three gauge groups as presented in figure 1. The gauge groups are

$$O(2N)_{-2k} \times U(2N)_{2k} \times SP(2N)_{-k} \tag{4.1}$$

and four bifundamental fields $\mathcal{Q}_1$, $\widetilde{\mathcal{Q}}_1$, $\mathcal{Q}_2$ and $\widetilde{\mathcal{Q}}_2$ transforming under the gauge groups as table 1.

The $\mathcal{N} = 1$ superpotential is

$$
\begin{aligned}
W = &-\mathcal{Q}_1 J \widetilde{\mathcal{Q}}_1^* \mathcal{Q}_1^* J \widetilde{\mathcal{Q}}_1 + \widetilde{\mathcal{Q}}_1^* \mathcal{Q}_2^* \widetilde{\mathcal{Q}}_2^* \mathcal{Q}_1^* - \mathcal{Q}_2^* \widetilde{\mathcal{Q}}_2 \mathcal{Q}_2 \widetilde{\mathcal{Q}}_2^* - \widetilde{\mathcal{Q}}_2 \mathcal{Q}_1 \widetilde{\mathcal{Q}}_1 \mathcal{Q}_2 \\
&- \frac{k}{\pi}(R_{SP}^2 + R_O^2 - R_U^2) + R_O \left(\mathcal{Q}_1^\dagger \mathcal{Q}_1 + \mathcal{Q}_1^T \mathcal{Q}_1^* - \widetilde{\mathcal{Q}}_1 \widetilde{\mathcal{Q}}_1^\dagger - \widetilde{\mathcal{Q}}_1^* \widetilde{\mathcal{Q}}_1^T \right) \\
&+ R_U \left(\mathcal{Q}_1^\dagger \widetilde{\mathcal{Q}}_1 - \mathcal{Q}_1 \mathcal{Q}_1^\dagger - \mathcal{Q}_2^\dagger \widetilde{\mathcal{Q}}_2 + \mathcal{Q}_2 \mathcal{Q}_2^\dagger \right) \\
&+ R_{SP} \left(\widetilde{\mathcal{Q}}_2 \widetilde{\mathcal{Q}}_2^\dagger - J \widetilde{\mathcal{Q}}_2^* \widetilde{\mathcal{Q}}_2^T J - \mathcal{Q}_2^\dagger \mathcal{Q}_2 + J \mathcal{Q}_2^T \mathcal{Q}_2^* J \right)
\end{aligned}
\tag{4.2}
$$

We claim that this model is Spin(7) dual to another $\mathcal{N} = 1$ CS matter theory with gauge groups:

$$
O(2N)_{2k} \times U(2N)_{-2k} \times SP(2N)_k \tag{4.3}
$$

with four bifundamental fields $\mathcal{Q}_{di}, \widetilde{\mathcal{Q}}_{di}$, $i = 1, 2$, as in figure 1, and the $\mathcal{N} = 1$ dual superpotential coincides with (4.2) with $k \to -k$. These two models can be obtained by projecting two toric dual parent $\mathcal{N} = 2$ theories.

$\mathcal{N} = 2$ parents. The parent $\mathcal{N} = 2$ theories are denoted as $\widetilde{L_{k_i}^{222}}$ theory. There are two possible quivers associated to this singularity, each with four gauge groups. One has eight bifundamentals and quartic couplings and the second one has eight bifundamentals and two adjoints. Here we analyze the moduli space for one M2 brane, where the gauge group is simply $U(1)^4$. The moduli space for the $U(N)$ case is the N-times symmetric product of the moduli space for a single brane.

At this point of the discussion we specify a choice of CS levels useful to perform the orientifold. We choose the levels as $\vec{k} = (k, -k, k, -k)$. The $\mathcal{N} = 2$ superpotential for the first phase is

$$
W_I = Q_{12}Q_{23}Q_{32}Q_{21} - Q_{23}Q_{34}Q_{43}Q_{32} + Q_{34}Q_{41}Q_{14}Q_{43} - Q_{41}Q_{12}Q_{21}Q_{14} \tag{4.4}
$$

The equations of motion are solved by

$$
Q_{12}Q_{21} = Q_{34}Q_{43} \quad , \quad Q_{23}Q_{32} = Q_{14}Q_{41} \tag{4.5}
$$

The operators gauge invariant with respect to the gauge factors orthogonal to the CS vector are

$$
\begin{aligned}
x_1 &= Q_{12}Q_{21} & x_2 &= Q_{23}Q_{32} & x_3 &= Q_{34}Q_{43} & x_4 &= Q_{14}Q_{41} \\
x_5 &= Q_{12}Q_{34} & x_6 &= Q_{21}Q_{43} & x_7 &= Q_{23}Q_{41} & x_8 &= Q_{32}Q_{14}
\end{aligned}
\tag{4.6}
$$

They are related by

$$
x_1 x_3 = x_1^2 = x_5 x_6 \qquad x_2 x_4 = x_2^2 = x_7 x_8 \tag{4.7}
$$

These equations define the CY$_4$ Y that has to be mod by the $\mathbb{Z}_k$ along the direction of the CS.
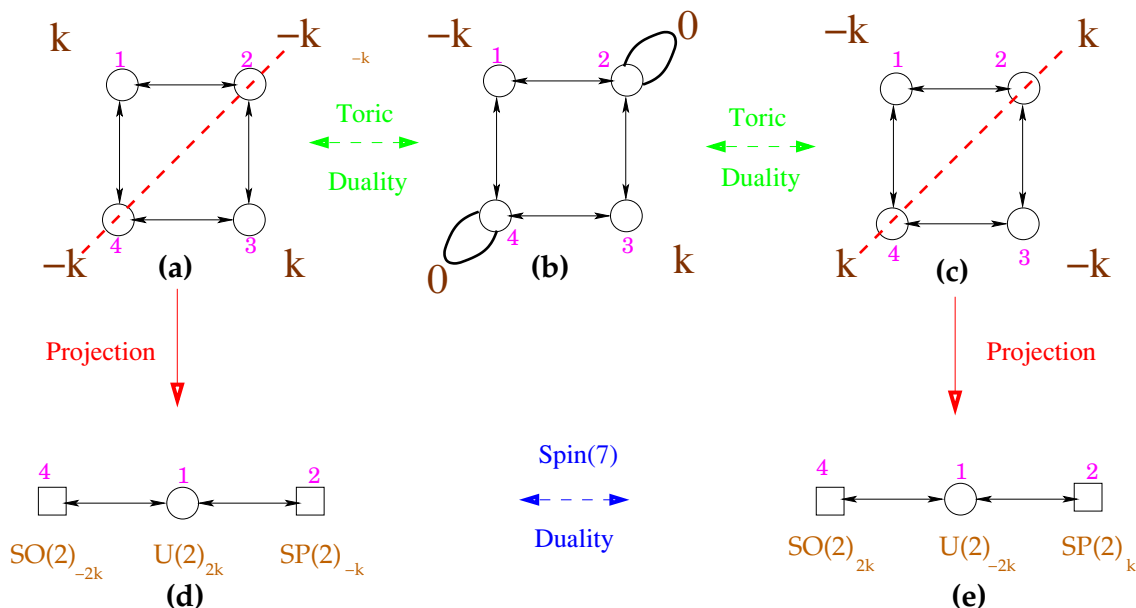
Figure 1. This picture represents schematically the relation between toric duality and Spin(7) duality. The models in (*a*), (*b*) and (*c*) represent three toric dual phases. In the cases (*a*) and (*c*) the orientifold projection acts by folding the quiver, along the dashed red lines. By projecting these models to $\mathcal{N} = 1$ Spin(7) cones we obtain the phases (*d*) and (*e*), that are related by Spin(7) duality.

The second quiver is represented in figure 1 (b). It has superpotential

$$W_{II} = Q_{12}\Phi_2 Q_{21} - Q_{32}\Phi_2 Q_{23} + Q_{23}Q_{34}Q_{43}Q_{32} + Q_{34}\Phi_4 Q_{43} - Q_{14}\Phi_4 Q_{41} + Q_{41}Q_{12}Q_{21}Q_{14} \tag{4.8}$$

The $U(1)$ gauge groups have CS levels $\vec{k} = (-k, 0, k, 0)$. One can check that this model describes the same CY$_4$ geometry (4.7) of the original theory, and the two phases are toric dual. One can build another dual phase with superpotential (4.4). The $U(1)$ gauge groups have CS levels $\vec{k} = (-k, k, -k, k)$. The quiver is represented in 1 (c). This is the other parent theory that we have to project to obtain the Spin(7) dual phase.

The next step consists of studying the orientifold projection of the dual models represented in figure 1 (a) and (c) to $\mathcal{N} = 1$ and check the Spin(7) duality between the models represented in figure 1 (d) and (e).

Projection to $\mathcal{N} = 1$. We start by analyzing the first case. The anti-holomorphic involution on the coordinates is

$$\begin{aligned} x_1 &\to -x_2^* & x_2 &\to x_1^* & x_3 &\to -x_4^* & x_4 &\to x_3^* \\ x_5 &\to x_8^* & x_6 &\to x_7^* & x_7 &\to -x_6^* & x_8 &\to -x_5^* \end{aligned} \tag{4.9}$$

this action does not have fixed points, except the origin, on the CY$_4$ geometry (4.7).

The anti-holomorphic involution acts on the gauge fields and on the scalars σ_i as

$$\begin{aligned} A_\mu^1 &\to -\Omega (A_\mu^3)^T \Omega^{-1} & A_\mu^{2,4} &\to -\Omega_{2,4}(A_\mu^{2,4})^T \Omega_{2,4}^{-1} \\ \sigma_1 &\to \Omega \sigma_3^T \Omega^{-1} & \sigma_{2,4} &\to \Omega_{2,4} \sigma_{2,4}^T \Omega_{2,4}^{-1} \end{aligned} \tag{4.10}$$

The anti-holomorphic involution acts on the bifundamental as

$$
\begin{aligned}
Q_{41} &\to \ \Omega_4 Q_{43}^* \Omega^{-1} & Q_{14} &\to \ \Omega Q_{34}^* \Omega_4^{-1} & Q_{43} &\to \ \Omega_4 Q_{41}^* \Omega^{-1} & Q_{34} &\to -\Omega Q_{14}^* \Omega_4^{-1} \\
Q_{23} &\to -\Omega_2 Q_{21}^* \Omega^{-1} & Q_{32} &\to -\Omega Q_{12}^* \Omega_2^{-1} & Q_{12} &\to -\Omega Q_{32}^* \Omega_2^{-1} & Q_{21} &\to \ \Omega_2 Q_{23}^* \Omega^{-1}
\end{aligned}
$$
$$(4.11)$$

The transformation corresponds to an orientifold projection, sending $\sigma \to \sigma$ and the angle $\psi \to -\psi$. Moreover this transformation is a symmetry of the $\mathcal{N} = 2$ lagrangian. Indeed one can check that the superpotential is sent into its complex conjugate and that the D-terms transform consistently with the constraint $D_a = \frac{k_a}{2\pi}\sigma_a$. From now on we fix $\Omega_4 = I_{2N}$ and $\Omega_2 = J_{2N}$, and the projected gauge theory is $O(2N)_{-2k} \times U(2N)_{2k} \times SP(2N)_{-k}$. This is precisely the gauge symmetry of the $\mathcal{N} = 1$ theory that we want to obtain.

The dual phase can be obtained analogously. The projection is obtained by flipping the sign of each CS level. This does not affect the ansatz but only the D terms.

Calculation of the $\mathcal{N} = 1$ moduli space. The next step consists of calculating the moduli space for a single M2 brane and show that they coincide. In this case even for the single brane the gauge group is non abelian, $O(2)_{-2k} \times U(2)_{2k} \times SP(2)_{-k}$.

We choose an ansatz for the $\mathcal{N} = 1$ fields $\mathcal{Q}_1$, $\widetilde{\mathcal{Q}}_1$, $\mathcal{Q}_2$ and $\widetilde{\mathcal{Q}}_2$ in terms of the $\mathcal{N} = 2$ bifundamentals as

$$
\begin{aligned}
\mathcal{Q}_1 &= \frac{Q_{12} + Q_{32}^*}{2} I + \frac{Q_{12} - Q_{32}^*}{2i} J & \widetilde{\mathcal{Q}}_1 &= \frac{Q_{21} + Q_{23}^*}{2} I + \frac{Q_{21} - Q_{23}^*}{2i} J \\
\widetilde{\mathcal{Q}}_2 &= \frac{Q_{41} + iQ_{43}^*}{2} I + \frac{Q_{41} - iQ_{43}^*}{2i} J & \mathcal{Q}_2 &= \frac{Q_{14} - iQ_{34}^*}{2} I + \frac{Q_{14} + Q_{34}^*}{2i} J
\end{aligned}
$$
$$(4.12)$$

The ansatz (4.12) exactly reproduces the equations of motion (3.2) of the parent $\mathcal{N} = 2$ theory. Moreover this ansatz exhausts the vacuum space of the $\mathcal{N} = 1$ theory. Indeed we checked that by fluctuating around the solution there are not other flat directions.

There are four residual abelian gauge factors on the moduli space, $SO(2) \in O(2)$, $SO(2) \in SP(2)$ and $U(1)^2 \in U(2)$. They act on the ansatz fields exactly as the $U(1)^4$ gauge group in the $\mathcal{N} = 2$ case. One can observe that one of them acts trivially, two combinations are used to mod the moduli space and the last factor is broken to $\mathbb{Z}_{2k}$ by the CS. In this way the moduli space exactly reproduces the geometry (4.7) modded by $\mathbb{Z}_{2k}$ as in the parent theory.

However there is still a residual discrete symmetry, Θ, generated by σ_3 in $O(2)$ and $i\sigma_3$ in $SU(2)$ and $U(2)$. It corresponds to the antiholomorphic involution. The moduli space of the $\mathcal{N} = 1$ theory is then the Spin(7) quotient geometry Y/Θ_k, where Θ_k is the combination of Θ in (4.9) with $\mathbb{Z}_{2k}$, and Y is the CY$_4$ in (4.7).

The analysis of the dual phase is similar. The moduli space of the two $\mathcal{N} = 1$ theories are coincident and this supports the Spin(7) duality for this first simple example. Observe that the duality can be understood as a parity transformation $k \to -k$. Although the simplicity of this duality, we studied this toy model because we believe that it could be useful for the reader to understand our general picture.

4.2 Second example

Let us now provide a more involved and interesting example of Spin(7) duality. In this case we consider $\mathcal{N} = 1$ models with only orthogonal or symplectic gauge groups. We consider a case with four gauge groups. We distinguish two possibilities

$$\begin{cases} (I) \ \ O(2N)_{2k} \times O(2N)_0 \times O(2N)_{-2k} \times O(2N)_0 \\ (II) \ SP(2N)_k \times SP(2N)_0 \times SP(2N)_{-k} \times SP(2N)_0 \end{cases} \tag{4.13}$$

In the rest of the section we study the case I but everything can be easily generalized to the case II. The $\mathcal{N} = 1$ superpotential is[8]

$$\begin{aligned}
W = \ & \mathcal{Q}_{12}\mathcal{Q}_{23}\mathcal{Q}_{32}\mathcal{Q}_{21} - \mathcal{Q}_{23}\mathcal{Q}_{34}\mathcal{Q}_{43}\mathcal{Q}_{32} + \mathcal{Q}_{34}\mathcal{Q}_{41}\mathcal{Q}_{14}\mathcal{Q}_{43} - \mathcal{Q}_{41}\mathcal{Q}_{12}\mathcal{Q}_{21}\mathcal{Q}_{14} \\
& + R_1(\mathcal{Q}_{12}\mathcal{Q}_{12}^T - \mathcal{Q}_{21}^T\mathcal{Q}_{21} + \mathcal{Q}_{14}\mathcal{Q}_{14}^T - \mathcal{Q}_{41}^T\mathcal{Q}_{41}) \\
& + R_2(\mathcal{Q}_{21}\mathcal{Q}_{21}^T - \mathcal{Q}_{12}^T\mathcal{Q}_{12} + \mathcal{Q}_{23}\mathcal{Q}_{23}^T - \mathcal{Q}_{32}^T\mathcal{Q}_{32}) \\
& + R_3(\mathcal{Q}_{32}\mathcal{Q}_{32}^T - \mathcal{Q}_{23}^T\mathcal{Q}_{23} + \mathcal{Q}_{34}\mathcal{Q}_{34}^T - \mathcal{Q}_{43}^T\mathcal{Q}_{43}) \\
& + R_4(\mathcal{Q}_{43}\mathcal{Q}_{43}^T + \mathcal{Q}_{34}^T\mathcal{Q}_{34} - \mathcal{Q}_{41}\mathcal{Q}_{41}^T + \mathcal{Q}_{14}^T\mathcal{Q}_{14}) \\
& + \frac{k}{2\pi}(R_1^2 - R_3^2)
\end{aligned} \tag{4.14}$$

The Spin(7) dual theories have gauge groups

$$\begin{cases} (I) \ \ O(2N)_{-2k} \times O(2N)_{2k} \times O(2N)_{-2k} \times O(2N)_{2k} \\ (II) \ SP(2N)_{-k} \times SP(2N)_k \times SP(2N)_{-k} \times SP(2N)_k \end{cases} \tag{4.15}$$

Here we still restrict to the first case. The $\mathcal{N} = 1$ superpotential becomes

$$\begin{aligned}
W = \ & \mathcal{Q}_{14}\mathcal{Q}_{44}\mathcal{Q}_{41} - \mathcal{Q}_{12}\mathcal{Q}_{22}\mathcal{Q}_{21} + \mathcal{Q}_{32}\mathcal{Q}_{22}\mathcal{Q}_{23} - \mathcal{Q}_{23}\mathcal{Q}_{34}\mathcal{Q}_{43}\mathcal{Q}_{32} + \mathcal{Q}_{34}\mathcal{Q}_{44}\mathcal{Q}_{43} \\
& + R_1(\mathcal{Q}_{12}\mathcal{Q}_{12}^T - \mathcal{Q}_{21}^T\mathcal{Q}_{21} + \mathcal{Q}_{14}\mathcal{Q}_{14}^T - \mathcal{Q}_{41}^T\mathcal{Q}_{41}) \\
& + R_2(\mathcal{Q}_{21}\mathcal{Q}_{21}^T - \mathcal{Q}_{12}^T\mathcal{Q}_{12} + \mathcal{Q}_{23}\mathcal{Q}_{23}^T - \mathcal{Q}_{32}^T\mathcal{Q}_{32}) \\
& + R_3(\mathcal{Q}_{32}\mathcal{Q}_{32}^T - \mathcal{Q}_{23}^T\mathcal{Q}_{23} + \mathcal{Q}_{34}\mathcal{Q}_{34}^T - \mathcal{Q}_{43}^T\mathcal{Q}_{43}) \\
& + R_4(\mathcal{Q}_{43}\mathcal{Q}_{43}^T + \mathcal{Q}_{34}^T\mathcal{Q}_{34} - \mathcal{Q}_{41}\mathcal{Q}_{41}^T + \mathcal{Q}_{14}^T\mathcal{Q}_{14}) \\
& + \frac{k}{2\pi}(R_4^2 - R_1^2 + R_2^2 - R_3^2)
\end{aligned} \tag{4.16}$$

In the rest of this section we study this duality as before. First we provide the $\mathcal{N} = 2$ dual parents, then we study the projection to $\mathcal{N} = 1$ and show that the moduli spaces match, supporting the Spin(7) duality.

$\mathcal{N} = 2$ parents. In this case the parent theories are $\widetilde{L_{k_i}^{222}}$ models in the $\mathcal{N} = 2$ case. The dual phase is obtained by dualizing the first gauge group. The quiver and the superpotential coincide with the ones studied in subsection 4.1.

---

[8]With abuse of notation we keep the same notation as before for the the matter fields $\mathcal{Q}_{ij}$ even if both indices i and j refer now to the fundamental representation because the gauge group is now real.

We study here the moduli space for one M2 brane where the gauge group is $U(1)^4$ gauge group. The CS levels are $\vec{k} = (k, 0, -k, 0)$. The gauge invariant combinations, orthogonal to the CS vector, are

$$x_1 = Q_{12}Q_{21} = Q_{34}Q_{43} \qquad x_2 = Q_{23}Q_{32} = Q_{14}Q_{41}$$
$$y_1 = Q_{12}Q_{23} \quad y_2 = Q_{21}Q_{32} \quad y_3 = Q_{34}Q_{41} \quad y_4 = Q_{43}Q_{14} \qquad (4.17)$$

They are related by

$$x_1 x_2 = y_1 y_2 = y_3 y_4 \qquad (4.18)$$

These equations define the CY$_4$ Y that has to be mod by the $\mathbb{Z}_k$.

The $U(1)$ gauge groups of the toric dual $\mathcal{N} = 2$ phase have CS levels $\vec{k} = (-k, k, -k, k)$. The gauge invariant combinations, orthogonal to the CS vector, are

$$x_1 = Q_{12}Q_{21} = Q_{23}Q_{32} = Q_{44} \qquad x_2 = Q_{34}Q_{43} = Q_{14}Q_{41} = Q_{22}$$
$$y_1 = Q_{12}Q_{34} \quad y_2 = Q_{21}Q_{43} \quad y_3 = Q_{23}Q_{41} \quad y_4 = Q_{32}Q_{14} \qquad (4.19)$$

They are related by

$$x_1 x_2 = y_1 y_2 = y_3 y_4 \qquad (4.20)$$

These equations define the CY$_4$ Y that has to be mod by the $\mathbb{Z}_k$. The moduli space of the two theories is then the same and they are indeed toric dual.

In the rest of this section we project the theories to $\mathcal{N} = 1$ to obtain the two models discussed above. We check that they reproduce the expected $\mathcal{N} = 1$ phases and compute the classical moduli space with our usual procedure. Eventually we match the two moduli spaces, supporting the Spin(7) duality.

Projection to $\mathcal{N} = 1$ of the electric phase. We choose the anti-holomorphic involution as

$$x_1 \to -x_1^* \quad , \quad x_2 \to -x_2^* \quad , \quad x_3 \to -x_3^* \quad , \quad x_4 \to -x_4^*$$
$$y_1 \to -y_1^* \quad , \quad y_2 \to -y_2^* \quad , \quad y_3 \to -y_3^* \quad , \quad y_4 \to -y_4^* \qquad (4.21)$$

this action has a real four dimensional locus of fixed points on the CY$_4$ geometry (4.20).

On the fields Q_{ij} this anti-involution becomes

$$
\begin{array}{llll}
Q_{12} \to -\Omega_1 Q_{12}^* \Omega_2^{-1} & Q_{21} \to \Omega_2 Q_{21}^* \Omega_1^{-1} & Q_{23} \to \Omega_2 Q_{23}^* \Omega_3^{-1} & Q_{32} \to -\Omega_3 Q_{32}^* \Omega_2^{-1} \\
Q_{34} \to -\Omega_3 Q_{34}^* \Omega_4^{-1} & Q_{43} \to \Omega_4 Q_{43}^* \Omega_3^{-1} & Q_{41} \to \Omega_4 Q_{41}^* \Omega_1^{-1} & Q_{14} \to -\Omega_1 Q_{14}^* \Omega_4^{-1}
\end{array}
\qquad (4.22)
$$

Here $\Omega_i = I_2$ or J_2 means that we project on an orthogonal or symplectic group. By choosing $\Omega_i = I_{2N}$ the gauge groups become $O(2N)$ with $\vec{k} = (2k, 0, -2k, 0)$ while choosing $\Omega_i = I_{2N}$ we have a product of $SP(2N)$ gauge groups with $\vec{k} = (k, 0, -k, 0)$. Also in this case the anti-holomorphic action is a symmetry of the full lagrangian. The transformation corresponds to an orientifold projection, sending $\sigma \to \sigma$ and the angle $\psi \to -\psi$.

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Moduli space of the $\mathcal{N} = 1$ electric phase. Here we compute the moduli space for a single M2 brane. We choose the ansatz for the $\mathcal{N} = 1$ fields as

$$\mathcal{Q}_{ij} = Re(Q_{ij})I + Im(Q_{ij})J \tag{4.23}$$

Once we plug these projection in the superpotential (4.14) they reproduce the equations of motion (3.2) of the $\mathcal{N} = 2$ case. Moreover this ansatz exhausts the vacuum space of the $\mathcal{N} = 1$ theory.

There are four residual abelian SO(2) gauge factors on the moduli space that act as the $U(1)$ gauge groups in the $\mathcal{N} = 2$ case. One of them acts trivially, two combinations are used to mod the moduli space and the last factor is broken to $\mathbb{Z}_{2k}$ by the CS. There is still a residual discrete symmetry, Θ, generated by σ_3 in $O(2)$ that corresponds to the antiholomorphic involution. The moduli space is the Spin(7) quotient Y/Θ_k, where Θ_k is the combination of the Θ action (4.21) with $\mathbb{Z}_{2k}$, and Y is the CY$_4$ in (4.18).

The magnetic phase. In this case we choose the anti-holomorphic involution as (4.21). On the fields Q_{ij} this anti-involution becomes

$$
\begin{array}{llll}
Q_{12} \to -\Omega_1 Q_{12}^* \Omega_2^{-1} & Q_{21} \to \Omega_2 Q_{21}^* \Omega_1^{-1} & Q_{23} \to \Omega_2 Q_{23}^* \Omega_3^{-1} & Q_{32} \to -\Omega_3 Q_{32}^* \Omega_2^{-1} \\
Q_{34} \to -\Omega_3 Q_{34}^* \Omega_4^{-1} & Q_{43} \to \Omega_4 Q_{43}^* \Omega_3^{-1} & Q_{41} \to \Omega_4 Q_{41}^* \Omega_1^{-1} & Q_{14} \to -\Omega_1 Q_{14}^* \Omega_4^{-1} \\
& Q_{22} \to -\Omega_2 Q_{22}^* \Omega_2^{-1} & Q_{44} \to -\Omega_4 Q_{44}^* \Omega_4^{-1} &
\end{array} \tag{4.24}
$$

where Ω_i and the ansatz for the bifundamentals are chosen as before. Also the adjoints become $\mathcal{Q}_{ii} = Re(Q_{11})I_2 + Im(Q_{ii})J_2$ and they do not contribute to the D-terms. The ansatz reproduces the equations of motion (3.2) of the $\mathcal{N} = 2$ case and it exhausts the vacuum space of the $\mathcal{N} = 1$ theory.

There are four residual abelian SO(2) gauge factors on the moduli space that act as in the $\mathcal{N} = 2$ case. One of them acts trivially, two combinations are used to mod the moduli space and the last factor is broken to $\mathbb{Z}_{2k}$ by the CS. There is still a residual discrete symmetry, Θ, generated by σ_3 in $O(2)$ that corresponds to the antiholomorphic involution. As in the electric phase the moduli space for the magnetic phase is the Spin(7) quotient Y/Θ_k, where Θ_k is the combination of the Θ action (4.21) with $\mathbb{Z}_{2k}$, and Y is the CY$_4$ in (4.18).

The two geometries coincide and this confirms that the two $\mathcal{N} = 1$ theories are Spin(7) dual.

As already remarked the Spin(7) duality is insensitive to the presence of fractional branes. The choice of equal rank, $2N$, for each gauge factor in the examples studied above comes naturally from the orientifold projection. However the Spin(7) duality would have been valid also for different choices of ranks for the projected theories. In the next section we explore the possibility to fix the ranks, and hence the number of fraction branes, using Seiberg-like dualities.

5 Spin(7) duality as Seiberg like duality

For $\mathcal{N} = 2$ CS-matter theories in [6] it has been shown that some toric dualities between $\widetilde{L_k^{aba}}$ theories are actually three dimensional Seiberg-like dualities. Namely that the two

different field theories not only have the same moduli space, but they are actually two different descriptions of the same IR conformal field theory that is holographic dual to the M theory background: $AdS_4 \times H_7$. It is maybe worth to underline the principal differences between toric (and similarly Spin(7)) duality and Seiberg-like duality. Toric duality is essentially the statement that the moduli space for one regular brane is the same for the dual pairs of theories. Seiberg-like duality is instead a non-abelian statement valid for the set of regular and fractional branes at the tip of a CY_4 or Spin(7) cone. Indeed, as we have previously explained, for $\mathcal{N} = 2 \ \widetilde{L_{k_i}^{aba}}$ theories the Seiberg-duality transforms the gauge groups as in (3.1). Anyway, in toric duality, the extra shift in the rank of the dual gauge group does not play any role. Indeed the moduli space for one M2 brane is obtained by setting $N = 1$ in all the gauge group factors and disregarding the rank difference among the various gauge group factors: only regular M2 branes can explore the geometry transverse to the brane, while the fractional branes are stacked at the singularity and do not contribute to the moduli space. Moreover for $\mathcal{N} = 2$ the moduli space of N regular branes is simply the N times symmetric product of the moduli space for one brane.

In analogy with the $\mathcal{N} = 2$ case, in this section we study examples of Spin(7) dual $\mathcal{N} = 1$ pairs of theories that are also Seiberg-like dual.

5.1 Example

Let us illustrate in detail a specific example to explain our general philosophy. We consider a three dimensional $\mathcal{N} = 1$ CS-matter theory with four gauge groups

$$U(2N)_{2k} \times U(2N)_{-2k} \times U(2N)_0 \times U(2N)_0 \tag{5.1}$$

and $\mathcal{N} = 1$ superpotential:

$$
\begin{aligned}
W = {}& \mathcal{Q}_{11}\mathcal{Q}_{12}\mathcal{Q}_{21}\mathcal{Q}_{11}^* - \mathcal{Q}_{12}\mathcal{Q}_{23}\mathcal{Q}_{32}\mathcal{Q}_{21} + \mathcal{Q}_{23}\mathcal{Q}_{34}\mathcal{Q}_{43}\mathcal{Q}_{32} - \mathcal{Q}_{34}\mathcal{Q}_{44}^*\mathcal{Q}_{44}\mathcal{Q}_{43} \\
& + \mathcal{Q}_{11}^*\mathcal{Q}_{12}^*\mathcal{Q}_{21}^*\mathcal{Q}_{11} - \mathcal{Q}_{12}^*\mathcal{Q}_{23}^*\mathcal{Q}_{32}^*\mathcal{Q}_{21}^* + \mathcal{Q}_{23}^*\mathcal{Q}_{34}^*\mathcal{Q}_{43}^*\mathcal{Q}_{32}^* - \mathcal{Q}_{34}^*\mathcal{Q}_{44}\mathcal{Q}_{44}^*\mathcal{Q}_{43}^* \\
& + R_1(\mathcal{Q}_{12}\mathcal{Q}_{12}^\dagger - \mathcal{Q}_{21}^\dagger\mathcal{Q}_{21}[\mathcal{Q}_{11}, \mathcal{Q}_{11}^\dagger]) + R_2(\mathcal{Q}_{21}\mathcal{Q}_{21}^\dagger - \mathcal{Q}_{12}^\dagger\mathcal{Q}_{12} + \mathcal{Q}_{23}\mathcal{Q}_{23}^\dagger - \mathcal{Q}_{32}^\dagger\mathcal{Q}_{32}) \\
& + R_3(\mathcal{Q}_{23}\mathcal{Q}_{23}^\dagger - \mathcal{Q}_{32}^\dagger\mathcal{Q}_{32} + \mathcal{Q}_{43}\mathcal{Q}_{43}^\dagger - \mathcal{Q}_{34}^\dagger\mathcal{Q}_{34}) + R_4(\mathcal{Q}_{34}\mathcal{Q}_{34}^\dagger - \mathcal{Q}_{43}^\dagger\mathcal{Q}_{43} + [\mathcal{Q}_{44}, \mathcal{Q}_{44}^\dagger] \\
& + \frac{k}{2\pi}(R_1^2 - R_2^2)
\end{aligned}
\tag{5.2}
$$

We claim that this theory is Seiberg-like dual to another $\mathcal{N} = 1$ CS-matter theory with gauge group and CS levels:

$$U(2N)_0 \times U(2(N + |k|))_{2k} \times U(2N)_{-2k} \times U(2N)_0 \tag{5.3}$$

and $\mathcal{N} = 1$ superpotential

$$
\begin{aligned}
W = {}& \mathcal{Q}_{11}(\mathcal{Q}_{12}\mathcal{Q}_{21} - \mathcal{X}_{11}\mathcal{X}_{11}^*) - \mathcal{Q}_{12}\mathcal{Q}_{23}\mathcal{Q}_{32}\mathcal{Q}_{21} + \mathcal{Q}_{33}(\mathcal{Q}_{32}\mathcal{Q}_{23} - \mathcal{Q}_{34}\mathcal{Q}_{43}) + \mathcal{Q}_{34}\mathcal{X}_{44}\mathcal{X}_{44}^*\mathcal{Q}_{43} \\
& + \mathcal{Q}_{11}^*(\mathcal{Q}_{12}^*\mathcal{Q}_{21}^* - \mathcal{X}_{11}^*\mathcal{X}_{11}) - \mathcal{Q}_{12}^*\mathcal{Q}_{23}^*\mathcal{Q}_{32}^*\mathcal{Q}_{21}^* + \mathcal{Q}_{33}^*(\mathcal{Q}_{32}^*\mathcal{Q}_{23}^* - \mathcal{Q}_{34}^*\mathcal{Q}_{43}^*) + \mathcal{Q}_{34}^*\mathcal{X}_{44}^\dagger\mathcal{X}_{44}\mathcal{Q}_{43}^* \\
& + R_1(\mathcal{Q}_{12}\mathcal{Q}_{12}^\dagger - \mathcal{Q}_{21}^\dagger\mathcal{Q}_{21} - [\mathcal{Q}_{11}, \mathcal{Q}_{11}^\dagger]) + R_2(\mathcal{Q}_{21}\mathcal{Q}_{21}^\dagger - \mathcal{Q}_{12}^\dagger\mathcal{Q}_{12} + \mathcal{Q}_{23}\mathcal{Q}_{23}^\dagger - \mathcal{Q}_{32}^\dagger\mathcal{Q}_{32}) \\
& + R_3(\mathcal{Q}_{23}\mathcal{Q}_{23}^\dagger - \mathcal{Q}_{32}^\dagger\mathcal{Q}_{32} + \mathcal{Q}_{43}\mathcal{Q}_{43}^\dagger - \mathcal{Q}_{34}^\dagger\mathcal{Q}_{34}) + R_4(\mathcal{Q}_{34}\mathcal{Q}_{34}^\dagger - \mathcal{Q}_{43}^\dagger\mathcal{Q}_{43} - [\mathcal{Q}_{44}, \mathcal{Q}_{44}^\dagger] \\
& + \frac{k}{2\pi}(R_2^2 - R_3^2)
\end{aligned}
\tag{5.4}
$$

We start showing that the two models can be obtained by projecting two toric dual $\mathcal{N} = 2$ parent theories of the $\widetilde{L_k^{444}}$ family. Then we study the projection and compute the $\mathcal{N} = 1$ moduli space for one M2 brane: namely when all the gauge groups are $U(2)$. By comparing the result in the two phases we show that the two models are indeed Spin(7) dual. Eventually we show that the brane description supports the claim that the two models are also Seiberg-like dual.

$\mathcal{N} = 2$ parents. The quivers for the parent theories are represented in figure 2. They have eight gauge groups, each associated to a $U(N_i)_{k_i}$ factor. We choose the ranks as $N_i = N$. In the first case represented in figure 2 (a) there is a pair bifundamental antibifundamental connecting each pair of consecutive nodes. The $\mathcal{N} = 2$ superpotential is

$$W = Q_{12}Q_{23}Q_{32}Q_{21} - Q_{23}Q_{34}Q_{43}Q_{32} + Q_{34}Q_{45}Q_{54}Q_{43} - Q_{45}Q_{56}Q_{65}Q_{54}$$
$$+Q_{56}Q_{67}Q_{76}Q_{65} - Q_{67}Q_{78}Q_{87}Q_{76} + Q_{78}Q_{81}Q_{18}Q_{87} - Q_{81}Q_{12}Q_{21}Q_{18} \quad (5.5)$$

We choose the CS levels as $\vec{k} = (-k, k, 0, 0, 0, 0, k, -k)$. Let us analyze the $\mathcal{N} = 2$ moduli space for one M2 regular brane, namely for the $U(1)^4$ gauge group. After solving the F-term equations the operators gauge invariant with respect to the gauge factor orthogonal to the CS vector are

$$x_1 = Q_{12}Q_{21} = Q_{34}Q_{43} = Q_{56}Q_{65} = Q_{78}Q_{87}$$
$$x_2 = Q_{23}Q_{32} = Q_{45}Q_{54} = Q_{67}Q_{76} = Q_{81}Q_{18}$$
$$y_1 = Q_{12}Q_{23}Q_{34}Q_{45}Q_{56}Q_{67}Q_{78}Q_{81}$$
$$y_2 = Q_{18}Q_{87}Q_{76}Q_{65}Q_{54}Q_{43}Q_{32}Q_{21}$$
$$t_1 = Q_{12}Q_{87} \qquad t_2 = Q_{21}Q_{78} \quad (5.6)$$

They are related by

$$x_1^4 x_2^4 = y_1 y_2 \qquad t_1 t_2 = x_1^2 \quad (5.7)$$

These equations define the CY$_4$ Y that has to be modded by the $\mathbb{Z}_k$ action.

The second parent is obtained by acting with two Seiberg-like dualities on $U(N_2)$ and $U(N_7)$ respectively. The dual quiver is represented in figure 2 (c). In this case there are four extra adjoint fields. The ranks of the dualized groups are

$$\widetilde{N_2} = N_1 + N_3 - N_2 + |k_2| = N + |k| \quad , \quad \widetilde{N_7} = N_6 + N_8 - N_7 + |k_7| = N + |k| \quad (5.8)$$

while all the other ranks remain the same. The CS levels of the dual phase are $\vec{k} = (0, -k, k, 0, 0, k, -k, 0)$. The dual $\mathcal{N} = 2$ superpotential is

$$W = Q_{11}(Q_{12}Q_{21} - Q_{18}Q_{81}) - Q_{12}Q_{23}Q_{32}Q_{21} + Q_{33}(Q_{32}Q_{23} - Q_{34}Q_{43}) + Q_{34}Q_{45}Q_{54}Q_{43}$$
$$-Q_{45}Q_{56}Q_{65}Q_{54} + Q_{66}(Q_{65}Q_{56} - Q_{67}Q_{76}) + Q_{67}Q_{78}Q_{87}Q_{76} - Q_{88}(Q_{87}Q_{78} - Q_{81}Q_{18})$$
$$(5.9)$$

Let us analyze the $\mathcal{N} = 2$ moduli space for one M2 regular brane, namely for the $U(1)^4$ gauge group. Where, as previously explained, we disregarded the presence of fractional
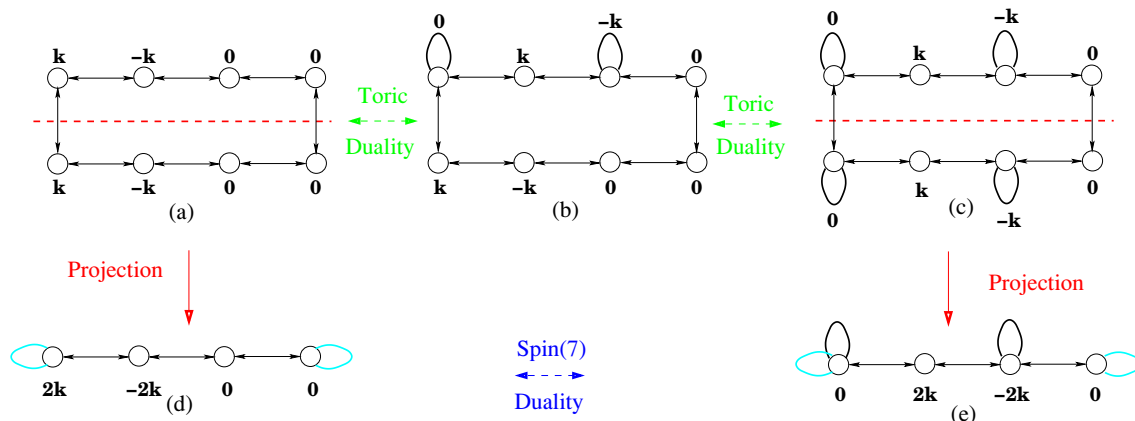
Figure 2. The models in (a), (b) and (c) represent three L^{444} toric dual phases. In the cases (a) and (c) the orientifold projection acts by folding the quiver, along the dashed red lines. By projecting these models to $\mathcal{N} = 1$ Spin(7) cones we obtain the phases (d) and (e), that are related by Spin(7) duality. These models are also Seiberg-like dual.

branes, because they are stacked at the origin and they do not explore the moduli space. After solving the F-term equations the gauge invariant operators orthogonal to the CS vector are

$$
\begin{aligned}
x_1 &= Q_{12}Q_{21} = Q_{33} = Q_{45}Q_{54} = Q_{66} = Q_{78}Q_{87} = Q_{81}Q_{18} \\
x_2 &= Q_{11} = Q_{23}Q_{32} = Q_{34}Q_{43} = Q_{56}Q_{65} = Q_{67}Q_{76} = Q_{88} \\
y_1 &= Q_{12}Q_{23}Q_{34}Q_{45}Q_{56}Q_{67}Q_{78}Q_{81} \\
y_2 &= Q_{18}Q_{87}Q_{76}Q_{65}Q_{54}Q_{43}Q_{32}Q_{21} \\
t_1 &= Q_{23}Q_{76} \qquad t_2 = Q_{32}Q_{67}
\end{aligned}
\tag{5.10}
$$

and they are related by

$$
x_1^4 x_2^4 = y_1 y_2 \quad , \quad t_1 t_2 = x_2^2
\tag{5.11}
$$

These equations define the CY$_4$ Y that has to be mod by the $\mathbb{Z}_k$. Equations (5.7) for the first phase and equations (5.11) for the second phase are equivalent: the two theories are indeed Seiberg and toric dual and they have the same moduli space for one regular brane.

Here we study the projection of the two phases to obtain the two $\mathcal{N} = 1$ theories introduced above.

Projection to $\mathcal{N} = 1$ of the electric theory. In the first case the anti-holomorphic involution on the coordinates is

$$
x_1 \to x_1^* \quad x_2 \to -x_2^* \quad y_1 \to y_2^* \quad y_2 \to y_1^* \quad t_1 \to t_1^* \quad t_2 \to t_2^*
\tag{5.12}
$$

This action has a real four dimensional locus of fixed points on the CY$_4$ (5.7) and it represents an orientifold projection that sends $\sigma \to \sigma$ and $\psi \to -\psi$. The associated

orientifold action on the fields is

$$
\begin{aligned}
&Q_{12} \to \quad \Omega_1 Q_{87}^* \Omega_2^{-1} \quad Q_{21} \to \quad \Omega_2 Q_{78}^* \Omega_1^{-1} \quad Q_{23} \to -\Omega_2 Q_{76}^* \Omega_3^{-1} \quad Q_{32} \to \quad \Omega_3 Q_{67}^* \Omega_2^{-1} \\
&Q_{34} \to -\Omega_3 Q_{65}^* \Omega_4^{-1} \quad Q_{43} \to \quad \Omega_4 Q_{56}^* \Omega_3^{-1} \quad Q_{45} \to \quad \Omega_4 Q_{54}^* \Omega_4^{-1} \quad Q_{54} \to \quad \Omega_4 Q_{45}^* \Omega_4^{-1} \\
&Q_{56} \to \quad \Omega_4 Q_{43}^* \Omega_3^{-1} \quad Q_{65} \to -\Omega_3 Q_{34}^* \Omega_4^{-1} \quad Q_{67} \to \quad \Omega_3 Q_{32}^* \Omega_2^{-1} \quad Q_{76} \to -\Omega_2 Q_{23}^* \Omega_3^{-1} \\
&Q_{78} \to \quad \Omega_2 Q_{21}^* \Omega_1^{-1} \quad Q_{87} \to \quad \Omega_1 Q_{12}^* \Omega_2^{-1} \quad Q_{81} \to \quad \Omega_1 Q_{18}^* \Omega_1^{-1} \quad Q_{18} \to \quad \Omega_1 Q_{81}^* \Omega_1^{-1} \\
&Q_{11} \to -\Omega_1 Q_{11}^* \Omega_1^{-1} \quad Q_{33} \to \quad \Omega_3 Q_{33}^* \Omega_3^{-1} \quad Q_{66} \to \quad \Omega_6 Q_{66}^* \Omega_6^{-1} \quad Q_{88} \to -\Omega_8 Q_{88}^* \Omega_8^{-1}
\end{aligned}
\tag{5.13}
$$

This action is a symmetry of the $\mathcal{N} = 2$ lagrangian. The superpotential is sent into its complex conjugate and once again the D-terms transform consistently with the constraint $D_a = \frac{k_a}{2\pi} \sigma_a$. The gauge groups after the projection become $U(2N)$ and the CS vector is $\vec{k} = (-2k, 2k, 0, 0)$.

Projection to $\mathcal{N} = 1$ of the magnetic theory. In the dual case the anti-holomorphic involution on the coordinates is still given by (5.12). On the matter fields is implemented as

$$
\begin{aligned}
&Q_{12} \to \quad \Omega_1 Q_{87}^* \Omega_2^{-1} \quad Q_{21} \to \quad \Omega_2 Q_{78}^* \Omega_1^{-1} \quad Q_{23} \to -\Omega_2 Q_{76}^* \Omega_3^{-1} \quad Q_{32} \to \quad \Omega_3 Q_{67}^* \Omega_2^{-1} \\
&Q_{34} \to -\Omega_3 Q_{65}^* \Omega_4^{-1} \quad Q_{43} \to \quad \Omega_4 Q_{56}^* \Omega_3^{-1} \quad Q_{45} \to \quad \Omega_4 Q_{54}^* \Omega_4^{-1} \quad Q_{54} \to \quad \Omega_4 Q_{45}^* \Omega_4^{-1} \\
&Q_{56} \to \quad \Omega_4 Q_{43}^* \Omega_3^{-1} \quad Q_{65} \to -\Omega_3 Q_{34}^* \Omega_4^{-1} \quad Q_{67} \to \quad \Omega_3 Q_{32}^* \Omega_2^{-1} \quad Q_{76} \to -\Omega_2 Q_{23}^* \Omega_3^{-1} \\
&Q_{78} \to \quad \Omega_2 Q_{21}^* \Omega_1^{-1} \quad Q_{87} \to \quad \Omega_1 Q_{12}^* \Omega_2^{-1} \quad Q_{81} \to \quad \Omega_1 Q_{18}^* \Omega_1^{-1} \quad Q_{18} \to \quad \Omega_1 Q_{81}^* \Omega_1^{-1} \\
&Q_{11} \to -\Omega_1 Q_{11}^* \Omega_1^{-1} \quad Q_{33} \to \quad \Omega_3 Q_{33}^* \Omega_3^{-1} \quad Q_{66} \to \quad \Omega_6 Q_{66}^* \Omega_6^{-1} \quad Q_{88} \to -\Omega_8 Q_{88}^* \Omega_8^{-1}
\end{aligned}
\tag{5.14}
$$

This action as a real four dimensional locus of fixed points on the CY$_4$ and it represents an orientifold projection that sends $\sigma \to \sigma$ and $\psi \to -\psi$. This action is a symmetry of the $\mathcal{N} = 2$ lagrangian. The superpotential is sent into its complex conjugate and once again the D-terms transform consistently with the constraint $D_a = \frac{k_a}{2\pi} \sigma_a$. The gauge groups after the projection become $U(2N)$ and the CS vector is $\vec{k} = (0, -2k, 2k, 0)$.

Moduli space of the electric $\mathcal{N} = 1$ theory. Here we study the moduli space for a single M2 brane. In the projected theory the gauge group is then a product of $U(2)$ factors and the fields are two by two matrices. To solve the zero potential condition for the scalar components of the fields of the $\mathcal{N} = 1$ theory we use the ansatz

$$
\begin{aligned}
\mathcal{Q}_{12} &= -\frac{Q_{12} - Q_{87}^*}{2}\sigma_2 + \frac{Q_{12} + Q_{87}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{21} = -\frac{Q_{21} + Q_{78}^*}{2}\sigma_2 + \frac{Q_{21} - Q_{78}^*}{2i}\sigma_1 \\
\mathcal{Q}_{23} &= \quad \frac{Q_{23} - Q_{76}^*}{2}\sigma_2 - \frac{Q_{23} + Q_{76}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{32} = \quad \frac{Q_{32} - Q_{67}^*}{2}\sigma_2 + \frac{Q_{32} + Q_{67}^*}{2i}\sigma_1 \\
\mathcal{Q}_{34} &= -\frac{Q_{34} - Q_{65}^*}{2}\sigma_2 - \frac{Q_{34} + Q_{65}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{43} = -\frac{Q_{43} + Q_{56}^*}{2}\sigma_2 + \frac{Q_{43} - Q_{56}^*}{2i}\sigma_1
\end{aligned}
\tag{5.15}
$$

For the other fields we have

$$
\mathcal{Q}_{11} = \frac{\sigma_1 + i\sigma_2}{2}Q_{81}^* + \frac{\sigma_1 - i\sigma_2}{2}Q_{18} \quad , \quad \mathcal{Q}_{44} = \frac{\sigma_1 - i\sigma_2}{2}Q_{45}^* + \frac{\sigma_1 + i\sigma_2}{2}Q_{54}
\tag{5.16}
$$

where now the σ_i are the two by two Pauli matrices, while the Q_{ij} are complex numbers.

By inserting this ansatz in the $\mathcal{N} = 1$ superpotential (5.2) we verify that it reproduces the equations of motion (3.2) of the parent $\mathcal{N} = 2$ theory. Moreover we explicitly verified that this ansatz exhausts the vacuum space condition.

To complete the analysis of the moduli space for the $\mathcal{N} = 1$ theory it is important to analyze the action of the gauge groups. The ansatz breaks the gauge group down to its abelian component. There are indeed eight residual $U(1)$ abelian gauge factors on the moduli space, an $U(1)^2$ in each $U(2)$. They act on the Q_{ij} as in the $\mathcal{N} = 2$ case. We can check this explicitly as follows.

$$
\begin{aligned}
\mathcal{Q}_{12} &\to \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} \mathcal{Q}_{12} \begin{pmatrix} e^{i\phi_7} & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix} &,& \quad
\mathcal{Q}_{21} \to \begin{pmatrix} e^{-i\phi_7} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \mathcal{Q}_{21} \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_8} \end{pmatrix} \\
\mathcal{Q}_{23} &\to \begin{pmatrix} e^{-i\phi_7} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \mathcal{Q}_{23} \begin{pmatrix} e^{-i\phi_3} & 0 \\ 0 & e^{i\phi_6} \end{pmatrix} &,& \quad
\mathcal{Q}_{32} \to \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{-i\phi_6} \end{pmatrix} \mathcal{Q}_{32} \begin{pmatrix} e^{i\phi_7} & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix} \\
\mathcal{Q}_{34} &\to \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{-i\phi_6} \end{pmatrix} \mathcal{Q}_{34} \begin{pmatrix} e^{i\phi_5} & 0 \\ 0 & e^{-i\phi_4} \end{pmatrix} &,& \quad
\mathcal{Q}_{43} \to \begin{pmatrix} e^{i\phi_5} & 0 \\ 0 & e^{-i\phi_4} \end{pmatrix} \mathcal{Q}_{43} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{-i\phi_6} \end{pmatrix} \\
\mathcal{Q}_{11} &\to \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} \mathcal{Q}_{11} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} &,& \quad
\mathcal{Q}_{44} \to \begin{pmatrix} e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_5} \end{pmatrix} \mathcal{Q}_{44} \begin{pmatrix} e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_5} \end{pmatrix}
\end{aligned}
\tag{5.17}
$$

where ϕ_i are the phases of the $U(1)$s. They are equivalent to

$$
\begin{aligned}
Q_{12} &\to e^{i(\phi_1-\phi_2)}Q_{12}, & Q_{21} &\to e^{i(\phi_2-\phi_1)}Q_{21}, & Q_{23} &\to e^{i(\phi_2-\phi_3)}Q_{23}, & Q_{32} &\to e^{i(\phi_3-\phi_2)}Q_{32} \\
Q_{34} &\to e^{i(\phi_3-\phi_4)}Q_{34}, & Q_{43} &\to e^{i(\phi_4-\phi_3)}Q_{43}, & Q_{45} &\to e^{i(\phi_4-\phi_5)}Q_{45}, & Q_{54} &\to e^{i(\phi_5-\phi_4)}Q_{54} \\
Q_{56} &\to e^{i(\phi_5-\phi_6)}Q_{56}, & Q_{65} &\to e^{i(\phi_6-\phi_5)}Q_{65}, & Q_{67} &\to e^{i(\phi_6-\phi_7)}Q_{67}, & Q_{76} &\to e^{i(\phi_7-\phi_6)}Q_{76} \\
Q_{78} &\to e^{i(\phi_7-\phi_8)}Q_{78}, & Q_{87} &\to e^{i(\phi_8-\phi_7)}Q_{87}, & Q_{81} &\to e^{i(\phi_8-\phi_1)}Q_{81}, & Q_{18} &\to e^{i(\phi_1-\phi_8)}Q_{18}
\end{aligned}
\tag{5.18}
$$

One can observe that one of them acts trivially, six combinations are used to mod the moduli space and the last factor is broken to $\mathbb{Z}_{2k}$ by the CS and consequently they reproduce exactly the the $\mathcal{N} = 2$ CY$_4$ geometry (5.7) quotiented by the same $\mathbb{Z}_{2k}$ action. Actually there is still a residual discrete symmetry, Θ, generated by $i\sigma_3$ in $U(2)$. It corresponds to the antiholomorphic involution (5.12) for which we need to mod the geometry. The moduli space is then the Spin(7) quotient Y/Θ_k, where Θ_k is the combination of Θ with $\mathbb{Z}_{2k}$. In this way the moduli space of the electric phase of the $\mathcal{N} = 1$ theory is exactly the Spin(7) geometry obtained by the anti-holomorphic involution on the CY$_4$ of the parent $\mathcal{N} = 2$ theory.

Moduli space of the magnetic $\mathcal{N} = 1$ theory. Here we study the moduli space for a single M2 brane in the dual phase. Also in this case the gauge group for the $\mathcal{N} = 1$ projected theory is the product of $U(2)$ factors, where, as before we disregarded the presence of additional fractional branes, that do not explore the moduli space. To solve the zero

potential condition for the $\mathcal{N} = 1$ theory we consider the ansatz for the scalar components of the $\mathcal{N} = 1$ projected bifundamental fields

$$\mathcal{Q}_{12} = \frac{Q_{12} - Q_{87}^*}{2}\sigma_2 - \frac{Q_{12} + Q_{87}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{21} = \frac{Q_{21} - Q_{78}^*}{2}\sigma_2 + \frac{Q_{21} + Q_{78}^*}{2i}\sigma_1$$

$$\mathcal{Q}_{23} = \frac{Q_{23} - Q_{76}^*}{2}\sigma_2 + \frac{Q_{23} + Q_{76}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{32} = -\frac{Q_{32} + Q_{67}^*}{2}\sigma_2 + \frac{Q_{32} - Q_{67}^*}{2i}\sigma_1$$

$$\mathcal{Q}_{34} = -\frac{Q_{34} - Q_{65}^*}{2}\sigma_2 + \frac{Q_{34} + Q_{65}^*}{2i}\sigma_1 \quad , \quad \mathcal{Q}_{43} = \frac{Q_{43} + Q_{56}^*}{2}\sigma_2 + \frac{Q_{43} - Q_{56}^*}{2i}\sigma_1$$

$$(5.19)$$

and

$$\mathcal{Q}_{11} = \frac{\sigma_1 + i\sigma_2}{2}Q_{81}^* + \frac{\sigma_1 - i\sigma_2}{2}Q_{18} \quad , \quad \mathcal{Q}_{44} = \frac{\sigma_1 - i\sigma_2}{2}Q_{45}^* + \frac{\sigma_1 + i\sigma_2}{2}Q_{54} \quad (5.20)$$

For the adjoints we have

$$\mathcal{X}_{11} = \frac{Q_{88}^* - Q_{11}}{2}I + \frac{Q_{88}^* + Q_{11}}{2}\sigma_3 \quad , \quad \mathcal{X}_{33} = \frac{Q_{33}^* + Q_{66}}{2}I - \frac{Q_{33}^* - Q_{66}}{2i}\sigma_3 \quad (5.21)$$

By inserting this ansatz on the $\mathcal{N} = 1$ superpotential (5.4) we verified that the ansatz exactly reproduces the equations of motion (3.2) of the parent $\mathcal{N} = 2$ theory. Moreover we verified that ansatz exhausts the vacuum space.

To compute the moduli space we still need for the residual gauge symmetries. There are eight residual $U(1)$ abelian gauge factors on the moduli space, an $U(1)^2$ in each $U(2)$. They act as in the $\mathcal{N} = 2$ case. We can check this explicitly as follows:

$$\mathcal{Q}_{12} \to \begin{pmatrix} e^{-i\phi_8} & 0 \\ 0 & e^{i\phi_1} \end{pmatrix} \mathcal{Q}_{12} \begin{pmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_7} \end{pmatrix} \quad , \quad \mathcal{Q}_{21} \to \begin{pmatrix} e^{i\phi_2} & 0 \\ 0 & e^{-i\phi_7} \end{pmatrix} \mathcal{Q}_{21} \begin{pmatrix} e^{i\phi_8} & 0 \\ 0 & e^{-i\phi_1} \end{pmatrix}$$

$$\mathcal{Q}_{23} \to \begin{pmatrix} e^{i\phi_2} & 0 \\ 0 & e^{-i\phi_7} \end{pmatrix} \mathcal{Q}_{23} \begin{pmatrix} e^{i\phi_6} & 0 \\ 0 & e^{-i\phi_3} \end{pmatrix} \quad , \quad \mathcal{Q}_{32} \to \begin{pmatrix} e^{-i\phi_6} & 0 \\ 0 & e^{i\phi_3} \end{pmatrix} \mathcal{Q}_{32} \begin{pmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_7} \end{pmatrix}$$

$$\mathcal{Q}_{34} \to \begin{pmatrix} e^{-i\phi_6} & 0 \\ 0 & e^{i\phi_3} \end{pmatrix} \mathcal{Q}_{34} \begin{pmatrix} e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_5} \end{pmatrix} \quad , \quad \mathcal{Q}_{43} \to \begin{pmatrix} e^{i\phi_4} & 0 \\ 0 & e^{-i\phi_5} \end{pmatrix} \mathcal{Q}_{43} \begin{pmatrix} e^{i\phi_6} & 0 \\ 0 & e^{-i\phi_3} \end{pmatrix}$$

$$\mathcal{Q}_{11} \to \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} \mathcal{Q}_{11} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} \quad , \quad \mathcal{Q}_{44} \to \begin{pmatrix} e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_5} \end{pmatrix} \mathcal{Q}_{44} \begin{pmatrix} e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_5} \end{pmatrix}$$

$$\mathcal{X}_{11} \to \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_8} \end{pmatrix} \mathcal{X}_{11} \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_8} \end{pmatrix} \quad , \quad \mathcal{X}_{33} \to \begin{pmatrix} e^{-i\phi_3} & 0 \\ 0 & e^{i\phi_6} \end{pmatrix} \mathcal{X}_{33} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{-i\phi_6} \end{pmatrix}$$

$$(5.22)$$

and these are equivalent to the $\mathcal{N} = 2$ action

$$Q_{12} \to e^{i(\phi_1 - \phi_2)}Q_{12}, \quad Q_{21} \to e^{i(\phi_2 - \phi_1)}Q_{21}, \quad Q_{23} \to e^{i(\phi_2 - \phi_3)}Q_{23}, \quad Q_{32} \to e^{i(\phi_3 - \phi_2)}Q_{32}$$

$$Q_{34} \to e^{i(\phi_3 - \phi_4)}Q_{34}, \quad Q_{43} \to e^{i(\phi_4 - \phi_3)}Q_{43}, \quad Q_{45} \to e^{i(\phi_4 - \phi_5)}Q_{45}, \quad Q_{54} \to e^{i(\phi_5 - \phi_4)}Q_{54}$$

$$Q_{56} \to e^{i(\phi_5 - \phi_6)}Q_{56}, \quad Q_{65} \to e^{i(\phi_6 - \phi_5)}Q_{65}, \quad Q_{67} \to e^{i(\phi_6 - \phi_7)}Q_{67}, \quad Q_{76} \to e^{i(\phi_7 - \phi_6)}Q_{76}$$

$$Q_{78} \to e^{i(\phi_7 - \phi_8)}Q_{78}, \quad Q_{87} \to e^{i(\phi_8 - \phi_7)}Q_{87}, \quad Q_{81} \to e^{i(\phi_8 - \phi_1)}Q_{81}, \quad Q_{18} \to e^{i(\phi_1 - \phi_8)}Q_{18}$$

$$Q_{11} \to \qquad Q_{11}, \quad Q_{33} \to \qquad Q_{33}, \quad Q_{66} \to \qquad Q_{66}, \quad Q_{88} \to \qquad Q_{88}$$

$$(5.23)$$

One of them acts trivially, six combinations are used to mod the moduli space and the last factor is broken to $\mathbb{Z}_{2k}$ by the CS. We hence obtain exactly the CY$_4$ moduli space (5.11) quotiented by the same $\mathbb{Z}_{2k}$ of the parent $\mathcal{N} = 2$ theory. Actually there is still a residual discrete symmetry, Θ, generated by $i\sigma_3$ in $U(2)$ that acts on the moduli space exactly as the anti-holomorphic involution (5.12). The moduli space is then the Spin(7) quotient Y/Θ_k, where Θ_k is the combination of Θ with $\mathbb{Z}_{2k}$. In this way we computed the Spin(7) geometry obtained by the anti-holomorphic involution on the CY$_4$. It coincides with the geometry of the other $\mathcal{N} = 1$ theory introduced above.

The two Spin(7) geometries Y/Θ_k computed by projecting the toric dual parent theories coincide, and we conclude that the two $\mathcal{N} = 1$ models are Spin(7) dual. We conclude this section by arguing that in this case the Spin(7) duality is actually a Seiberg-like duality.

Brane construction and Seiberg-like duality. In this case we can support the duality between the two $\mathcal{N} = 1$ theories by using the brane construction. One can observe from figure 2 that the orientifold projection in this case folds the quiver by identifying pairs of $U(N)$ gauge groups. At the level of type IIB brane description the orientifolded theory is locally $\mathcal{N} = 2$. We can then exchange without problem the $(1, p_i)$ branes at the boundaries of the D3s associated to the second gauge group of the projected $\mathcal{N} = 1$ theory. This operation generates the Seiberg-dual phase exactly as in the parent theory, where the Seiberg duality is implemented at the same time on the two identified gauge groups. The $|p_i - p_{i+1}|$ fractional branes, created during the exchange, modify the dual ranks $\widetilde{N}_2$, and we have $\widetilde{N}_2 = 2(N + |k|)$. The CS levels transform as discussed above and the superpotential transforms according to the usual rules of Seiberg-like duality.[9] We conclude that in this case the Spin(7) duality is a Seiberg-like duality.

5.2 An infinite family

In this section we propose a generalization of the Seiberg-like duality discussed above, for an infinite family of $\mathcal{N} = 1$ gauge theories. The two dual phases are represented in figure 3 (c) and (d). They can be obtained by projecting the $\widetilde{L^{aba}}_{k_i}$ theories represented in figure 3 (a) and (b). We fix both a and b to be even. We choose the CS levels of the model in figure 3 (a) as:

$$\begin{cases} k_i = k & i = a - 1 & i = a + 2 \\ k_i = -k & i = a & i = a + 1 \\ k_i = 0 & \text{otherwise} \end{cases} \tag{5.24}$$

We can describe the geometry of these models in a unified way. The gauge invariant operators orthogonal to the CS vector are

$$\begin{aligned}
x_1 &= Q_{12}Q_{21} = Q_{34}Q_{43} = \cdots = Q_{2a-1,2a}Q_{2a,2a-1} = Q_{2a+1,2a+1} = \cdots = Q_{b+a,b+a} \\
x_2 &= Q_{23}Q_{32} = Q_{45}Q_{54} = \cdots = Q_{2a,2a+1}Q_{2a+1,2a+1} = Q_{2a+1,2a+2}Q_{2a+2,2a+1} = \cdots \\
&\cdots = Q_{b+a-1,b+a}Q_{b+a,b+a-1} = Q_{b+a,1}Q_{1,b+a} \\
y_1 &= Q_{12}Q_{23}\ldots Q_{b+a,1}, \quad y_2 = Q_{1,b+a}Q_{b+a,b+a-1}\cdots Q_{21} \\
t_1 &= Q_{a-1,a}Q_{a+2,a+1}, \quad t_1 = Q_{a+1,a+1}Q_{a,a-1}
\end{aligned} \tag{5.25}$$

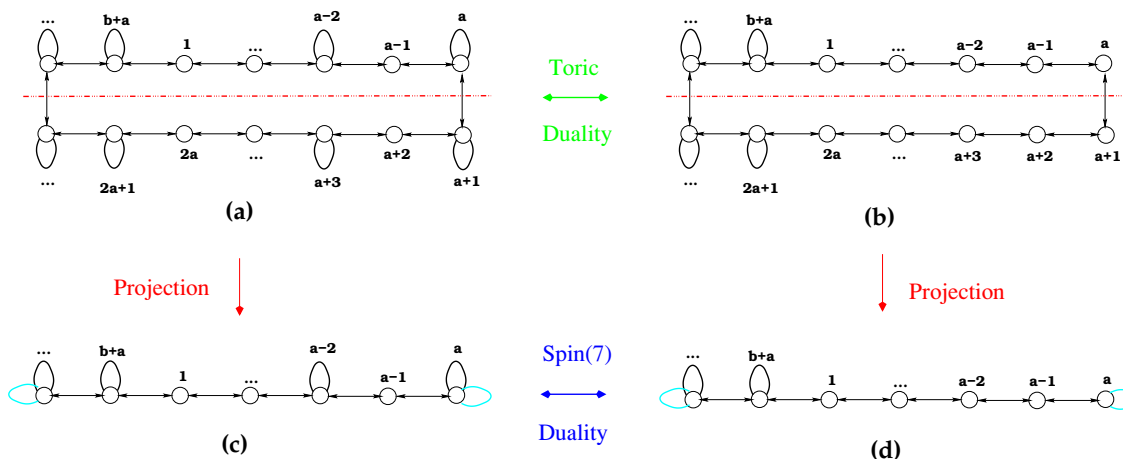[9]We will come back to this issue in section 6.

Figure 3. The models in (a) and (b) represent two L^{aba} toric dual phases with both a and b even. The orientifold projection acts by folding the quiver, along the dashed red lines. By projecting these models to $\mathcal{N} = 1$ Spin(7) cones we obtain the phases (c) and (d), that are related by Spin(7) duality. These models are also Seiberg-like dual.

The CY$_4$ Y geometry that has to be mod by the $\mathbb{Z}_k$ is

$$y_1 y_2 = x_1^a x_2^b \quad , \quad t_1 t_2 = x_2^2 \tag{5.26}$$

We choose the anti-holomorphic involution by generalizing the choice of section 5.1. On the Y coordinates it is

$$x_1 \to x_1^* \quad x_2 \to -x_2^* \quad y_1 \to y_2^* \quad y_2 \to y_1^* \quad t_1 \to t_1^* \quad t_2 \to t_2^* \tag{5.27}$$

This action has fixed points and it can be translated on the vector and matter multiplets as usual. It sends $\sigma \to \sigma$ and $\psi \to -\psi$ and it hence acts as an orientifold projection on the quiver field theory. It can indeed be realized quotienting by an antiholomorphic orientifold symmetry of the $\mathcal{N} = 2$ lagrangian that identifies i-th group with the $i + (a+b)/2$-th. The rank of each gauge group and the CS level are doubled after the identification. Finally we obtain the $\mathcal{N} = 1$ theory represented in figure 3 (c).

We choose to dualize this family by acting on nodes $a - 1$ and $a + 2$. Other choices are possible. The quiver is depicted in figure 3 (b). The new CS levels are

$$\begin{cases} k_i = -k & i = a - 1 & i = a + 2 \\ k_i = -k & i = a - 2 & i = a + 3 \\ k_i = 0 & \text{otherwise} \end{cases} \tag{5.28}$$

This theory has the same CY$_4$ geometry Y as before. One can verify that, by implementing the same anti-holomorphic involution on the coordinates as before, we obtain the quiver in figure 3 (d).

We computed the Spin(7) geometries Y/Θ_k of both the $\mathcal{N} = 1$ models and we showed that they coincide. This confirms that they are Spin(7) dual. As in the subsection 5.1 the Spin(7) duality is also in this case a Seiberg-like duality. Indeed one can embed the

$\mathcal{N} = 1$ theories in a IIB brane setup and observe that locally the Seiberg-like duality can be performed ignoring the effect of the orientifold.

6 Comments on $\mathcal{N} = 1$ Seiberg-like duality

As discussed in the previous section in some cases we can claim that the proposed Spin(7) duality coincides with three dimensional Seiberg-like duality for $\mathcal{N} = 1$ theories. We used a brane description to support this idea.

In this section we translate this brane description in a field theoretical language. We propose a procedure to obtain the Seiberg-like dual in the $\mathcal{N} = 1$ case. We provide the transformation rules on the superpotential, on the field content and on the gauge groups by extracting them from the example in subsection 5.1.

We can summarize the procedure as follows. First the gauge invariant operators of the electric theory appear as mesons in the magnetic theory. Then the ranks of the gauge groups and the CS levels transform as in (3.1). Moreover we distinguish three terms in the dual $\mathcal{N} = 1$ superpotential. The first term is a holomorphic contribution that we call W_{holo}. It is cubic and involves the coupling of the dual quarks with the mesons. The second part is non holomorphic, and it is obtained from the $\mathcal{N} = 1$ superpotential after a proper substitution of the electric quarks with the mesons. In the case of the Spin(7) duality this term corresponds to the $\mathcal{N} = 2$ superpotential of the parent theory, projected to $\mathcal{N} = 1$. The last term is obtained by coupling the dual R fields with the D terms. The masses of the R fields are proportional to the dual CS levels.

In the first part of the section we show that these rules reproduce the dual theory studied in subsection 5.1. Then we apply this procedure to one $\mathcal{N} = 1$ theory with $U(N_c)_k$ gauge group, N_f flavors and a quartic, non holomorphic, superpotential. This theory is not associated to a CY_4 and we cannot use the Spin(7) duality. In any case we propose a possible Seiberg-like dual description. We obtain a dual $U(N_f + |k| - N_c)_{-k}$ gauge theory with N_f flavors and the same quartic, non holomorphic, superpotential.

6.1 Revisiting the $\widetilde{L}_{k_i}^{444}$ theory

Here we reconsider the models studied in section 5.1. We start from the electric theory, with superpotential (5.2).

Here we follow the procedure sketched above to obtain the dual phase. First we identify the group to dualize. This group and its neighbours are modified as (3.1). Then we can build the dual superpotential. We start from the holomorphic term:

$$W_{holo} = \mathcal{X}_{11}\mathcal{Q}_{12}\mathcal{Q}_{21} + \mathcal{X}_{33}\mathcal{Q}_{32}\mathcal{Q}_{23} + \mathcal{X}_{13}\mathcal{Q}_{32}\mathcal{Q}_{21} + \mathcal{X}_{31}\mathcal{Q}_{12}\mathcal{Q}_{23} + h.c. \qquad (6.1)$$

The next step consists of contracting the fields charged under the dualized gauge groups into mesons $\mathcal{X}_{ij}$. They are

$$\begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{13} \\ \mathcal{X}_{31} & \mathcal{X}_{33} \end{pmatrix} = \begin{pmatrix} Q_{12}Q_{21} & Q_{12}Q_{23} \\ Q_{32}Q_{21} & Q_{32}Q_{23} \end{pmatrix} \qquad (6.2)$$

We substitute the mesons in the first two lines of (5.2) and integrate them out the massive fields. This procedure reproduces the first two lines of (5.4). The other terms of (5.4) are obtained by reintroducing the D-terms and the R fields. The mass terms for R_i are obtained by transforming the CS levels with the usual rule (3.1). In this way we reproduced the dual theory discussed in section 5.1.

6.2 $U(N)_k$ SQCD with quartic superpotential

Even if we derived the rules of the $\mathcal{N} = 1$ Seiberg-like duality from a specific set of theories, describing M2 branes probing Spin(7) singularities, we can try to push further in the field theoretical direction. Here we apply these rules to a SQCD like model, that does not have a known AdS$_4$ dual. We propose a dual version of $U(2N_c)_{2k}$ SQCD with $2N_f$ flavors and a non-holomorphic quartic superpotential

$$W = Q\tilde{Q}Q^*\tilde{Q}^* + \frac{k}{2\pi}R^2 + R\left(QQ^\dagger - \widetilde{Q}^\dagger\widetilde{Q}\right) \tag{6.3}$$

We study the dual of the non-holomorphic superpotential (6.3) and the quarks are $\mathcal{N} = 1$ complex scalar superfields. The dual theory is obtained by applying the rules explained above in the case of the quiver gauge theories. The dual gauge group is expected to be $U(2N_f - 2N_c + 2|k|)_{-2k}$. There are $2N_f$ dual flavors and the meson $M = Q\tilde{Q}$. The holomorphic part of the dual superpotential is

$$W_{holo} = Mq\tilde{q} + M^*q^*\tilde{q}^* \tag{6.4}$$

By considering the deformation $Q\tilde{Q}Q^*\tilde{Q}^* = MM^*$ we can integrate out the meson M. By turning on the contributions of the D terms and of the R field, with $k_{CS} = -2k$ we have

$$W = q\tilde{q}q^*\tilde{q}^* - \frac{k}{2\pi}\widetilde{R}^2 + \widetilde{R}\left(\widetilde{q}\widetilde{q}^\dagger - q^\dagger q\right) \tag{6.5}$$

As in the quartic $\mathcal{N} = 2$ SQCD in three dimensions or the quartic $\mathcal{N} = 1$ SQCD in four dimensions we observe that our procedure predicts the self duality for $\mathcal{N} = 1$ three dimensional CS-SQCD with a quartic interaction. Anyway, in general, this theory is not superconformal and moreover it is not protected against quantum corrections. It would be interesting to provide some checks of this duality, by engineering a brane realization and by computing the Witten index, by first lifting the moduli space in both phases consistently. We leave this analysis to future investigations.

7 Discussion and further developments

In this paper we proposed a generalization of $\mathcal{N} = 2$ toric duality for M2 branes probing toric CY$_4$ singularities to $\mathcal{N} = 1$ models of M2 branes probing Spin(7) singularities. We called this generalization Spin(7) duality. This proposal has been supported by the AdS/CFT correspondence. Indeed we matched the moduli space of $\mathcal{N} = 1$ Spin(7) dual models by orientifolding $\mathcal{N} = 2$ toric dual pairs. In some cases, with the help of the brane

picture, we argued that the Spin(7) duality is also a Seiberg-like duality. Finally we proposed a generalization of this $\mathcal{N} = 1$ Seiberg-like duality for models without a known AdS dual description.

The main problem in the study of a supersymmetric, but non holomorphic, duality is its validity at quantum level. In the near horizon limit the AdS/CFT correspondence provides some arguments to protect the validity of the duality beyond the classical level. The strongly coupled phases of the dual pair of theories are conjectured to describe the QFT of M2 branes probing the same Spin(7) cone. By considering the near horizon geometry the models are superconformal invariant and represent two dual descriptions of the same singularity that should hence be valid in the strong coupling region. Planar equivalence moreover supports the duality between the pairs for large N.

However other checks are necessary. For example one should compute the Witten index [46–48] to match the number of supersymmetric vacua. Moreover it would be interesting to study other partition functions, by localizing the $\mathcal{N} = 1$ models on more complicate manifolds, like the three sphere S^3. In the $\mathcal{N} = 2$ case [49, 50] toric duality on the three sphere has been checked for the $\widetilde{L^{aba}}_{k_i}$ theories in [51–54]. A generalization of this analysis to Seiberg-like duality for these theories appeared in [55]. In the $\mathcal{N} = 1$ case the calculations may be very involved, because of the absence of holomorphy and of a continuous R-symmetry, but they can potentially provide strong checks of the dualities.

Another interesting aspect regards other possible models. Here we discussed only vector like models, but there are also $\mathcal{N} = 2$ chiral models with an AdS$_4$ dual [14]. They correspond to quiver gauge theories with vector-like bifundamentals and chiral flavors. It should be interesting to extend the orientifold projection to these models and study the Spin(7) duality for those cases.

Acknowledgments

A $\mathcal{N} = 1$ formalism

In this appendix we quickly report some known results about the $\mathcal{N} = 1$ superspace obtained by setting to zero some of the Grassmann variables of the $\mathcal{N} = 2$ case in three dimensions.

First we start by reviewing the $\mathcal{N} = 2$ case. There are two possible multiplets involved in a $U(N)$ quiver, the vector multiplet and the bifundamental chiral multiplet. In a quiver with G nodes the a-th vector multiplet V_a contains a three dimensional gauge field A_μ, a two component Dirac spinor and two real scalars, σ_a and D_a. A chiral bifundamental X_{ab} connecting the a-th and the b-th node (if $a = b$ we have a chiral field in the adjoint representation) than consists of two complex scalars and a two dimensional Dirac spinor. The $\mathcal{N} = 1$ superspace is obtained by decomposing the $\mathcal{N} = 2$ case in two copies of $\mathcal{N} = 1$ and by projecting out one of them [56–58]. The decomposition is obtained by splitting the θ variables and the super-derivatives as

$$\theta_\alpha = \theta_{1\alpha} + i\theta_{2\alpha} \quad , \quad D_\alpha = \frac{1}{2}\left(D_{1\alpha} + iD_{2\alpha}\right) \quad , \quad \overline{D}_\alpha = \frac{1}{2}\left(D_{1\alpha} - iD_{2\alpha}\right) \tag{A.1}$$

and the projection to $\mathcal{N} = 1$ is performed by setting $\theta_2 = 0$ in the lagrangian. In terms of superfields the $\mathcal{N} = 2$ vector multiplet decomposes into a $\mathcal{N} = 1$ spinor superfield Γ_α^a and an $\mathcal{N} = 1$ auxiliary real scalar superfield R_a. The chiral multiplet X_{ab} decomposes into two $\mathcal{N} = 1$ real scalar superfields, $Re(X_{ab})$ and $Im(X_{ab})$ that can be combined into a single complex scalar superfield Y_{ab}.

By acting on the $\mathcal{N} = 2$ lagrangian the $\mathcal{N} = 1$ superpotential has three different contributions. They are

$$\frac{k_a}{8\pi} S_{CS_a}^{\mathcal{N}=2} \rightarrow -\frac{k_a}{4\pi}\int d^2\theta_1 R_a^2$$

$$-\int d^4\theta X_{ab}^\dagger e^{-V_a} X_{ab} e^{V_a} \rightarrow \int d^2\theta_1 \left(Y_{ab} Y_{ab}^\dagger R_a - Y_{ab}^\dagger Y_{ab} R_b\right)$$

$$\int d^2\theta W(X_{ab} + c.c. \rightarrow \int d^2\theta_1 \left(W(Y_{ab}) + W(Y_{ab}^*)\right)$$

B $\mathcal{N} = 1$ superconformal algebra

In this appendix we provide the generic structure of the superconformal algebra in three dimensional $\mathcal{N} = 1$ theories [59, 60].

We define the two dimensional Gamma matrices γ^μ, $\mu = 0, 1, 2$, satisfying the relations

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho \tag{B.1}$$

with $\eta^{\mu\nu} = (1, -1, -1)$. The three dimensional $\mathcal{N} = 1$ superconformal algebra is

$$\begin{aligned}
&[P_\mu, P_\nu] = [P_\mu, Q] = [k_\mu, k_\nu] = [D, M_{\mu\nu}] = [D, D] = [K_\mu, S] = 0 \\
&[M_{\mu\nu}, P_\lambda] = i(\eta_{\mu\nu}P_\nu - \eta_{\nu\lambda}P_\mu), \quad [M_{\mu\nu}, M_{\lambda\rho}] = i\left(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda}\right), \\
&\{Q, Q\} = 2\gamma^\mu P_\mu, \quad [M_{\mu\nu}, Q] = \frac{i}{2}\gamma_{[\mu}\gamma_{\nu]}Q, \quad [M_{\mu\nu}, K_\lambda] = i(\eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu), \\
&\{S, S\} = 2\gamma^\mu K_\mu, \quad [M_{\mu\nu}, S] = \frac{i}{2}\gamma_{[\mu}\gamma_{\nu]}S, \quad [P_\mu, K_\nu] = 2i(M_{\mu\nu} + \eta_{\mu\nu}D), \\
&[P_\mu, S] = -\gamma_\mu Q, \quad [K_\mu, Q] = -\gamma_\mu S, \quad \{Q, S\} = -i\left(2D + \gamma^{[\mu}\gamma^{\nu]}M_{\mu\nu}\right), \\
&[D, P_\mu] = -iP_\mu, \quad [D, S] = \frac{i}{2}S, \quad [D, K_\mu] = iK_\mu, \quad [D, Q] = -\frac{i}{2}Q
\end{aligned} \tag{B.2}$$

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