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Supersymmetric backgrounds from 5d $\mathcal{N}=1$ supergravity

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Abstract: We construct curved backgrounds with Euclidean signature admitting rigid supersymmetry by using a 5d $\mathcal{N}=1$ off-shell Poincaré supergravity. We solve the conditions for the background Weyl multiplet and vector multiplets that preserve at least one supersymmetry parameterized by a symplectic Majorana spinor, and represent the solution in terms of several independent fields. We also show that the partition function does not depends on the local degrees of freedom of the background fields. Namely, as far as we focus on a single coordinate patch, we can freely change the independent fields by combining Q-exact deformations and gauge transformations. We also discuss realization of several known examples of supersymmetric theories in curved backgrounds by using the supergravity.

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1 Introduction

Recent progress in non-perturbative aspects of supersymmetric field theories owes a great deal to the construction of rigid supersymmetry on curved backgrounds. Pestun [1] constructed $\mathcal{N} = 2$ supersymmetric theories on S^4 and computed the partition function and the expectation values of circular Wilson loops. The results play a crucial role in the AGT conjecture [2], which relates 4d $\mathcal{N} = 2$ theories and 2d conformal field theories. The partition function of supersymmetric theories on S^3 [3–5] enables us to perform quantitative checks of dualities among 3d theories and the AdS$_4$/CFT$_3$ correspondence. Supersymmetric theories on 3d and 4d squashed spheres [6–9] and manifolds with other topologies [10–12] are also constructed, and the exact partition functions for those provide useful information about supersymmetric field theories.

Also in 5d, supersymmetric field theories are constructed on various curved manifolds. Theories on round and squashed S^5 are constructed in [13–17]. The perturbative [13, 15, 16, 18, 19] and instanton [20] partition functions on S^5 are computed, and N^3 behavior of the free energy of the maximally supersymmetric Yang-Mills theory (SYM) is confirmed [21]. This is a strong evidence of the close relation [22, 23] between the 5d SYM and the 6d

(2,0) theory realized on a stack of N M5-branes. Supersymmetric theories on $\boldsymbol{S}^4 \times \boldsymbol{S}^1$ are constructed in [24, 25], and the partition function [24] (superconformal index) provides an evidence of the existence of non-trivial fixed points with enhanced global symmetries [26]. Theories on $\boldsymbol{S}^3 \times \Sigma$, the product of three-sphere and a Riemann surface Σ, are constructed in [27, 28], and used to study a conjectured relation between the 6d (2,0) theory and a q-deformed 2d Yang-Mills theory. Supersymmetric theories on $\boldsymbol{S}^2 \times M_3$ [29–31] are used to confirm predictions of the 3d/3d correspondence [32].

A systematic construction of rigid supersymmetric field theories on curved backgrounds was started in [33].[1] To obtain rigid supersymmetry on a curved manifold, we couple matter fields to a background off-shell supergravity multiplet, and require the supersymmetry transformation of the gravitino, $\delta_Q \psi_\mu$, to vanish. If the gravity multiplet contains other fermions their supersymmetry transformation should also vanish. By solving these conditions, we obtain backgrounds that admit rigid supersymmetry. In 4d, the analysis by using the new minimal supergravity [35] shows that we can realize at least one rigid supersymmetry on backgrounds with Hermitian metrics [36, 37]. With the old minimal supergravity [38, 39] we can realize a supersymmetry in (squashed) $\boldsymbol{S}^4$ or backgrounds with Hermitian metrics [40]. (See also [41, 42] for studies of 4d supersymmetric theories on curved background with the help of supergravity.) The analysis in [36, 43] using the 3d version of the new minimal supergravity shows that the manifold is required to have the almost contact metric structure which satisfies a certain integrability condition. The existence of two or more supersymmetries imposes stronger restrictions. See [36, 44] for analysis from the holographic viewpoint.

In this paper, we realize rigid supersymmetry on a 5d manifold $\mathcal{M}$ with Euclidean signature by using the 5d $\mathcal{N}=1$ off-shell Poincaré supergravity [45–47] whose Weyl multiplet has $40+40$ degrees of freedom [48]. The first step of the analysis with this supergravity is taken in [49], where the condition associated with the gravitino, $\delta_Q \psi_\mu = 0$, is focused on. (See also [50] for supersymmetric backgrounds in the minimal gauged supergravity without auxiliary fields [51], and [52] for an analysis with off-shell conformal supergravity with a smaller Weyl multiplet with $32+32$ degrees of freedom [53–56].) There is actually another fermion, which we denote by η, in the Weyl multiplet, and thus we should also consider the condition $\delta_Q \eta = 0$. One of the purposes of this paper is to complete this analysis and to give the solution to the supersymmetry conditions.

Another purpose of this paper is to study supersymmetry-preserving deformations of the background. It is often happens that the partition functions for different backgrounds are the same. For example, in the case of $\boldsymbol{S}^3$ partition function, a certain squashed $\boldsymbol{S}^3$ gives the same partition function as the round $\boldsymbol{S}^3$ [6]. The partition function of another squashed $\boldsymbol{S}^3$ [7] is the same as that for an ellipsoid [6]. These facts suggests that the partition function depends only on a small part of the data of the background. This is confirmed in [57] by showing that the partition function of a supersymmetric theory on manifolds with $\boldsymbol{S}^3$ topology depends on the background manifolds only through a single

[1] See also [34] for construction of supersymmetric theories on AdS_4 by taking the decoupling limit of supergravity.

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parameter. Furthermore, [58] shows for 3d and 4d cases that although supersymmetric backgrounds have functional degrees of freedom almost all deformations of the background correspond to Q-exact deformations of the action, and do not affect the partition function. We perform similar analysis in 5d.

This paper is organized as follows. In the next section we solve the conditions $\delta_Q\psi = \delta_Q\eta = 0$ and derive the restrictions for the background fields under the assumption of the existence of at least one rigid supersymmetry parameterized by a symplectic Majorana spinor. In section 3, we show that all supersymmetry-preserving deformations of the background fields can be realized by Q-exact deformations and gauge transformations as far as we focus on a single coordinate patch. We also study supersymmetric backgrounds of vector multiplets in section 4. In section 5 we discuss realization of some known examples of supersymmetric theories in curved manifolds by using the supergravity. Section 6 is devoted to discussion. Notation and conventions are summarized in appendix.

2 Supersymmetric backgrounds

2.1 5d $\mathcal{N} = 1$ off-shell supergravity

The 5d $\mathcal{N} = 1$ off-shell supergravity constructed in [45–47] has the following local bosonic symmetries.

- The general coordinate invariance
- $\mathrm{Sp}(2)_L$: the local Lorentz symmetry
- $\mathrm{Sp}(1)_R$: the local R-symmetry
- $\mathrm{U}(1)_Z$: the gauge symmetry associated with the central charge

In addition to these, the formulation in [46, 47] has the local dilatation symmetry. The corresponding gauge field $b_\mu = \alpha^{-1}\partial_\mu\alpha$ is pure-gauge, and in this paper we fix the gauge by the condition $b_\mu = 0$.

The Weyl multiplet consists of the fields shown in table 1. In particular, it contains two fermions: ψ_μ and η. A supersymmetric background is defined as a configuration of the Weyl multiplet that is invariant under the supersymmetry transformation with a non-vanishing parameter ξ. If we assume $\psi_\mu = \eta = 0$ in the background the transformations of the bosonic components automatically vanish, and the nontrivial conditions are $\delta_Q\psi_\mu = \delta_Q\eta = 0$. The transformation laws of the fermions are [45, 46]

$$\begin{aligned} \delta_Q(\xi)\psi_\mu &= D_\mu\xi - f_{\mu\nu}\gamma^\nu\xi + \frac{1}{4}\gamma_{\mu\rho\sigma}v^{\rho\sigma}\xi - t\gamma_\mu\xi, \\ \delta_Q(\xi)\eta &= -2\gamma_\nu\xi D_\mu v^{\mu\nu} + \xi C + 4(\not{D}t)\xi + 8(\not{f} - \not{v})t\xi + \gamma^{\mu\nu\rho\sigma}\xi f_{\mu\nu}f_{\rho\sigma}. \end{aligned} \tag{2.1}$$

See appendix for the notation of $\mathrm{Sp}(1)_R$ and spinor indices. We treat the transformation parameter ξ as a Grassmann-even variable. $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the $\mathrm{U}(1)_Z$ field strength and D_μ is the covariant derivative defined by

$$D_\mu = \partial_\mu + \delta_M(\omega_{\mu\widehat{\rho}\widehat{\sigma}}) - \delta_U(V^a_\mu) - \delta_Z(a_\mu), \tag{2.2}$$

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	fields	dof	$Sp(1)_R$	ours	Zucker	KO
bosons	vielbein	10	**1**	$e_\mu^{\widehat{\nu}}$	$e_\mu^{\widehat{\nu}}$	$e_\mu^{\widehat{\nu}}$
	$U(1)_Z$ gauge field	4	**1**	a_μ	$\frac{\kappa}{\sqrt{3}}A_\mu$	$-\frac{1}{2\alpha}A_\mu$
	anti-sym. tensor	10	**1**	$v^{\mu\nu}$	$2\kappa v^{\mu\nu}$	$2v^{\mu\nu}$
	$Sp(1)_R$ triplet scalars	3	**3**	t_a	$-2i\kappa t_a$	t_a
	$Sp(1)_R$ gauge field	12	**3**	V_μ^a	$\frac{\kappa i}{2}V_\mu^a$	$-V_\mu^a$
	scalar	1	**1**	C	$16\kappa C$	$-4C$
fermions	gravitino	32	**2**	$\psi_{I\mu\alpha}$	$\frac{\kappa}{\sqrt{2}}\psi_{I\mu\alpha}$	$\psi_{I\mu\alpha}$
	fermion	8	**2**	$\eta_{I\alpha}$	$8\sqrt{2}\kappa\lambda_{I\alpha}$	$-8\widetilde{\chi}_{I\alpha}$
	supersymmetry parameter	8	**2**	ξ	$\frac{1}{\sqrt{2}}\varepsilon$	ε
	fermion bilinears			$(\psi\chi)$	$i(\overline{\psi}\chi)$	$i(\overline{\psi}\chi)$

Table 1. Component fields in the Weyl multiplet. The last two columns show the relation to Zucker's [45] and Kugo-Ohashi's [46] conventions. We also show the relations among supersymmetry parameters and fermion bilinears in the three conventions in the last two lines.

where δ_M, δ_U, and δ_Z are $Sp(2)_L$, $Sp(1)_R$, and $U(1)_Z$ transformations, respectively. The explicit form of $D_\mu\xi$ is

$$D_\mu\xi = \partial_\mu\xi + \frac{1}{4}\omega_{\mu\widehat{\rho}\widehat{\sigma}}\gamma^{\widehat{\rho}\widehat{\sigma}}\xi - V_\mu\xi. \tag{2.3}$$

In this paper terms in transformation laws (Lagrangians) including two (three) or more fermions are always omitted.

2.2 Spinor bilinears and orthonormal frame

In a 5d spacetime with Lorentzian signature, the parameter ξ of the local $\mathcal{N}=1$ supersymmetry transformation is a symplectic Majorana spinor. Although we can impose the symplectic Majorana condition on ξ also in the Euclidean space, the condition is not the same as that for Lorentzian signature, and we do not have to impose it. To study most general case it is desirable to consider complex spinor without symplectic Majorana condition imposed. In this paper, however, we restrict ourselves to the case with ξ satisfying the symplectic Majorana condition. This is just for simplicity of the analysis.

Following a standard strategy, we define the bilinears of the spinor ξ:

$$S = (\xi\xi), \quad R^\mu = (\xi\gamma^\mu\xi), \quad J^a_{\mu\nu} = \frac{1}{S}(\xi\tau_a\gamma_{\mu\nu}\xi). \tag{2.4}$$

By a Fierz's identity, we can show

$$\gamma_\mu\xi R^\mu = \xi S. \tag{2.5}$$

The following equations are easily derived from this:

$$R_\mu R^\mu = S^2, \quad J^a_{\mu\nu}R^\nu = 0, \quad -\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\rho\sigma}R_\lambda J^a_{\rho\sigma} = SJ^a_{\mu\nu}. \tag{2.6}$$

Because ξ is a solution to the first order differential equation $\delta_Q\psi_\mu = 0$, it is nowhere vanishing and so are the bilinears. In particular, $S > 0$ everywhere. We assume the

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vielbein $e_{\mu}^{\widehat{\nu}}$ is real, and then R^{μ} is real, too. The existence of the non-vanishing real vector field R^{μ} enables us to treat the background manifold $\mathcal{M}$ as a fibration over a base manifold $\mathcal{B}$, at least locally. In this paper we will not discuss global issues and focus only on a single coordinate patch. Let us define the fifth coordinate x^5 by

$$R^{\mu}\partial_{\mu} = \partial_5, \tag{2.7}$$

and use a local frame with

$$e^{\widehat{m}} = e_n^{\widehat{m}} dx^n, \quad e^{\widehat{5}} = S(dx^5 + \mathcal{V}_m dx^m). \tag{2.8}$$

With this frame R^{μ} has the local components

$$R_{\widehat{5}} = S, \quad R_{\widehat{m}} = 0. \tag{2.9}$$

The second and third equations in (2.6) can be rewritten as

$$J^a_{\widehat{m}\widehat{5}} = 0, \quad -\frac{1}{2}\epsilon^{(4)}_{\widehat{m}\widehat{n}\widehat{k}\widehat{l}} J^a_{\widehat{k}\widehat{l}} = J^a_{\widehat{m}\widehat{n}}, \tag{2.10}$$

where $\epsilon^{(4)}_{\widehat{m}\widehat{n}\widehat{k}\widehat{l}} = \epsilon_{\widehat{m}\widehat{n}\widehat{k}\widehat{l}\widehat{5}}$. The equation (2.5) means that ξ has positive chirality with respect to $\gamma_{\widehat{5}} = S^{-1}R^{\mu}\gamma_{\mu}$;

$$\gamma_{\widehat{5}}\xi = \xi. \tag{2.11}$$

A symplectic Majorana spinor χ belongs to the $(\mathbf{4}, \mathbf{2})$ representation of $\mathrm{Sp}(2)_L \times \mathrm{Sp}(1)_R$. Because $\mathrm{Sp}(k) = \mathrm{U}(k, \mathbb{H})$, we can treat χ as a vector with two quaternionic components. If we use the matrix representation of quaternions we can represent χ as a 4×2 matrix in the form

$$\chi = (\chi_{\alpha}{}^{I}) = \begin{pmatrix} U \\ D \end{pmatrix}, \quad U = U_0\mathbf{1}_2 + iU_a\tau_a, \quad D = D_0\mathbf{1}_2 + iD_a\tau_a, \quad U_i, D_i \in \mathbb{R}. \tag{2.12}$$

The vector $R_{\widehat{m}}$ breaks the local Lorentz symmetry $\mathrm{Sp}(2)_L$ to its subgroup $\mathrm{Sp}(1)_l \times \mathrm{Sp}(1)_r$, where $\mathrm{Sp}(1)_l$ and $\mathrm{Sp}(1)_r$ act on the upper and lower blocks of the matrix (2.12), respectively.

The chirality condition (2.5) implies that the spinor ξ has the upper block only. Furthermore, we can choose a gauge such that $U \propto \mathbf{1}_2$, and then ξ is given by

$$\xi = (\xi_{\alpha}{}^{I}) = \sqrt{\frac{S}{2}} \begin{pmatrix} \mathbf{1}_2 \\ 0 \end{pmatrix}, \tag{2.13}$$

where the normalization is fixed by $S = (\xi\xi)$. This gauge choice breaks $\mathrm{Sp}(1)_l \times \mathrm{Sp}(1)_R$ into its diagonal subgroup $\mathrm{Sp}(1)_D$. It is obvious in this frame that the following eight spinors form a basis of the space of symplectic spinors:

$$\xi_{\alpha}{}^{I}, \quad (\gamma_{\widehat{m}})_{\alpha}{}^{\beta}\xi_{\beta}{}^{I}, \quad \xi_{\alpha}{}^{J}(\tau_a)_J{}^{I}. \tag{2.14}$$

An arbitrary spinor can be expanded by this basis. For example, $\gamma_{\widehat{m}\widehat{n}}\xi$ is related to $\xi\tau_a$ by

$$\gamma_{\widehat{m}\widehat{n}}\xi = -\xi\tau_a J^a_{\widehat{m}\widehat{n}}, \quad \xi\tau_a = \frac{1}{4}J^a_{\widehat{m}\widehat{n}}\gamma^{\widehat{m}\widehat{n}}\xi. \tag{2.15}$$

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The second relation in (2.15) implies that the three matrices J^a satisfy the same algebra with the Pauli matrices τ_a;

$$J^a_{\widehat{m}\widehat{k}}J^b_{\widehat{k}\widehat{n}} = \delta_{ab}\delta_{\widehat{m}\widehat{n}} + i\epsilon_{abc}J^c_{\widehat{m}\widehat{n}}. \tag{2.16}$$

Namely, J^a enjoy the quaternion algebra.[2]

2.3 $\delta_Q\psi_\mu = 0$

Let us first solve the condition $\delta_Q\psi_\mu = 0$, which is also investigated in [49]. Using the basis $(\xi, \gamma_{\widehat{m}}\xi, \tau_a\xi) = (\gamma_{\widehat{\mu}}\xi, \tau_a\xi)$ in (2.14) we decompose $\delta_Q\psi_\mu = 0$ into the following conditions:

$$0 = (\xi\gamma_{\widehat{\lambda}}\delta_Q\psi_{\widehat{\mu}}) = \frac{1}{2}D_{\widehat{\mu}}R_{\widehat{\lambda}} - Sf_{\widehat{\mu}\widehat{\lambda}} - \frac{S}{4}\epsilon_{\widehat{5}\widehat{\mu}\widehat{\lambda}\widehat{\rho}\widehat{\sigma}}v^{\widehat{\rho}\widehat{\sigma}} + St_aJ^a_{\widehat{\mu}\widehat{\lambda}}, \tag{2.17}$$

$$0 = (\xi\tau_a\delta_Q\psi_{\widehat{\mu}}) = (\xi\tau_aD_{\widehat{\mu}}\xi) + \frac{1}{4}(\xi\tau_a\gamma_{\widehat{\mu}\widehat{\rho}\widehat{\sigma}}\xi)v^{\widehat{\rho}\widehat{\sigma}} - R_{\widehat{\mu}}t_a. \tag{2.18}$$

The symmetric part of (2.17), $D_{\{\widehat{\mu}}R_{\widehat{\lambda}\}} = 0$, means that R^μ is a Killing vector. We can take an $\mathrm{Sp}(1)_D \times \mathrm{Sp}(1)_r$ gauge such that

$$\partial_5 e_n^{\widehat{m}} = \partial_5 S = \partial_5\mathcal{V}_m = 0, \tag{2.19}$$

and then $e_n^{\widehat{m}}$, S, and $\mathcal{V}_m$ can be treated as fields on the base manifold $\mathcal{B}$. The $(\widehat{\lambda}, \widehat{\mu}) = (5, m)$ components of (2.17) give

$$f_{m5} = \frac{1}{2}\partial_m S. \tag{2.20}$$

From the integrability condition $\partial_n f_{m5} = \partial_m f_{n5}$ and the Bianchi identity for $f_{\mu\nu}$ we obtain $\partial_5 f_{mn} = 0$. This means that the $\mathrm{U}(1)_Z$ gauge field a_μ is essentially a gauge field on $\mathcal{B}$. (2.20) can be solved, up to $\mathrm{U}(1)_Z$ gauge transformation, by

$$a = a_m dx^m + \frac{1}{2}Sdx^5, \quad \partial_5 a_m = 0. \tag{2.21}$$

For later use we give the non-vanishing components of the spin connection.

$$\omega_{\widehat{k}-\widehat{m}\widehat{n}} = \omega^{(4)}_{\widehat{k}-\widehat{m}\widehat{n}}, \quad \omega_{\widehat{m}-\widehat{n}\widehat{5}} = \omega_{\widehat{5}-\widehat{n}\widehat{m}} = \frac{S}{2}\mathcal{W}_{\widehat{m}\widehat{n}} = \frac{1}{S}D_{\widehat{m}}R_{\widehat{n}}, \quad \omega_{\widehat{5}-\widehat{5}\widehat{m}} = \frac{1}{S}\partial_{\widehat{m}}S = 2f_{\widehat{m}\widehat{5}}. \tag{2.22}$$

$\omega^{(4)}_{\widehat{k}-\widehat{m}\widehat{n}}$ is the spin connection in the base manifold $\mathcal{B}$ defined with the vielbein $e_m^{\widehat{n}}$ and $\mathcal{W}_{mn} = \partial_m\mathcal{V}_n - \partial_n\mathcal{V}_m$ is the field strength of $\mathcal{V}_m$.

The anti-symmetric part of (2.17) can be used to represent the horizontal part of $v^{\mu\nu}$ in terms of other fields;

$$v_{\widehat{p}\widehat{q}} = \epsilon^{(4)}_{\widehat{p}\widehat{q}\widehat{m}\widehat{n}}\left(\frac{S}{4}\mathcal{W}_{\widehat{m}\widehat{n}} - f_{\widehat{m}\widehat{n}} + t_aJ^a_{\widehat{m}\widehat{n}}\right). \tag{2.23}$$

By using (2.13) we obtain

$$(\xi\tau_aD_\mu\xi) = \frac{S}{4}\omega_{\mu\widehat{p}\widehat{q}}J^a_{\widehat{p}\widehat{q}} - V^a_\mu S, \tag{2.24}$$

[2] The definition of J^a differs from the usual definition of the quaternion basis by factor i.

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and we can solve (2.18) with respect to V^a_μ and obtain

$$V^a_{\hat{5}} = \frac{1}{4}\omega_{\hat{5}\hat{p}\hat{q}} J^a_{\hat{p}\hat{q}} + \frac{1}{4} J^a_{\hat{p}\hat{q}} v^{\hat{p}\hat{q}} - t_a, \quad V^a_{\hat{m}} = \frac{1}{4}\omega_{\hat{m}\hat{p}\hat{q}} J^a_{\hat{p}\hat{q}} + \frac{1}{2} J^a_{\hat{m}\hat{p}} v^{\hat{p}\hat{5}}. \tag{2.25}$$

Now we have completely solved $\delta_Q \psi_\mu = 0$. Independent fields are

$$e_n^{\hat{m}}(x^m), \quad S(x^m), \quad \mathcal{V}_m(x^m), \quad a_m(x^m), \quad v_{\hat{m}\hat{5}}(x^m, x^5), \quad t_a(x^m, x^5), \quad C(x^m, x^5), \tag{2.26}$$

and the other fields are represented by these fields.

2.4 $\delta_Q \eta = 0$

By using the spinor basis (2.14) we decompose $\delta_Q \eta = 0$ into the following equations.

$$0 = S^{-1}(\xi \delta_Q \eta) = -2D_\mu v^{\mu\hat{5}} + C + 4t_a J^a_{\hat{m}\hat{n}}(f^{\hat{m}\hat{n}} - v^{\hat{m}\hat{n}}) + \epsilon^{(4)}_{\hat{m}\hat{n}\hat{p}\hat{q}} f^{\hat{m}\hat{n}} f^{\hat{p}\hat{q}}, \tag{2.27}$$

$$0 = S^{-1}(\xi\gamma_{\hat{m}} \delta_Q \eta) = -2D^\lambda v_{\lambda\hat{m}} + 4J^a_{\hat{m}\hat{n}} D^{\hat{n}} t_a + 8t_a J^a_{\hat{m}\hat{p}}(f^{\hat{p}\hat{5}} - v^{\hat{p}\hat{5}}) + 4\epsilon^{(4)}_{\hat{m}\hat{p}\hat{q}\hat{r}} f^{\hat{p}\hat{q}} f^{\hat{r}\hat{5}}, \tag{2.28}$$

$$0 = S^{-1}(\xi\tau_a \delta_Q \eta) = 4D_{\hat{5}} t_a + 4i\epsilon_{abc} J^c_{\hat{m}\hat{n}} t_b (f - v)^{\hat{m}\hat{n}}. \tag{2.29}$$

(2.27) is the only condition including C, and can be used to determine C. (2.28) and (2.29) are drastically simplified if we substitute the solution of $\delta_Q \psi_\mu = 0$;

$$0 = S^{-1}(\xi\gamma_{\hat{m}} \delta_Q \eta) = 2\partial_{\hat{5}} v_{\hat{m}\hat{5}}, \quad 0 = S^{-1}(\xi\tau_a \delta_Q \eta) = 4\partial_{\hat{5}} t_a. \tag{2.30}$$

Namely, $v_{\hat{m}\hat{5}}$ and t_a are x^5 independent. After all, we have obtained the following solution:

$$\begin{aligned}
\xi_\alpha{}^I &= \sqrt{\frac{S}{2}} \begin{pmatrix} \mathbf{1}_2 \\ 0 \end{pmatrix}, \\
e_n^{\hat{m}} &= \text{(indep.)}, \\
e_5^{\hat{5}} &= S \quad \text{(indep.)}, \\
\mathcal{V}_m &= \text{(indep.)}, \\
a_{\hat{m}} &= \text{(indep.)}, \\
a_5 &= \frac{1}{2} S, \\
v_{\hat{p}\hat{q}} &= \epsilon^{(4)}_{\hat{p}\hat{q}\hat{m}\hat{n}} \left(\frac{S}{4} \mathcal{W}_{\hat{m}\hat{n}} - f_{\hat{m}\hat{n}} + t_a J^a_{\hat{m}\hat{n}} \right), \\
v^{\hat{m}\hat{5}} &= \text{(indep.)}, \\
t_a &= \text{(indep.)}, \\
V^a_{\hat{m}} &= \frac{1}{4}\omega^{(4)}_{\hat{m}\hat{p}\hat{q}} J^a_{\hat{p}\hat{q}} + \frac{1}{2} J^a_{\hat{m}\hat{p}} v^{\hat{p}\hat{5}}, \\
V^a_{\hat{5}} &= \frac{1}{2} J^a_{\hat{m}\hat{n}} \left(f_{\hat{m}\hat{n}} - \frac{S}{2} \mathcal{W}_{\hat{m}\hat{n}} \right) + t_a, \\
C &= 2D^{(4)}_{\hat{m}} v^{\hat{m}\hat{5}} + 4t_a J^a_{\hat{m}\hat{n}} f_{\hat{m}\hat{n}} + 32 t_a t_a - \epsilon^{(4)}_{\hat{m}\hat{n}\hat{p}\hat{q}} \left(f^{\hat{m}\hat{n}} - \frac{S}{2} \mathcal{W}^{\hat{m}\hat{n}} \right) \left(f^{\hat{p}\hat{q}} - \frac{S}{2} \mathcal{W}^{\hat{p}\hat{q}} \right).
\end{aligned} \tag{2.31}$$

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"(indep.)" means that the field is an independent field. All the fields are x^5-independent. This is in fact a direct consequence of the algebra. From the commutation relation (2.45) in [46], we obtain

$$\begin{aligned}\delta_Q(\xi)^2 &= R^\mu D_\mu - \delta_M\left(2Sf_{\widehat{\mu}\widehat{\nu}} + \frac{1}{2}\epsilon_{\widehat{\mu}\widehat{\nu}\lambda\rho\sigma}R^\lambda v^{\rho\sigma} - 2SJ^a_{\widehat{\mu}\widehat{\nu}}t_a\right)\\ &\quad + \delta_Z\left(\frac{1}{2}S\right) + \delta_U\left(-3St_a - \frac{S}{2}J^a_{\widehat{m}\widehat{n}}(f^{\widehat{m}\widehat{n}} - v^{\widehat{m}\widehat{n}})\right)\\ &\quad + (\text{terms with } \eta \text{ or } \psi_\mu).\end{aligned} \tag{2.32}$$

In the background (2.31), the right hand side reduces to the x^5 derivative;

$$\delta_Q(\xi)^2 = R^\mu D_\mu - \delta_M(R^\lambda\omega_{\lambda-\widehat{\mu}\widehat{\nu}}) + \delta_Z(R^\mu a_\mu) + \delta_U(R^\mu V^a_\mu) = \partial_5. \tag{2.33}$$

Therefore, a $\delta_Q(\xi)$-invariant background is also invariant under the isometry ∂_5.

3 Q-exact deformations

The solution obtained in the previous section depends on several functions and has large degrees of freedom. However, as we will show in this section, only small part of them can affect the partition function.

Let S_0 be the action of a supersymmetric theory on a supersymmetric background given by the solution (2.31). A small deformation around the background induces the change of the action

$$S_1 = \int d^5x\sqrt{g}\left[-\delta e^{\widehat{\nu}}_\mu T^\mu_{\widehat{\nu}} + \delta V^a_\mu R^\mu_a + (\delta\psi_\mu S^\mu) - \delta a_\mu J^\mu + \delta v^{\mu\nu}M_{\mu\nu} + \delta C\Phi + (\delta\eta\chi) + \delta t_a X_a\right], \tag{3.1}$$

where the set of the operators

$$\begin{aligned}&R^\mu_a(12), \quad S^\mu_{I\alpha}(32), \quad T^{\mu\nu}(10), \quad J^\mu(4),\\ &M^{\mu\nu}(10), \quad \Phi(1), \quad \chi_{I\alpha}(8), \quad X_a(3),\end{aligned} \tag{3.2}$$

forms the supercurrent multiplet with 40+40 degrees of freedom. $S^\mu_{I\alpha}$ and $\chi_{I\alpha}$ are fermionic and the others are bosonic. The numbers in the parenthesis represent the degrees of freedom of the operator.

If the change of the background fields are consistent to the solution (2.31) S_1 is Q-invariant and the deformed action $S_0 + S_1$ gives the supersymmetric theory on the deformed background. We would like to consider the problem whether such a supersymmetric deformation affects the partition function. If the deformation S_1 is Q-exact as well as Q-invariant, it does not change the partition function. A Q-exact deformation that is regarded as a change of the bosonic background fields in general has the form

$$\delta_Q(\xi)\int\sqrt{g}d^5x\left[H_\mu S^\mu + K\chi\right], \tag{3.3}$$

where H_μ and K are vectorial-spinor and spinor coefficient functions. Both H_μ and K are Grassmann-even. Because $\delta_Q(\xi)^2 = \partial_5$ for the action (3.3) to be Q-invariant the functions H_μ and K should be x^5-independent.

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$\delta_Q S^\mu$ and $\delta_Q\chi$ are determined as follows. For an arbitrary deformation that may not preserve the supersymmetry S_1 is invariant under the supersymmetry if we transform both the Weyl multiplet and matter fields. The transformation laws of the bosonic components of the Weyl multiplet are [45, 46]

$$\begin{aligned}
\delta_Q e_\mu^{\widehat{\nu}} &= -2(\xi\gamma^{\widehat{\nu}}\psi_\mu),\\
\delta_Q a_\mu &= -(\xi\psi_\mu),\\
\delta_Q V_\mu^a &= -\frac{1}{4}(\xi\tau_a\gamma_\mu\eta) + (\xi\tau_a\gamma^\lambda R_{\lambda\mu}(Q)) + (\xi\tau_a\gamma^{\rho\sigma}f_{\rho\sigma}\psi_\mu) - (\xi\tau_a\gamma^{\rho\sigma}v_{\rho\sigma}\psi_\mu) + 6(\xi\psi_\mu)t_a,\\
\delta_Q t_a &= -\frac{1}{4}(\xi\tau_a\eta),\\
\delta_Q v_{\widehat{\mu}\widehat{\nu}} &= \frac{1}{2}(\xi\gamma_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}R^{\widehat{\rho}\widehat{\sigma}}(Q)) + \frac{1}{2}(\xi\gamma_{\widehat{\mu}\widehat{\nu}}\eta),\\
\delta_Q C &= -(\xi\widehat{\not{D}}\eta) - 11(\xi t\eta) - \frac{3}{4}(\xi\gamma_{\mu\nu}v^{\mu\nu}\eta) - 4(\xi t\gamma^{\mu\nu}R_{\mu\nu}(Q)),
\end{aligned}\tag{3.4}$$

where $R_{\mu\nu}(Q)$ and $\widehat{D}_\mu\eta$ are defined by

$$\begin{aligned}
R_{\mu\nu}(Q) &= 2D_{[\mu}\psi_{\nu]} + \frac{1}{2}\gamma_{\rho\sigma[\mu}\psi_{\nu]}v^{\rho\sigma} + 2\gamma^\rho\psi_{[\mu}f_{\nu]\rho} - 2\gamma_{[\mu}t\psi_{\nu]},\\
\widehat{D}_\mu\eta &= D_\mu\eta - \delta_Q(\psi_\mu)\eta.
\end{aligned}\tag{3.5}$$

By requiring the Q-invariance of S_1 we can determine the transformation laws of the fields in (3.2). For example, the transformation of χ is

$$\delta_Q\chi = \frac{1}{4}\tau_a\gamma_\mu\xi R_a^\mu + \frac{1}{4}\tau_a\xi X_a - \frac{1}{2}\gamma_{\mu\nu}\xi M^{\mu\nu} + \gamma^\mu\xi D_\mu\Phi + f_{\mu\nu}\gamma^{\mu\nu}\xi\Phi + 16t\xi\Phi. \tag{3.6}$$

Let us consider the second term in the Q-exact action (3.3). It is convenient to expand the spinor function K by the basis in (2.14) as

$$K = k\xi + \frac{4}{S}k^a\xi\tau_a - \frac{2}{S}k^{\widehat{m}}\xi\gamma_{\widehat{m}}. \tag{3.7}$$

The first term in (3.7), $k\xi$, gives the action

$$\delta_Q\int d^5x\sqrt{g}k(\xi\chi) = \int d^5x\sqrt{g}k\partial_5\Phi. \tag{3.8}$$

This is total derivative, and does not give a non-trivial deformation of the theory.

The second term in (3.7) gives

$$\begin{aligned}
&\delta_Q\int d^5x\sqrt{g}\frac{4}{S}k^a(\xi\tau_a\chi)\\
&= \int d^5x\sqrt{g}\left(k^aR_a^{\widehat{5}} + k^aX_a - 2k^aJ^a_{\widehat{m}\widehat{n}}M^{\widehat{m}\widehat{n}} + 4k^aJ^a_{\widehat{m}\widehat{n}}f^{\widehat{m}\widehat{n}}\Phi + 64k^at_a\Phi\right).
\end{aligned}\tag{3.9}$$

Comparing this to (3.1), we find that the addition of (3.9) to the action is equivalent to the background deformation

$$\begin{aligned}
&\delta t_a = k^a, \quad \delta V^a_{\widehat{5}} = k^a, \quad \delta v^{\widehat{m}\widehat{n}} = -2k^aJ^a_{\widehat{m}\widehat{n}}, \quad \delta C = 4k^aJ^a_{\widehat{m}\widehat{n}}f^{\widehat{m}\widehat{n}} + 64k^at_a,\\
&\delta e_\mu^{\widehat{\nu}} = \delta a_\mu = \delta v^{\widehat{m}\widehat{5}} = \delta V^a_{\widehat{m}} = 0.
\end{aligned}\tag{3.10}$$

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These variations are consistent to the solution (2.31). We obtain (3.10) by shifting t_a by

$$t_a \to t_a + k^a, \tag{3.11}$$

and keeping other independent fields intact.

Similarly, the addition of the Q-exact action

$$\begin{aligned} &\delta_Q \int d^5x\sqrt{g}\left(-\frac{2}{S}k^{\widehat{m}}(\xi\gamma_{\widehat{m}}\chi)\right) \\ &= \int d^5x\sqrt{g}\left(-\frac{1}{2}k^{\widehat{m}}J^a_{\widehat{m}\widehat{n}}R^{\widehat{n}}_a + 2k^{\widehat{m}}M^{\widehat{m}5} + 2(D^{(4)}_{\widehat{m}}k^{\widehat{m}})\Phi\right) \end{aligned} \tag{3.12}$$

corresponding to the third term in (3.7) is equivalent to the changes of the background fields

$$\begin{aligned} &\delta v^{\widehat{m}\widehat{5}} = k^{\widehat{m}}, \quad \delta V^a_{\widehat{m}} = \frac{1}{2}J^a_{\widehat{m}\widehat{n}}k^{\widehat{n}}, \quad \delta C = 2D^{(4)}_{\widehat{m}}k^{\widehat{m}}, \\ &\delta e^{\widehat{\nu}}_\mu = \delta a_\mu = \delta v^{\widehat{m}\widehat{n}} = \delta t_a = \delta V^a_{\widehat{5}} = 0. \end{aligned} \tag{3.13}$$

These variations are again consistent to the solution (2.31), and generated by the shift of the independent field $v^{\widehat{m}\widehat{5}}$ by

$$v^{\widehat{m}\widehat{5}} \to v^{\widehat{m}\widehat{5}} + k^{\widehat{m}}. \tag{3.14}$$

Before considering Q-exact terms made from the supersymmetry current S^μ, which are expected to be more complicated, it is convenient to simplify the prescription used above to obtain the Q-exact deformations. A small deformation of the theory is schematically expressed as

$$S_1 = A^B_i J^B_i + A^F_i J^F_i, \tag{3.15}$$

where (A^B_i, A^F_i) is a small variation of the Weyl multiplet around a supersymmetric background and (J^B_i, J^F_i) is the multiplet of currents. The superscripts 'B' and 'F' indicate the bosonic and fermionic statistics, respectively. The index i collectively represents all indices of fields including the coordinates x^μ. The transformation laws of the fermionic components J^F_i of the current multiplet are obtained by requiring the cancellation

$$\delta_Q A^B_i J^B_i - A^F_i \delta_Q J^F_i = 0. \tag{3.16}$$

We only need to consider linear order terms with respect to fermions, and the transformation of bosonic components A^B_i of the Weyl multiplet can be written as

$$\delta_Q A^B_i = A^F_j M_{ji}, \tag{3.17}$$

where M_{ji} are functions of bosonic fields. Then the transformation of J^F_i is

$$\delta_Q J^F_j = M_{ji} J^B_i, \tag{3.18}$$

and the general Q exact term can be written as

$$\delta_Q (f_j J^F_j) = f_j M_{ji} J^B_i, \tag{3.19}$$

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where f_j are Grassmann-even deformation parameters. This can be interpreted as the following deformation of the background.

$$A_i^B = f_j M_{ji}. \tag{3.20}$$

This is nothing but the supersymmetry transformation (3.17) with the fermion fields A_j^F replaced by the parameters f_j. Namely, changes of the background which are realized by Q-exact deformations are obtained from the supersymmetry transformation laws (3.4) by replacing fermions by deformation parameters. Indeed, the deformations (3.10) and (3.13) are respectively obtained from (3.4) by the replacements

$$(\psi_\mu, \eta) \to \left(0, -\frac{4}{S}k^a\tau_a\xi\right), \quad (\psi_\mu, \eta) \to \left(0, -\frac{2}{S}k^{\widehat{m}}\gamma_{\widehat{m}}\xi\right). \tag{3.21}$$

Now let us consider Q-exact terms including $\delta_Q S^\mu$ by using this method. The corresponding background deformation can be obtained from the transformation laws (3.4) by the replacement

$$(\psi_\mu, \eta) \to (H_\mu, 0). \tag{3.22}$$

We expand the function H_μ by the spinor basis as

$$H_\mu = -\frac{1}{2S}h_\mu\xi + \frac{1}{S}h_\mu^a\tau_a\xi - \frac{1}{2S}h_\mu^{\widehat{m}}\gamma_{\widehat{m}}\xi. \tag{3.23}$$

The deformation parameters h_μ, h_μ^a, and $h_\mu^{\widehat{m}}$ are arbitrary functions on the base manifold $\mathcal{B}$.

The variations of the independent fields in the deformation by the parameter h_μ are

$$\delta S = h_5, \quad \delta\mathcal{V}_{\widehat{m}} = \frac{1}{S}h_{\widehat{m}}, \quad \delta a_{\widehat{m}} = \delta v_{\widehat{m}\widehat{5}} = \delta t_a = 0. \tag{3.24}$$

The variations of the dependent fields are obtained from the solution (2.31). By this Q-exact deformation we can freely change the functions S and $\mathcal{V}_{\widehat{m}}$.

The deformation by the parameter h_μ^a is

$$\delta v_{\widehat{m}\widehat{5}} = 4iJ^a_{\widehat{m}\widehat{n}}h^b_{\widehat{n}}t_c\epsilon_{abc}, \quad \delta S = \delta\mathcal{V}_m = \delta a_m = \delta t_a = 0. \tag{3.25}$$

This is not independent of the deformation (3.14). Finally, the deformation by the parameter $h_\mu^{\widehat{m}}$ is

$$\begin{aligned}
&\delta e_\mu^{\widehat{m}} = h_\mu^{\widehat{m}}, \quad \delta e_\mu^{\widehat{5}} = \delta a_\mu = \delta t_a = 0,\\
&\delta v_{\widehat{m}\widehat{5}} = \epsilon^{(4)}_{\widehat{m}\widehat{p}\widehat{q}\widehat{k}}\left(\frac{1}{2}D^{(4)}_{\widehat{p}}h_{\widehat{q}}{}^{\widehat{k}} + \frac{1}{4}\mathcal{W}_{\widehat{m}\widehat{n}}h_5{}^{\widehat{k}} - h_{\widehat{q}}{}^{\widehat{k}}v^{\widehat{p}\widehat{5}}\right) + h_{\widehat{q}}{}^{\widehat{m}}v^{\widehat{q}\widehat{5}} - h_{\widehat{q}}{}^{\widehat{q}}v^{\widehat{m}\widehat{5}}.
\end{aligned} \tag{3.26}$$

The change of $v_{\widehat{m}\widehat{5}}$ can be absorbed by the deformation (3.14), and we are not interested in it. By using the parameter $h_n^{\widehat{m}}$ we can freely change the vielbein $e_m^{\widehat{n}}$ of the base manifold $\mathcal{B}$. The deformation with the parameter $h_5^{\widehat{m}}$ breaks the choice of the gauge (2.8) for the vielbein. To recover $e_5^{\widehat{m}} = 0$, we should perform the compensating local Lorentz transformation

$$\delta_M(\lambda_{\widehat{\mu}\widehat{\nu}})e_5^{\widehat{m}} = -h_5^{\widehat{m}}, \quad \lambda_{\widehat{m}\widehat{5}} = -\frac{1}{S}h_5^{\widehat{m}}, \quad \lambda_{\widehat{m}\widehat{n}} = 0. \tag{3.27}$$

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	fields	dof	$Sp(1)_R$	ours	KO
bosons	gauge field	4	**1**	A_μ	$-igW_\mu$
	scalar	1	**1**	ϕ	gM
	auxiliary fields	3	**3**	D_a	$2gY_a$
fermion	gaugino	8	**2**	λ	$-2ig\Omega$
	prepotential			$\mathcal{F}$	$-\frac{1}{2}\mathcal{N}$

Table 2. Component fields in a vector multiplet. Relation to Kugo-Ohashi's convention is also shown.

This transformation, in turn, changes the vector field $a_{\widehat{m}}$ by

$$\delta_M(\lambda_{\widehat{\mu}\widehat{\nu}})a_{\widehat{m}} = -\frac{1}{2S}h_5^{\widehat{m}}. \tag{3.28}$$

As a result, we can freely change $a_{\widehat{m}}$ by using the combination of the Q-exact deformation with the parameter $h_5^{\widehat{m}}$ and the compensating δ_M transformation.

After all, by using Q-exact deformations and gauge transformations, we can freely change all the independent fields. Of course this does not mean that the partition function does not depend on the background at all. To clarify the background dependence of the partition function, careful analysis of the global structure of the background is needed.

4 Background vector multiplets

In addition to the Weyl multiplet, we can introduce vector multiplets as background fields coupling to global symmetry currents. A vector multiplet consists of the component fields shown in table 2. The transformation laws for those are (eq. (3.2) in [46])

$$\begin{aligned}
\delta_Q(\xi)\lambda &= -\not{F}\xi + 2i\phi\not{f}\xi + i(\not{D}\phi)\xi + iD\xi, \\
\delta_Q(\xi)A_\mu &= -(\xi\gamma_\mu\lambda) - 2i(\xi\psi_\mu)\phi, \\
\delta_Q(\xi)\phi &= i(\xi\lambda), \\
\delta_Q(\xi)D_a &= i(\xi\tau_a\widehat{\not{D}}\lambda) - i(\xi\tau_a[\phi,\lambda]) - \frac{i}{2}(\xi\tau_a\not{v}\lambda) - i(\xi\tau_a t\lambda) + 4i(\xi\lambda)t_a,
\end{aligned} \tag{4.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the gauge field strength and $\widehat{D}_\mu\lambda$ is the supercovariant derivative

$$\widehat{D}_\mu\lambda = D_\mu\lambda - \delta_Q(\psi_\mu)\lambda. \tag{4.2}$$

In the presence of the background vector multiplets we should impose the condition $\delta_Q\lambda = 0$. For simplicity, we consider a U(1) vector multiplet. We decompose the condition into the following two.

$$0 = (\xi\gamma_\mu\delta_Q\lambda) = -F_{\mu\nu}R^\nu + 2i\phi f_{\mu\nu}R^\nu + iSD_\mu\phi, \tag{4.3}$$

$$0 = (\xi\tau_a\delta_Q\lambda) = -\frac{S}{2}(F^{\mu\nu} - 2i\phi f^{\mu\nu})J^a_{\mu\nu} + iSD_a. \tag{4.4}$$

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From (4.3) we obtain

$$D_5\phi = 0, \quad F_{m5} = iD_m(S\phi). \tag{4.5}$$

The solution of (4.5) together with D_a represented in terms of other fields by solving (4.4) is

$$\begin{aligned}
\phi &= (\text{indep.}),\\
A_5 &= iS\phi,\\
A_m &= (\text{indep.}),\\
D_a &= -\frac{i}{2}(F_{\widehat{m}\widehat{n}} - 2i\phi f_{\widehat{m}\widehat{n}})J^a_{\widehat{m}\widehat{n}},
\end{aligned} \tag{4.6}$$

up to gauge transformation.

Next, let us specify the degrees of freedom realized by Q-exact deformations. As explained in the previous section, such deformations can be easily obtained from the transformation laws of the bosonic components in (4.1) by replacing the fermion λ by deformation parameters. The replacement $\lambda \to -S^{-1}f_\mu\gamma^\mu\xi$ gives

$$\delta A_\mu = f_\mu, \quad \delta\phi = -iS^{-1}f_5, \tag{4.7}$$

while $\lambda \to f^a\tau_a\xi$ does not give non-trivial deformation. By using (4.7) we can freely change the independent fields in (4.6), at least locally.

5 Examples

5.1 Conformally flat backgrounds

For a given superconformal field theory on the flat background, it is straightforward to obtain the theory in a conformally flat background by a Weyl transformation that maps the flat space to the conformally flat background. The parameter ξ of the superconformal transformation on the background satisfies the Killing spinor equation

$$D_\mu\xi = \gamma_\mu\kappa \quad \exists\kappa. \tag{5.1}$$

For vector multiplets, the Weyl transformation gives the superconformal transformation laws

$$\begin{aligned}
\delta_{\rm SC}A^i_\mu &= -(\xi\gamma_\mu\lambda^i),\\
\delta_{\rm SC}\phi^i &= i(\xi\lambda^i),\\
\delta_{\rm SC}\lambda^i &= -\not{F}^i\xi + i(\not{D}\phi^i)\xi + iD'^i\xi + 2i\kappa\phi^i,\\
\delta_{\rm SC}D'^i_a &= i(\xi\tau_a\gamma^\mu D_\mu\lambda^i) - i(\xi\tau_a[\phi,\lambda]^i) - i(\kappa\tau_a\lambda^i),
\end{aligned} \tag{5.2}$$

where D' is the auxiliary field, whose definition is different from the previous auxiliary field D. The relation between D and D' will be shown later. We use $i, j, \ldots$ for adjoint indices of the gauge group. The superconformal Lagrangian is

$$e^{-1}\mathcal{L}^{(V)}_{\rm SC} = e^{-1}\mathcal{L}^{(V)}_0|_{\rm conf} + \frac{R}{4}\mathcal{F}, \tag{5.3}$$

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where $\mathcal{L}_0^{(V)}$ is the superconformal Lagrangian on the flat background covariantized with respect to the local symmetries listed in 2.1:

$$\begin{aligned}e^{-1}\mathcal{L}_0^{(V)} =& -\frac{1}{2}\mathcal{F}_i[\lambda,\lambda]^i \\ &+\mathcal{F}_{ij}\left(\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu j}+\frac{1}{2}D_\mu\phi^i D^\mu\phi^j-\frac{1}{2}D'^i_a D'^j_a-\frac{1}{2}\lambda^i \not{D}\lambda^j\right) \\ &+\mathcal{F}_{ijk}\left(\frac{i}{6}[\mathrm{CS}]_5^{ijk}+\frac{1}{4}\lambda^i(i\not{F}^j+D'^j)\lambda^k\right).\end{aligned} \tag{5.4}$$

$\mathcal{L}_0^{(V)}$ depends on the background Weyl multiplet, and $(\cdots)|_{\mathrm{conf}}$ in (5.3) represents the substitution of the conformally flat background. In particular, the $\mathrm{Sp}(1)_R$ gauge field V_μ^a vanishes in (5.3). $[\mathrm{CS}]_5^{ijk}$ is the 5d Chern-Simons term defined by

$$[\mathrm{CS}]_5^{ijk}=\epsilon^{\lambda\mu\nu\rho\sigma}A^i_\lambda\partial_\mu A^j_\nu\partial_\rho A^k_\sigma \tag{5.5}$$

for Abelian gauge fields. For non-Abelian gauge fields A^3dA and A^5 terms should be appropriately supplemented. The prepotential $\mathcal{F}(\phi)$ is a homogeneous cubic polynomial of the scalar fields ϕ^i, and $\mathcal{F}_i$, $\mathcal{F}_{ij}$, and $\mathcal{F}_{ijk}$ are its derivatives:

$$\mathcal{F}_i=\frac{\partial\mathcal{F}}{\partial\phi^i},\quad \mathcal{F}_{ij}=\frac{\partial^2\mathcal{F}}{\partial\phi^i\partial\phi^j},\quad \mathcal{F}_{ijk}=\frac{\partial^3\mathcal{F}}{\partial\phi^i\partial\phi^j\partial\phi^k}. \tag{5.6}$$

If all the vector multiplets are not backgrounds but dynamical the theory is conformal. We will mention the non-conformal case later. The second term in (5.3) is the curvature coupling of the scalar fields.

We would like to reproduce these transformation laws and the Lagrangian by the supergravity. In the 5d $\mathcal{N}=1$ supergravity, the Killing equation (5.1) is realized if

$$V_\mu^a=0, \tag{5.7}$$

$$v_{\mu\nu}+2f_{\mu\nu}=0. \tag{5.8}$$

Indeed, if these are satisfied, the transformation law of gravitino in (2.1) becomes

$$\delta_Q\psi_\mu=D_\mu\xi-\gamma_\mu(\not{f}+t)\xi, \tag{5.9}$$

and the condition $\delta_Q\psi_\mu=0$ gives the Killing equation with

$$\kappa=(\not{f}+t)\xi. \tag{5.10}$$

It is easy to confirm that the transformation laws (5.2) agree with (4.1) if we shift the auxiliary fields by

$$D'^i_a=D^i_a-2\phi^i t_a. \tag{5.11}$$

We can also show that the Lagrangian (5.3) is reproduced from the supergravity Lagrangian. In the 5d $\mathcal{N}=1$ supergravity vector multiplets couple to the Weyl multiplet through the Lagrangian ((2.11) in [47])

$$e^{-1}\mathcal{L}^{(V)}_{\mathrm{SUGRA}}=e^{-1}\mathcal{L}_0^{(V)}+e^{-1}\mathcal{L}_1^{(V)}, \tag{5.12}$$

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where $\mathcal{L}_0^{(V)}$ is the Lagrangian in (5.4), and $\mathcal{L}_1^{(V)}$ is given by

$$\begin{aligned} e^{-1}\mathcal{L}_1^{(V)} = \mathcal{F}P & \\ & - i\mathcal{F}_i F^i_{\mu\nu}(v^{\mu\nu} + 2f^{\mu\nu}) + \frac{1}{4}\mathcal{F}_{ij}\lambda^i(\not{v} + 2\not{f})\lambda^j \\ & + (\text{terms with } \psi_{\mu I} \text{ or } \eta_I). \end{aligned} \tag{5.13}$$

P in the first line is defined by

$$P = C - 20t_a t_a - 4f_{\mu\nu}v^{\mu\nu} - 6f_{\mu\nu}f^{\mu\nu}. \tag{5.14}$$

In the background (2.31) this is rewritten as

$$\begin{aligned} P = & \, 3\,(J^a_{\widehat{m}\widehat{n}} f_{\widehat{m}\widehat{n}} + 2t^a)^2 - \frac{S^2}{4}\epsilon_{\widehat{m}\widehat{n}\widehat{p}\widehat{q}}\mathcal{W}_{\widehat{m}\widehat{n}}\mathcal{W}_{\widehat{p}\widehat{q}} \\ & + 2D^{(4)}_{\widehat{m}} v^{\widehat{m}\widehat{5}} - \frac{4}{S}(\partial_{\widehat{m}} S)v^{\widehat{m}\widehat{5}} - \frac{3}{S^2}(\partial_{\widehat{m}} S)^2. \end{aligned} \tag{5.15}$$

The two terms in the second line in (5.13) contain $v^{\mu\nu} + 2f^{\mu\nu}$, and vanish if (5.8) holds. What we need to show is that the first line in (5.13) is the same as the curvature coupling in (5.3). This is easily shown by using the condition $\delta_Q \eta = 0$. If (5.8) holds, we can rewrite $\delta_Q \eta$ in (2.1) as

$$\delta_Q \eta = 4[\not{D}(\not{f} + t)]\xi + 4\gamma_\mu(\not{f} + t)\gamma^\mu(\not{f} + t)\xi + (C - 20t_a t_a + 2f_{\mu\nu}f^{\mu\nu})\xi. \tag{5.16}$$

Using this and $D_\mu \xi = \gamma_\mu \kappa$ with κ in (5.10), we obtain

$$P\xi = (C - 20t_a t_a + 2f_{\mu\nu}f^{\mu\nu})\xi = -4D_\mu D^\mu \xi = \frac{R}{4}\xi. \tag{5.17}$$

The third equality is shown by using $D_\mu \xi = \gamma_\mu \kappa$ as follows:

$$\begin{aligned} \frac{1}{4}R\xi &= -\frac{1}{8}\gamma^{\mu\nu}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\xi \\ &= -\gamma^{\mu\nu}D_\mu D_\nu \xi \\ &= -\not{D}\not{D}\xi + D_\mu D^\mu \xi \\ &= -5\not{D}\kappa + D_\mu D^\mu \xi \\ &= -5D_\mu D^\mu \xi + D_\mu D^\mu \xi \\ &= -4D_\mu D^\mu \xi. \end{aligned} \tag{5.18}$$

We used the flatness of the $\mathrm{Sp}(1)_R$ connection between the first and the second lines. (5.17) shows that the first line in (5.13) is precisely the same as the curvature coupling in (5.3).

Next, let us consider hypermultiplets. For simplicity we consider a neutral on-shell hypermultiplet that is not coupled by vector multiplets. A hypermultiplet consists of scalar fields $\mathcal{A}^I_A$ and a symplectic Majorana fermion field ζ_A.[3] (See table 3.) $A = 1, 2$ is an

[3]We use the convention in [46] for hypermultiplets.

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	fields	$\mathrm{Sp}(1)_R$	$\mathrm{Sp}(1)_F$	
bosons	scalar fields	**2**	**2**	$\mathcal{A}^I_A$
fermion	symplectic Majorana	**1**	**2**	ζ_A

Table 3. Component fields in a hypermultiplet.

$\mathrm{Sp}(1)_F$ flavor index. The local supersymmetry transformation laws for the hypermultiplet are ((4.4) in [46])

$$\begin{aligned} \delta_Q \mathcal{A}^I_A &= 2(\xi^I \zeta_A), \\ \delta_Q \zeta_A &= -(\not{D}\mathcal{A}^I_A)\xi_I + \mathcal{A}^I_A(-3t\xi - \not{f}\xi + \not{v}\xi)_I, \end{aligned} \tag{5.19}$$

and the Lagrangian is ((3.1) in [47])

$$e^{-1}\mathcal{L}^{(H)}_{\mathrm{SUGRA}} = e^{-1}\mathcal{L}^{(H)}_0 + e^{-1}\mathcal{L}^{(H)}_1, \tag{5.20}$$

where $\mathcal{L}^{(H)}_0$ and $\mathcal{L}^{(H)}_1$ are given by

$$\begin{aligned} e^{-1}\mathcal{L}^{(H)}_0 &= D_\mu \mathcal{A}^A_I D^\mu \mathcal{A}^I_A - 2(\zeta^A \not{D}\zeta_A), \\ e^{-1}\mathcal{L}^{(H)}_1 &= \left(\frac{1}{4}R - \frac{1}{4}P - \frac{1}{4}(v_{\mu\nu} + 2f_{\mu\nu})^2\right)\mathcal{A}^A_I \mathcal{A}^I_A \\ &\quad - \frac{1}{2}(\zeta^A \gamma_{\mu\nu}\zeta_A)(v_{\mu\nu} + 2f_{\mu\nu}) \\ &\quad + (\text{terms with } \psi_{\mu I} \text{ or } \eta_I). \end{aligned} \tag{5.21}$$

By substituting (5.7) and (5.8) into the transformation laws (5.19) we obtain the superconformal transformation laws

$$\begin{aligned} \delta_{\mathrm{SC}} \mathcal{A}^I_A &= 2(\xi^I \zeta_A), \\ \delta_{\mathrm{SC}} \zeta_A &= -(\not{D}\mathcal{A}^I_A)\xi_I - 3\mathcal{A}^I_A \kappa_I, \end{aligned} \tag{5.22}$$

which are obtained from those in the flat background by the Weyl transformation. For the Lagrangian, the Weyl transformation gives

$$e^{-1}\mathcal{L}^{(H)}_{\mathrm{SC}} = e^{-1}\mathcal{L}^{(H)}_0|_{\mathrm{conf}} + \frac{3R}{16}\mathcal{A}^A_I \mathcal{A}^I_A, \tag{5.23}$$

and the curvature coupling of the scalar fields $\mathcal{A}^I_A$ is reproduced by substituting (5.7) and (5.8) into $\mathcal{L}^{(H)}_1$ in the same way as the vector multiplets.

Notice that the number of the solutions to $\delta_Q \psi_\mu = 0$ is at most 8, and the formulation with the Poincaré supergravity cannot reproduce all the 16 supersymmetries in the 5d superconformal algebra. This is because the relation (5.10) partially breaks the supersymmetry in the superconformal theory. This can be interpreted as the supersymmetry breaking by a mass deformation. Mass deformations in the superconformal theory can be realized by coupling global symmetry currents to the central charge vector multiplet [46]:

a background vector multiplet with a constant scalar component. The components of the central charge vector multiplet are

$$(\phi, A_\mu, \lambda, D_a) = (1, 2ia_\mu, 0, 0). \tag{5.24}$$

If we substitute this into $\delta_{\rm SC}\lambda = 0$, we obtain

$$0 = \delta_{\rm SC}\lambda = 2i[\kappa - (\not{X} + t)\xi], \tag{5.25}$$

and this is nothing but the relation (5.10). Even if we consider a conformal theory, the Weyl multiplet of the Poincare supergravity contains the central charge vector multiplet as a submultiplet, and it breaks a part of the superconformal symmetry.

It is shown in [55] that we can construct a conformal supergravity by separating the central charge vector multiplet from the Weyl multiplet. In the context of the conformal supergravity κ can be regarded as the parameter of the S-transformation. The Poincare supergravity is reproduced from the conformal supergravity by fixing the S and K symmetries. (See appendix D in [55].) The S symmetry is gauge fixed by setting the fermion component of the central charge vector multiplet to be 0, and (5.25) defines the compensating S-transformation necessary to keep the S-gauge fixing condition invariant under the Q-transformation in the Poincare supergravity.

5.2 $\boldsymbol{S}^5$

The supersymmetric theories on the round $\boldsymbol{S}^5$ and the corresponding supergravity background are given in [14]. Let us confirm that this is a special case of the solution (2.31).

The $\boldsymbol{S}^5$ metric represented as the Hopf fibration over $\mathbb{C}P_2$ is

$$ds^2 = ds^2_{\mathbb{C}P_2} + e^{\widehat{5}}e^{\widehat{5}}, \quad ds^2_{\mathbb{C}P_2} = e^{\widehat{m}}e^{\widehat{m}}, \quad e^{\widehat{5}} = r(dx^5 + \mathcal{V}), \tag{5.26}$$

where r is the radius of $\boldsymbol{S}^5$ and $\mathcal{V}$ is a one-form on $\mathbb{C}P_2$. We take a local frame such that J^3 is the complex structure of the $\mathbb{C}P_2$, and then the following relations hold.

$$S = r, \quad \mathcal{W} = \frac{2i}{r^2}J^3. \tag{5.27}$$

Due to the Kählerity, the holonomy of $\mathbb{C}P_2$ is $\mathrm{U}(2) = \mathrm{Sp}(1)_r \times \mathrm{U}(1)_l$ where $\mathrm{U}(1)_l \subset \mathrm{Sp}(1)_l$ is the stabilizer subgroup of the complex structure J^3. The spin connection of $\mathbb{C}P_2$ commutes with J^3, and takes the form

$$\omega^{\mathbb{C}P_2}_{\widehat{m}\widehat{n}} = \frac{3i}{2}\mathcal{V}J^3_{\widehat{m}\widehat{n}} + (\mathrm{Sp}(1)_r \text{ part}). \tag{5.28}$$

Let us assume the invariance of the matter Lagrangians $\mathcal{L}^{(V)}_{\rm SUGRA}$ and $\mathcal{L}^{(H)}_{\rm SUGRA}$ under the SO(6) rotational symmetry of the $\boldsymbol{S}^5$. The second line of $\mathcal{L}^{(V)}_1$ in (5.13) and the second line of $\mathcal{L}^{(H)}_1$ in (5.21) depend on the tensor fields $f_{\mu\nu}$ and $v_{\mu\nu}$ through the combination

$$v'_{\mu\nu} = v_{\mu\nu} + 2f_{\mu\nu}. \tag{5.29}$$

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The SO(6) invariance requires $v'^{\mu\nu} = 0$, and the independent fields should satisfy

$$v_{\widehat{m}\widehat{5}} = 0, \quad f_{\widehat{m}\widehat{n}} J^a_{\widehat{m}\widehat{n}} + 2t^a = -\frac{i}{r}\delta^{a3}. \tag{5.30}$$

The components of the $Sp(1)_R$ gauge field are

$$V^a_{\widehat{m}} = -\frac{3i}{2}\mathcal{V}_{\widehat{m}}\delta^a_3, \quad V^a_{\widehat{5}} = \frac{3i}{2r}\delta^a_3. \tag{5.31}$$

This is a flat connection and can be gauged away. Then this solution becomes a special case of the conformally flat background we considered in 5.1. Although (5.30) do not completely fix the background fields the ambiguity does not affect the Lagrangians $\mathcal{L}^{(V)}_{\rm SUGRA}$ and $\mathcal{L}^{(H)}_{\rm SUGRA}$, and they are given by (5.3) and (5.23) with $R = 20/r^2$.

For a mass deformed theory the Lagrangian depends on the tensor field $f_{\mu\nu}$ through the central charge vector multiplet (5.24). Then the SO(6) invariance requires $f_{\mu\nu} = 0$, and (5.30) is replaced by the stronger conditions

$$v_{\widehat{m}\widehat{5}} = 0, \quad f_{\widehat{m}\widehat{n}} = 0, \quad t^a = -\frac{i}{2r}\delta^{a3}. \tag{5.32}$$

This agree with the background fields given in [14].

Although a superconformal theory on the round $\boldsymbol{S}^5$ has 16 supersymmetries, as we mentioned in 5.1, the supergravity formulation reproduces only a part of them. For the background specified by (5.32) $\delta_Q\eta = 0$ is automatically holds and $\delta_Q\psi_\mu = 0$ gives

$$D_\mu\xi = -\frac{i}{2r}\tau_3\gamma_\mu\xi. \tag{5.33}$$

This has eight solutions belonging to the real representation $(\mathbf{4},\mathbf{2})+(\overline{\mathbf{4}},\mathbf{2})$ of $SO(6)\times Sp(1)_R$.

If we choose another background satisfying (5.30) we obtain a different Killing spinor equation. Although different backgrounds give the same superconformal Lagrangians $\mathcal{L}^{(V)}_{\rm SC}$ and $\mathcal{L}^{(H)}_{\rm SC}$, the number of supersymmetries which are realized by the supergravity in general depends on the choice of the background fields.

5.3 $\boldsymbol{S}^4 \times \mathbb{R}$

A supersymmetric theory on $\boldsymbol{S}^4\times\mathbb{R}$ can be easily obtained by using Weyl rescaling from the theory on the flat background, and is used in [24] for the computation of the superconformal index. Although we can easily construct a supersymmetric background with the geometry $\boldsymbol{S}^4 \times \mathbb{R}$ by using the solution (2.31) it gives a theory different from the Weyl-rescaled one.

Let us identify $\mathbb{R}$ with the fifth direction. S, $e^{\widehat{n}}_m$, and $\mathcal{V}_m$ are given by

$$S = (\text{positive constant}), \quad e^{\widehat{n}}_m = (\text{vielbein of round } \boldsymbol{S}^4), \quad \mathcal{V}_{\widehat{m}} = 0. \tag{5.34}$$

We assume the SO(5) rotational invariance of the Lagrangians of vector and hypermultiplets. As in the case of $\boldsymbol{S}^5$, this requires $v'_{\mu\nu} \equiv v_{\mu\nu} + 2f_{\mu\nu} = 0$ for a conformal theory and $v_{\mu\nu} = f_{\mu\nu} = 0$ for a mass deformed theory. For independent fields these are rewritten as

$$v^{\widehat{m}\widehat{5}} = f_{\widehat{m}\widehat{n}} J^a_{\widehat{m}\widehat{n}} + 2t^a = 0, \tag{5.35}$$

for the conformal case and

$$v^{\widehat{m}\widehat{5}} = f_{\widehat{m}\widehat{n}} = t^a = 0, \tag{5.36}$$

for the mass deformed case. The latter background is given in [49]. In both cases

$$P = 0 \tag{5.37}$$

and the $\mathrm{Sp}(1)_R$ connection is the instanton configuration related to the spin connection on $\boldsymbol{S}^4$ by

$$V^a = \frac{1}{4}\omega^{(\boldsymbol{S}^4)}_{\widehat{p}\widehat{q}} J^a_{\widehat{p}\widehat{q}}. \tag{5.38}$$

(5.37) and (5.38) are different from what are expected in a Weyl-rescaled theory: $P = R/4 = 3/r^2$ and flat V^a_μ. Actually it is impossible to realize a flat $\mathrm{Sp}(1)_R$ connection in the solution (2.31) because $\boldsymbol{S}^4$ does not admit an almost complex structure. It is necessary to turn on a non-trivial $\mathrm{Sp}(1)_R$ flux for the existence of $J^a_{\widehat{m}\widehat{n}}$.

This result does not change even if we take a different x^5 direction. Because an arbitrary rotation of $\boldsymbol{S}^4$ has fixed points and R^μ is nowhere vanishing, we cannot take x^5 within $\boldsymbol{S}^4$ and R^μ necessarily has the component along $\mathbb{R}$. Then the topology of the base manifold $\mathcal{B}$ is $\boldsymbol{S}^4$, and the existence of $J^a_{\widehat{m}\widehat{n}}$ requires non-trivial $\mathrm{Sp}(1)_R$ flux. Therefore, we cannot realize the Weyl-rescaled theory on $\boldsymbol{S}^4 \times \mathbb{R}$ as a special case of the background (2.31).

The reason for this impossibility may be the symplectic Majorana condition imposed on ξ. We have imposed this condition only for simplicity of the analysis, and it may be possible to realize $\boldsymbol{S}^4 \times \mathbb{R}$ background without $\mathrm{Sp}(1)_R$ flux by relaxing this condition. In the 3d case, it is shown in [58] that $\boldsymbol{S}^2 \times \boldsymbol{S}^1$ backgrounds with and without $\mathrm{U}(1)_R$ flux can be both realized in the framework of the 3d new minimal supergravity. It would be interesting to study whether this is the case in 5d by considering general ξ.

5.4 $\boldsymbol{S}^3 \times \Sigma$

The last example we consider is $\boldsymbol{S}^3 \times \Sigma$, the direct product of three-sphere $\boldsymbol{S}^3$ with radius r and a Riemann surface Σ. A supersymmetric theory on this background is constructed in [27] for $\Sigma = \mathbb{R}^2$ and in [28] for general Σ. It can be reproduced by the solution (2.31) as is shown below.

We treat $\boldsymbol{S}^3$ as the Hopf fibration over $\boldsymbol{S}^2$, and identify the Hopf fiber direction with x^5. The metric of $\boldsymbol{S}^3 \times \Sigma$ is

$$ds^2 = ds^2_\Sigma + ds^2_{\boldsymbol{S}^2} + e^{\widehat{5}}e^{\widehat{5}}, \quad ds^2_\Sigma = e^{\widehat{1}}e^{\widehat{1}} + e^{\widehat{2}}e^{\widehat{2}}, \quad ds^2_{\boldsymbol{S}^2} = e^{\widehat{3}}e^{\widehat{3}} + e^{\widehat{4}}e^{\widehat{4}}, \quad e^{\widehat{5}} = r(dx^5 + \mathcal{V}), \tag{5.39}$$

where $\mathcal{V}$ is a one-form on $\boldsymbol{S}^2$. The following equations hold.

$$S = r, \quad \omega^{\boldsymbol{S}^2}_{\widehat{3}\widehat{4}} = 2\mathcal{V}, \quad \mathcal{W} = \frac{2}{r^2} e^{\widehat{3}} \wedge e^{\widehat{4}}. \tag{5.40}$$

We can take a local frame such that J^3 is the complex structure of $\boldsymbol{S}^2 \times \Sigma$, which is the summation of the complex structures of $\boldsymbol{S}^2$ and Σ.

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Let us assume that the Lagrangians $\mathcal{L}^{(V)}_{\rm SUGRA}$ and $\mathcal{L}^{(H)}_{\rm SUGRA}$ are invariant under the SO(4) isometry of $\boldsymbol{S}^3$. As in previous subsections, all components of $v'_{\mu\nu}$ should vanish except for $v'_{\widehat{1}\widehat{2}}$ for the SO(4) invariance. This requires that the independent fields satisfy

$$v^{\widehat{m}\widehat{5}} = f_{\widehat{m}\widehat{n}} J^a_{\widehat{m}\widehat{n}} + 2t^a = 0, \tag{5.41}$$

and then the non-vanishing component of $v'_{\mu\nu}$ is

$$v'_{\widehat{1}\widehat{2}} = \frac{1}{r}. \tag{5.42}$$

The $\mathrm{Sp}(1)_R$ connection is

$$V^a_{\widehat{m}=\widehat{1},\widehat{2}} = -\frac{i}{2}\delta^{a3}\omega^{(\Sigma)}_{\widehat{m}\widehat{1}\widehat{2}}, \quad V^a_{\widehat{m}=\widehat{3},\widehat{4}} = i\delta^{a3}\mathcal{V}_{\widehat{m}}, \quad V^a_{\widehat{5}} = -\frac{i}{r}\delta^{a3}. \tag{5.43}$$

The $\boldsymbol{S}^3$ part of the connection (5.43)

$$V^{(\boldsymbol{S}^3)a} = V^a_{\widehat{3}} e^{\widehat{3}} + V^a_{\widehat{4}} e^{\widehat{4}} + V^a_{\widehat{5}} e^{\widehat{5}} = -i\delta^a_3 dx^5 \tag{5.44}$$

is flat, and can be gauged away. This guarantees the SO(4) invariance of $\mathcal{L}^{(V)}_0$ and $\mathcal{L}^{(H)}_0$. The $\mathrm{Sp}(1)_R$ connection on Σ is topologically twisted in such a way that a covariantly constant spinor on Σ exists.

If the conditions in (5.41) are satisfied, $\mathcal{L}^{(V)}_1$ and $\mathcal{L}^{(H)}_1$ are given by

$$\begin{aligned} e^{-1}\mathcal{L}^{(V)}_1 &= -\frac{2i}{r}\mathcal{F}_i F^i_{\widehat{1}\widehat{2}} + \frac{1}{4r}\mathcal{F}_{ij}(\lambda^i\gamma_{\widehat{1}\widehat{2}}\lambda^j), \\ e^{-1}\mathcal{L}^{(H)}_1 &= \frac{1}{r^2}\mathcal{A}^A_I\mathcal{A}^I_A - \frac{1}{r}(\zeta^A\gamma_{\widehat{1}\widehat{2}}\zeta_A). \end{aligned} \tag{5.45}$$

Although (5.41) does not completely determine the background fields, the ambiguity does not affect the Lagrangians in the absence of mass deformations with the central charge vector multiplet. The hypermultiplet Lagrangian $\mathcal{L}^{(H)}_{\rm SUGRA}$ for this background agrees with the Lagrangian in [28] up to field redefinition.

In the mass-deformed case for the SO(4) invariance only non-vanishing component of $f_{\mu\nu}$ should be $f_{\widehat{1}\widehat{2}}$ which is related to t^3 by

$$t^3 = if_{\widehat{1}\widehat{2}}. \tag{5.46}$$

If we take the prepotential $\mathcal{F} = (1/2g^2_{\rm YM})\phi^0\mathrm{tr}(\phi)^2$ with $\phi^0 = 1$ being the scalar component of the central charge vector multiplet, $\mathcal{L}^{(V)}_0$ and $\mathcal{L}^{(V)}_1$ are given by

$$\begin{aligned} e^{-1}\mathcal{L}^{(V)}_0 &= \frac{1}{g^2_{\rm YM}}\mathrm{tr}\Bigg[\frac{1}{4}F^2_{\widehat{\mu}\widehat{\nu}} + \frac{1}{2}(D_{\widehat{\mu}}\phi)^2 - \frac{1}{2}D'^2_a - \frac{1}{2}(\lambda \not{D}\lambda) + \frac{1}{2}(\lambda[\phi,\lambda]) \\ &\qquad + f_{\widehat{1}\widehat{2}}\left(2i\phi F_{\widehat{1}\widehat{2}} + 2i\phi D'_3 - \frac{1}{2}(\lambda\gamma_{\widehat{1}\widehat{2}}\lambda) - \frac{i}{2}(\lambda\tau_3\lambda) - [\mathrm{CS}]_3\right)\Bigg], \\ e^{-1}\mathcal{L}^{(V)}_1 &= \frac{1}{g^2_{\rm YM}}\mathrm{tr}\left[-\frac{2i}{r}\phi F_{\widehat{1}\widehat{2}} + \frac{1}{4r}(\lambda\gamma_{\widehat{1}\widehat{2}}\lambda) + \frac{2}{r}f_{\widehat{1}\widehat{2}}\phi^2\right], \end{aligned} \tag{5.47}$$

where $[\mathrm{CS}]_3$ is the Chern-Simons term on $\boldsymbol{S}^3$

$$[\mathrm{CS}]_3 = \epsilon^{\widehat{1}\widehat{2}\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right). \tag{5.48}$$

(5.47) gives a family of the supersymmetric Yang-Mills Lagrangian parameterized by $f_{\widehat{1}\widehat{2}}$, which is a function on Σ. For the gauge invariance of the Chern-Simons term, the $\mathrm{U}(1)_Z$ flux on Σ should be quantized as

$$\frac{1}{g_{\mathrm{YM}}^2}\int_\Sigma f \in \frac{i}{4\pi}\mathbb{Z}. \tag{5.49}$$

The supersymmetric Yang-Mills Lagrangian in [28] is obtained up to a field redefinition by setting

$$f_{\widehat{1}\widehat{2}} = -it^3 = \frac{1}{2r}. \tag{5.50}$$

6 Discussion

We constructed supersymmetric backgrounds of a 5d $\mathcal{N}=1$ supergravity. We solved the supersymmetry conditions $\delta_Q\psi_\mu = \delta_Q\eta = 0$, and obtained the solution that depends on the independent fields

$$S(x^m), \quad \mathcal{V}_m(x^m), \quad e_m^{\widehat{n}}(x^m), \quad a_m(x^m), \quad v_{\widehat{m}\widehat{5}}(x^m), \tag{6.1}$$

on which no local constraints are imposed. A supersymmetric background is specified by choosing these functions. We also showed that the independent fields in the solution can be freely changed by combining Q-exact deformations and gauge transformations. This means that the partition function does not affected by the local degrees of freedom.

We should emphasize that we did not take care about global issues. In order to determine the parameter dependence of the partition function, we need to investigate global obstructions carefully. For example, for a compact background manifold, we cannot freely change the fifth component of a gauge field by gauge transformations and it may affect the partition function. Similarly, if the manifold has non-trivial two-cycles we have the restriction that a flux through the cycles should be appropriately quantized. This prohibit continuous deformations of background gauge fields, and may cause background dependence of the partition function. Detailed analysis of these restrictions is necessary to understand parameter dependence of the partition function. We hope we could return to this problem in near future.

Important feature of the solution is the existence of the isometry. This suggests a close relation to four-dimensional supersymmetric backgrounds. It would be interesting to study supersymmetric configurations of 4d $\mathcal{N}=2$ off-shell supergravity [59, 60] and their relation to the solution obtained in this paper.

In section 5 we reproduced some known examples as special cases of the general solution. We also found that our solution does not include all the known supersymmetric backgrounds. A possible reason for this is that we assumed for simplicity that the supersymmetry parameter ξ satisfies the symplectic Majorana condition. Another possibility

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is that the choice of the supergravity is not suitable to realize some of supersymmetric backgrounds.

Our analysis was based on a Poincaré supergravity. As is mentioned in 5.1 we cannot reproduce all supersymmetries of a superconformal theory in the framework of Poincaré supergravity. To realize a superconformal theory it would be more suitable to use a conformal supergravity to describe curved backgrounds. As is shown in [55], the Weyl multiplet shown in table 1 is obtained by fixing a part of the local superconformal symmetry by using a vector multiplet as a compensator. It is also possible to write down the gauge fixing condition by using a hypermultiplet [56] or a linear multiplet [61, 62] instead of a vector multiplet. It may be possible to obtain a more general class of solutions by considering a system consisting of a superconformal Weyl multiplet and different kinds of matter multiplets without gauge fixing conditions imposed.

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A Notations and conventions

We use Greek letters $\mu, \nu, \ldots = 1, \ldots, 5$ for 5d world indices, and hatted Greek letters $\widehat{\mu}, \widehat{\nu}, \ldots = \widehat{1}, \ldots, \widehat{5}$ for orthonormal indices. Roman letters $m, n, \ldots$ and $\widehat{m}, \widehat{n}, \ldots$ are vector indices running over $1, \ldots, 4$ or $\widehat{1}, \ldots, \widehat{4}$.

The 5d anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma\tau}$ is defined by

$$\gamma^{\mu\nu\rho\sigma\tau} = \epsilon^{\mu\nu\rho\sigma\tau} \mathbf{1}_4. \tag{A.1}$$

We use $\alpha, \beta, \ldots = 1, 2, 3, 4$ for $\mathrm{Sp}(2)_L$ spinor indices and $I, J, \ldots = 1, 2$ for $\mathrm{Sp}(1)_R$ doublet indices. They are raised and lowered by $\mathrm{Sp}(2)_L$ and $\mathrm{Sp}(1)_R$ invariant anti-symmetric tensors $\epsilon_{IJ} = \epsilon^{IJ}$ and $C_{\alpha\beta} = C^{\alpha\beta}$ satisfying

$$\epsilon^{IK} \epsilon_{JK} = \delta^I_J, \quad C^{\alpha\gamma} C_{\beta\gamma} = \delta^\alpha_\beta. \tag{A.2}$$

We use NW-SE convention for implicit contraction of these indices. For example, $(\eta\chi) \equiv \eta^{\alpha I} \chi_{\alpha I} \equiv C^{\alpha\beta} \epsilon^{IJ} \eta_{\beta J} \chi_{\alpha I}$.

For a rank n anti-symmetric tensor $A_{\mu_1 \cdots \mu_n}$ we define

$$\not{A} = \frac{1}{n!} A_{\mu_1 \cdots \mu_n} \gamma^{\mu_1 \cdots \mu_n}. \tag{A.3}$$

For $\mathrm{Sp}(1)_R$ triplet fields we use the matrix notation

$$t_I{}^J \equiv t_a (\tau_a)_I{}^J \tag{A.4}$$

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where τ_a $(a = 1, 2, 3)$ are the Pauli matrices. As an example, we present $\delta_Q \eta$ in (2.1) with all indices explicit;

$$\begin{aligned}\delta_Q \eta_{I\alpha} &= -2(\gamma_\nu)_\alpha{}^\beta \xi_{I\beta} D_\mu v^{\mu\nu} + \xi_{I\alpha} C + 4(D_\mu t^a)(\gamma^\mu)_\alpha{}^\beta (\tau_a)_I{}^J \xi_{J\beta} \\ &\quad + 8\left(\frac{1}{2} f_{\mu\nu}(\gamma^{\mu\nu})_\alpha{}^\beta - \frac{1}{2} v_{\mu\nu}(\gamma^{\mu\nu})_\alpha{}^\beta\right) t^a (\tau_a)_I{}^J \xi_{J\beta} + (\gamma^{\mu\nu\rho\sigma})_\alpha{}^\beta \xi_{I\beta} f_{\mu\nu} f_{\rho\sigma}. \end{aligned} \tag{A.5}$$

We use a convention in which a symplectic Majorana spinor $\chi_\alpha{}^I$ is expressed in the form

$$\chi = (\chi_\alpha{}^I) = \begin{pmatrix} U \\ D \end{pmatrix}, \quad U = U_0 \mathbf{1}_2 + i U_a \tau_a, \quad D = D_0 \mathbf{1}_2 + i D_a \tau_a, \tag{A.6}$$

with real U_i and D_i $(i = 0, 1, 2, 3)$, and the scalar product of two symplectic Majorana spinors are given by

$$(\chi^{(1)} \chi^{(2)}) = 2 U_i^{(1)} U_i^{(2)} + 2 D_i^{(1)} D_i^{(2)}. \tag{A.7}$$

Therefore, $(\chi\chi) > 0$ for a non-vanishing Grassmann-even symplectic Majorana spinor χ. The following formulas for Grassmann-even spinors η and χ are useful.

$$(\eta\chi) = (\chi\eta), \quad (\eta\gamma_\mu\chi) = (\chi\gamma_\mu\eta), \quad (\eta\tau_a\chi) = -(\chi\tau_a\eta). \tag{A.8}$$

For Grassmann-odd spinors, the signs in (A.8) are flipped.

We do not rely on a particular choice of $\gamma^{\widehat{m}}$, C, and ϵ except in 5.4, where we use the following matrices

$$\gamma^{\widehat{1},\widehat{2},\widehat{3}} = \begin{pmatrix} & -i\tau_{1,2,3} \\ i\tau_{1,2,3} & \end{pmatrix}, \quad \gamma^{\widehat{4}} = \begin{pmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix}, \quad \gamma^{\widehat{5}} = \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_2 \end{pmatrix}, \tag{A.9}$$

$$\epsilon_{12} = \epsilon^{12} = +1, \quad C_{\alpha\beta} = C^{\alpha\beta} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{A.10}$$

With this choice of the matrices, $\epsilon_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}\widehat{\tau}}$ and $J^a_{\widehat{m}\widehat{n}}$ have the components

$$\begin{aligned} &\epsilon_{\widehat{1}\widehat{2}\widehat{3}\widehat{4}\widehat{5}} = +1, \\ &J^a_{\widehat{b}\widehat{c}} = -i\epsilon_{abc}, \quad J^a_{\widehat{b}\widehat{4}} = i\delta^a_b \quad (a, b, c = 1, 2, 3). \end{aligned} \tag{A.11}$$

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