The Hesse potential, the c-map and black hole solutions

T. Mohaupt[a] **and O. Vaughan**[a,b]

[a]*Department of Mathematical Sciences, University of Liverpool,*
Peach Street, Liverpool L69 7ZL, U.K.

[b]*Department of Mathematics and Center for Mathematical Physics, University of Hamburg,*
Bundesstraße 55, D-20146 Hamburg, Germany

E-mail: Thomas.Mohaupt@liv.ac.uk, Owen.Vaughan@liv.ac.uk

ABSTRACT: We present a new formulation of the local c-map, which makes use of a symplectically covariant real formulation of special Kähler geometry. We obtain an explicit and simple expression for the resulting quaternionic, or, in the case of reduction over time, para-quaternionic Kähler metric in terms of the Hesse potential, which is similar to the expressions for the metrics obtained from the rigid r- and c-map, and from the local r-map.

As an application we use the temporal version of the c-map to derive the black hole attractor equations from geometric properties of the scalar manifold, without imposing supersymmetry or spherical symmetry. We observe that for general (non-symmetric) c-map spaces static BPS solutions are related to a canonical family of totally isotropic, totally geodesic submanifolds. Static non-BPS solutions can be obtained by applying a field rotation matrix which is subject to a non-trivial compatibility condition. We show that for a class of prepotentials, which includes the very special ('cubic') prepotentials as a subclass, axion-free solutions always admit a non-trivial field rotation matrix.

Contents

1 Introduction

The special Kähler geometry of $\mathcal{N} = 2$ vector multiplets [1] plays a central role in the study of the non-perturbative properties of gauge theories [2, 3], string compactifications [4–6], and of black holes, in particular the attractor mechanism [7], black hole entropy [8–11] and the OSV conjecture [12–14]. Its distinguished feature is the existence of a single holomorphic function, the prepotential, which encodes all vector multiplet couplings. The power of holomorphicity is a key property, which sets $\mathcal{N} = 2$ theories apart from $\mathcal{N} = 1$ theories where the Kähler potential is not related to an underlying holomorphic function. While at first glance our knowledge of special Kähler geometry appears to be comprehensive, there are still aspects which deserve further study.

1.1 Projective special Kähler geometry in real coordinates

It is known that effective supergravity actions are subject to non-holomorphic corrections [15], which enter into the relation between the supergravity effective action and string amplitudes. This has consequences for black hole entropy and the OSV conjecture [13, 14, 16–18]. In this context it became clear that it is sometimes preferable to formulate special Kähler geometry in terms of special real instead of special holomorphic coordinates [14, 19]. This real formulation has been used to develop a manifestly duality covariant approach to the OSV conjecture [14, 20, 21].

While the real formulation of the affine special Kähler geometry of rigid vector multiplets is straightforward, the real formulation of the projective special Kähler geometry of local vector multiplets leaves room for improvements. For affine special Kähler manifolds N the special real coordinates are Darboux coordinates, and the special Kähler metric is Hessian [19]. The Hesse potential is obtained by applying a Legendre transformation to the imaginary part of the prepotential [22]. Electric-magnetic duality, which is a central feature of $N = 2$ vector multiplets, acts by symplectic transformations. While the prepotential is not a symplectic function, the Hesse potential is, and the special real coordinates form a symplectic vector. In [25] a real formulation of projective special Kähler geometry was worked out, and it was shown that only part of the symplectic covariance of the underlying affine manifold could be kept manifest. However, in applications such as black hole solutions and the study of non-holomorphic corrections one would like to have the full symplectic covariance manifest.

In this paper we obtain a real formulation of projective special Kähler geometry which is symplectically covariant. We make use of the superconformal formalism which employs the gauge equivalence between a theory of $n + 1$ superconformal vector multiplets with scalar manifold N and a theory of n vector multiplets coupled to Poincaré supergravity, with scalar manifold $\bar{N} = N/\mathbb{C}^* = M//\mathrm{U}(1)$, see for example [26] for a review. The main idea is to keep the U(1) gauge invariance of the superconformal formulation intact, which amounts to working on N or on the associated Sasakian S, which is a U(1) principal bundle over $\bar{N}$, instead of working on $\bar{N}$ itself. We derive explicit expressions for the scalar and vector kinetic terms as real symmetric tensor fields on N. These tensor fields can be expressed in terms of the Hesse potential and are related to one another and to the metric

of the associated superconformal theory by adding differentials dual to the vector fields generating the $\mathbb{C}^*$-action.

1.2 The c-map

The special geometries of five-dimensional vector multiplets [27], four-dimensional vector multiplets [1] and of hypermultiplets [28] are related to one another by dimensional reduction. The corresponding maps between the scalar manifolds are called the r-map and the c-map respectively [29–32]. Both maps have rigid and local versions, depending on whether rigid or local supersymmetry is considered. Moreover, by reducing over time rather than space one obtains 'temporal' versions of the r- and c-map [24, 33, 34, 45, 51], which can be used for generating stationary solitonic solutions by lifting Euclidean, instantonic solutions [24, 33–37, 37, 47], and to study the radial quantization of BPS solutions [33, 35]. The local c-map is also an important tool for investigating the non-perturbative dynamics of hypermultiplets [40–42], which shows interesting phenomena such as wall crossing [43, 44].

The geometry underlying the rigid r-map and rigid c-map [29, 45] is well understood: for both maps the scalar manifold of the higher-dimensional theory is simply replaced by its tangent bundle (or, equivalently, its cotangent bundle) and the special structures on both manifolds are related in a canonical way. The metric induced on the (co-)tangent bundle is a version of the so-called Sasaki metric, where the special connection rather than the Levi-Civita connection is used to define the vertical distribution. To be specific, the rigid r-map between the scalar manifolds $M, N \simeq TM$ of five- and four-dimensional rigid vector multiplets takes the following form in terms of adapted coordinates σ^i, b^i on TM:

$$ds_M^2 = H_{ij}(\sigma)d\sigma^i d\sigma^j \to ds_N^2 = H_{ij}(\sigma)(d\sigma^i d\sigma^j + db^i db^j) \,. \tag{1.1}$$

The geometry of the local r-map and c-map is more complicated because the supergravity multiplet contributes additional degrees of freedom to the scalar manifold. For the local r-map the metric on the vector multiplet manifold $\bar{N}$ can nevertheless be brought to the above Sasaki form [26, 46, 47]. The reason is that the Kaluza-Klein scalar combines with the five-dimensional scalars precisely in such a way that the scalar manifold $\bar{M}$ of five-dimensional vector multiplets coupled to supergravity is extended to the scalar manifold M of the associated superconformal theory, but with the superconformal Hesse potential replaced by its logarithm. The scalar manifold $\bar{N}$ of the four-dimensional vector multiplet theory is then obtained by applying the rigid r-map to M.

The local c-map [29, 30] has an even more complicated structure. It relates projective special Kähler manifolds $\bar{N}$ of dimension $2n$ to quaternion-Kähler manifolds $\bar{Q}$ of dimension $4n + 4$. In three dimensions abelian gauge fields, including the Kaluza-Klein vector can be dualized into scalars, which become part of the scalar manifold $\bar{Q}$. Using special holomorphic coordinates on $\bar{N}$, the metric on $\bar{Q}$ was obtained in [30]. While completely explicit, this expression is rather complicated, and not covariant with respect to the symplectic transformations of the underlying vector multiplet theory.

In this paper we reformulate the local c-map and obtain an explicit expression for the metric in terms of the Hesse potential of the associated vector multiplet theory which is

symplectically covariant and only differs from the Sasaki form by simple universal terms. This is done using the ideas introduced above: (i) we show that the Kaluza-Klein scalar can be identified with the radial direction of the $\mathbb{C}^*$-bundle N over $\bar{N}$. Thus as in the case of the local r-map there is a natural way to combine the four-dimensional scalars with the Kaluza-Klein scalar. (ii) To preserve symplectic covariance we avoid U(1) gauge fixing, which amounts to working on a principal U(1) bundle $\hat{Q}$ over the quaternion-Kähler manifold $\bar{Q}$. In complete analogy to the vector multiplet case, the metric of $\bar{Q}$ is lifted horizontally to a symmetric (degenerate) tensor field on the total space of $\hat{Q}$. (iii) We use our real formulation of projective Kähler geometry to express everything in terms of real coordinates and the Hesse potential.

Our construction is different from other 'covariant' c-maps, which use the hyper-Kähler cone and twistor space associated to every quaternion-Kähler manifold [34, 38, 39]. In particular, the U(1)-bundle $\hat{Q}$ and the systematic use of horizontal lifts and of special real coordinates are specific to our approach. One advantage of our formulation is that we obtain an explicit and relatively simple expression for the quaternion-Kähler metric itself. In contrast, other constructions provide expressions for the hyper-Kähler potential of the hyper-Kähler cone, or for the Kähler potential of the twistor space, in terms of either the holomorphic prepotential [39] or the Hesse potential [34]. This leaves the still complicated step of lifting data from $\bar{Q}$ to the hyper-Kähler cone or twistor space, or projecting data down from there to $\bar{Q}$. Being able to work directly on $\bar{Q}$ has immediate advantages for constructing solutions, as we will explain below.

1.3 Solitons and instantons

Dimensional reduction is a standard tool for generating solutions with (at least) one Killing vector field [50]. In particular, dimensional reduction over time allows to lift Euclidean, instantonic solutions to stationary, solitonic solutions. Therefore we include the case of time-like reduction when working out the c-map. For temporal reduction the resulting manifold has split signature and is expected on general grounds to be para-quaternion Kähler [45, 51].

Our main motivation in studying solutions is to develop a formalism which does not depend on supersymmetry (Killing spinor equations), and applies to general c-map spaces, without the assumption that the scalar manifold is symmetric or homogeneous. This continues work done previously in [24, 47–49] for five-dimensional vector multiplets. For symmetric spaces group theoretical methods have led to a detailed understanding of extremal BPS and non-BPS solutions [36, 37]. For general c-map spaces such methods are not applicable and need to be replaced by other methods. Solving the reduced, three-dimensional equations of motion is equivalent to finding a harmonic map from the three-dimensional base space (i.e. the reduced space-time) into the scalar target space. Particular solutions to this problem are given by harmonic maps onto totally geodesic submanifolds [24, 50]. The simplest choice for the base manifold is to take it to be flat, which for non-rotating black hole solutions corresponds to imposing extremality. In this case the scalar submanifold must be totally isotropic, so that the classification of BPS and non-BPS non-rotating solutions corresponds to the classification of totally geodesic, totally isotropic submanifolds.

In this paper we only consider three-dimensional base spaces which are Ricci-flat, and, hence, flat. We do not impose spherical symmetry, unless when considering specific examples. One advantage of our approach is that, for flat base spaces, spherical symmetry is not needed to solve the field equation, i.e. it is as easy to obtain multi-centered solutions as single-centered solutions. This is different in the approach of [34], where only single centered BPS black holes were constructed, while multi-centered solutions were left as an open problem. For this type of problem it is advantageous that we do not need to lift solutions to the twistor space or to the hyper-Kähler cone.

The structure of our expression for the (para-)quaternion-Kähler metric immediately suggests that in order to restrict fields to a totally isotropic submanifold we should make an ansatz of the form $\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$, where the two sets of scalars correspond to the positive and negative directions of the scalar metric. By lifting to four dimensions we recognize that this is equivalent to the BPS condition imposed by the vanishing of the gaugino variation, and we can also verify that in this case the ADM mass is equal to the central charge. Thus we have identified totally isotropic submanifolds which exist for any $\bar{Q}$ and correspond to BPS field configurations. As further part of the ansatz we can specify whether the solution is rotating or non-rotating. While the non-rotating solutions include BPS black holes, the rotating solutions are over-extremal, as expected for rotating BPS solutions in four dimensions. By introducing dual coordinates q_a the remaining field equations can be brought to the form of decoupled linear harmonic equations, $\Delta q_a = 0$. Upon dimensional lifting these equations are recognized as the black hole attractor equations, which express all fields in terms of a set of harmonic functions. This is completely analogous to the five-dimensional case. To illustrate how the formalism works we include several examples of rotating and non-rotating solutions. The rotating solutions we find include those described in [11, 52–54]. For static axion free solutions we show that the solutions previously known for 'very special' prepotentials (those which can be obtained by dimensional reduction from five dimensions) can be generalized to a larger class of prepotentials. The reason is that the ability to solve the attractor equations only depends on certain homogeneity properties of the prepotential. A similar observation allowed the construction of new solutions in five dimensions [47].

Extremal non-BPS solutions are associated to totally geodesic, totally isotropic submanifolds different from the universal ones described above. Since we want to include target spaces which are not symmetric, we cannot use the group theoretical methods of [36, 37] to find non-BPS solutions. Another method is to replace the central charge by a 'fake superportential' by applying a charge rotation matrix [63, 64]. Within our approach we can modify the ansatz by allowing a constant field rotation matrix, $\partial_\mu q^a = R^a{}_b \partial_\mu \hat{q}^b$, as was done for the local r-map in [47]. For non-rotating solutions we show that this generalized ansatz works, but only if a compatibility condition between the field rotation matrix and the metric is satisfied. At first glance this makes it hard to say anything about the existence of non-BPS extremal solutions for general, non-symmetric target spaces, without considering specific models. However, for the class of prepotentials already mentioned above, which includes the very special prepotential as a subclass, we can demonstrate the existence of a non-trivial field rotation matrix for axion-free solutions. In contrast, for rotating solutions the presence of a non-trivial field rotation matrix always requires to generalize the ansatz by admitting a curved three-dimensional base space.

2 Review of rigid vector multiplets

2.1 Rigid vector multiplets and the rigid c-map

To set the scene, we will review rigid $\mathcal{N} = 2$ vector multiplets and the rigid c-map [45, 51]. We will use the conventions of [45], except for a relative minus sign in the relation between the scalar metric N_{IJ} and $\text{Im}F_{IJ}$.[1]

Vector multiplets $(A_{\hat{\mu}}^I, \lambda_i^I, X^I)$ contain vector fields, a doublet of fermions, and complex scalars. Here and in the following $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3$ are Lorentz indices, $i = 1, 2$ is the $SU(2)_R$-index, and I labels the vector multiplets. The relevant terms in the bosonic Lagrangian are

$$\mathcal{L}_4 \sim -N_{IJ}(X, \bar{X})\partial_{\hat{\mu}} X^I \partial^{\hat{\mu}} \bar{X}^J \qquad (2.1)$$
$$+i\left(F_{IJ}(X)F_{\hat{\mu}\hat{\nu}}^{I|-} F^{J|-|\hat{\mu}\hat{\nu}} - \bar{F}_{IJ}(\bar{X})F_{\hat{\mu}\hat{\nu}}^{I|+} F^{J|+|\hat{\mu}\hat{\nu}} \right) \,,$$

where $F_{\hat{\mu}\hat{\nu}}^{I|\pm} = \frac{1}{2}\left(F_{\hat{\mu}\hat{\nu}}^I \mp i\tilde{F}_{\hat{\mu}\hat{\nu}}^I \right)$ are the (anti-)selfdual projections of the field strengths $F_{\hat{\mu}\hat{\nu}}^I = 2\partial_{[\hat{\mu}} A_{\hat{\nu}]}^I$. The Hodge-dualization of field strength is given by $\tilde{F}_{\hat{\mu}\hat{\nu}}^I = \frac{1}{2}\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}F^{I|\hat{\rho}\hat{\sigma}}$.

All couplings in the Lagrangian can be expressed in terms of the holomorphic prepotential $F(X^I)$.[2] Denoting the derivatives of the prepotential as

$$F_I = \frac{\partial F}{\partial X^I} \,, \quad \bar{F}_I = \frac{\partial \bar{F}}{\partial \bar{X}^I} \,, \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J} \,, \cdots$$

the scalar metric is

$$N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ}) = 2\text{Im}(F_{IJ}) \,.$$

This is an affine special Kähler metric, because the Kähler potential $K(X, \bar{X})$ for the metric

$$N_{IJ} = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J}$$

can be expressed in terms of the holomorphic prepotential,

$$K = i(X^I \bar{F}_I - F_I \bar{X}^I) \,. \qquad (2.2)$$

The additional, 'special' structure of the scalar geometry is a consequence of the electric-magnetic duality transformations, which leave the field equations (but not the action) invariant. Electric-magnetic duality acts by symplectic transformations, see [56] for a concise summary. The quantities

$$(X^I, F_I)^T \,, \quad (F_{\hat{\mu}\hat{\nu}}^{I|\pm}, G_{I|\hat{\mu}\hat{\nu}}^{\pm})^T \,,$$

where the dual gauge fields are defined by

$$G_{I|\hat{\mu}\hat{\nu}}^- = F_{IJ}F_{\hat{\mu}\hat{\nu}}^{J|-} \,,$$

---

[1]In our present convention the kinetic terms for scalar and vector fields are positive definite if $\text{Im}F_{IJ}$ is positive definite. Note that if we use the superconformal approach to construct a supergravity theory, N_{IJ} must be chosen indefinite, with the negative directions corresponding to conformal compensators.

[2]For non-generic choices of a symplectic frame the prepotential might not exist, but then one can always perform a symplectic transformation to a frame where a prepotential exists [55].

transform as symplectic vectors, while the second derivatives F_{IJ} of the prepotential transform fractionally linearly. The prepotential itself does not transform covariantly, i.e. it is not a symplectic function (scalar).

Upon dimensional reduction the components of the gauge fields along the reduced direction become scalars. After dualizing the three-dimensional gauge fields into scalars, one is left with a theory of scalars and fermions, which organize themselves into hypermultiplets. The dimensional reductions with respect to a space-like and a time-like directions differ by relative signs. We can discuss both reductions in parallel by introducing the parameter ϵ, where $\epsilon = -1$ for space-like and $\epsilon = +1$ for time-like reductions. We denote scalars descending from four-dimensional gauge fields by $p^I = A^{I|*}$, where $* = 3$ for space-like and $* = 0$ for time-like reductions. The scalars obtained by dualizing the three-dimensional gauge fields are denoted s_I. The scalar part of the three-dimensional Lagrangian takes the form [45]

$$
\begin{aligned}
\mathcal{L}_3 \sim &-N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^J + \epsilon N_{IJ}\partial_\mu p^I \partial^\mu p^J \\
&+\epsilon N^{IJ}(\partial_\mu s_I + R_{IK}\partial_\mu p^K)(\partial^\mu s_I + R_{JL}\partial^\mu p^L) \,.
\end{aligned}
\tag{2.3}
$$

Here N^{IJ} is the inverse of N_{IJ}, and $\mu, \nu, \ldots = 0, 1, 2$ for space-like and $\mu, \nu, \ldots = 1, 2, 3$ for time-like reductions.

The map induced by dimensional reduction between the respective scalar manifolds M and N is called the rigid c-map. For space-like reductions N is hyper-Kähler [29], as required for rigid hypermultiplets [57]. For time-like reductions one obtains a para-hyper-Kähler manifold, as required for Euclidean hypermultiplets [45]. In both cases the manifold N can be interpreted as the cotangent bundle of M, $N = T^*M$, equipped with a natural metric, which one might call the '∇-Sasaki' metric.[3] This becomes manifest if one uses special real coordinates instead of special holomorphic coordinates on M, see (2.9) below. Since special real coordinates will play an important role in the following, we will review them in some detail.

2.2 The real formulation of affine special Kähler geometry

The intrinsic definition of affine special Kähler geometry [19] states that a Kähler manifold is affine special Kähler[4] if it is equipped with a flat, torsion-free, symplectic connection ∇, such that the complex structure I satisfies $d^\nabla I = 0$. The affine coordinates (x^I, y_I) of this flat connection are Darboux coordinates, and are called special real coordinates in the following. They are related to the special holomorphic coordinates X^I by:

$$
x^I = \mathrm{Re}(X^I)\,, \quad y_I = \mathrm{Re}(F_I)\,.
$$

Conversely, the special holomorphic coordinates X^I and the quantities F_I, which complete them into a complex symplectic vector, can be decomposed as

$$
\begin{aligned}
X^I &= x^I + iu^I(x, y)\,, \\
F_I &= y_I + iv_I(x, y)\,.
\end{aligned}
$$

_____

[3]In contrast to the Sasaki metric, we use the special connection ∇ instead of the Levi-Civita connection to pick a horizontal distribution on TM.

[4]This definition can be generalized to pseudo-Kähler and adapted to para-Kähler manifolds [51].

We remark that we could also take the imaginary parts u^I, v_I as real coordinates and x^I, y_I to be functions of u^I, v_I. More generally we could take the real parts of $e^{i\alpha}(X^I, F_I)$ as Darboux coordinates. Affine special Kähler manifolds always admit not just one special connection, but an S^1-family which is generated by [19]

$$\nabla^{(\alpha)} = e^{\alpha I} \circ \nabla \circ e^{-\alpha I} \ .$$

Neither physics, nor geometry depends on the choice of the special connection from this family, but each connection in the family has its own system of special real coordinates. The 'dual' special real coordinates u^I, v_I are flat Darboux coordinates with respect to the special connection $\nabla^{(\pi/2)}$. By computing the Jacobian of the coordinate transformation $(X, \bar{X}) \leftrightarrow (x, y)$, and using $F_{IJ} = F_{JI}$, one obtains the following relations:

$$\frac{\partial v_I}{\partial x^J} = \frac{\partial v_J}{\partial x^I} \ , \qquad\qquad \frac{\partial v_I}{\partial u^J} = \tfrac{1}{2} R_{IJ} \ ,$$
$$\frac{\partial v_I}{\partial y_J} = -\frac{\partial u^J}{\partial x^I} \ , \qquad\qquad \frac{\partial y_I}{\partial u^J} = -\tfrac{1}{2} N_{IJ} \ ,$$
$$\frac{\partial u^I}{\partial y_J} = \frac{\partial u^J}{\partial y_I} \ , \qquad\qquad \frac{\partial u^I}{\partial x^J} = N^{IK} R_{KJ} \ .$$

Affine special Kähler manifolds are Hessian manifolds, and the Hesse potential is proportional to the Legendre transform of the imaginary part of the holomorphic prepotential [22]. This transformation replaces $u^I = \text{Im} X^I$ by $y_I = \text{Re} F_I$ as independent variables:

$$H(x, y) = 2\text{Im} F(X(x, y)) - 2 y_I u^I(x, y) \ .$$

Taking derivatives of $2\, \text{Im}(F)$ with respect to (x, y) we find

$$\begin{aligned}
\frac{\partial}{\partial x^I} 2\text{Im}(F)\Big|_{x, u(x, y)} &= \left(\frac{\partial}{\partial x^I} + \frac{\partial u^J}{\partial x^I} \frac{\partial}{\partial u^J} \right) 2\text{Im}(F)\Big|_{x, u} \\
&= \left[\left(\frac{\partial}{\partial X^I} + \frac{\partial}{\partial \bar{X}^I} \right) + i \frac{\partial u^J}{\partial x^I} \left(\frac{\partial}{\partial X^J} - \frac{\partial}{\partial \bar{X}^J} \right) \right] 2\text{Im}(F)\Big|_{X, \bar{X}} \\
&= 2 v_I + 2 y_I \frac{\partial u^J}{\partial x^I} \ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial y_I} 2\text{Im}(F)\Big|_{x, u(x, y)} &= \left(\frac{\partial u^J}{\partial y_I} \frac{\partial}{\partial u^J} \right) 2\text{Im}(F)\Big|_{x, u} \\
&= i \frac{\partial u^J}{\partial y_I} \left(\frac{\partial}{\partial X^J} - \frac{\partial}{\partial \bar{X}^J} \right) 2\text{Im}(F)\Big|_{X, \bar{X}} \\
&= 2 y_J \frac{\partial u^J}{\partial y_I} \ .
\end{aligned}$$

Using these results, we find that the derivatives of the Hesse potential are proportional to the dual real coordinates:

$$H_a = \left(\frac{\partial H}{\partial q^a} \right) = \left(\frac{\partial H}{\partial x^I}, \frac{\partial H}{\partial y_I} \right) = \left(2 v_I, -2 u^I \right) \ . \tag{2.4}$$

Taking second derivatives we find

$$\frac{\partial^2 H}{\partial x^I \partial x^J} = N_{IJ} + R_{IK} N^{KL} R_{LJ} \; ,$$

$$\frac{\partial^2 H}{\partial x^I \partial y_J} = -2N^{IK} R_{KJ} \; ,$$

$$\frac{\partial^2 H}{\partial y_I \partial y_J} = 4N^{IJ} \; .$$

This allows us to express the Hessian metric H_{ab} in terms of the second derivatives of the prepotential:

$$(H_{ab}) = \left(\frac{\partial^2 H}{\partial q^a \partial q^b} \right) = \begin{pmatrix} N + RN^{-1}R & -2RN^{-1} \\ -2N^{-1}R & 4N^{-1} \end{pmatrix} \; . \tag{2.5}$$

We will also need the relation between the differentials of the special holomorphic and the special real coordinates:

$$\begin{aligned} dX^M &= dx^M + idu^M \\ &= dx^M + i \left(\frac{\partial u^M}{\partial x^K} dx^K + \frac{\partial u^M}{\partial y_I} dy_I \right) \\ &= dx^M + i \left(N^{MI} R_{IK} dx^K - 2N^{MI} dy_I \right) \; . \end{aligned} \tag{2.6}$$

Next, we compute the derivatives of the Hesse potential with respect to the special holomorphic coordinates. This is not needed for the real formulation of affine special Kähler geometry, but will be important later for the real formulation of projective affine special Kähler geometry.

$$\begin{aligned} \frac{\partial H}{\partial X^I} &= \frac{\partial x^J}{\partial X^I} \frac{\partial H}{\partial x^J} + \frac{\partial y_J}{\partial X^I} \frac{\partial H}{\partial y_J} \\ &= 2v_J \frac{\partial x^J}{\partial X^I} - 2u^J \frac{\partial y_J}{\partial X^I} = v_I - F_{IJ} u^J \\ &= v_I - \tfrac{1}{2} (R_{IJ} + iN_{IJ}) u^J \; , \end{aligned} \tag{2.7}$$

and by a similar calculation

$$\frac{\partial H}{\partial \bar{X}^I} = v_I - \tfrac{1}{2} (R_{IJ} - iN_{IJ}) u^J \; .$$

Taking second derivatives we find

$$\frac{\partial^2 H}{\partial \bar{X}^J \partial X^I} = \tfrac{1}{2} N_{IJ} \; . \tag{2.8}$$

Using equations (2.5) and (2.6) it is straightforward to verify verify that

$$ds_M^2 = N_{IJ} dX^I d\bar{X}^J = H_{ab} dq^a dq^b \; ,$$

which shows that N_{IJ} and H_{ab} represent the same metric in terms of special holomorphic and special real coordinates, respectively. It is easy to show that the inverse of the Hessian metric is given by

$$(H^{-1})^{ab} = (H^{ab}) = \begin{pmatrix} N^{-1} & \frac{1}{2}N^{-1}R \\ \frac{1}{2}RN^{-1} & \frac{1}{4}(N + RN^{-1}R) \end{pmatrix} .$$

Moreover, it is useful to note that

$$H_{ab}\Omega^{bc}H_{cd} = -4\Omega_{ad} ,$$

where

$$\Omega_{ab} := \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

is the matrix representing the fundamental form (Kähler form) in special real coordinates.[5] With these results it is straightforward to express the reduced Lagrangian (2.3) in terms of special real coordinates. Defining $(\hat{q}_a) = (s_I, 2p^I)$, we find [45, 58]

$$\mathcal{L}_3 \sim - \left(H_{ab}(q)\partial_\mu q^a \partial^\mu q^b - \epsilon H^{ab}(q)\partial_\mu \hat{q}_a \partial^\mu \hat{q}_b \right) . \tag{2.9}$$

It is now manifest that the metric on N is the canonical positive definite (for $\epsilon = -1$) and split signature (for $\epsilon = 1$) metric on the cotangent bundle of M, respectively. Using special real coordinates has further advantages. All objects appearing in the above Lagrangian transform linearly under symplectic transformations: q^a, $\hat{q}_a$ are contravariant and covariant vectors, respectively, while H_{ab} and H^{ab} are symmetric tensors [46]. In contrast, $F_{IJ} = \frac{1}{2}(R_{IJ} + iN_{IJ})$ transforms fractionally linearly under symplectic transformations.

3 Vector multiplets coupled to 4d supergravity

The coupling of vector multiplets to supergravity can be constructed using the superconformal calculus, which exploits the gauge equivalence between a locally superconformal theory of $n + 1$ vector multiplets and n vector multiplets coupled to Poincaré supergravity.[6] This is reviewed, for example, in [59, 60]. We will use elements of this approach, and focus on the bosonic fields and the underlying scalar geometry. The first step in the construction is to write down a theory of $n + 1$ rigidly superconformal vector multiplets. Compared to the previous section, this amounts to the additional constraint that the prepotential is homogeneous of degree two. The resulting scalar manifold is a conical affine special Kähler manifold [19, 24], which is an affine special Kähler manifold with a holomorphic homothetic action of $\mathbb{C}^* = \mathbb{R}^{>0} \cdot U(1)$:

$$X^I \to \lambda X^I ,$$

where $\lambda = |\lambda|e^{i\phi} \in \mathbb{C}^*$. Both the scale transformation and the U(1) phase transformation are part of the superconformal algebra. The scale transformations act as homotheties, and

---

[5]The fundamental form has constant coefficients because special real coordinates are Darboux coordinates.

[6]This requires the presence of a further auxiliary multiplet, which will not be relevant for our discussion.

give the scalar manifold N the structure of a Riemannian cone over a Sasakian manifold S. The U(1) transformations act isometrically on both N and S.

The next step in the superconformal construction is to gauge the superconformal transformations. For our purposes, the relevant part of the resulting bosonic action is

$$\mathcal{L}_4 \sim -\tfrac{1}{2}e^{-\mathcal{K}}R_4 - N_{IJ}\mathcal{D}_{\hat{\mu}}X^I\mathcal{D}^{\hat{\mu}}\bar{X}^J + \tfrac{1}{4}\mathcal{I}_{IJ}F^I_{\hat{\mu}\hat{\nu}}F^{J\hat{\mu}\hat{\nu}} + \tfrac{1}{4}\mathcal{R}_{IJ}F^I_{\hat{\mu}\hat{\nu}}\tilde{F}^{J\hat{\mu}\hat{\nu}} \,, \qquad (3.1)$$

where the indices run from $I = 0, \ldots, n$. This Lagrangian contains the space-time Ricci scalar R_4 as a result of the gauging. It is invariant under local dilatations and U(1) dilatations. The U(1) covariant derivatives are defined by

$$\mathcal{D}_{\hat{\mu}}X^I = (\partial_{\hat{\mu}} + iA_{\hat{\mu}})X^I \,,$$
$$\mathcal{D}_{\hat{\mu}}\bar{X}^I = (\partial_{\hat{\mu}} - iA_{\hat{\mu}})\bar{X}^I \,,$$

where $A_{\hat{\mu}}$ is the U(1) connection. In principle we should also include the connection $b_{\hat{\mu}}$ of local dilatations, but it is known that the terms containing this connection cancel within the Lagrangian. Alternatively, one can impose the gauge condition $b_{\hat{\mu}} = 0$, known as the K-gauge. The gravitational term is not canonical, since the Ricci scalar is multiplied by the dependent field

$$e^{-\mathcal{K}} = -N_{IJ}X^I\bar{X}^J = -i(X^I\bar{F}_I - F_I\bar{X}^I) \,, \qquad (3.2)$$

which acts as a compensator for local dilatations.

The gauge couplings are given by the real and imaginary parts of the complex matrix

$$\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} = \bar{F}_{IJ} + i\frac{(NX)_I(NX)_J}{XNX} \,. \qquad (3.3)$$

This differs from the gauge couplings $F_{IJ} = \tfrac{1}{2}(R_{IJ} + iN_{IJ})$ of the rigid theory by terms which arise from integrating out an auxiliary field (the tensor field of the Weyl multiplet). Note that $\mathcal{N}_{IJ}$ is manifestly U(1) invariant, so that by imposing the D-gauge we obtain tensor fields on S and $\bar{N}$.

The locally superconformal Lagrangian, of which we have displayed only the pieces relevant for our purposes, is gauge equivalent to a Lagrangian of vector multiplets coupled to Poincaré supergravity. The Poincaré supergravity Lagrangian is obtained by imposing conditions which gauge fix the additional transformations which extend the Poincaré supersymmetry algebra to the superconformal algebra. For our purposes the relevant transformations are the dilatations and U(1) transformations. The dilatations are gauge fixed by imposing the D-gauge $e^{-\mathcal{K}} = 1$, which brings the gravitational term to its canonical, Einstein-Hilbert form. Geometrically, this restricts the scalar fields to a hypersurface $\mathcal{H} \subset N$ in the conical affine special Kähler manifold. This hypersurface can be identified with the Sasakian S, which forms the basis of the Riemannian cone. Similarly, one can impose a U(1) gauge condition to obtain the scalar manifolds $\bar{N}$ of the Poincaré supergravity theory. In practice, one often prefers to work in terms of U(1) invariant quantities instead of imposing an explicit gauge fixing condition. Since the U(1) transformations act isometrically on S, this corresponds to taking a quotient $S/U(1)$. Moreover, since the function $e^{-\mathcal{K}}$ used to define the D-gauge is the moment map of the U(1) isometry, the scalar

manifolds N and $\bar{N}$ of the superconformal and super-Poincaré theories are related by a symplectic quotient

$$\bar{N} \simeq N/\mathbb{C}^* \simeq N//\mathrm{U}(1) \ .$$

This is in fact a Kähler quotient, because $\bar{N}$ inherits a Kähler metric from N. Manifolds $\bar{N}$, which are obtained by this construction from conical affine special Kähler manifolds, are called projective special Kähler manifolds.

It is well known from work on black hole solutions that it is often advantageous to use the gauge equivalence, and to work on the larger space N rather than on the physical scalar manifold $\bar{N}$. One particular advantage is that this keeps symplectic covariance manifest. Fixing a $\mathrm{U}(1)$ gauge corresponds to selecting a hypersurface of the Sasakian S, which can be done by choosing any condition which is transversal to the $\mathrm{U}(1)$ action (for example $\mathrm{Im}X^0 = 0$). However, choosing a symplectically invariant condition corresponds to selecting, at each point, the direction orthogonal to the $\mathrm{U}(1)$ action. But this is the contact distribution of the Sasakian and therefore not integrable. For this reason a hypersurface corresponding to a $\mathrm{U}(1)$ gauge cannot be selected in a symplectically invariant way.[7] In the following we will keep the local $\mathrm{U}(1)$ gauge invariance intact, and for reasons that will become clear later we also postpone imposing the D-gauge.

The above Lagrangian contains the $\mathrm{U}(1)$ gauge field, which makes its local $\mathrm{U}(1)$ invariance manifest. However, the $\mathrm{U}(1)$ connection is a non-dynamical, auxiliary field, and we now eliminate it by its equation of motion

$$A_{\hat{\mu}} = -\tfrac{i}{2}e^{\mathcal{K}}\left[(\partial_{\hat{\mu}}X)N\bar{X} - XN(\partial_{\hat{\mu}}\bar{X})\right] \ .$$

Now the gauged sigma model is replaced by the ungauged sigma model

$$-N_{I\bar{J}}\mathcal{D}_{\hat{\mu}}X^I\mathcal{D}^{\hat{\mu}}\bar{X}^J = -\left(N_{IJ} - \frac{(NX)_I(N\bar{X})_J}{XN\bar{X}}\right)\partial_{\hat{\mu}}X^I\partial^{\hat{\mu}}\bar{X}^J + \tfrac{1}{4}e^{-\mathcal{K}}\partial_{\hat{\mu}}\mathcal{K}\partial^{\hat{\mu}}\mathcal{K} \ ,$$
$$= -e^{-\mathcal{K}}g_{IJ}\partial_{\hat{\mu}}X^I\partial^{\hat{\mu}}\bar{X}^J + \tfrac{1}{4}e^{-\mathcal{K}}\partial_{\hat{\mu}}\mathcal{K}\partial^{\hat{\mu}}\mathcal{K} \ ,$$

where $g_{IJ} = \partial_I\partial_{\bar{J}}\mathcal{K}$,

$$\mathcal{K} = -\log[-i(X^I\bar{F}_I - F_I\bar{X}^I)] \ . \tag{3.4}$$

We have used that the prepotential is homogeneous of degree 2 and therefore $X(\partial_{\hat{\mu}}N)\bar{X} = 0$. The Lagrangian still contains terms proportional to $\partial_{\hat{\mu}}\mathcal{K}$ because we have not yet imposed the D-gauge. Observe that the tensor field g_{IJ} is degenerate on the large space N, because

$$X^I g_{IJ} = 0 = g_{IJ}\bar{X}^J \ .$$

This is not a problem, because the directions along which g_{IJ} is degenerate correspond to the unphysical degrees of freedom normal to $\bar{N} \subset N$. Geometrically, these are the vertical directions of the $\mathbb{C}^*$-bundle N over $\bar{N}$, i.e. the radial direction of the Riemannian cone and the orbits of the $\mathrm{U}(1)$ isometry. While g_{IJ} is not a metric on N, we obtain a non-degenerate metric by projecting it $\bar{N}$. In other words, g_{IJ} is the horizontal lift of the projective special Kähler metric $g_{\bar{N}}$ to N, and, if we impose the D-gauge, to S.

---

[7]We thank Vicente Cortés for an illuminating discussion of this point.

The well known formula for the Kähler potential of the projective special Kähler manifold $\bar{N}$ can be obtained by using coordinates X^0, z^i on N, where $z^i = X^i/X^0$ are special coordinates on $\bar{N}$. Rewriting $\mathcal{K}$ given in (3.4) as a function of X^0, z^i, one finds that the dependence on X^0 can be removed by a Kähler transformation. This shows explicitly that the tensor g_{IJ} is degenerate on the two vertical directions. Defining $\mathcal{F}(z) = (X^0)^{-2} F(X^I)$, we obtain the Kähler potential of the projective special Kähler metric of $\bar{N}$:

$$\mathcal{K} = -\log(-i[(\mathcal{F} - \bar{\mathcal{F}}) - (z^i - \bar{z}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)]) \,, \quad \mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial z^i} \,.$$

To obtain a theory with positive definite kinetic terms for the physical scalars, the projection of g_{IJ} onto $\bar{N}$ must be positive definite, while positive definite kinetic terms for the vector fields require that $\mathcal{I}_{IJ}$ is negative definite, see (3.1). It is known that both conditions are satisfied if the metric N_{IJ} of the conical affine special Kähler manifold N has complex Lorentz signature $(- - + \cdots +)$ [46, 61]. The negative directions, which are the directions normal to $\bar{N} \subset N$, correspond to conformal compensators. We remark that $-\mathcal{I}_{IJ}$ can be interpreted as a positive definite metric on N, and that the relation between the indefinite metric N_{IJ} and the definite metric $-\mathcal{I}_{IJ}$ has a natural geometric interpretation, which is analogous to the relation between the Griffith and Weil intermediate Jacobians for Calabi-Yau threefolds [46].

4 The real formulation of projective special Kähler geometry

In section 2 we have reviewed the real formulation of affine special Kähler geometry. It is not straightforward to obtain a real formulation of projective special Kähler geometry which preserves symplectic covariance. The reason is that the physical scalars of the super-Poincaré theory correspond to special coordinates $z^i = \frac{X^i}{X^0}$ on $\bar{N}$. While (X^I, F_I) is a symplectic vector, the (z^i) is not, and only part of the symplectic covariance can be kept manifest [25].

In this section we show how a manifestly symplectic real formulation can be obtained by preserving the U(1) gauge invariance. This amounts to expressing the degenerate tensor g_{IJ} and the vector kinetic matrix $\mathcal{N}_{IJ}$ in terms of special real coordinates on N and in terms of the Hesse potential H. In doing so we will get a clearer understanding of the geometrical meaning of these tensor fields.

Since the theory associated with N is now superconformal, we have additional relations in addition to those derived in section 2. The prepotential and the Hesse potential are now homogeneous of degree two in special holomorphic and special real coordinates, respectively. This implies

$$2H = H_a q^a = H_{ab} q^a q^b \,. \tag{4.1}$$

Also note that

$$2(y_I u^I - x^I v_I) = -2H = -i(X^I \bar{F}_I - F_I \bar{X}^I) = -N_{IJ} X^I \bar{X}^J = e^{-\mathcal{K}} \,. \tag{4.2}$$

The affine special Kähler manifold is now a complex cone, at least locally. This means that there is a homothetic and holomorphic action of $\mathbb{C}^*$, which is given by the the homothetic Killing vector field ξ and the U(1) Killing vector field $I\xi$, where I is the complex

structure. The explicit expressions with respect to special holomorphic and special real coordinates are:

$$\xi = X^I \frac{\partial}{\partial X^I} + \bar{X}^I \frac{\partial}{\partial \bar{X}^I} = q^a \frac{\partial}{\partial q^a} \ ,$$

$$I\xi = iX^I \frac{\partial}{\partial X^I} - i\bar{X}^I \frac{\partial}{\partial \bar{X}^I} = \tfrac{1}{2} H_a \Omega^{ab} \frac{\partial}{\partial q^b} \ .$$

In special real coordinates the complex structure itself is given by $I^a{}_c = \frac{1}{2}\Omega^{ab} H_{bc}$ in terms of the Kähler form Ω_{ab} and the metric H_{ab}.

We remark that the q^a are special real coordinates with respect to a fixed, but arbitrary special connection. For conical affine special Kähler manifolds the U(1) gauge transformations preserve the metric, the symplectic and the complex structure, but they rotate the special connections, and the associated special real coordinates, among themselves.

Our first task is to rewrite the tensor

$$g_{IJ} = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} = -\frac{Y_{I\bar{J}}}{Y} + \frac{Y_I Y_{\bar{J}}}{Y^2} \ , \tag{4.3}$$

where

$$\mathcal{K} = -\log Y \ , \quad Y = -i(X^I \bar{F}_I - F_I \bar{X}^I) = -2H \ ,$$

in terms of special real coordinates. Using (2.7) and (2.8) we find

$$g_{IJ} = -\frac{1}{2H} N_{IJ} + \frac{1}{H^2} \left(v_I - \tfrac{1}{2} (R_{IK} + iN_{IK}) u^K \right) \left(v_J - \tfrac{1}{2} (R_{JL} - iN_{JL}) u^L \right) \ . \tag{4.4}$$

Using (4.4), we find

$$K_{I\bar{J}} dX^I d\bar{X}^J = -\frac{1}{2H} N_{IJ} dX^I d\bar{X}^J \tag{4.5}$$

$$+\frac{1}{H^2} \left(v_I - \frac{1}{2}(R_{IK} + iN_{IK})u^K \right) \left(v_J - \frac{1}{2}(R_{JL} - iN_{JL})u^L \right) dX^I d\bar{X}^J \ .$$

By the results of section 2, the first term gives

$$-\frac{1}{2H} N_{IJ} dX^I d\bar{X}^J = -\frac{1}{2H} H_{ab} dq^a dq^b \ .$$

To evaluate the second term, we observe that

$$(2v_I, -2u^I) = (H_a) = (H_{ab} q^b) \ ,$$

where we used (2.4) together with homogeneity. Using further results from section 2, this implies

$$x^I = 2N^{IJ} v_J - N^{IJ} R_{JK} u^I \ ,$$

$$y_I = R_{IJ} N^{JK} v_K - \frac{1}{2}(N_{IJ} + R_{IK} N^{KL} R_{LJ}) u^J \ .$$

To proceed, we substitute (2.6) into the second term on the right hand side of (4.5), with the result

$$\frac{1}{H^2}\left(\left(v_I - \frac{1}{2}(R_{IK} + iN_{IK})u^K\right)\left(v_J - \frac{1}{2}(R_{JL} - iN_{JL})u^L\right)\right)dX^I d\bar{X}^J$$

$$= \frac{1}{H^2}\Big(v_I v_J + y_I y_J)dx^I dx^J - (v_I u^J + y_I x^J)dx^I dy_J$$

$$- (u^I v_J + x^I y_J)dy_I dx^J + (u^I u^J + x^I x^J)dy_I dy_J\Big) .$$

We now observe that

$$(H_a H_b) = 4\begin{pmatrix} v_I v_J & -v_I u^J \\ -u^I v_J & u^I u^J \end{pmatrix}$$

and

$$(\Omega_{ac}q^c \Omega_{bd}q^d) = \begin{pmatrix} y_I y_J & -y_I x^J \\ -x^I y_J & x^I x^J \end{pmatrix} .$$

Using this, the second term becomes

$$\frac{1}{H^2}\Big((v_I v_J + y_I y_J)dx^I dx^J - (v_I u^J + y_I x^J)dx^I dy_J$$

$$- (u^I v_J + x^I y_J)dy_I dx^J + (u^I u^J + x^I x^J)dy_I dy_J\Big)$$

$$= \left(\frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c \Omega_{bd}q^d)\right)dq^a dq^b .$$

Combining the two terms, we find that

$$g_{IJ}dX^I d\bar{X}^J = \left[-\frac{1}{2H}H_{ab} + \frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c \Omega_{bd}q^d)\right]dq^a dq^b =: H_{ab}^{(0)}dq^a dq^b , \quad (4.6)$$

where $H_{ab}^{(0)}$ is the horizontal lift of the projective special Kähler metric, expressed in special real coordinates.

Before we proceed to express $\mathcal{N}_{IJ}$ in real coordinates, let us analyze what the above calculation tells us about the underlying geometry. Solving (4.6) for the affine special Kähler metric H_{ab}, we obtain:

$$H_{ab} = -2H H_{ab}^{(0)} + \frac{1}{2H}H_a H_b + \frac{2}{H}\Omega_{ac}q^c \Omega_{bd}q^d$$

This is a decomposition of H_{ab} into the horizontal component $H_{ab}^{(0)}$, which by projection gives the projective special Kähler metric, and two negative definite terms[8] which correspond to the directions generated by $\xi = q^a\partial_a$ and $I\xi = \frac{1}{2}H_a\Omega^{ab}\partial_b$. As we will see, all relevant tensor fields on N are related to the metric H_{ab} by adding terms proportional to the squares of the one-forms H_a and $\Omega_{ac}q^c$. These one forms are obtained by contracting the homothety ξ with the metric and the Kähler form, respectively (equivalently by

---

[8]With our conventions H is negative definite, see (4.2).

contracting ξ and $I\xi$ with the metric). It is an advantage of the real formalism that the directions generated by ξ and $I\xi$ can be described in such a simple way.

We now introduce one further tensor field on N, which will play an important role for the c-map. As we have seen before, the Kähler potential (3.4) of the supergravity theory is obtained by taking the logarithm of the Kähler potential (2.2) of the corresponding superconformal theory. The logarithm effectively encodes the superconformal quotient. This motivates us introduce the tensor obtained by taking the second derivatives of the logarithm of the Hesse potential H of the rigid theory. Specifically, we set $\tilde{H} = -\frac{1}{2}\log(-2H)$ and $\tilde{H}_{ab} = \partial^2_{a,b}\tilde{H}$. Then

$$g_{IJ}dX^I d\bar{X}^J = \left[\tilde{H}_{ab} - \frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c)(\Omega_{bd}q^d)\right]dq^a dq^b . \qquad (4.7)$$

Since we know that the right hand side is positive definite in the horizontal directions and degenerate in the vertical directions generated by ξ and $I\xi$, it follows immediately that $\tilde{H}_{ab}$ is a non-degenerate Hessian metric which is negative definite along the U(1) direction generated by $I\xi$ and positive definite in all other directions. The homogeneity properties of the Hesse potential (4.1) also imply that the matrix $\tilde{H}$ satisfies the identity

$$q^a q^b \tilde{H}_{ab} = 1 .$$

This will not be used in this paper, but may be useful for produce 4d non-extremal black hole solutions as a similar identity was needed in the 5d case [49].

We now turn to the vector kinetic matrix $\mathcal{N}_{IJ}$. It is known how to express this complex matrix, which transforms fractionally linearly under symplectic transformations in terms of a real matrix $\hat{H}_{ab}$, which transforms as a symmetric tensor of rank 2. In the conventions of [46], the relation is

$$\hat{H}_{ab} := \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} .$$

It is known that the tensor $-\hat{H}_{ab}$ is positive definite, given that H_{ab} has complex Lorentz signature, and therefore it can be interpreted as a positive definite metric on N. In [46] it was shown that in terms of complex geometry the indefinite and definite metric are related by a transformation that exchanges Griffith and Weyl flags. We would now like to relate $\hat{H}_{ab}$ to the other tensor fields in terms of real coordinates.

Below we will prove that the tensors $\tilde{H}_{ab}$, H_{ab} and $\hat{H}_{ab}$ are related by:

$$\begin{aligned}
\tilde{H}_{ab} &= -\frac{1}{2H}H_{ab} + \frac{1}{2H^2}H_a H_b \\
&= \frac{1}{H}\hat{H}_{ab} - \frac{2}{H^2}(\Omega_{ac}q^c)(\Omega_{bd}q^d) . \qquad (4.8)
\end{aligned}$$

Given that H_{ab} has complex Lorentz signature, it is manifest that $-\hat{H}_{ab}$ is positive definite. In contrast to the indefinite metrics H_{ab} and $\tilde{H}_{ab}$, the definite metric $\hat{H}_{ab}$ is not Hessian. However it is uniquely determined by the Hesse potential H. It is the above identity which

will be critical in matching up moduli fields with the electric/magnetic potentials in order to produce black hole solutions.

It remains to prove (4.8), which requires some effort. To start we need the explicit relations between the real and imaginary parts of $F_{IJ} = \frac{1}{2}(R_{IJ} + iN_{IJ})$ and $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$:

$$\mathcal{R}_{IJ} = \frac{1}{2}R_{IJ} + \frac{i}{2}\left(\frac{N_{IK}X^K N_{JL}X^L}{(XNX)} - \frac{N_{IK}\bar{X}^K N_{JL}\bar{X}^L}{(\bar{X}N\bar{X})}\right)$$
$$\mathcal{I}_{IJ} = -\frac{1}{2}N_{IJ} + \frac{1}{2}\left(\frac{N_{IK}X^K N_{JL}X^L}{(XNX)} + \frac{N_{IK}\bar{X}^K N_{JL}\bar{X}^L}{(\bar{X}N\bar{X})}\right) ,$$

where $(XNX) = N_{MN}X^M X^N$, etc. It is straightforward to verify that the inverse of $\mathcal{I}_{IJ}$ is

$$\mathcal{I}^{IJ} = -2N^{IJ} + \frac{2}{(XN\bar{X})}\left(X^I\bar{X}^J + \bar{X}^I X^J\right) = -2N^{IJ} + \frac{2}{H}\left(x^I x^J + u^I u^J\right) ,$$

where we used $2H = (XN\bar{X})$ and the decomposition $X^I = x^I + iu^I$, $F_I = y_I + iv_I$. Next, one can verify

$$-\mathcal{I}^{IK}\mathcal{R}_{KJ} = N^{IK}R_{KJ} - \frac{2}{(XN\bar{X})}(X^I\bar{F}_J + \bar{X}^I F_J) = N^{IK}R_{KJ} - \frac{2}{H}(x^I y_J + u^I v_J) .$$

Finally, one can also verify that

$$\mathcal{I}_{IJ} + \mathcal{R}_{IK}\mathcal{I}^{KL}\mathcal{R}_{LJ} = -\frac{1}{2}N_{IJ} - \frac{1}{2}R_{IK}N^{KL}R_{LJ} + \frac{2}{(XN\bar{X})}(F_I\bar{F}_J + \bar{F}_I F_J)$$
$$= -\frac{1}{2}N_{IJ} - \frac{1}{2}R_{IK}N^{KL}R_{LJ} + \frac{2}{H}(y_I y_J + v_I v_J)$$

Putting everything together we have

$$\begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}N - \frac{1}{2}RN^{-1}R & RN^{-1} \\ RN^{-1} & -2N^{-1} \end{pmatrix}$$
$$+ \frac{2}{H}\begin{pmatrix} y_I y_J + v_I v_J & -(y_I x^J + v_I u^J) \\ -(x^I y_J + u^I v_J) & x^I x^J + u^I u^J \end{pmatrix} .$$

Expressing this in terms of the special real coordinates q^a using $\hat{H}_{ab}$, H_{ab} and Ω_{ab} this becomes

$$\hat{H}_{ab} = -\frac{1}{2}H_{ab} + \frac{2}{H}\left(\frac{1}{4}H_a H_b + \Omega_{ac}q^c \Omega_{bd}q^d\right) .$$

which proves (4.8).

In summary, we have found the real tensor fields $H_{ab}^{(0)}$ and $\hat{H}_{ab}$ which lift the scalar metric and vector kinetic matrix of the super-Poincaré theory associated to $\bar{N}$ to the Sasakian S and the complex cone N. This provides a real formulation of projective special Kähler geometry as long as we do not gauge fix the U(1) transformations.

For later use we now derive the expression for the graviphoton in terms of real coordinates. The graviphoton is the vector field which in the Poincaré supergravity theory belongs

to the supergravity multiplet and therefore is invariant under symplectic transformations. Its field strength is given by

$$T_{\hat\mu\hat\nu}^- = -X^I G_{I|\hat\mu\hat\nu}^- + F_I F_{\hat\mu\hat\nu}^{I|-} \ ,$$

where the dual field strength are

$$G_{\hat\mu\hat\nu}^- = \bar{\mathcal{N}}_{IJ} F_{\hat\mu\hat\nu}^{J|-} \ .$$

Adding the self-dual part and expressing everything in real variables, we obtain

$$T_{\hat\mu\hat\nu} = T_{\hat\mu\hat\nu}^+ + T_{\hat\mu\hat\nu}^- = -x^I G_{I|\hat\mu\hat\nu} + y_I F_{\hat\mu\hat\nu}^I + u^I \tilde{G}_{I|\hat\mu\hat\nu} - v_I \tilde{F}_{\hat\mu\hat\nu}^I \ .$$

These terms are not independent, we can either use the real coordinates (x^I, y_I) together with the field strength $F_{\hat\mu\hat\nu}^I, G_{I|\hat\mu\hat\nu}$, or the dual real coordinates (u^I, v_I) together with the Hodge-dual field strength $\tilde{F}_{\hat\mu\hat\nu}^I, \tilde{G}_{I|\hat\mu\hat\nu}$. Using the definitions of these quantities, one can verify that

$$x^I G_{I|\hat\mu\hat\nu} - y_I F_{\hat\mu\hat\nu}^I = v_I \tilde{F}_{\hat\mu\hat\nu}^I - u^I \tilde{G}_{I|\hat\mu\hat\nu} \ ,$$

so that

$$T_{\hat\mu\hat\nu} = -2 \left(x^I G_{I|\hat\mu\hat\nu} - y_I F_{\hat\mu\hat\nu}^I \right) = 2 \left(u^I \tilde{G}_{I|\hat\mu\hat\nu} - v_I \tilde{F}_{\hat\mu\hat\nu}^I \right) \ .$$

5   The local c-map and the Hesse potential

We now turn to the dimensional reduction of four-dimensional vector multiplets coupled to supergravity. We perform the reduction using the complex formulation of the four-dimensional scalars, and use the gauge equivalence to describe them in terms of the scalars X^I taking values in N. The reductions over space and time are performed in parallel. After dualizing the three-dimensional vector fields we systematically express all quantities in terms of special real coordinates. Our overall strategy is to obtain an expression which comes as close to the 'metric on the cotangent bundle form' (2.9) of the rigid c-map. Therefore we express the couplings in terms of the logarithm of the Hesse potential. We will see that all terms that cannot be brought to this form are universal, in the sense that their couplings only contain constant matrices and the Kaluzu-Klein scalar.

5.1   Dimensional reduction

Our starting point is the Lagrangian representing the bosonic part of four-dimensional $\mathcal{N} = 2$ supergravity coupled to n vector multiplets,

$$\begin{aligned}
\mathcal{L}_4 \sim &-\tfrac{1}{2}e^{-\mathcal{K}} R_4 - e^{-\mathcal{K}} g_{IJ} \partial_{\hat\mu} X^I \partial^{\hat\mu} \bar{X}^J + \tfrac{1}{4} e^{-\mathcal{K}} \partial_{\hat\mu}\mathcal{K} \partial^{\hat\mu}\mathcal{K} \\
&+ \tfrac{1}{4} \mathcal{I}_{IJ} F_{\hat\mu\hat\nu}^I F^{J\hat\mu\hat\nu} + \tfrac{1}{4} \mathcal{R}_{IJ} F_{\hat\mu\hat\nu}^I \tilde{F}^{J\hat\mu\hat\nu} \ ,
\end{aligned} \tag{5.1}$$

where $g_{IJ} = \partial_I \partial_{\bar{J}} \mathcal{K}$. We have eliminated the U(1) gauge field by its equation of motion, thus replacing the gauged sigma model by a sigma model with a degenerate 'metric'. Since we postpone imposing the D-gauge, this Lagrangian contains the non-constant, but dependent field $e^{-\mathcal{K}}$

$$e^{-\mathcal{K}} = -N_{IJ} X^I \bar{X}^J \ .$$

We perform the reduction of the Lagrangian over a time-like and space-like dimension simultaneously, differentiating between the two cases by

$$\epsilon = \begin{cases} -1, & \text{spacelike} , \\ +1, & \text{timelike} . \end{cases}$$

In order to reduce directly into the Einstein frame we decompose the metric as

$$ds_4^2 = -\epsilon \, e^{\phi} \, (dy + V_{\mu}dx^{\mu})^2 + e^{-\phi}g_{\mu\nu}dx^{\mu}dx^{\nu} ,$$

which implicitly defines $(e^{\phi}, V_{\mu}, g_{\mu\nu})$ in terms of the four-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$. The reduced Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_3 \sim & -\tfrac{1}{2}e^{-\mathcal{K}} \left(R_3 + \tfrac{1}{2}\partial_{\mu}\phi \, \partial^{\mu}\phi - \tfrac{1}{4}\epsilon \, e^{2\phi}V^{\mu\nu}V_{\mu\nu} - \partial_{\mu}\mathcal{K}\partial^{\mu}\phi \right) \\
& - e^{-\mathcal{K}}g_{IJ}\partial_{\mu}X^I\partial^{\mu}\bar{X}^J + \tfrac{1}{4}e^{-\mathcal{K}}\partial_{\mu}\mathcal{K}\partial^{\mu}\mathcal{K} \\
& + \tfrac{1}{4}e^{\phi}\mathcal{I}_{IJ}(F^I_{\mu\nu} - 2\partial_{[\mu}\zeta^I V_{\nu]})(F^{J\mu\nu} - 2\partial^{[\mu}\zeta^J V^{\nu]}) \\
& - \tfrac{1}{2}\epsilon \, e^{-\phi}\mathcal{I}_{IJ}\partial_{\mu}\zeta^I\partial^{\mu}\zeta^J - \tfrac{1}{2}\epsilon \, \mathcal{R}_{IJ}\varepsilon^{\mu\nu\rho}F^I_{\mu\nu}\partial_{\rho}\zeta^J ,
\end{aligned}$$

where the terms descending from the four-dimensional metric appear in the first line, the four-dimensional scalars in the second line, and the gauge fields in the third and fourth line. We have denoted the field strength of the Kaluza Klein-vector by $V_{\mu\nu}$, and the scalar fields $\zeta^I = A_0^I \, (A_3^I)$ are the components of the four dimensional vectors along the reduced timelike (spacelike) direction. The Lagrangian at present still contains the bare Kaluza Klein-vector V_{μ}, which prevents the associated abelian gauge symmetry from being manifest. Therefore we make the field redefinition

$$(A^I_{\mu})' := A^I_{\mu} - \zeta^I V_{\mu} , \qquad \Longrightarrow \qquad (F^I_{\mu\nu})' + \zeta^I V_{\mu\nu} = F^I_{\mu\nu} - 2\partial_{[\mu}\zeta^I V_{\nu]} .$$

The Lagrangian now takes the manifestly gauge invariant form,

$$\begin{aligned}
\mathcal{L}_3 \sim & -\tfrac{1}{2}e^{-\mathcal{K}} \left(R_3 - \tfrac{1}{4}\epsilon \, e^{2\phi}V^{\mu\nu}V_{\mu\nu} \right) \\
& - e^{-\mathcal{K}}g_{IJ}\partial_{\mu}X^I\partial^{\mu}\bar{X}^J - \tfrac{1}{4}e^{-\mathcal{K}}(\partial_{\mu}\phi - \partial_{\mu}\mathcal{K})(\partial^{\mu}\phi - \partial^{\mu}\mathcal{K}) + \tfrac{1}{4}e^{-\mathcal{K}}\partial_{\mu}\mathcal{K}\partial^{\mu}\mathcal{K} \\
& + \tfrac{1}{4}e^{\phi}\mathcal{I}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})(F^{J\mu\nu} + \zeta^J V^{\mu\nu}) \\
& - \tfrac{1}{2}\epsilon \, e^{-\phi}\mathcal{I}_{IJ}\partial_{\mu}\zeta^I\partial^{\mu}\zeta^J - \tfrac{1}{2}\epsilon \, \mathcal{R}_{IJ}\varepsilon^{\mu\nu\rho}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})\partial_{\rho}\zeta^J ,
\end{aligned}$$

where we have dropped the primes and gathered together like terms.

Conformal rescaling. In order to obtain a canonical Einstein-Hilbert term we perform the conformal rescaling

$$g_{\mu\nu} = e^{2\mathcal{K}}\tilde{g}_{\mu\nu} .$$

The various terms in the Lagrangian have the following transformation rules in three dimensions:

$$\begin{aligned}
\sqrt{g} &= \sqrt{\tilde{g}}e^{3\mathcal{K}} \\
\sqrt{g}\,g^{\mu\nu} &= \sqrt{\tilde{g}}\,\tilde{g}^{\mu\nu}e^{\mathcal{K}} \\
\sqrt{g}\,g^{\mu\nu}g^{\rho\sigma} &= \sqrt{\tilde{g}}\,\tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}e^{-\mathcal{K}} \\
R_3 &= e^{-2\mathcal{K}}\left[\tilde{R}_3 - 4\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\mathcal{K} + 2\tilde{g}^{\mu\nu}\partial_{\mu}\mathcal{K}\partial_{\nu}\mathcal{K} \right] .
\end{aligned}$$

The Lagrangian itself becomes

$$\mathcal{L}_3 \sim -\tfrac{1}{2}\tilde{R}_3 + \tfrac{1}{8}\epsilon\, e^{2(\phi-\mathcal{K})}V^{\mu\nu}V_{\mu\nu}$$
$$- g_{IJ}\partial_\mu X^I\partial^\mu \bar{X}^J - \tfrac{1}{4}(\partial_\mu\phi - \partial_\mu\mathcal{K})(\partial^\mu\phi - \partial^\mu\mathcal{K}) - \tfrac{1}{2}\partial_\mu\mathcal{K}\partial^\mu\mathcal{K}$$
$$+ \tfrac{1}{4}e^{(\phi-\mathcal{K})}\mathcal{I}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})(F^{J\mu\nu} + \zeta^J V^{\mu\nu})$$
$$- \tfrac{1}{2}\epsilon\, e^{(\mathcal{K}-\phi)}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J - \tfrac{1}{2}\epsilon\,\mathcal{R}_{IJ}\varepsilon^{\mu\nu\rho}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})\partial_\rho\zeta^J \ .$$

One can see that by redefining the KK-scalar $\phi' = \phi - \mathcal{K}$, the field $\mathcal{K}$ decouples from all other fields besides gravity. We will now set this field to be constant, and drop the primes.[9] This amounts to imposing the D-gauge. We could of course have done this at an earlier stage, but we found it instructive to demonstrate how the radial degree of freedom $\mathcal{K}$ of the cone N decouples.

Dualization of vector fields. Since we are working in three dimensions, and the vector fields in the Lagrangian only appear via their field strengths, it is possible to dualise the vector fields into scalar fields $(A^I, V) \sim (\tilde{\zeta}_I, \tilde{\phi})$. This is achieved by adding the Lagrange multiplier

$$\mathcal{L}_{\mathrm{Lm}} \sim \tfrac{1}{2}\epsilon\,\varepsilon^{\mu\nu\rho}(F^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\partial_\rho(\tilde{\phi} - \tfrac{1}{2}\zeta^I\tilde{\zeta}_I)) \ .$$

The variation of $\mathcal{L}_3 + \mathcal{L}_{\mathrm{Lm}}$ gives the algebraic equations of motion (note that $\varepsilon^{\mu\nu\rho}\varepsilon_{\mu\nu\sigma} = 2\epsilon\,\delta^\rho_\sigma$)

$$V_{\mu\nu} = 2e^{-2\phi}\varepsilon_{\mu\nu\rho}(\partial^\rho\tilde{\phi} + \tfrac{1}{2}(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I)) \ ,$$
$$F^I_{\mu\nu} = -\epsilon\, e^{-\phi}\mathcal{I}^{IJ}\varepsilon_{\mu\nu\rho}(\partial^\rho\tilde{\zeta}_J - \mathcal{R}_{JK}\partial^\rho\zeta^K) - \zeta^I V_{\mu\nu} \ .$$

Substituting the above expressions back into $\tilde{\mathcal{L}}_3 = \mathcal{L}_3 + \mathcal{L}_{\mathrm{Lm}}$ we are left with the dual Lagrangian

$$\tilde{\mathcal{L}}_3 \sim -\tfrac{1}{2}\tilde{R}_3 - g_{I\bar{J}}\partial_\mu X^I\partial^\mu \bar{X}^J - \tfrac{1}{4}\partial_\mu\phi\partial^\mu\phi \tag{5.2}$$
$$- e^{-2\phi}\left(\partial_\mu\tilde{\phi} + \tfrac{1}{2}(\zeta^I\partial_\mu\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\mu\zeta^I)\right)^2$$
$$- \tfrac{1}{2}\epsilon\, e^{-\phi}\left[\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J + \mathcal{I}^{IJ}\left(\partial_\mu\tilde{\zeta}_I - \mathcal{R}_{IK}\partial_\mu\zeta^K\right)\left(\partial^\mu\tilde{\zeta}_J - \mathcal{R}_{JL}\partial^\mu\zeta^L\right)\right] \ .$$

5.2 A field redefinition

We would now like to bring the Lagrangian into a form that resembles (2.9) more closely. From [46] we know that by setting $(\hat{q}^a) = \tfrac{1}{2}(\zeta^I, \tilde{\zeta}_I)$ and using the real tensor $\hat{H}_{ab}$, the terms in the third line of (5.2) are proportional to $\tfrac{1}{H}\hat{H}_{ab}\partial_\mu\hat{q}^a\partial^\mu\hat{q}^b$.[10] Using (4.8) we can express this in terms of the Hessian metric $\tilde{H}_{ab}$ up to model independent terms. To proceed, we need to re-organize the remaining scalars $X^I, \phi, \tilde{\phi}$ into $2n+2$ real scalars q^a which transform as a symplectic vector and balance the $2n + 2$ real scalars $\hat{q}^a$. The counting of degree of freedom works out, because the $n + 1$ complex scalars are subject to two conditions, and

---

[9] When computing the tensor g_{IJ}, it is understood that $\mathcal{K}$ is set constant after computing the derivatives.

[10] Actually, in [46] the dual coordinates $\hat{q}_a$ and the inverse metric $\hat{H}^{ab}$ were used, but this is simply a different parametrization.

therefore correspond to $2n$ independent real scalar fields. Moreover, by going to special real coordinates on N, we can relate them to a symplectic vector. But what about ϕ and $\tilde{\phi}$?

We proceed by making use of an observation that was made in the context of the local r-map, which relates the scalar manifolds of five-dimensional and four-dimensional vector multiplets [47]. There it is possible to absorb the Kaluza Klein-scalar into the manifold parametrized by the higher-dimensional (in the case of the r-map, the five-dimensional) scalars. This amounts to lifting the constraint imposed by the D-gauge. The Kaluza Klein-scalar is identified with the radial direction of the cone N over S, which is promoted from a gauge degree of freedom to a dynamical degree of freedom. This idea can be implemented in the four-dimensional setting by defining a new set of complex scalars Y^I by

$$Y^I = e^{\frac{\phi}{2}} X^I \ , \qquad \bar{Y}^I = e^{\frac{\phi}{2}} \bar{X}^I \ .$$

The Kaluza Klein-scalar is now a dependent field, determined by the expression

$$e^\phi = -i(Y^I \bar{F}_I - F_I \bar{Y}^I) \ . \tag{5.3}$$

Since ϕ transforms by a shift under the global isometry group, we find that the new scalar fields must transform by a scale factor under these isometries

$$Y^I \longrightarrow e^{\frac{\lambda}{2}} Y^I \ , \qquad \bar{Y}^I \longrightarrow e^{\frac{\lambda}{2}} \bar{Y}^I \ .$$

The Lagrangian can now be written as

$$\begin{aligned}
\tilde{\mathcal{L}}_3 \sim \ & -\tfrac{1}{2}\tilde{R}_3 - g_{IJ}\partial_\mu Y^I \partial^\mu \bar{Y}^J - \tfrac{1}{4}\partial_\mu \phi \, \partial^\mu \phi \\
& -e^{-2\phi}\left[\partial_\mu \tilde{\phi} + \tfrac{1}{2}(\phi^I \partial_\mu b_I - b_I \partial_\mu \phi^I)\right]^2 \\
& -\tfrac{1}{2}\epsilon \, e^{-\phi}\Big[\mathcal{I}_{IJ}\partial_\mu \phi^I \partial^\mu \phi^J + \mathcal{I}^{IJ}\left(\partial_\mu b_I - \mathcal{R}_{IK}\partial_\mu \phi^K\right)\left(\partial^\mu b_J - \mathcal{R}_{JL}\partial^\mu \phi^L\right)\Big] \ ,
\end{aligned} \tag{5.4}$$

where ϕ is a dependent field.

The Lagrangian is still invariant under local U(1) transformations of the fields $(Y, \bar{Y})$, and the equations of motion transform by an overall phase factor. This is shown using that the tensor g_{IJ} has two null directions

$$Y^I g_{I\bar{J}} = 0 = g_{I\bar{J}}\bar{Y}^J \ .$$

By differentiation we obtain the identities

$$\begin{aligned}
Y^I \partial_K g_{I\bar{J}} &= -g_{K\bar{J}} \ , & \partial_K g_{I\bar{J}}\bar{Y}^J &= 0 \ , \\
y^I \partial_{\bar{K}} g_{I\bar{J}} &= 0 \ , & \partial_{\bar{K}} g_{I\bar{J}}\bar{Y}^J &= -g_{I\bar{K}} \ .
\end{aligned}$$

Under phase transformations the derivatives of the metric transform by a phase and the Kaluza Klein-scalar is invariant

$$\begin{aligned}
\partial_K g_{I\bar{J}} &\longrightarrow e^{-i\alpha}\partial_K g_{I\bar{J}} \ , & \phi &\longrightarrow \phi \ , \\
\partial_{\bar{K}} g_{I\bar{J}} &\longrightarrow e^{i\alpha}\partial_{\bar{K}} g_{I\bar{J}} \ .
\end{aligned}$$

It follows that the Lagrangian is U(1) invariant, the $(Y, \bar{Y})$ equations of motion transform by an overall phase, and all other equations of motion are invariant.

When comparing (5.2) to (5.4) both Lagrangians look identical, except that X^I have been replaced by Y^I. It is instructive to check that substituting $X^I = e^{-\phi/2} Y^I$ into (5.2) gives indeed (5.4). Due to the peculiar properties of the degenerate tensor g_{IJ}, no derivative terms involving ϕ are generated by the substitution. All factors $e^{\phi/2}$ cancel, because g_{IJ} is homogeneous of degree -2, and because $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ are homogeneous of degree 0. Of course the essential difference between (5.2) and (5.4) is that ϕ is now a dependent field.

One might wonder whether the dualized Kaluza Klein-vector $\tilde{\phi}$ could be treated in a similar way as the Kaluza Klein scalar ϕ. This is not so, because the symmetries carried by the reduced gravitational degrees of freedom $\phi, \tilde{\phi}$ do not match with the symmetries of the affine special Kähler manifold N. The fields $\phi, \tilde{\phi}$ parametrize the coset

$$M_{\phi, \tilde{\phi}} = \frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)}$$

with isometry group $\mathrm{SL}(2, \mathbb{R})$. A two-dimensional solvable subgroup generated by shifts in ϕ and $\tilde{\phi}$ extends to a symmetry of the full Lagrangian once the $\hat{q}^a$ are included. In contrast, the manifold N has a homothetic action of $\mathbb{C}^*$. Upon taking the logarithm of the Hesse potential, the dilatation becomes an isometry (rather than a homothety) of the 'metric' g_{IJ}.[11] Above we observed that the fields Y^I transform under shifts of ϕ, and we might think of these continuous global symmetries as residual symmetries left after we have eliminated the local dilatation symmetry by absorbing the KK scalar into N. The fields Y^I are still subject to U(1) gauge transformations, and one is tempted to identify $\tilde{\phi}$ with the U(1) gauge degree of freedom. If this was the case one could absorb $\tilde{\phi}$ into the Y^I, thus making the gauge degree of freedom a physical one. However, the global continuous shift symmetry of $\tilde{\phi}$ do act differently from U(1) gauge transformations, and therefore there is no way of absorbing $\tilde{\phi}$ into Y^I such that the new variable transforms naturally under the global symmetry. Therefore we proceed differently, by keeping $\tilde{\phi}$ as an independent field, and, consequently, keeping the local U(1) gauge invariance. We will see later that when we construct solutions, the local U(1) gauge symmetry is gauge fixed while preserving symplectic covariance and the isometries of scalar metric. We will also see that for our solutions it will always be possible to express $\tilde{\phi}$ in terms of the other fields.

We can interpret our treatment of the scalar fields geometrically as follows. If we freeze the scalars $\hat{q}^a$ descending from the four-dimensional gauge fields, then the scalar manifold parametrized by the physical four-dimensional scalars z^i, and by ϕ and $\tilde{\phi}$ is

$$M_{z, \phi, \tilde{\phi}} = \bar{N} \times \frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)} \ .$$

Using the gauge equivalence, we can describe $\bar{N}$ in terms of the fields X^I using the Kähler

---

[11]This works as in [47]: if the Hesse potential is homogeneous, and we take its logarithm as the new Hesse potential, then the new metric is homogeneous of degree zero (as a tensor, i.e. the metric coefficients are homogeneous of degree -2) irrespective of the degree of homogeneity of the original Hesse potential.

quotient:

$$M_{z,\phi,\tilde{\phi}} = N//\mathrm{U}(1) \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \ .$$

Now we absorb ϕ into N. This 'un-does' the D-gauge, and re-institutes the radial degree of freedom of the cone N over S, while leaving the U(1) isometry intact. The coset SL(2$\mathbb{R}$)/SO(2) is broken up and the remaining one-dimensional piece is parametrized by the scalar $\tilde{\phi}$, with a metric depending on ϕ. The scalar manifold can be represented as a deformed product

$$M_{z,\phi,\tilde{\phi}} = N/\mathrm{U}(1) \times_\phi \mathbb{R}_{\tilde{\phi}} \ .$$

Here $N/\mathrm{U}(1)$ is the quotient of N with respect to its U(1) isometry rather than the Kähler quotient. The advantage of this way of organising the fields becomes apparent once we use real special coordinates on N.

5.3 The real parametrization

The kinetic term of the complex scalar fields takes precisely the same form as considered previously in section 4

$$\mathcal{L}_3 \sim -g_{IJ}\partial_\mu Y^I \partial^\mu \bar{Y}^J + \cdots \ .$$

We make the real decomposition

$$Y^I = x^I + iu^I(x,y) \qquad F_I = y_I + iv_I(x,y) \ ,$$

and use the previous results to write this term in the Lagrangian as

$$\mathcal{L}_3 \sim -\left[\tilde{H}_{ab} - \frac{1}{4H^2}H_a H_b + \frac{1}{H^2}(\Omega_{ac}q^c)(\Omega_{bd}q^d) \right] \partial_\mu q^a \partial^\mu q^b + \cdots \ , \tag{5.5}$$

where $q^a = (x^I, y_I)^T$. Note that our previous calculations are still applicable after the replacement $X^I \to Y^I$, due to homogeneity.

The Kaluza Klein-scalar is given in terms of the real variables by

$$e^\phi = -2H = -2(x^I v_I(x,y) - y_I u^I(x,y)) \ , \tag{5.6}$$

which is homogeneous of degree two in $q^a = (x^I, y_I)^T$. The kinetic term for the Kaluza Klein-scalar can then be written as

$$\tfrac{1}{4}\partial_\mu \phi \, \partial^\mu \phi = \frac{1}{4H^2}H_a H_b \partial_\mu q^a \partial^\mu q^b \ ,$$

and this term cancels against the second term in (5.5). When rewriting the terms descending from the four-dimensional gauge fields using the variables $\hat{q}^a = (\frac{1}{2}\zeta^I, \frac{1}{2}\tilde{\zeta}_I)^T$, they take the form

$$\mathcal{L}_{3\mathrm{gauge}} \sim \epsilon\, \tilde{H}_{ab}\partial_\mu \hat{q}^a \partial^\mu \hat{q}^b + \epsilon\, \frac{2}{H^2}\left(q^a \Omega_{ab}\partial_\mu \hat{q}^b \right)^2 - \frac{1}{4H^2}\left[\partial_\mu \tilde{\phi} - 2\left(\hat{q}^a \Omega_{ab}\partial_\mu \hat{q}^b \right) \right]^2 \ .$$

We can now put together all terms and write the Lagrangian in terms of real fields as

$$\tilde{\mathcal{L}}_3 \sim -\tfrac{1}{2}\tilde{R}_3 - \tilde{H}_{ab}\left(\partial_\mu q^a \partial^\mu q^b - \epsilon\, \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b\right)$$
$$-\frac{1}{H^2}\left(q^a \Omega_{ab}\partial_\mu q^b\right)^2 + \epsilon\,\frac{2}{H^2}\left(q^a \Omega_{ab}\partial_\mu \hat{q}^b\right)^2$$
$$-\frac{1}{4H^2}\left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab}\partial_\mu \hat{q}^b\right)^2 . \tag{5.7}$$

This formula is one of our main results, and provides a new formulation of the supergravity c-map (and its temporal version) in terms of real variables and the Hesse potential. It comes surprisingly close to the Sasaki-type form of the rigid and local r-map and rigid c-map. The scalar term in the first line has precisely the form found for the local r-map, a Sasaki-type metric with the Hesse potential of the rigid theory being replaced by its logarithm. The terms in second and third line are simple and universal, they only depend on the constant matrix Ω_{ab} and the Hesse potential H (identified with the Kaluza-Klein scalar). We can also understand the origin of these additional terms. First, there is one term involving the dualized Kaluza-Klein vector $\tilde{\phi}$. This field plays a special role because we could not absorb it into N in the same way as the Kaluza-Klein scalar. The other terms can be understood from our real formulation of projective special Kähler geometry. They arise from rewriting the tensor fields $H_{ab}^{(0)}$ and $\hat{H}_{ab}$ in terms of $\tilde{H}_{ab}$. In the analogous case of the r-map such terms are absent, because there the analogues of $\tilde{H}_{ab}$ and $\hat{H}_{ab}$ coincide, and because the scalar metric becomes the analogue of $\tilde{H}_{ab}$ after absorbing the Kaluza-Klein scalar.

The fields in (5.7) are still subject to U(1) gauge transformations, and therefore the quaternion-Kähler metric on the physical scalar manifold is obtained by a U(1) quotient. One could impose a gauge fixing condition and eliminate one of the scalar fields. Since the metric on the U(1) bundle parametrized by $q^a, \hat{q}^a, \tilde{\phi}$ is degenerate along the direction generated by the U(1), we can choose any condition which is transverse to the U(1) action (such as $q^1 = 0$) and then restrict the (degenerate) metric on the bundle to the resulting hypersurface to obtain the positive definite quaternion-Kähler metric (or split signature para-Quaternion-Kähler metric). Since the U(1) action relates the members of the S^1 family of special connections to one another, the U(1) bundle can be viewed as the bundle of special connections, and a U(1) gauge fixing as picking a special connection.

We prefer not to fix the U(1) gauge and to work on the U(1) bundle, because, as we explained in section 3, a U(1) gauge fixing would spoil the manifest symplectic covariance. In the following section we will show that instantonic solutions can be constructed and be lifted to solitonic solutions, such as black holes, while preserving symplectic covariance. We will then revisit the issue of U(1) gauge fixing.

6 Stationary solutions

We now turn to finding stationary solutions of the four-dimensional Lagrangian. Four-dimensional stationary BPS solutions for general vector multiplet couplings have been constructed some time ago by imposing invariance under part of the supersymmetry transformations [11, 53, 54, 62]. We expect to recover these solutions and to obtain further

non-BPS solutions. To this end we reduce over a time-like dimension, therefore making the choice $\epsilon = 1$ in the formula for the reduced Lagrangian (5.7). We will find that in flat backgrounds we can give solutions to generic models in terms of harmonic functions.

Before embarking into the details, let us explain the overall strategy. Since the three-dimensional Lagrangian is a combination of perfect squares, we will try to reduce the field equations to Bogomol'nyi equations which follow from imposing that the squares vanish individually. We will focus on solutions where the three-dimensional metric is Ricci-flat, and, hence, flat. This restricts the fields to take values in totally isotropic submanifold, and therefore we will call the corresponding ansatz the isotropic ansatz. After lifting to four dimensions we will obtain four-dimensional extremal static black hole solutions as well as over-extremal (singular) rotating solutions. The structure of the Bogomol'nyi equations can be read off from the Lagrangian (5.7). One of the Bogomol'nyi equations results from imposing that the first line of (5.7) vanishes, which gives a relation of the form

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$$

between the q^a and the $\hat{q}^a$, which is identical to the relation found for five-dimensional black holes [47]. If the scalar metric satisfies a certain compatibility condition, one can instead impose the more general condition

$$\partial_\mu q^a = R^a{}_b \partial_\mu \hat{q}^b \, ,$$

where $R^a{}_b$ is a constant 'field rotation matrix'. Such solutions are non-BPS, and will be discussed in a separate section. Once either of these condition is imposed, the terms in the second line combine into one term, which, however, has a similar structure as a term within the square in the third line. The most general ansatz only requires that the second and third line vanish in combination, while a more restricted ansatz requires that the second and third line vanish independently. The restricted ansatz corresponds to static solutions, because imposing that the third line vanishes is equivalent to the vanishing of the field strength of the Kaluza-Klein vector. Without this restriction, we obtain stationary rotating solutions. We will refer to solutions obtained from our isotropic ansatz as isotropic solutions. Note that they will in general neither be BPS (since we admit a field rotation matrix), nor extremal (since rotating solutions are over-extremal).

As in the five-dimensional case [47], we will be able to demonstrate that the equations of motion can be reduced to decoupled harmonic equations by choosing suitable 'dual' coordinates. Therefore the solution will be given in terms of a set of harmonic functions. We will also see that this way we naturally obtain the generalized stabilization equations of four-dimensional black holes, in their algebraic and manifestly symplectic form.

6.1 Equations of motion

We will now derive all the field equations of the Lagrangian (5.7) and show explicitly how they are solved by imposing Bogomol'nyi equations.

Firstly, we perform the variation of the equation (5.7) with respect to the field q^a to obtain the equation of motion

$$2\nabla^\mu\Big[\tilde{H}_{ab}\partial_\mu q^b\Big] - \partial_a\tilde{H}_{bc}\Big(\partial_\mu q^b\partial^\mu q^c - \partial_\mu\hat{q}^b\partial^\mu\hat{q}^c\Big) + 2\nabla^\mu\left[\frac{1}{H^2}q^c\Omega_{ca}\Big(q^d\Omega_{de}\partial_\mu q^e\Big)\right] +$$

$$-2\partial_a\left(\frac{1}{H}q^c\right)\left[\Omega_{cb}\partial_\mu q^b\frac{1}{H}\Big(q^d\Omega_{de}\partial_\mu q^e\Big) - 2\Omega_{cb}\partial_\mu\hat{q}^b\frac{1}{H}\Big(q^d\Omega_{de}\partial_\mu\hat{q}^e\Big)\right] \quad (6.1)$$

$$-\partial_a\left(\frac{1}{4H^2}\right)\Big(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b\Big)^2 = 0\,.$$

Next, the equation of motion for the $\hat{q}^a$ fields

$$-2\nabla^\mu\Big[\tilde{H}_{ab}\partial_\mu\hat{q}^b\Big] - 4\nabla^\mu\left[\frac{1}{H^2}q^c\Omega_{ca}\Big(q^d\Omega_{de}\partial_\mu\hat{q}^e\Big)\right] + \nabla^\mu\left[\frac{1}{H^2}\hat{q}^b\Omega_{ba}\Big(\partial_\mu\tilde{\phi} + 2\hat{q}^c\Omega_{cd}\partial_\mu\hat{q}^d\Big)\right] \quad (6.2)$$

$$-\frac{1}{H^2}\Omega_{ab}\partial_\mu\hat{q}^b\Big(\partial_\mu\tilde{\phi} + 2\hat{q}^c\Omega_{cd}\partial_\mu\hat{q}^d\Big) = 0\,.$$

The equation of the field $\tilde{\phi}$, which descends from the Kaluza Klein-vector, is given by

$$\nabla^\mu\left[\frac{1}{4H^2}\Big(\partial_\mu\tilde{\phi} + 2\hat{q}^c\Omega_{cd}\partial_\mu\hat{q}^d\Big)\right] = 0\,. \quad (6.3)$$

This equation is nothing but the Bianchi identity for $V_{\mu\nu}$, the field strength of the Kaluza Klein-vector, which allow us to write the field strength in terms of a gauge potential $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. Finally, from the variation of the metric we find the Einstein equations

$$-\tfrac{1}{2}\tilde{R}_{3\mu\nu} - \tilde{H}_{ab}\Big(\partial_\mu q^a\partial_\nu q^b - \partial_\mu\hat{q}^a\partial_\nu\hat{q}^b\Big)$$

$$-\frac{1}{H^2}\Big(q^a\Omega_{ab}\partial_\mu q^b\Big)\Big(q^c\Omega_{cd}\partial_\nu q^d\Big) + \frac{2}{H^2}\Big(q^a\Omega_{ab}\partial_\mu\hat{q}^b\Big)\Big(q^c\Omega_{cd}\partial_\nu\hat{q}^d\Big)$$

$$-\frac{1}{4H^2}\Big(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b\Big)\Big(\partial_\nu\tilde{\phi} + 2\hat{q}^c\Omega_{cd}\partial_\nu\hat{q}^d\Big) = 0\,. \quad (6.4)$$

Dual coordinates. The Hessian matrix $\tilde{H}_{ab}$ allows us to define a natural set of dual coordinates

$$q_a := \tilde{H}_a = -\tilde{H}_{ab}q^b$$

$$= -\frac{H_a}{2H} = \frac{1}{H}\begin{pmatrix} -v_I \\ u^I \end{pmatrix}\,.$$

By the chain rule we find the expression for the derivative of the dual coordinates

$$\tilde{H}_{ab}\partial_\mu q^b = \partial_\mu q_a = \partial_\mu\left[\frac{1}{H}\begin{pmatrix} -v_I \\ u^I \end{pmatrix}\right]\,.$$

The existence of these dual coordinates is critical for obtaining solutions to generic models in terms of harmonic functions. Note that the definition of the dual coordinates is completely analogous to the five-dimensional case [47].

6.2 The isotropic ansatz

A flat three-dimensional geometry requires that the energy-momentum tensor must vanish identically. To achieve this we must impose an appropriate ansatz for the fields, which consists of two distinct parts. The first part of our ansatz is to identify the vectors $\partial_\mu q^a$ and $\partial_\mu \hat{q}^a$ up to an overall sign

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a \ . \tag{6.5}$$

Upon imposing this ansatz the vacuum Einstein equations reduce to

$$\frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^q \Omega_{ab} \partial_\mu q^b \right)^2 = \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right)^2 \ .$$

The second part of our ansatz is now clear: we must make the identification

$$\tfrac{1}{2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right) = q^a \Omega_{ab} \partial_\mu q^b \ , \tag{6.6}$$

where the choice of sign is important. One can interpret this as fixing $\tilde{\phi}$ in terms of other fields which are independent

$$\tilde{\phi} = 2(q^a \mp \hat{q}^a) \Omega_{ab} q^b \ .$$

Note that our first ansatz means that $q^q \mp \hat{q}^a$ is a constant in spacetime. By construction the ansätze (6.5) and (6.6) solve the Einstein equations with a flat spacetime metric. This means that the scalar fields take values in a totally isotropic submanifold of the target space of the non-linear sigma model described by the Lagrangian (5.7).

Next, we need to consider the effect this ansatz has on the other equations of motion. Firstly, from the $\tilde{\phi}$ equation of motion we find the condition

$$\nabla^\mu \left[\frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right) \right] = 0 \ . \tag{6.7}$$

Turning our attention to the q^a equation of motion, we see that the second term will drop out, the third line will simplify, and due to (6.7) the derivative in the second line will only act on q^c. We are left with

$$2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] + \frac{2}{H^2} \partial_\mu q^c \Omega_{ca} \left(q^d \Omega_{de} \partial_\mu q^e \right) + 2\partial_a \left(\frac{1}{H} q^c \right) \Omega_{cb} \partial_\mu q^b \frac{1}{H} \left(q^d \Omega_{de} \partial_\mu q^e \right)$$
$$-2\partial_a \left(\frac{1}{H} \right) q^c \Omega_{cb} \partial_\mu q^b \frac{1}{H} \left(q^d \Omega_{de} \partial_\mu q^e \right) = 0 \ .$$

The fourth term then cancels with the derivative acting on the Hesse potential in the third term

$$2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] + \left(\frac{2}{H^2} \partial_\mu q^c \Omega_{ca} + 2\frac{1}{H^2} \Omega_{ab} \partial_\mu q^b \right) \left(q^d \Omega_{de} \partial_\mu q^e \right) = 0 \ .$$

Since Ω_{ab} is antisymmetric the second term cancels with the third term, and writing the first term in terms of the dual coordinates q_a we are finally left with

$$\Delta q_a = 0 \ . \tag{6.8}$$

This is the Laplace equation for the dual coordinates q_a, with respect to the flat Euclidean three-dimensional metric. Solutions are given by harmonic functions.

Now let us consider the $\hat{q}^a$ equation of motion. From (6.7) we see that the derivative in the second term will only act on q^c, and the second and third term will simplify to give

$$-2\nabla^\mu\left[\tilde{H}_{ab}\partial_\mu\hat{q}^b\right] - \left(\frac{2}{H^2}\partial_\mu q^c\Omega_{ca} + \frac{2}{H^2}\Omega_{ab}\partial_\mu q^b\right)\left(q^d\Omega_{de}\partial_\mu q^e\right) = 0 \,.$$

The second and third term cancel due to antisymmetry of Ω_{ab}, and we again get the Laplace equation on the dual coordinates (6.8).

Let us finally check that the solutions to the q and $\hat{q}$ equations of motions are consistent with the $\tilde{\phi}$ equation of motion (6.7). Using the identity $q^a\Omega_{ac} = -\frac{1}{4}H_a\Omega^{ab}H_{bc}$, we can write the l.h.s. of (6.7) in terms of dual coordinates as

$$
\begin{aligned}
\nabla^\mu\left[\frac{1}{H^2}\left(q^a\Omega_{ab}\partial_\mu q^b\right)\right] &= -\tfrac{1}{4}\nabla^\mu\left[\frac{1}{H^2}\left(H_a\Omega^{ab}H_{bc}\partial_\mu q^c\right)\right] \\
&= -\nabla^\mu\left[\tilde{H}_a\Omega^{ab}\tilde{H}_{bc}\partial_\mu q^c\right] \\
&= -\nabla^\mu\left[q_a\Omega^{ab}\partial_\mu q_b\right] \\
&= -q_a\Omega^{ab}\Delta q_b \,.
\end{aligned}
$$

It is clear that for solutions satisfying the Laplace equation the r.h.s. will vanish. We conclude that upon imposing our ansätze all equations of motion reduce to the Laplace equation on the dual coordinates (6.8).

When rewriting the isotropic ansatz (6.5) in terms of four-dimensional quantities one recovers a well known relation which for four-dimensional BPS solutions follows from supersymmetry. First, it is useful to note that the three-dimensional scalars $\hat{q}^a = \frac{1}{2}(\zeta^I, \tilde{\zeta}_I)$ are related to four-dimensional field strength by

$$\partial_\mu\zeta^I = F_{\mu 0}^I \,, \quad \partial_\mu\tilde{\zeta}_I = G_{I|\mu 0} \,.$$

While the first relation holds by definition, the second requires one to combine and manipulate various of the relations in this section. The above relations show that the scalar fields $\zeta^I, \tilde{\zeta}_I$ can be interpreted as electro-static potentials for the field strength and Hodge-dual field strength. Combining this with $(q^a) = \frac{1}{2}\left(Y^I + \bar{Y}^I, F_I(Y) + \bar{F}_I(\bar{Y})\right) = \frac{1}{2}e^{\phi/2}\left(X^I + \bar{X}^I, F_I(X) + \bar{F}_I(\bar{X})\right)$, the isotropic ansatz (6.5) becomes

$$\partial_\mu(e^{\phi/2}(X^I + \bar{X}^I)) = \pm F_{\mu 0}^I = \pm(F_{\mu 0}^{I|+} + F_{\mu 0}^{I|-}) \,, \tag{6.9}$$

$$\partial_\mu(e^{\phi/2}(F_I + \bar{F}_I)) = \pm G_{I|\mu 0} = \pm(G_{I|\mu 0}^+ + G_{I|\mu 0}^-) \,. \tag{6.10}$$

Thus the isotropic ansatz implies that the real part of the symplectic vector (X^I, F_I) is proportional to the gauge potentials. For supersymmetric solutions this follows from the gaugino variation, see for example [11], while here we obtain it as the Bogomol'nyi equation associated to the first line of (5.7).

– 28 –

Remarks on the local U(1) symmetry. The ansatz (6.5), (6.6) for stationary BPS solutions breaks the manifest local U(1) invariance of the equations of motion. This is obvious since we equate quantities which transform under the U(1) to quantities which don't. In other words the ansatz implicitly fixes the U(1) gauge. Since symplectic covariance and the global isometries are respected by the ansatz, the gauge fixing respects these symmetries. Moreover, once the equations of motion have been solved, we can specify the gauge fixing condition explicitly. Re-writing the solutions $q_a = \mathcal{H}_a$, where $\mathcal{H}_a$ are harmonic functions, in terms of the complex variables, this becomes

$$e^{-\phi}(Y^I - \bar{Y}^I) = -i\mathcal{H}^I , \quad e^{-\phi}(F_I - \bar{F}_I) = -i\mathcal{H}_I , \tag{6.11}$$

where $\mathcal{H}^I, \mathcal{H}_I$ are harmonic functions. Rewriting this in terms of the original four-dimensional fields X^I, we obtain

$$e^{-\phi/2}(X^I - \bar{X}^I) = -i\mathcal{H}^I , \quad e^{-\phi/2}(F_I - \bar{F}_I) = -i\mathcal{H}_I . \tag{6.12}$$

Using the D-gauge $-i(X^I\bar{F}_I - F_I\bar{X}^I) = 1$, we can verify that

$$X^I\mathcal{H}_I - F_I\mathcal{H}^I = e^{-\phi/2} . \tag{6.13}$$

This relation is clearly not U(1) invariant and can be viewed as the U(1) gauge fixing implied by our ansatz. It reflects that the fields Y^I only correspond to $2n + 1$ independent scalar fields. This missing real scalar, the dualized Kaluza-Klein vector $\tilde{\phi}$, is determined by its own equation of motion. Note that we could not gauge fix the U(1) by a symplectically invariant condition of the form (6.13) without imposing part of the field equations. As explained in section 3 a condition of this type forces the fields to be orthogonal to the U(1) action. Since this distribution is the contact distribution of the Sasakian, it is not integrable, and cannot be used to realize $\bar{N}$ as a hypersurface in the Sasakian (or the (para-)quaternion-Kähler manifold as a hypersurface in the principal bundle parametrized by $q^a, \hat{q}^a, \tilde{\phi}$). However, solutions to the field equations correspond to maps into lower-dimensional submanifolds of the scalar target space, and integral manifolds of lower dimension do exist.

We remark that there is an alternative description which allows to keep the U(1) invariance manifest and effectively decouples the U(1) gauge degree of freedom. As in [11], one can modify the definition of Y^I as

$$Y^I = e^{\phi/2}\bar{h}X^I , \tag{6.14}$$

where h is a phase factor which transforms with the same charge under U(1) as X^I. Note that in [11] a different convention is used, which corresponds in our notation to taking $Y^I = e^{-\phi/2}\bar{h}X^I$, instead of the above relation. This only alters how e^ϕ depends on the independent coordinates Y^I, but has no baring on our discussion. When comparing to [11], note that the Kaluza-Klein scalar e^ϕ is related to the functions f, g used there by $e^\phi = e^{-2f} = e^{2g}$.

The effect of the modified definition (6.14) is that Y^I is now a U(1) invariant field. Due to the degeneracy of g_{IJ} and the homogeneity properties of the functions involved, this

modification does not change the calculations presented above. In particular, no derivative term for the field h is generated. When rewriting (6.11), by replacing the U(1) invariant variables Y^I by the original variables X^I, which are subject to U(1) transformations, we obtain:

$$e^{-\phi/2}(\bar{h}X^I - h\bar{X}^I) = -i\mathcal{H}^I , \quad e^{-\phi/2}(\bar{h}F_I - h\bar{F}_I) = -i\mathcal{H}_I ,$$

as in [11], except for a different normalization of the Y^I.[12] Using again the D-gauge $-i(X^I\bar{F}_I - F_I\bar{X}^I) = 1$, this implies

$$X^I\mathcal{H}_I - F_I\mathcal{H}^I = he^{-\phi/2} ,$$

which determines the compensating phase h for our solution.

Remarks on attractor behaviour and gradient flow equations. The equations (6.11), (6.12) are the well known black hole attractor equations. To be precise the term attractor equations is applied in the literature to both the equations which determine the values of the scalars on the horizon, and to the more general equations which determine the scalars globally in terms of harmonic functions. Here we have recovered the global version, the horizon version can be obtained by taking the near horizon limit. The equations (6.11) are algebraic equations, and they are symplectically covariant. Another formulation of the attractor equations takes the form of gradient flow equations driven by a so-called 'fake superpotential' [63–65]. Most of the literature on gradient flow equations focuses on spherically symmetric solutions and uses the physical scalars z^i, so that the resulting equations are not symplectically covariant. Recently the BPS equations for four-dimensional $\mathcal{N} = 2$ gauge theories were reformulated, using the Hesse potential, in symplectically covariant form, for general non-spherical solutions [66].

Our formalism by-passes the gradient flow equations and we directly obtain solutions in terms of harmonic functions. While we leave a comprehensive discussion of the relation between our approach and gradient flow equations for future work, we would like to expand a little on the discussion given in [47], where we observed that the field equation can be recast in first order form. One way of re-writing the second order equations of motion into first order form is to rewrite the Lagrangian as a (possibly alternating) sum of squares. This can be done systematically within our formalism, as follows. Upon inspection of the Lagrangian (5.7) we see that the second and third line are already written as the sum of square terms. We then only need to consider the first line, which we can write as

$$\begin{aligned}
\tilde{H}_{ab}(\partial_\mu q^a \partial^\mu q^b - \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b) = {} & \tilde{H}_{ab}(\partial_\mu q^a \pm \tilde{H}^{ac}\partial_\mu \mathcal{H}_c)(\partial^\mu q^b \pm \tilde{H}^{bd}\partial^\mu \mathcal{H}_d) \\
& - \tilde{H}_{ab}(\partial_\mu \hat{q}^a - \tilde{H}^{ac}\partial_\mu \mathcal{H}_c)(\partial^\mu \hat{q}^b - \tilde{H}^{bd}\partial^\mu \mathcal{H}_d) \\
& + \text{Total derivatives} ,
\end{aligned}$$

where $\mathcal{H}_a$ are harmonic functions. In the spherically symmetric case one can dimensionally reduce the Lagrangian to one dimension, where derivatives of harmonic functions are just constants, which can be identified with the conserved charges carried by the solution. One

---

[12]And, of course, in the present paper we do not consider higher derivative terms.

then obtains gradient flow equations, which are driven by the central charge in the super-symmetric case and by a fake superpotential in general. We refer to [47] for a discussion of the spherically symmetric case and proceed without imposing spherical symmetry.

The first part $\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$ of the isotropic ansatz can be seen as imposing that the squares displayed above vanish. The second part of the isotropic ansatz matches the remaining squares, which appear with a relative sign difference, and, hence, the sum of all squares vanishes. The reduces the field equations of the three-dimensional scalars to first order equations, which become the usual flow equations upon imposing spherical symmetry. By eliminating the fields $\hat{q}^a$ by their equations of motion, we are left with (generalized) flow equations for the fields q^a, which are the four-dimensional scalars combined with the Kaluza-Klein scalar, i.e. a component of the four-dimensional metric.

When we instead eliminate the harmonic functions, we recover the isotropic ansatz. We can also make contact with relations recently found in [66] by contracting

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a \ .$$

with $q_a = \tilde{H}_a$. Then the left-hand side is related to the gradient of the Hesse potential,

$$\partial_\mu \tilde{H} = \tilde{H}_a \partial_\mu q^a = q_a \partial_\mu q^a \ ,$$

while the right-hand side is

$$\begin{aligned} q_a \partial_\mu \hat{q}^a &= e^{-\phi/2} \left((\mathrm{Im} F_I(X)) \partial_\mu \zeta^I - (\mathrm{Im} X^I) \partial_\mu \tilde{\zeta}_I \right) \\ &= e^{-\phi/2} \left((\mathrm{Im} F_I(X)) F^I_{\mu 0} - (\mathrm{Im} X^I) G_{I|\mu 0} \right) \ . \end{aligned}$$

This can be related to the expression for the graviphoton in terms of real coordinates by Hodge-dualizing the field strength

$$q_a \partial_\mu \hat{q}^a = +\frac{1}{2} \epsilon_{0\mu\nu\rho} e^{-\phi/2} \left(\mathrm{Im} F_I \tilde{F}^{I|\nu\rho} - \mathrm{Im} X^I \tilde{G}_I^{\nu\rho} \right) = \frac{1}{4} e^{-\phi/2} \epsilon_{0\mu\nu\rho} T^{\nu\rho} \ .$$

Thus we obtain a relation between the gradient of the Hesse potential and the magnetic components of the graviphoton, or, equivalently, the electric components of the Hodge-dual of the graviphoton

$$\partial_\mu \tilde{H} = \pm \frac{1}{4} e^{-\phi/2} \epsilon_{0\mu\nu\rho} T^{\nu\rho} = \pm \frac{1}{2} e^{-\phi/2} \tilde{T}_{0\mu} \ .$$

This relation appears to be the local analogue of an equation for the gradient of the Hesse potential recently found in [66] for BPS dyons in rigid $\mathcal{N} = 2$ theories. As the unique symplectically invariant contraction between scalars and gauge fields, the graviphoton plays the role of the central charge vector field used in [66].

6.3 Rotating solutions

We now have an ansatz for finding stationary isotropic solutions (flat 3d metric) to completely generic models in terms of the dual coordinates. However, in order to write down these solutions explicitly in terms of the four-dimensional fields one must disentangle them

from the dual coordinates. This is equivalent to solving the generalised stabilisation equations, and is not always possible in closed form. In this section we will discuss solutions which lift to rotating over-extremal solutions in four-dimensions, with the STU model as an explicit example. These solutions are characterised by axial symmetry and the requirement that they are asymptotic to Minkowski space at infinity.

The results of the previous section show that upon imposing our isotropic ansatz (6.5) and (6.6), the equations of motion reduce to $\Delta q_a = 0$, and solutions are given in terms of the dual coordinates by harmonic functions

$$q_a = \frac{1}{H} \begin{pmatrix} -v_I \\ u^I \end{pmatrix} = \begin{pmatrix} -\mathcal{H}_I \\ \mathcal{H}^I \end{pmatrix} = \mathcal{H}_a \ . \tag{6.15}$$

We wish to disentangle the four-dimensional metric from this solution, and show that it corresponds to a rotating solution. We can do this by retracing our steps in the dimensional reduction procedure to find

$$g_{\mu\nu} = \delta_{\mu\nu} \ , \qquad e^\phi = -2H \ , \qquad \partial_\mu V_\nu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho} \left(\mathcal{H}_I \partial^\rho \mathcal{H}^I - \mathcal{H}^I \partial^\rho \mathcal{H}_I \right) \ . \tag{6.16}$$

The first equation is trivial; the second is model dependent and we will look into it in more detail later. For now let us focus on the third equation, or more accurately set of equations. These are entirely independent of the details of the model, i.e. choice of prepotential. Following the method for producing rotating isotropic solutions used in [53, 54], we impose that solutions are axially symmetry about the coordinate φ in an oblate spheroidal coordinate system, defined by

$$\begin{aligned} x &= \sqrt{r^2 + \alpha^2} \sin\theta \cos\varphi \ , \\ y &= \sqrt{r^2 + \alpha^2} \sin\theta \sin\varphi \ , \\ z &= r \cos\theta \ . \end{aligned}$$

The (flat) three-dimensional Euclidean metric is given in these coordinates by

$$ds_3^2 = \left(\frac{r^2 + \alpha^2 \cos^2\theta}{r^2 + \alpha^2} \right) dr^2 + (r^2 + \alpha^2 \cos^2\theta) d\theta^2 + (r^2 + \alpha^2) \sin^2\theta d\varphi^2 \ .$$

In this coordinate system the third set of equations in (6.16) become

$$\frac{1}{(r^2 + \alpha^2)\sin\theta} \partial_\theta V_\varphi = \tfrac{1}{2} \left(\mathcal{H}_I \partial_r \mathcal{H}^I - \mathcal{H}^I \partial_r \mathcal{H}_I \right) \ , \tag{6.17}$$

$$-\frac{1}{\sin\theta} \partial_r V_\varphi = \tfrac{1}{2} \left(\mathcal{H}_I \partial_\theta \mathcal{H}^I - \mathcal{H}^I \partial_\theta \mathcal{H}_I \right) \ . \tag{6.18}$$

Since solutions should be asymptotically flat, we must require that $\partial_{[\mu} V_{\nu]} \longrightarrow 0$ as $r \longrightarrow \infty$. We will come back to this shortly. Single-centred harmonic functions in oblate spheroidal coordinates can be written as

$$\begin{aligned} \mathcal{H}^I &= h^I + \frac{p^I r + m^I \alpha \cos\theta}{R} \ , \\ \mathcal{H}_I &= h_I + \frac{q_I r + m_I \alpha \cos\theta}{R} \ , \end{aligned}$$

where $R = r^2 + \alpha^2 \cos^2 \theta$. It is understood that $(h^I, h_I, m^I, m_I, p^I, q_I)$ are all independent integration constants. While h^I, h_I determine the values of the scalars at infinity and p^I, q_I are the magnetic and electric charges, m^I, m_I are the dipole momenta [54]. In [54] a restricted class of harmonic functions was considered, which corresponds to switching off half of the integration constants appearing in the expressions above. This restricted class of solutions was taken in order to satisfy the condition that the field strength of the U(1) connection vanishes. In our formalism it is clear that we do not need impose this condition to produce solutions.

Integrating the equations (6.17) and (6.18) we find an explicit model independent expression for the only non-zero component of the KK-vector

$$V_\varphi = \frac{1}{2}(h_I p^I - h^I q_I) \cos\theta \left(\frac{r^2 + \alpha^2}{R}\right) + \frac{\alpha}{2}(m_I h^I - m^I h_I) \sin^2\theta \left(\frac{r}{R}\right)$$
$$+ \frac{\alpha}{4}(m_I p^I - m^I q_I) \sin^2\theta \left(\frac{1}{R}\right) + C \,, \qquad (6.19)$$

where C is an arbitrary constant. We observe that all three independent symplectic constructions of the vectors (h^I, h_I), (p^I, q_I) and (m^I, m_I) of integration constants appear in this expression. The term in the second line is the angular momentum of the black hole, while $n = \frac{1}{2}(h_I p^I - h^I q_I)$ is the NUT charge, as can be seen by comparison with [67, 68]. The term in the third line does not carry a particular name, but is known to occur in rotating solutions [67]. For static solutions all these terms are absent, which beside $m^I = m_I = 0$ imposes the constraint $h_I p^I - h^I q_I = 0$ on the integration constants. Note that upon imposing this condition the KK-vector reduces to

$$V_\varphi = \frac{\alpha \sin^2\theta}{R} \left[\tfrac{1}{2}\Big(m_I h^I - m^I h_I\Big)r + \tfrac{1}{4}\Big(m_I p^I - m^I q_I\Big)\right] \,. \qquad (6.20)$$

Since this is proportional to α it will vanish in the static limit. In the general case V_φ does note vanish for $r \to \infty$ unless we impose $h_I p^I - h^I p_I = 0$ (and $C = 0$). However, since the field strength $\partial_{[\mu} V_{\nu]}$ goes to zero, such a term could be eliminated by a coordinate transformation. In addition to requiring the KK-vector to vanish asymptotically, we also need to ensure the KK-scalar behaves appropriately, i.e. $e^\phi \longrightarrow 1$ as $r \longrightarrow \infty$. This will place one more restriction on the integration constants (h^I, h_I). Since the KK-scalar is a model dependent field we will need to look at specific examples if we wish to write this constraint explicitly.

The formula for the ADM mass for axially symmetric solutions is given by

$$16\pi M_{\text{ADM}} = 2 \oint_{S^2_\infty} d^2\Sigma^r e^{-\phi} \partial_r \phi \,.$$

Expanding in descending orders of r we have

$$d^2\Sigma^r = (r^2 + \mathcal{O}(r)) \sin\theta \, d\theta d\varphi \,, \qquad e^{-\phi} = 1 + \mathcal{O}\left(\frac{1}{r}\right) \,.$$

Computing the ADM mass one finds a particularly simple dependence on the Hesse potential

$$M_{\text{ADM}} = -\lim_{r\to\infty} r^2 \partial_r \tilde{H} \ .$$

We would now like to investigate the relation between the mass and central charge. For solutions with vanishing NUT charge one has $r^2 q^a \Omega_{ab} \partial_r q^b \to 0$ asymptotically, which implies that $r^2 q_a \Omega^{ab} \partial_r q_b \to 0$ asymptotically. We can then write the mass as

$$
\begin{aligned}
M_{\text{ADM}} &= \lim_{r\to\infty} r^2 \Big(q^a - iH\Omega^{ab} q_b \Big) \partial_r q_a \ , \\
&= \lim_{r\to\infty} \big| X^I q_I - F_I p^I \big| = \lim_{r\to\infty} |\mathcal{Z}| \ .
\end{aligned}
\tag{6.21}
$$

This confirms that these solutions are BPS.

Before we enter into a discussion of specific models, we need to make a few comments about this class of rotating solutions. It contains the rotating supersymmetric solutions of [54], which are not black holes but have naked singularities. As is well known, for rotating four-dimensional solutions the extremality bound is higher than the supersymmetric mass bound, so that rotating supersymmetric solutions are necessarily singular. Besides the ring singularity at $r = 0$, a non-vanishing NUT charge can introduce further singularities [68]. We also remark that time-independence might imply further constraints on the allowed charges [62, 66]. Due to such constraints and the presence of naked singularities, the physical relevance of these rotating solutions is not immediately clear, in contrast to the static solutions to be considered later. For us they are interesting for technical reasons, because they show how rotating solutions can be obtained within the framework of dimensional reduction over time. To obtain physically relevant rotating solutions without naked singularity our method needs to be extended to solutions which take values along non-isotropic submanifolds. This is similar to the problem of deforming static extremal into non-extremal black holes, and both problems will be addressed in future work. We conclude this section by giving the explicit solution for the STU model.

6.3.1 The *STU* model

For the STU model we can find solutions explicitly in closed form. The model is characterised by the prepotential

$$F = -\frac{Y^1 Y^2 Y^3}{Y^0} \ .$$

The name *STU*-models derives from the conventional notation $S, T, U = \frac{Y^i}{Y^0}$, $i = 1, 2, 3$ for the physical scalars. The corresponding Hesse potential is given in terms of the imaginary parts of Y^I, F_I by

$$H = -2\sqrt{-(u^I v_I)^2 + d_{ABC} u^B u^C d^{ADE} v_D v_e + 4u^0 v_1 v_2 v_3 - 4v_0 u^1 u^2 u^3} \ , \tag{6.22}$$

where $d_{ABC} = |\epsilon_{ABC}|$. A detailed derivation of this expression is given in appendix A.1.

Rotating isotropic solutions to this model correspond to taking $\frac{1}{H}u^I = \mathcal{H}^I$ and $\frac{1}{H}v_I = \mathcal{H}_I$. Using the expression $e^\phi = -2H$ we can write the KK-scalar for the STU model explicitly in terms of harmonic functions

$$e^{-\phi} = \sqrt{-(\mathcal{H}^I\mathcal{H}_I)^2 + d_{ABC}\mathcal{H}^B\mathcal{H}^C d^{ADE}\mathcal{H}_D\mathcal{H}_e + 4\mathcal{H}^0\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3 - 4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3} .$$

In order that the solution is asymptotically Minkowski space we must impose a constraint on the integration constants

$$-(h^I h_I)^2 + d_{ABC}h^B h^C d^{ADE}h_D h_e + 4h^0 h_1 h_2 h_3 - 4h_0 h^1 h^2 h^3 = 1 .$$

At first glance it also appears that the KK-vector (6.19) will not vanish asymptotically, as is required for Minkowski space. However, since the field strength of the KK-vector vanishes asymptotically we can make a change of coordinates so that spacetime is Minkowski.

For completeness, let us remark on the remaining four-dimensional fields for this solution. The original complex scalar fields are given by

$$X^I = e^{-\frac{\phi}{2}}Y^I , \qquad \bar{X}^I = e^{-\frac{\phi}{2}}\bar{Y}^I , \qquad (6.23)$$

where Y^I are given in terms of u^I, v_I through

$$Y^0 = \frac{1}{U + \bar{U}}\left(2u^3 + i2u^0\bar{U}\right) , \qquad Y^1 = \frac{1}{U + \bar{U}}\left(-2v_2 + i2u^1\bar{U}\right) ,$$
$$Y^2 = \frac{1}{U + \bar{U}}\left(-2v_1 + i2u^2\bar{U}\right) , \qquad Y^3 = iUY^0 , \qquad (6.24)$$

with

$$U = i\frac{v_0 u^0 + v_1 u^1 + v_2 u^2 - v_3 u^3}{2\left(v_3 u^0 + u^1 v^1\right)}$$
$$\pm \sqrt{\frac{v_1 v_2 - v_0 u^3}{v_3 u^0 + u^1 u^2} - \frac{(v_0 u^0 + v_1 u^1 + v_2 u^2 - v_3 u^3)^2}{4(v_3 u^0 + u^1 u^2)^2}} . \qquad (6.25)$$

These expressions have been adapted from similar expressions derived in [78]. One can substitute $u^I = -\frac{1}{2}e^{\phi/2}\mathcal{H}^I$ and $v_I = -\frac{1}{2}e^{\phi/2}\mathcal{H}_I$ to obtain the solution explicitly in terms of harmonic functions. The gauge fields are given by the expressions (6.9), (6.10).

7 Static solutions

7.1 General discussion

When we impose that solutions are static and not only stationary, the isotropic ansatz provides us with extremal black hole solutions. This class is therefore of imminent physical importance. Static backgrounds are characterised by a vanishing KK-vector $V_\mu = 0$, which in dualised fields corresponds to

$$\frac{1}{2H}\left(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b\right) = 0 .$$

To obtain static solutions we will impose precisely the same isotropic ansatz as for stationary solutions, but in order to link to previous work we will reverse the order in which we apply the two parts of the ansatz. We first impose only the second part of the isoptropic ansatz (6.6), which in this case is simply

$$q^a \Omega_{ab} \partial_\mu q^b = \pm q^a \Omega_{ab} \partial_\mu \hat{q}^b = \tfrac{1}{2} \left(\partial_\mu \tilde{\phi} + 2 \hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right) = 0 \; . \tag{7.1}$$

It is then clear that the equations of motion simplify considerably. Only the first line of each equation is relevant, and we are have left with

$$\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] - \tfrac{1}{2} \partial_a \tilde{H}_{bc} \left(\partial_\mu q^b \partial^\mu q^c - \partial_\mu \hat{q}^b \partial^\mu \hat{q}^c \right) = 0 \; , \tag{7.2}$$

$$\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu \hat{q}^b \right] = 0 \; , \tag{7.3}$$

$$\tilde{H}_{ab} \left(\partial_\mu q^a \partial_\nu q^b - \partial_\mu \hat{q}^a \partial_\nu \hat{q}^b \right) = -\tfrac{1}{2} \tilde{R}_{3\mu\nu} \; . \tag{7.4}$$

The equation of motion corresponding to the KK-vector is clearly solved automatically. The effective action for these equations is given by the first line of (5.7)

$$\tilde{\mathcal{L}}_3 \sim -\tfrac{1}{2} \tilde{R}_3 - \tilde{H}_{ab} \left(\partial_\mu q^a \partial^\mu q^b - \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b \right) \; .$$

The equations of motion (7.2), (7.3) and (7.4) take precisely the same form as when one reduces five-dimensional vector-multiplets over a timelike dimension in static, purely electric backgrounds. Both isotropic and non-isotropic solutions have been found in this case, and can be shown to lift to electrically charged extremal black holes [47] and non-extremal black holes respectively [49]. In order to obtain non-isotropic solutions one must modify (6.5), the part of our ansatz that relates $\partial_\mu q$ and $\partial_\mu \hat{q}$, by a universal 'non-extremality' factor. In this case the three-dimensional spacetime metric is no longer flat but conformally flat. The machinery for producing these non-isotropic solutions takes a slightly different form than in the isotropic case, and for that reason we will not consider these solutions in this paper. We remark that is possible to use the techniques established in [49] to produce non-isotropic solutions which lift to non-extremal black holes in four-dimensions, which we have found for particular models, but we leave a detailed discussion of this topic to future work.

In order to produce isotropic solutions to these equations of motion in flat three-dimensional backgrounds we must again impose the ansatz

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a \; . \tag{7.5}$$

It is clear by inspection that in this case all equations of motion reduce to the Laplace equation for the dual coordinates

$$\Delta q_a = 0 \; .$$

In this case condition (7.1) places one constraint on the integration constants of q_a.

The formula for the ADM mass for is given by

$$16\pi M_{\text{ADM}} = 2 \oint_{S^2_\infty} d^2\Sigma^\mu e^{-\phi} \partial_\mu \phi \; .$$

Since $e^\phi \to 1$ at spatial infinity we can write this as

$$M_{\text{ADM}} = -\frac{1}{4\pi} \oint_{S^2_\infty} d^2\Sigma^\mu \partial_\mu \tilde{H} \ ,$$

Using the fact that the NUT charge vanishes $q^a \Omega_{ab} \partial_\mu q^b = 0$, which implies that $q_a \Omega^{ab} \partial_\mu q_b = 0$, we can write this as

$$\begin{aligned}
M_{\text{ADM}} &= \frac{1}{4\pi} \oint_{S^2_\infty} d^2\Sigma^\mu \left(q^a - iH\Omega^{ab} q_b \right) \partial_\mu q_a \ , \\
&= \frac{1}{4\pi} \oint_{S^2_\infty} d^2\Sigma^\mu \left| X^I \partial_\mu \mathcal{H}_I - F_I \partial_\mu \mathcal{H}^I \right| = |\mathcal{Z}_\infty| \ .
\end{aligned} \tag{7.6}$$

These extremal black hole solutions therefore satisfy the BPS bound.

7.2 Examples of extremal black hole solutions

We will now consider explicit solutions to the equations of motion in static backgrounds. We impose the ansätze (7.1) and (7.5) and solutions are again given by harmonic functions, but in this case they are not bound by any symmetry constraints. Solutions correspond to extremal black holes in four-dimensions in the sense they have finite horizons, are asymptotically Minkowski, and saturate a bound on the mass and charge.

We will first consider a class of extremal black hole solutions of the STU model that are obtained by taking the static limit of the rotating solutions discussed in the previous section. We will then present axion-free solutions to a wider class of models which have prepotentials of the form $F(Y) = \frac{f(Y^1,\ldots,Y^n)}{Y^0}$. This class of models includes those that have a 'very special' form, where $f(Y^1,\ldots,Y^n)$ is a homogeneous cubic polynomial. Such models can which be obtained by the dimensional reduction of five-dimensional theories. While axion-free solutions for very special prepotentials are well known [52], our derivation shows that to obtain solutions it is enough to assume that f is homogeneous, and so we can obtain axion-free solutions for a larger class of prepotentials.

We end by giving explicit solutions to models where $f = STU + aU^3$. This is a deformation of the STU-model which is still of the very special form, but the target space is no longer symmetric. The model with $a = \frac{1}{3}$ corresponds to a particular Calabi-Yau compactification and its heterotic dual [79, 80].

7.2.1 The STU model

We first consider the static limit of the rotating solutions found in the previous section. This will give us extremal black hole solutions to the STU model. Taking the static limit amounts to setting $\alpha \to 0$ and imposing the constraint

$$h^I q_I - h_I p^I = 0 \ ,$$

which ensures the KK-vector vanish identically. The dipole momenta m^I, m_I completely vanish from the solution, along with angular momentum and NUT charge, and we are

left with a spherically symmetric configuration. The expression for the KK-scalar remains unchanged, and we obtain the solution

$$e^{-\phi} = \sqrt{-(\mathcal{H}^I \mathcal{H}_I)^2 + (d_{ABC} \mathcal{H}^B \mathcal{H}^C d^{ADE} \mathcal{H}_D \mathcal{H}_e) + 4\mathcal{H}^0 \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 - 4\mathcal{H}_0 \mathcal{H}^1 \mathcal{H}^2 \mathcal{H}^3} \ ,$$

$$g_{\mu\nu} = \delta_{\mu\nu} \ , \qquad V_\mu = 0 \ .$$

where the harmonic functions are given by

$$\mathcal{H}^I = h^I + \frac{p^I}{r} \ ,$$

$$\mathcal{H}_I = h_I + \frac{q_I}{r} \ ,$$

The asymptotic integration constants h^I, h_I satisfy the two constraints

$$-(h^I h_I)^2 + (d_{ABC} h^B h^C d^{ADE} h_D h_e) + 4h^0 h_1 h_2 h_3 - 4h_0 h^1 h^2 h^3 = 1 \ ,$$

$$h_I p^I - h^I q_I = 0 \ .$$

Like in the case for rotating solutions, using the expression (5.6) we can write u^I, v_I explicitly in terms of harmonic functions by

$$u^I = -\tfrac{1}{2} e^\phi \mathcal{H}^I \ , \qquad v_I = -\tfrac{1}{2} e^\phi \mathcal{H}_I. \tag{7.7}$$

The original four-dimensional scalar fields are given by

$$X^I = e^{-\frac{\phi}{2}} Y^I \ ,$$

where Y^I are given in terms of harmonic functions through (6.24) and (7.7). Again, the gauge fields are given by the expressions (6.9), (6.10).

The above extremal black hole solutions of the STU model are spherically symmetric as they were obtained by taking the static limit of axially-symmetric rotating solutions, but this need not be the case in general. If we do not impose any symmetry constraints on spacetime then we will obtain the same expressions for the four-dimensional fields, but with the harmonic functions which are completely general. Multi-centered black hole solutions with centers are at x_α correspond to the choice

$$\mathcal{H}^I = h^I + \sum_\alpha \frac{p_\alpha^I}{|x - x_\alpha|} \ ,$$

$$\mathcal{H}_I = h_I + \sum_\alpha \frac{q_{I\alpha}}{|x - x_\alpha|} \ .$$

7.2.2   Models of the form $F = \frac{f(Y^1,...,Y^n)}{Y^0}$

A class of models for which we can find explicit extremal black hole solutions are those where the prepotential takes the form

$$F(Y) = \frac{f(Y^1, \ldots, Y^n)}{Y^0} \ ,$$

where f is real when evaluated on real fields. Since F is a homogeneous function of degree 2 it follows that f is a homogeneous function of degree 3. If f is in particular a cubic polynomial $f = C_{ABC}Y^AY^BY^C$ with real C_{ABC}, then this is of the 'very special' form which derives from five-dimensional supergravity by reduction.

We will consider a restricted set of solutions that are characterised by the requirement that Y^A are purely imaginary and Y^0 is purely real, which implies that the four-dimensional scalars $Z^A = Y^A/Y^0$ are purely imaginary. For very special prepotentials, where the real part of Z^A corresponds to a five-dimensional gauge potential, this means that such solutions are 'axion-free.' For $f = C_{ABC}Y^AY^BY^C$ it follows that F_0 is imaginary while F_A are real. If we replace $C_{ABC}Y^AY^BY^C$ by a general homogeneous function f of degree three this remains true only if we impose that f is real when evaluated on real fields Y^A (and, by homogeneity, imaginary when evaluated on imaginary fields Y^A). Therefore we , we impose this condition in the following, and 'axion-free solutions' are characterized by the consistent reality condition on the fields which impose that Y^0 and F_A are purely real while Y^A and F_0 are purely imaginary. In terms of real variables this corresponds to imposing that

$$x^1 = \ldots = x^n = y_0 = 0 , \tag{7.8}$$

which defines a particular submanifold of the scalar manifold. In terms of dual coordinates (7.8) is equivalent to

$$v_1 = \ldots = v_n = u^0 = 0 . \tag{7.9}$$

With the above assumptions we can write Y^I, F_I in terms of the dual real coordinates as

$$Y^0 = \lambda , \qquad F_0 = iv_0 ,$$
$$Y^A = iu^A , \quad F_A = -\frac{f_A(u^1, \ldots, u^n)}{\lambda} , \tag{7.10}$$

where

$$\lambda = -\sqrt{\frac{f(u^1, \ldots, u^n)}{v_0}} ,$$

and $f_A = \frac{\partial f}{\partial Y^A}$. Using (4.2) and (5.6) we obtain expressions for the KK-scalar and Hesse potential (evaluated on axion-free configurations)

$$e^\phi = -2H = -i(Y^I\bar{F}_I - F_I\bar{Y}^I) = 8\sqrt{v_0 f(u^1, \ldots, u^n)} .$$

The real parts of Y^I, F_I can be read off from (7.10) as

$$x^0 = \lambda , \qquad y_A = \frac{f_A(u^1, \ldots, u^n)}{\lambda} .$$

This amounts to solving the generalised stabilisation equations, and is the reason why we can find solutions explicitly in closed form.

Solutions to these models are given in terms of harmonic functions by

$$e^{-\phi} = \sqrt{4\mathcal{H}_0 f(\mathcal{H}^1, \ldots, \mathcal{H}^n)} ,$$
$$g_{\mu\nu} = \delta_{\mu\nu} , \qquad V_\mu = 0 .$$

The harmonic functions are given by

$$\mathcal{H}_0 = h_0 + \sum_\alpha \frac{q_{0\alpha}}{|x - x_\alpha|} \;, \qquad \mathcal{H}^A = h^A + \sum_\alpha \frac{p_\alpha^A}{|x - x_\alpha|} \;,$$

with $\mathcal{H}_A = \mathcal{H}^0 = 0$.

The asymptotic integration constants h^I, h_I must satisfy only one constraint

$$4h_0 f(h^1, \ldots, h^n) = 1 \;.$$

We can write v_0, u^A explicitly in terms of harmonic functions by

$$v_0 = -\tfrac{1}{2} e^\phi \mathcal{H}_0 \;, \qquad u^A = -\tfrac{1}{2} e^\phi \mathcal{H}^A \;. \tag{7.11}$$

The original four-dimensional scalar fields are given by

$$X^I = e^{-\frac{\phi}{2}} Y^I \;,$$

which can be written in terms of harmonic functions using (7.10) and (7.11). The gauge fields are given by the expressions (6.9), (6.10).

7.2.3 The $STU + aU^3$ model

We now turn to a specific one-parameter family of models of the form $F = \frac{f(Y^1, Y^2, Y^3)}{Y^0}$, which are characterised by the prepotential

$$F(Y) = -\frac{Y^1 Y^2 Y^3 + a(Y^1)^3}{Y^0} \;.$$

This is a deformation of the STU-model where the target space is no longer symmetric. Specialising to solutions with $x^1 = x^2 = x^3 = 0$ and $y_0 = 0$ we have

$$
\begin{aligned}
Y^0 &= \lambda \;, & F_0 &= iv_0 \;, \\
Y^1 &= iu^1 \;, & F_1 &= \frac{u^2 u^3 + 3a(u^1)^2}{\lambda} \;, \\
Y^2 &= iu^2 \;, & F_2 &= \frac{u^1 u^3}{\lambda} \;, \\
Y^3 &= iu^3 \;, & F_3 &= \frac{u^1 u^2}{\lambda} \;,
\end{aligned}
\tag{7.12}
$$

where

$$\lambda = -\sqrt{-\frac{u^1 u^2 u^3 + a(u^1)^3}{v_0}} \;.$$

Solutions are given in terms of harmonic functions by

$$e^{-\phi} = \sqrt{-4\mathcal{H}_0(\mathcal{H}^1 \mathcal{H}^2 \mathcal{H}^3 + a(\mathcal{H}^1)^3)} \;,$$
$$g_{\mu\nu} = \delta_{\mu\nu} \;, \qquad V_\mu = 0 \;,$$

where the harmonic functions are again defined to be

$$\mathcal{H}_0 = h_0 + \sum_\alpha \frac{q_{0\alpha}}{|x - x_\alpha|} \ , \qquad \mathcal{H}^A = h^A + \sum_\alpha \frac{p_\alpha^A}{|x - x_\alpha|} \ ,$$

with $\mathcal{H}_A = \mathcal{H}^0 = 0$. The asymptotic integration constants h^I, h_I must satisfy the constraint

$$-4h_0(h^1 h^2 h^3 + a h^{1^3}) = 1 \ .$$

We can write v_0, u^A explicitly in terms of harmonic functions by

$$v_0 = -\tfrac{1}{2} e^\phi \mathcal{H}_0 \ , \qquad u^A = -\tfrac{1}{2} e^\phi \mathcal{H}^A \ . \tag{7.13}$$

The original four-dimensional scalar fields can be determined through the expressions

$$X^I = e^{-\frac{\phi}{2}} Y^I \ ,$$

which one can write explicitly in terms of harmonic functions using (7.12) and (7.13). The gauge fields are given by the expressions (6.9), (6.10).

7.3   Field rotations and non-BPS solutions

Four-dimensional extremal non-BPS have been studied in the past [69–71], and more recently there has been increased interest in this topic, starting from [72–74]. As in the five-dimensional case [47], the ansatz $\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$ with a universal sign does not necessarily exhaust all solutions. To obtain further solutions we can adapt the observation that new solutions can be generated by flipping signs of charges [70], or, more generally, by 'rotating charges' [63, 64]. BPS solutions correspond to particular combinations of signs, while other choices lead to non-BPS solutions.

As we have seen above the ansatz $\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$ leads to BPS solutions. For static solutions we can use the same generalization of the ansatz as in five-dimensions [47] and introduce a constant field rotation matrix

$$\partial_\mu q^a = R^a{}_b \partial_\mu \hat{q}^b \ . \tag{7.14}$$

This is the analogue of 'rotating charges' in our framework. By inspection of the field equations, we find that this ansatz only works if the following compatibility condition between the scalar metric and the field rotation matrix holds

$$\tilde{H}_{ab} R^a{}_c R^b{}_d = \tilde{H}_{cd} \ . \tag{7.15}$$

If this condition is satisfied, then the solution for the dual scalar fields $q_a, \hat{q}_a$ is again given by harmonic functions, but now the harmonic functions for q_a are related to those for $\hat{q}_a$ through the constant matrix $R_a{}^b$, which is the transposed of the inverse of $R^a{}_b$:

$$\partial_\mu q_a = \partial_\mu \mathcal{H}_a = R_a{}^b \partial_\mu \hat{q}_b = R_a{}^b \partial_\mu \hat{H}_b \ , \quad R^a{}_b R_a{}^c = \delta_b^c \ .$$

Equivalently, the relations (6.9) and (6.10) between four-dimensional scalars and gauge fields are modified by the presence of this matrix. Decomposing the field rotation matrix $R^{T,-1}$ into blocks

$$R^{T,-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ,$$

the expressions for the gauge fields become

$$\partial_\mu(e^{\phi/2}(X^J + \bar{X}^J))A_J{}^I + \partial_\mu(e^{\phi/2}(F_J + \bar{F}_J))C^{JI} = \pm(F_{0\mu}^{I|+} + F_{0\mu}^{I|-}) , \tag{7.16}$$

$$\partial_\mu(e^{\phi/2}(X^J + \bar{X}^J))B_{JI} + \partial_\mu(e^{\phi/2}(F_J + \bar{F}_J))D^J{}_I = \pm(G_{I|0\mu}^+ + G_{I|0\mu}^-) . \tag{7.17}$$

In particular, the electric and magnetic charges appear rotated relative to the solutions of the scalar fields. Note that in general not only the charges but also the asymptotic behaviour of solutions changes [75]. This is necessary in order to avoid introducing naked singularities.[13]

The presence of a non-trivial field rotation matrix also modifies the ADM mass (7.6):

$$M_{\text{ADM}} = \frac{1}{4\pi} \oint_{S_\infty^2} d^2\Sigma^\mu \left| X^I \left(A_I{}^J \partial_\mu \hat{\mathcal{H}}_J + B_{IJ} \partial_\mu \hat{\mathcal{H}}^J \right) - F_I \left(C^{IJ} \partial_\mu \hat{\mathcal{H}}_J + D^I{}_J \partial_\mu \hat{\mathcal{H}}^J \right) \right| .$$

This makes it manifest that such solutions are not BPS. Note that we saw above that the $R = \pm\text{Id}$ leads precisely to the relation between four-dimensional scalars and gauge fields which is implied by the BPS condition.

Since a field rotation matrix only provides a solution if the compatibility condition (7.15) is satisfied, it is in general not clear that non-BPS solutions can be obtained by this ansatz. For symmetric spaces non-BPS solutions can be obtained in a systematic way using group-theoretical methods [36, 37]. For non-symmetric target spaces these methods do not apply, and therefore it is interesting to ask under which conditions one can guarantee the existence of a non-trivial field rotation matrix which satisfies (7.15).

Geometrically, this is equivalent to the problem of identifying totally geodesic, totally isotropic submanifolds of the scalar target space. We have seen that for c-map spaces there is a universal solution, given by the ansatz $\partial_\mu q^a = \pm\partial_\mu \hat{q}^a$, which corresponds to BPS solutions. Finding non-BPS solutions amounts to finding further such submanifolds, which correspond to the non-BPS branches that one can identify in symmetric target spaces by group theoretical methods.

In the following section we establish that a non-trivial field rotation matrix exists for non-axionic solutions of models with a prepotential of the form $F = \frac{f(Y^1,...,Y^n)}{Y^0}$ where f is real when evaluated on real fields, i.e. for the class of examples considered above. Before we turn to the details, we remark that we do not only need to impose a condition on the model (i.e. on the form of the prepotential), but also on the field configurations, by restricting to axion-free solutions. This corresponds to restricting to lower-dimensional submanifolds of the scalar manifold. If no such restriction is imposed, the compatibility condition (7.15) implies that the field rotation matrix acts by an isometry. Requiring the existence of such an isometry imposes a condition on the prepotential. By restricting to

---

[13]We thank the authors of [75] for bringing this to our attention.

field configurations which are axion free, the compatibility condition (7.15) becomes less restrictive and we can establish the existence of a field rotation matrix under much milder assumptions on the form of the prepotential. But the resulting totally geodesic, totally isotropic submanifold corresponding to the axion-free non-BPS solution is of lower dimension than the submanifold corresponding to BPS solutions, which has maximal dimension. It would be interesting to clarify whether this is a generic feature of non-BPS solutions in models with non-symmetric target spaces.

Finally, we mention that in the rotating case one cannot simply adapt the isotropic ansatz in the same way when a field rotation matrix is available, as this no longer produces a solution to the equations of motion. In order to produce non-BPS rotating solutions one needs to relax the condition that three-dimensional metric is flat. We will not consider such solutions in this paper, and leave the investigation of such solutions to future work.

7.3.1 Non-BPS solutions to $F = \frac{f(Y^1,...,Y^n)}{Y^0}$ models

For models with prepotentials of the form $F = \frac{f(Y^1,...,Y^n)}{Y^0}$, where f is real when evaluated on real fields, there always exists a non-trivial field rotation matrix for solutions satisfying the conditions (7.8). For the remainder of this section we will focus on the specific case where $n = 3$, but the solutions can be extended to arbitrary $n \geq 1$ without loss of generality.

To see why a field rotation matrix always exists for this class of models we must analyze the matrix $\tilde{H}_{ab}$ in some detail. Firstly, one observes that the conditions (7.8) imply that the matrix $\tilde{H}_{ab}$ decomposes into

$$
\tilde{H}_{ab} = \left(\begin{array}{c|cccc|ccc}
* & 0 & 0 & 0 & 0 & * & * & * \\
\hline
0 & * & * & * & * & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 \\
0 & * & * & * & * & 0 & 0 & 0 \\
\hline
* & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & * & * & *
\end{array}\right) , \tag{7.18}
$$

where a $*$ represents a possible non-zero entry. To see why this is the case, consider, for example, the matrix element $\tilde{H}_{10}$. Let us denote by $\sharp$ the restriction of solutions to (7.8). We can write $\tilde{H}_{10}$ as

$$
\begin{aligned}
\tilde{H}_{10}\Big|_{\sharp} &= \left(\frac{\partial}{\partial x^0} \frac{\partial \tilde{H}}{\partial x^1}\right)\Bigg|_{\sharp} \\
&= \frac{\partial}{\partial x^0} \left(\frac{\partial \tilde{H}}{\partial x^1}\Bigg|_{\sharp}\right) \\
&= \frac{\partial}{\partial x^0}(0) = 0 .
\end{aligned}
$$

In the second line we used that the variable x^0 does not enter into the axion-free condition $\sharp$, which amounts to setting other variables to constant (zero) values. Therefore we can

– 43 –

take the derivative with respect to x^0 after imposing the axion free condition $\sharp$. In the third line we used the fact that $\frac{\partial \tilde{H}}{\partial x^1} = \frac{-v_1}{H}$. This is valid irrespective of the condition $\sharp$ by definition of the dual coordinates. The same argument is true for any matrix element containing one index in $\{0, 5, 6, 7\}$ and one index in $\{1, 2, 3, 4\}$.

When expressed in terms of u^I, v_I, the axion free ansatz (7.8) implies that $v_1 = v_2 = v_3 = u^0 = 0$. Consequently, the corresponding harmonic functions vanish $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}^0 = 0$, and the central 4×4 block appearing in (7.18) completely decouples from the equations of motion, and is of no relevance to the remaining discussion.

Actually, the matrix $\tilde{H}_{ab}$ decomposes even further. Using the formula for the Hesse potential (A.6) for this class of solution, which is derived in appendix A.2 one observes that $\tilde{H}_{ab}$ takes the more restrictive form

$$
\tilde{H}_{ab} = \begin{pmatrix}
\frac{1}{4(x^0)^2} & 0\ 0\ 0\ 0 & 0\ 0\ 0 \\
\hline
0 & *\ *\ *\ * & 0\ 0\ 0 \\
0 & *\ *\ *\ * & 0\ 0\ 0 \\
0 & *\ *\ *\ * & 0\ 0\ 0 \\
0 & *\ *\ *\ * & 0\ 0\ 0 \\
\hline
0 & 0\ 0\ 0\ 0 & *\ *\ * \\
0 & 0\ 0\ 0\ 0 & *\ *\ * \\
0 & 0\ 0\ 0\ 0 & *\ *\ *
\end{pmatrix} , \tag{7.19}
$$

where the entries in the bottom-right block depend only on y_1, y_2, y_3.

For such solutions, these modes always admit a non-trivial field rotation matrix of the form

$$
R^a{}_b = \pm \begin{pmatrix}
-1 & 0 \\
\hline
0 & \mathbb{I}_{2n+1}
\end{pmatrix} . \tag{7.20}
$$

One can therefore find non-BPS solutions to these models generically.

7.3.2 Non-BPS solutions to $STU + aU^3$ model

Since this model falls into the category of $F = \frac{f(Y^1, Y^2, Y^3)}{Y^0}$ it admits the non-trivial field rotation matrix given by (7.20), and we can obtain non-BPS solutions.

The non-BPS solutions are given explicitly by

$$
e^{-\phi} = \sqrt{4\mathcal{H}_0(\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3 + a(\mathcal{H}^1)^3)} ,
$$
$$
g_{\mu\nu} = \delta_{\mu\nu} , \qquad V_\mu = 0 ,
$$

where the harmonic functions are again given by

$$
\mathcal{H}_0 = h_0 + \sum_\alpha \frac{q_{0\alpha}}{|x - x_\alpha|} , \qquad \mathcal{H}^A = h^A + \sum_\alpha \frac{p_\alpha^A}{|x - x_\alpha|} ,
$$

with $\mathcal{H}_A = \mathcal{H}^0 = 0$. The asymptotic integration constants h^I, h_I satisfy the constraint

$$
4h_0(h^1 h^2 h^3 + a(h^1)^3) = 1 .
$$

We can write v_0, u^A explicitly in terms of harmonic functions by

$$v_0 = \tfrac{1}{2} e^\phi \mathcal{H}_0 \ , \qquad u^A = -\tfrac{1}{2} e^\phi \mathcal{H}^A \ . \tag{7.21}$$

The original four-dimensional scalar fields can be determined through the expressions

$$X^I = e^{-\frac{\phi}{2}} Y^I \ ,$$

which one can write explicitly in terms of harmonic functions using (7.12) and (7.21). The expressions for the gauge fields remain unchanged, and are given by (6.9), (6.10).

8 Conclusions and outlook

In this paper we have shown how four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to supergravity can be described in terms of a real formulation of special Kähler geometry using the gauge equivalence with conformal supergravity. Key technical points, which allowed us to preserve symplectic covariance, were to avoid U(1) gauge fixing, and the use of the degenerate metric obtained by integrating out the auxiliary U(1) gauge field. Geometrically this corresponds to working on the Sasakian S or the conical affine special Kähler manifold N, and to use a horizontal lift for the metric. We expect that this formulation will be useful for studying non-holomorphic corrections.

By dimensional reduction we have obtained a new formulation of the supergravity c-map, which is complementary to other existing formulations and offers new insights into the geometry as well as practical advantages for some types of problems. In our formulation the local c-map comes very close to the Sasaki form of the rigid r- and c-map, and of the local r-map. It is manifestly symplectically invariant with respect to both vector and hypermultiplets, it is completely formulated in terms of real variables, and it provides a simple and explicit expression for the quaternion-Kähler metric in terms of the Hesse potential. We have introduced a new geometrical object, a principal U(1) bundle over the quaternion-Kähler manifold, and work with the horizontal lift of the metric to the total space of this bundle. We are currently investigating the deeper geometrical interpretation of our results and expect that this will be useful for understanding the dynamics of hypermultiplets in string compactifications. One obvious question is the relation of our construction to the hyper-Kähler cone and twistor space, which could lead to a more complete picture of the c-map, hypermultiplets, and black hole and instanton solutions.

When applied to the temporal version of the c-map, the new parametrization makes it easy to find instanton solutions which are restricted to totally isotropic submanifolds. By dimensional lifting we have obtained extremal black holes and over-extremal rotating solutions. Since the equations of motion are reduced to decoupled harmonic equations, multi-centered solutions can be obtained as easily as single centered ones. The flexibility in choosing harmonic functions at the very end is an advantage of the method, which was further illustrated by constructing rotating solutions. Since the method does not rely on Killing spinors it is not restricted to supersymmetric solutions. The black hole attractor equations and other relations known from supersymmetric solutions are derived from geometric properties of the scalar manifold and take a manifestly symplectically covariant

form. For static extremal solutions we thus obtain a full generalization of the previous results on five-dimensional black holes [47].

While the canonical version $\partial_\mu q^a = \pm \partial_\mu \hat{q}^a$ of the ansatz always works and leads to BPS solutions, non-BPS solutions can be obtained if a non-trivial field rotation matrix exists, which must satisfy a compatibility condition with the metric. For non-symmetric target spaces the existence of such a matrix is non-trivial, but we were able to show that it exists for axion-free solutions for a class of prepotentials, which contains the very special ones as a subclass. An interesting future direction is to develop the understanding of non-BPS solutions for non-symmetric target spaces. Since symmetric spaces are contained in our formalism as special cases, one promising strategy is to translate the group-theoretical characterisations of BPS and non-BPS solutions into geometrical properties of totally geodesic, totally isotropic submanifolds and then to investigate whether these conditions have natural generalizations for non-symmetric spaces.

Another direction is the generalization to non-extremal static black holes, which for the five-dimensional case was discussed in [49], and, more recently, in [76]. Deforming extremal into non-extremal solutions corresponds to deforming isotropic into non-isotropic submanifolds. It is currently not clear to us to which extent this can be done in a universal way. However, specific examples suggest that our method can be generalized, and we plan to report on this in a future publication. Non-extremal four-dimensional black holes in $\mathcal{N} = 2$ supergravity have been recently discussed in [75] from a different though related point of view. For $\mathcal{N} = 4$ supergravity the full class of stationary point-like solutions is known [77].

We have also shown how rotating solutions can be obtained, and recovered the known rotating supersymmetric solutions. In this case the use of field rotation matrices to produce non-BPS solutions requires to generalize the ansatz and to admit a curved three-dimensional base space. Moreover, these solutions have naked singularities, and making non-singular will also require to go beyond the isotropic ansatz considered in the second part of this paper.

Acknowledgments

A   Hesse potentials

A.1   Hesse potential for STU model

In this section we derive the Hesse potential for the STU model. Due to the relation between the Hesse potential and BPS black hole entropy [14], this is equivalent to solving the attractor equations. However, the relation between Hesse potential and prepotential is

'off-shell', and does not require to impose a particular background solution. Therefore we find it instructive to present the derivation in a form where this is manifest. Technically we closely follow [52], but instead of charges and horizon values of fields we use fields without imposing supersymmetry or any of the field equations.

The STU model in special complex coordinates is characterised by the holomorphic prepotential

$$F(Y) = -\frac{Y^1 Y^2 Y^3}{Y^0} \ .$$

Introducing the inhomogeneous coordinates $Z^A = Y^A / Y^0$ one can write the Kähler potential for the STU model as

$$e^{-\mathcal{K}} = -i \left(Y^I \bar{F}_I - F_I \bar{Y}^I \right) = 8 Y^0 \bar{Y}^0 \text{Im}(Z^1) \text{Im}(Z^2) \text{Im}(Z^3) \ . \tag{A.1}$$

Our strategy will be to write the individual fields Y^0, Z^1, Z^2, Z^3 in terms of $x^I = \text{Re}(Y^I)$ and $y_I = \text{Re}(F_I)$.

Firstly, by direct calculations one can show that

$$-\bar{Z}^2 \bar{Z}^3 = \frac{y_1 Z^1 + y_0}{x^0 Z^1 - x^1} \ , \qquad \bar{Z}^2 = \frac{x^2 Z^1 + y_3}{x^0 Z^1 - x^1} \ , \qquad \bar{Z}^3 = \frac{x^3 Z^1 + y_2}{x^0 Z^1 - x^1} \ .$$

Combining these three expressions one gets the quadratic equation for Z^1:

$$(Z^1)^2 + \frac{y.x - 2y_1 x^1}{y_1 x^0 + x^2 x^3} Z^1 + \frac{y_2 y_3 - y_0 x^1}{y_1 x^0 + x^1 x^3} = 0 \ ,$$

where $y.x = y_I x^I$. Solving this we find an expression for Z^1 purely in terms of x^I, y_I:

$$Z^1 = -\frac{y.x - 2y_1 x^1}{2(y_1 x^0 + x^2 x^3)} \pm i \frac{\sqrt{W}}{2(y_1 x^0 + x^2 x^3)} \ ,$$

where

$$W = -(y.x)^2 + 4y_1 x^1 y_2 x^2 + 4y_1 x^1 y_3 x^3 + 4y_2 x^2 y_3 x^3 + 4x^0 y_1 y_2 y_3 - 4y_0 x^1 x^2 x^3 \ .$$

By identical calculations, or simply by noting the symmetry between Z^1, Z^2, Z^3, we obtain similar expressions for Z^2, Z^3:

$$Z^2 = -\frac{y.x - 2y_2 x^2}{2(y_2 x^0 + x^1 x^3)} \pm i \frac{\sqrt{W}}{2(y_2 x^0 + x^1 x^3)} \ ,$$

$$Z^3 = -\frac{y.x - 2y_3 x^3}{2(y_3 x^0 + x^1 x^2)} \pm i \frac{\sqrt{W}}{2(y_3 x^0 + x^1 x^2)} \ .$$

Next, again by direct calculation one obtains the expression

$$\bar{Y}^0 = -\frac{2(x^0 Z^1 - x^1)}{Z^1 - \bar{Z}^1} \ ,$$

and, hence,

$$Y^0 \bar{Y}^0 = \frac{1}{W} \left(x^{0^2} W + \left(x^0 (y.x) + 2x^1 x^2 x^3 \right)^2 \right) \ .$$

Also by direct calculation one can show that

$$(y_1 x^0 + x^2 x^3)(y_2 x^0 + x^1 x^3)(y_3 x^0 + x^1 x^2) = \tfrac{1}{4}\left((x^0)^2 W + \left(x^0(y.x) + 2x^1 x^2 x^3\right)^2\right) \, .$$

Substituting the above expressions into (A.1), we obtain

$$e^{-\mathcal{K}} = \pm 4 W^{1/2} \, .$$

We now restrict ourselves to physically relevant configurations, where the r.h.s. is strictly positive. Since $H = -\tfrac{1}{2} e^{-\mathcal{K}}$ we can write the Hesse potential explicitly in terms of x^I, y_I as

$$H(x,y) = -2\Big(- (y.x)^2 + 4y_1 x^1 y_2 x^2 + 4y_1 x^1 y_3 x^3 + 4y_2 x^2 y_3 x^3$$
$$+ 4x^0 y_1 y_2 y_3 - 4y_0 x^1 x^2 x^3 \Big)^{1/2} \, . \tag{A.2}$$

One can use a similar procedure to determine the Hesse potential in terms of the imaginary parts of Y^I, F_I, which we denote by $u^I = \mathrm{Im}(Y^I)$ and $v_I = \mathrm{Im}(F_I)$. What one obtains is precisely the same expression:

$$H(u,v) = -2\Big(- (v.u)^2 + 4v_1 u^1 v_2 u^2 + 4v_1 u^1 v_3 u^3 + 4v_2 u^2 v_3 u^3$$
$$+ 4u^0 v_1 v_2 v_3 - 4v_0 u^1 u^2 u^3 \Big)^{1/2} \, . \tag{A.3}$$

The reason why we obtain the same result is that the Hesse potential is independent of the phase of Y^I, i.e. it is invariant under U(1) transformations $Y^I \to e^{i\alpha} Y^I$. The imaginary parts of Y^I, F_I are simply the real parts of $e^{-i\pi/2} Y^I, e^{-i\pi/2} F_I$, which describe the same Hesse potential.

A.2 Hesse potential for models of form $F = \frac{f(Y^1,\dots,Y^n)}{Y^0}$

We now extend the previous discussion to models with a prepotential of the form

$$F(Y) = \frac{f(Y^1,\dots,Y^n)}{Y^0} \, .$$

Since F is a homogeneous function of degree 2 it follows that f is homogeneous function of degree 3. In this case it is not possible to obtain an expression for the Hesse potential in closed form. However, one can still show that the Hessian metric $\tilde{H}_{ab}$ takes the from (7.19) when restricting to axion-free field configurations (7.8). The part of the Hessian metric relevant for this subspace can consistently be obtained by setting half of the variables of the Hesse potential to zero, and for this truncated Hesse potential we can obtain an explicit expression.

Recall the definition of x^I, y_I and u^I, v_I:

$$x^I + iu^I := Y^I = \begin{pmatrix} Y^0 \\ Y^1 \\ \vdots \\ Y^n \end{pmatrix},$$

$$y_I + iv_I := F_I = \begin{pmatrix} -\frac{f(Y^1,...,Y^n)}{Y^{02}} \\ \frac{f_1(Y^1,...,Y^n)}{Y^0} \\ \vdots \\ \frac{f_n(Y^1,...,Y^n)}{Y^0} \end{pmatrix}.$$

We will now impose the conditions (7.8), which restrict us to the particular class of solutions for which Y^A are purely imaginary, Y^0 is purely real and F_0 is purely imaginary. In this case the fields x^I, y_I can be given explicitly in terms of u^I, v_I by

$$\begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \\ y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{f_1(u^1,...,u^n)}{\lambda} \\ \vdots \\ -\frac{f_n(u^1,...,u^n)}{\lambda} \end{pmatrix}, \tag{A.4}$$

where

$$\lambda = -\sqrt{\frac{f(u^1, \ldots, u^n)}{v_0}} .$$

One must choose the negative sign in the expression for λ in order to ensure that the Hesse potential is strictly negative. The Kähler potential can be written as

$$e^{-\mathcal{K}} = -i \left(Y^I \bar{F}_I - F_I \bar{Y}^I \right) = 8\sqrt{v_0 f(u^1, \ldots, u^n)} ,$$

and since $e^{-\mathcal{K}} = -2H$ we have the following explicit expression for the Hesse potential in terms of u^I, v_I:

$$H(u,v) = -4\sqrt{v_0 f(u^1, \ldots, u^n)} . \tag{A.5}$$

We would now like to find an equivalent expression for the Hesse potential in terms of x^I, y_I. Here we cannot use the same trick of making a U(1) rotation as in the STU model, since imposing the conditions (7.8) implicitly brakes the U(1) invariance of the system. Geometrically the condition selects a lower dimensional hypersurface which no longer has this U(1) isometry.

Finding an explicit expression for the Hesse potential in terms of x^I, y_I would involve inverting the relations (A.4), which in general cannot be calculated in closed form. However,

we will now show that the Hesse potential can be consistently restricted to the subspace of axion-free solutions, where it separates into two distinct factors:

$$H(x, y) = \sqrt{x^0} \, h(y_1, \ldots, y_n) \,, \qquad (A.6)$$

where h is some homogeneous function of degree $3/2$. This property is crucial in demonstrating the existence of non-BPS solutions to such models.

Firstly, on the subspace of axion-free solutions, half of the variables x^I, y_I are zero. We denote the restricted Hesse potential by

$$H(x, y) = H(x^0, y_1, \ldots, y_n) \,.$$

Next, observe that for axion-free field configurations

$$x^0 v_0 = -\sqrt{v_0 f(u^1, \ldots, u^n)} \,, \qquad \text{and} \qquad H = -4\sqrt{v_0 f(u^1, \ldots, u^n)} \,,$$
$$\Rightarrow \qquad H = 4x^0 v_0 \,.$$

Taking partial derivatives with respect to x^0 we find

$$\frac{\partial H}{\partial x^0} = 4v_0 + 4x^0 \frac{\partial v_0}{\partial x^0} \,.$$

Note that it does not make a difference whether we impose the axion-free condition before or after taking derivatives with respect to x^0, because the axion-free condition does not involve this variable. But we know from (2.4) that

$$\frac{\partial H}{\partial x^0} = 2v_0 \,,$$

and, hence,

$$x^0 \frac{\partial v_0}{\partial x^0} = -\frac{1}{2} v_0 \,, \qquad \Rightarrow \quad v_0 = \frac{1}{\sqrt{x^0}} \frac{1}{4} h(y_1, y_2, y_3) \,,$$

for some specific, but as yet undetermined, function h. The restriction of the Hesse potential to axion-free configurations is therefore given by (A.6). This allows us to determine components of $\tilde{H}_{ab}$ which we need to go from (7.18) to (7.19).

References

[1] B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged N = 2 Supergravity: Yang-Mills Models*, *Nucl. Phys.* **B 245** (1984) 89 [INSPIRE].

[2] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in N = 2 supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **B 430** (1994) 485-486] [hep-th/9407087] [INSPIRE].

[3] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N = 2 supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [INSPIRE].

[4] S. Kachru and C. Vafa, *Exact results for N = 2 compactifications of heterotic strings*, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105] [INSPIRE].

[5] S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, *Phys. Lett.* **B 361** (1995) 59 [`hep-th/9505162`] [ɪɴSPIRE].

[6] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of $N = 2$ heterotic string compactifications*, *Nucl. Phys.* **B 459** (1996) 537 [`hep-th/9508155`] [ɪɴSPIRE].

[7] S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ *extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [`hep-th/9508072`] [ɪɴSPIRE].

[8] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [`hep-th/9601029`] [ɪɴSPIRE].

[9] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [`hep-th/9711053`] [ɪɴSPIRE].

[10] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [`hep-th/9812082`] [ɪɴSPIRE].

[11] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions*, *JHEP* **12** (2000) 019 [`hep-th/0009234`] [ɪɴSPIRE].

[12] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [`hep-th/0405146`] [ɪɴSPIRE].

[13] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [`hep-th/0412287`] [ɪɴSPIRE].

[14] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Black hole partition functions and duality*, *JHEP* **03** (2006) 074 [`hep-th/0601108`] [ɪɴSPIRE].

[15] L.J. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of string loop corrections to gauge coupling constants*, *Nucl. Phys.* **B 355** (1991) 649 [ɪɴSPIRE].

[16] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortsch. Phys.* **48** (2000) 49 [`hep-th/9904005`] [ɪɴSPIRE].

[17] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [`hep-th/9906094`] [ɪɴSPIRE].

[18] G. Cardoso, B. de Wit and S. Mahapatra, *Subleading and non-holomorphic corrections to $N = 2$ BPS black hole entropy*, *JHEP* **02** (2009) 006 [`arXiv:0808.2627`] [ɪɴSPIRE].

[19] D.S. Freed, *Special Kähler manifolds*, *Commun. Math. Phys.* **203** (1999) 31 [`hep-th/9712042`] [ɪɴSPIRE].

[20] G. Cardoso, B. de Wit and S. Mahapatra, *Black hole entropy functions and attractor equations*, *JHEP* **03** (2007) 085 [`hep-th/0612225`] [ɪɴSPIRE].

[21] G. Cardoso, B. de Wit and S. Mahapatra, *BPS black holes, the Hesse potential and the topological string*, *JHEP* **06** (2010) 052 [`arXiv:1003.1970`] [ɪɴSPIRE].

[22] V. Cortés, in *Proceedings of the international conference "PDEs, Submanifolds and Affine Differential Geometry". Vol. 57*, Warsaw Poland (2000), B. Opozda, U. Simon and M. Wiehe eds., Banach Center Publications (Polish Academy of Sciences, Institute of Mathematics), Warsaw Poland (2000), pg. 11–16.

[23] D. Alekseevsky, V. Cortés and C. Devchand, *Special complex manifolds*, *J. Geom. Phys.* **42** (2002) 85 [math/9910091] [INSPIRE].

[24] V. Cortés and T. Mohaupt, *Special geometry of Euclidean supersymmetry III: the local r-map, instantons and black holes*, *JHEP* **07** (2009) 066 [arXiv:0905.2844] [INSPIRE].

[25] S. Ferrara and O. Macia, *Real symplectic formulation of local special geometry*, *Phys. Lett.* **B 637** (2006) 102 [hep-th/0603111] [INSPIRE].

[26] T. Mohaupt and O. Vaughan, *Developments in special geometry*, *J. Phys. Conf. Ser.* **343** (2012) 012078 [arXiv:1112.2873] [INSPIRE].

[27] M. Günaydin, G. Sierra and P. Townsend, *The geometry of N = 2 Maxwell-Einstein supergravity and Jordan algebras*, *Nucl. Phys.* **B 242** (1984) 244 [INSPIRE].

[28] J. Bagger and E. Witten, *Matter Couplings in N = 2 Supergravity*, *Nucl. Phys.* **B 222** (1983) 1 [INSPIRE].

[29] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475 [INSPIRE].

[30] S. Ferrara and S. Sabharwal, *Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces*, *Nucl. Phys.* **B 332** (1990) 317 [INSPIRE].

[31] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, *Commun. Math. Phys.* **149** (1992) 307 [hep-th/9112027] [INSPIRE].

[32] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, *Nucl. Phys.* **B 400** (1993) 463 [hep-th/9210068] [INSPIRE].

[33] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *BPS black holes, quantum attractor flows and automorphic forms*, *Phys. Rev.* **D 73** (2006) 084019 [hep-th/0512296] [INSPIRE].

[34] A. Neitzke, B. Pioline and S. Vandoren, *Twistors and black holes*, *JHEP* **04** (2007) 038 [hep-th/0701214] [INSPIRE].

[35] M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, *Quantum Attractor Flows*, *JHEP* **09** (2007) 056 [arXiv:0707.0267] [INSPIRE].

[36] D. Gaiotto, W. Li and M. Padi, *Non-supersymmetric attractor flow in symmetric spaces*, *JHEP* **12** (2007) 093 [arXiv:0710.1638] [INSPIRE].

[37] G. Bossard, Y. Michel and B. Pioline, *Extremal black holes, nilpotent orbits and the true fake superpotential*, *JHEP* **01** (2010) 038 [arXiv:0908.1742] [INSPIRE].

[38] M. Roček, C. Vafa and S. Vandoren, *Hypermultiplets and topological strings*, *JHEP* **02** (2006) 062 [hep-th/0512206] [INSPIRE].

[39] M. Roček, C. Vafa and S. Vandoren, *Quaternion-Kahler spaces, hyperkahler cones, and the c-map*, math/0603048.

[40] D. Robles-Llana, F. Saueressig and S. Vandoren, *String loop corrected hypermultiplet moduli spaces*, *JHEP* **03** (2006) 081 [hep-th/0602164] [INSPIRE].

[41] D. Robles-Llana, M. Roček, F. Saueressig, U. Theis and S. Vandoren, *Nonperturbative corrections to 4D string theory effective actions from* $SL(2, \mathbb{Z})$ *duality and supersymmetry*, *Phys. Rev. Lett.* **98** (2007) 211602 [hep-th/0612027] [INSPIRE].

[42] S. Alexandrov, B. Pioline, F. Saueressig and S. Vandoren, *D-instantons and twistors*, *JHEP* **03** (2009) 044 [arXiv:0812.4219] [INSPIRE].

[43] D. Gaiotto, G.W. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010) 163 [arXiv:0807.4723] [INSPIRE].

[44] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin Systems and the WKB Approximation*, arXiv:0907.3987 [INSPIRE].

[45] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of euclidean supersymmetry. II. Hypermultiplets and the c-map*, *JHEP* **06** (2005) 025 [hep-th/0503094] [INSPIRE].

[46] V. Cortes, T. Mohaupt and H. Xu, *Completeness in supergravity constructions*, arXiv:1101.5103 [INSPIRE].

[47] T. Mohaupt and K. Waite, *Instantons, black holes and harmonic functions*, *JHEP* **10** (2009) 058 [arXiv:0906.3451] [INSPIRE].

[48] T. Mohaupt and K. Waite, *Euclidean Actions, Instantons, Solitons and Supersymmetry*, *J. Phys.* **A 44** (2011) 175403 [arXiv:1011.6301] [INSPIRE].

[49] T. Mohaupt and O. Vaughan, *Non-extremal Black Holes, Harmonic Functions and Attractor Equations*, *Class. Quant. Grav.* **27** (2010) 235008 [arXiv:1006.3439] [INSPIRE].

[50] P. Breitenlohner, D. Maison and G.W. Gibbons, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, *Commun. Math. Phys.* **120** (1988) 295 [INSPIRE].

[51] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. 1. Vector multiplets*, *JHEP* **03** (2004) 028 [hep-th/0312001] [INSPIRE].

[52] K. Behrndt et al., *Classical and quantum $N = 2$ supersymmetric black holes*, *Nucl. Phys.* **B 488** (1997) 236 [hep-th/9610105] [INSPIRE].

[53] E. Bergshoeff, R. Kallosh and T. Ortín, *Stationary axion/dilaton solutions and supersymmetry*, *Nucl. Phys.* **B 478** (1996) 156 [hep-th/9605059] [INSPIRE].

[54] K. Behrndt, D. Lüst and W.A. Sabra, *Stationary solutions of $N = 2$ supergravity*, *Nucl. Phys.* **B 510** (1998) 264 [hep-th/9705169] [INSPIRE].

[55] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *On electromagnetic duality in locally supersymmetric $N = 2$ Yang-Mills theory*, hep-th/9412200 [INSPIRE].

[56] B. de Wit, *N=2 electric-magnetic duality in a chiral background*, *Nucl. Phys. Proc. Suppl.* **49** (1996) 191 [hep-th/9602060] [INSPIRE].

[57] L. Alvarez-Gaume and D.Z. Freedman, *Kahler geometry and the renormalization of supersymmetric σ-models*, *Phys. Rev.* **D 22** (1980) 846.

[58] T. Mohaupt, *Special Geometry, Black Holes and Euclidean Supersymmetry*, in *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, V. Cortés eds., European Mathematical Society Publishing House, Zürich Switzerland (2010) [hep-th/0703037] [INSPIRE].

[59] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortsch. Phys.* **49** (2001) 3 [hep-th/0007195] [INSPIRE].

[60] B. de Wit, *Supergravity*, hep-th/0212245 [INSPIRE].

[61] E. Cremmer et al., *Vector Multiplets Coupled to $N = 2$ Supergravity: SuperHiggs Effect, Flat Potentials and Geometric Structure*, *Nucl. Phys.* **B 250** (1985) 385 [INSPIRE].

[62] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [`hep-th/0005049`] [ɪɴSPIRE].

[63] A. Ceresole and G. Dall'Agata, *Flow equations for non-BPS extremal black holes*, *JHEP* **03** (2007) 110 [`hep-th/0702088`] [ɪɴSPIRE].

[64] G. Lopes Cardoso, A. Ceresole, G. Dall'Agata, J.M. Oberreuter and J. Perz, *First-order flow equations for extremal black holes in very special geometry*, *JHEP* **10** (2007) 063 [`arXiv:0706.3373`] [ɪɴSPIRE].

[65] J. Perz, P. Smyth, T. Van Riet and B. Vercnocke, *First-order flow equations for extremal and non-extremal black holes*, *JHEP* **03** (2009) 150 [`arXiv:0810.1528`] [ɪɴSPIRE].

[66] D. Van den Bleeken, *BPS Dyons and Hesse Flow*, *JHEP* **02** (2012) 067 [`arXiv:1111.6979`] [ɪɴSPIRE].

[67] J. Bellorín, P. Meessen and T. Ortín, *Supersymmetry, attractors and cosmic censorship*, *Nucl. Phys.* **B 762** (2007) 229 [`hep-th/0606201`] [ɪɴSPIRE].

[68] G. Bossard, H. Nicolai and K. Stelle, *Gravitational multi-NUT solitons, Komar masses and charges*, *Gen. Rel. Grav.* **41** (2009) 1367 [`arXiv:0809.5218`] [ɪɴSPIRE].

[69] R.R. Khuri and T. Ortín, *A Nonsupersymmetric dyonic extreme Reissner-Nordstrom black hole*, *Phys. Lett.* **B 373** (1996) 56 [`hep-th/9512178`] [ɪɴSPIRE].

[70] T. Ortín, *Extremality versus supersymmetry in stringy black holes*, *Phys. Lett.* **B 422** (1998) 93 [`hep-th/9612142`] [ɪɴSPIRE].

[71] T. Ortín, *Nonsupersymmetric (but) extreme black holes, scalar hair and other open problems*, `hep-th/9705095` [ɪɴSPIRE].

[72] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Non-supersymmetric attractors*, *Phys. Rev.* **D 72** (2005) 124021 [`hep-th/0507096`] [ɪɴSPIRE].

[73] P.K. Tripathy and S.P. Trivedi, *Non-supersymmetric attractors in string theory*, *JHEP* **03** (2006) 022 [`hep-th/0511117`] [ɪɴSPIRE].

[74] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022 [`hep-th/0510024`] [ɪɴSPIRE].

[75] P. Galli, T. Ortín, J. Perz and C.S. Shahbazi, *Non-extremal black holes of $N = 2$, $D = 4$ supergravity*, *JHEP* **07** (2011) 041 [`arXiv:1105.3311`] [ɪɴSPIRE].

[76] P. Meessen and T. Ortín, *Non-Extremal Black Holes of $N = 2$, $D = 5$ Supergravity*, *Phys. Lett.* **B 707** (2012) 178 [`arXiv:1107.5454`] [ɪɴSPIRE].

[77] E. Lozano-Tellechea and T. Ortín, *The general, duality invariant family of nonBPS black hole solutions of $N = 4$, $D = 4$ supergravity*, *Nucl. Phys.* **B 569** (2000) 435 [`hep-th/9910020`] [ɪɴSPIRE].

[78] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, *STU black holes and string triality*, *Phys. Rev.* **D 54** (1996) 6293 [`hep-th/9608059`] [ɪɴSPIRE].

[79] J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, *Nonperturbative properties of heterotic string vacua compactified on $K3 \times T^2$*, *Nucl. Phys.* **B 480** (1996) 185 [`hep-th/9606049`] [ɪɴSPIRE].

[80] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2.*, *Nucl. Phys.* **B 476** (1996) 437 [`hep-th/9603161`] [ɪɴSPIRE].