

Triality in minimal model holography

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ABSTRACT: The non-linear $\mathcal{W}_\infty[\mu]$ symmetry algebra underlies the duality between the \mathcal{W}_N minimal model CFTs and the $hs[\mu]$ higher spin theory on AdS_3 . It is shown how the structure of this symmetry algebra at the *quantum* level, i.e. for finite central charge, can be determined completely. The resulting algebra exhibits an exact equivalence (a ‘triality’) between three (generically) distinct values of the parameter μ . This explains, among other things, the agreement of symmetries between the \mathcal{W}_N minimal models and the bulk higher spin theory. We also study the consequences of this triality for some of the simplest $\mathcal{W}_\infty[\mu]$ representations, thereby clarifying the analytic continuation between the ‘light states’ of the minimal models and conical defect solutions in the bulk. These considerations also lead us to propose that one of the two scalar fields in the bulk actually has a non-perturbative origin.

KEYWORDS: Field Theories in Lower Dimensions, AdS-CFT Correspondence, Conformal and W Symmetry, $1/N$ Expansion

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1 Introduction and summary

Symmetry plays a very powerful role in the AdS/CFT correspondence. The presence of large symmetries in both the bulk and the boundary can, in some instances, effectively constrain the dynamics so that the equivalence between the two descriptions is largely a consequence of the matching of the symmetries. Such examples, in turn, can help in deciphering the holographic dictionary better. The developments which have uncovered the planar integrability of $\mathcal{N} = 4$ Super Yang-Mills theory and the related integrability of the

string sigma model on $\text{AdS}_5 \times \text{S}^5$ go in this direction. Interestingly, these enlarged symmetries are usually not very manifest and their matching on both sides is a nontrivial fact.

The presence of supersymmetry is usually a necessary prerequisite for these larger symmetries, and most of the well studied examples of AdS/CFT exploit the power of supersymmetry. It has gradually been realised that higher spin symmetries can play an analogous role in effectively constraining the dynamics in non-supersymmetric contexts. These symmetries might essentially govern the vicinity of the tensionless limit of string theory on AdS, or equivalently the weak coupling limit of gauge theories. Even when the symmetries are (mildly) broken they can provide constraints on the form of the correlation functions [1, 2]. While for CFTs in $d > 2$ higher spin symmetries can only be realised exactly in free boson/fermion theories,¹ in $d = 2$ this conclusion may be evaded (as in the Coleman-Mandula theorem). Indeed, there is a large class of interacting two dimensional CFTs which have (holomorphic) conserved currents of arbitrarily high spin. These symmetries usually define so-called \mathcal{W} -algebras. They are generically nonlinear in the sense that the generators do not form a conventional Lie algebra, but that the commutators of two generators can only be expressed in terms of quadratic (or even higher order) products of the generators.

An important step was taken in [3, 4], where it was realised that the asymptotic symmetry algebras of higher spin theories on AdS_3 are such \mathcal{W} -algebras. The particular cases studied in [3, 4] actually belong to a one-parameter family of higher spin theories whose symmetry algebras could be identified with $\mathcal{W}_\infty[\mu]$ [5] (see also [6]). These generalisations of the Brown-Henneaux result opened the possibility of a 2d CFT with \mathcal{W} -symmetry being dual to some higher spin theory on AdS_3 . A concrete proposal was then made in [7], relating the so-called $\mathcal{W}_{N,k}$ family of unitary CFTs (in a certain large N , k 't Hooft limit) to a specific higher spin theory (coupled to additional scalar fields) on AdS_3 . This duality has been further investigated in [8–15].

However, the symmetries on the bulk and boundary theories are not obviously the same, though they are both \mathcal{W} -algebras. At any fixed N , the minimal model CFTs have \mathcal{W}_N symmetry, which corresponds to $\mathcal{W}_\infty[\mu]$ with $\mu = N$; at these values the algebra truncates consistently to one with currents of a maximum spin $s = N$. On the other hand, the bulk theory is based on the $\text{hs}[\lambda]$ higher spin theory with λ identified with the 't Hooft coupling $\lambda = \frac{N}{N+k}$, and its asymptotic symmetry algebra is $\mathcal{W}_\infty[\mu]$ with $\mu = \lambda$ [5]. At first sight this appears to be rather different from the symmetries of a theory with $\mu = N$.

The crucial point, however, is that the Brown-Henneaux like analysis of [3–5] is ‘classical’, i.e. it determines the Poisson brackets of the \mathcal{W} -generators, and is only valid at large $c = \frac{3\ell}{2G_N}$; in order to emphasize this aspect, we shall sometimes denote this *classical* $\mathcal{W}_\infty[\mu]$ algebra by $\mathcal{W}_\infty^{\text{cl}}[\mu]$. When the central charge is finite and Poisson brackets are replaced by commutators, the non-linear nature of $\mathcal{W}_\infty[\mu]$ leads to additional terms arising from the normal ordering of products and the requirement to satisfy the Jacobi identities. As a consequence, the *quantum* $\mathcal{W}_\infty[\mu]$ algebra, $\mathcal{W}_\infty^{\text{qu}}[\mu]$, is a significant deformation of the classical algebra.

¹This conclusion relies on a number of general assumptions, one of which is that the number of degrees of freedom, N , is finite.

Therefore, while the classical algebras $\mathcal{W}_\infty^{\text{cl}}[\mu]$ for $\mu = N$ and $\mu = \lambda$ are certainly not isomorphic, this does not preclude that there exists a non-trivial equivalence of the corresponding quantum \mathcal{W} -algebras. In fact, heuristic considerations [8] based on generalised level-rank dualities for coset CFTs [16, 17] suggest such a relation; this will be reviewed at the beginning of section 2 below. In this paper, we put this equivalence on a firmer footing by giving compelling evidence for a general triad of isomorphic $\mathcal{W}_\infty^{\text{qu}}[\mu]$ algebras for (three generically) different values of μ . As we will see, this triality implies the desired equivalence in the case when one of the values of μ is $\mu = N$.

More specifically, as we shall explain in section 2.1, we can determine the structure of the quantum $\mathcal{W}_\infty^{\text{qu}}[\mu]$ algebra completely, using two constraints. Starting from the classical algebra $\mathcal{W}_\infty^{\text{cl}}[\mu]$, the requirement that the Jacobi identities are satisfied fixes the correct form of the normal ordered products, as well as the finite shifts in the coefficients of the non-linear terms. This determines the algebra up to the form of some structure constants that are only known from $\mathcal{W}_\infty^{\text{cl}}[\mu]$ to leading order in $\frac{1}{c}$. The complete c -dependence of these structure constants can then be determined by requiring that the representation theory of $\mathcal{W}_\infty^{\text{qu}}[\mu]$ matches that of \mathcal{W}_N for $\mu = N$. In general, the resulting quantum algebra does not actually contain $\text{hs}[\mu]$ as a subalgebra; instead $\text{hs}[\mu]$ is only a subalgebra in the $c \rightarrow \infty$ limit.

As it turns out, the quantum algebra $\mathcal{W}_\infty^{\text{qu}}[\mu]$ is more invariantly parametrised in terms of two numbers: these are c , the central charge, and γ , the structure constant which captures the leading nontrivial higher spin coupling (of the spin four current in the OPE of two spin three currents). All other structure constants appear to be fixed in terms of these two parameters. Furthermore, μ is determined by a cubic equation which depends only on (c, γ) , and therefore there are three values of μ which correspond to isomorphic algebras. This effectively proves the quantum equivalence of these three $\mathcal{W}_\infty^{\text{qu}}[\mu]$ algebras.

In section 3 we then go on to study some of the simplest representations of $\mathcal{W}_\infty^{\text{qu}}[\mu]$ which we call minimal representations. These are the representations which have the fewest number of low-lying states. It turns out that there are three of them for fixed values of (c, γ) . We verify that the quantum numbers (dimensions, low spin charges) of these three representations are indeed consistent with the above triality.² For the case of the $\mathcal{W}_{N,k}$ minimal models, the two physical representations correspond to the basic coset primaries labelled as $(0; f)$ and $(f; 0)$ (and their complex conjugates). The fusion of these two representations produces other non-minimal representations such as $(f; f)$ — the lightest of the light states (in the large N 't Hooft limit).

Therefore, in section 4 we revisit these light states and their relation to semi-classical solutions (conical defects) [12] of the bulk $\text{SL}(N)$ higher spin theory. Since our analysis of the minimal representations and their fusion holds for all values of c , we can study the representation theory at fixed N (which then determines $\gamma = \gamma(N, c)$), and hence understand the behaviour of the various representations as a function of c . In particular, we consider the analytic continuation of the $(f; f)$ state from $c = c_{N,k} \leq (N - 1)$ (the value for the minimal models), to the semi-classical regime $c \rightarrow \infty$. In the latter regime,

²Interestingly, one of the three is not a representation of $\text{hs}[\mu]$, even in the large c limit.

its quantum numbers match (to leading order in c) those of the conical defect solution of the bulk theory. We can similarly look at all the other light states $(\Lambda; \Lambda)$ of the CFT and continue the representations (at fixed N) to large c . The expressions for charges and dimensions are smooth functions of c , and their leading behavior matches with those calculated for the conical defects [12]. This shores up the identification of the latter with the light states in a precise analytic continuation in c of representations of $\mathcal{W}_\infty^{\text{qu}}[\mu]$.

If we consider the similar analytic continuation in c (again at fixed N) of the two minimal representations $(0; \mathfrak{f})$ and $(\mathfrak{f}; 0)$, we find that the dimension of the former is proportional to c (for large c) while that of the latter is of order one. Thus the two states are on a different footing, and it appears to be more natural to consider the former as non-perturbative (or solitonic), whereas the latter can be viewed as a perturbative excitation. Therefore, in section 5 we are led to refine the conjecture of [7], and propose that the bulk theory should be considered to be the $\text{hs}[\lambda]$ theory with only *one* complex scalar (with $m^2 = -1 + \lambda^2$), and quantized in the standard way (the + quantization). The other primary corresponding to $(0; \mathfrak{f})$ is to be viewed as an excited state of the lightest conical defect. We believe this alternative picture explains some of the puzzling aspects of the light states and their relation to the perturbative excitations. We summarise the current status of the duality and interesting avenues for further work in section 6.

2 The structure of the $\mathcal{W}_\infty[\mu]$ -algebra

Let us begin by motivating why there should be non-trivial identifications among the quantum $\mathcal{W}_\infty[\mu]$ algebras. Recall that by construction, \mathcal{W}_N agrees with $\mathcal{W}_\infty[\mu]$ for integer $\mu = N$. Indeed $\mathcal{W}_\infty[\mu]$ is the Drinfeld-Sokolov reduction of $\text{hs}[\mu]$, and $\text{hs}[\mu]$ reduces³ to $\mathfrak{su}(N)$ for $\mu = N$, whose Drinfeld-Sokolov reduction is \mathcal{W}_N ; thus we have the relation

$$\mathcal{W}_\infty[\mu]_{\mu=N} \cong \mathcal{W}_N . \tag{2.1}$$

However, there is also a second, somewhat more subtle, relation between $\mathcal{W}_\infty[\mu]$ and \mathcal{W}_N . It was conjectured in [16, 17] that the coset models

$$\mathcal{W}_{N,k} \equiv \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \cong \frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}} \equiv \mathcal{W}_{M,l} \tag{2.2}$$

are related to one another by some sort of level-rank duality, where the relation between the parameters is

$$k = \frac{N}{M} - N, \quad l = \frac{M}{N} - M . \tag{2.3}$$

Here M and N are taken to be positive integers, whereas k and l are fractional (real) numbers, and the central charges of both sides are equal to

$$c_{N,k} \equiv (N-1) \left[1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right] = (M-1) \left[1 - \frac{M(M+1)}{(M+l)(M+l+1)} \right] \equiv c_{M,l} . \tag{2.4}$$

³Strictly speaking, the relation is that $\text{hs}[N]$ contains a large ideal, and that the quotient of $\text{hs}[N]$ by this ideal is equivalent to $\mathfrak{su}(N)$. We will come back to this in section 3.3.

However, it seems reasonable to assume that this level-rank duality will also hold if instead of integer N , M , we consider the situation where N and k are integers. Then we can solve (2.3) for M to obtain

$$M \equiv \lambda = \frac{N}{N+k}, \tag{2.5}$$

while l is determined by the condition that both sides have the same central charge. Next we observe that we have also quite generically that

$$\frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}} \cong \text{Drinfeld-Sokolov reduction of } \mathfrak{su}(M) \text{ at level } \hat{l}, \tag{2.6}$$

where again \hat{l} is determined so as to have the same central charge as the left-hand-side. For non-integer M we can think of

$$\mathfrak{su}(\lambda) \cong \text{hs}[\lambda], \tag{2.7}$$

and the Drinfeld-Sokolov reduction of $\text{hs}[\lambda]$ equals $\mathcal{W}_\infty[\lambda]$. Combining these statements then leads to the claim that we have an isomorphism of algebras

$$\mathcal{W}_{N,k} \equiv \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \cong \mathcal{W}_\infty[\lambda] \quad \text{with } \lambda = \frac{N}{N+k}. \tag{2.8}$$

Here the central charge of $\mathcal{W}_\infty[\lambda]$ is taken to agree with that of $\mathcal{W}_{N,k}$, i.e. with $c_{N,k}$ defined in (2.4). This relation should be true not just in the 't Hooft limit, but also for finite N and k (and hence finite central charge).

Actually, there is a second variant of this relation. The \mathcal{W}_N algebra at level k is identical to the \mathcal{W}_N algebra at level

$$\hat{k} = -2N - k - 1 \tag{2.9}$$

since the central charges of the two algebras agree, i.e. $c_{N,k} = c_{N,\hat{k}}$. Incidentally, this identification has a natural interpretation from the Drinfeld-Sokolov (DS) point of view. Recall that the cosets $\mathcal{W}_{N,k}$ in (2.2) are equivalent to the DS reduction of $\mathfrak{su}(N)$ at level k_{DS} , where the two levels are related as (see e.g. [18] for a review of these matters)

$$\frac{1}{k+N} = \frac{1}{k_{\text{DS}}+N} - 1. \tag{2.10}$$

From the DS point of view, replacing $k \mapsto \hat{k}$ as in (2.9) is equivalent to replacing k_{DS} by \hat{k}_{DS} with

$$\hat{k}_{\text{DS}} + N = \frac{1}{k_{\text{DS}} + N}. \tag{2.11}$$

In terms of the underlying free field description, this corresponds to exchanging (see e.g. [18] or [8, Section 6.2.2]) the roles of α_\pm , i.e. to define $(\hat{\alpha}_+, \hat{\alpha}_-) = (-\alpha_-, -\alpha_+)$. This is an obvious symmetry of the DS reduction under which the representations are related as $\Lambda_+ \leftrightarrow \Lambda_-^*$, see also the discussion in section 4.3.

Thus we can repeat the above analysis with \hat{k} in place of k , to conclude that $\mathcal{W}_{N,k}$ is also equivalent to $\mathcal{W}_\infty[\mu]$ with $\mu = -\frac{N}{N+k+1}$. Altogether this suggests that we have the ‘triatlity’

$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty[\frac{N}{N+k}] \cong \mathcal{W}_\infty[-\frac{N}{N+k+1}] \quad \text{at } c = c_{N,k}. \quad (2.12)$$

In the following we want to give highly non-trivial evidence for this claim. (Actually, as we shall see, a somewhat stronger statement appears to be true in that we need not even assume that N is integer.) In order to discuss these issues, however, we first need to understand the explicit structure of $\mathcal{W}_\infty[\mu]$ in more detail.

2.1 Explicit form of the algebra

We can derive the commutation relations of the quantum $\mathcal{W}_\infty[\mu]$ algebra by starting with the classical $\mathcal{W}_\infty[\mu]$ algebra that can be defined as the asymptotic symmetry algebra of Chern-Simons theory based on $\mathfrak{hs}[\mu]$. The finite c corrections to the non-linear terms can then be determined recursively by solving the Jacobi identities. Using the results of [5] (see also [19]) we have worked this out explicitly for the first few terms, and the resulting commutation relations are given in appendix A. To the order to which we have studied this problem, these considerations fix the entire structure of the commutators completely, except for the full c -dependence of the structure constant C_{33}^4 . Schematically, this is the structure constant appearing in the OPE

$$W \cdot W \sim C_{33}^4 U + \dots, \quad (2.13)$$

where W, U are the spin three and spin four currents respectively. It follows from the analysis of appendix A (see eq. (A.25)) that

$$C_{33}^4 = 8 \sqrt{\frac{1}{5} \frac{\mu^2 - 9}{\mu^2 - 4}} + \mathcal{O}\left(\frac{1}{c}\right). \quad (2.14)$$

In order to determine the full c dependence of this structure constant, we can study the representation theory of the resulting algebra, and demand that it is compatible, for $\mu = N$, with the known results for \mathcal{W}_N ; this is sketched below in section 3.1. Actually, effectively the same analysis was already done in [20, 21] (and later in [22]), leading to

$$(C_{33}^4)^2 \equiv \gamma^2 = \frac{64(c+2)(\mu-3)(c(\mu+3) + 2(4\mu+3)(\mu-1))}{(5c+22)(\mu-2)(c(\mu+2) + (3\mu+2)(\mu-1))}. \quad (2.15)$$

Note that there is a sign ambiguity in the definition of C_{33}^4 since the normalisation convention of [23] is defined by fixing the OPE of the spin s field $W^{(s)}$ with itself

$$W^{(s)} \cdot W^{(s)} \sim \frac{c}{s} \cdot \mathbf{1} + \dots, \quad (2.16)$$

and hence only determines the normalisation of each field up to a sign. The same comment also applies to the other structure constants (see below).

In [20–22] a few of the other low-lying structure constants of $\mathcal{W}_\infty[\mu]$ were also derived; in the conventions of [23] and using our notation they are explicitly equal to (see eqs.

(2.1.25a/b) of [22] as well as [24])

$$C_{33}^4 C_{44}^4 = \frac{48(c^2(\mu^2 - 19) + 3c(6\mu^3 - 25\mu^2 + 15) + 2(\mu - 1)(6\mu^2 - 41\mu - 41))}{(\mu - 2)(5c + 22)(c(\mu + 2) + (3\mu + 2)(\mu - 1))} \quad (2.17)$$

$$(C_{34}^5)^2 = \frac{25(5c + 22)(\mu - 4)(c(\mu + 4) + 3(5\mu + 4)(\mu - 1))}{(7c + 114)(\mu - 2)(c(\mu + 2) + (3\mu + 2)(\mu - 1))} \quad (2.18)$$

$$C_{45}^5 = \frac{15}{8(\mu - 3)(c + 2)(114 + 7c)(c(\mu + 3) + 2(4\mu + 3)(\mu - 1))} C_{33}^4 \times \left[c^3(3\mu^2 - 97) + c^2(94\mu^3 - 467\mu^2 - 483) + c(856\mu^3 - 5192\mu^2 + 4120) + 216\mu^3 - 6972\mu^2 + 6756 \right]. \quad (2.19)$$

These expressions look very complicated, but as we will see momentarily, they actually exhibit a very nice structure.

2.2 Triality in $\mathcal{W}_\infty[\mu]$

Our first observation is that, for fixed c , there are three values of μ (which we label as $\mu_{1,2,3}$) for which the structure constant γ in (2.15) is actually the same. Indeed, for given c and γ , it follows directly from (2.15) that the three values are the roots of the cubic equation

$$(3\tilde{\gamma}^2 - 8)\mu^3 + (\tilde{\gamma}^2(c - 7) + (26 - c))\mu^2 - (4\tilde{\gamma}^2(c - 1) - 9(c - 2)) = 0, \quad (2.20)$$

where we have defined $\tilde{\gamma}^2 = \gamma^2 \frac{5c+22}{64(c+2)}$. Note that the cubic equation does not have a linear term in μ ; thus the three solutions satisfy

$$\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 0, \quad (2.21)$$

which is equivalent to $\sum_{i=1}^3 \frac{1}{\mu_i} = 0$ provided that all $\mu_j \neq 0$.

The analysis from the beginning of this section suggests that actually the full $\mathcal{W}_\infty[\mu]$ algebra should exhibit this triality symmetry, i.e. that *all* structure constants are the same for $\mu_{1,2,3}$. At least for the known structure constants in eqs. (2.17)–(2.19) this is true; one way to see this, is to observe that they can all be expressed in terms of γ and c as

$$C_{44}^4 = \frac{9(c + 3)}{4(c + 2)} \gamma - \frac{96(c + 10)}{(5c + 22)} \gamma^{-1} \quad (2.22)$$

$$(C_{34}^5)^2 = \frac{75(c + 7)(5c + 22)}{16(c + 2)(7c + 114)} \gamma^2 - 25 \quad (2.23)$$

$$C_{45}^5 = \frac{15(17c + 126)(c + 7)}{8(7c + 114)(c + 2)} \gamma - 240 \frac{(c + 10)}{(5c + 22)} \gamma^{-1}. \quad (2.24)$$

Incidentally, the structure of these identities suggests that these higher OPE coefficients are completely determined from C_{33}^4 by the Jacobi identities, and this appears indeed to be true [25]. Since the three values of $\mu_{1,2,3}$ lead to the same value of γ (at a given c),

these structure constants are then also equal for the three values of μ . This is a very strong indication that the quantum $\mathcal{W}_\infty[\mu]$ algebras are actually equivalent for these three (generically distinct) values of μ , i.e. that

$$\mathcal{W}_\infty[\mu_1] \cong \mathcal{W}_\infty[\mu_2] \cong \mathcal{W}_\infty[\mu_3] \quad \text{at fixed } c \tag{2.25}$$

where $\mu_{1,2,3}$ are the roots of the cubic equation (2.20), evaluated for a given γ .

Note that these algebras look very different from the point of view of $\text{hs}[\mu]$ or even at the classical level. In fact, at very large c , eq. (2.20) reduces to a linear equation in μ^2 , and hence reduces to the familiar equivalence between the classical $\mathcal{W}_\infty[\mu]$ algebras for $\pm\mu$ — this property is directly inherited from $\text{hs}[\mu]$. The statement in (2.25) is a very nontrivial generalisation to the quantum level (finite c), where the equivalence is a triality between the three values $\mu_{1,2,3}$. There are three special cases where the cubic equation eq. (2.20) degenerates: for $\mu = 0$ we have $\tilde{\gamma}^2 = \frac{9(c-2)}{4(c-1)}$, and the constant term in (2.20) vanishes. Then $\mu = 0$ is a double zero, and the other solution simply becomes

$$\mathcal{W}_\infty[\mu = 0] \cong \mathcal{W}_\infty[\mu = c + 1] . \tag{2.26}$$

For $\mu = 1$, on the other hand, we have $\tilde{\gamma}^2 = \frac{8}{3}$, and the cubic power vanishes; then we have the equivalences

$$\mathcal{W}_\infty[\mu = 1] \cong \mathcal{W}_\infty[\mu = -1] \cong \mathcal{W}_\infty[\mu = \infty] . \tag{2.27}$$

The fact that for $\mu = 1$ the symmetry $\mu \mapsto -\mu$ survives at the quantum level is a direct consequence of the fact that, for this value of μ , $\mathcal{W}_\infty[\mu]$ is a linear \mathcal{W} -algebra whose structure constants are simply the (analytic continuation) of the $\text{hs}[\mu]$ structure constants.

Finally, the coefficient in front of the μ^2 term in (2.20) vanishes for $\tilde{\gamma}^2 = \frac{(c-26)}{(c-7)}$, when the equation becomes $\mu^3 = (c + 1)$. Thus the three cubic roots of $(c + 1)$ define equivalent $\mathcal{W}_\infty[\mu]$ algebras.

2.3 Truncation to finite N

In order to clarify the analytic continuation of [12] in section 4 we will be interested in the case where the algebra $\mathcal{W}_\infty[\mu]$ truncates to \mathcal{W}_N . In that case, the coset level-rank duality from the beginning of this section suggests that we have the equivalences (2.12). We now want to show that they are a special case of (2.25).

In order to see this we take one of the roots of (2.20) to be $\mu_1 = N$. Then this determines $\gamma = \gamma(\mu = N, c)$, and hence the other two roots $\mu_{2,3}$. It follows from (2.20) that they satisfy the quadratic equation

$$\mu^2(N^2 - 1) - \mu(N - 1 - c) - N(N - 1 - c) = 0 , \tag{2.28}$$

whose solutions are

$$\mu_{2,3}(N, c) = \frac{1}{2(N^2 - 1)} \left[(N - 1 - c) \pm \sqrt{(N - 1 - c)(4N^3 - 3N - c - 1)} \right] . \tag{2.29}$$

For the particular value $c = c_{N,k}$ defined in (2.4), we then find

$$\mu_2(N, c_{N,k}) = \frac{N}{N + k} \quad \text{and} \quad \mu_3(N, c_{N,k}) = -\frac{N}{N + k + 1} , \tag{2.30}$$

thus reproducing precisely (2.12). In particular, this therefore gives strong evidence for the equivalence of the \mathcal{W}_N minimal model at level k , with the $\mathcal{W}_\infty[\mu]$ theory at $\mu = \lambda \equiv \frac{N}{N+k}$. This symmetry is crucial for the duality to the bulk Vasiliev theory proposed in [7].

Moreover, we see that there is another value of μ , namely $\mu_3 = -\frac{N}{N+k+1}$, which is also equivalent to the other two descriptions. In the large N 't Hooft limit, $\mu_2 = -\mu_3$ and this is just the statement about the classical equivalence of the $\text{hs}[\pm\mu]$ theories.

3 Minimal representations of $\mathcal{W}_\infty[\mu]$

In this section, we will study a special class of representations of $\mathcal{W}_\infty[\mu]$ and see how the results are consistent with the above equivalences. These considerations will also play an important role for the analysis of the analytic continuation in section 4.

3.1 Determining the c -dependence of the structure constants

Let us consider the representation of $\mathcal{W}_\infty[\mu]$ that has the fewest number of low-lying states. Leaving aside the vacuum representation, this ‘minimal’ representation has then the character

$$\chi = \frac{q^h}{(1-q)} \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)} = q^h (1 + q + 2q^2 + \dots). \quad (3.1)$$

Note that if the conformal dimension of the ground state is non-zero, then the L_{-1} descendant is necessarily non-trivial, and hence the representation contains at least one state at level one; for the ‘minimal’ representation this is the only non-trivial state at level one, i.e. all other states are proportional to it (modulo null states). Similarly, in the minimal representation there are only two descendants at level two, which we may take to be the L_{-1}^2 and L_{-2} descendants of the ground state; all other descendants at level two are again equal to linear combinations of them (modulo null states).

Thus the minimal representation must contain many (sic!) null-vectors, and as a consequence its structure is completely determined. In particular, it follows from the analysis of appendix B that the conformal dimension of the ground state must satisfy the cubic equation

$$7(5c + 22)(16h^2 + 2ch + c - 10h)(2ch - 3c - 2h)N_4 + 150(18h^2c + c^2 - 12ch + c^2h + 36h^2 + 2c - 28h)(hc - 2h - 2c)N_3^2 = 0. \quad (3.2)$$

We can then turn the logic around, and *use* this identity to determine the full c -dependence of the structure constants N_3 and N_4 .⁴ To this end we recall that $\mathcal{W}_\infty[\mu = N]$ truncates to \mathcal{W}_N , and that the $\mathcal{W}_{N,k}$ theories have the minimal representations $(\mathfrak{f}; 0)$ and $(0; \mathfrak{f})$ (or their conjugates), where \mathfrak{f} denotes the fundamental representation of $\mathfrak{su}(N)$. The corresponding conformal dimensions equal

$$h(\mathfrak{f}; 0) = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k}\right), \quad h(0; \mathfrak{f}) = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1}\right). \quad (3.3)$$

⁴Actually, as is clear from the structure of the $\mathcal{W}_\infty[\mu]$ algebra, there is an overall normalisation freedom (which is described by q^2 in (A.20) and (A.21) and which corresponds to rescaling the primary field of spin s by q^{s-2}), and only N_4/N_3^2 has any independent meaning.

Either of these values must therefore be a solution of (3.2) for finite N and k ; expressing k in terms of c and $N = \mu$, the first value of h in (3.3) implies that we have

$$\frac{N_4}{N_3^2} = \frac{75(c+2)(\mu-3)(c(\mu+3)+2(4\mu+3)(\mu-1))}{14(5c+22)(\mu-2)(c(\mu+2)+(3\mu+2)(\mu-1))}. \quad (3.4)$$

It is then a non-trivial consistency check that, with this expression for the structure constants, also the second value of (3.3) solves (3.2). In order to relate this to the conventions of [23], we choose $N_3 = \frac{2}{5}$ so that the OPE of two W -fields is as in (A.22), and rescale U as in (A.24); then (3.4) implies that

$$(C_{33}^4)^2 = \frac{16 \cdot 56}{75} \frac{N_4}{N_3^2} = \frac{64(c+2)(\mu-3)(c(\mu+3)+2(4\mu+3)(\mu-1))}{(5c+22)(\mu-2)(c(\mu+2)+(3\mu+2)(\mu-1))}, \quad (3.5)$$

thus leading to (2.15).

3.2 Structure of solutions

Plugging in the explicit expressions for $\frac{N_4}{N_3^2}$, the cubic equation in (3.2) factorises into a linear equation

$$2h(1-\mu+c) - (1+\mu)c = 0 \quad \implies \quad h = h^{(1)}(\mu, c) \equiv \frac{(1+\mu)c}{2(1+c-\mu)}, \quad (3.6)$$

as well as the quadratic equation

$$4h^2\mu^2 + 2h(1+c+\mu-2\mu^2) - c(1-\mu) = 0 \quad (3.7)$$

with solutions

$$h = h_{\pm}^{(2)}(\mu, c) \equiv \frac{1}{4\mu^2} \left[-(1+c+\mu-2\mu^2) \pm \sqrt{(c+1-\mu)(c+1+3\mu-4\mu^3)} \right]. \quad (3.8)$$

Note that the cubic equation (3.2), once we substitute (3.4), is actually triality invariant. As a consequence the roots in (3.7) and (3.8) are permuted among each other under a triality transformation.

For $c \rightarrow \infty$ (and μ fixed), the three solutions behave as

$$h^{(1)} \simeq \frac{1}{2}(1+\mu), \quad h_+^{(2)} \simeq \frac{1}{2}(1-\mu), \quad h_-^{(2)} \approx -\frac{c}{2\mu^2} + \frac{\mu^3 + \mu^2 - \mu - 1}{2\mu^2}. \quad (3.9)$$

For $\mu = \lambda \equiv \frac{N}{N+k}$, the first two are the familiar solutions for the scalar fields in the duality of [7], while the last solution does not correspond to a representation of $\text{hs}[\lambda]$. The reason for this is that, as discussed in detail in [5], $\text{hs}[\lambda]$ is only a subalgebra of $\mathcal{W}_{\infty}[\lambda]$ for $c \rightarrow \infty$, but the third representation in (3.9) decouples in this limit.

We will see in the next subsection that for $\mu = N$ and c taking one of the values $c_{N,k}$ of the $\mathcal{W}_{N,k}$ minimal models, $h_{\pm}^{(2)}$ correspond to the physical representations $(f; 0)$ and $(0; f)$, respectively. Note the different behavior of the dimensions of the two representations at large central charge, suggesting that the two scalar excitations appear on a very different footing. We will return to this important distinction in section 5.

3.3 The minimal model parametrisation

For the following it will sometimes be useful to parametrise the $\mathcal{W}_\infty[\mu]$ algebra in terms of N and k , rather than c and μ , where the relation between the two parameters is that $\mu = N$, and $c = c_{N,k}$, with $c_{N,k}$ defined in (2.4). Obviously, this is a useful parametrisation if we are interested in the truncation of $\mathcal{W}_\infty[\mu]$ to finite N (as will be the case in section 4). However, we may also take the more general point of view that N and k are not necessarily integers, and then this is just a useful parametrisation (covering all $\mathcal{W}_\infty[\mu]$ algebras at all values of the central charge).

We should note, however, that this parametrisation exhibits a six-fold ambiguity. The reason for this is that, for a given c and N , the equation $c = c_{N,k}$ has two solutions for k ; if $k = k'$ is one solution, then the second solution is $k = -(2N + k' + 1)$. Together with (2.30), the six equivalent pairs are therefore

$$\begin{aligned}
 (N, k) & & (N, -2N - 1 - k) \\
 \left(\frac{N}{N+k}, \frac{1-N}{N+k} \right) & & \left(\frac{N}{N+k}, -\frac{2N+k+1}{N+k} \right) \\
 \left(-\frac{N}{N+k+1}, -\frac{k}{N+k+1} \right) & & \left(-\frac{N}{N+k+1}, \frac{N-1}{N+k+1} \right).
 \end{aligned} \tag{3.10}$$

In this parametrisation the eigenvalues of the minimal representations are

$$h(\mathfrak{f}; 0) = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k} \right) = h_+^{(2)}(\mu = N, c = c_{N,k}) \tag{3.11}$$

$$h(0; \mathfrak{f}) = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1} \right) = h_-^{(2)}(\mu = N, c = c_{N,k}), \tag{3.12}$$

while the third solution, eq. (3.6), equals

$$h^{(1)}(\mu = N, c = c_{N,k}) = -k \frac{(2N+k+1)}{2N}. \tag{3.13}$$

This last representation does not appear to be (and in fact is not) a representation of \mathcal{W}_N (at level k). It may therefore seem that we have a contradiction with (2.12).

In order to understand why this is not the case, we need to be more precise about the nature of the truncation, say in (2.1). This identity is only true after quotienting $\mathcal{W}_\infty[\mu]$ by the non-trivial ideal that appears for $\mu = N$; this just mirrors the fact that $\text{hs}[\mu = N]$ is not identically equal to $\mathfrak{su}(N)$ either since $\text{hs}[\mu]$ is infinite dimensional, whereas $\mathfrak{su}(N)$ is finite-dimensional. Rather, for $\mu = N$ the algebra $\text{hs}[\mu]$ develops an (infinite-dimensional) ideal, and if we divide $\text{hs}[\mu]$ by this ideal, the resulting Lie algebra is isomorphic to $\mathfrak{su}(N)$.

Similarly, for (2.1), (2.8) or (2.12), the two algebras are only isomorphic if we quotient $\mathcal{W}_\infty[\mu]$ by the relevant ideal (that appears for these special values of μ). But then it is not guaranteed that the representations of $\mathcal{W}_\infty[\mu]$ are compatible with this quotienting.

For example, if we set $\mu = 3$, then $\mathcal{W}_\infty[3]$ should truncate to \mathcal{W}_3 . This requires that we set all higher spin fields with spin greater than 3 (such as U and X) to zero. But then only those ‘minimal’ representations of $\mathcal{W}_\infty[\mu]$ define (minimal) representations of \mathcal{W}_3 for

which $u = x = 0$, see (B.14) and (B.15). In particular, the numerator of u in (B.14) leads to a quadratic relation for h , that is satisfied for $h = h_{\pm}^{(2)}$, but not for $h = h^{(1)}$. [Similarly, (B.15) then also follows since $N_4 = 0$ for $\mu = 3$, see eq. (3.4).] This explains why (3.13) is *not* a representation of \mathcal{W}_N at $N = 3$, and we expect that a similar argument will apply for any integer N . Similarly, if we set

$$\mu = \frac{3}{3+k} \quad \text{with} \quad c = c_{3,k} = 2 - \frac{24}{(3+k)(4+k)}, \quad (3.14)$$

corresponding to \mathcal{W}_3 at level k , then $h^{(1)}$ and $h_+^{(2)}$ describe the actual representations of \mathcal{W}_3 at that central charge, whereas $h_-^{(2)}$ does not, as can be seen by the same reasoning. Note that also for these values of μ and c , N_4 in (3.4) vanishes.

4 Analytic continuation

Let us now apply the insights of the previous sections to shed some more light on the analytic continuation proposed in [12]. Recall that this analytic continuation related a class of states in the $\mathcal{W}_{N,k}$ minimal models (the ‘light states’) to certain classical solutions in the (euclidean) higher spin theory based on the gauge group $\text{SL}(N, \mathbb{C})$. In the process N was kept fixed and finite, while k , which is a positive integer in the minimal models, was taken to the value $k = -(N + 1)$. The expressions for the dimensions as well as spin 3 and spin 4 charges of all these states in the minimal model were formally found to match (in a fairly non-trivial way) with those of the bulk solutions for any value of N .

It is not immediately obvious whether the formal procedure of taking k from positive integer values to the negative value $k = -(N + 1)$ makes any sense, and indeed, in the analysis of [12], it was not entirely clear what precisely was being kept fixed in the process. With our improved understanding of the structure of the quantum $\mathcal{W}_{\infty}[\mu]$ algebras we can now give a clear interpretation of these results. As we shall explain below, the correct way to describe this analytic continuation is to consider the family of \mathcal{W}_N theories at fixed finite N , and vary c from the minimal model value $c \leq (N - 1)$ to the semiclassical case where $c \rightarrow \infty$. Since we now understand how to describe the algebras $\mathcal{W}_{\infty}[\mu]$ for arbitrary μ and c , this analytic continuation is well-defined, and it induces a corresponding analytic continuation on all representations.

In the following we shall first study this in detail for the lightest of the light states, the representation labelled by (f, f) . This then suggests a natural generalisation for all the light states; this will be described in section 4.2. Incidentally, our analysis also implies that the two minimal representations (that play an important role in the duality of [7]) behave rather differently as we take $c \rightarrow \infty$. This suggests that one of them should probably not be interpreted as a ‘perturbative’ state of the higher spin gravity theory; we will come back to this issue in section 5.

4.1 The (f, f) states

Recall that the primaries of the minimal models are labelled by two integrable representations $(\Lambda_+; \Lambda_-)$ of $\mathfrak{su}(N)_k$ and $\mathfrak{su}(N)_{k+1}$, respectively. The set of light states that were

considered in [12] are of the form $\Lambda_+ = \Lambda_- = \Lambda$, and their conformal dimension equals

$$h(\Lambda, \Lambda) = \frac{c_2(\Lambda)}{(N+k)(N+k+1)}. \tag{4.1}$$

Here $c_2(\Lambda)$ is the quadratic Casimir of the representation Λ . If we were to consider the 't Hooft limit where $(N, k) \rightarrow \infty$, the states where Λ has a finite number of Young tableau boxes have conformal dimension $h \propto \frac{1}{N}$, and hence were dubbed 'light states'. In the following we want to follow these states as we change the conformal dimension from the minimal model value $c = c_{N,k}$ to the quasiclassical regime where c is large. Let us first explain this in detail for the lightest of the light states, the one corresponding to $\Lambda = f$.

In order to do so we note that we can think of $(f; f)$ as the fusion of $(0; f) \otimes (f; 0)$. We can thus repeat the fusion analysis of [8], but now done for $\mathcal{W}_\infty[\mu = N]$ at finite c , using the explicit form of the commutation relations of appendix A; some details of this calculation are given in appendix C. Provided that c is finite, the resulting fusion product is irreducible,⁵ and the conformal dimension of the resulting highest weight state equals exactly

$$h(f; f) = h(f; 0) + h(0; f) - \frac{N-1}{N}. \tag{4.2}$$

Given the identifications (3.11) and (3.12), we know how to analytically continue both $(f; 0)$ and $(0; f)$; for $c \rightarrow \infty$, it then follows that their conformal dimensions behave as (see eq. (3.9))

$$h(f; 0) \sim -\frac{(N-1)}{2}, \quad h(0; f) \sim -\frac{c}{2N^2} + \frac{N^3 + N^2 - N - 1}{2N^2}, \quad \text{as } c \rightarrow \infty. \tag{4.3}$$

Thus it follows that the analytic continuation of the $(f; f)$ representation has conformal dimension

$$h(f; f) \sim -\frac{c}{2N^2} + \frac{N-1}{2N^2} \quad \text{as } c \rightarrow \infty. \tag{4.4}$$

This then reproduces precisely the observation of [12].

4.2 Light states and conical surpluses

One can actually generalise the above discussion to all light states. To this end one observes that the conformal dimension, as well as the eigenvalues of the spin 3 zero mode of the state $(\Lambda; \Lambda)$, can be written as

$$\begin{aligned} h &= \alpha_0^2 c_2(\Lambda) \\ w &= \alpha_0^3 c_3(\Lambda + \rho), \end{aligned} \tag{4.5}$$

where ρ denotes the Weyl vector of the algebra $\mathfrak{su}(N)$, c_s are the various Casimir operators — for the precise definitions see eqs. (4.12) and (5.10) of [12] — and α_0 is defined by

$$\alpha_0^2 = \frac{1}{(N+k)(N+k+1)}. \tag{4.6}$$

⁵This is different to what happened in [8] where the $c \rightarrow \infty$ limit was considered. There the fusion product turned out to be indecomposable.

Note that the entire k dependence of the expressions in (4.5) is contained in the prefactor α_0 . We should mention in passing that in [12] a similar statement was also made (to leading order in $\frac{1}{c}$) for the spin 4 zero mode. This (as well as corresponding statements for the higher spin charges) can be deduced from the Drinfeld-Sokolov description, using the simple formula for the eigenvalue in the non-primary basis, see for example eq. (6.50) of [18]. However, the field redefinition that is required for going from this non-primary basis to the corresponding primary basis is only known to leading order in $\frac{1}{c}$ [26], and is likely to receive non-trivial quantum corrections (coming from the normal ordering) for higher spins. Thus the simple statement corresponding to (4.5) will, for spins greater than three, only hold in a suitable non-primary basis.

Returning to (4.5), it is now natural to believe that the analytic continuation simply consists of writing these expressions in terms of N and c , rather than N and k . Since

$$c = c_{N,k} = (N - 1) \left[1 - \frac{N(N + 1)}{(N + k)(N + k + 1)} \right] = (N - 1) \left(1 - N(N + 1) \alpha_0^2 \right), \quad (4.7)$$

this amounts to writing

$$\alpha_0^2 = \frac{(N - 1 - c)}{N(N^2 - 1)}. \quad (4.8)$$

The analytic continuation is then straightforward: we keep N fixed, and vary c continuously from $c = c_{N,k}$ to the semiclassical regime $c \rightarrow \infty$. Note that for the case of $\Lambda = f$ this then reproduces indeed (4.4) since $c_2(f) = \frac{N^2 - 1}{2N}$. Similarly, it follows from the analysis of appendix C that the same holds for $w(f; f)$, see eq. (C.17).

Since the entire c dependence of the eigenvalues (4.5) is carried by their dependence on α_0 , and since $\alpha_0^2 \sim c$ for large c , all the eigenvalues in (4.5) become proportional to some positive power of c . Thus these states can be interpreted in terms of ‘classical solutions’ in this limit.

More concretely, the c dependence of α_0 (at fixed N) contains only a linear and constant term

$$\alpha_0^2(c) = -\frac{c}{N(N^2 - 1)} + \frac{1}{N(N + 1)}. \quad (4.9)$$

In the semiclassical limit we can drop the second (constant term), and with the resulting value of α_0^2 , the spectrum and charges were matched with that of the conical defect solutions in the bulk higher spin theory. This match is only true to leading order in the central charge;⁶ of course, that is the best one could hope for from the classical solutions which are not sensitive to $\frac{1}{c}$ corrections. In fact, our analysis gives a prediction that the energies of the conical defects only get an $\mathcal{O}(1)$ positive correction

$$\delta h(\Lambda, \Lambda) = \frac{c_2(\Lambda)}{N(N + 1)}, \quad (4.10)$$

without any further $\frac{1}{c}$ corrections. However, the higher spin charges given in (4.5) do generically have higher order corrections. But perhaps all these corrections are best viewed as a quantum renormalisation of the central charge $c \rightarrow c - (N - 1)$ (or equivalently Newton’s constant G_N) in the bulk theory.

⁶We thank Joris Raeymakers for sharing closely related observations.

4.3 Analytic continuation of the minimal states

Actually, the above considerations also apply directly to the ‘minimal’ representations; indeed, this is already implicit in what was done above in section 4.1. As we explained there, the conformal dimension of the two minimal representations that exist at finite integer N behave as (4.3). Note that the semiclassical interpretation of these two states (at fixed N) is quite different: the conformal dimension of the state $(f; 0)$ remains finite, while that of $(0; f)$ is proportional to c . We are therefore led to the point of view that the only true perturbative states are those corresponding to $(f; 0)$ and $(\bar{f}; 0)$, making up one complex scalar. What was earlier interpreted as another perturbative scalar, namely, that corresponding to $(0; f)$ and its conjugate $(0; \bar{f})$, is perhaps more naturally thought of as a solitonic state that just happens to have a finite dimension in the ’t Hooft limit. This is also in line with the observations made at the end of section 3.2. We shall come back to the implications of this for the duality proposed in [7] in section 5.

Incidentally, eq. (4.2) also leads to a somewhat different point of view. Since in the large c limit $h(f; 0) = h(\bar{f}; 0) < 0$, it is more natural to rewrite (4.2) as

$$h(0; f) = h(f; f) - h(\bar{f}; 0) + \frac{N-1}{N} . \tag{4.11}$$

This suggests that $(0; f)$ should be interpreted as some kind of bound state of $(f; f)$ with a perturbative excitation $(\bar{f}; 0)$. Actually, a similar statement holds for all representations of the form $(0; \Lambda_-)$. To see this, recall that the conformal dimension of the representation $(\Lambda_+; \Lambda_-)$ equals

$$h(\Lambda_+; \Lambda_-) = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0\rho) , \quad \text{where} \quad \Lambda = \alpha_+\Lambda_+ + \alpha_-\Lambda_- , \tag{4.12}$$

the inner product (\cdot, \cdot) is the usual inner product on the weight space, and

$$\alpha_+ = \frac{1}{\sqrt{k_{\text{DS}} + N}} , \quad \alpha_- = -\sqrt{k_{\text{DS}} + N} , \quad \alpha_0 = \alpha_+ + \alpha_- . \tag{4.13}$$

Note that, using the relation to the coset labels of eq. (2.10), α_0^2 agrees then precisely with (4.6). Since $\alpha_+ \cdot \alpha_- = -1$, it now follows that

$$h(\Lambda_+; \Lambda_-) = h(\Lambda_+; 0) + h(0; \Lambda_-) - (\Lambda_+, \Lambda_-) . \tag{4.14}$$

In particular, we can apply this to the case $\Lambda_+ = \Lambda_-$ and conclude that

$$h(0; \Lambda_-) = h(\Lambda_-; \Lambda_-) - h(\Lambda_-^*; 0) + (\Lambda_-, \Lambda_-) . \tag{4.15}$$

The last term is positive and purely group-theoretic, i.e. it does not depend on k (or c), but only on N (as well as Λ_-). We can thus think of $(0; \Lambda_-)$ as a bound state of $(\Lambda_-; \Lambda_-)$ with the perturbative excitation $(\Lambda_-^*; 0)$. Note that, again, $h(\Lambda; 0)$ becomes negative in the semiclassical limit, since we have

$$h(\Lambda; 0) = \alpha_+^2 c_2(\Lambda) - (\Lambda, \rho) , \tag{4.16}$$

where

$$\alpha_+^2 = \frac{k + N + 1}{k + N} \rightarrow 0 \quad \text{as} \quad k \rightarrow -N - 1 . \tag{4.17}$$

5 Refining the minimal model holography conjecture

One of the striking consequences of the analysis of the quantum $\mathcal{W}_\infty[\mu]$ algebra is the very different nature of the two minimal representations which correspond to $(f; 0)$ and $(0; f)$, respectively. This difference was not at all obvious in the 't Hooft limit where they have dimensions $\frac{1}{2}(1 \pm \lambda)$, and appear to be on a similar footing. However, we now see that this is, in a sense, an artifact of the 't Hooft limit: for any finite N (i.e. $\mu = N$) the two states have conformal dimensions with a very different dependence on c ; in particular, in the semi-classical limit they behave as

$$h(f; 0) \sim \frac{(1 - N)}{2}, \quad h(0; f) \sim -\frac{c}{2N^2}, \quad \text{as } c \rightarrow \infty, \quad (5.1)$$

see eq. (4.3). Since $h(0; f) \propto c$, this now suggests that $(0; f)$ is more naturally thought of as a non-perturbative state or soliton, rather than a perturbative state. We should mention that both $h(f; 0)$ and $h(0; f)$ turn negative in this limit (i.e. for $c \rightarrow \infty$ at fixed N), thus signalling that the theory becomes non-unitary. However, for the purpose of identifying the semiclassical interpretation of the various states, this should be immaterial. (Obviously, this problem is absent in the actual 't Hooft limit since it is a sequence of unitary minimal models; it corresponds to taking both N and c to infinity in a 't Hooft like manner.)

It is therefore natural to propose that the state $(0; f)$ (and its conjugate) should *not* be thought of as corresponding to a perturbative scalar mode in the bulk. Instead, there is only one (complex) perturbative scalar dual to $(f; 0)$ (and its conjugate). We should then think of the state $(0; f)$ as being on the same footing as the light states such as $(f; f)$. Indeed, as we saw in the previous section, in the large c (finite N) limit it makes a lot of sense to view $(0; f)$ as a bound state of $(f; f)$ with the perturbative state $(\bar{f}; 0)$. Since $(f; f)$, was already identified with a semiclassical solution (a conical surplus), we then also have a candidate bulk interpretation for the state $(0; f)$ as an excitation of this semi-classical solution.

This reinterpretation also makes sense of the observations in [10, 14], where it was argued that the double trace operator corresponding to the two particle state of $(0; f)$ and $(\bar{f}; 0)$ (whose conformal dimension equals $h = 1$ in the 't Hooft limit) is a descendant of the light state $(f; f)$. This would be very strange if one were to interpret both $(0; f)$ and $(\bar{f}; 0)$ as perturbative states while viewing $(f; f)$ as non-perturbative. The interpretation we are proposing here, on the other hand, makes this quite natural since we now consider $(f; f)$ as the basic non-perturbative object, which has $(0; f)$ as an excited state.

Finally, this also fits in with the fact that the most natural (unambiguous) bulk higher spin $hs[\lambda]$ theory is the one with a single complex scalar. This was one of the motivations for the proposal of [9] that this higher spin theory describes, say, the $(\Lambda; 0)$ ⁷ sector of the minimal models. In a similar vein, at $\lambda = 0$ it is the theory with a single complex scalar that is dual to the singlet sector of a free theory [27].

The final picture is therefore one in which the bulk theory has a perturbative sector consisting of one complex scalar (dual to $(f; 0)$ and its conjugate) together with a tower

⁷In their analysis it was ambiguous whether to consider the $(\Lambda; 0)$ or $(0; \Lambda)$ subsector of the minimal models, whereas we see here a basic distinction between the two.

of higher spin fields. The scalar is quantised in the standard way. At finite N the tower of higher spins is truncated nonperturbatively to a maximal spin N . This truncation is a consequence of the equivalence of the $\mathcal{W}_\infty^{\text{qu}}[\mu]$ algebra to \mathcal{W}_N when $\mu = \frac{N}{N+k}$ and $c = c_{N,k}$. This is a bit like a stringy exclusion principle (for a similar truncation in a higher spin context see [28]). This sector is closed under the OPE with itself and is a consistent subsector of the theory at large N . It is however, not modular invariant by itself. For this to be restored we need the nonperturbative states which correspond to the scalar dual to $(0; f)$ and the light states $(f; f)$ (or more generally $(\Lambda; \Lambda)$) which are nontrivial classical configurations in the bulk — the analytic continuations of the conical defects in the $\text{SL}(N)$ theory. While it is unusual to have a large number of light nonperturbative states, the selection rules of the CFT seem to ensure a good large N behavior of correlators [10, 14]. The upshot seems to be that the perturbative Vasiliev theory is highly incomplete at the quantum level and requires the various nonperturbative excitations to be taken into account for a consistent completion. The novel feature is that these nonperturbative states are not decoupled from the perturbative states by virtue of being highly energetic, but rather because of the special nature of the interactions — the fusion rules of the CFT.

6 Conclusions

In this paper we have shown how to determine the quantum algebra $\mathcal{W}_\infty^{\text{qu}}[\mu]$ underlying the $\text{hs}[\mu]$ higher spin theory on AdS_3 explicitly. In particular, we have managed to find the exact form of the finite c corrections to the commutation relations of the classical $\mathcal{W}_\infty^{\text{cl}}[\mu]$ algebra. This quantum deformation is essentially uniquely determined by consistency conditions, in particular the Jacobi identity [25].

While we have not managed to give a closed form expression for the full quantum algebra, our results are for example sufficient to determine the structure of the ‘minimal’ representations exactly. They also give very strong evidence for the claim that the resulting quantum algebra, $\mathcal{W}_\infty^{\text{qu}}[\mu]$, exhibits an exact ‘trinality’ symmetry, relating in particular the algebra with $\mu = \lambda \equiv \frac{N}{N+k}$ to the $\mathcal{W}_{N,k}$ minimal model algebra (at finite N and k). Given that $\mathcal{W}_\infty^{\text{qu}}[\mu]$ is the only consistent quantum deformation of $\mathcal{W}_\infty^{\text{cl}}[\mu]$, this shows that the symmetries of the $\mathcal{W}_{N,k}$ minimal model agree with those of the quantum higher spin theory based on $\text{hs}[\lambda]$. Since the relevant symmetry algebras constrain the theories very significantly, this goes a long way towards proving the duality at finite N and k .

We should stress that the quantum algebra $\mathcal{W}_\infty^{\text{qu}}[\mu]$ generically does not contain $\text{hs}[\mu]$ as a subalgebra, and as a consequence, the representations of $\mathcal{W}_\infty^{\text{qu}}[\mu]$ cannot necessarily be described in terms of representations of $\text{hs}[\mu]$. (For example, this is the case for the third ‘minimal’ representation, see the comments at the end of section 3.2.) The higher spin algebra $\text{hs}[\mu]$ only emerges as a subalgebra for $c \rightarrow \infty$. This is somewhat reminiscent of the result of Maldacena and Zhiboedov [1, 2] who showed that, in higher dimensions, the higher spin symmetry is necessarily broken by $1/N$ corrections (unless the theory is free).⁸ In our case, however, while $\text{hs}[\mu]$ generically is no longer a symmetry of the quantum

⁸Note that for $\mu = 1$, which corresponds to a free theory, $\text{hs}[\mu = 1] \subset \mathcal{W}_\infty^{\text{qu}}[\mu = 1]$ is an actual subalgebra, and hence $\text{hs}[\mu = 1]$ remains a genuine symmetry at finite c .

theory, it is replaced by the even larger $\mathcal{W}_\infty^{\text{qu}}[\mu]$ algebra that remains a true symmetry in the quantum theory.

In this paper we have determined the quantum corrections to the classical $\mathcal{W}_\infty^{\text{cl}}[\mu]$ algebra using indirect methods, such as the Jacobi identity as well as the representation theory of the \mathcal{W}_N minimal models. It would be very interesting to calculate these corrections directly in the higher spin gravity theory. For example, our analysis makes a specific prediction for the $1/c$ corrections to the conformal dimension of the perturbative scalar, see eq. (3.6), and it would be very interesting to rederive this using perturbation theory of the higher spin theory.

It would also be interesting to study the quantum \mathcal{W}_∞ -algebra in the supersymmetric case, following [29–32]; this is currently under investigation [25].

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A Explicit commutation relations

Let us denote the modes of the stress tensor, as usual, by L_n , while the modes of the spin 3, 4 and 5 fields are called W_n , U_n and X_n , respectively. Using the ansatz of [5, 19] and requiring the Jacobi identities

$$[L_m, [L_n, W_l]] + \text{cycl.} = [L_m, [W_n, W_l]] + \text{cycl.} = [U_m, [W_n, L_l]] + \text{cycl.} = 0, \quad (\text{A.1})$$

we can determine the finite c corrections of the commutation relations. The resulting structure takes then the form

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n} \quad (\text{A.2})$$

$$[L_m, W_n] = (2m - n)W_{m+n} \quad (\text{A.3})$$

$$[L_m, U_n] = (3m - n)U_{m+n} \quad (\text{A.4})$$

$$[L_m, X_n] = (4m - n)X_{m+n} \quad (\text{A.5})$$

$$[W_m, W_n] = 2(m - n)U_{m+n} + \frac{N_3}{12}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \quad (\text{A.6})$$

$$+ \frac{8N_3}{(c + \frac{22}{5})}(m - n)\Lambda_{m+n}^{(4)} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

$$\begin{aligned}
[W_m, U_n] &= (3m - 2n)X_{m+n} - \frac{N_4}{15N_3}(n^3 - 5m^3 - 3mn^2 + 5m^2n - 9n + 17m)W_{m+n} \\
&+ \frac{208N_4}{25N_3(c + \frac{114}{7})}(3m - 2n)\Lambda_{m+n}^{(5)} + \frac{84N_4}{25N_3(c + 2)}\Theta_{m+n}^{(6)}. \tag{A.7}
\end{aligned}$$

The modes of the composite fields are defined by

$$\Lambda_n^{(4)} = \sum_p : L_{n-p}L_p : + \frac{1}{5}x_nL_n \tag{A.8}$$

$$\Lambda_n^{(5)} = \sum_p : L_{n-p}W_p : + \frac{1}{14}y_nW_n \tag{A.9}$$

$$\Theta_n^{(6)} = \sum_p (\frac{5}{3}p - n) : L_{n-p}W_p : + \frac{1}{6}z_nW_n, \tag{A.10}$$

where

$$x_{2l} = (l + 1)(1 - l), \quad x_{2l-1} = (l + 1)(2 - l), \tag{A.11}$$

$$y_{2l} = (l + 2)(3 - 5l), \quad y_{2l-1} = 5(l + 1)(2 - l), \tag{A.12}$$

$$z_{2l} = l(l + 2), \quad z_{2l-1} = 0. \tag{A.13}$$

With this definition we then have the commutation relations

$$[L_m, \Lambda_n^{(4)}] = (3m - n)\Lambda_{m+n}^{(4)} + (\frac{c}{6} + \frac{11}{15})m(m^2 - 1)L_{m+n} \tag{A.14}$$

$$[L_m, \Lambda_n^{(5)}] = (4m - n)\Lambda_{m+n}^{(5)} + (\frac{c}{12} + \frac{19}{14})m(m^2 - 1)W_{m+n} \tag{A.15}$$

$$[L_m, \Theta_n^{(6)}] = (5m - n)\Theta_{m+n}^{(6)} + (\frac{c}{36} + \frac{1}{18})m(m^2 - 1)(5m + 2n)W_{m+n}. \tag{A.16}$$

The corresponding states are all quasiprimary, and are explicitly given as

$$\Lambda^{(4)} = (L_{-2}L_{-2} - \frac{3}{5}L_{-4})\Omega \tag{A.17}$$

$$\Lambda^{(5)} = (L_{-2}W_{-3} - \frac{3}{7}W_{-5})\Omega \tag{A.18}$$

$$\Theta^{(6)} = (L_{-3}W_{-3} - \frac{2}{3}L_{-2}W_{-4} + \frac{1}{2}W_{-6})\Omega. \tag{A.19}$$

The above commutation relations then satisfy the Jacobi identities (A.1); this is true for any value of N_3 and N_4 . It follows from the classical analysis [5] that, to leading order in $1/c$, the structure constants take the form

$$N_3 = \frac{16}{5}q^2(\mu^2 - 4) \tag{A.20}$$

$$N_4 = \frac{384}{35}q^4(\mu^2 - 4)(\mu^2 - 9). \tag{A.21}$$

Here q is an arbitrary normalisation constant; we can choose it so that $N_3 = \frac{2}{5}$, i.e. $q^2 = \frac{1}{8(\mu^2 - 4)}$, in which case the OPE of the W -field takes the form

$$W \cdot W \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + \frac{32}{(5c + 22)} \cdot \Lambda^{(4)} + 4 \cdot U. \tag{A.22}$$

Then the leading term in the OPE of W with U equals

$$W \cdot U \sim \frac{56}{25} \frac{N_4}{N_3^2} W + \dots = \frac{12}{5} \frac{\mu^2 - 9}{\mu^2 - 4} W + \dots \quad (\text{A.23})$$

In order to compare this to [20–22, 24], let us define

$$\hat{U} = \beta^{-1} U \quad \text{with} \quad \beta^2 = \frac{56}{75} \frac{N_4}{N_3^2} = \frac{4}{5} \frac{\mu^2 - 9}{\mu^2 - 4}; \quad (\text{A.24})$$

then the OPEs are of the form

$$W \cdot W \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + 8 \sqrt{\frac{1}{5} \frac{\mu^2 - 9}{\mu^2 - 4}} \cdot \hat{U} + \dots \quad (\text{A.25})$$

$$W \cdot \hat{U} \sim + 6 \sqrt{\frac{1}{5} \frac{\mu^2 - 9}{\mu^2 - 4}} \cdot W + \dots, \quad (\text{A.26})$$

i.e. in the conventions of [23], the structure constant C_{33}^4 equals (2.14).

B Representation theory of $\mathcal{W}_\infty[\mu]$

In this appendix we want to study the minimal representation of $\mathcal{W}_\infty[\mu]$, whose character is given in eq. (3.1).

B.1 Relations at level one

The minimal representation has only a single state at level one, which we may take to be the L_{-1} descendant of the ground state (which we shall denote by ϕ). Thus we must have the null relations

$$\mathcal{N}_{1W} = \left(W_{-1} - \frac{3w}{2h} L_{-1} \right) \phi \quad (\text{B.1})$$

$$\mathcal{N}_{1U} = \left(U_{-1} - \frac{2u}{h} L_{-1} \right) \phi \quad (\text{B.2})$$

$$\mathcal{N}_{1X} = \left(X_{-1} - \frac{5x}{2h} L_{-1} \right) \phi. \quad (\text{B.3})$$

Here w , u and x are the eigenvalues of the zero modes on ϕ , i.e.

$$W_0 \phi = w \phi, \quad U_0 \phi = u \phi, \quad X_0 \phi = x \phi, \quad (\text{B.4})$$

and the relative normalisations in \mathcal{N}_{1W} , \mathcal{N}_{1U} and \mathcal{N}_{1X} are determined from the condition that L_1 annihilates these states. Actually, if we denote by $V_n^{(s)}$ the modes of the primary spin s field, then the commutation relations with the Virasoro algebra take the form

$$[L_m, V_n^{(s)}] = ((s-1)m - n) V_{m+n}^{(s)}, \quad (\text{B.5})$$

and hence the corresponding null-vector must be

$$\mathcal{N}_{1s} = \left(V_{-1}^{(s)} - \frac{sv(s)}{2h} L_{-1} \right) \phi, \quad \text{where} \quad V_0^{(s)} \phi = v(s) \phi. \quad (\text{B.6})$$

Again this guarantees that \mathcal{N}_{1s} is annihilated by L_1 . Note that (B.6) generalises the form of the null-vectors (B.1)–(B.3) to arbitrary spin s .

These null-vectors must obviously not just be annihilated by L_1 , but also by the other positive modes, i.e. by W_1 , U_1 , etc., and this will give rise to relations between the eigenvalues $v(s)$ of the zero modes. For example, from $W_1\mathcal{N}_{1W} = 0$ we deduce that

$$4u - \frac{N_3}{2}h + \frac{16N_3}{(c + \frac{22}{5})}(h^2 + \frac{1}{5}h) - \frac{9}{2}\frac{w^2}{h} = 0, \quad (\text{B.7})$$

while $W_1\mathcal{N}_{1U} = 0$ leads to

$$5x + \frac{4N_4}{5N_3}w - \frac{208N_4}{5N_3(c + \frac{114}{7})}w(h + \frac{3}{7}) - 6\frac{wu}{h} = 0. \quad (\text{B.8})$$

Incidentally, this equation can also be obtained from demanding that $U_1\mathcal{N}_{1W} = 0$.

B.2 Relations at level two

At level two we may take the linearly independent states to be $L_{-1}^2\phi$ and $L_{-2}\phi$. In particular, we must therefore be able to express $W_{-2}\phi$ in terms of these two states. From the requirement that the corresponding null-vector must be annihilated by L_1^2 and L_2 one finds that it must take the form

$$\mathcal{N}_{2W} = \left(W_{-2} + aL_{-1}^2 + bL_{-2}\right)\phi, \quad (\text{B.9})$$

with

$$a = -\frac{3w(2h + c)}{h(16h^2 + 2ch + c - 10h)}, \quad b = -\frac{24w(h - 1)}{(16h^2 + 2ch + c - 10h)}. \quad (\text{B.10})$$

Then we get relations from the requirement that $W_2\mathcal{N}_{2W} = 0$, and that $W_1\mathcal{N}_{2W}$ must be a linear combination of the null-vectors \mathcal{N}_{1*} . The former condition leads to

$$8u + 4N_3h + \frac{32N_3}{(c + \frac{22}{5})}(h^2 + \frac{1}{5}h) + 12aw + 6bw = 0, \quad (\text{B.11})$$

while the latter condition turns out to require

$$\frac{12u}{h} + N_3 + \frac{24N_3}{(c + \frac{22}{5})}(2h + \frac{2}{5}) + a\left(9\frac{w}{h} + 6w\right) + \frac{15}{2}b\frac{w}{h} = 0. \quad (\text{B.12})$$

Actually, these relations are linearly dependent, and thus we cannot determine all eigenvalues directly, but we can express both w and u as functions of h ; indeed we can eliminate u by combining (B.7) and (B.11), and then obtain

$$w(h) = \pm \frac{h}{3} \sqrt{\frac{-5N_3(16h^2 + 2ch + c - 10h)}{(2ch - 3c - 2h)}}. \quad (\text{B.13})$$

Similarly, we can determine $u \equiv u(h)$ as

$$u = -hN_3 \frac{c^2 - 12ch + c^2h + 18h^2c + 2c + 36h^2 - 28h}{(c + \frac{22}{5})(2ch - 3c - 2h)}. \quad (\text{B.14})$$

Once we have these relations, we can then determine all eigenvalues $v(s)$ recursively as a function of h . To this end we consider the relations coming from $W_1\mathcal{N}_{1s} = 0$ for $s = 4, 5, 6, \dots$. For example, for $s = 4$, this is just (B.8), which we can solve for x as

$$x = w \left(-\frac{4N_4}{25N_3} + \frac{208N_4}{25N_3(c + \frac{114}{7})} \left(h + \frac{3}{7} \right) + \frac{6u}{5h} \right). \quad (\text{B.15})$$

Since we already know w and u as a function of h , this therefore leads to an expression for $x \equiv x(h)$. For the general case, we note that the OPE of the spin 3 field W with $V^{(s)}$ will only involve simple fields of spin at most $s + 1$. Thus $W_1\mathcal{N}_{1s} = 0$ will lead to a relation between $v(s + 1)$ and $v(t)$ with $t \leq s$. Recursively, we can therefore determine all $v(s)$ in terms of h .

B.3 The final equation

Thus it only remains to find one final equation which will allow us to also determine h . To find this equation we now evaluate the condition $U_2\mathcal{N}_{2W} = 0$, which leads to

$$10x - 4w \frac{N_4}{N_3} - \frac{416N_4}{5N_3(c + \frac{114}{7})} w \left(h + \frac{3}{7} \right) + 20au + 8bu = 0. \quad (\text{B.16})$$

Instead of demanding that $U_2\mathcal{N}_{2W} = 0$ we may also study the condition that $U_1\mathcal{N}_{2W}$ is a linear combination of the null-vectors at level one, i.e. the vectors \mathcal{N}_{1*} given in (B.1)–(B.3). This leads to

$$10x - w \frac{6N_4}{5N_3} - \frac{416N_4}{5N_3(c + \frac{114}{7})} w \left(h + \frac{3}{7} \right) + 12au + 4auh + 7bu = 0. \quad (\text{B.17})$$

Together with (B.8) these three equations are indeed linearly dependent. In order to solve for h , we now equate (B.15) with (B.17) to obtain

$$4a \frac{u}{w} (3 + h) + 7b \frac{u}{w} + \frac{12u}{h} - \frac{14N_4}{5N_3} = 0, \quad (\text{B.18})$$

where a and b are defined as in (B.10). Using the expressions for u from (B.14) we then get the cubic equation for h given in eq. (3.2).

C The fusion of (f; f) at finite N and c

The fusion analysis of $(f; 0) \otimes (0; f)$ can essentially be done using the steps described in [8], so we shall be somewhat brief and only stress the main differences. Let us denote the highest weight states of the relevant representations by $\phi_1 = (f; 0)$ and $\phi_2 = (0; f)$. Following the discussion of appendix B, we then have the null-vectors

$$\begin{aligned} (W_{-1} - \frac{3w_1}{2h_1} L_{-1})\phi_1 &= 0 & (W_{-1} - \frac{3w_2}{2h_2} L_{-1})\phi_2 &= 0 \\ (W_{-2} + a_1 L_{-1}^2 + b_1 L_{-2})\phi_1 &= 0 & (W_{-2} + a_2 L_{-1}^2 + b_2 L_{-2})\phi_2 &= 0, \end{aligned} \quad (\text{C.1})$$

where $a_j = a(h_j, w_j)$ and $b_j = b(h_j, w_j)$, $j = 1, 2$, are defined by (B.10), and in the parametrisation of section 3.3, the eigenvalues equal (for $N_3 = \frac{2}{5}$)

$$\begin{aligned} h_1 &= \frac{(N-1)(2N+k+1)}{2N(N+k)} & w_1 &= -\frac{\sqrt{2}(N-1)(2N+k+1)}{6N(N+k)} \sqrt{\frac{(N-2)(3N+2k+2)}{N(N+2k)}} \\ h_2 &= \frac{(N-1)k}{2N(N+k+1)} & w_2 &= \frac{\sqrt{2}(N-1)k}{6N(N+k+1)} \sqrt{\frac{(N-2)(N+2k)}{N(3N+2k+2)}}. \end{aligned} \quad (\text{C.2})$$

Note that in the 't Hooft limit, $N, k \rightarrow \infty$ with $\lambda = \frac{N}{N+k}$, we have the familiar relations

$$\begin{aligned} h_1 &\simeq \frac{1}{2}(1 + \lambda) & w_1 &\simeq -\frac{\sqrt{2}}{6}(1 + \lambda) \sqrt{\frac{(2+\lambda)}{(2-\lambda)}} \\ h_2 &\simeq \frac{1}{2}(1 - \lambda) & w_2 &\simeq \frac{\sqrt{2}}{6}(1 - \lambda) \sqrt{\frac{(2-\lambda)}{(2+\lambda)}}. \end{aligned} \quad (\text{C.3})$$

Furthermore, the parameters a_j and b_j simplify in that limit to

$$a_j \simeq \frac{3w_j}{h_j(2h_j+1)} = (-1)^j \sqrt{\frac{2}{(4-\lambda^2)}}, \quad b_j \simeq 0, \quad (\text{C.4})$$

since $c \rightarrow \infty$ in that limit.

Let us study the highest weight space of the fusion product. Then we have the relations

$$\begin{aligned} 0 \cong \Delta(W_{-1}) &= (W_{-2} \otimes \mathbf{1}) + (W_{-1} \otimes \mathbf{1}) + (\mathbf{1} \otimes W_{-1}) \\ &= -a_1(L_{-1}^2 \otimes \mathbf{1}) - b_1(L_{-2} \otimes \mathbf{1}) + \frac{3w_1}{2h_1}(L_{-1} \otimes \mathbf{1}) + \frac{3w_2}{2h_2}(\mathbf{1} \otimes L_{-1}) \\ &= -a_1(L_{-1}^2 \otimes \mathbf{1}) + \left(\frac{3w_1}{2h_1} - \frac{3w_2}{2h_2} + b_1\right)(L_{-1} \otimes \mathbf{1}) - b_1 h_2 (\mathbf{1} \otimes \mathbf{1}), \end{aligned} \quad (\text{C.5})$$

where we have used that on the highest weight space

$$(\mathbf{1} \otimes L_{-1}) \cong -(L_{-1} \otimes \mathbf{1}) \quad (\text{C.6})$$

$$(L_{-2} \otimes \mathbf{1}) \cong (\mathbf{1} \otimes L_{-1}) + (\mathbf{1} \otimes L_0). \quad (\text{C.7})$$

Thus we obtain the identity

$$a_1(L_{-1}^2 \otimes \mathbf{1}) = \left(\frac{3w_1}{2h_1} - \frac{3w_2}{2h_2} + b_1\right)(L_{-1} \otimes \mathbf{1}) - b_1 h_2 (\mathbf{1} \otimes \mathbf{1}). \quad (\text{C.8})$$

Incidentally, in the 't Hooft limit, this reduces to the identity $(L_{-1}^2 \otimes \mathbf{1}) \cong -2(L_{-1} \otimes \mathbf{1})$, see eq. (5.17) of [8], since $b_j \rightarrow 0$ in that limit, see (C.4).

A second identity can be obtained from

$$\begin{aligned} 0 \cong -\Delta(W_{-2}) &= -(W_{-2} \otimes \mathbf{1}) - (\mathbf{1} \otimes W_{-2}) \\ &= a_1(L_{-1}^2 \otimes \mathbf{1}) + b_1(L_{-2} \otimes \mathbf{1}) + a_2(\mathbf{1} \otimes L_{-1}^2) + b_2(\mathbf{1} \otimes L_{-2}) \\ &= (a_1 + a_2)(L_{-1}^2 \otimes \mathbf{1}) - (b_1 + b_2)(L_{-1} \otimes \mathbf{1}) + (b_1 h_2 + b_2 h_1)(\mathbf{1} \otimes \mathbf{1}), \end{aligned} \quad (\text{C.9})$$

where we have used (C.6), (C.7) as well as

$$(\mathbf{1} \otimes L_{-2}) \cong -(L_{-1} \otimes \mathbf{1}) + (L_0 \otimes \mathbf{1}). \quad (\text{C.10})$$

Thus we obtain a second identity, namely

$$(a_1 + a_2)(L_{-1}^2 \otimes \mathbf{1}) = (b_1 + b_2)(L_{-1} \otimes \mathbf{1}) - (b_1 h_2 + b_2 h_1)(\mathbf{1} \otimes \mathbf{1}). \quad (\text{C.11})$$

Note that this identity becomes trivial in the 't Hooft limit (since $b_j \rightarrow 0$ and $a_1 \rightarrow -a_2$, see (C.4)), but for finite c , we can deduce from (C.8) and (C.11) the relation

$$\left[\left(\frac{3w_1}{2h_1} - \frac{3w_2}{2h_2} \right) (a_1 + a_2) + (b_1 a_2 - b_2 a_1) \right] (L_{-1} \otimes \mathbf{1}) \cong (a_2 b_1 h_2 - a_1 b_2 h_1) (\mathbf{1} \otimes \mathbf{1}), \quad (\text{C.12})$$

which simplifies to

$$(L_{-1} \otimes \mathbf{1}) \cong -\frac{N-1}{N} (\mathbf{1} \otimes \mathbf{1}). \quad (\text{C.13})$$

Thus for finite c , the highest weight space is one-dimensional, and the L_0 eigenvalue of (f, f) becomes indeed just (4.2).

One can also determine the W_0 eigenvalue of the ground state of the fusion product; using

$$\Delta(W_0) = (W_{-2} \otimes \mathbf{1}) + 2(W_{-1} \otimes \mathbf{1}) + (W_0 \otimes \mathbf{1}) + (\mathbf{1} \otimes W_0) \quad (\text{C.14})$$

one finds that

$$\begin{aligned} w(f; f) &= w_1 + w_2 - \frac{N-1}{N} \left(\frac{3w_1}{2h_1} + \frac{3w_2}{2h_2} \right) \\ &= -\sqrt{\frac{2(N-2)}{N}} \frac{(N^2-1)(N+2)}{6N} \frac{1}{(N+k)(N+k+1)} \sqrt{\frac{1}{(N+2k)(3N+2k+2)}}. \end{aligned} \quad (\text{C.15})$$

In order to bring this into the form of (4.5), we have to rescale w , i.e. we have to work with

$$\hat{N}_3 = \frac{2}{5} ((N+2)c + (3N+2)(N-1)) = \frac{2}{5} \frac{(N^2-1)(N+2k)(3N+2k+2)}{(N+k)(N+k+1)} \quad (\text{C.16})$$

instead. Then the corresponding \hat{w} eigenvalue equals

$$\hat{w}(f; f) = -\sqrt{\frac{2(N-2)(N^2-1)}{N}} \frac{(N^2-1)(N+2)}{6N} \left(\frac{1}{(N+k)(N+k+1)} \right)^{\frac{3}{2}}, \quad (\text{C.17})$$

and is hence of the form (4.5).

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