

Massless particles on supergroups and $AdS_3 \times S^3$ supergravity

Jan Troost

*Laboratoire de Physique Théorique,¹ Ecole Normale Supérieure,
24 rue Lhomond, F-75231 Paris Cedex 05, France*

E-mail: troost@lpt.ens.fr

ABSTRACT: Firstly, we study the state space of a massless particle on a supergroup with a reparameterization invariant action. After gauge fixing the reparameterization invariance, we compute the physical state space through the BRST cohomology and show that the quadratic Casimir Hamiltonian becomes diagonalizable in cohomology. We illustrate the general mechanism in detail in the example of a supergroup target $GL(1|1)$. The space of physical states remains an indecomposable infinite dimensional representation of the space-time supersymmetry algebra. Secondly, we show how the full string BRST cohomology in the particle limit of string theory on $AdS_3 \times S^3$ renders the quadratic Casimir diagonalizable, and reduces the Hilbert space to finite dimensional representations of the space-time supersymmetry algebra. Our analysis provides an efficient way to calculate the Kaluza-Klein spectrum for supergravity on $AdS_3 \times S^3$. It may also lead to the identification of an interesting and simpler subsector of logarithmic supergroup conformal field theories, relevant to string theory.

KEYWORDS: Conformal Field Models in String Theory, Extended Supersymmetry

¹Unité Mixte du CNRS et de l'Ecole Normale Supérieure associée à l'université Pierre et Marie Curie 6, UMR 8549.

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1 Introduction

It is believed that the holographic nature of quantum gravity [1, 2] renders anti-de Sitter compactifications of string theory equivalent to dual gauge theories [3]. Many pairs of dual theories have been proposed. They often involve anti-de Sitter backgrounds of string theory

with Ramond-Ramond flux which arise in the near-brane limit of backreacted D-branes [4]. That makes it desirable to compute the spectrum of string theory on these Ramond-Ramond backgrounds. The light-cone gauge is the most efficient gauge choice to determine these spectra at present. Nevertheless it remains interesting to further our understanding of the calculation of the spectrum in conformal gauge, for instance in a Berkovits formulation of the worldsheet string action in these backgrounds (see e.g. [5] and references thereto). These worldsheet conformal field theories involve supergroup or supercoset target spaces in many interesting examples. The whole of target space (super)symmetry is manifestly realized in these models.

Two-dimensional conformal field theories with supergroup or coset targets are also interesting in their own right. They have been studied from various perspectives (see e.g. [6–15]). One crucial feature of these theories is that they are logarithmic. The scaling operator is not diagonalizable on the state space. Moreover, this feature already manifests itself in the one-dimensional limit of these models. Indeed, the Laplacian on a supergroup is typically not diagonalizable on the space of quadratically integrable functions [7, 8, 10, 16]. Moreover, the space of functions is typically an infinite dimensional indecomposable representation of the supersymmetry algebra. Although space-time superisometries are manifest, their representation is intricate.

String theory in particular Ramond-Ramond backgrounds and in conformal gauge will be built using such a conformal field theory, but it will only make use of a physical state space determined by a BRST cohomology. It is insensitive to BRST exact features of the worldsheet conformal field theory. For complicated target spaces though, it can be hard to discern what the BRST exact data in the worldsheet conformal field theory are that one may wish to ignore. To gain insight into this question, we study simpler models that exhibit some of the same crucial features.

Concretely, in this paper we compute the BRST cohomology for a reparameterization invariant particle living on a supergroup manifold, and investigate to what extent the curious features of the space of quadratically integrable functions survive in the physical state space. In section 2 we show that implementing reparameterization invariance on the physical state space is enough to render the quadratic Casimir diagonalizable. We illustrate the details of the structure of the representation space in the case of the supergroup $GL(1|1)$ in section 3. In section 4 we compute the full string BRST cohomology for compactification independent states in $AdS_3 \times S^3$ string theory with Ramond-Ramond and Neveu-Schwarz-Neveu-Schwarz flux, and show that due to the more refined cohomology, the space of physical states decomposes into finite dimensional representation spaces of the supersymmetry algebra. As a byproduct, we show that this gives an efficient derivation of the equivalence of this subsector of string theory to supergravity, as well as a brief and manifestly supersymmetric derivation of the Kaluza-Klein spectrum. Finally, we draw general lessons for applications of logarithmic conformal field theories to string theory.

2 A massless particle on a supergroup

In this section we study a massless particle on a supergroup G and argue that its Hamiltonian becomes diagonalizable in cohomology.

2.1 The action

Our model for a massless particle on a supergroup G is defined in terms of a reparameterization invariant action. Due to the fermionic directions in target space, the model will be non-unitary. The action is:

$$S = \int_L d\tau e^{-1} \langle g^{-1} \partial_\tau g, g^{-1} \partial_\tau \rangle, \tag{2.1}$$

where the map $g : L \rightarrow G : \tau \mapsto g(\tau)$ maps the worldline L of the massless particle into the group manifold G , and $\langle \cdot, \cdot \rangle$ denotes an invariant metric on a Lie super algebra \mathfrak{g} . We choose it to be proportional to the supertrace in a matrix representation of the algebra. The field e is an einbein on the worldline of the particle. After gauge fixing the reparameterization invariance through the gauge choice $e = 1$, we find the gauge fixed action:

$$S = \int d\tau \langle g^{-1} \partial_\tau g, g^{-1} \partial_\tau \rangle + \int d\tau b \partial_\tau c, \tag{2.2}$$

where we introduced the (b, c) ghosts to take into account the measure factor arising from gauge fixing. The physical state space in the quantum theory will be determined by the cohomology of the BRST operator:

$$Q_B = c C_2 \tag{2.3}$$

where the quadratic Casimir C_2 equals the Hamiltonian of the system.

2.2 The quadratic Casimir in cohomology

When we consider a particle on a supergroup, its wave-function will correspond to a function on the supergroup. Functions on a supergroup can be expanded in the fermionic coordinates on which they depend. We will study a space of functions such that the coefficients in the fermionic coordinate expansion are quadratically integrable. We refer to this space as the space of quadratically integrable functions on the supergroup G and will loosely denote it by $\mathcal{L}^2(G)$. When we consider the space of functions on a supergroup, the group invariant Laplacian, which is equal to the quadratic Casimir operator C_2 , acts on the space. The operator turns out to have a non-trivial Jordan form — it is generically not diagonalizable [7, 8, 10, 16].

The first point we wish to make is that in our model the quadratic Casimir is diagonalizable in cohomology. When we impose the Siegel condition $b = 0$ on physical states, we will also need to impose that the quadratic Casimir C_2 annihilates physical states. For simplicity, let's suppose first that the quadratic Casimir C_2 has the form:

$$C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4}$$

in a certain sector of the state space. Its square is zero on this two-dimensional generalized eigenspace. It is then clear that the state $(0 \ 1)^t$ will not be annihilated by C_2 , while on the space spanned by the state $(1 \ 0)^t$ alone, the quadratic Casimir *is* diagonalizable. It is diagonalizable in cohomology. This argument also holds when the quadratic Casimir has a more elaborate Jordan form.

We think it is interesting to flesh out this general observation in a simple concrete example. This will give us the opportunity to see how this simplification of the physical state space relates to other algebraic properties of the state space. It will also allow us to find the classical origin for the elimination of some generalized eigenvalue zero states from the physical state space.

3 A massless particle on the supergroup $GL(1|1)$

In this section, we will analyze in some detail the example of a particle on the supergroup $G = GL(1|1)$. This simple example has the advantage that the decomposition of the space of functions in terms of representations of the left and right regular action of the group on itself is known [7, 8, 16], and not difficult to rederive. All calculations can be done explicitly, and they illustrate concretely the more advanced algebra that we will use in a later section.

The super Lie algebra $\mathfrak{g} = gl(1|1)$ is an algebra that can be represented in terms of 2×2 supermatrices. These supermatrices have bosonic diagonal entries and fermionic off-diagonal elements. We can write the super Lie algebra in terms of generators $h_{1,2}, e_1, f_1$ which satisfy the commutation relations:

$$\begin{aligned} \{e_1, f_1\} &= h_2 \\ [h_1, e_1] &= 2e_1 \\ [h_1, f_1] &= -2f_1, \end{aligned} \tag{3.1}$$

while all other commutation relations are zero. The fermionic annihilation and creation operators e_1 and f_1 anti-commute into the central bosonic operator h_2 . We also have an operator h_1 whose eigenvalue is raised or lowered when we annihilate or create a fermion

When we have in mind applications to backgrounds of string theory with superconformal symmetries, it can be useful to think of the $gl(1|1)$ algebra as embedded into a superconformal algebra. The generators of the $gl(1|1)$ subalgebra can then be identified with a subset of the superconformal generators and are more conventionally denoted as:

$$\begin{aligned} h_2 &= \frac{1}{2}(J - \Delta) = P_- \\ h_1 &= \Delta + J = 2P_+ \\ e_1 &= Q^+ \\ f_1 &= Q^-, \end{aligned} \tag{3.2}$$

where Δ measures the conformal dimension and J the R-charge while Q^\pm are supercharges of R-charge ± 1 with correlated conformal dimension ± 1 . We note that the difference of the

R-charge and the conformal dimension is a central generator in this subalgebra. In many contexts, it is natural to take an exponentiation of the algebra in which the R-charge J is the generator of a compact $U(1)$ subgroup, while the scaling direction Δ is taken to be non-compact. A quadratic Casimir is given by the expression $C_2 = \frac{1}{2}(J^2 - \Delta^2) + 2Q^-Q^+$. If we add any function of the central generator, the resulting operator remains central.

3.1 The classical foreshadowing

In this subsection, we will show that the reduction of the physical state space has a counterpart in the classical theory. For our concrete calculations, it is convenient to choose a matrix realization of the group elements as follows (see e.g. [7]):

$$g = e^{+i\sqrt{2}\eta_-Q^-} e^{ix^-P_-+ix^+P_+} e^{+i\sqrt{2}\eta_+Q^+} = \begin{pmatrix} e^{\frac{i}{2}(-x^++x^-)} + \eta_+\eta_-e^{\frac{i}{2}(x^++x^-)} & i\eta_-e^{\frac{i}{2}(x^++x^-)} \\ i\eta_+e^{\frac{i}{2}(x^++x^-)} & e^{\frac{i}{2}(x^++x^-)} \end{pmatrix}, \tag{3.3}$$

where x^\pm are light-cone coordinates in $\mathbb{R} \times \mathbb{R}$.¹ We have two Grassmann variables η_\pm . The classical action for a massless particle (in equation (2.1)) in this parameterization is:

$$S = \int (\partial_\tau x^+ \partial_\tau x^- + 2e^{ix^+} \partial_\tau \eta_+ \partial_\tau \eta_-) d\tau, \tag{3.4}$$

and it is supplemented with the constraint equation

$$\partial_\tau x^+ \partial_\tau x^- + 2e^{ix^+} \partial_\tau \eta_+ \partial_\tau \eta_- = 0. \tag{3.5}$$

When one solves the classical equations of motion, one finds that one needs to distinguish two solution sets. The first solution set is parameterized by integration constants p_\pm, x_0^\pm as well as $\pi_\pm, \eta_{\pm,0}$:

$$\begin{aligned} x^+ &= p_- \tau + x_0^+ \\ \eta_\pm &= \frac{i}{p_-} e^{-ip_- \tau - ix_0^+} \pi_\pm + \eta_{\pm,0} \\ x^- &= -i \frac{2}{p_-^2} e^{-ip_- \tau - ix_0^+} \pi_+ \pi_- + p_+ \tau + x_0^-, \end{aligned} \tag{3.6}$$

while the second one arises when the x^+ -momentum p_- is zero, and it reads:

$$\begin{aligned} x^+ &= x_0^+ \\ \eta_\pm &= e^{-ix_0^+} \pi_\pm \tau + \eta_{\pm,0} \\ x^- &= 2ie^{-ix_0^+} \pi_+ \pi_- \frac{\tau^2}{2} + p_+ \tau + x_0^-. \end{aligned} \tag{3.7}$$

The constraint equation (3.5) on the first set reads:

$$p_+ p_- = 0, \tag{3.8}$$

¹It is easy to adapt our analysis to the case where a bosonic direction of the group manifold is compact.

or in other words the momentum $p_+ = 0$ (since $p_- \neq 0$ for the first set). More interestingly, for the solution with zero light-cone momentum p_- , we find that the constraint equation remains non-trivial:

$$\pi_+ \pi_- = 0, \tag{3.9}$$

and we therefore find a second constraint, on top of the fact that the momentum p_- is zero. Thus we find that when the momentum satisfies $p_- = 0$, the space of classical solutions is smaller than when $p_- \neq 0$. We have two types of trajectories. One is where the momentum p_+ is zero and the momenta p_- and π_{\pm} (as well as $x_0^+, x_0^-, \eta_{\pm,0}$) are arbitrary. The other type of trajectories is where the momentum p_- is zero and the product of π_+ with π_- is also zero.

The origin of the surprising structure of the solution space is a fermionic contribution to the length of the curve. We chose to parameterize the curve by the proper time. The length of the curve is always zero since we study a massless particle. The length of the curve is ordinarily the product of momenta $p_+ p_-$. However, when the momentum p_- is zero, we get a fermionic contribution to the (generalized) length from the product $\pi_+ \pi_-$. We have to put the latter combination to zero to obtain a curve of length zero. In subsection 3.3, we will see that this reduction of the classical phase space is a foreshadowing of the reduction of the state space in the quantum theory.

3.2 The space of functions on $GL(1|1)$

In this subsection, we review properties of the space of quadratically integrable functions on the supergroup $GL(1|1)$ [7, 16]. There is a (delta-function normalizable) basis of the space of integrable functions given by the exponentials:

$$\begin{aligned} e_0(p_-, p_+) &= e^{ip_- x^- + ip_+ x^+} \\ e_{\pm}(p_-, p_+) &= e^{ip_- x^- + ip_+ x^+} \eta_{\pm} \\ e_2(p_-, p_+) &= e^{ip_- x^- + ip_+ x^+} \eta_- \eta_+. \end{aligned} \tag{3.10}$$

We denote the superspace functions with quadratically integrable component functions by $\mathcal{L}^2(GL(1|1))$. For future purposes, we review the decomposition of the space of functions in terms of representations of the left and right regular action of the group [7, 16]. The infinitesimal group actions on the function space can be given in terms of differential operators. The left-invariant vector fields act on the wave-functions as [7]:

$$L_{P_-} = i\partial_{x^-} \quad L_{P_+} = i\partial_{x^+} - \eta_+ \partial_+ \quad L_{Q_+} = -i\partial_+ \quad L_{Q_-} = ie^{ix^+} \partial_- - \eta_+ \partial_{x^-}, \tag{3.11}$$

and the right invariant vector fields by:

$$R_{P_-} = -i\partial_{x^-} \quad R_{P_+} = -i\partial_{x^+} + \eta_- \partial_- \quad R_{Q_+} = ie^{ix^+} \partial_+ + \eta_- \partial_{x^-} \quad R_{Q_-} = -i\partial_-. \tag{3.12}$$

In this example, the structure of the space of functions can easily be determined by explicit calculation. It is useful to distinguish the typical and the atypical sectors of the state space (where the nomenclature originates in reference [17]).

3.2.1 Typical sector

In the typical case where $p_- \neq 0$, we have the following group matrix elements corresponding to a direct summand $M_{\langle p_-, p_+ \rangle}$ in the decomposition of the quadratically integrable functions as a representation space under the left-right regular action [7]:

$$M_{\langle p_-, p_+ \rangle}(g) = \begin{pmatrix} e^{ip_-x^- + i(p_+ - 1)x^+} & i\eta_- e^{ip_-x^- + i(p_+ - 1)x^+} \\ ip_- \eta_+ e^{ip_-x^- + i(p_+ - 1)x^+} & p_- \eta_- \eta_+ e^{ip_-x^- + i(p_+ - 1)x^+} + e^{ip_-x^- + ip_+x^+} \end{pmatrix}. \quad (3.13)$$

The functions in the first row of the above matrix form a basis of the summand $H_{\langle p_-, p_+ \rangle}^R$ in the right regular representation, where the space $H_{\langle p_-, p_+ \rangle}^R$ is a typical graded representation space labeled by eigenvalues $\langle p_-, p_+ \rangle$ (of one state — the other state in the representation has eigenvalues p_- and $p_+ - 1$). This can be checked by acting with the right generators given in equation (3.12). The second row forms a basis of the summand $H_{\langle p_-, p_+ \rangle}^{R'}$ which is the same typical representation, with opposite grading. When we consider the left regular action, we note that it mixes the two representation spaces with each other, to form a tensor product representation $H_{\langle -p_-, -p_+ \rangle}^L \otimes H_{\langle p_-, p_+ \rangle}^R$ of the left-right group actions. The part of the function space with momentum $p_- \neq 0$ decomposes as a direct sum of these tensor product representations. This type of structure is familiar from the Peter-Weyl theorem for compact Lie groups. If we fix the momentum p_- to be non-zero, we can draw a picture of the action of the left and right generators on the summands of the state space (see figure 1). In the diagram, we drop the common value of $-i\partial_{x^-}$ from the notation, write the eigenvalue of $-i\partial_{x^+}$ as a superscript and denote the function $p_- e_2(p_-, p_+ - 1) + e_0(p_-, p_+)$ by $p_- e_2^{p_+ - 1}$. For ease of illustration, we arbitrarily took the spacing between consecutive values of $-i\partial_{x^+}$ to be one in the diagram.

Figure 1 represents the fact that each summand contains four states, that pair up two-by-two to form representations of either the left or the right group action. The fermionic creation and annihilation operations are invertible in this sector of the state space. The action of the quadratic Casimir is diagonal and can be taken to be proportional to the product of lightcone momenta $p_- p_+$.

3.2.2 Atypical sectors

The structure of the state space is more interesting when the lightcone momentum p_- satisfies $p_- = 0$. Because we concentrated on a $\mathfrak{gl}(1|1)$ algebra, this condition has a particular chirality. It corresponds to demanding that the difference of twice the conformal dimension and the R-charge is equal to zero. In this chiral subsector, the action of the generators on a basis of states is as in figure 2.

We see that the vectors $e_0^{p_+}, e_+^{p_+ - 1}, e_-^{p_+}, e_2^{p_+ - 1}$ make up an indecomposable right representation. The vectors $e_0^{p_+}, e_+^{p_+}, e_-^{p_+ - 1}, e_2^{p_+ - 1}$ generate an indecomposable left representation. Note that the action of the fermionic generators is no longer invertible. The quadratic Casimir acts as the differential operator $2e^{ix^+} \partial_- \partial_+$ in these representations, i.e. it maps the state $e_2^{p_+}$ to the state $e_0^{p_+ + 1}$ (for all momenta p_+) and it annihilates all other states. This sector forms an infinite dimensional indecomposable representation of the left-right action of the group. We denote this summand of the representation space by C_{p_+} .

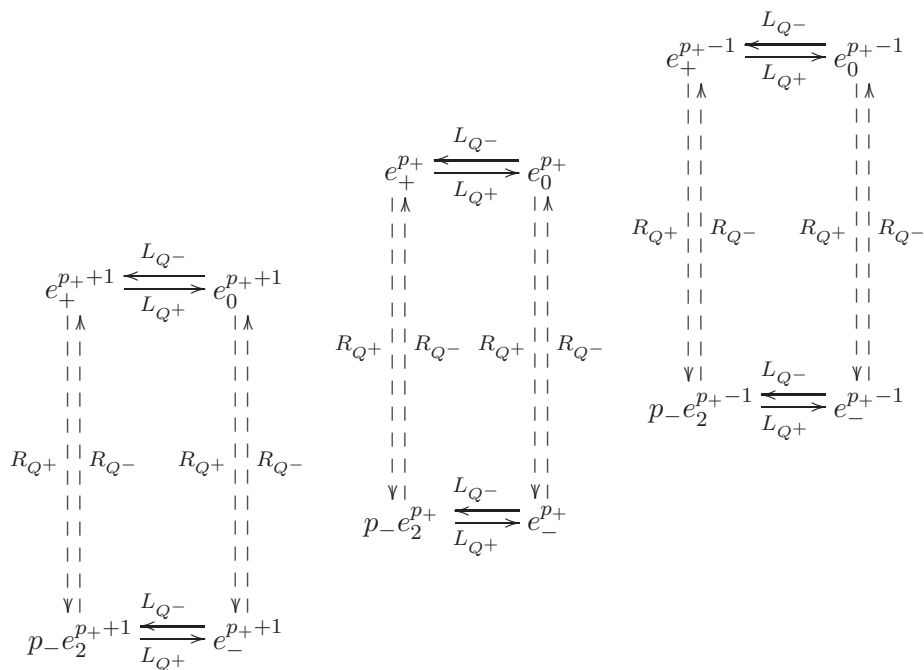


Figure 1. The typical sector of the space of functions on $GL(1|1)$ as a representation of the left-right regular action of the group. The left action is indicated with solid lines, the right action with stripes.

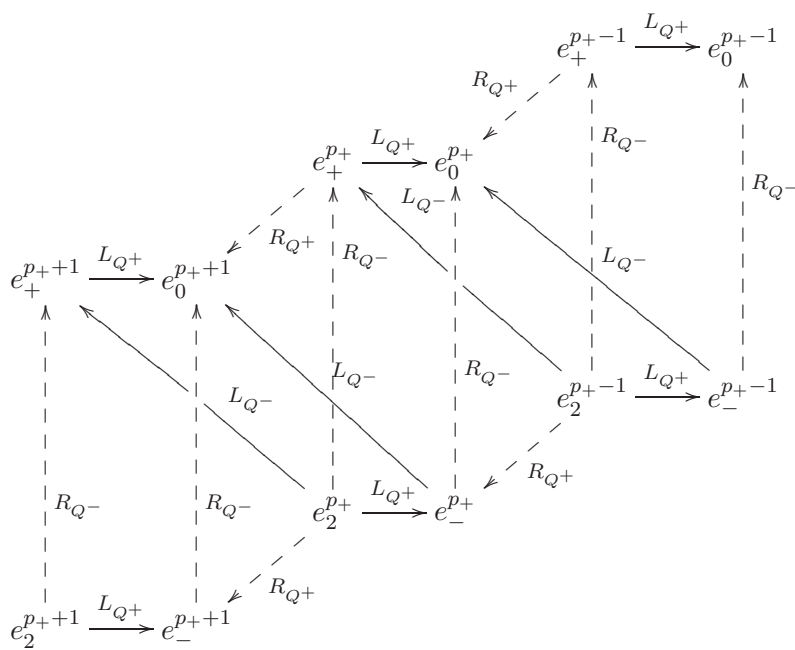


Figure 2. The left-right group action on functions on $GL(1|1)$ transforming in atypical representations.

In summary, if we take the momentum p_+ to be continuous, the following decomposition for the space of quadratically integrable functions holds [7, 16]:

$$\mathcal{L}^2(GL(1|1)) = \int_{p_- \neq 0} \int dp_+ H_{\langle -p_-, -p_+ \rangle}^L \otimes H_{\langle p_-, p_+ \rangle}^R \oplus \int_0^1 dp_+ C_{p_+}. \quad (3.14)$$

The representation space C_{p_+} decomposes with respect to either the left or the right action only as an infinite sum of projective four-dimensional representations [7, 16]. Note that projective representations cannot be further extended without introducing direct summands. The quadratic Casimir is diagonalizable, except in the atypical sector. There it maps top states to bottom states, and is otherwise zero.

3.3 The BRST cohomology

In the quantum theory for the gauge fixed particle action, we tensor the state space with a (b, c) ghost system. We take the latter to be a two-state system on which the quantum operators act as:

$$\begin{aligned} b|\uparrow\rangle &= |\downarrow\rangle, & b|\downarrow\rangle &= 0 \\ c|\downarrow\rangle &= |\uparrow\rangle, & c|\uparrow\rangle &= 0. \end{aligned} \quad (3.15)$$

3.3.1 Typical sector

In the typical sector where $p_- \neq 0$, the quadratic Casimir acts diagonally with eigenvalues $p_+ p_-$. The cohomology localizes on momentum p_+ equal to zero. This is the familiar on-shell condition for a massless particle.

3.3.2 Atypical sector

When the momentum p_- is zero, the calculation of the cohomology is more interesting. In this case, the quadratic Casimir is zero on a large part of the space, but it maps top states $e_2^{p_+}$ to bottom states $e_0^{p_++1}$. The BRST closed states are the up states, as well as the down states annihilated by the quadratic Casimir. The BRST exact states are the up e_0 states. Therefore, the closed non-exact states are the up states tensored with e_{\pm}, e_2 , and the e_{\pm}, e_0 down states.

The physical states satisfying the Siegel condition $b|\text{phys}\rangle = 0$, namely the down states, are the states e_{\pm}, e_0 . Dually, the up states that are physical are e_{\pm}, e_2 . In many contexts, like the model of a particle in flat space, where the Hamiltonian acts diagonally, the Siegel condition lifts a two-fold degeneracy in the physical state space cohomology. Here, because the quadratic Casimir is not diagonalizable, this is not the case. The cohomology on up states is dual to that on down states (under a duality that maps all fermionic occupied states to unoccupied states and vice versa).

The action of the quadratic Casimir. In this example, we can see by inspection that the action of the quadratic Casimir on the BRST cohomology has become diagonal. This is a hands-on illustration of the general argument given in subsection 2.2. We also saw in subsection 3.1 that this feature has a classical counterpart.

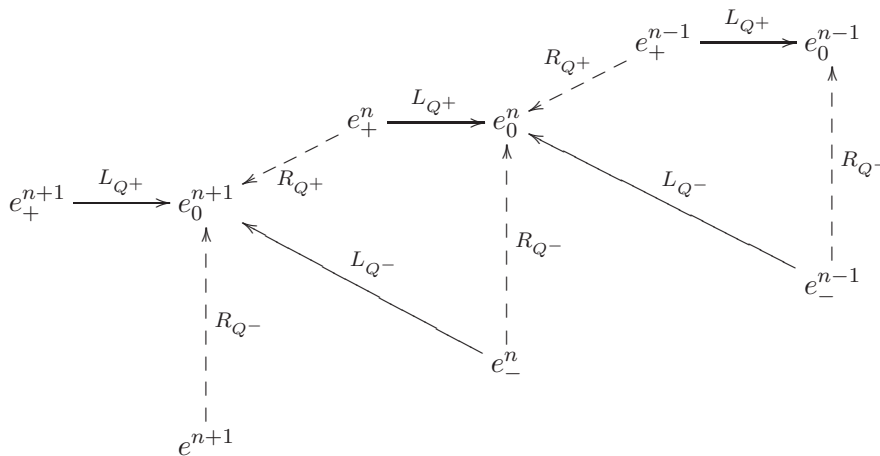


Figure 3. The left-right action of the superalgebra $\mathfrak{gl}(1|1)$ on the down physical state space in the atypical sector.

3.4 The action of space-time supersymmetry

We now wish to point out a further algebraic property arising in this and more elaborate examples. First of all, we note that the space-time isometries commute with the BRST operator. The BRST cohomology is therefore again a representation space of the supersymmetry algebra. What is the structure of this representation space?

3.4.1 Typical

In this sector, the cohomology coincides with the function space, and the representation space of the left-right supercharges are ordinary tensor products of long multiplets (i.e. typical Kac modules).

3.4.2 Atypical

In the atypical sector, we will draw the representation space for the down states. The top states are not present in the cohomology of down states. Removing them from the diagram of states gives rise to the supercharge actions on the physical state space depicted in figure 3.

We obtain a (non-unitary) indecomposable and infinite dimensional representation space of the space-time super isometry algebra.

3.5 Summary

The non-diagonalizability of the Laplacian on the function space was removed in the physical state cohomology. We were motivated to analyze this phenomenon because in an embedding of the particle model in a conformal field theory, the non-diagonalizability of the Laplacian on the function space is inherited by the scaling operator [7, 8], and it makes these conformal field theories with supergroup targets logarithmic in nature. It is interesting to note that worldline reparameterization invariance reduces the physical state space such that the quadratic Casimir becomes diagonalizable.

In these particle models, we will still be left with a non-unitary theory in cohomology. We could define further space-time supercharge cohomologies, on which the calculation of certain chiral correlation functions localizes to unitarize these models. Alternatively, one may simply analyze further features of these interesting non-unitary theories.

In the following however, we will explore further how to reconcile these models with expected properties of stringy space-time physics. The low-dimensional example we studied up to now corresponds to the supersymmetrization of two light-cone directions. To make progress, we consider an example with more space-time directions, in which physical fluctuations can survive. And we will study a more refined cohomology that will alter the structure of the physical state space more drastically than the cohomology associated to reparameterization invariance alone.

4 A massless particle on $AdS_3 \times S^3$

A central building block in the Berkovits formulation of string theory on $AdS_3 \times S^3$ with Neveu-Schwarz Neveu-Schwarz and Ramond-Ramond fluxes is a conformal field theory with $PSU(1,1|2)$ supergroup target [18]. The physical state space is determined by computing a cohomology on a large space of conformal field theory states [18, 19]. In this section, we wish to solve for an important subset of the physical state space of this model. It corresponds to the point particle limit for string theory on $AdS_3 \times S^3$, in which we moreover concentrate on compactification independent excitations. In other words, we solve for the physical supergravity modes which correspond to massless particle excitations on the supergroup. In this limit, the subspace of physical states is determined by a cohomology on the space of functions on the supergroup. We will compute this cohomology, and show how it simplifies the function space as a representation space of the super isometry group. We also compare the physical states we obtain to the result of lengthy calculations linking the Berkovits formulation to supergravity [19], as well as the Kaluza-Klein reduction of supergravity on $AdS_3 \times S^3$ [20]. The cohomological method for determining the physical excitations, as well as their supermultiplet structure will turn out to be efficient. In this section we will draw on more advanced (super) algebra techniques which the reader can study for instance from the references [17, 21–24].

4.1 The space of functions on the group $PSU(2|2)$

We will mostly work with the version of the supergroup target which has a compact maximal bosonic subgroup. We consider the space of quadratically integrable functions on $G = PSU(2|2)$. The super Lie algebra \mathfrak{g} corresponding to the group consists of four by four hermitian matrices with zero trace and supertrace. To compute the physical state space, it is useful to understand how the function space decomposes into representations of the left and right regular action of the group on itself. We will start by analyzing the left regular action. To that end, we think of the function space as consisting of a component function (which we take to be the top component of a superfield on the supergroup) on the bosonic subgroup $G_0 = SU(2) \times SU(2)$ acted upon by all fermionic generators through

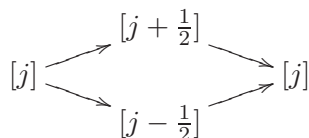


Figure 4. The short multiplet composition series of atypical Kac modules.

the left action of the group on itself. The function on the bosonic subgroup can be decomposed by the Peter-Weyl theorem into representations of $\mathfrak{g}_0 = su(2) \oplus su(2)$ of the type $\sum_{j_1, j_2=0, \frac{1}{2}, \dots} M_{j_1, j_2}^L \otimes M_{j_1, j_2}^{*R}$ where M_{j_1, j_2}^L is a representation of spins j_1 and j_2 under the left action of the group on itself. Therefore the space of functions on the supergroup splits into a sum of representations of the left regular action of the type [9]:

$$A = U(\mathfrak{g}) \otimes_{\mathfrak{g}_0} M_{j_1, j_2}^L, \tag{4.1}$$

where $U(\mathfrak{g})$ indicates the universal enveloping algebra of the super Lie algebra \mathfrak{g} . We have kept the tensor product with the representation space of the right action of the bosonic subgroup implicit for the moment. By the Poincaré-Birkhoff-Witt theorem for the universal enveloping algebra $U(\mathfrak{g})$, the representation space A consists of the states in the $su(2) \oplus su(2)$ representation, acted upon by all eight fermionic operators. The representation space A has dimension $2^8(2j_1 + 1)(2j_2 + 1)$. We would like to decompose it with respect to the left action of the superalgebra.

4.1.1 Representations of the algebra $psu(2|2)$

In order to present the solution to this problem, we briefly review some Lie superalgebra representation theory (see e.g. [9]). We recall from the representation theory of $psl(2|2)$ that atypical Kac modules $K(j, j) = [j, j]$ (composed by acting with all creation operators on a highest weight state with spins (j, j)) are composed from short multiplets $L(j, j) = [j]$ as in figure 4 [9].

while a short multiplet $[j]$ contains the following $sl(2) \oplus sl(2)$ representations [9]:

$$[j]_{\mathfrak{g}_0} \equiv (j + 1/2, j - 1/2) \oplus 2(j, j) \oplus (j - 1/2, j + 1/2). \tag{4.2}$$

If we would draw the action of fermionic generators on bosonic multiplets within a short representation, it would be isomorphic to the diagram of the composition series of atypical Kac modules. We note that the diagrams that we draw are strictly speaking only valid for j larger than a particular lower bound at which exceptions to the above diagrams occur. Those are not essential to our discussion, and we will ignore them throughout.

The projective representations are the largest indecomposable covers of these modules. To describe them, we need more mathematical results, and we wish to discuss a subtlety in their description. It has been proven by algebraic means [21, 22] that for representations of type I supergroups that satisfy that the multiplicity of the simple quotient $L(\lambda)$ of a Kac module $K(\lambda)$ in a composition series is no more than one, there is a Bernstein-Gelfand-Gelfand reciprocity formula that holds. Namely, the multiplicity of the Kac module $K(\lambda)$

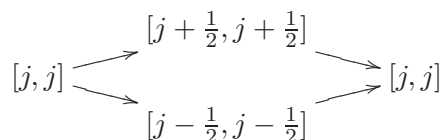


Figure 5. The Kac composition of the projective representation $P(j, j)$.

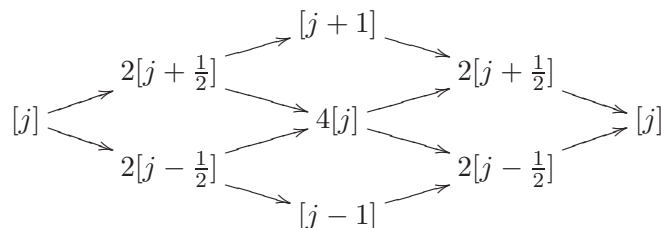


Figure 6. The composition of the projective representation $P(j, j)$ in terms of short multiplets.

in the projective representation $P(\mu)$ associated to the highest weight μ , is equal to the multiplicity of the simple module $L(\mu)$ in the Kac module $K(\lambda)$. We can state this briefly and roughly as the fact that the Kac module is covered as it is composed. In the case at hand, there is a slight complication which is (as we saw) that the multiplicity of the short representation in the composition series of the atypical Kac module is not equal to one, and that therefore the theorem quoted above cannot be applied directly. We circumvent this complication by lifting the $psl(2|2)$ representation to a representation of the algebra $gl(2|2)$. In other words, we provide the representation space with an extra grading that keeps track of the number of fermionic creation minus the number of fermionic annihilation operators that act on a ground state. Thus we lift the degeneracy of the short multiplet in the composition series, and we can apply the result on the multiplicities of Kac modules in the projective cover to confirm that the projective representation $P(j, j)$ is composed out of Kac modules as in figure 5 [9].

Moreover, we can use the results on how projective representations of the algebra $gl(2|2)$ are composed of short multiplets [24] to rederive that the projective representation of $P(j, j)$ has the structure of figure 6 [9].

For our purposes, it will be useful to have more explicit information about the grading. It can be gleaned from the results of [24] which we recall in appendix A that the additional $u(1)$ grading that an embedding in $gl(2|2)$ provides will partially lift the degeneracies in the above diagram to those in figure 7.

This result is also coded in the earlier work [23] in the action of the outer automorphisms of $psl(2|2)$ on the short multiplets. An intuitive reading of the diagram goes as follows. In the projective representation $P(j, j)$, we have two annihilation operators that act non-trivially on a generating vector on the left, to generate two new top representations in Kac composition factors (of charge -1 say). When both act consecutively, we have the fourth and last top representation in a Kac composition factor (of charge -2). All other short representations are obtained through the action of fermionic creation operators (of

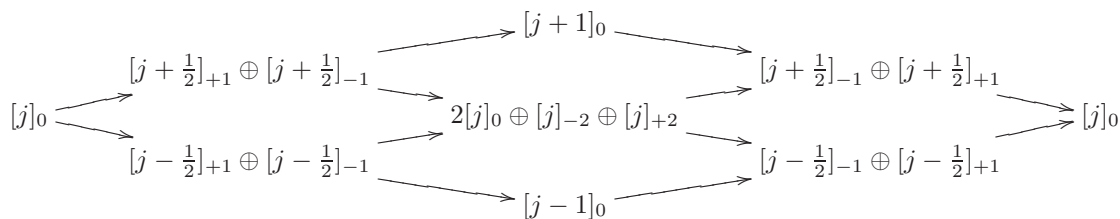


Figure 7. The graded composition of the projective representation $P(j, j)$ in terms of short multiplets.

charge +1) within a given Kac composition factor.

The method we used in this subsection to lift $sl(n|n)$ or $psl(n|n)$ representations in order to be able to apply results for $gl(n|n)$ works generically.

4.1.2 The left regular representation on the superfield

With these prerequisites in hand, we can argue how the representation space A of equation (4.1) contained within a superfield V on the supergroup decomposes with respect to the left regular action. First of all, it is known [21] that the representation space A permits a Kac composition series. Moreover, it can be reconstructed as in the proof of Lemma 2.3 of [21]. In short, the Kac composition factors correspond one to one to the representations of the bosonic subalgebra appearing in a Kac module. When the two spins $j_{1,2}$ of the bosonic representation space M_{j_1, j_2}^L on which the representation A is built are equal, $j_1 = j_2$, we have eight Kac composition modules that are atypical which combine four by four into two projective representations $P(j_1, j_1)$. When the spins satisfy $j_1 = j_2 + 1$, we have four Kac composition modules that are atypical which combine into one projective representation $P(j_1 - 1/2, j_1 - 1/2)$, and for spins $j_1 = j_2 - 1$ we obtain the projective representation $P(j_1 + 1/2, j_1 + 1/2)$. All other Kac composition modules that appear in the representation space A are typical (and therefore projective). They are direct summands in the representation A . That characterizes fully the representation space A .

When we tensor back in the right representation space M_{j_1, j_2}^{*R} , and concentrate on the atypical summands in the superfield V , we obtain the atypical part of the space of functions:

$$V_{atyp} = \sum_{j_1=0}^{\infty} P(j_1, j_1)_L \otimes (2(j_1, j_1)_R \oplus (j_1 + 1/2, j_1 - 1/2)_R \oplus (j_1 - 1/2, j_1 + 1/2)_R) \quad (4.3)$$

The right representations of the Lie algebra \mathfrak{g}_0 necessarily combine into a right short multiplet:

$$V_{atyp} = \sum_{j_1=0}^{\infty} P(j_1, j_1)_L \otimes [j_1]_R. \quad (4.4)$$

The formula gives the decomposition of V_{atyp} under the left regular action (and not under the full right regular action as we will see in detail). Again, we remind of the caveat that

these results are strictly valid only for spins j_1 large enough. We only wrote the atypical part of the representation space since we will later be interested in the space of states with quadratic Casimir generalized eigenvalue equal to zero. The typical part decomposes as in the Peter-Weyl theorem for compact groups.

4.1.3 The big picture

We have fully characterized the left regular representation on the space of functions. We want to combine it into one big picture with the action of the right generators on the representation space. The picture is big since we have sixteen short multiplets in each indecomposable representation on the left, and because the right action further mixes left representation spaces. A partial diagram of the left and right actions on left-right short multiplets that compose the representation space in the atypical sector is sketched in figures 8 and 9. We drew the left projective representations (with full lines), and (a small part of) the right action in striped lines. Figure 8 is a detail of figure 9. In the second figure, we left out the labeling of the grid by tensor products of left-right short multiplets. From these two pictures one can reconstruct the diagram extending towards higher and lower spins. The picture degenerates near spin zero (in a way that can be derived from the results in [24]). It is straightforward to further split and grade the picture with an additional $u(1)$ left and $u(1)$ right grading (as we did in figure 7). We invite the reader to picture the resulting diagram.

4.2 The cohomology defined

We have understood the structure of the space of functions on the supergroup, and can now determine which states are physical. The space of physical particle states which are compactification independent is obtained by imposing constraints. These constraints were derived in the Berkovits formulation of string theory on $AdS_3 \times S^3$ with Neveu-Schwarz Neveu-Schwarz and Ramond-Ramond fluxes [18]. First of all, it was argued that the physical cohomology is coded in a single function V on the supergroup [18]. The square of the quadratic Casimir should vanish on the function.² Thus we can restrict our analysis to generalized eigenspaces of eigenvalue zero.

It is convenient to express the further constraints in terms of generators that make the $so(4)$ representation content of the adjoint of the $psl(2|2)$ algebra manifest. We can take generators such that they satisfy the commutation relations [18]:

$$\begin{aligned}
 [K_{ab}, K_{cd}] &= \delta_{ac}K_{bd} + \delta_{bd}K_{ac} - \delta_{ad}K_{bc} - \delta_{bc}K_{ad} \\
 [K_{ab}, F_c] &= \delta_{ac}F_b - \delta_{bc}F_a, \quad [K_{ab}, E_c] = \delta_{ac}E_b - \delta_{bc}E_a \\
 \{E_a, F_b\} &= \frac{1}{2}\epsilon_{abcd}K^{cd},
 \end{aligned}
 \tag{4.5}$$

while all other commutators are zero. The index $a \in \{1, 2, 3, 4\}$ is an $so(4)$ vector index and the bosonic generators K_{ab} are in the (anti-symmetric) adjoint. The further constraint equations on the superfield V derived in [19] are that $F^4V = 0$ as well as $K_{ab}F^aF^bV = 0$.

²That does not imply that the quadratic Casimir is zero, since it is not diagonalizable.

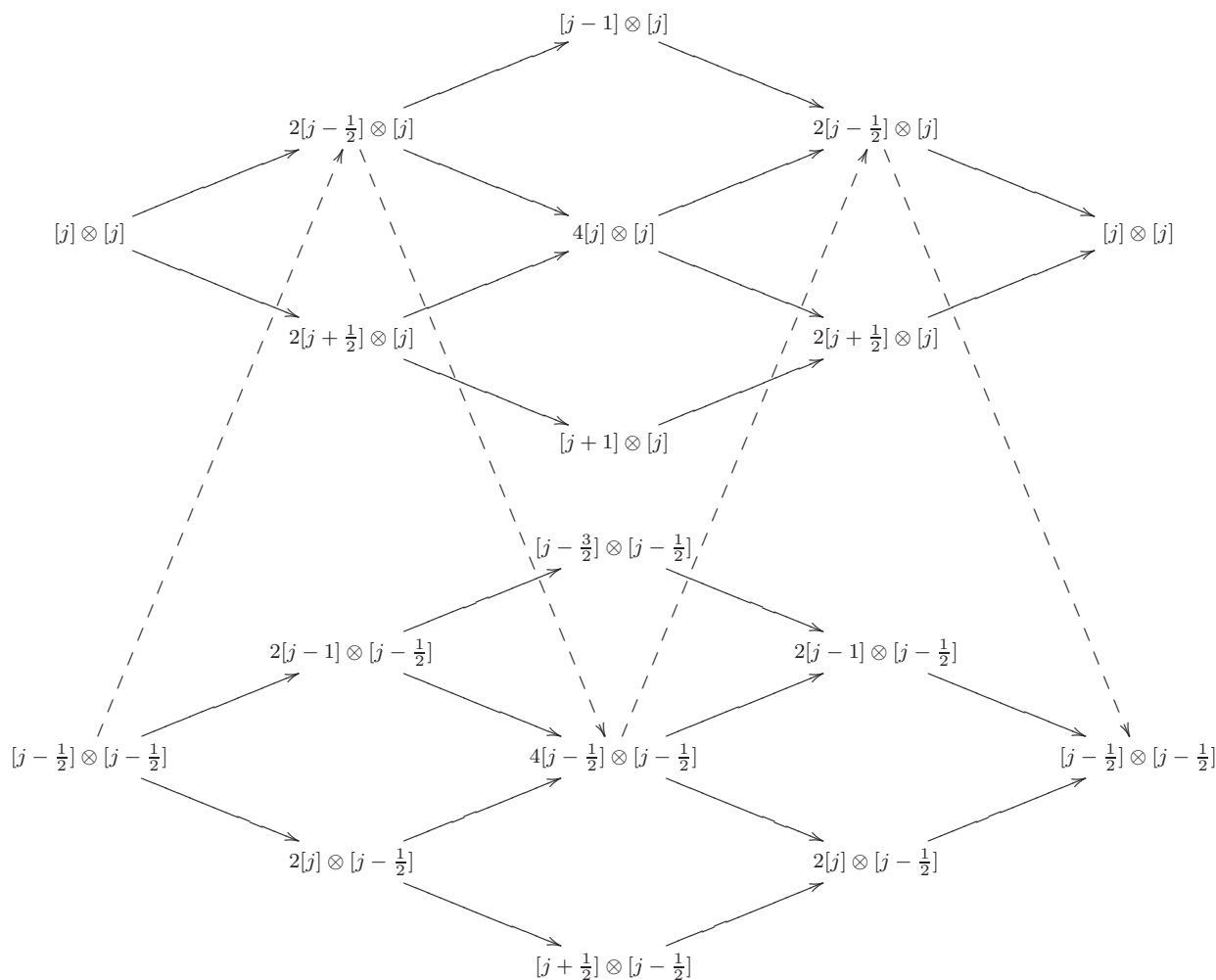


Figure 8. Detail of the decomposition of the atypical sector of the function space as a sum of left projective representations interconnected by the right group action.

Moreover, the functions of the form $K_{ab}F^a F^b W$ are gauge trivial. These constraints are valid for both the left and the right actions on the function space, and are moreover $psl(2|2)$ covariant in a subtle way spelled out in [19]. Below we will concentrate on the cohomology of the operator $K_{ab}F^a F^b$ in the space of generalized eigenfunctions of eigenvalue zero. All other left constraints will then automatically be satisfied in this cohomology.

We will see that on the cohomology of the operator $K_{ab}F^a F^b$ the quadratic Casimir vanishes. In particular, this implies that the model will be reparameterization invariant (as for the massless particle on the supergroup in section 2). However, the string cohomology is more refined and in particular it will also eliminate some unphysical fermionic directions in space-time. The underlying idea is that the string cohomology must arise from a model which also has fermionic reparameterization invariances.

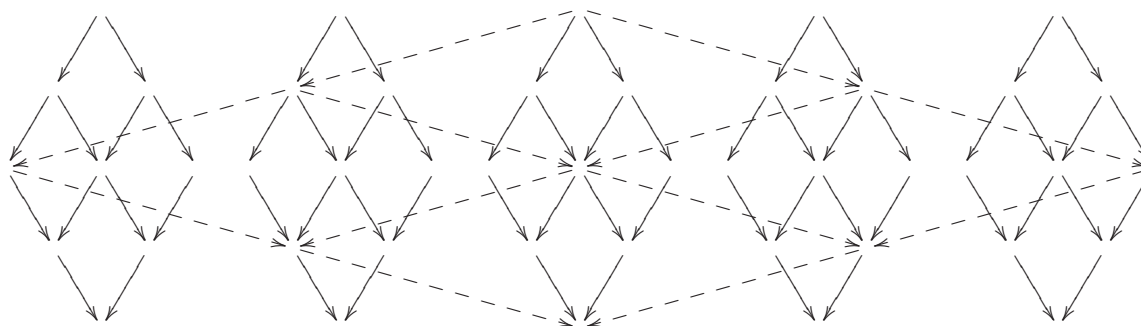


Figure 9. A selection and a slice out of an even bigger picture, showing how one (striped) right projective representation interconnects with (solid) left projective representations in the atypical sector of the space of functions on the supergroup. The figure has been rotated ninety degrees clockwise with respect to the previous one.

4.3 The left cohomology in a projective summand

First we analyze the cohomology for the constraints associated to the left action of the group on itself. Since the generalized eigenspaces of eigenvalue zero correspond to a sum of atypical projective modules for the left action, we will work in one direct summand projective module. The constraint for a state in the projective module to be physical is $K_{ab}F^aF^b|\text{phys}\rangle = 0$. All states in the projective module can be generated from a single state. It is a highest weight state of spins (j, j) with respect to the bosonic subalgebra $\mathfrak{g}_0 = sl(2) \oplus sl(2)$, and, as we saw before, it can be acted upon by up to two fermionic annihilation operators E to give new top states for Kac composition factors. On those states we can act with any number of fermionic creation operators F to fill out a Kac module. We analyze the physical state condition level by level in the number of fermionic creation operators F acting on the generator of the module. Here, a top state is a state at level F^0 .

We have that states obtained by the action of four creation operators F satisfy the constraint automatically, as do states at level F^3 . When two creation operators F act, we must take into account the following facts. The constraint equation $K_{ab}F^aF^b|\text{phys}\rangle = 0$ is scalar in terms of the bosonic subalgebra. It generates $(2j' + 1)^2$ independent constraints in Kac modules built on (j', j') representations. In other words, for each Kac composition factor in the projective module, the constraint equations eliminate one (bosonic) (j', j') representation at the middle level. We are left with states in the $(2 - 1)(j', j') \oplus (j' \pm 1, j') \oplus (j', j' \pm 1)$ \mathfrak{g}_0 -representations that satisfy the constraint equation at level F^2 , in each composite Kac module of spin (j', j') . For future purposes, we note that the states $K_{ab}F^aF^b$ acting on the top state in any Kac composition factor satisfy the constraint automatically. That is because the constraint acting on such a state gives rise to the bosonic quadratic Casimir operator. The bosonic quadratic Casimir evaluated on a top state in an atypical Kac module is zero.

The analysis at first order in the operators F is a little more intricate. As an inter-

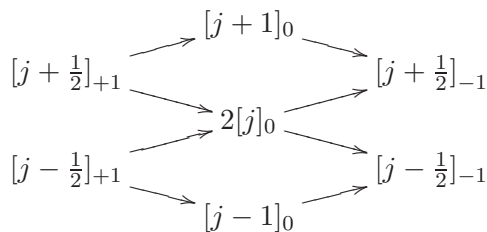


Figure 10. The left cohomology in a graded atypical projective module.

mediate step, it will be useful to compute the action of the bosonic quadratic Casimir on a generic state at level F^3 . Since the bosonic quadratic Casimir C_2^{bos} satisfies the following relation with the total quadratic Casimir C_2^{tot} : $E_a F^a = C_2^{tot} - C_2^{bos}$, we will start by computing the action of $E_a F^a$ on a state at level three:

$$\begin{aligned}
 & E_e F^e c_d \epsilon^{abcd} F_a F_b F_c |top\rangle = \\
 & (3c^{[a} K^{bc]} F_a F_b F_c + \frac{1}{4} c^d \epsilon^{abce} F_e F_b F_c F_a E_d) |top\rangle = \\
 & (3c^{[a} K^{bc]} F_a F_b F_c - c_a \epsilon^{abce} F_e F_b F_c E^d F_d) |top\rangle = \\
 & (3c^{[a} K^{bc]} F_a F_b F_c + c_a \epsilon^{abce} F_e F_b F_c (C_2^{bos} - C_2^{tot})) |top\rangle = \\
 & (3c^{[a} K^{bc]} F_a F_b F_c + C_2^{tot} c_d \epsilon^{abcd} F_a F_b F_c) |top\rangle. \quad (4.6)
 \end{aligned}$$

We conclude that we have that the operator $C_2^{bos} = C_2^{tot} - E^a F_a$ acting on a state at level three is zero if and only if $c^a K^{bc} F_a F_b F_c |top\rangle$ is equal to zero. This implies that there is a state $c^a F_a |top\rangle$ at level one which satisfies the constraint equation for every state at level three whose bosonic quadratic Casimir is zero. That implies that the physical states at level one in a Kac composition factor built on top states with spin (j', j') are the $sl(2) \oplus sl(2)$ representations $(j' + 1/2, j' + 1/2)$ and $(j' - 1/2, j' - 1/2)$. Finally, at level zero, there are no solutions to the constraint equation. Thus, we have found all closed states.

The gauge trivial or $K^{ab} F_a F_b$ exact states are found as follows. At level F^4 , all states are gauge trivial. At level F^3 , we use again the calculation above that says that we can reach all level three states whose bosonic quadratic Casimir is non-zero. We are left with the states $(j' \pm 1/2, j' \pm 1/2)$ at level three. At level two, the states $K_{ab} F^a F^b |top\rangle$ are gauge trivial, and form a (j', j') representation which is different from the one excluded by the physical constraint condition (as follows by the remark made previously on the states at level two). Thus, in each Kac composition factor $[j', j']$, we are left with the representation content of the two middle short multiplets $[j' \pm 1/2]$. We apply this reasoning to all Kac composition factors in a left projective module and find that starting from figure 7, we are left with a representation of the left $psl(2|2)$ action on the cohomology described by figure 10.

4.4 The full cohomology

We have just computed the cohomology with respect to the generators of the left action of the supergroup on itself. We now need to further compute the cohomology with respect

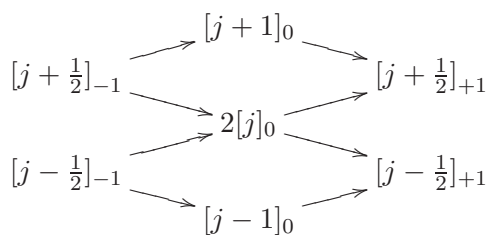


Figure 11. The right cohomology in a graded atypical projective module.

to the right action of the supergroup. Since the cohomological operators commute (since the left and the right action of the supergroup on itself commute), we can compute the cohomologies independently, and then restrict to the representations which are non-trivial in both complexes.

Since the right action of the group on itself is isomorphic to the left action of the group on itself, we have a very similar answer for the right-moving cohomology in the right atypical projective modules. There is one important difference, which is that we assign the opposite grading to the right fermionic creation and annihilation operators.³ To make this concrete, let's first define the algebra of generators of the right action of the group on itself to be again a $psl(2|2)$ algebra as in equations (4.5). We denote all of them with an extra bar.

The right cohomology is now taken with respect to an operator $\bar{K}_{ab}\bar{F}_a\bar{F}_b$ of opposite $u(1)$ grading. Thus, where the representations $[j \pm 1/2]_{+1}$ survived in the left cohomology, the representations $[j \pm 1/2]_{-1}$ will survive in the right cohomology, and vice versa. The resulting right cohomology in a graded atypical right projective module will be as in figure 11.

Therefore, in each projective module, after taking both left and right cohomologies into account (combining figures 10 and 11 with figures 8 and 9), we will only be left with the middle short multiplets $[j+1]_0 \oplus 2[j]_0 \oplus [j-1]_0$ of zero grading. Indeed, the grade +1 representations one removed from the top level are eliminated by the right cohomology (see figure 11) while the grade -1 representations one removed from the top level are eliminated by the left cohomology (see figure 10). A similar reasoning, exchanging left and right cohomologies in the argument, shows that the level one removed from the bottom level is also entirely eliminated in the double complex.

The full solution to our cohomological problem is then a sum over the spin j of the representations $([j+1]_L \oplus 2[j]_L \oplus [j-1]_L) \otimes [j]_R$, where we tensored in the right short multiplet of equation (4.4). In conclusion, we found the physical state space:

$$V_{phys} = \sum_{j=0}^{\infty} ([j+1]_L \otimes [j]_R \oplus 2[j]_L \otimes [j]_R \oplus [j]_L \otimes [j+1]_R), \tag{4.7}$$

³This is dictated for instance by the demand that one recuperates flat space supergravity in the infinite radius limit.

where we have written the solution in a manifestly left-right symmetric manner. We note that the full cohomology has reduced to a direct sum of tensor product spaces of short representations of the left and right supersymmetry algebra. The big infinite dimensional indecomposable structure has been cut into finite dimensional and unitary representations by the sharp scissors of physical cohomology.

Remarks. Since the final result localized on middle short multiplets in the Kac composition factors, it should be clear that the quadratic Casimir itself vanishes in cohomology (since the quadratic Casimir acts diagonally up to a term that changes the level of short multiplets). All constraints on physical states are automatically satisfied once we restrict to the above left-right cohomology.

An important difference with the reparameterization invariant superparticle on $GL(1|1)$ is the fact that the physical cohomology consists of finite dimensional representations of the supergroup. The origin of this further reduction lies in the fact that the Berkovits cohomology is more refined, and in particular eliminates all fermionic target space directions, rendering the model unitary. To obtain a similar finding in the $GL(1|1)$ case, one would need to refine the cohomology beyond the quadratic Casimir operator, for instance by introducing a BRST operator proportional to a space-time supercharge.

4.5 The comparison with Kaluza-Klein supergravity results

We can compare our final answer to two related results in the literature. Firstly, in [19] it was shown that the physical state conditions agree with the linearized supergravity equations of motion, by explicitly realizing the action of the symmetry algebra as differential operators acting on the component fields. Secondly, in [20] the Kaluza-Klein reduction of $(2, 0)$ chiral supergravity on $AdS_3 \times S^3$ was performed in terms of the component fields. The final result of this two-step analysis of physical states can be seen in figures 1, 2 and 3 in [20]. In our compact notation, the figures 2 and 3 correspond to two $[j]_L \times [j]_R$ representations of the algebra $psu(2|2)$.⁴ Similarly, by rendering the $su(2) \oplus su(2)$ representation content of the multiplets $[j-1]_L \times [j]_R$ and $[j]_L \times [j-1]_R$ manifest, we can match them onto the multiplet visualized in figure 1 of [20], and its conjugate multiplet. We have found full agreement.

In passing we note that the technique used in [20] of comparing Kaluza-Klein reduction on a sphere to Kaluza-Klein reduction on AdS_3 , by analytic continuation, precisely agrees with the analytic continuation technique used here. We claim therefore that the analysis in the case of $PSU(1, 1|2)$ runs along precisely the same lines as the analysis performed in this paper. The crucial technical aspect of the analysis will be that the weight spaces of the representations that arise are all finite dimensional. It will be interesting to confirm this expectation by explicitly analyzing the extension of the results of [24] on the structure of projective representations to the case of projective representations built on discrete lowest and highest weight representations, and to carefully state the mathematical and cohomological results in the context of the category of representations with finite dimensional weight spaces.

⁴We have that $n = 1$ in [20] since we only have a single tensor multiplet in our supergravity theory [18].

In conclusion, we observe that we not only coded the supergravity equations of motion [19] algebraically, but we also immediately obtained their solution upon Kaluza-Klein reduction of supergravity on $AdS_3 \times S^3$ [20]. By keeping the space-time super isometries manifest, we were able to calculate very efficiently. We thus reaped a reward for working in the Berkovits formalism.

5 Conclusions

The space of functions on a supergroup has a quadratic Casimir, or Laplacian, with non-trivial Jordan form. That property is inherited by conformal two-dimensional sigma-models with supergroup target. We showed that for a massless particle on a supergroup with reparameterization invariant action, the quadratic Casimir operator becomes diagonalizable in cohomology.

Secondly, to analyze further how the on-shell spectrum of string theory in AdS backgrounds with RR flux unitarizes in conformal gauge, we studied the stringy physical state space cohomology for a particle on the supergroup $PSU(1,1|2)$. By keeping space-time supersymmetry manifest at all stages, we were able to efficiently compute the Kaluza-Klein supergravity spectrum (corresponding to the particle limit), and to understand algebraically how unitary superconformal multiplets arise in cohomology.

We believe our kinematical analysis shows that we should make an effort to isolate those properties of the logarithmic conformal field theories arising on supergroups and their cosets that will survive in the physical state space of string theory. From our study it is clear that a lot of the intricate properties of correlation functions associated to the logarithmicity of the conformal field theories will not survive in the BRST cohomology, simply because the states involved in those intricate correlators are not physical. It is an important open problem to thoroughly understand how to efficiently isolate the stringy data within these logarithmic conformal field theories.

As a byproduct of our analysis of these questions, we showed that by using super algebra we can very efficiently compute Kaluza-Klein supergravity spectra on maximal bosonic subgroups of supergroups. Our technique generalizes to cosets of supergroups, like $AdS_5 \times S^5$ or $AdS_2 \times S^2$, etcetera, and is likely to provide a very efficient calculation of the full Kaluza-Klein spectrum.

We find it interesting that working in a manifestly supersymmetric formalism requires one to adopt super algebra representations that are considerably more intricate than Kac modules.

Acknowledgments

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A A few results in $gl(2|2)$ representation theory

In the bulk of the paper, we refer to some results in the representation theory of the superalgebra $gl(2|2)$ of four-by-four super matrices. The superalgebra $sl(2|2)$ is a subalgebra of $gl(2|2)$, consisting of matrices of zero supertrace, and the superalgebra $psl(2|2)$ is an ideal of $sl(2|2)$ where we mod out by the identity matrix. As a consequence, a representation of $psl(2|2)$ lifted to a representation of $gl(2|2)$ will have a trivial representation of the identity matrix, while the representation of the other extra $u(1)$ is not determined uniquely. Note that under the extra anti-diagonal $u(1)$, the fermionic entries of the supermatrix, corresponding to the fermionic generators, are charged. The point of embedding the $psl(2|2)$ representations in $gl(2|2)$ representations is that we have this extra $u(1)$ charge grading at our disposal in order to distinguish various representation spaces.

Lifting representations. The super algebra $gl(2|2)$ has a Cartan subalgebra \mathfrak{h} of diagonal matrices $H = \text{diag}(h_1, h_2, h_3, h_4)$. We define linear functionals $\epsilon_i(H) = h_i$ and $\delta_j(H) = h_{2+j}$ for $i \in \{1, 2\}$. The algebra has the roots $\epsilon_i - \epsilon_j, \delta_i - \delta_j$ for $i \neq j$ and $\epsilon_i - \delta_j, \delta_i - \epsilon_j$ for $i, j \in \{1, 2\}$. Here we follow [24] closely, and denote the weights λ of a $gl(2|2)$ representation by $\lambda = (a_1, a_2|b_1, b_2)$ for the weight $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + b_1\delta_1 + b_2\delta_2$. We have that the weight $\lambda = (a_1, a_2|b_1, b_2)$ is atypical when one of the numbers $a_1 + b_1 + 1, a_1 + b_2, a_2 + a_1, a_2 + b_2 - 1$ is zero. It is maximally atypical when the weight is of the form $\lambda = (a_1, a_2| -a_2, -a_1)$.

We want to lift representations of the super algebra $psl(2|2)$ to representations of the super algebra $gl(2|2)$. To that end, we demand first of all that the identity matrix in $gl(2|2)$ be represented trivially, namely that the coefficients of the weight λ satisfy $\sum_{i=1}^2 (a_i + b_i) = 0$. It should also be clear that the spins j_1, j_2 of the $sl(2) \oplus sl(2)$ subalgebras of both $gl(2|2)$ and $psl(2|2)$ are associated to the coefficients of the weights $\epsilon_1 - \epsilon_2$ and $\delta_1 - \delta_2$ in the weight λ while there is also another overall anti-diagonal $u(1)$ associated to the coefficient of the weight $\sum_i (\epsilon_i - \delta_i)$ in the weight λ . Therefore, a possible choice of lift of a $psl(2|2)$ representation characterized by spins $j_{1,2}$ is to take the weight of the lifted representation of $gl(2|2)$ to be $\lambda = j_1(\epsilon_1 - \epsilon_2) + j_2(\delta_1 - \delta_2) = (j_1, -j_1|j_2, -j_2)$. If we consider positive spins only, we have an atypical weight when $j_1 = j_2$. Indeed, the Kac module built on a ground state with spins $j_1 = j_2$ is atypical. When the spins are equal, we automatically have maximal atypicality from the perspective of the algebra $gl(2|2)$.

The Kac composition series for maximally atypical modules. We concentrate on the relevant case of the atypical $psl(2|2)$ modules, which lift to maximally atypical modules of the algebra $gl(2|2)$. Moreover, we will focus on spins $j_1 = j_2 = j$ which are not too small, to avoid exceptional cases. We then have from the results of ([24] theorem 4.1.5), that the Kac modules that appear in the Kac decomposition series of the projective cover are the modules $K(j, -j|j, -j), K(j, -j + 1|j - 1, -j), K(j + 1, -j|j, -j - 1)$ as well as $K(j + 1, -j + 1|j - 1, -j - 1)$. When we restrict to the $psl(2|2)$ action, these Kac modules of $gl(2|2)$ correspond to $psl(2|2)$ Kac modules $K(j, j), K(j - 1/2, j - 1/2), K(j + 1/2, j + 1/2)$ and $K(j, j)$. Their anti-diagonal $U(1)$ charges (divided by two) distinguish the first and last $K(j, j)$ representations. Their anti-diagonal gradings are 0, 1, 1, 2 respectively.

The composition series. We can also borrow the result for the composition series of the projective cover in terms of short multiplets from the $gl(2|2)$ result. Indeed, the result of ([24] corollary 4.1.5 and lemma 4.1.6) is used in figure 7 representing the composition series of the projective representation in terms of irreducible modules drawn in the bulk of the paper (and [24] contains even more detail). Thus, through the embedding, we gained that we are able to distinguish short representations by their anti-diagonal $u(1)$ charge, and that we can borrow freely from $gl(2|2)$ representation theory where we can apply the Berenstein-Gelfand-Gelfand duality theorem.

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