Effective conformal theory and the flat-space limit of AdS

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ABSTRACT: We develop the idea of an effective conformal theory describing the low-lying spectrum of the dilatation operator in a CFT. Such an effective theory is useful when the spectrum contains a hierarchy in the dimension of operators, and a small parameter whose role is similar to that of $1/N$ in a large N gauge theory. These criteria insure that there is a regime where the dilatation operator is modified perturbatively. Global AdS is the natural framework for perturbations of the dilatation operator respecting conformal invariance, much as Minkowski space naturally describes Lorentz invariant perturbations of the Hamiltonian. Assuming that the lowest-dimension single-trace operator is a scalar, $\mathcal{O}$, we consider the anomalous dimensions, $\gamma(n,l)$, of the double-trace operators of the form $\mathcal{O}(\partial^2)^n(\partial)^l\mathcal{O}$. Purely from the CFT we find that perturbative unitarity places a bound on these dimensions of $|\gamma(n,l)| < 4$. Non-renormalizable AdS interactions lead to violations of the bound at large values of n. We also consider the case that these interactions are generated by integrating out a heavy scalar field in AdS. We show that the presence of the heavy field "unitarizes" the growth in the anomalous dimensions, and leads to a resonance-like behavior in $\gamma(n,l)$ when n is close to the dimension of the CFT operator dual to the heavy field. Finally, we demonstrate that bulk flat-space S-matrix elements can be extracted from the large n behavior of the anomalous dimensions. This leads to a direct connection between the spectrum of anomalous dimensions in d-dimensional CFTs and flat-space S-matrix elements in $d+1$ dimensions. We comment on the emergence of flat-space locality from the CFT perspective.

Contents

1 Introduction

One of the central puzzles of the AdS/CFT correspondence [1–3] concerns determining which CFTs have well-behaved AdS descriptions. A well-behaved description is usually taken to mean an effective theory containing several AdS fields whose interactions allow a perturbative description over a range of scales. Thus, bulk theories typically contain fields whose masses are of order the AdS curvature scale, while their non-renormalizable

interactions are suppressed by a scale much larger than the curvature scale. In particular, the bulk Planck scale must also be large compared to the AdS curvature scale. Local bulk scattering of the light fields then satisfies perturbative unitarity until one reaches the scale of non-renormalizable operators. Though high-energy scattering appears to violate unitarity, the expectation is that the infinitely many heavy AdS fields will ultimately "unitarize" this scattering, much as QCD resonances lead to sensible scattering of pions. The low-energy bulk description is therefore valid as long as tree level processes are far from violating the bounds of perturbative unitarity.

From the AdS effective theory perspective, it appears therefore that what is essential for the simplicity of description is simply the existence of a small sector of the theory that is lighter than the Planck scale and most other states.[1] Since the AdS/CFT dictionary relates dimensions of operators to masses of fields in the bulk, a natural conjecture, proposed by [4], is that any CFT with a few low dimension operators separated by a hierarchy from the dimension of other operators will have a well-behaved dual. However, as any CFT contains an energy-momentum tensor (dual to the graviton in AdS), there must also be an additional condition to suppress gravitational interactions in the bulk. In most known cases this condition follows from the existence of a large number of degrees of freedom in the CFT (typically, one takes the large N limit of an $SU(N)$ gauge theory). The large N limit suppresses the connected pieces of higher-point correlation functions as compared to two-point functions. $1/N$ thus behaves as a natural expansion parameter for bulk interactions, and allows one to distinguish between operators dual to single-particle bulk states, and those dual to multiple-particle bulk states. The idea suggested by [4] is that having a hierarchy in dimensions and a parameter such as N in a CFT is sufficient to construct a sensible AdS effective theory. This theory describes well the correlation functions of low-dimension operators.

A natural question to ask is then what is the CFT interpretation of the bulk effective field theory. In particular, there must be an effective conformal theory (ECT) description which includes only low-dimension CFT operators as states. This ECT must be able to distinguish between renormalizable and non-renormalizable bulk interactions. It must also obey a condition equivalent to bulk perturbative unitarity which sets the range of its validity. Finally, following standard effective field theory mythology, it would be satisfying, if in the case that the non-renormalizable terms come from "integrating out" a high-dimension operator with renormalizable interactions, that perturbative unitarity is restored on the CFT side. We will see that the ECT indeed has these features once we determine the appropriate CFT condition for perturbative unitarity.

For simplicity, following [4], we will consider a scenario where the lowest-dimension operator is a scalar operator, $\mathcal{O}(x)$, with dimension Δ. We will refer to $\mathcal{O}(x)$ as a "single-trace operator" in analogy to large N gauge theories with adjoint representations, but it is not necessary for the operator to have this origin. Other single-trace operators are taken to have much larger dimensions. We assume that there is a parameter such as N so that

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[1]For instance, supersymmetry does not appear to have a direct role in ensuring that the bulk effective theory is well behaved, although it might be important for selecting which low-energy bulk descriptions have actual UV completions.

at zero-th order in $1/N$ the primary operators appearing in the $\mathcal{O} \times \mathcal{O}$ operator product expansion (OPE) are the "double-trace operators", which have the schematic form

$$\mathcal{O}_{n,l}(x) \equiv \mathcal{O}(\overleftrightarrow{\partial}_\nu \overleftrightarrow{\partial}^\nu)^n \overleftrightarrow{\partial}_{\mu_1} \ldots \overleftrightarrow{\partial}_{\mu_l} \mathcal{O}(x) - \text{traces}. \tag{1.1}$$

Here, $\overleftrightarrow{\partial} = \overleftarrow{\partial} - \overrightarrow{\partial}$, where the arrows indicate which of the two operators the derivative acts upon. At zero-th order in $1/N$ the dimension of this operator is given by $2\Delta + 2n + l$. We will be interested in computing the correction to this dimension, $\gamma(n, l)$, arising from bulk interactions. For previous work on computing the anomalous dimensions of double-trace operators in the context of AdS/CFT, see e.g. [4–15].

In order to develop an ECT, we need to specify a notion of energy in the CFT that will map nicely to energies in the bulk theory. As the ECT is supposed to describe low-dimension operators, a natural notion of energy is the dimension itself. The Hamiltonian for which we are developing the ECT is the dilatation operator, and the ECT is then intended to capture its low-lying spectrum. In that sense, for fixed spin, one can think of energy, E, as $E \sim 2n$. It will be important to keep in mind that this notion of energy corresponds to the dimensions of CFT operators and is distinct from Poincaré energy. From the CFT perspective, the task is to start from a dilatation operator, $D^{(0)}$, whose spectrum contains a hierarchy, and perturb it by adding a small correction, V, suppressed by N. The new dilatation operator, $D = D^{(0)} + V$, is taken to act on the low-dimension sector of $D^{(0)}$. In our simplified scenario, this includes multi-trace operators containing only $\mathcal{O}$ and derivatives. Calculating $\gamma(n, l)$ thus amounts to diagonalizing $D^{(0)} + V$ in perturbation theory. Purely from the CFT, we will show that perturbative unitarity places a bound on the anomalous dimensions of $|\gamma(n, l)| < 4$. We will then turn to calculating the anomalous dimensions for particular choices of V, corresponding to local bulk interactions in AdS. For such calculations we find it most natural to work in global AdS, since the energy conjugate to global time is associated with the dilatation operator. Indeed, we will show that local bulk interactions in global AdS automatically lead to a V which is consistent with conformal symmetry. We will then demonstrate that using old-fashioned perturbation theory in global AdS gives a very efficient method of computing the anomalous dimensions $\gamma(n, l)$. This is because these anomalous dimensions are just the correction to the energy in global coordinates[2] of two-particle AdS states due to bulk interactions. Previously, obtaining $\gamma(n, l)$, involved extracting the anomalous dimensions from four-point correlation functions using sophisticated techniques limited to even CFT dimensions. Our method is simpler and applies for any dimension.

As expected from AdS, the above unitarity bound will be violated by terms in V coming from non-renormalizable bulk interactions. Indeed, as would follow from the above identification of n with energy, we find that a local bulk term suppressed by Λ^p, will lead to a growth in $\gamma(n, l) \sim n^p$.[3] Thus, the value of n at which the bound is violated sets a natural boundary for the validity of the ECT. The existence of a useful ECT description

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[2]The Hamiltonian of AdS in global coordinates is more useful for our purposes than the Hamiltonian in the Poincaré patch. This is because translations in global AdS time correspond to dilatations in the CFT, whereas time in the Poincaré patch corresponds to Poincaré time in the CFT.

[3]This growth was found earlier by [4] using other methods.

is then the statement that perturbative unitarity is not violated over a wide range of n's. This is related to locality of interactions which include only the field dual to operator $\mathcal{O}$ in the bulk theory.

To make connection with the conjecture of [4], and to verify standard effective theory lore, we also consider the generation of non-renormalizable bulk interactions via the exchange of a heavy scalar, dual to a CFT operator $\mathcal{O}_{\text{Heavy}}$ (where $\Delta_{\text{Heavy}} \gg \Delta$). At $n \ll \Delta_{\text{Heavy}}$ we reproduce the exact contributions to $\gamma(n, l)$ one would expect from the leading non-renormalizable interactions generated by integrating out the heavy state, suppressed by the appropriate powers of Δ_{Heavy}. This result is suggestive that a hierarchy in the dimension of operators leads to a large range for the ECT. This example also shows explicitly how putting a large-dimension operator back into the ECT "unitarizes" $\gamma(n, l)$. In fact, just as one would expect from effective field theory, we will see that the growth in $\gamma(n, l)$ turns into a resonance at $n \sim \Delta_{\text{Heavy}}/2$, before decreasing at large n.

At energies much larger than the inverse AdS radius it is expected that one can make contact with flat-space scattering amplitudes. An important goal that has been pursued using a variety of methods [16–24] is to understand how these amplitudes arise from CFT data. Here we will show that it is in fact possible to extract the flat-space S-matrix elements of the bulk theory from the large n behavior of $\gamma(n, l)$. Stated simply, we will argue that at leading order for bulk ϕ-particle scattering,

$$\mathcal{M}(s, t, u)_{\text{flat space}}^{d+1} \sim \frac{E_n}{(E_n^2 - 4\Delta^2)^{\frac{d-2}{2}}} \sum_l [\gamma(n, l)]_{n \gg l} \; r_l P_l^{(d)} (\cos \theta), \qquad (1.2)$$

where $r_l P_l^{(d)} (\cos \theta)$ are the appropriate polynomials in d-dimensions, the total flat-space energy, E_n, is given in units of the AdS radius by $E_n = 2\Delta + 2n$, and $[\gamma(n, l)]_{n \gg l}$ indicates that one needs to take the large n limit of $\gamma(n, l)$, keeping l fixed. In other words, the $\gamma(n, l)$'s form the partial wave expansion of the higher dimensional flat-space S-matrix.[4] By "flat-space S-matrix", one means simply the scattering amplitudes one obtains from the Lagrangian of the bulk theory, but applied in Minkowski space. It is interesting that there seems to be such a direct connection between CFT quantities and flat-space matrix elements. Note that this connection is only possible if the ECT including $\mathcal{O}$ and $\mathcal{O}_{\text{Heavy}}$ obeys perturbative unitarity for n sufficiently large. Therefore, a hierarchy in dimensions and a parameter such as N are essential for flat space to emerge.

This paper is organized as follows. In section 2 we will introduce the general formalism concerning perturbations of the dilatation operator and discuss the constraints arising from perturbative unitarity. We will then review the construction of scalar wavefunctions in global AdS, and discuss why local bulk interactions lead to a sensible perturbation of the dilatation operator. In section 3 we will derive the general form of the wavefunctions corresponding to primary operators in the CFT, and use this to calculate the anomalous dimensions of primary double-trace operators arising from various bulk quartic interactions. In section 4, we will consider integrating out a heavy scalar field in AdS, and we will compare

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[4]This sharpens the relation between $\mathcal{M}$ and γ found previously for local bulk operators and neglecting mass terms [4].

the resulting anomalous dimensions to the leading-order contributions from the low-energy effective field theory. In section 5, we will explore the flat-space limit of AdS, and show how flat-space S-matrix elements can be determined from the large n behavior of the anomalous dimensions. We conclude in section 6.

2   Formalism

2.1   Algebra constraints

In quantum field theory, free fields provide a fundamental starting point for perturbation theory because they have a solvable Hamiltonian and simple dynamics corresponding to multi-particle states. In conformal field theory, the dual role is played by "mean fields", which have a Gaussian partition function and a simple spectrum of operator dimensions corresponding to multi-trace operators. For CFTs arising from a gauge group with a large rank N, corrections to three- and higher n-point correlation functions of canonically normalized primary operators are expected in general to be suppressed by powers of N, so that the mean field theory correlation functions are a good approximation. In this case, the dilatation operator D of the CFT may be split into a mean-field piece $D^{(0)}$ that survives as N is taken to infinity, and a perturbation $V \equiv D - D^{(0)}$ that is suppressed by some power of N. In radial quantization, where one studies radial evolution rather than time evolution of the CFT, D plays the role of a Hamiltonian, and so V plays the role of an interaction. However, this procedure is not limited to CFTs arising from large-rank gauge groups; we may perform perturbation theory in this way any time the CFT reduces to a mean field theory when some small parameter or parameters vanish. Thus, we shall follow [4] and use N in this more general sense, as the formal parameter suppressing V. Of course, we are not interested in general perturbations around mean field theory, but rather only in those where the perturbed theory is also conformal. A great strength of AdS/CFT is that local AdS-Lorentz invariant interactions generate perturbations in the CFT of exactly this form.

We will write the conformal algebra as

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\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), & [M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[M_{\mu\nu}, D] &= 0, & [P_\mu, K_\nu] &= -2(\eta_{\mu\nu}D + iM_{\mu\nu}), \\
[D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu.
\end{aligned}
\tag{2.1}
$$

Note that we have chosen our convention for D so that it is Hermitian, which differs from the most common convention by a factor of i. The requirement that this algebra is held fixed is then a non-trivial constraint on the form of possible perturbations to the generators.

For simplicity, we will start by specializing to the case of 2d CFTs, where the algebra can be divided into left and right pieces using the decomposition $SO(2,2) = SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. In particular, the generators $M_{\mu\nu}, P_\mu, K_\mu, D$ of the algebra are all linear combinations of operators that act non-trivially on left-moving states only and right-moving

states only

$$K = \frac{K_1 + iK_2}{2}, \quad P = \frac{P_1 - iP_2}{2}, \quad L_0 = \frac{D - M_{12}}{2}, \quad \text{(left-moving)},$$
$$\overline{K} = \frac{K_1 - iK_2}{2}, \quad \overline{P} = \frac{P_1 + iP_2}{2}, \quad \overline{L}_0 = \frac{D + M_{12}}{2}, \quad \text{(right-moving)}. \tag{2.2}$$

The left-moving generators then satisfy the algebra

$$[L_0, P] = P, \quad [L_0, K] = -K, \quad [P, K] = -2L_0, \tag{2.3}$$

and the right-moving generators satisfy the same algebra, with $K, P, L_0 \to \overline{K}, \overline{P}, \overline{L}_0$.

Focusing on the left-moving algebra, we can now split the generators into mean field theory generators and $O(1/N^2)$ corrections. In general, the perturbations will be constructed so that $M_{\mu\nu}$ is unmodified, so that both L_0 and $\overline{L}_0$ get corrected by $\frac{1}{2}V$:

$$\begin{aligned} L_0 &= L_0^{(0)} + \frac{1}{2}V, \\ P &= P^{(0)} + P^{(1)}, \\ K &= K^{(0)} + K^{(1)}. \end{aligned} \tag{2.4}$$

The constraint that the theory is still conformal then implies the following relations at $O(1/N^2)$ among the perturbations to the generators:

$$\begin{aligned} \left[\frac{1}{2}V, K^{(0)}\right] + \left[L_0^{(0)}, K^{(1)}\right] &= -K^{(1)}, \\ \left[\frac{1}{2}V, P^{(0)}\right] + \left[L_0^{(0)}, P^{(1)}\right] &= P^{(1)}, \\ \left[P^{(1)}, K^{(0)}\right] + \left[P^{(0)}, K^{(1)}\right] &= -V. \end{aligned} \tag{2.5}$$

These relations turn out to be extremely useful. To derive their implications for the matrix elements of the perturbed generators, let us choose our basis to be the eigenstates of $L_0^{(0)}$. As usual, the left-moving states are classified as primary states, which are annihilated by K, or descendant states, which are obtained from the primary states by acting repeatedly with P. In this subsection, we will denote a primary state with $L_0^{(0)}$ eigenvalue α as $|\alpha, 0\rangle$, and its normalized m-th descendant as $|\alpha, m\rangle$. It is then straightforward using the algebra to work out the action of the zero-th order generators on any state. In particular,

$$\begin{aligned} L_0^{(0)}|\alpha, m\rangle &= (\alpha + m)|\alpha, m\rangle, \\ P^{(0)}|\alpha, m\rangle &= \sqrt{(m+1)(2\alpha + m)}|\alpha, m+1\rangle \equiv c_m^\alpha|\alpha, m+1\rangle, \\ K^{(0)}|\alpha, m\rangle &= \sqrt{m(2\alpha + m - 1)}|\alpha, m-1\rangle = c_{m-1}^\alpha|\alpha, m-1\rangle. \end{aligned} \tag{2.6}$$

By taking matrix elements of eqs. (2.5) between zero-th order states $\langle\alpha, m|$ and $|\beta, m'\rangle$, we obtain three separate equations. The first can be written as

$$2K_{\alpha,m;\beta,m'}^{(1)} = \frac{c_m^\alpha V_{\alpha,m+1;\beta,m'} - c_{m'-1}^\beta V_{\alpha,m;\beta,m'-1}}{1 + \alpha + m - \beta - m'}, \tag{2.7}$$

where $\mathcal{O}_{\alpha,m;\beta,m'}$ denotes $\langle \alpha, m | \mathcal{O} | \beta, m' \rangle$. The second condition in eq. (2.5) becomes

$$2P^{(1)}_{\alpha,m;\beta,m'} = \frac{c^{\alpha}_{m-1} V_{\alpha,m-1;\beta,m'} - c^{\beta}_{m'} V_{\alpha,m;\beta,m'+1}}{-1 + \alpha + m - \beta - m'}, \tag{2.8}$$

which follows from the first one using $P = K^{\dagger}, V = V^{\dagger}$. The third condition of eq. (2.5) also follows from the first two. Thus, all of the perturbed generators can be determined from the matrix elements of the dilatation operator. One of our major goals will be to calculate and study the behavior of these matrix elements.

The above relations will be extremely important when we use time-independent perturbation theory to construct the dilatation eigenstates at first order. Naïvely, a straightforward construction is impossible in practice because of the enormous zero-th order degeneracy between multi-trace states. Thus, one would expect to have to diagonalize V within the space of degenerate states, which would be intractable for the vast majority of states of interest.

Fortunately, this is not the case, a fact that follows from the above relations under the assumption that K, P have finite matrix elements between zero-th order dilatation eigenstates.[5] Specifically, taking $m' = 0$ in eq. (2.7) we see that matrix elements of V between a primary state $|\beta, 0\rangle$ and a descendant $|\alpha, m + 1\rangle$ with the same dimension must vanish! The reason is that $V_{\alpha,m;\beta,-1}$ must vanish since $|\beta, 0\rangle$ is primary, and the denominator $1 + \alpha + m - \beta$ also vanishes under the assumption that that the states have the same dimension. Thus there is no possible cancellation between the two terms in the numerator, and since $K^{(1)}$ is assumed to be finite, we necessarily have $V_{\alpha,m+1;\beta,0} = 0$. This is very useful, since it means that we do not have to do degenerate perturbation theory in order to construct the first-order primary states.

It will be helpful to discuss the space of states further, and to establish some more notation. We will be focusing on the simplest possible CFTs, where the only single-trace primary operator is a scalar operator $\mathcal{O}$ with dimension Δ. Following [4], we will be ignoring the role of the energy-momentum tensor $T_{\mu\nu}$ in the majority of our analysis, which formally corresponds to taking the limit of very large central charge $c \gg N$. In a sense, therefore, we will be studying toy models, though we believe our results are rather general and would apply to theories with a $T_{\mu\nu}$ as well. Out of $\mathcal{O}$, one can make many double-trace primary operators. In mean field theory, one knows their form explicitly. Adopting the notation of [4], they are schematically

$$\mathcal{O}_{n,l} = \mathcal{O} \overleftrightarrow{\partial}_{\mu_1} \ldots \overleftrightarrow{\partial}_{\mu_l} (\overleftrightarrow{\partial}_{\nu} \overleftrightarrow{\partial}^{\nu})^n \mathcal{O} - \text{traces}, \tag{2.9}$$

and they have dimension $E_{n,l} = 2\Delta + 2n + l$ and spin l. Inserting one of these operators at the origin creates a double-trace primary state $\mathcal{O}_{n,l}(0)|0\rangle = |n, l\rangle_2$, which we will label by their n and l values.

When we perturb the mean-field theory dilatation operator by an interaction V, the eigenstates of the perturbed dilatation operator acquire the anomalous dimensions

$$\Delta_{n,l} = E_{n,l} + \gamma(n, l). \tag{2.10}$$

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[5]This assumption is satisfied by perturbations generated by local interactions in AdS, except at particular fractional values of α.

It is then relatively straightforward to calculate $\gamma(n,l)$ using old-fashioned perturbation theory

$$\gamma(n,l) = {}_2\langle n,l|V|n,l\rangle_2 + \sum_\alpha \frac{|\langle\alpha|V|n,l\rangle_2|^2}{E_{n,l} - E_\alpha} + \ldots, \qquad (2.11)$$

where E_α is the leading order dimension of $|\alpha\rangle$. In this paper we will give a number of concrete examples which demonstrate how to calculate $\gamma(n,l)$ using the above method.

Of course, not every choice of V will lead to a well-behaved perturbative expansion for all n and l. This is quite similar to the statement that not every interaction in flat space leads to a perturbatively calculable S-matrix for all choices of external energy. In particular, non-renormalizable interactions lead to a violation of perturbative unitarity when scattering at sufficiently high energies. In the next subsection we will show that large N CFTs have a similar constraint from perturbative unitarity, which can be stated quite simply in terms of the large n behavior of $\gamma(n,l)$. We will later show that this constraint is satisfied if V arises from renormalizable local bulk interactions in AdS, and is violated if V arises from non-renormalizable bulk interactions.

2.2 Unitarity limit

The requirement that scattering amplitudes in flat-space field theory be unitary means that contributions from higher-dimensional operators cannot continue to grow indefinitely, and eventually the validity of the effective theory breaks down. One expects that before this happens, heavy fields will appear to unitarize the theory. The systematic description of such constraints is through the optical theorem, and more generally through the cutting rules, which will appear to be violated at tree-level if one considers sufficiently high energy scattering. It was demonstrated in [4] that all $O(1/N^2)$ CFT perturbations that satisfy crossing symmetry can be generated by local operators in AdS. Most of these AdS operators will be non-renormalizable, and we would like to derive something like an optical theorem which is violated by conformal theories with perturbations generated by higher-dimensional operators in AdS. Naïvely, there can be no such limit. At tree-level, a generic AdS action essentially defines a CFT at $O(1/N^2)$, and the correlation functions are perfectly well-behaved. Indeed, since there is no scale in the CFT, there would appear to be ipso facto no scale where the theory could break down. However, the point is that the CFT secretly does have something that plays the role of a scale: the n in the double-trace primary operators $\mathcal{O}_{n,l}$.

By considering a scattering thought experiment in AdS and relating it to CFT correlation functions, [4] found that the anomalous dimension $\gamma(n,l)$ of $\mathcal{O}_{n,l}$ generated by a non-renormalizable interaction in AdS_{d+1} of scaling dimension p must grow like $n^{p-(d+1)}$. As a result, regardless of how small $1/N$ is, for $p > d+2$ there will be some n above which the $O(1/N^2)$ corrections to the dimension of a double-trace primary operator is larger than the leading term $2\Delta + 2n + l$.[6] Our goal in this section will be to find a sharp limit

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[6]We thank João Penedones for pointing this out to us, and for noting that the dimensions of double-trace operators will become negative if the sign of the AdS interaction is chosen incorrectly.

where this growth leads to problems, and in the process tighten the constraint to apply to non-renormalizable operators with $p > d + 1$.

We can try to set up something like an optical theorem in terms of CFT quantities. The dilatation eigenstates $|A\rangle$ of the perturbed theory will be related to those of the unperturbed theory through a transition matrix T

$$|A\rangle = (\delta_{AB} + T_{AB}) |B\rangle^{(0)}. \qquad (2.12)$$

The optical theorem in quantum field theory follows just from completeness of the "in" and "out" states, and the fact that the S-matrix is just a change of basis. The most similar condition we can build out of the CFT quantities at hand is the completeness of the perturbed and unperturbed eigenstates

$$\delta_{AB} = \sum_C \langle A|C\rangle^{(0)\,(0)}\langle C|B\rangle. \qquad (2.13)$$

Here, it is important to note that we will be interested in applying this completeness relation to the low-lying states of the dilatation operator. Indeed, changing N in the full CFT will in general modify the Hilbert space and therefore the eigenstates of D and $D^{(0)}$ are not strictly describing the same space.[7] However, at large N it will be the large-dimension operators (with dimensions of $O(N)$) that will be sensitive to such changes in the Hilbert space, not the low-dimension ones. This is very similar to the situation in large N QCD where one is similarly changing the Hilbert space by varying N. At large N, however, the subspace of low-mass meson states (of mass, $m \ll N\Lambda_{QCD}$) is not changing significantly. In fact, perturbative unitarity of the S-matrix is precisely the criterion one uses to determine the range of energy and mass over which a change in N is not modifying the space of states.

Let us then find the implication of the above completeness relation. If we insert (2.12) and take $A = B$, we find

$$-(T + T^*)_{AA} = \sum_C |T_{AC}|^2. \qquad (2.14)$$

It is clear from this relation that $\mathcal{R}e(T)_{AA} < 0$, which one should keep in mind in the following manipulations. Using that $\sum_C |T_{AC}|^2 > |\mathcal{R}e(T)_{AA}|^2$, one obtains the constraint on $|\mathcal{R}e(T)_{AA}|$ that

$$|\mathcal{R}e(T_{AA})| < 2. \qquad (2.15)$$

This limit must be satisfied, and we will refer to it as the unitarity limit since it followed from the fact that $\langle A|C\rangle^{(0)}$ is a unitary matrix. Consider now the condition that it be satisfied in perturbation theory. The first contribution to $\mathcal{R}e(T_{AA})$ occurs at $O(V^2)$ from the renormalization $|A\rangle \to Z_A^{-\frac{1}{2}}|A\rangle$, where $Z_A = 1 + \sum_{B \neq A} |V_{AB}|^2/(E_A - E_B)^2 + O(V^3)$ and E_A denotes the zero-th order dimension of $|A\rangle$. Thus, at $O(V^2)$, we have

$$2 > |\mathcal{R}e(T_{AA})| = \frac{1}{2} \sum_{B \neq A} \frac{|V_{AB}|^2}{(E_A - E_B)^2}. \qquad (2.16)$$

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[7]We thank Joe Polchinski for bringing up this point.

Let us now take $|A\rangle$, $|B\rangle$ to be neighboring double-trace primary states $|n,l\rangle_2$ and $|n+1,l\rangle_2$, respectively. The difference in their mean field dimensions is exactly 2, so the above relation implies $|V_{n,l;n+1,l}| < 4$, since every other term on the right hand side is positive. At large n, there is not much difference between $V_{n,l;n+1,l}$ and $V_{n,l;n,l}$. Both can be calculated from the overlap of wavefunctions in AdS, and the difference between wavefunctions for $|n,l\rangle_2$ and $|n+1,l\rangle_2$ is $O(1/n)$ at large n. This will become especially obvious when we consider example calculations of matrix elements of V. But, $V_{n,l;n,l}$ is just the leading order anomalous dimension $\gamma(n,l)$ of the state $|n,l\rangle_2$. Thus, we can state a very simple necessary condition in order to maintain perturbative unitarity in the CFT $1/N^2$ expansion

$$|\gamma(n,l)| < 4 \quad (n \gg 1) \tag{2.17}$$

What this says is that perturbation theory fails when the anomalous dimensions $\gamma(n,l)$ become much greater than 1.[8] In fact, tracing back the steps that lead to this break-down, we see that the states $|n,l\rangle_2$ have negative norm at $O(1/N^2)$ when the above condition is not satisfied. When this happens, the description of the CFT must be modified to maintain unitarity, and if this is to occur before the n where perturbation theory fails then one must have new large-dimension single trace operators that contribute to $\gamma(n,l)$ and unitarize the transition matrix. Even if new single-trace operators do not appear before $|\gamma(n,l)| > 4$, the theory becomes "strongly coupled" at that point, in the sense that V is large, and the standard lore is that the modified description of the theory at large n will contain additional heavy states.

Consequently, the implications of large n growth are fairly striking. Naïvely, effective field theories in AdS are dual to a very limited class of CFTs. In order for the AdS EFT to be calculable, all non-renormalizable operators must be suppressed at least by appropriate powers of some scale Λ, the cut-off of the theory. For example, consider all possible four-point contact interactions of a scalar field $\phi(x)$, dual to a CFT operator $\mathcal{O}$. Such four-ϕ interactions are in one-to-one correspondence with all different possible crossing-symmetric contributions to the $\mathcal{O}$ four-point function [4]. Thus, we appear to require an infinite number of conditions on the CFT four-point function, one for each non-renormalizable operator in AdS. What the above discussion says is that all of these apparently independent conditions are simply the condition of a hierarchy in the dimensions of operators in the CFT, with no new single-trace primary operators appearing below some dimension Δ_{Heavy}. Furthermore, the suppression of the perturbations dual to non-renormalizable AdS interactions is given by appropriate powers of Δ_{Heavy}. This is exactly dual to the condition in AdS that there is a hierarchy in scales between the mass of ϕ (and whatever other fields appear in our effective theory) and the new physics that appears around the cut-off Λ. In the following sections, we will explore this relation further, and in particular the description within the CFT of the transition at low n below Δ_{Heavy} to large n, where the heavy conformal sector is "integrated in" to restore unitarity.

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[8]We note that the above bound is not as general as those derived in [25–27], which are valid also when both n and N are small, and are thus non-perturbative statements.

2.3 Review of AdS global coordinate wavefunctions

Next we will turn to the concrete construction of effective field theories in AdS. The connection between fields in AdS and operators with definite scaling dimension in the CFT is significantly more transparent in global coordinates than in Poincaré coordinates. For completeness and to establish notation, we will now review this connection in detail [3, 28], as well as the construction of the canonical field operators in AdS global coordinates.

To begin, we work in global coordinates in AdS_{d+1}, with the metric

$$ds^2 = \frac{1}{\cos^2 \rho}\left(-dt^2 + d\rho^2 + \sin^2 \rho\, d\Omega^2\right). \tag{2.18}$$

We will work in units of the AdS radius $R_{\text{AdS}} \to 1$. The center of AdS lies at $\rho = 0$, and the boundary at $\rho = \pi/2$. The boundary manifold is $\mathbb{R} \times S^{d-1}$, where translations in global coordinate time generate dilatations in the CFT.

We will now consider a bulk scalar field $\phi(x)$, dual to a single-trace scalar operator $\mathcal{O}(0)$ and its descendants in the boundary CFT. The free field wavefunctions in AdS_{d+1} satisfy $(\nabla^2 - m^2)\phi = 0$, which has the solutions (keeping only the modes which are well-behaved at $\rho = 0, \pi/2$)

$$\phi_{nlJ}(x) = \frac{1}{N_{\Delta,n,l}} e^{iE_{n,l}t} Y_{lJ}(\Omega) \sin^l \rho \cos^\Delta \rho\, F\left(-n, \Delta + l + n, l + \frac{d}{2}, \sin^2 \rho\right)$$
$$E_{n,l} \equiv \Delta + 2n + l, \quad m^2 = \Delta(\Delta - d), \tag{2.19}$$

where $F = {}_2F_1$ is the Gauss hypergeometric function, $Y_{lJ}(\Omega)$ are normalized eigenstates of the Laplacian on S^{d-1} with eigenvalue $-l(l+d-2)$, and J denotes all the angular quantum numbers other than l. In many formulae, dependence on the J index will be clear from context and we will often suppress it. The canonical field operators are then constructed in terms of the wavefunctions and creation/annihilation operators

$$\phi(x) = \sum_{n,l,J} \phi_{nlJ}(x) a_{nlJ} + \phi_{nlJ}^*(x) a_{nlJ}^\dagger. \tag{2.20}$$

We will denote the one-particle states created by $a_{nlJ}^\dagger$ as $|\phi; n, l, J\rangle$, where indices after the semi-colon indicate descendants. They are in one-to-one correspondence with the states created at the origin by the single-trace operator $\mathcal{O}(0)$ and its descendants, since both are simply the eigenstates of the dilatation and rotation operators with energy $\Delta + 2n + l$. This is what makes AdS global coordinates a natural place to work when studying anomalous dimensions of operators.

Using the norm $(\phi_1, \phi_2) \equiv \int d^d x \sqrt{-g} g^{00} \phi_1(x)^* \overleftrightarrow{\partial_0} \phi_2(x)$, the wavefunctions are properly normalized when

$$N_{\Delta,n,l} = (-1)^n \sqrt{\frac{n! \Gamma^2(l + \frac{d}{2}) \Gamma(\Delta + n - \frac{d-2}{2})}{\Gamma(n + l + \frac{d}{2}) \Gamma(\Delta + n + l)}}, \tag{2.21}$$

where we have chosen the n-dependent phase for later convenience.

In addition to the one-particle wavefunctions, we will be interested in more general wavefunctions (e.g., two-particle wavefunctions) in AdS that are dual to primary states in the CFT. In order to study this we will need to understand the action of the conformal generators on functions of AdS global coordinates. This is most easily determined by going to the embedding space of AdS_{d+1}, which we will write as

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{\mu=1}^{d} dX_\mu^2, \qquad -1 = X_M X^M. \tag{2.22}$$

The embedding space coordinates are then related to global coordinates through the identifications

$$X_0 = \frac{\cos t}{\cos \rho}, \qquad X_{d+1} = \frac{\sin t}{\cos \rho}, \qquad X_\mu = \tan \rho \, \Omega_\mu. \tag{2.23}$$

The generators of the $\mathrm{SO}(d,2)$ algebra are simply represented in the embedding space as $J_{MN} = -i(X_M \partial_N - X_N \partial_M)$. In particular, the conformal algebra eq. (2.1) is correctly reproduced if we identify

$$P_\mu = J_{\mu,d+1} - iJ_{\mu,0} \qquad K_\mu = J_{\mu,d+1} + iJ_{\mu,0} \qquad D = -J_{0,d+1} \qquad M_{\mu\nu} = J_{\mu\nu}. \tag{2.24}$$

It is then straightforward to work out their corresponding action in terms of global coordinates. For example, in general $D = -i\partial_t$, and in AdS$_3$ the left- and right-moving generators act as

$$K_\pm = ie^{-it\pm i\varphi} \left(\sin \rho \partial_t + i \cos \rho \partial_\rho \mp \frac{1}{\sin \rho} \partial_\varphi \right)$$

$$P_\pm = ie^{it\pm i\varphi} \left(\sin \rho \partial_t - i \cos \rho \partial_\rho \pm \frac{1}{\sin \rho} \partial_\varphi \right) \tag{2.25}$$

where $K_\pm = K_1 \pm iK_2, P_\pm = P_1 \pm iP_2$.

We are now in position to construct the wavefunctions in AdS that are dual to the double-trace primary operators $\mathcal{O}_{n,l}(0)$. We will do this in detail in section 3. Afterwords we will consider adding local bulk interactions $\mathcal{V}(x)$, treating $V = \int d^d x \mathcal{V}(x)$ as a perturbation to the dilatation operator of the CFT. We will then use old-fashioned perturbation theory in order to calculate the corrections to the anomalous dimensions $\gamma(n,l)$ arising from V. However, first we would like to consider more carefully why the integral of a local bulk interaction in AdS leads to a sensible perturbation of the dilatation operator in the dual CFT.

2.4 Locality and microcausality in AdS

In the case of a Lorentz invariant theory in flat space, it is well known that if the interaction part of the Hamiltonian, V, can be written in terms of local interaction density $\mathcal{V}(x)$ integrated over space, then Lorentz invariance requires that $[\mathcal{V}(x), \mathcal{V}(y)] = 0$ for $(x-y)^2 < 0$. Thus, in order to build Lorentz-invariant interactions for a particular particle, the standard procedure is to take the creation and annihilation operators for that particle and assemble

them into a field $\phi(x)$. $\phi(x)$ transforms simply under Lorentz transformations, and in addition obeys $[\phi(x), \phi(y)] = 0$ for $(x - y)^2 < 0$. We then build $\mathcal{V}(x)$ as a scalar operator made up of $\phi(x)$ and its derivatives, $\mathcal{V}(x) = \mathcal{V}(\phi(x), \partial_\mu \phi(x), \partial_\mu \partial_\nu \phi(x), \ldots)$. Such a $\mathcal{V}(x)$ automatically obeys microcausality and leads to a Lorentz-invariant theory.

In many ways, the procedure in AdS is similar to the Lorentz-invariant case. We are interested in constructing the interaction part of the dilatation operator, V, in a way which gives a conformally invariant theory. In the previous section we reviewed how to assemble the creation and annihilation operators associated with a primary operator in the CFT and its descendants into an AdS field, $\phi(x, t)$. (Note that here x denotes all coordinates other than the global time t.) Under the AdS isometries $\phi(x, t)$ transforms in a simple way, and it also obeys $[\phi(x, t), \phi(y, t)] = 0$ for $x \neq y$ by construction. If we now build $\mathcal{V}(x, t)$ as an AdS scalar made out of $\phi(x)$ and its derivatives, it will also obey $[\mathcal{V}(x, t), \mathcal{V}(y, t)] = 0$. We will now check that the AdS microcausality condition on $\mathcal{V}(x, t)$ is sufficient to insure that $D = D^{(0)} + V$ is a sensible dilatation operator. Along the way, we will see explicitly that the operator $K^{(1)}$ has non-singular matrix elements as discussed in section 2.1.

We will make our argument for the case of AdS$_3$ for simplicity, although it naturally generalizes to higher dimensions. Let $V = \int d^2x \sqrt{-g}\, \mathcal{V}(x)$, where $\mathcal{V}(x)$ is a local interaction density. Then the leading order special conformal transformation, $K^{(0)}$, acts on the scalar $\mathcal{V}(x)$ simply through the corresponding isometry (2.25) of AdS

$$[K_\pm^{(0)}, \frac{V}{2}] = -\frac{i}{2} \int d^2x \sqrt{-g}\, e^{-it \pm i\varphi} \left(\sin\rho \partial_t + i \cos\rho \partial_\rho \mp \frac{1}{\sin\rho} \partial_\varphi \right) \mathcal{V}(x, t). \quad (2.26)$$

Here, $\mathcal{V}(x, t)$ is evolved using $D^{(0)}$, and so $\partial_t \mathcal{V}(x, t) = -i[D^{(0)}, \mathcal{V}(x, t)]$. Consequently, upon integrating the last two terms in the above expression by parts, one obtains

$$[K_\pm^{(0)}, \frac{V}{2}] = -\frac{1}{2} \int d^2x \sqrt{-g} \sin\rho\, e^{-it \pm i\varphi} \left([D^{(0)}, \mathcal{V}(x, t)] + \mathcal{V}(x, t) \right). \quad (2.27)$$

Comparing the above expression with eqs. (2.5),[9] we can identify $K^{(1)}$ as

$$K_\pm^{(1)} = \int d^2x \sqrt{-g} \sin\rho\, e^{-it \pm i\varphi}\, \mathcal{V}(x, t). \quad (2.28)$$

This operator clearly has non-singular matrix elements between states. With this identification of $K^{(1)}$, we get the proper conformal algebra at $O(V^2)$ only if in addition we impose the requirement that $[K_\pm^{(1)}, V] = 0$. For a generic interaction, this is possible only if $[\mathcal{V}(x, t), \mathcal{V}(y, t)] = 0$. A coordinate-invariant version of this condition is that whenever one can chosoe a space-like surface containing the two points (x, x_0) and (y, y_0), that $[\mathcal{V}(x, x_0), \mathcal{V}(y, y_0)] = 0$.

This discussion makes it clear that any local interaction terms, constructed from AdS fields obeying canonical commutation relations, will lead to a sensible conformally-invariant theory. Unitarity then places additional constraints on these local interaction terms. In particular, if we require perturbative unitarity for all operator dimensions $\Delta < \Delta_{\text{Heavy}}$,

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[9]Eqs. (2.5) are used along with $[K_\pm^{(0)}, \frac{V}{2}] = [\overline{L}_0^{(0)}, K_\pm^{(1)}]$ from the fact that left- and right-moving sectors commute.

then as discussed in section 2.2, local non-renormalizable interactions must be suppressed by powers of $1/\Delta_{\text{Heavy}}$. In order to understand this matching in more detail, we now turn to developing the tools needed to efficiently calculate the CFT anomalous dimensions induced by various local bulk interactions.

3 Dilatation matrix elements in low-energy ECT

3.1 Primary wavefunctions

At leading order in perturbation theory, corrections to anomalous dimensions are matrix elements of V between primary states. In many cases of interest, the building blocks of these matrix elements are amplitudes $\langle 0|\Phi(x)|\psi\rangle$ for a bulk operator $\Phi(x)$ to annihilate a primary state $|\psi\rangle$. For example, in computing the anomalous dimensions of the two-particle primary states $|n,0\rangle_2$ in ϕ^4-theory, we must evaluate $_2\langle n,0|\phi^4(x)|n,0\rangle_2 = 6|\langle 0|\phi^2(x)|n,0\rangle_2|^2$. These "primary wavefunctions" are highly constrained by symmetry, and we can often compute them very efficiently. In this section, we will discuss their general form, and in the next section we will show how to determine their normalizations.

Scalar primary wavefunctions in AdS_{d+1} are extremely simple. Note first that any function annihilated by all the K_μ must be of the form $f(e^{it}\cos\rho)$. This is clearest in the embedding space construction, where $(e^{it}\cos\rho)^{-1} = X_0 - iX_{d+1}$, which is the only linear combination of X's that is killed by all the rotation generators $K_\mu = J_{\mu,d+1} + iJ_{\mu,0}$. Thus, for scalar $\Phi(x)$, a primary wavefunction for a state $|\psi\rangle$ with definite energy ω is proportional to

$$\langle 0|\Phi(x)|\psi\rangle \propto \left(e^{it}\cos\rho\right)^\omega, \tag{3.1}$$

where the constant of proportionality vanishes if $|\psi\rangle$ has nonzero spin. Related arguments were used in [29, 30].

More generally, we might be interested in the wavefunction of a tensor operator $\Phi^{a_1\ldots a_n}(x)$ in a primary state $|\psi_{\mu_1\ldots\mu_l}\rangle$ with energy ω and spin l.[10] To determine its general form, we can start by writing down a basis of tensor fields in AdS_{d+1} that are invariant under the action of K_μ. Since special conformal transformations commute, the associated vector fields $\xi_\mu^a \equiv (K_\mu)^a$ are trivially invariant under Lie derivatives $\mathcal{L}_{K_\nu}$. Together with $\zeta^a \equiv \partial^a(e^{it}\cos\rho)^{-1}$, they form a K_μ-invariant basis for the tangent space at each point in AdS_{d+1}.[11] A general primary tensor is therefore just a product of ξ_μ's and ζ's, times a function $f(e^{it}\cos\rho)$. Note further that

$$h^{ab} = (e^{it}\cos\rho)^2(\xi_\mu^a\xi^{\mu b} + \zeta^a\zeta^b), \tag{3.2}$$

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[10]We use Roman indices $a,b,c,\cdots = 1,\ldots,d+1$ for the tangent space in global AdS_{d+1}, and Greek indices $\mu,\nu,\cdots = 1,\ldots,d$ for the Euclidean coordinates of the embedding space. In particular, $g_{\mu\nu} = \delta_{\mu\nu}$. Here, we are writing an element of the spin-l representation of $\text{SO}(d)$ as a traceless symmetric tensor with l μ-indices.

[11] We could have chosen ζ^a to be a derivative of any function of $e^{it}\cos\rho$, since the K_μ's would annihilate it. The choice $(e^{it}\cos\rho)^{-1}$ is convenient since then ζ^a and ξ_μ^a have the same scaling dimension.

where h^{ab} is the metric on AdS_{d+1}, so we can trade traces $g^{\mu\nu}\xi^a_\mu \xi^b_\nu$ for factors of $\zeta^a \zeta^b$ and h^{ab}. Finally since ξ_μ and ζ are lowering operators for the dilatation generator D, a basis for wavefunctions of states $|\psi_{\mu_1\ldots\mu_l}\rangle$ with definite energy ω and spin l is given by

$$\langle 0|\Phi^{a_1\ldots b_1\cdots}(x)|\psi_{\mu_1\ldots\mu_l}\rangle \sim (e^{it}\cos\rho)^{\omega+n+l}\,\zeta^{a_1}\cdots\zeta^{a_n}\left(\xi^{b_1}_{(\mu_1}\cdots\xi^{b_l}_{\mu_l)} - \text{traces with } g^{\mu\nu}\right) \quad (3.3)$$

(up to possible factors of h^{ab}). Here, the states $|\psi_{\mu_1\ldots\mu_l}\rangle$ have been labeled so that their wavefunctions are grouped together into tensors like the right-hand side of eq. (3.3), but one is usually interested in states with definite angular quantum numbers. One can obtain the wavefunction for such a state by projecting the above wavefunctions onto the appropriate polarization. For instance, in AdS_4 we obtain the unique $l = m = 2$ two-index wavefunction by projecting eq. (3.3) onto the polarization tensor $\epsilon^{(2,2)}_{\mu\nu}$:

$$\langle 0|\Phi^{b_1 b_2}(x)|2,2\rangle \propto (e^{it}\cos\rho)^{\omega+2}\left(\xi^{b_1}_{(\mu}\xi^{b_2}_{\nu)} - \frac{1}{3}\xi^{b_1}_\sigma \xi^{\sigma b_2}g_{\mu\nu}\right)\epsilon^{(2,2)\mu\nu}, \quad \epsilon^{(2,2)}_{\mu\nu} = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.4)$$

In AdS_3, this basis simplifies slightly. In light-cone coordinates on the boundary, the special conformal generators $\xi^a_\pm$ are given in eq. (2.25), and we have $g^{\mu\nu}\xi^a_\mu\xi^b_\nu = \xi^{(a}_+\xi^{b)}_-$. Thus, a basis for tensor wavefunctions is given by

$$(e^{it}\cos\rho)^{\omega+n+l}\xi^{a_1}_+\cdots\xi^{a_l}_+\zeta^{b_1}\cdots\zeta^{b_n}, \qquad (l > 0)$$
$$(e^{it}\cos\rho)^{\omega+n-l}\xi^{a_1}_-\cdots\xi^{a-l}_-\zeta^{b_1}\cdots\zeta^{b_n}, \qquad (l < 0) \quad (3.5)$$

(up to possible factors of h^{ab}). For example, to write the two-index spin-2 wavefunction in AdS_3, we can use $(\xi_\pm)_a = -i\frac{\sin\rho}{\cos^2\rho}e^{-it\pm i\varphi}(1, \pm 1, -i\cot\rho)$ (in coordinates t, φ, ρ), and find

$$\langle 0|\Phi_{ab}(x)|\pm 2\rangle \propto (e^{it}\cos\rho)^{\omega+2}(\xi_\pm)_a(\xi_\pm)_b$$
$$\propto e^{i\omega t \pm 2i\varphi}\cos^\omega\rho\tan^2\rho \begin{pmatrix} 1 & \pm 1 & -i\cot\rho \\ \pm 1 & 1 & \mp i\cot\rho \\ -i\cot\rho & \mp i\cot\rho & -\cot^2\rho \end{pmatrix}. \quad (3.6)$$

3.2 Normalization of primary two-particle wavefunctions

We can extract normalizations of primary wavefunctions by a procedure analogous to the conformal block decomposition of CFT correlators. Consider the contribution of a scalar primary state $|\psi\rangle$ of dimension ω and its descendants to the two-point function of a bulk scalar operator $\Phi(x)$,

$$\sum_{\alpha=\psi,\ \text{desc.}}\langle 0|\Phi(x)|\alpha\rangle\langle\alpha|\Phi(x')|0\rangle. \quad (3.7)$$

We know $\langle 0|\Phi(x)|\psi\rangle$ is determined by symmetry. In particular, up to a normalization factor it is the same as the primary wavefunction of a free field,

$$\langle 0|\Phi(x)|\psi\rangle = \frac{1}{N^\Phi_\psi}(e^{it}\cos\rho)^\omega$$
$$= \frac{N_{\omega,0,0}}{N^\Phi_\psi}\text{vol}(S^{d-1})^{1/2}\phi_{00}(x), \quad (3.8)$$

where $N_{\omega,0,0}$ is given in eq. (2.21). But descendant wavefunctions are determined by the primary wavefunction, so all the $\langle 0|\Phi(x)|\alpha\rangle$ are also proportional to wavefunctions of a free field, given in eq. (2.19) with $\Delta \to \omega$. Note that $\Phi(x)$ itself need not be a free field, and $|\psi\rangle$ need not be a single-particle state — conformal symmetry determines everything up to normalization. Consequently the sum in eq. (3.7) is precisely the same as the sum over modes in a free-field two-point function, and the answer is simply a constant times the bulk propagator,

$$\sum_{\alpha=\psi,\ \text{desc.}} \langle 0|\Phi(x)|\alpha\rangle\langle\alpha|\Phi(x')|0\rangle = \frac{N_{\omega,0,0}^2}{(N_\psi^\Phi)^2}\text{vol}(S^{d-1})K_B(x,x')$$
$$\equiv \frac{G_\omega(z)}{(N_\psi^\Phi)^2}, \tag{3.9}$$

where

$$G_\omega(z) = z^{\omega/2}F\left(\omega,\frac{d}{2},\omega+1-\frac{d}{2},z\right) \tag{3.10}$$

and $z = e^{-2\sigma(x,x')}$, with $\sigma(x,x')$ the geodesic distance between x and x'. Summing over primary states $|\psi\rangle$, we find

$$\langle 0|\Phi(x)\Phi(x')|0\rangle = \sum_{\psi\ \text{primary}} \frac{G_\omega(z)}{(N_\psi^\Phi)^2}, \tag{3.11}$$

so we can extract the normalizations N_ψ^Φ by decomposing $\langle 0|\Phi(x)\Phi(x')|0\rangle$ into bulk propagators. To do this in practice, it is useful to exploit the Klein-Gordon equation for the propagator as a function of z,

$$\frac{z^{d/2+1}}{(1-z)^d}\frac{d}{dz}\left(\frac{(1-z)^d}{z^{d/2-1}}\frac{d}{dz}G_\omega(z)\right) = \frac{1}{4}\omega(\omega-d)G_\omega(z). \tag{3.12}$$

This implies the orthogonality relation,

$$\oint \frac{dz}{2\pi i}\frac{(1-z)^d}{z^{1+d/2}}G_{d-\alpha}(z)G_\beta(z) = \delta_{\alpha\beta}, \tag{3.13}$$

where the right-hand side uses the fact that the $G_\omega(z)$ are already normalized with respect to this inner product. As an example that will be relevant shortly, let us find the normalization of the wavefunction of $\phi^2(x)$ in the scalar two-particle primary state $|n,0\rangle_2$ of dimension $2\Delta + 2n$. The two-point function $\langle 0|\phi^2(x)\phi^2(x')|0\rangle$ is easily computed from Wick contractions:

$$\langle 0|\phi^2(x)\phi^2(x')|0\rangle = 2K_B(x,x')^2 = \frac{2}{N_{\Delta,0,0}^4\text{vol}(S^{d-1})^2}G_\Delta(z)^2. \tag{3.14}$$

Applying our orthogonality relation, we get

$$\frac{1}{(N_{n,0}^{\phi^2})^2} = \frac{2}{N_{\Delta,0,0}^4\text{vol}(S^{d-1})^2}\oint\frac{dz}{2\pi i}\frac{(1-z)^d}{z^{1+d/2}}G_\Delta(z)^2G_{d-(2\Delta+2n)}(z)$$
$$= \frac{\Gamma(n+\frac{d}{2})\Gamma(\Delta+n)^2\Gamma(2\Delta+n-\frac{d}{2})\Gamma(2\Delta+2n-d+1)}{2\pi^d n!\Gamma(\frac{d}{2})\Gamma(\Delta+n-\frac{d-2}{2})^2\Gamma(2\Delta+n-d+1)\Gamma(2\Delta+2n-\frac{d}{2})}. \tag{3.15}$$

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Though we have given the general answer, the above integral tends to be particularly simple in even dimensions where $G_\Delta(z)$ is an elementary function. For instance, in $d = 2$, we have $G_\Delta(z) = z^{\Delta/2}(1-z)^{-1}$, and the contour integral essentially just computes coefficients in the Taylor expansion of $(1-z)^{-1}$ around $z = 0$. For use in later sections, let us quote the result in $d = 2$ and $d = 4$:

$$\langle 0|\phi^2(x)|n,0\rangle_2 = \frac{1}{\sqrt{2\pi}}(e^{it}\cos\rho)^{2\Delta+2n} \qquad (d = 2), \quad (3.16)$$

$$\langle 0|\phi^2(x)|n,0\rangle_2 = \frac{(\Delta+n-1)}{\sqrt{2}\pi^2}\sqrt{\frac{(n+1)(2\Delta+n-3)}{2\Delta+2n-3}}(e^{it}\cos\rho)^{2\Delta+2n} \quad (d = 4). \quad (3.17)$$

3.3 Example calculation of V_{nm}

We are now in a position to easily calculate the matrix elements of V for various local AdS bulk interactions. Let us begin with the simplest example, which is a quartic interaction in AdS$_3$,

$$V = \frac{\mu}{4!}\int d^2x\sqrt{-g}\phi^4(x). \quad (3.18)$$

We are specifically interested in the matrix elements

$$\begin{aligned}
V_{nm} &= \frac{\mu}{4!}{}_2\langle n,0|\int d^2x\sqrt{-g}\phi^4(x)|m,0\rangle_2 \\
&= \frac{\mu}{4!}\int d^2x\sqrt{-g}\,{}_2\langle n,0|:\left(\sum_{n,l}\phi_{nl}(x)a_{nl} + \phi^*_{nl}(x)a^\dagger_{nl}\right)^4:|m,0\rangle_2, \quad (3.19)
\end{aligned}$$

where $:(\dots):$ denotes normal ordering, which we will not write explicitly from now on. There are 4! possible contractions of the external states, each of which gives the same contribution, summing to

$$V_{nm} = \frac{\mu}{4}\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\rho\frac{\sin\rho}{\cos^3\rho}{}_2\langle n,0|\phi^2(x)|0\rangle\langle 0|\phi^2(x)|m,0\rangle_2. \quad (3.20)$$

Now we can apply the results of the previous two subsections, namely that the wavefunctions $\langle 0|\phi^2(x)|n,0\rangle_2$ are completely determined by conformal symmetry! Plugging in (3.16), we can trivially perform the integration above to obtain

$$V_{nm} = \frac{\mu}{8\pi(2\Delta+n+m-1)}. \quad (3.21)$$

Of course, the anomalous dimension $\gamma(n,0)$ of $|n,0\rangle_2$ is just V_{nn}, so we have

$$\gamma(n,0) = \frac{\mu}{8\pi(2\Delta+2n-1)}, \quad (3.22)$$

which reproduces the result in [4] based on analysis of the four-point AdS boundary correlator. Note that this provides a simple example of why in section 2.2 we could take $V_{n,n+1} \approx \gamma(n,0)$ at large n — the wavefunctions for $|n,0\rangle_2$ and $|n+1,0\rangle_2$ are negligibly

different at large n, so the matrix element of V between them is nearly the same as the matrix element between $|n, 0\rangle_2$ and itself.

Let us pause to emphasize the simplicity of this calculation. The integrations we had to do above were extremely simple. Even the machinery developed in the previous sections, which was designed solely to construct the two-particle wavefunctions and was not specific to any individual AdS bulk interaction, required little calculation. The form of the wavefunctions followed very simply from the property of the states being primary and scalar, and their normalization followed essentially from expanding $(1-z)^{-1}$ around $z = 0$. Nowhere did we have to calculate a four-point boundary correlation function in AdS, or to extract log terms. It is also completely manifest that no primary state with spin $l > 0$ can get a contribution from $\phi^4(x)$; there simply is no spin-l primary wavefunction that can be constructed without AdS-Lorentz indices unless $l = 0$.

By projecting onto the double-trace primary states at the very beginning of the calculation, rather than near the end, one can circumvent having to deal with significantly more complicated structures which are not particularly relevant to the calculation of anomalous dimensions. This should make it clear that the present approach is capable of greatly simplifying the analysis of the behavior of anomalous dimensions in the $1/N$ expansion. In particular, we will now turn to a discussion of the scaling behavior of $\gamma(n, l)$ for various AdS interactions. We will see that dimensionless quantities like n and Δ can in fact be interpreted as dimensionful quantities when they are large (compared to 1), and that they obey their own rules of dimensional analysis.

3.4 Dimensional analysis with n

The interaction ϕ^4 in AdS$_3$ we considered in the previous section was renormalizable, i.e. μ had mass-dimension 1, and the anomalous dimension $\gamma(n, 0)$ decreased like $\sim n^{-1}$ at large n. This suggests that we should assign mass-dimension zero to $\gamma(n, 0)$ and mass-dimension 1 to n, so that at large n dimensional analysis forces the correct n-dependence $\gamma(n, 0) \sim \mu/n$. How does this work for other examples, in particular non-renormalizable operators? Consider the first few non-renormalizable four-point interactions in AdS$_3$: $\mu^{-1}\phi^2(\nabla\phi)^2$, $\mu^{-3}(\nabla\phi)^4$, and $\mu^{-5}(\nabla_\mu\nabla_\nu\phi)^2$. In all these cases, $\gamma(n, l)$ was calculated in [4] based on four-point correlators; we show in appendix A how to reproduce these results using the present methods. The first is accidentally renormalizable, since it may be reduced to $-\frac{m^2}{3\mu}\phi^4$ by integration by parts and using the equations of motion. However, when we calculate V_{nn} from this operator, its accidental renormalizability arises from a cancellation among the different contractions of the ϕ's, and it is illuminating to consider them separately,

$$_2\langle n, 0|(\nabla\phi)^2\phi^2|n, 0\rangle_2 = 2\,_2\langle n, 0|(\nabla\phi)^2|0\rangle\,\langle 0|\phi^2|n, 0\rangle_2 + \nabla^\mu\,_2\langle n, 0|\phi^2|0\rangle\,\nabla_\mu\langle 0|\phi^2|n, 0\rangle_2.$$
$$(3.23)$$

The first of these may easily be evaluated, since $(\nabla\phi)^2 = \frac{1}{2}\nabla^2\phi^2 - \phi\nabla^2\phi \cong (\frac{1}{2}m_n^2 - m^2)\phi^2$, where $m_n^2 = 4(\Delta + n)(\Delta + n - 1)$ is the effective mass of the two-particle primary operator (i.e., its scalar wavefunction obeys $(\nabla^2 - m_n^2)\phi^2 = 0$). The second term is only slightly more involved. In both cases, one can clearly see the additional powers of $\Delta + n$ being

pulled down from the ∂_t and ∂_ρ derivatives to make the contribution at large n behave like n^2 times the ϕ^4 result. The reduction to a lower-dimensional operator due to the equations of motion is specific to $(\nabla\phi)^2\phi^2$, and in general additional derivatives behave like additional powers of n, exactly as is necessary for dimensional analysis with n's to work. It follows that any four-point interaction in AdS$_3$ with dimension p leads to growth in $\gamma(n,0)$ like $\sim n^{p-3}$.

We can generalize these results to a quartic ϕ interaction in any dimension, using our previous results for the scalar two-particle wavefunctions. To consider the large n behavior arising from an arbitrary quartic interaction, it suffices to calculate the scaling of $\gamma(n,0)$ for ϕ^4, since as we have seen above, additional derivatives in the interaction just pull down more powers of $\Delta + n$. More concretely, if we consider a quartic interaction in AdS$_{d+1}$

$$V = \frac{\mu^{3-d}}{4!} \int d^d x \sqrt{-g} \phi^4(x), \tag{3.24}$$

using the general 2-particle wavefunctions we can readily calculate

$$\gamma(n,0) = \frac{\mu^{3-d}}{4} \int d\Omega \int_0^{\pi/2} d\rho \frac{\sin^{d-1}\rho}{\cos^{d+1}\rho} {}_2\langle n,0|\phi^2(x)|0\rangle\langle 0|\phi^2(x)|n,0\rangle_2$$
$$= \frac{\mu^{3-d}\pi^{d/2}}{4(N_{n,0}^{\phi^2})^2} \frac{\Gamma(2\Delta + 2n - \frac{d}{2})}{\Gamma(2\Delta + 2n)}. \tag{3.25}$$

Now from eq. (3.15), we can read off that the wavefunction coefficient-squared $(N_{n,0}^{\phi^2})^{-2}$ grows like $[n(n+\Delta)(n+2\Delta)]^{(d-2)/2}$ at large n, whereas the ratio of gamma functions in eq. (3.25) scales like $(n+\Delta)^{-d/2}$. Consequently, we have that $\gamma(n,0)$ for ϕ^4 at large n scales like

$$\gamma(n,0) \sim \mu^{3-d} \frac{[n(n+2\Delta)]^{(d-2)/2}}{\Delta + n} \xrightarrow{n \gg \Delta} \left(\frac{\mu}{n}\right)^{3-d}, \tag{3.26}$$

which verifies explicitly that dimensional analysis works with any quartic scalar contact interaction in any dimension. Note that we could have easily predicted this behavior simply by demanding that $\gamma(n,0)$ is proportional to the "dimensionless" combination $(\mu/n)^{3-d}$ built out of the "dimensionful" parameters μ and n, since the μ scaling is just determined by the dimension of the interaction. Roughly speaking, $\Delta+n$ is an "energy" and $[n(n+2\Delta)]^{1/2}$ is a "momentum", and the scaling simplifies when $n \gg \Delta$ because energy and momentum become the same in this "relativistic" limit. We will discuss this connection in detail in section 5.

4   Heavy field exchange

Finally we will turn to the exchange of a heavy scalar in AdS, which will help to illustrate the real power of the techniques developed in the previous sections and will let us further explore the meaning of AdS effective field theory in terms of CFT quantities. Heavy scalar exchange contributions to CFT four-point functions have been studied using a variety of techniques (see e.g. [5, 6, 9, 13, 14, 31–33]), but extracting information about anomalous

dimensions has proven to be relatively difficult using the standard methods. Here we will see that the formalism developed above is well suited to studying this problem.

To be concrete, we will consider the bulk interaction

$$V = \frac{\mu^{\frac{5-d}{2}}}{2} \int d^d x \sqrt{-g} \phi^2(x) \chi(x) \tag{4.1}$$

between massive scalars $\phi(x)$ and $\chi(x)$ in AdS_{d+1}. We will focus on the case of $d < 6$ so that this interaction is a renormalizable operator. In the limit that $m_\chi \gg m_\phi$ we can integrate out χ and obtain an effective field theory with contact terms

$$V_{\mathrm{eff}} \sim \frac{\mu^{5-d}}{m_\chi^2} \int d^d x \sqrt{-g} \phi(x)^4 + \dots \tag{4.2}$$

Below we will compare the contributions to the anomalous dimensions of the ϕ double-trace operators from the full interaction eq. (4.1) to the contributions from the effective field theory truncation eq. (4.2). We will find that the effective Lagrangian indeed approximates the full result when $n \ll \Delta_\chi$, but deviates from it when $n \sim \Delta_\chi$, eventually growing and violating the unitarity constraint discussed in section 2.2. In the full theory this growth is cut off by considering more and more terms in the effective Lagrangian, and in the CFT this amounts to "integrating in" the operator sourced by χ. In fact, as we will see shortly, one can even observe the appearance of a resonance in $\gamma(n, 0)$ near $n \sim \Delta_\chi$, completely analogous to the resonance that appears in scattering amplitudes! We will have more to say about this below, but it should be clear that much of the intuition gained from thinking about effective field theories can be directly carried over to effective conformal theories.

4.1 S-channel scalar exchange

In order to simplify the problem we will start by focusing on scalar exchange in the s-channel, which only contributes to the $l = 0$ anomalous dimensions $\gamma(n, 0)$. Since it is straightforward to identify the s-channel contractions of the quartic operators in the low-energy theory, we will be able to compare the full s-channel scalar exchange contribution at all energies to the low-energy effective theory.

Now let us compute the corrections to the anomalous dimensions $\gamma(n, 0)$ using old-fashioned perturbation theory. Since scalar exchange requires two insertions of the interaction in eq. (4.1) we must go to second order in perturbation theory. The anomalous dimensions are then given by

$$\gamma(n, 0) = \sum_\alpha \frac{|\langle \alpha | V | n, 0 \rangle_2|^2}{E_n - E_\alpha}, \tag{4.3}$$

where $E_n \equiv E_{n,0} = 2\Delta + 2n$ and α runs over all states with one χ particle and either zero, two, or four ϕ particles.

S-channel exchange corresponds to intermediate states with one χ particle as well as the "time reversed" intermediate states with four ϕ particles and one χ particle (see figure 1).
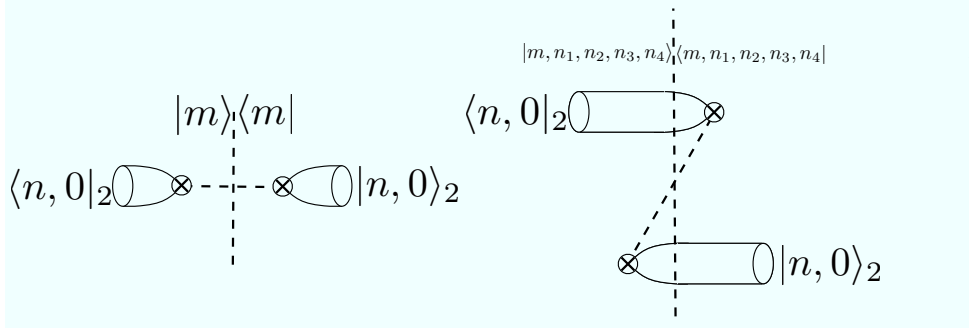
Figure 1. One-particle (left) and five-particle (right) intermediate state diagrams contributing to the "s-channel".

Since time reversal is equivalent to taking $E_n \to -E_n$, the full s-channel contribution is given by a sum over one-particle states

$$\gamma(n,0) = \sum_{m=0}^{\infty} |\langle \chi; m, 0|V|n,0\rangle_2|^2 \left(\frac{2E_m^{\chi}}{E_n^2 - E_m^{\chi 2}} \right) \tag{4.4}$$

where $E_m^{\chi} = \Delta_{\chi} + 2m$, and we have used the fact that angular momentum conservation allows only $l = 0$ states to contribute.

Now we can easily calculate the needed matrix element using the explicit form of the one-particle and two-particle states obtained in the previous sections

$$\langle \chi; m, 0|V|n,0\rangle_2 = \frac{\mu^{\frac{5-d}{2}}}{2} \int d^d x \sqrt{-g} \langle \chi; m, 0|\chi(x)|0\rangle \langle 0|\phi^2(x)|n,0\rangle_2$$

$$= \frac{\mu^{\frac{5-d}{2}}}{2\sqrt{\mathrm{vol}(S^{d-1})} N_{m,0}^{\chi} N_{n,0}^{\phi^2}} \times$$

$$\times \int d\Omega \int_0^{\pi/2} d\rho \frac{\sin^{d-1}\rho}{\cos^{d+1}\rho} \cos^{E_n + \Delta_{\chi}} \rho F\left(-m, \Delta_{\chi} + m, \frac{d}{2}, \sin^2 \rho\right)$$

$$= \frac{\mu^{\frac{5-d}{2}}}{N_{n,0}^{\phi^2}} \sqrt{\frac{\pi^{d/2}\Gamma(\frac{d}{2}+m)\Gamma(\Delta_{\chi}+m)}{8\Gamma(\frac{d}{2})m!\Gamma(\Delta_{\chi}-\frac{d-2}{2}+m)}} \frac{\Gamma(\frac{\Delta_{\chi}+E_n-d}{2})\Gamma(\frac{E_m^{\chi}-E_n+2}{2})}{\Gamma(\frac{\Delta_{\chi}-E_n+2}{2})\Gamma(\frac{E_m^{\chi}+E_n}{2})}. \tag{4.5}$$

Finally, we can square this and perform the sum over m in eq. (4.4), which for general d may be written in terms of $_4F_3$ hypergeometric functions

$$\gamma(n,0) = -\frac{\mu^{5-d}\pi^{d/2}}{8(N_{n,0}^{\phi^2})^2} \frac{\Gamma(\Delta_{\chi})\Gamma(\frac{\Delta_{\chi}+E_n-d}{2})^2}{\Gamma(\Delta_{\chi}-\frac{d-2}{2})\Gamma(\frac{\Delta_{\chi}+E_n}{2})^2} \tag{4.6}$$

$$\times \left[\frac{_4F_3\left(\left\{\frac{\Delta_{\chi}-E_n}{2}, \frac{\Delta_{\chi}-E_n+2}{2}, \Delta_{\chi}, \frac{d}{2}\right\}, \left\{\frac{\Delta_{\chi}+E_n}{2}, \frac{\Delta_{\chi}+E_n}{2}, \Delta_{\chi}-\frac{d-2}{2}\right\}, 1\right)}{\Delta_{\chi} - E_n} \right.$$

$$\left. + \frac{_4F_3\left(\left\{\frac{\Delta_{\chi}-E_n+2}{2}, \frac{\Delta_{\chi}-E_n}{2}, \Delta_{\chi}, \frac{d}{2}\right\}, \left\{\frac{\Delta_{\chi}+E_n}{2}, \frac{\Delta_{\chi}+E_n+2}{2}, \Delta_{\chi}-\frac{d-2}{2}\right\}, 1\right)}{\Delta_{\chi} + E_n} \right]$$
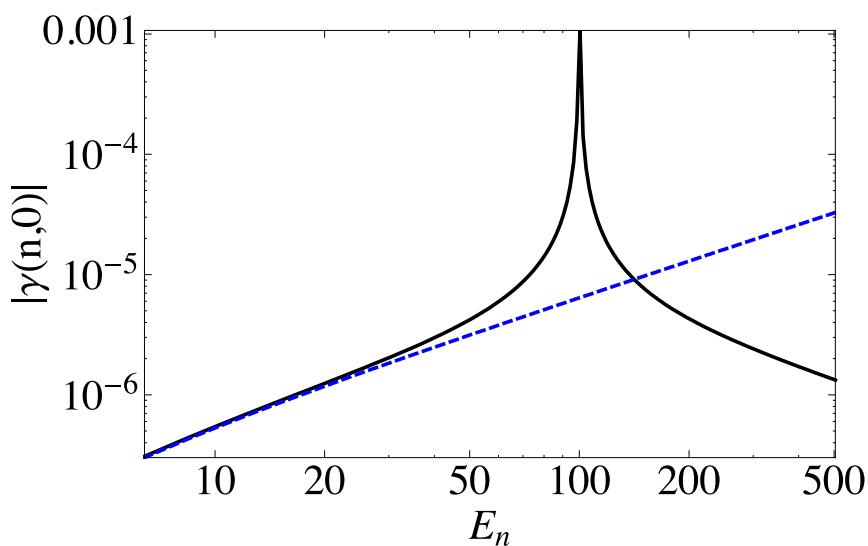
Figure 2. Plotted are the contributions to $|\gamma(n,0)|$ from s-channel scalar exchange (solid line) and s-channel contractions of the low-energy ϕ^4 interaction (dashed line) in AdS_5 with $\Delta_\chi = 100.1$ and $\Delta = 2.2$.

It is easy to see that this expression has a pole at $E_n = \Delta_\chi$, and close to this value there is a resonance-like enhancement of $\gamma(n,0)$. We can clearly see this behavior in figure 2, where we have specialized to AdS_5 and chosen $\Delta = 2.2$ and $\Delta_\chi = 100.1$ for illustrative purposes. Actually, while the expression we derived blows up at $E_n = \Delta_\chi$, if we were to go to higher order in perturbation theory we would see that the resonance gets smoothed out and has a finite width $\Gamma \sim \sum |\langle \chi | V | \phi^2 \rangle|^2$, corresponding to the fact that χ has a finite lifetime in AdS due to the trilinear interaction.

At large n we see that $\gamma(n,0)$ has a $1/n$ falloff in AdS_5, and more generally the large n behavior scales like $1/n^{5-d}$. This is precisely what we would expect based on our "dimensional analysis" discussion in the previous section, since $\gamma(n,0)$ should be proportional to the "dimensionless" combination $(\mu/n)^{5-d}$.

4.2 Matching between low and high energies

On the other hand, at small n there is another "scale" in the problem (namely Δ_χ), and the behavior is dominated by the bulk contact interactions in the effective field theory suppressed by this scale. We can concretely see this behavior in figure 2, where we have in addition plotted the contribution to $\gamma(n,0)$ from the s-channel contractions of the low-energy bulk contact interaction term ϕ^4. At smaller values of n, both functions behave roughly like $\sim n$ (as expected from dimensional analysis of the ϕ^4 interaction), but while the full correction then passes through a resonance at $E_n = \Delta_\chi$ and transitions to its large n behavior, the contribution from the ϕ^4 interaction continues to simply rise like $\sim n$. Because this operator is non-renormalizable, we see continued growth in $\gamma(n,0)$ as n increases; however, rather than continuing indefinitely and violating unitarity, the growth

is cut off in the full theory by "integrating in" the heavy primary, exactly as we would expect from effective field theory in AdS.

To better understand the matching to low energies let us try to analytically extract the leading low-n behavior of $\gamma(n,0)$ by taking the large Δ_χ limit. To do this we can approximate the Γ functions in the sum using the expansion

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + \frac{(a+b-1)(a-b)}{2z} + O\left(\frac{1}{z^2}\right)\right). \tag{4.7}$$

Also in this limit we can take $E_m^\chi/(E_m^{\chi 2} - E_n^2) \approx 1/E_m^\chi$. Finally, the sum over m can be approximated as an integral in the limit of large Δ_χ using an Euler-Maclaurin expansion. Putting everything together, we have the limiting behavior

$$\gamma(n,0) \approx -\frac{\mu^{5-d}\pi^{d/2}}{4(N_{n,0}^{\phi^2})^2\Delta_\chi^2}\left(\frac{2^d\Delta_\chi^{2E_n-d}}{\Gamma(\frac{d}{2})}\int_0^\infty dm\,\frac{\Gamma(\frac{d}{2}+m)\,(\Delta_\chi+m)^{d/2-1}}{\Gamma(1+m)\,(\Delta_\chi+2m)^{2E_n-1}} + \frac{2^{d-1}}{\Delta_\chi^{d/2}} + \dots\right)$$

$$\approx -\frac{\mu^{5-d}\pi^{d/2}}{4(N_{n,0}^{\phi^2})^2\Delta_\chi^2}\left(\frac{2^d}{\Gamma(\frac{d}{2})}\int_0^\infty dx\,\frac{(x(1+x))^{d/2-1}}{(1+2x)^{2E_n-1}}\right) + O\left(\frac{1}{\Delta_\chi^3}\right)$$

$$\approx -\frac{\mu^{5-d}\pi^{d/2}}{4(N_{n,0}^{\phi^2})^2\Delta_\chi^2}\frac{\Gamma(E_n-\frac{d}{2})}{\Gamma(E_n)} + O\left(\frac{1}{\Delta_\chi^3}\right) \tag{4.8}$$

which is precisely the form that we found in eq. (3.25) corresponding to a ϕ^4 interaction in AdS_{d+1}.

4.3 T- and U-channels

The remaining contributions to V_{nn} for scalar exchange come from three-particle intermediate states, where the $\phi^2\chi$ interaction creates a χ particle and both creates and destroys a ϕ particle. Note that while the s-channel contribution may be alternatively written in terms of an integral over the primary wavefunctions of local operators

$$\gamma_s(n,0) \propto \int d^dx\,d^{d+1}x'\sqrt{-g}\sqrt{-g'}\,_2\langle n,0|\phi^2(x)|0\rangle K_B^\chi(x,x')\langle 0|\phi^2(x')|n,0\rangle_2, \tag{4.9}$$

the t- and u-channels depend on non-local primary wavefunctions,

$$\gamma_{t,u}(n,l) \propto \int d^dx\,d^{d+1}x'\sqrt{-g}\sqrt{-g'}\,_2\langle n,l|\phi(x)\phi(x')|0\rangle K_B^\chi(x,x')\langle 0|\phi(x)\phi(x')|n,l\rangle_2 \tag{4.10}$$

which are not completely fixed by symmetry. Symmetry does imply, e.g., that

$$\langle 0|\phi(x)\phi(x')|n,0\rangle_2 \sim (e^{it}\cos\rho)^{\Delta+n}(e^{it'}\cos\rho')^{\Delta+n}f(\sigma,y), \tag{4.11}$$

where $y = (e^{it}\cos\rho)/(e^{it'}\cos\rho')$ and σ is the geodesic distance between x and x'. We could then use the Klein-Gordon equation in x or x' to solve for the function f. However, we will not continue with this analysis in the present paper. The s-channel contains most of the interesting physics, including the resonance effect discussed above. Further, we will develop a full understanding of all channels at large n (with Δ_χ, Δ arbitrary) in the next section.

5 Emergence of the flat-space S-matrix from $\gamma(n, l)$

An important goal of the AdS/CFT correspondence that has been pursued since its proposal [16–24] is to learn how information about the S-matrix of the bulk theory may be extracted from knowledge of the CFT. This is significantly more complicated than gaining information in the other direction, largely because it is difficult to eliminate the boundary effects of the AdS curvature when the theory being used to probe the S-matrix lives solely on the boundary. Various approaches have been taken to get around this issue, frequently employing the construction in the CFT of wavepackets designed to collide in the interior of AdS and extract information about divergences in the resulting interactions.

Here we will take a different approach, based on anomalous dimensions of primary operators, which are more natural quantities from the point of view of the CFT. We have seen in the preceding sections that $\gamma(n, l)$ can be computed directly via an AdS scattering process with particular external wavefunctions. Thus, it is reasonable to expect that we can extract information about the flat-space S-matrix from $\gamma(n, l)$ in the limit that the energy of this scattering process becomes much larger than the AdS curvature scale. Remarkably, it turns out this information is encoded very simply. In the limit $n \gg 1$, two-particle primary states just become flat-space spherical waves with opposite spatial momentum in the frame of the center of AdS. Consequently, matrix elements ${}_2\langle n, l | V | n, l \rangle_2$ literally become the partial wave expansion of the flat-space S-matrix up to a normalization factor,

$$\mathcal{M}(s, t, u)^{d+1}_{\text{flat space}} = \frac{(4\pi)^d}{\text{vol}(S^{d-1})} \frac{E_n}{(E_n^2 - 4\Delta^2)^{\frac{d-2}{2}}} \sum_l [\gamma(n, l)]_{n \gg l} \, r_l P_l^{(d)}(\cos\theta), \quad (5.1)$$

where the total flat-space energy is $2E = E_n = 2\Delta + 2n$ (still in units of $R = 1$), and the Mandelstam variables are defined in the usual way with $s = (2E)^2$, $t = -2p^2(1-\cos\theta)$, $u = -2p^2(1+\cos\theta)$, and $p^2 = E^2 - \Delta^2$. One must formally take the large n limit of $\gamma(n, l)$ before substituting into the above formula when constructing the flat-space amplitudes.[12] This correspondence between $\mathcal{M}(s, t, u)^{d+1}_{\text{flat space}}$ and $\gamma(n, l)$ holds whenever n is much greater than 1. In particular, it allows us to probe the S-matrix even away from singularities in the four-point function, as was done previously. For instance, in section 4.1, we saw that the anomalous dimensions are sensitive to the behavior of the S-matrix for scalar exchange at all energies, from far below the intermediate particle mass, through the resonance, and to far above it. Singularities in the four-point function from non-renormalizable interactions in the bulk are unlikely to occur in isolation in an effective AdS theory, since all non-renormalizable operators tend to become important at around the same scale. So we

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[12]When there are additional CFT parameters such as Δ (or Δ_χ in section 4) that correspond to mass terms, one must take these to be large as well in order to see their effects in the scattering amplitude. More formally, one takes $E_n = 2(\Delta + n)k$, $p^2 = n(n + 2\Delta)k^2$, $m = \Delta k$, ... and takes $k \to 0$ with E_n, p, m, ... fixed. There may not always be a free parameter within the CFT that allows one to take Δ large; in such cases, eq. (5.1) obtains the amplitude $\mathcal{M}$ with $m = 0$. There is an important caveat here; the presence of such massless fields in the bulk theory can lead to infrared-divergent scattering amplitudes, and for such quantities the left-hand side of eq. (5.1) would have to be modified to include AdS boundary effects. Thus, if one is not free to dial $m \gg k$ in the CFT, then one should apply (5.1) only to amplitudes that are infrared-safe in the $m \to 0$ limit.

expect it will prove convenient to have a method for extracting S-matrix elements that does not depend on isolating such singularities.

Why should we expect $\gamma(n,l)$ to probe flat space at large n? To a large extent, it is because the primary wavefunctions $\sim \cos^{2\Delta+2n}\rho$ are extremely peaked near $\rho \sim 0$ in this limit. Since the contribution to $\gamma(n,l)$ is dominated by the interior of AdS, we expect that the AdS radius R will become negligible, and the dynamics will be increasingly well described by flat-space scattering. More precisely, $\cos^{2\Delta+2n}\rho$ becomes proportional to a delta function at $\cos\rho = 1$ as n is taken to ∞, so the integral over the bulk may be restricted to smaller and smaller regions around $\rho = 0$. One may take the large n, small ρ limit by restoring factors of R (as well as $k \equiv 1/R$) and taking $R \to \infty$ with n/R and $r = \rho R$ fixed. The metric in the new coordinates (also making the replacement $t \to t/R$) is

$$ds^2 = \frac{1}{\cos^2(r/R)}(-dt^2 + dr^2 + R^2\sin^2(r/R)d\Omega^2), \tag{5.2}$$

which approaches the flat-space metric for small r/R. The primary wavefunction then becomes suppressed by an exponential damping term $\cos^{2n}\rho \sim e^{-n(kr)^2}$ at the scale $r \sim \frac{1}{k\sqrt{n}}$.

However, we can also represent the two-particle primary wavefunctions as a sum over products of one-particle wavefunctions. Moreover, deep in the interior of AdS, it is straightforward to see that the one-particle wavefunctions in eq. (2.19) can be approximated by flat-space spherical waves (see e.g. [20]). That is, the one-particle wavefunctions become

$$
\begin{aligned}
\phi_{nlJ}(x) &= \frac{1}{N_{\Delta,n,l}}e^{iE_{n,l}kt}Y_{lJ}(\Omega)\sin^l(kr)\cos^\Delta(kr)F\left(-n, \Delta+l+n, l+\frac{d}{2}, \sin^2(kr)\right) \\
&\overset{kr\ll1}{\propto} \frac{1}{(kr)^{\frac{d-2}{2}}}e^{iE_{n,l}kt}Y_{lJ}(\Omega)J_{l+(d-2)/2}(E_{n,l}kr),
\end{aligned}
\tag{5.3}
$$

which is a flat-space spherical wave in $d+1$ dimensions with energy $E_{n,l}k$ and angular momentum l. Thus, we expect the two-particle primary wavefunctions in this limit to look like a sum over products of flat-space spherical waves (or alternatively plane waves, using the standard decomposition).

In the next two subsections we will explore more carefully the way in which momentum conservation emerges at large n, forcing these waves to have opposite spatial momentum so that matrix elements of V look precisely like flat-space scattering amplitudes in the center-of-mass frame. We will approach this question from both the CFT and bulk perspectives. This will eventually lead to a derivation of eq. (5.1), and we will then check it in a number of examples.

5.1 Emergence of momentum conservation

Translation invariance and momentum conservation of amplitudes must emerge in the flat-space limit. In particular, one would like to see how delta-functions of the total momentum emerge in the overlap between two-particle primary states with one-particle states. Since a primary state with large n carries zero momentum (as we will see explicitly in the next subsection), what must emerge is something like the flat-space relation $\langle P|p_1, p_2\rangle \propto \delta(\vec{p}_1 + $

$\vec{p}_2$) for $\vec{P} = 0$, where $|P\rangle$ denotes a two-particle state with center-of-mass four-momentum P, and $|p_1, p_2\rangle$ is a tensor product of one-particle states with four-momenta p_1 and p_2. We can look for this behavior in the explicit form of the overlap of two-particle primary states with one-particle states. To begin, let us consider more carefully what these overlaps look like in flat space. Since we are interested in primary states, we will consider a flat-space two-particle state $|2E, 0\rangle_2$ with zero momentum and energy $2E$, which we can decompose into one-particle states as

$$|2E, 0\rangle_2 = \int \frac{d^d p_1}{2E_1 (2\pi)^d} \frac{d^d p_2}{2E_2 (2\pi)^d} \sqrt{\frac{E_1 E_2}{E p_1^{d-2}}} (2\pi)^{d+1} \delta(2E - E_1 - E_2) \delta^d(\vec{p}_1 + \vec{p}_2) |E_1, p_1\rangle |E_2, p_2\rangle$$

$$= \frac{1}{(2\pi)^{d-1}} \frac{p^{\frac{d-2}{2}}}{8E^{\frac{1}{2}}} \int d\Omega \, |E, p\widehat{e}\rangle \, |E, -p\widehat{e}\rangle, \tag{5.4}$$

where in the last line, $p = \sqrt{E^2 - m^2}$, and the factor $\sqrt{\frac{E_1 E_2}{E p_1^{d-2}}}$ is inserted to give the state the norm $_2\langle 2E', p'|2E, 0\rangle_2 \sim \delta(E - E') \delta^d(p')$. Note that the energies of the one-particle states are the same, as a consequence of momentum conservation. In addition, one can see here a general factor $p^{\frac{d-2}{2}} / E^{\frac{1}{2}}$ that is responsible for the E_n-dependence of the normalization factor in eq. (5.1).

Now we would like to consider the analogous decomposition in the CFT, where we can write the double-trace primary states $|n, l\rangle_2$ in terms of products of single-trace states. We should be able to see that the overlaps at large n are very narrowly peaked on products of single-trace states that have nearly equal weights, just as in eq. (5.4). We can extract this overlap without too much difficulty by considering the two-point and three-point functions in the CFT, the form of which is fixed up to an overall constant coefficient. We will show this explicitly in 2d, where we will not have to deal with additional angular variables, but the arguments are essentially the same and can be carried out explicitly in any dimension.

To simplify the discussion even further we will focus our attention on just the left-moving sector. More precisely, we will consider holomorphic operators $\mathcal{O}(z)$ that depend on z, but not $\overline{z}$. Let us take $\mathcal{O}(z)$ to be a single-trace such operator with left-moving weight $h = \Delta/2$. Also, in analogy with the double-trace primary operators $\mathcal{O}_{n,l}(x)$ discussed in the rest of the paper, let us take $\mathcal{O}_n(z)$ to be a double-trace left-moving primary with weight $2h + n$. Then $\mathcal{O}(z)$ and its descendants are in one-to-one correspondence with the one-particle states $|h; s\rangle$, and $\mathcal{O}_n(z)$ with the primary state $|2h + n\rangle_2$. We will now proceed to compute the overlap $\langle h; s|\langle h; n - s|2h + n\rangle_2$ in order to compare with our expectations from flat space.

First we will perform the usual Laurent expansion of the operator $\mathcal{O}(z)$ in terms of creation and annihilation operators.[13] Taking $z = e^\tau$, we have

$$\mathcal{O}(\tau) = \sum_{s=0}^{\infty} N_s(h) e^{\tau(h+s)} a_s^\dagger + N_s(h) e^{-\tau(h+s)} a_s, \tag{5.5}$$

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[13]Technically, this is a Laurent expansion only for even integer Δ, but for convenience we will abuse terminology somewhat and use this term for general Δ.

where $a_s^\dagger$ creates the one-particle s-th descendant state $|h; s\rangle$. The $N_s(h)$ factors are the Laurent coefficients, which can easily be extracted from the two-point function:

$$\langle \mathcal{O}(\tau)\mathcal{O}(0)\rangle = \frac{e^{\tau h}}{(e^\tau - 1)^{2h}} = \sum_s \frac{\Gamma(2h+s)}{\Gamma(2h)s!}e^{-\tau(h+s)} = \sum_s N_s^2(h)e^{-\tau(h+s)}. \quad (5.6)$$

We can obtain the overlap of $|2h+n\rangle_2$ with the tensor product of one particle states $|h; m\rangle|h; n-m\rangle$ by considering a similar expansion of the three-point function. To do this, we can first evaluate the correlator $\langle \mathcal{O}(\tau)\mathcal{O}(0)\mathcal{O}_n(-T)\rangle$ with $T \to \infty$ using the Laurent expansion, which gives

$$\langle \mathcal{O}(\tau)\mathcal{O}(0)\mathcal{O}_n(-T)\rangle e^{(2h+n)T} \overset{T \to \infty}{=} \sum_s N_s(h)N_{n-s}(h)e^{-\tau(h+s)}\langle h; s|\langle h; n-s|2h+n\rangle_2. \quad (5.7)$$

Alternatively, we can use the explicit form determined by conformal symmetry:

$$\langle \mathcal{O}(\tau)\mathcal{O}(0)\mathcal{O}_n(-T)\rangle e^{(2h+n)T} \overset{T \to \infty}{=} c_n e^{-\tau h}(1 - e^{-\tau})^n = c_n \sum_s (-1)^s \binom{n}{s}e^{-\tau(h+s)}, \quad (5.8)$$

where c_n is the OPE coefficient for $\mathcal{O}_n$ inside $\mathcal{O} \times \mathcal{O}$. Together, these imply that

$$\langle h; s|\langle h; n-s|2h+n\rangle_2 = \binom{n}{s}\frac{(-1)^s c_n}{N_s(h)N_{n-s}(h)}. \quad (5.9)$$

The right-moving sector essentially just introduces additional quantum numbers for the states and an additional overlap factor symmetric with the above one.

Now let us return to the issue of momentum conservation. For large n, the overlap factors between two-particle primaries and single-particle states are strongly peaked at $s = n/2$, which is exactly where the one-particle momenta are equal in magnitude, corresponding to the expected delta function $\delta(\vec{p}_1 + \vec{p}_2)$. In fact, by expanding $s = n/2 + m$ in m, one obtains the combinatoric suppression factor $\binom{n}{s} \sim e^{-m^2/n}$, so that momentum conservation emerges with a fuzziness proportional to $\sqrt{n}$.[14]

We can see a similar phenomenon in matrix elements of V, which are also expected to conserve momentum at large n. For example, let us consider the matrix elements corresponding to the $\phi^2\chi$ interaction considered in section 4. As $n \to \infty$, the overlap eq. (4.5) (in d=2) can be approximated as

$$\langle \chi; m, 0|V|n, 0\rangle_2 \to \frac{(-1)^m \mu^{3/2}\pi^{1/2}}{nN_{n,0}^{\phi^2}2\sqrt{2}}\exp\left(\frac{-m(m+\Delta_\chi) - \Delta - \frac{\Delta_\chi}{2} + 1}{n}\right), \quad (5.10)$$

which is peaked at $m = 0$ (zero χ-momentum), again with fuzziness $\sim \sqrt{n}$.

Curiously, though we do indeed find momentum conservation at large n, we also find violations that *grow* with n. This is not a contradiction. In fact, $\sqrt{n}$ growth is exactly what is needed for emergence of the flat-space S-matrix. To see this, let us restore the

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[14]In 2d, the Laurent coefficients $N_s(h)$ have trivial s-dependence at large s and may be neglected. For instance, at $h = \frac{1}{2}$, $N_s(h) = 1$ identically for any s.

AdS curvature scale k, writing the energy as $E = nk$ and the momentum as $p = mk$. The typical momentum spread is then

$$\delta p \sim \sqrt{kE}. \tag{5.11}$$

At a fixed curvature scale the "uncertainty" in momentum grows with E, reflecting the fact that primary wavefunctions become more and more localized in position space,

$$(\cos kr)^{2E/k} \sim e^{-(kE)r^2}. \tag{5.12}$$

However, relative to the scale E of our scattering process, the momentum spread goes to zero at high energies

$$\frac{\delta p}{E} \sim \sqrt{\frac{k}{E}} \to 0, \tag{5.13}$$

so the amplitude is momentum-conserving to leading order in E. In other words, as $n \to \infty$, the primary wavefunctions simultaneously become localized at the center of AdS (and thus insensitive to the global geometry), and approach flat-space momentum eigenstates with translationally-invariant interactions.

5.2 Two-particle primaries at large n in AdS_{d+1}

We have seen how a $\sqrt{n}$ fuzziness in momentum conservation emerges from the CFT perspective. Now we will try to see this behavior emerge directly in AdS_{d+1}, and solve for the behavior of two-particle primary wavefunctions at large R. In the coordinates (5.2), the AdS isometries (2.24) become

$$K_\mu = -R\frac{\partial}{\partial x^\mu} + ix_\mu\frac{\partial}{\partial t} + it\frac{\partial}{\partial x^\mu} + O(t/R, x/R) \tag{5.14}$$

$$P_\mu = +R\frac{\partial}{\partial x^\mu} + ix_\mu\frac{\partial}{\partial t} + it\frac{\partial}{\partial x^\mu} + O(t/R, x/R) \tag{5.15}$$

where $x_\mu = r\Omega_\mu$. Here we see that at leading order, $K_\mu \sim -R\frac{\partial}{\partial x_\mu}$ is just the flat-space translation generator, so the leading order condition for a two-particle state to be primary is simply that it have zero total spatial momentum. Hence, near the center of AdS, if we take a two-particle primary wavefunction $\langle 0|\phi(x_1)\phi(x_2)|\psi\rangle_2$ to have definite energy $2E$ and definite momentum $\vec{p}$ in the x_1 coordinate, it should behave like a superposition of plane waves in the center of mass frame

$$\langle 0|\phi(x_1)\phi(x_2)|\psi\rangle_2 \sim e^{iE(t_1+t_2)+ip\cdot(x_1-x_2)} + O(x/R), \tag{5.16}$$

where $E = E_p \equiv \sqrt{p^2 + m^2}$.

This is almost enough to understand why matrix elements between primaries are so closely related to the flat-space S-matrix. One might worry that primary states behave less like plane waves away from the center of AdS, and that their matrix elements could be sensitive to these effects. However, by solving for the two-particle primaries at the next

order in $1/R$, we will start to see the position-space localization observed in the previous section, which implies that global geometry becomes irrelevant at high energies.

Let us begin with the zero-th order solution eq. (5.16), and allow a small perturbation q around zero total spatial momentum,

$$\langle 0|\phi(x_1)\phi(x_2)|\psi\rangle_2 \sim \int d^d q\, f(q) e^{iE_{p+q}t_1 + iE_{p-q}t_2 + i(p+q)\cdot x_1 - i(p-q)\cdot x_2}. \tag{5.17}$$

Requiring that this be killed by the $O(R)$ and $O(1)$ terms in eq. (5.14) then implies

$$\left(E\frac{\partial}{\partial q_\mu} + 2Rq_\mu + O(q/E) \right) f(q) = 0. \tag{5.18}$$

Finally, dropping the $O(q/E)$ terms, this has the solution

$$f(q) = \frac{1}{(\pi kE/2)^{d/4}} e^{-q^2/kE}, \tag{5.19}$$

where the normalization has been chosen so that $\int d^d q\, f(q)^2 = 1$. We have thus rederived what we observed in the previous section. Two-particle primaries at large n approach flat-space plane waves, with opposing momenta peaked at $p \sim \sqrt{E^2 - m^2}$, up to an uncertainty $\delta p \sim \sqrt{kE}$.

An important point is that this momentum uncertainty only occurs in the center of mass degree of freedom. Performing the q-integration, we see that the wavefunction is proportional to $e^{-kE(x_1+x_2)^2/4}$. In particular, it is not necessarily suppressed when $x_1 \sim -x_2 \sim R$. In this regime, $O(x/R)$ corrections could become important, and to fully understand the wavefunctions we would have to solve for these corrections. However, for the cases we will be considering, the interactions are either completely local, or we have the exchange of a massive particle, with mass $M \gg 1/R$. Therefore, the propagator will suppress the amplitude when $|x_1 - x_2| \gg 1/M$. Thus, the combination of the localization of the center of mass as well as the short range of propagation ensures that as E becomes large the dominant contribution to the amplitude comes from the flat region in middle of AdS. When the angular momentum, l, of the state is also large, there is a danger that the wave function is no longer fully localized in the center of mass coordinate. For $l = 2$, this lack of localization can already be seen explicitly in eq. (3.6) when ω is small. For the large l cases, we therefore require in addition that $ER \gg l$. In terms of CFT quantities this requirement amounts to $n \gg l$, which we assume in following.

Localization near the center of AdS in both x_1 and x_2 means that when we compute matrix elements, the integrals over spatial slices (coming from our interaction $V = \int d^d x \sqrt{-g}\, \mathcal{V}(x)$) will always converge before $O(x/R)$ effects become important. More precisely, we can split up the integration over r into three different regions: flat-space scales $0 < r \lesssim y_f E^{-1}$, large scales $y_l \sqrt{RE^{-1}} \lesssim r < R\pi/2$ containing the boundary of AdS, and the remaining intermediate region, containing the transition scale $\sqrt{RE^{-1}}$. As R and n increase, we may increase y_l to obtain arbitrarily good exponential damping of the AdS boundary effects from large scales. Then, the wavefunctions in the remaining regions are described by flat-space plane waves, times the exponential envelope factor that essentially

puts the plane waves in finite volume. As a result, all the important dynamics are taking place in a regime where they can be described in terms of single-particle, flat-space plane waves.

Now projecting (5.17) onto states with definite angular momentum, the correct flat-space states[15] corresponding to two-particle primaries are

$$|n, lJ\rangle_2 = \frac{|2p|^{\frac{d-2}{2}}}{(2\pi)^d \sqrt{2RE}} \int d\widehat{p}\, Y_{lJ}(\widehat{p}) \int d^d q \, f(q) |p+q\rangle |-p+q\rangle \qquad (n \gg 1, l), \quad (5.20)$$

where we have fixed the normalization by requiring that $_2\langle n, lJ|n', l'J'\rangle_2 = \delta_{nn'}\delta_{ll'}\delta_{JJ'}$, which approaches $R^{-1}\delta(E - E')\delta_{JJ'}$ in the continuum limit. Taking matrix elements of both sides, we find that the leading large $n \gg l, 1$ behavior of $\gamma(n, l)$ matches $\mathcal{M}(s, t, u)_{\text{flat space}}^{d+1}$ after taking $2E = E_n k = (2\Delta + 2n)k$, $p^2 = n(n + 2\Delta)k^2$ according to the relation

$$\gamma(n, l) = \frac{\text{vol}(S^{d-2})}{(2\pi)^d} \frac{|p|^{d-2}}{8E} \int d\theta \sin^{d-2}\theta \, P_l^{(d)}(\cos\theta)\mathcal{M}(s, t, u)_{\text{flat space}}^{d+1}$$

$$= \frac{\text{vol}(S^{d-2})}{(4\pi)^d} \frac{(E_n^2 - 4\Delta^2)^{\frac{d-2}{2}}}{E_n} \int d\theta \sin^{d-2}\theta \, P_l^{(d)}(\cos\theta)\mathcal{M}(s, t, u)_{\text{flat space}}^{d+1}, \quad (5.21)$$

where we have introduced the angular polynomials $P_l^{(d)}(\cos\theta)$, defined by $P_l^{(d)}(\widehat{e} \cdot \widehat{e}') = \frac{1}{r_l}\text{vol}(S^{d-1})\sum_J Y_{lJ}(\widehat{e})Y_{lJ}^*(\widehat{e}')$, where r_l is the dimension of the spin-l representation of $SO(d)$. Finally we can invert this relation using the completeness relation

$$\text{vol}(S^{d-2})\sin^{d-2}\theta \sum_l r_l P_l(\cos\theta)P_l(\cos\theta') = \text{vol}(S^{d-1})\delta(\theta - \theta') \quad (5.22)$$

to obtain the result given in eq. (5.1).

5.3 Examples

5.3.1 Example 1: ϕ^4

Now we will turn to a number of checks that the flat-space S-matrix does indeed emerge from $\gamma(n, l)$ at large n, as described in eqs. (5.1) and (5.21). We will return to units of $R = 1$ for simplicity, since factors of R cannot appear in the flat space amplitude anyway. Our first check is the simplest case, a $\mu^{3-d}\phi^4/4!$ interaction in AdS_{d+1}, which has simply $\mathcal{M}_{\text{flat space}} = \mu^{3-d}$. We have essentially already computed the anomalous dimensions in eq. (3.26); keeping track of the $O(1)$ coefficients, one finds that the large n, Δ limit of $\gamma(n, 0)$ is

$$\gamma(n, 0) = \mu^{3-d}\frac{\text{vol}(S^{d-1})}{8(2\pi)^d}\left(\frac{[n(n+2\Delta)]^{\frac{d-2}{2}}}{\Delta + n}\right). \quad (5.23)$$

We recognize the factor $(\Delta + n)$ as the energy E in global coordinates of each one-particle state, and similarly the momentum is $p^2 = E^2 - m^2 = (\Delta + n)^2 - \Delta^2 = n(n + 2\Delta)$. Finally, since $P_0^{(d)}(\cos\theta) = 1/r_0$, we see this exactly agrees with eq. (5.1).

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[15]More precisely, eq. (5.20) should be understood to be true when it is acted on from the left by $\langle 0|\phi(x_1)\phi(x_2)$ for any $|x_1|, |x_2| \ll R$.

5.3.2 Example 2: $(\nabla\phi)^4$

Our second example is $\mu^3(\nabla\phi)^4/4!$ in AdS$_3$, where explicit formulae for $\gamma(n, l)$ are known. The flat-space amplitude for this operator is

$$\mathcal{M}_{\text{flat space}} = \mu^3 \left(E^4 + \frac{2}{3}p^2 E^2 + p^4 \left(\frac{1}{3} + \frac{2}{3}\cos^2\varphi \right) \right). \tag{5.24}$$

The appropriate angular polynomials $P_l^{(2)}(\cos\varphi)$ in 2d are $P_0^{(2)}(x) = 1$ and $P_2^{(2)}(x) = 2(x^2 - \frac{1}{2})$ for $l = 0$ and $l = 2$, respectively. Projecting $\mathcal{M}_{\text{flat space}}(\cos\varphi)$ onto these polynomials gives

$$\mathcal{M}_{\text{flat space}}(x) = \frac{\mu^3}{3}(3E^4 + 2E^2 p^2 + 2p^4)P_0^{(2)}(x) + \frac{\mu^3}{3}p^4 P_2^{(2)}(x). \tag{5.25}$$

In order to bring the explicit expressions for $\gamma(n,0)$ and $\gamma(n,2)$ given in equations (A.8, A.9) into this form, we can take the leading terms at large Δ, n and replace $\Delta \to \sqrt{E^2 - p^2}, n \to E - \sqrt{E^2 - p^2}$:

$$6\pi\mu^3\gamma(n,0) \overset{n,\Delta \gg 1}{\longrightarrow} \frac{7n^4 + 28n^3\Delta + 36n^2\Delta^2 + 16n\Delta^3 + 3\Delta^4}{8(\Delta + n)} \to \frac{3E^4 + 2E^2 p^2 + 2p^4}{8E}, \tag{5.26}$$

$$6\pi\mu^3\gamma(n,2) \overset{n,\Delta \gg 1}{\longrightarrow} \frac{n^2(n + 2\Delta)^2}{16(\Delta + n)} \to \frac{p^4}{16E}. \tag{5.27}$$

This again agrees with the flat-space scattering partial wave amplitude (5.25) upon substituting into (5.1).

5.3.3 Example 3: $\gamma(n, L)$ at maximum spin L

Contact quartic interactions have a maximum spin L for the primary operators to which they contribute anomalous dimensions; for instance, $(\nabla\phi)^4$ has $L = 2$. In [4], a general form for such contributions $\widetilde{\gamma}(n, L)$ for any operator was obtained, and its dependence on n and Δ is fixed by L. Since the overall constant coefficient is undetermined and so cannot be checked anyway, we will neglect many proportionality constants in this subsection. Consider first $d = 2$, where we may take the explicit expression for $\widetilde{\gamma}(n, L)$ in the large n, Δ limit, and replace them by the appropriate energy and momentum as above:

$$\begin{aligned}
\widetilde{\gamma}(n, L) &= \pi \frac{\Gamma(n + L + 1)\Gamma(2\Delta + n + L - 1)\Gamma(\Delta + n - \frac{1}{2})\Gamma(\Delta + n + L)}{4\Gamma(1 + n)\Gamma(\Delta + n)\Gamma(\Delta + n + L + \frac{1}{2})\Gamma(2\Delta + n - 1)} \\
&\overset{n,\Delta \gg 1}{\longrightarrow} \frac{\pi}{4} \frac{[n(n + 2\Delta)]^L}{\Delta + n} \to \frac{\pi}{4}\frac{p^{2L}}{E}.
\end{aligned} \tag{5.28}$$

To compare this with flat space, we may use the amplitude for the exchange of a heavy scalar with mass M as a trick to generate the correct quartic-interaction amplitude. Specifically, one may expand in $1/M$ and take the leading non-zero term for a given L. This then corresponds to the lowest-dimensional effective operator that contributes to $\gamma(n, L)$. But, for that operator, L is the largest spin that gets a correction, so the leading non-zero term in the $1/M^2$ series is $\widetilde{\gamma}(n, L)$.

Let $\mathcal{A}$ denote the amplitude for scalar exchange:

$$\mathcal{A} \equiv \mu^{5-d} \left(\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2} \right),$$
$$s = (2E)^2, \qquad t, u = -2(E^2 \pm p^2 \cos\varphi - m^2). \tag{5.29}$$

The angular polynomials in 2d are just $P_l(\varphi) = \cos(l\varphi)$, so we can project the scalar exchange amplitude as

$$2 \int_0^\pi d\varphi \cos(L\varphi) \mathcal{A} \supset -(1 + (-1)^L) 2^{L+1} \mu^3 \int_0^\pi d\varphi \cos(L\varphi) \left(\frac{p^{2L} \cos^L \varphi}{M^{2L+2}} \right)$$
$$= -(1 + (-1)^L) 2\pi \mu^3 \frac{p^{2L}}{M^{2L+2}}, \tag{5.30}$$

which, after dividing by the normalization factor $\propto E$ from eq. (5.1), matches the behavior from $\widetilde{\gamma}(n, L)$ above.

Similarly, in $d = 4$, the large Δ, n limit of $\widetilde{\gamma}(n, L)$ is

$$\widetilde{\gamma}(n, L) \overset{n, \Delta \gg 1}{\propto} \frac{[n(n + 2\Delta)]^{L+1}}{\Delta + n} \to \frac{p^{2(L+1)}}{E}. \tag{5.31}$$

The angular polynomials in 4d are $P_l(\varphi) \propto \frac{\sin((l+1)\varphi)}{\sin\varphi}$, and when we project the scalar exchange amplitude onto them, we find

$$\int_0^\pi d\varphi \sin\varphi \sin((L+1)\varphi) \mathcal{A} \supset -(1 + (-1)^L) 2^L \mu \int_0^\pi d\varphi \sin\varphi \sin((L+1)\varphi) \left(\frac{p^{2L} \cos^L \varphi}{M^{2L+2}} \right)$$
$$= -(1 + (-1)^L) \frac{\pi\mu}{2} \frac{p^{2L}}{M^{2L+2}}. \tag{5.32}$$

In $d = 4$, the wavefunction overlap factor is $\propto p^2/E$, which again accounts for the difference between $\gamma(n, L)$ and the flat-space amplitude.

5.3.4 Example 4: scalar exchange in $d = 2$

Finally, we will compare the anomalous dimensions arising from the scalar exchange calculation done in section 4.1 with the flat-space amplitude. In this section we will obtain from flat-space scattering the complete scalar exchange contribution to $\gamma(n, 0)$ at large n and Δ, but due to the difficulty of evaluating the t- and u-channels in the CFT, we will be able to check explicitly only the s-channel. As a partial check of the t- and u-channels, we will expand in inverse powers of the exchanged scalar mass and compare to the known form of $\gamma(n, 0)$ from operators in the low-energy theory; however, strictly speaking, this is a check only of the form of $\gamma(n, 0)$ at n's below the dimension of the exchanged scalar primary operator. For simplicity we will focus on scalar exchange in $d = 2$.

Projecting the amplitude $\mathcal{A}$ onto spin-0 modes in 2d, we have

$$\mathcal{A}_0 = \mu^3 \left(\frac{1}{4E^2 - M^2} \right) - \mu^3 \left(\frac{2}{M\sqrt{M^2 + 4p^2}} \right). \tag{5.33}$$

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We have explicitly separated out the first term in brackets as the s-channel contribution to the spin-0 amplitude. In order to compare to $\gamma(n,0)$, we need to take the large n, Δ limit from section 4.1. In $d=2$, the expression simplifies to

$$\gamma(n,0) = \frac{\mu^3}{(4\pi)(\Delta_\chi + E_n - 2)^2} \left[\frac{{}_3F_2\left(\left\{1, \frac{\Delta_\chi - E_n}{2}, \frac{\Delta_\chi - E_n + 2}{2}\right\}, \left\{\frac{\Delta_\chi + E_n}{2}, \frac{\Delta_\chi + E_n}{2}\right\}, 1\right)}{E_n - \Delta_\chi} \right.$$
$$\left. - \frac{{}_3F_2\left(\left\{1, \frac{\Delta_\chi - E_n + 2}{2}, \frac{\Delta_\chi - E_n + 2}{2}\right\}, \left\{\frac{\Delta_\chi + E_n}{2}, \frac{\Delta_\chi + E_n + 2}{2}\right\}, 1\right)}{E_n + \Delta_\chi} \right]. \quad (5.34)$$

To take the appropriate limit of the hypergeometric functions, we can use the integral representation

$$_3F_2\left(\{\delta, \delta+1, 1\}, \{M+2, M+2\}, 1\right) = \quad (5.35)$$
$$\frac{\Gamma^2(M+2)}{\Gamma(M+2-\delta)\Gamma(M+1-\delta)\Gamma(\delta+1)\Gamma(\delta)} \int_0^1 dt \int_0^1 ds \frac{t^\delta (1-t)^{M-\delta} s^{\delta-1}(1-s)^{M+1-\delta}}{1-st},$$

taking $M+2 = \frac{\Delta_\chi + E_n}{2}$ and $\delta = \frac{\Delta_\chi - E_n}{2}$. The integral has a saddle point near $s, t = \delta/M$ at large δ, M.[16] Around this point, all the factors in the integral except for $(1-st)$ simply contribute to cancel the Γ-function prefactors, leaving behind just the value $M^2/(M^2 - \delta^2)$ of $(1-st)$. The same argument applies to both ${}_3F_2$'s. Thus, we obtain

$$\gamma(n,0) \approx \frac{\mu^3}{(4\pi)(\Delta_\chi + E_n)^2} \left[\left(\frac{M^2}{M^2 - \delta^2}\right) \frac{2\Delta_\chi}{E_n^2 - \Delta_\chi^2} \right] = \frac{\mu^3}{(8\pi)E_n(E_n^2 - \Delta_\chi^2)}. \quad (5.36)$$

This matches the s-channel amplitude using (5.21).

To consider the full amplitude with all channels included, we can expand in $1/M$ and match to contributions from local operators in AdS$_3$. We are not interested in operators of the form $(\phi(\partial^2)^n \phi)(\phi(\partial^2)^m \phi)$, since these may be related to ϕ^4 by the equations of motion and therefore do not give new forms of momentum-dependence. To throw these out, we simply take $s = 2(E^2 + p^2), t, u = -2(E^2 \pm p^2 \cos^2 \varphi)$ in eq. (5.29), i.e. we replace terms like $(p_i + p_j)^2$ with $2p_i \cdot p_j$. The remaining s-wave amplitude is then

$$\mathcal{A}_{0, \text{no } m^2} = \mu^3 \left(\frac{1}{2(E^2 + p^2) - M^2}\right) - \mu^3 \left(\frac{2}{\sqrt{(2E^2 + M^2)^2 - 4p^2}}\right) \quad (5.37)$$

The first few terms $\sim 1/M^2, 1/M^4, 1/M^6$ are just the ϕ^4, $(\nabla\phi)^2\phi^2$, and $(\nabla\phi)^4$ contributions that we have already checked. The first new piece appears at $O(1/M^8)$:

$$\mathcal{A}_{0, \text{no } m^2} \supset 8\frac{\mu^3}{M^8}\left(E^6 - 3E^4 p^2 - p^6\right). \quad (5.38)$$

This needs to match the contribution from the local operator $(\nabla_\mu \nabla_\nu \phi)^2 (\nabla\phi)^2$. Performing the explicit computation (see appendix A), we obtain

$$\gamma(n,0) \propto \frac{\widetilde{P}_8(n)}{(2\Delta + 2n - 3)(2\Delta + 2n - 1)(2\Delta + 2n + 1)}, \quad (5.39)$$

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[16] The hypergeometric function ${}_3F_2$ is analytic in all of its arguments, so we may perform the integral assuming $\delta > 0$, and then obtain the result at $\delta < 0$ by analytic continuation.

where $\widetilde{P}_8(n)$ is an eighth-order polynomial in n, whose explicit form is given in equation (A.11). At large n, this expression for $\gamma(n, 0)$ simply approaches

$$\gamma(n, 0) \propto \frac{E^6 - 3E^4 p^2 - p^6}{E}, \tag{5.40}$$

as we expect.

6 Conclusion

In this paper we have argued that whenever the dilatation operator of a CFT admits a perturbative expansion, local interactions in global AdS provide a natural framework for organizing such a perturbation theory. This is particularly true if for dimensions $\Delta < \Delta_{\text{Heavy}}$ there are only a few single-trace primary operators, in which case AdS contains only a few fields. This is analogous to the statement that whenever a Lorentz-invariant theory describes weakly-interacting particles, local interactions in Minkowski space are a convenient way of organizing perturbation theory. This is especially useful if for energies $E < M_{\text{Heavy}}$ the Lorentz-invariant theory describes only a few particles. However, there was an important difference in the Lorentz-invariant case. Namely, for Lorentz-invariant theories there is a clear argument that perturbative unitarity (for all $E < M_{\text{Heavy}}$) requires that the scale Λ suppressing non-renormalizable interactions should satisfy $\Lambda \gtrsim M_{\text{Heavy}}$. Such an argument was previously lacking in connecting theories in AdS to their dual CFTs. Indeed, in AdS theories with suppressed higher-dimensional bulk terms there appear to be non-trivial constraints on CFT correlation functions. For example, in correlation functions involving conserved currents, such as $T_{\mu\nu}$ or J_μ, only certain polarization structures will appear — those which follow from the lowest-dimension AdS bulk terms [34]. From the CFT side it seemed strange that one polarization structure would be preferred over another. It was therefore unclear whether CFTs needed to satisfy multiple independent requirements in order to have well-behaved AdS duals.

Our results suggest that there is a single requirement that naturally suppresses non-renormalizable interactions in the bulk. Demanding perturbative unitarity for all operator dimensions $\Delta < \Delta_{\text{Heavy}}$ places a bound on the scale suppressing non-renormalizable AdS interactions of $\Lambda \gtrsim \Delta_{\text{Heavy}}/R_{\text{AdS}}$. Moreover, the dimension of non-renormalizable operators is directly related to the rate of growth in the anomalous dimensions of double-trace operators $\gamma(n, l)$ as n is increased. It would be interesting to repeat our analysis for the case of a bulk gauge field or graviton, and verify that indeed requiring CFT perturbative unitarity up to some large dimension Δ_{Heavy} leads to the suppression of certain polarization structures by appropriate powers of Δ_{Heavy}. Extending the approach to fermions would also be desirable.

As supersymmetry did not appear to a play a role in the analysis, it may be possible that there are condensed matter systems which enjoy conformal symmetry, for which the notion of an effective conformal theory might be useful. In particular, if one could find a system with even a mild hierarchy in the dimension of operators, there might be a useful AdS dual which includes only the order parameter, a few relevant deformations, and possible conserved currents. A possible way of detecting such a system could be to look for

the suppression of particular polarization structures in correlation functions. An outstanding question is the role of naturalness in determining which types of operators may have low dimensions in theories with a hierarchy. A related question concerns the cosmological constant itself, and whether getting a large hierarchy is possible in non supersymmetric theories. Finding a condensed matter system with a hierarchy might shed some light on these questions.

In addition to local bulk interactions, we have also considered probing bulk scalar exchange in AdS through the CFT anomalous dimensions $\gamma(n, l)$. In doing so we have found evidence that these anomalous dimensions behave very much like S-matrix elements, displaying a resonance-like behavior as n passes through the dimension corresponding to the exchanged scalar. More generally, for $n \gg 1$ we have shown that the anomalous dimensions simply turn into the partial wave expansion for the flat-space amplitudes of the higher-dimensional bulk theory. It would be interesting to further explore this correspondence in other examples of CFTs where the anomalous dimensions are calculable. It would also be very interesting to extend this analysis beyond tree level, where one could for example study the effect of renormalization group running in n.

It might also be useful to explore locality further by explicitly constructing bulk states localized in the extra dimension ρ and study their evolution. By superimposing multiple-particle states (or considering operators without a definite number of traces) one can also construct classical field states. These might lead to a better understanding of classical backgrounds such as small black holes at the center of AdS. These and related investigations are left to future work.

Acknowledgments

A Check of $\gamma(n, 0)$ for $(\nabla\phi)^4$ and $(\nabla_\mu\nabla_\nu\phi)^2(\nabla\phi)^2$

As a check of our methods, and to demonstrate them in a slightly more involved example, we will show that they reproduce the scalar anomalous dimensions $\gamma(n, 0)$ calculated in [4] for a $(\nabla\phi)^4$ interaction in $d = 2$. From perturbation theory, we have

$$
\begin{aligned}
\gamma(n, 0) &= \frac{1}{4!\mu^3} \int d^2x \sqrt{-g}\, {}_2\langle n, 0|(\nabla\phi)^4|n, 0\rangle_2 \qquad\qquad\qquad\qquad\qquad (A.1)\\
&= \frac{1}{6\mu^3} \int d^2x \sqrt{-g} \times \\
&\quad \times \left[\frac{1}{2}\, {}_2\langle n, 0|(\nabla\phi)^2|0\rangle\langle 0|(\nabla\phi)^2|n, 0\rangle_2 + {}_2\langle n, 0|\nabla^\mu\phi\nabla^\nu\phi|0\rangle\langle 0|\nabla_\mu\phi\nabla_\nu\phi|n, 0\rangle_2 \right].
\end{aligned}
$$

Using the identity $\langle 0|(\nabla\phi)^2|n,0\rangle_2 = (\frac{1}{2}m_n^2 - m^2)\langle 0|\phi^2|n,0\rangle_2$ with $m_n^2 = 4(\Delta+n)(\Delta+n-1)$, the first term is easily reduced to an integral in terms of $\langle 0|\phi^2|n,0\rangle_2 = \frac{1}{\sqrt{2}\pi}(e^{it}\cos\rho)^{E_n}$, which is straightforward to compute. The second term is more complicated. Since we are currently looking only at the dimensions of the scalar states $|n,0\rangle_2$, we want to decompose the operator $\nabla_\mu\phi\nabla_\nu\phi$ into its scalar pieces. The only primary wavefunctions with two Lorentz indices that transform like scalars ($l=0$) come from

$$\nabla_\mu\phi\nabla_\nu\phi \supset \alpha g_{\mu\nu}\phi^2 + \beta\nabla_\mu\nabla_\nu\phi^2. \tag{A.2}$$

To determine the values of α and β, we will manipulate $\nabla_\mu\phi\nabla_\nu\phi$ to get linear combinations of just the scalar pieces. That is, $\nabla_\mu\phi\nabla_\nu\phi$ also contains a spin-2 piece $H_{\mu\nu}$ ($H_\mu^\mu = 0, \nabla_\mu H_\nu^\mu = 0$) which needs to be projected out.[17] The first projection is obtained by taking the trace, which yields

$$(\frac{1}{2}m_n^2 - m^2) = 3\alpha + m_n^2\beta. \tag{A.3}$$

The second projection is obtained by acting with ∇_μ, which picks out a different linear combination of α and β,

$$\frac{1}{4}m_n^2\nabla_\nu\phi^2 = (\alpha + (m_n^2 - 2)\beta)\nabla_\nu\phi^2, \tag{A.4}$$

where we have used $[\nabla_\mu, \nabla_\nu]v^\mu = -2v_\nu$. Equations (A.3) and (A.4) have the solution

$$\alpha = \frac{m_n^2(m_n^2 - 4) - 4m^2(m_n^2 - 2)}{8(m_n^2 - 3)}, \qquad \beta = \frac{(4m^2 + m_n^2)}{8(m_n^2 - 3)}. \tag{A.5}$$

The second term in our original integral then becomes

$$|\langle 0|\nabla_\mu\phi\nabla_\nu\phi|n,0\rangle_2|^2 = |g_{\mu\nu}\alpha\langle 0|\phi^2|n,0\rangle_2 + \beta\nabla_\mu\nabla_\nu\langle 0|\phi^2|n,0\rangle_2|^2 \tag{A.6}$$
$$= \frac{1}{2\pi^2}\Big[(3\alpha^2 + 2\alpha\beta m_n^2)\cos^{2E_n}\rho$$
$$+ \beta^2\nabla^\mu\nabla^\nu(e^{-it}\cos\rho)^{E_n}\nabla_\mu\nabla_\nu(e^{it}\cos\rho)^{E_n}\Big].$$

Finally, integrating all terms over $\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\rho\sqrt{-g}$ we obtain

$$6\mu^3\pi\gamma(n,0) = \frac{\widetilde{P}_6(n)}{(2n+2\Delta-3)(2n+2\Delta-1)(2n+2\Delta+1)}, \tag{A.7}$$

where $\widetilde{P}_6(n)$ is the polynomial

$$\widetilde{P}_6(n) = 7n^6 + 21(2\Delta-1)n^5 + \left(99\Delta^2 - 93\Delta + 16\right)n^4 + (2\Delta-1)\left(58\Delta^2 - 46\Delta - 3\right)n^3$$
$$+ \left(71\Delta^4 - 110\Delta^3 + 31\Delta^2 + 11\Delta - 5\right)n^2 + \Delta^3(2\Delta-1)(11\Delta-14)n$$
$$+ \frac{1}{4}\Delta^3(2\Delta-3)\left(6\Delta^2 - 5\Delta + 4\right). \tag{A.8}$$

---

[17]Note that there is no possible spin-1 piece since the only vector that could possibly enter is $\nabla_\mu\phi^2$, which has $l=0$ on primary states.

Compared to [4], this agrees up to a term proportional to the contribution from a $\int d^2x \sqrt{-g}\phi^4$ interaction in the bulk, which was intentionally dropped in their calculation.

For reference we will also write down the contribution to the spin-2 primary operators computed in [4]

$$6\pi\mu^3\gamma(n,2) = \frac{(n+1)(n+2)(n+\Delta)(n+\Delta+1)(n+2\Delta-1)(n+2\Delta)}{2(2n+2\Delta-1)(2n+2\Delta+1)(2n+2\Delta+3)}. \quad \text{(A.9)}$$

Although we will not rederive this result here, it is straightforward to do so using the present method after determining the spin-2 primary wavefunctions in AdS$_3$.

Finally, we have used similar manipulations to those above in order to compute the scalar anomalous dimensions $\gamma(n,0)$ from the dimension-five operator $(\nabla_\mu\nabla_\nu\phi)^2(\nabla\phi)^2$ in $d=2$. The result we find is

$$\gamma(n,0) \propto \frac{\widetilde{P}_8(n)}{(2\Delta+2n-3)(2\Delta+2n-1)(2\Delta+2n+1)}, \quad \text{(A.10)}$$

where $\widetilde{P}_8(n)$ is an eighth-order polynomial in n,

$$\begin{aligned}
\widetilde{P}_8(n) = &-3n^8 + (-24\Delta+12)\,n^7 + \left(-78\Delta^2+72\Delta-24\right)n^6 \\
&+ \left(-132\Delta^3+162\Delta^2-108\Delta+30\right)n^5 \\
&+ \left(-123\Delta^4+162\Delta^3-171\Delta^2+108\Delta-15\right)n^4 \\
&+ \left(-60\Delta^5+54\Delta^4-108\Delta^3+144\Delta^2-36\Delta-6\right)n^3 \\
&+ \left(-11\Delta^6-24\Delta^5-9\Delta^4+88\Delta^3-33\Delta^2-12\Delta+6\right)n^2 \\
&+ \left(2\Delta^7-25\Delta^6+24\Delta^5+14\Delta^4-10\Delta^3\right)n \\
&+ \frac{1}{4}\left(4\Delta^8-28\Delta^7+45\Delta^6-14\Delta^5-22\Delta^4+24\Delta^3\right). \quad \text{(A.11)}
\end{aligned}$$

Again this agrees with the results in [4], up to subtracting off contributions from lower-dimensional operators.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [hep-th/9711200] [SPIRES].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [SPIRES].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [SPIRES].

[4] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, *Holography from conformal field theory*, *JHEP* **10** (2009) 079 [arXiv:0907.0151] [SPIRES].

[5] H. Liu, *Scattering in Anti-de Sitter space and operator product expansion*, *Phys. Rev.* **D 60** (1999) 106005 [hep-th/9811152] [SPIRES].

[6] E. D'Hoker and D.Z. Freedman, *General scalar exchange in AdS(d + 1)*, *Nucl. Phys.* **B 550** (1999) 261 [hep-th/9811257] [SPIRES].

[7] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Graviton exchange and complete 4-point functions in the AdS/CFT correspondence*, *Nucl. Phys.* **B 562** (1999) 353 [hep-th/9903196] [SPIRES].

[8] E. D'Hoker, S.D. Mathur, A. Matusis and L. Rastelli, *The operator product expansion of N = 4 SYM and the 4-point functions of supergravity*, *Nucl. Phys.* **B 589** (2000) 38 [hep-th/9911222] [SPIRES].

[9] L. Hoffmann, A.C. Petkou and W. Rühl, *Aspects of the conformal operator product expansion in AdS/CFT correspondence*, *Adv. Theor. Math. Phys.* **4** (2002) 571 [hep-th/0002154] [SPIRES].

[10] G. Arutyunov, S. Frolov and A.C. Petkou, *Operator product expansion of the lowest weight CPOs in N = 4 SYM(4) at strong coupling*, *Nucl. Phys.* **B 586** (2000) 547 [hep-th/0005182] [SPIRES].

[11] L. Hoffmann, L. Mesref and W. Rühl, *AdS box graphs, unitarity and operator product expansions*, *Nucl. Phys.* **B 589** (2000) 337 [hep-th/0006165] [SPIRES].

[12] G. Arutyunov, S. Penati, A.C. Petkou, A. Santambrogio and E. Sokatchev, *Non-protected operators in N = 4 SYM and multiparticle states of AdS_5 SUGRA*, *Nucl. Phys.* **B 643** (2002) 49 [hep-th/0206020] [SPIRES].

[13] L. Cornalba, M.S. Costa, J. Penedones and R. Schiappa, *Eikonal approximation in AdS/CFT: conformal partial waves and finite N four-point functions*, *Nucl. Phys.* **B 767** (2007) 327 [hep-th/0611123] [SPIRES].

[14] L. Cornalba, M.S. Costa and J. Penedones, *Eikonal approximation in AdS/CFT: resumming the gravitational loop expansion*, *JHEP* **09** (2007) 037 [arXiv:0707.0120] [SPIRES].

[15] I. Heemskerk and J. Sully, *More holography from conformal field theory*, *JHEP* **09** (2010) 099 [arXiv:1006.0976] [SPIRES].

[16] J. Polchinski, *S-matrices from AdS spacetime*, hep-th/9901076 [SPIRES].

[17] L. Susskind, *Holography in the flat space limit*, hep-th/9901079 [SPIRES].

[18] V. Balasubramanian, S.B. Giddings and A.E. Lawrence, *What do CFTs tell us about Anti-de Sitter spacetimes?*, *JHEP* **03** (1999) 001 [hep-th/9902052] [SPIRES].

[19] S.B. Giddings, *The boundary S-matrix and the AdS to CFT dictionary*, *Phys. Rev. Lett.* **83** (1999) 2707 [hep-th/9903048] [SPIRES].

[20] S.B. Giddings, *Flat-space scattering and bulk locality in the AdS/CFT correspondence*, *Phys. Rev.* **D 61** (2000) 106008 [hep-th/9907129] [SPIRES].

[21] A. Jevicki and H. Nastase, *Towards S matrices on flat space and pp waves from SYM*, hep-th/0501013 [SPIRES].

[22] M. Gary, S.B. Giddings and J. Penedones, *Local bulk S-matrix elements and CFT singularities*, *Phys. Rev.* **D 80** (2009) 085005 [arXiv:0903.4437] [SPIRES].

[23] M. Gary and S.B. Giddings, *The flat space S-matrix from the AdS/CFT correspondence?*, *Phys. Rev.* **D 80** (2009) 046008 [arXiv:0904.3544] [SPIRES].

[24] T. Okuda and J. Penedones, *String scattering in flat space and a scaling limit of Yang-Mills correlators*, *Phys. Rev.* **D 83** (2011) 086001 [arXiv:1002.2641] [SPIRES].

[25] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [arXiv:0807.0004] [SPIRES].

[26] V.S. Rychkov and A. Vichi, *Universal constraints on conformal operator dimensions*, *Phys. Rev.* **D 80** (2009) 045006 [arXiv:0905.2211] [SPIRES].

[27] F. Caracciolo and V.S. Rychkov, *Rigorous limits on the interaction strength in quantum field theory*, *Phys. Rev.* **D 81** (2010) 085037 [arXiv:0912.2726] [SPIRES].

[28] P. Breitenlohner and D.Z. Freedman, *Positive energy in Anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett.* **B 115** (1982) 197 [SPIRES].

[29] J.M. Maldacena and A. Strominger, *AdS$_3$ black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085] [SPIRES].

[30] W. Li, W. Song and A. Strominger, *Chiral gravity in three dimensions*, *JHEP* **04** (2008) 082 [arXiv:0801.4566] [SPIRES].

[31] E. D'Hoker, D.Z. Freedman and L. Rastelli, *AdS/CFT 4-point functions: how to succeed at z-integrals without really trying*, *Nucl. Phys.* **B 562** (1999) 395 [hep-th/9905049] [SPIRES].

[32] L. Hoffmann, A.C. Petkou and W. Rühl, *A note on the analyticity of AdS scalar exchange graphs in the crossed channel*, *Phys. Lett.* **B 478** (2000) 320 [hep-th/0002025] [SPIRES].

[33] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys.* **B 599** (2001) 459 [hep-th/0011040] [SPIRES].

[34] D.M. Hofman and J. Maldacena, *Conformal collider physics: energy and charge correlations*, *JHEP* **05** (2008) 012 [arXiv:0803.1467] [SPIRES].