Published for SISSA by Springer

Received: June 8, 2010
Accepted: July 3, 2010
Published: July 20, 2010

Geometry of Schrödinger space-times II: particle and field probes of the causal structure

Matthias Blau, Jelle Hartong and Blaise Rollier

Albert Einstein Center for Fundamental Physics,
Institute for Theoretical Physics, Bern University,
Sidlerstrasse 5, CH-3012 Bern, Switzerland

E-mail: blau@itp.unibe.ch, hartong@itp.unibe.ch, rollier@itp.unibe.ch

Abstract: We continue our study of the global properties of the $z = 2$ Schrödinger space-time. In particular, we provide a codimension 2 isometric embedding which naturally gives rise to the previously introduced global coordinates. Furthermore, we study the causal structure by probing the space-time with point particles as well as with scalar fields. We show that, even though there is no global time function in the technical sense (Schrödinger space-time being non-distinguishing), the time coordinate of the global Schrödinger coordinate system is, in a precise way, the closest one can get to having such a time function. In spite of this and the corresponding strongly Galilean and almost pathological causal structure of this space-time, it is nevertheless possible to define a Hilbert space of normalisable scalar modes with a well-defined time-evolution. We also discuss how the Galilean causal structure is reflected and encoded in the scalar Wightman functions and the bulk-to-bulk propagator.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence, Space-Time Symmetries

ArXiv ePrint: 1005.0760

doi:10.1007/JHEP07(2010)069

JHEP07(2010)069

Contents

1 Introduction

Recently, following [1–6], non-relativistic variants of the AdS/CFT correspondence have attracted considerable attention. This has brought to prominence deformations of (asymptotically) AdS space-time geometries that exhibit (asymptotic) isometry groups which are suitable Galilean counterparts of the relativistic conformal group, such as the Schrödinger group. These space-time geometries are interesting for at least three reasons:

1. First of all, of course, they are candidate gravitational duals to non-relativistic strongly coupled (scale or conformally invariant) condensed matter and other physical systems (for reviews see e.g. [7] and [8]). This has led to new ways of looking

at well-known (if not well understood) physical phenomena, but concrete and quantitative progress along these lines is currently hampered by the lack of precise dual pairs, and by the fact that the holographic dictionary in these space-times is still not nearly as well understood as in the AdS case.

2. Secondly, and related to the issue just raised, this set-up potentially provides one with a novel implementation of holography which requires one to suitably generalise and modify the standard AdS/CFT procedure. While holography is (on fairly general and convincing grounds) expected to be a generic feature of a quantum theory of gravity, currently the only case that is reasonably well understood is that of asymptotically AdS space-times. Attempts to generalise this to the assymptotically flat or dS situation are fraught with technical and conceptual complications. On the other hand, encouragingly some of the AdS/CFT recipes *do* appear to "carry over" in a simple-minded way to the Schrödinger case. What is required now is a more systematic understanding and underpinning of the calculational procedures, analogous to that for AdS based on a suitable notion of conformal boundary, the Fefferman-Graham expansion, and holographic renormalisation (for some concrete work along these lines see e.g. [9, 10]).

3. As a precursor to this, one needs to gain an as precise understanding as possible of the properties of the model Schrödinger space-time that are shared with AdS, and those that set it apart from AdS [11]. In particular, thirdly, Schrödinger space-time provides us with an interesting and physically well-motivated example of a relativistic space-time that exhibits a rather peculiar (and almost pathological) causal structure, quite different from that of AdS (whose lack of global hyperbolicity is its only mild, and well understood, potential source of pathology). It is thus of interest, both in its own right and for the reasons mentioned above, to study to which extent the behaviour of scalar fields, say, on such a space-time is sensitive to, or reflects, the Galilean oddities of this causal structure (as conventionally defined in terms of point particle probes and concepts).[1]

In this paper we will discuss in some detail the issues raised in (3.) in the case of the Schrödinger space-time with critical exponent $z = 2$. In Poincaré-like coordinates its metric takes the form

$$ds^2 = -\beta^2 \frac{dt^2}{r^4} + \frac{1}{r^2}\left(-2dtd\xi + dr^2 + d\vec{x}^2\right) \tag{1.1}$$

(reducing to the AdS metric in Poincaré coordinates for $\beta = 0$). This coordinate system is geodesically incomplete (particles can reach $r = \infty$ in finite proper time without encountering a singularity) [11]. While some causal properties can be (and have been [3, 14]) reliably read off from the Poincaré patch metric (1.1), a more detailed understanding of the global and causal properties requires a more global presentation and picture of the

[1]This is similar in spirit to the question to what extent (quantum) fields are sensitive to point particle notions of singularities, see e.g. [12, 13].

Schrödinger space-time. This is provided by the global coordinates introduced in [11] in which the metric takes the form

$$ds^2 = -\beta^2 \frac{dT^2}{R^4} + \frac{1}{R^2}(-2dTdV - \omega^2(R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2) \ . \tag{1.2}$$

This reduces to (1.1) for $\omega = 0$, and also gives a (somewhat unusual) global coordinatisation of AdS for $\beta = 0, \omega \neq 0$ (plane wave AdS). As a consequence, this allows us to directly compare and contrast certain global properties of AdS (and of scalar propagation in this background) with those of the Schrödinger space-time.

To set the stage, in section 2 we discuss various aspects of Schrödinger geometry related to the global coordinates (1.2). In particular, in section 2.2 we provide a codimension 2 isometric embedding of the Schrödinger space-time which naturally gives rise to these global cooordinates. This embedding turns out to be not equivariant (i.e. not all isometries are introduced from the isometries of the flat embedding space-time), and in appendix A we prove, using some group theory arguments, that indeed there are no codimension 2 equivariant isometric embeddings of the Schrödinger space-time.

In section 3 we study various aspects of the causal structure of Schrödinger space-time. In section 3.1 we focus on those properties that are common to the Schrödinger and plane wave AdS geometries. Primarily these are properties of the global time-coordinate T of (1.2). In particular, we highlight the fact that T, in spite of being a global function with ∂_T everywhere timelike, is *not* a time function in the strict sense. The difference between AdS and Schrödinger is that the former is stably causal and has a time function (the global time coordinate τ of the usual global AdS coordinates, for instance) while Schrödinger is not stably causal and hence admits no time function whatsoever. In this sense, T turns out to be the closest one can get to having a time function because it only fails to label causally related events that lie on a $T = \text{cst}$ slice Σ_T. In section 3.2 we discuss those aspects of the causal structure that are peculiar to $\beta \neq 0$, in particular the non-distinguishing character of this space-time already noted in [3, 14], and the ensuing strongly Galilean character of its causal structure.

Among the myriad of definitions and concepts related to the causal properties of a space-time (see e.g. [15, 16]) we have chosen to focus on those aspects of the causal structure that we found to have some counterpart in the analysis of scalar fields in the subsequent section 4. Here we will in particular address the question to which extent time-evolution of a scalar field is affected by the absence of a global time-function, and to which extent the Galilean causal structure of the space-time is reflected and encoded in the Green's functions and propagators of the theory. Some technical details have been relegated to the appendices.

2 Global Schrödinger geometry

In this section, we briefly recall some basic features of the geometry of Schrödinger space-times and record some obervations regarding timelike Killing vectors. We also introduce a codimension 2 isometric flat space embedding and discuss some aspects of Schrödinger

JHEP07(2010)069

geometry that are particularly transparent from this embedding point of view, in particular global coordinates.

2.1 Isometries, timelike Killing vectors and global coordinates

The metric

$$ds^2 = -\beta^2 \frac{dt^2}{r^4} + \frac{1}{r^2}\left(-2dtd\xi + dr^2 + d\vec{x}^2\right) \tag{2.1}$$

($d\vec{x}^2 = (dx^1)^2 + \cdots (dx^d)^2$) is that of the $(d+3)$-dimensional $z = 2$ Schrödinger spacetime Sch_{d+3} in Poincaré-like coordinates for $\beta^2 > 0$ and that of AdS_{d+3} in null Poincaré coordinates for $\beta = 0$.

For $\beta \neq 0$ this has the characteristic transitive Schrödinger isometry algebra $\mathfrak{sch}(d)$ consisting of spatial rotations M_{ab} and translations P_a and Galilean boosts V_a (which we will not make use of explicitly in the following, for details see appendix A) and a central element $N = \partial_\xi$ of null translations, as well as of an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra formed by time-translations H, anisotropic dilatations D and special conformal transformations C,

$$H = \partial_t \qquad D = 2t\partial_t + r\partial_r + x^a\partial_a \qquad C = t^2\partial_t + tr\partial_r + tx^a\partial_a + \frac{1}{2}(r^2 + \vec{x}^2)\partial_\xi \ . \tag{2.2}$$

In particular, the algebra generated by the $\mathfrak{so}(d)$-singlets $\{H, C, D, N\}$, i.e. the isometry algebra of 3-dimensional Schrödinger spacetime, is isomorphic to

$$\mathfrak{sch}(d = 0) \cong \mathfrak{so}(2, 1) \oplus \mathbb{R}_N \ . \tag{2.3}$$

For $\beta = 0$ the Schrödinger isometry algebra $\mathfrak{sch}(d)$ is enhanced to the AdS isometry algebra $\mathfrak{so}(2, d + 2)$, with $\dim \mathfrak{so}(2, d + 2) - \dim \mathfrak{sch}(d) = 2(d + 1)$.

The above Poincaré coordinate system is geodesically incomplete as $r \to \infty$ (geodesics can reach $r = \infty$ in finite affine parameter, and the geometry is non-singular there) [11]. This points to the necessity of introducing coordinates that also cover the region beyond the Poincaré coordinate patch. A hint as to go about this for $\beta \neq 0$ comes from analysing the timelike Killing vectors of this metric. For instance, the Killing vector $H = \partial_t$ becomes lightlike at the Poincaré horizon $r = \infty$, and is therefore not a suitable candidate for a global definition of time. If one considers, more generally, the linear combination

$$\tilde{H} = a_H H + a_C C + a_N N + a_D D \tag{2.4}$$

(these are the only relevant Killing vectors for these purposes), and calculates its norm in Poincaré coordinates, one finds (for simplicity in the 3-dimensional case $d = 0$ since nothing essential changes for $d > 0$)

$$||\tilde{H}||^2 = -\frac{\beta^2}{r^4}(a_H + 2a_D t + a_C t^2)^2 - \frac{2a_N(a_H + 2a_D t + a_C t^2)}{r^2} + a_D^2 - a_H a_C \ . \tag{2.5}$$

Thus a necessary condition for $\tilde{H}$ to be timelike beyond the Poincaré horizon is $a_D^2 - a_H a_C > 0$. The choice made in [11] based on these and other considerations was $\tilde{H} = H + \omega^2 C$. Introducing coordinates (T, V) adapted to $\tilde{H}$ and the central element N,

$$\tilde{H} = \partial_T = H + \omega^2 C, \qquad N = \partial_V \ , \tag{2.6}$$

the global metric reads

$$ds^2 = -\beta^2 \frac{dT^2}{R^4} + \frac{1}{R^2}(-2dTdV - \omega^2(R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2) \ . \tag{2.7}$$

This coordinate system, in which the metric simply has the form of a plane wave deformation of the Poincaré-like metric (2.1), is geodesically complete for $\omega > 0$ and reduces to the incomplete Poincaré-patch metric for $\omega = 0$. In [11] it was moreover shown that this metric is closely related to the harmonic trapping of non-relativstic CFTs that plays an important role in the non-relativistic operator-state correspondence [17]. In particular, time evolution with respect to the global time T (∂_T is everywhere timelike) is time-evolution with respect to the trapped Hamiltonian $H + \omega^2 C$, and the harmonic oscillator potential in the metric corresponds to the trapping potential of the boundary theory.

One could, without loss of generality, choose $\omega = \beta = 1$ for the global metric, but we will keep ω and β explicit in order to facilitate the comparison of the properties of the global Schrödinger metric with those of the Poincaré patch metric and, in particular, with those of the AdS metric in global plane wave coordinates (plane wave AdS) [4, 11, 18] one obtains for $\beta = 0$,

$$ds^2 = \frac{1}{R^2}(-2dTdV - \omega^2(R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2) \ . \tag{2.8}$$

One other thing that one can read off and learn from (2.5) is that the metric (2.1) for $\beta^2 < 0$ has no timelike Killing vectors for $r \to 0$ (since the first, now positive, term will dominate as $r \to 0$). As a consequence, even though the $\beta^2 < 0$ metric has Schrödinger isometry, it is not isometric to any patch of the global Schrödinger metric (2.7) (which has an everywhere timelike Killing vector). This illustrates that geometries with Schrödinger isometry are not locally unique.

In Poincaré coordinates and in global coordinates the metric is stationary (time-independent) but not static and one may wonder whether there is (at least locally) any timelike Killing vector that is hypersurface-orthogonal. To analyse this question, let us once again consider the linear combinations $\tilde{H}$ (2.4). Imposing the integrability condition $\tilde{H}_{[\mu}\nabla_\nu \tilde{H}_{\rho]} = 0$, one finds

$$-\frac{\beta^2}{r^4}(a_H + 2a_D t + a_C t^2)^2 + (a_H a_C - a_D^2) = 0 \ . \tag{2.9}$$

For $\beta \neq 0$ the only solution is $a_C = a_D = a_H = 0$ so that $\tilde{H} \sim N$ which is not timelike but null (and hypersurface orthogonal to the null surfaces t = const.). An analysis in global coordinates leads to exactly the same result, and we can conclude that Schrödinger space-times are globally stationary but admit no static patch. For $\beta = 0$, on the other hand, one only finds the constraint $a_H a_C - a_D^2 = 0$. A typical time-like solution is $a_C = a_D = 0$ and $\tilde{H} = H + N$ which corresponds to choosing $x^0 = t + \xi$ as the new (and standard and obviously hypersurface-orthogonal) Poincaré time-coordinate. Of course, for $\beta = 0$ there are other Killing vectors, and global (and also hypersurface orthogonal) time τ in the usual global coordinates for AdS corresponds to the linear combination $\partial_\tau = P_0 + K_0$, K_0 a special conformal transformation.

2.2 Isometric embeddings and global coordinates

For the AdS space-time AdS_{d+3} there exists a codimension 1 isometric embedding into the pseudo-Euclidean space $\mathbb{R}^{2,d+2}$. It is relatively easy to see that no such embedding exists for Sch_{d+3}, more specifically that any hypersurface with Schrödinger isometry is actually AdS-invariant. Moreover, similar arguments show that there are no codimension 2 equivariant isometric embeddings, i.e. isometric embeddings for which all the isometries are induced by isometries of the flat embedding space. We will establish these results in appendix A.

However, a codimension 2 isometric (but not equivariant) embedding into $\mathbb{R}^{2,d+3}$ equipped with the metric

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^{d+2})^2 - (dX^{d+3})^2 + (dX^{d+4})^2 \tag{2.10}$$

exists and is given by

$$\begin{aligned} (X^0, X^1) &= \frac{\xi \pm \frac{t}{2} + \frac{t}{2}\beta^2 f(t,r)}{r} \\ (X^{d+2}, X^{d+3}) &= \frac{1}{2r}\left[\pm 1 + 2\xi t - \vec{x}^2 - r^2 - \beta^2 f(t,r)\right] \\ X^{1+a} &= \frac{x^a}{r} \qquad\qquad X^{d+4} = \frac{\sqrt{3}}{2}\beta f(t,r) \end{aligned} \tag{2.11}$$

where $a = 1, \ldots, d$ and where $f(t,r) = \frac{t^2+1}{r^2}$. Indeed, the metric induced by this embedding on the codimension 2 surface parametrised by $(t, \xi, r, \vec{x})$ is precisely the Schrödinger/AdS metric in Poincaré coordinates (2.1). Explicitly, the $(d+5)$ coordinates are related by the two constraints

$$\begin{aligned} -(X^0)^2 + (X^1)^2 + \sum_a (X^{1+a})^2 + (X^{d+2})^2 - (X^{d+3})^2 &= -1 - \frac{4}{3}(X^{d+4})^2 \\ \beta\left[\left(X^0 - X^1\right)^2 + (X^{d+2} - X^{d+3})^2\right] &= \frac{2}{\sqrt{3}}X^{d+4} \end{aligned} \tag{2.12}$$

and the inverse transformation, subject to these constraints, is

$$\begin{aligned} t &= \frac{X^0 - X^1}{X^{d+2} - X^{d+3}} \qquad r = \frac{1}{X^{d+2} - X^{d+3}} \qquad x^a = \frac{X^{1+a}}{X^{d+2} - X^{d+3}} \\ \xi &= \frac{1}{2}\left[\left(\frac{X^0 + X^1}{X^{d+2} - X^{d+3}}\right) - \frac{2\beta X^{d+4}}{\sqrt{3}}\left(\frac{X^0 - X^1}{X^{d+2} - X^{d+3}}\right)\right] \end{aligned} \tag{2.13}$$

The parameter β describes the deformation away from AdS_{d+3} and for $\beta = 0$ one reproduces the standard codimension 1 embedding into the hyperplane $\mathbb{R}^{2,d+2} \subset \mathbb{R}^{2,d+3}$ given by $X^{d+4} = 0$. For $\beta \neq 0$, X^{d+4} is non-trivial and the first constraint equation describes a surface that can be viewed as AdS_{d+3} space-time of variable AdS radius where the radius is a function of X^{d+4}. Just as for AdS, in order not to have closed time-like curves we work with the universal cover.

As already alluded to above, the above isometric embedding has the property that not all the Schrödinger isometries are actually induced by the $\mathrm{ISO}(2, d+3)$-isometries of the embedding space $\mathbb{R}^{2,d+3}$. Indeed, for $\beta \neq 0$ the isometries that embed into $\mathrm{SO}(2, d+3)$

(all the translational symmetries are manifestly broken by the constraints) are those of the constant X^{d+4} slices, namely M_{ab}, P_a, V_a, N and $H+C$, while the "accidental" additional isometries are D and $H-C$. For instance, a shift in t (generated by H) is induced by a non-linear transformation of the coordinates X^A for $\beta \neq 0$ whereas it is realised by a linear SO(2, 2)-transformation in the $(X^0, X^1, X^{d+2}, X^{d+3})$-plane for $\beta = 0$, as it should be.

The geodesic distance between two points (relevant for our discussion of scalar fields and Green's functions in section 4) is invariant under the (simultaneous) action of the isometry group of a space-time on the two points. If we had an equivariant isometric embedding, we could introduce at least one isometry-invariant notion of the distance between two points in terms of the standard pseudo-Euclidean distance between two points in the embedding space. In the AdS case $\beta = 0$ this gives rise to the usual chordal distance and its relation with the geodesic distance. For $\beta \neq 0$, however, this option is not available (the induced distance function is not a Schrödinger invariant object). We will discuss and construct these invariants (it turns out that there are two independent such functions) in appendix B.

In spite of its shortcomings, the above embedding is quite useful for a number of things. For instance, the constraints (2.12) suggest a natural parametrisation of the form

$$
\begin{aligned}
X^0 - X^1 &= \frac{\sin T}{R} & X^0 + X^1 &= \frac{1}{R}\left(2V\cos T + b\sin T\right) \\
X^{d+2} - X^{d+3} &= \frac{\cos T}{R} & X^{d+2} + X^{d+3} &= \frac{1}{R}\left(2V\sin T - b\cos T\right) \\
X^{1+a} &= \frac{X^a}{R} & X^{d+4} &= \frac{\beta\sqrt{3}}{2R^2}
\end{aligned}
\tag{2.14}
$$

with which the first constraint reduces to $b = R^2 + \vec{X}^2 + \frac{\beta^2}{R^2}$. Then the induced metric is precisely the $\omega = 1$ case of the global plane wave Schrödinger metric (2.7). From the present embedding point of view we learn that this parametrisation indeed covers the entire space-time (both for the codimension 1 embedding of AdS for $\beta = 0$ and for $\beta \neq 0$), and that what was a geodesically complete coordinate system in [11] is now also global from the embedding point of view. It follows from (2.13) that the Poincaré patch only covers the region $X^{d+2} - X^{d+3} > 0$. This isometric embedding generalises the embedding of plane waves found a long time ago in [19, 20] (see also [21]), and correspondingly the equivariantly realised isometries M_{ab}, P_a, V_a, N and $H+C$ form the isometry algebra of an isotropic symmetric plane wave [21, 22].

Another issue that is particularly transparent from the embedding point of view is that of potential conical singularities that arise if one compactifies the V (or, equivalently, ξ) direction [4, 5]. The situation turns out to be identical for AdS and Schrödinger. First of all we note that the shift $V \to V + \alpha$ is a symmetry of both the two constraint equations as well as of the embedding space-time for any $\alpha \in \mathbb{R}$. We want to see what happens if we identify $V \sim V + 2\pi L$.

Using that

$$
V = \frac{1}{2}\left(\frac{(X^0 - X^1)(X^{d+2} + X^{d+3}) + (X^0 + X^1)(X^{d+2} - X^{d+3})}{(X^0 - X^1)^2 + (X^{d+2} - X^{d+3})^2}\right) \tag{2.15}
$$

we see that the identification of V with $V + 2\pi L$ leads to the identifications

$$\begin{aligned} X^0 + X^1 &\sim X^0 + X^1 + 4\pi L(X^{d+2} - X^{d+3}) \\ X^{d+2} + X^{d+3} &\sim X^{d+2} + X^{d+3} + 4\pi L(X^0 - X^1) \ . \end{aligned} \tag{2.16}$$

We therefore conclude that there are two conical singularities:

1. At $(X^0 + X^1, X^{d+2} - X^{d+3}) = (0, 0)$ which can be reached in the limit $R \to \infty$ with $\sin T = 0$ fixed (for $\omega = 1$) and V, X finite but arbitrary.

2. At $(X^{d+2} + X^{d+3}, X^0 - X^1) = (0, 0)$ which can be reached in the limit $R \to \infty$ with $\cos T = 0$ fixed (for $\omega = 1$) and V, X finite but arbitrary.

In Poincaré coordinates the limit mentioned in point 1 corresponds to the limits $r \to \infty$, $t/r \to 0$ and $\vec{x}/r \to 0$. This is in agreement with the comments made in [5] regarding the conical singularity after compactification of ξ. The singular locus of point 2, on the other hand, lies outside the Poincaré patch.

3 Point particle probes of the causal structure

In this section we will discuss the causal structure associated with point particles moving along causal curves in the space-time with global metric (2.7),

$$ds^2 = -\left(\frac{\beta^2}{R^4} + \frac{\omega^2}{R^2}(R^2 + \vec{X}^2)\right) dT^2 + \frac{1}{R^2}\left(-2dTdV + d\vec{X}^2 + dR^2\right) . \tag{3.1}$$

We will start with β-independent properties, i.e. those that also hold in the geodesically complete plane wave AdS space-time. We then explore causality statements which are specific to the Schrödinger space-time. We will focus on those aspects of the causal structure that are relevant to our analysis of scalar fields in section 4. Definitions follow the standard reference [15] and the more recent review [16].

3.1 Time functions and time coordinates

First of all, let us collect some basic properties of the global coordinate T:

1. T is a globally defined smooth function.

2. The vector field ∂_T is an everywhere timelike Killing vector. In particular, it provides a time orientation.

3. The gradient of T is null.

4. T is strictly increasing along any future-directed timelike curve.

5. T is non-decreasing along any future-directed null curve.

JHEP07(2010)069

The first two points are trivial and follow from the fact that (3.1) is a global coordinate system. The third follows from $g_{\rm VV} = 0$. The last two points can be summarised saying that $\dot{T} \geq 0$ along any future-directed causal curve, with $\dot{T} > 0$ for timelike curves, implying that the space-time is chronological (no closed timelike curves can occur). This can be seen as follows: take any curve $\gamma(\lambda)$ with tangent $\left(\dot{T}, \dot{V}, \dot{R}, \dot{\vec{X}}\right)$ and require it to be causal,

$$\left(\frac{\beta^2}{R^2} + \omega^2(R^2 + \vec{X}^2)\right)\dot{T}^2 + 2\dot{T}\dot{V} \geq \dot{R}^2 + \dot{\vec{X}}^2\,, \tag{3.2}$$

and future-directed[2] with respect to the timelike vector field $\left(\frac{\partial}{\partial T}\right)^\mu$,

$$\left(\frac{\beta^2}{R^4} + \frac{\omega^2}{R^2}(R^2 + \vec{X}^2)\right)\dot{T} + \frac{\dot{V}}{R^2} > 0\,. \tag{3.3}$$

Then, since the right hand side of equation (3.2) is greater than or equal to zero it follows that

$$\dot{T}\left(\left(\frac{\beta^2}{R^2} + \omega^2(R^2 + \vec{X}^2)\right)\dot{T} + \dot{V}\right) \geq -\dot{T}\dot{V}\,. \tag{3.4}$$

Now we prove that $\dot{T} \geq 0$ by arguing that $\dot{T} < 0$ leads to a contradiction. Suppose $\dot{T} < 0$, then equations (3.3) and (3.4) implies $\dot{T}\dot{V} > 0$ but since $\dot{T} < 0$ it must be that $\dot{V} < 0$ which is then in contradiction with equation (3.3). Hence, we must have $\dot{T} \geq 0$ along all future-directed causal curves. Similarly, by restricting (3.2) to timelike curves one shows that $\dot{T} > 0$ along all future-directed timelike curves (statement 4). Furthermore, one observes from (3.3) that if $\dot{T} = 0$ then one necessarily has $\dot{V} > 0$ so that no closed causal curve can ever be formed for non-compact V. This shows that the space-time is causal. In the compact V case, closed causal curves exist (by construction), and the space-time is only chronological.

A time function is a globally defined continuous function that is strictly increasing along all future-directed causal curves. It therefore provides an ordering, as all causally related events can then be labeled by different values of T. The existence of a time-function is equivalent to the space-time being stably causal, and this in turn is equivalent to the existence of a (not necessarily the same) globally defined function whose gradient is everywhere timelike [15, 16]. Because there exist future-directed causal curves for which $\dot{T} = 0$, T is not a time function (and neither is the gradient of T everywhere timelike; in fact, as mentioned above, it is everywhere null). So what about stable causality of these space-times?

AdS is well known to be stably causal; thus even though T is not a time function one *can* find a time function for $\beta = 0$ (this global time function can be taken to be τ, the time coordinate of the usual global AdS coordinate system). But, as we will see in section 3.2, the Schrödinger space-time ($\beta \neq 0$) is not stably causal and hence it does not admit any time function. In that respect, T is the closest one can get to having a time function because it only fails to distinguish causally related events that lie on a $T = \text{cst}$ slice Σ_T.

[2] A curve γ is future-directed with respect to a timelike vector field X^μ if $g_{\mu\nu}X^\mu\dot{\gamma}^\nu < 0$.

JHEP07(2010)069

Such causally related events with the same T are related by so-called lightlike lines.[3] Indeed, a chronological space-time without such lightlike lines would be stably causal [23]. Lightlike lines are always null geodesics but the converse is generally not true. In space-times such as Minkowski and AdS all null geodesics are lightlike lines, so that the existence of lightlike lines alone does not signal any pathology.

From what we proved so far (statement 4) it follows that the surfaces Σ_T are achronal but not acausal. Hence in our context the lightlike lines are given by

$$\gamma(\lambda) = (T_0, V(\lambda), R_0, \vec{X}_0)\,, \tag{3.5}$$

where $V(\lambda)$ is a strictly monotonically increasing function of λ. These are precisely the null geodesics (affinely parametrised for $V(\lambda) \sim \lambda$) with zero lightcone momentum $P_V \equiv P_- = 0$ (cf. appendix D). The tangent is $u^\mu = (0, \dot{V}, 0, \vec{0})$, so that we have $g_{\mu\nu} \left(\frac{\partial}{\partial T}\right)^\mu u^\nu < 0$ from which it follows that γ is a future-directed null geodesic along which the time coordinate T remains constant.

Finally, let us us note that, as a consequences of the existence of these lightlike lines, the future domain of dependence of a constant time slice Σ_T, denoted by $D^+(\Sigma_T)$, is empty.[4] This has to be contrasted with AdS in the usual global coordinates where the future domain of dependence of a global time slice τ is not empty. Actually there are two distinct sources for the emptiness of the future and past domain of dependence: one has $D^\pm(\Sigma_{T_0}) = \emptyset$ because

1. for each time $T > T_0$ there exists a future and past inextendible null geodesic that has $\dot{T} = 0$, those are the lightlike lines (3.5);

2. for any arbitrary point $P = (T_0 \pm \delta, V_0, R_0, \vec{X}_0)$, say, with $\delta > 0$, that lies to the future (+) or past (-) of T_0 there exists a, respectively, past or future inextendible timelike curve that goes all the way to the boundary at $R = 0$ without crossing the slice Σ_{T_0}. An example of such a timelike curve is given by

$$\gamma_{past}(\lambda) = \begin{pmatrix} T_0 - \frac{\delta}{2}e^{-2\lambda} - \frac{\delta}{2} \\ \frac{R_0^2}{\delta}\lambda + V_0 \\ R_0 e^{-\lambda} \\ \vec{X}_0 \end{pmatrix} \qquad \gamma_{future}(\lambda) = \begin{pmatrix} T_0 + \frac{\delta}{2}e^{2\lambda} + \frac{\delta}{2} \\ \frac{R_0^2}{\delta}\lambda + V_0 \\ R_0 e^{\lambda} \\ \vec{X}_0 \end{pmatrix} \tag{3.6}$$

3.2 Galilean-like causal structure

The dramatic effect of having a non-zero β is that it makes the space-time non-distinguishing[5] whereas it is stably causal for AdS. This has already been proven in [14] for the $z = 3$ Schrödinger space-time using the Poincaré patch (and the possible connection

[3] A lightlike line is an achronal inextendible causal curve [23]. A set S is called achronal resp. acausal if no two distinct points of S can be connected by a timelike resp. causal curve.

[4] The future domain of dependence $D^+(S)$ of a set S is the set of points p such that every past inextendible causal curve through p intersects S.

[5] A space-time is called non-distinguishing if there exist two distinct points that have identical past and future. Non-distinguishing space-times do not admit any time function.

JHEP07(2010)069

of this property with a Galilean-like causal structure was noted in [3]). The proof is based on the existence of a causal curve that connects any two points whose time interval is infinitesimally small. It can be shown that such a curve can be constructed for any $z > 1$ and because there exists a local defining property for a space-time to be distinguishing [16], being non-distinguishing in the Poincaré patch is enough to assure that the space is also non-distinguishing globally.[6] For $z = 2$, such a curve can also be constructed directly in global coordinates, leading to the same conclusions (see appendix C).

In appendix C, we show explicitly that the chronological future (past) $I^{\pm}(p_0)$ of any point $p_0 = (T_0, V_0, R_0, \vec{X}_0)$ on the slice Σ_{T_0} is the set of all points with $T > T_0$ ($T < T_0$). Therefore, for any point p_0 one has the decomposition

$$\mathsf{Sch} = I^{-}(p_0) \cup \Sigma_{T_0} \cup I^{+}(p_0) \tag{3.7}$$

of the Schrödinger space-time. Since all points on a constant time slice share the same future and past, the space-time is in a sense "maximally non-distinguishing".

This is strongly reminiscent of a Galilean causal structure and Galilean relativity. In order to sharpen this analogy, we need an appropriate notion of spacelike separation. We will call two points x and x' *spacelike separated* if there is no causal curve connecting them. It is perhaps worth pointing out that this notion of spacelike separation does not imply that two points are spacelike separated when they can be connected by a spacelike geodesic: there are spacelike geodesics along which $\dot{T} \neq 0$, while we already know that any two points with $T \neq T'$ can be connected by a timelike curve. This means that spacelike separated points necessarily lie on an equal-time slice Σ_T.

This appears to be completely Galilean, since in Galilean relativity any two non-simultaneous events can be connected by the worldline of a (sufficiently fast moving) particle, and the only events for which no such curve exists are those that are simultaneous. However, the novel and non-Galilean feature of the causal structure of Schrödinger space-times is the presence of lightlike lines. Indeed, on a Schrödinger space-time all points with the same value of T are either spacelike separated or separated by a lightlike line and conversely all points that are either spacelike separated or separated by a lightlike line lie on an equal time T surface. While any time coordinate on the Schrödinger space-time whose values label the slices Σ_T plays the role of some absolute (Galilean) time, the null coordinate V (affinely) parametrises the lightlike lines and thus that part of the surfaces Σ_T that has no Galilean counterpart.

This Galilean-like structure is preserved by the subgroup

$$(T', V', R', \vec{X}') = (T'(T), V'(T, V, R, \vec{X}), R'(T, R, \vec{X}), \vec{X}'(T, R, \vec{X})) \tag{3.8}$$

of the full group of space-time diffeomorphisms. Indeed, any set of coordinates $(T', V', R', \vec{X}')$ obtained by acting on the global coordinates $(T, V, R, \vec{X})$ with such a

[6] Alternatively, one can prove that the $z > 1$ Schrödinger space-times are non-distinguishing by observing i) that in the Poincaré-like coordinate system they are conformal to a class of pp-wave space-times that in [24] have been proven to be non-distinguishing, and ii) that being non-distinguishing is a local property of a space-time which is therefore preserved under conformal transformations.

diffeomorphism is such that T', the new time coordinate, labels surfaces of spacelike and lightlike line separated events while any new V' coordinate parametrises the lightlike lines. The normal to a constant T' slice $\Sigma_{T'}$ is proportional to the null Killing vector N, and the (degenerate) induced metric on $\Sigma_{T'}$ agrees with the Galilean metric measuring the distance between simultaneous (spacelike) separated events. This special class of diffeomorphisms consists precisely of the double foliation preserving diffeomorphisms discussed in a related context in [25]. Here the double foliation refers to the foliations associated with the equal time surfaces and the lightlike lines.

4 Scalar field probes of the causal structure

In this section we will study the causal structure of Schrödinger space-times as seen by scalar field probes and show that, even though the causal structure seen by point particles is close to pathological, this is not so from the point of view of the scalars.

4.1 Canonical analysis

The action for a massive complex scalar field ϕ is

$$S = -\int d^{d+3}x\sqrt{-g}\left(\partial_\mu\phi^*\partial^\mu\phi + m_0^2\phi^*\phi\right) + \cdots\,, \tag{4.1}$$

where m_0 is a mass parameter and the dots refer to intrinsic boundary terms, e.g. terms that only involve the scalar ϕ, its tangential derivatives along the boundary and the induced boundary metric. We will consider scalar fields ϕ that are eigenstates of the central element ∂_V of the Schrödinger algebra, i.e.

$$\phi(T,V,R,\vec{X}) = e^{-imV}\psi(T,R,\vec{X})\,, \tag{4.2}$$

in which $m \neq 0$, and we will decompose solutions to the scalar field equation formally as

$$\phi = \sum_M a_M u_M\,, \tag{4.3}$$

where the $u_M(T,V,R,\vec{X})$ form a complete set of modes with a fixed momentum m in the V direction, $u_M(T,V,R,\vec{X}) = e^{-imV}v_M(T,R,\vec{X})$. These states furnish a unitary irreducible representation of the Schrödinger group with respect to the inner product

$$\langle u_M|u_{M'}\rangle = \frac{i}{2}\int_{\Sigma_T} d\Sigma^\mu u_M^*\overleftrightarrow{\partial_\mu}u_{M'}\,. \tag{4.4}$$

The $T = \text{cst}$ slice Σ_T is a lightlike surface whose normal is $\left(\frac{\partial}{\partial V}\right)^\mu = \delta^\mu_V$. The integration measure is $d\Sigma^\mu = \delta^\mu_V R^{-(d+1)}dRd^d\vec{X}dV$. Irreducibility follows from irreducibility with respect to the centrally extended Galilean subgroup.

We denote the Killing vectors of the Schrödinger metric collectively by $k_A = k_A^\mu\partial_\mu$. From the Noether theorem one obtains the corresponding conserved currents

$$j_A^\mu = \sqrt{-g}\,k_A^\nu T^\mu{}_\nu\,, \tag{4.5}$$

where $T_{\mu\nu}$ is the energy momentum tensor. We define the corresponding charges K_A by

$$K_A = \int_{\Sigma_T} dV d^d\vec{X} dR \, j_A^T = \int_{\Sigma_T} d\Sigma^\mu k_A^\nu T_{\mu\nu} \,. \tag{4.6}$$

For fields ϕ of the form $\phi = e^{-imV}\psi(T, R, \vec{X})$ the charges K_A can be written as

$$K_A = \int_{\Sigma_T} dV d^d\vec{X} dR \left(\pi k_A\phi + \pi^* k_A\phi^* - k_A^T\mathcal{L}\right) \,, \tag{4.7}$$

where k_A^T is the T component of the Killing vector k_A, $\mathcal{L}$ denotes the scalar field bulk Lagrangian and where π denotes the canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_T\phi)} = R^{-(d+1)}\partial_V\phi^* \,. \tag{4.8}$$

The canonical momentum is thus not independent of the initial data $\phi^*(T, V, R, \vec{X})$ specified at some equal time T surface, and imposing vanishing equal time Poisson brackets between $\phi(T, V, R, \vec{X})$ and $\phi^*(T, V', R', \vec{X}')$ would be inconsistent with a non-vanishing Poisson bracket $\{\phi(T, V, R, \vec{X}), \pi(T, V', R', \vec{X}')\} \neq 0$. This problem is resolved by taking the following Poisson bracket (for fields with the same nonzero m):

$$\{\phi(T, V, R, \vec{X}), \phi^*(T, V', R, \vec{X}')\} = f(V - V')R^{d+1}\delta(\vec{X} - \vec{X}')\delta(R - R') \,. \tag{4.9}$$

The Poisson bracket for fields with different m is taken to vanish. The function $f(V - V')$ will be chosen such that

$$\{K_A, \phi\} = -k_A\phi \,, \tag{4.10}$$

upon use of the Euler-Lagrange equations of the Lagrangian given in (4.1). If we consider the Hamiltonian H_T associated with the Killing vector $\partial/\partial T$ then this requirement means that the Hamilton equations and the Euler-Lagrange equations coincide. In the definition of the charges K_A (4.6), the dV integral ranges from V_1 to V_2 where $V_1 \neq V_2$ are arbitrary finite points.

There exists a unique function $f(V - V')$ which is such that (4.10) holds true for any choice of V_1 and V_2. This function is given by

$$f(V - V') = \frac{-i}{2m(V_2 - V_1)} e^{-im(V - V')} \,. \tag{4.11}$$

When we compactify V by identifying $V \sim V + 2\pi L$ then we should replace in the function f the momentum m by the discrete momentum m/L with $m \in \mathbb{Z}$ and write $V_2 - V_1 = 2\pi L$, so that we get

$$f(V - V') = \frac{-i}{4\pi m} e^{-im(V - V')/L} \,. \tag{4.12}$$

For the case of a Schrödinger space-time with a non-compact V we will from now on take $V_2 - V_1 = 2\pi$. This choice will prove useful later on. It is the value for which results obtained for the free scalar field on a Schrödinger space-time after integration over m gives us (for $\beta = 0$) the corresponding result on plane wave AdS (2.8). Also, for this value the Poisson brackets for compact and non-compact V are identical.

The form of the Schrödinger Poisson bracket can also be understood by starting with the Poisson bracket for scalar fields in plane wave AdS and decomposing it into modes with a fixed momentum in the V direction. If we then fix the momentum m the resulting Poisson bracket takes the Schrödinger form. To see this consider plane wave AdS with a compact V coordinate. Because T is like a lightcone time coordinate we must once again define equal T Poisson brackets for $\phi(T,V,R,\vec{X})$ and $\phi^*(T,V',R',\vec{X}')$. Borrowing from the result used in lightcone quantisation in Minkowski space-time with a compact null circle [26], appropriately generalised to AdS, the Poisson brackets turn out to also have the form (4.9), with

$$f(V-V') = -\frac{1}{2}\left(\frac{1}{2}\text{sign}(V-V') - \frac{V-V'}{2\pi L}\right) \quad . \tag{4.13}$$

The sign function can be decomposed into Fourier modes as

$$\frac{1}{2}\text{sign}(V-V') - \frac{V-V'}{2\pi L} = \sum_{m\neq 0} \frac{i}{2\pi m} e^{-im(V-V')/L} \quad , \tag{4.14}$$

and substituting the corresponding mode decomposition

$$\phi(T,V,R,\vec{X}) = \psi_0(T,R,\vec{X}) + \sum_{m\neq 0} \psi_m(T,R,\vec{X}) e^{-imV/L} \tag{4.15}$$

into the Poisson bracket, we find that the functions $\psi_{m\neq 0}$ satisfy the Schrödinger Poisson bracket of (4.9) with the function f precisely as in (4.12).

As regards the $m = 0$ modes, let us first note that they have an arbitrary time-dependence that is not fixed by the Klein-Gordon equation. Since these are the modes with zero lightcone momentum, $P_-\phi = 0$, they can be thought of as the precise scalar field counterparts of the lightlike lines discussed in section 3. It turns out that these modes vanish for plane wave AdS with a compact V coordinate and for a free non-interacting theory (see [26] for an explanation of this fact in Minkowski space-time with a compact null circle). This follows from substituting the decomposition (4.15) into the Hamiltonian. One of Hamilton's equations is then the statement that $\psi_0 = 0$. The problems encountered with the $m = 0$ modes in [27] appear when one studies loop corrections in an interacting theory. This lies beyond the scope of our work and it might be interesting to see what kind of interacting theories on a Schrödinger space-time with a compact lightlike circle are perturbatively well-defined.

To obtain the normalisable as well as the non-normalisable modes we impose the condition that solutions are regular everywhere in the bulk. The normalisable modes must furthermore satisfy the boundary condition that the inner product (4.4) is time independent. This will be the case provided we have

$$\lim_{\varepsilon\to 0} \int_{R=\varepsilon} R^{-(d+1)} u_M^* \overleftrightarrow{\partial_R} u_{M'} dV d\vec{X} = 0 \, . \tag{4.16}$$

This is the condition that the flux of the current $u_M^* \overleftrightarrow{\partial_\mu} u_{M'}$ through the boundary at $R = 0$ vanishes. Imposing this boundary condition requires that ν defined by

$$\nu = \sqrt{\frac{(d+2)^2}{4} + m_0^2 + \beta^2 m^2} \tag{4.17}$$

JHEP07(2010)069

is real so that all normalisable modes respect the Breitenlohner-Freedman bound [28]. There are two set of modes compatible with this boundary condition.[7] They are given by

$$\begin{aligned}\phi_\pm &= e^{-imV}\sum_{L,n,k} a^\pm_{L,n,k} v^\pm_{L,n,k}\\ &= e^{-imV}\sum_{L,n,k} C^\pm_{L,n,k} a^\pm_{L,n,k} e^{-iE^\pm_{L,n,k}T} Y_L e^{-\frac{1}{2}\omega|m|(\rho^2+R^2)}\rho^L R^{\Delta_\pm}\times\\ &\qquad\times L_n^{L-1+d/2}(\omega|m|\rho^2) L_k^{\pm\nu}(\omega|m|R^2)\,,\end{aligned} \tag{4.18}$$

where

$$\Delta_\pm = \frac{d+2}{2} \pm \nu\,. \tag{4.19}$$

The energy of the $+/-$ modes is given by

$$E^\pm_{L,n,k} = \mathrm{sign}(m) 2\omega\left(n+k+\frac{L}{2}+\frac{\Delta_\pm}{2}\right)\,, \tag{4.20}$$

with $L, n, k = 0, 1, 2, \ldots$. For the minus modes we must assume $0 < \nu < 1$ while for the plus modes we must assume that $\nu > 0$. The cases $\nu = 0, 1, 2, \ldots$ have to be dealt with separately because they involve logarithmic solutions. Here we will always assume that $\nu \neq 0, 1, 2, \ldots$.

The constant $C^\pm_{L,n,k}$ will be chosen such that upon quantisation the creation and annihilation operators $a^\pm_{L,n,k}$ and $a^{\pm\dagger}_{L,n,k}$ satisfy the commutation relation

$$[a^\pm_{L,n,k}, a^{\pm\dagger}_{L',n',k'}] = \frac{1}{2}\mathrm{sign}(m)\delta_{LL'}\delta_{nn'}\delta_{kk'}\,. \tag{4.21}$$

The constant $C^\pm_{L,n,k}$ can be taken to be real and positive and is found to be

$$(C^\pm_{L,n,k})^2 = \frac{2(\omega|m|)^{L+\Delta_\pm}}{|m|\pi}\frac{n!k!}{\Gamma(n+L+\frac{d}{2})\Gamma(1+k\pm\nu)}\,. \tag{4.22}$$

The sign function on the right hand side of (4.21) can be understood as follows. The Fock space vacuum $|0\rangle$ is defined by $a^\pm_{L,n,k}|0\rangle = 0$ for $m > 0$ and $a^{\pm\dagger}_{L,n,k}|0\rangle = 0$ for $m < 0$. The interpretation of the latter statement is that $a^{\pm\dagger}_{L,n,k}$ for $m < 0$ is the annihilation operator for the antiparticle making $a^\pm_{L,n,k}$ for $m < 0$ the creation operator for the antiparticle. In lightcone quantisation it is common practise to rename the creation and annihilation operators for $m < 0$ by $a^\pm_{-m,L,n,k} = b^{\pm\dagger}_{m,L,n,k}$ and likewise $a^{\pm\dagger}_{-m,L,n,k} = b^\pm_{m,L,n,k}$ and restrict m to only take positive values. Here we will not use this notation because m is not summed over anyway. We could always restrict m to be positive; however, to test results we find it useful to keep track of the sign of m. One other motivation for keeping both signs of m comes from the fact that from these results one can obtain the results for scalar field propagation on AdS in plane wave coordinates (after setting $\beta = 0$ and summing over all values of m).

[7] These $\pm$ normalisable modes have also been discussed in global coordinates in [29] and in Poincaré coordinates in [1]. For a different class of solutions, with cut-off dependent boundary conditions allowing for imaginary ν (violating the Breitenlohner-Freedman bound) see [29].

JHEP07(2010)069

For the normalisable modes ϕ_+ the Hamiltonian H_T is conserved in time. For the normalisable modes ϕ_- this is not the case and for these modes the action (4.1) and the charges K_A (4.6) are not appropriate. For the normalisable modes ϕ_- following [28, 30] we expect it to be necessary to introduce non-minimal coupling terms for the scalar field ϕ. This being said we stress that the condition (4.16) is only a condition on the modes and is therefore insensitive to the addition of non-minimal coupling terms in the bulk and boundary action of (4.1) since on-shell the Ricci scalar is constant and can be absorbed into the definition of m_0.

4.2 Time evolution

We show in this section that even though the future domain of dependence $\mathcal{D}^+(\Sigma_{T_0})$ of a constant T slice Σ_{T_0} is empty,

$$\mathcal{D}^+(\Sigma_{T_0}) = \emptyset \tag{4.23}$$

(section 3.1), the scalar field has a unique time evolution that is fully predictable given appropriate initial data.

In the previous subsection we have identified two inequivalent Hilbert spaces, those of the plus and minus modes $\phi^\pm$ (4.18) respectively. Both of the spaces satisfy the property that any element is an eigenfunction of N. Let us denote these two Hilbert spaces by $\mathcal{H}_m^+$ and $\mathcal{H}_m^-$.

We will show that for the Hilbert spaces with $m \neq 0$ there exists a well-posed initial value problem in the sense that given initial data for a scalar field in $\mathcal{H}_m^\pm$ at some time $T = T_0$ it is possible to uniquely predict the future dependence. To see this one just has to note that from $\phi(T = T_0, V, R, \vec{X})$ and the mode decomposition (4.18) it is possible to read off the coefficients $a_{L,n,k}$ via[8]

$$\langle e^{-imV} v^\pm_{L,n,k} | \phi(T = T_0)\rangle = \text{sign}(m) a^\pm_{L,n,k} \,. \tag{4.24}$$

Knowing all the $a^\pm_{L,n,k}$ determines the full future dependence of the function ϕ (from (4.18)). Note that in order to have a well-defined time evolution we only need to specify the values of the field ϕ at time $T = T_0$ and not its T-derivative.

This structure and property of the initial value problem and time-evolution of scalar fields on Schrödinger space-times is preserved by the foliation-preserving diffeomorphisms (3.8). In any coordinate system obtained in this way, the Klein-Gordon equation is a 1st order differential equation in the new time coordiante T', and the evolution of the Klein-Gordon field ϕ is determined by the value of the field on the null surface $\Sigma_{T'}$ (and the momentum in the V'-direction, the mass).

We have thus resolved the problem associated with the emptiness of the future domain of dependence $\mathcal{D}^+(\Sigma_{T_0})$. The emptiness of $\mathcal{D}^+(\Sigma_{T_0})$ resulted from 1) the existence of lightlike lines and 2) from the existence of curves that reach the boundary before crossing the equal time surface. The way the scalars get around this potential unpredictability follows from i) the restriction to modes with $m \neq 0$ (as explained above, the $m = 0$ modes

[8]We have used the following two orthogonality relations: $\int d\Omega_{d-1} Y_{L'}(\Omega) Y_L(\Omega) = \delta_{LL'}$ and $\int_0^\infty dx x^a e^{-x} L_n^a(x) L_{n'}^a(x) = \frac{\Gamma(n+a+1)}{n!}\delta_{nn'}$ where $\text{Re}\, a > -1$.

are the scalar analogues of lightlike lines, and the restriction to $m \neq 0$ indeed avoids the problems associated with these lightlike lines), and ii) from imposing suitable boundary conditions which forbid information exchange with the boundary.

4.3 Wightman functions and Green's functions

We will now first study the positive and negative frequency Wightman functions, $G^{\pm}(x,x')$, and then from those build the bulk-to-bulk propagator in global coordinates. We have

$$G^{+}(x,x') = \langle 0|\phi(x)\phi^{\dagger}(x')|0\rangle\,, \tag{4.25}$$

$$G^{-}(x,x') = \langle 0|\phi^{\dagger}(x')\phi(x)|0\rangle\,. \tag{4.26}$$

Our conventions for the creation and annihilation operators are given in (4.21). The positive and negative frequency Wightman functions, denoted by G^{+} and G^{-} respectively, can be defined for both Hilbert spaces $\mathcal{H}_m^{\pm}$ where the $\pm$ refer to the two different sets of normalisable modes in (4.18). We will write the expressions for G^{+} and G^{-} on $\mathcal{H}_m^{\pm}$ simultaneously, hoping that this does not cause any confusion. Using the mode decompositions (4.18) we obtain for the Wightman functions the expressions

$$\begin{aligned} G^{+}(x,x') &= \frac{1}{2}\theta(m)e^{-im(V-V')}\sum_{L,n,k}(C^{\pm}_{L,n,k})^2 e^{-i2\omega\left(n+k+\frac{L}{2}+\frac{\Delta_{\pm}}{2}\right)(T-T')}\times \\ &\quad\times Y_L(\Omega)Y_L^{*}(\Omega')\varphi_{L,n}(\rho)\varphi_{L,n}(\rho')\phi_k^{\pm}(R)\phi_k^{\pm}(R')\,, \end{aligned} \tag{4.27}$$

$$\begin{aligned} G^{-}(x,x') &= \frac{1}{2}\theta(-m)e^{-im(V-V')}\sum_{L,n,k}(C^{\pm}_{L,n,k})^2 e^{i2\omega\left(n+k+\frac{L}{2}+\frac{\Delta_{\pm}}{2}\right)(T-T')}\times \\ &\quad\times Y_L(\Omega)Y_L^{*}(\Omega')\varphi_{L,n}(\rho)\varphi_{L,n}(\rho')\phi_k^{\pm}(R)\phi_k^{\pm}(R')\,. \end{aligned} \tag{4.28}$$

Both $G^{\pm}$ are solutions to the homogeneous Klein-Gordon equation. We have under complex conjugation $(G^{\pm}(x,x'))^{*} = G^{\pm}(x',x)$. As it stands the sums in the expressions for $G^{\pm}$ are not convergent in the sense of functions. If we consider the various sums as series in the parameter $s = \exp[-i2\omega(T-T')]$ then the series only converges if $|s|<1$. Thus, in order to have convergent series we replace $T-T'$ in G^{+} by $T-T'-i\epsilon$ and $T-T'$ in G^{-} by $T-T'+i\epsilon$, with $\epsilon>0$ infinitesimal. In terms of

$$s_\epsilon = e^{-i2\omega(T-T')-2\omega\epsilon}\,. \tag{4.29}$$

the regulated G^{+} is then a series in s_ϵ and the regulated G^{-} is a series in s_ϵ^{*}.

In order to evaluate the sums we use the following generating function for the Laguerre polynomials (see e.g. [31, Theorem 69] or [32])

$$\sum_{n=0}^{\infty} e^{-\frac{1}{2}(x+y)}\frac{(xy)^{\frac{a}{2}}s^n n!}{\Gamma(n+a+1)}L_n^a(x)L_n^a(y) = \frac{s^{-\frac{a}{2}}}{1-s}\exp[-\frac{1}{2}(x+y)\frac{1+s}{1-s}]e^{-i\frac{\pi}{2}a}J_a\left(2i\frac{\sqrt{xys}}{1-s}\right)\,. \tag{4.30}$$

We will also need the decomposition of a plane wave into spherical harmonics which is given by (see e.g. [33])

$$e^{iz\hat{n}\cdot\hat{n}'} = (2\pi)^{\frac{d}{2}}\sum_{L} i^L z^{-\frac{d-2}{2}}J_{L+\frac{d-2}{2}}(z)Y_L^{*}(\hat{n})Y_L(\hat{n}')\,, \tag{4.31}$$

where $\hat{n}$ and $\hat{n}'$ are unit vectors on S^{d-1} that are parametrised by Ω and Ω', respectively. In fact, the unit vector $\hat{n}$ is nothing but the Cartesian vector $\vec{X}$ that appears in global Schrödinger metric normalised to unit length.

Armed with these two expressions we can evaluate the sums that define the Wightman functions. The result is

$$G^+(x,x') = \theta(m)\frac{i^{-\Delta_\pm}}{(2\pi)^{\frac{d}{2}}4\pi m}(m\zeta_{-\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(m\zeta_{-\epsilon})e^{im\eta_{-\epsilon}}\,, \tag{4.32}$$

$$G^-(x,x') = -\theta(-m)\frac{i^{\Delta_\pm}}{(2\pi)^{\frac{d}{2}}4\pi m}(-m\zeta_{+\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(-m\zeta_{+\epsilon})e^{im\eta_{+\epsilon}}\,, \tag{4.33}$$

where $\zeta_{\pm\epsilon}$ and $\eta_{\pm\epsilon}$ are ϵ-deformations of the invariant functions $\zeta(x,x')$ and $\eta(x,x')$ (B.2) expressed in global coordinates. We have

$$\zeta_{\pm\epsilon} = \frac{\omega RR'}{\sin\omega(T-T'\pm i\epsilon)}\,, \tag{4.34}$$

$$\eta_{\pm\epsilon} = -(V-V') + \frac{\omega(\vec{X}^2+\vec{X}'^2+R^2+R'^2)}{2\tan\omega(T-T'\pm i\epsilon)} - \frac{\omega\vec{X}\cdot\vec{X}'}{\sin\omega(T-T'\pm i\epsilon)}\,. \tag{4.35}$$

It can be checked that, apart from the $i\epsilon$ and the overall constant, the result for the Wightman functions agrees with the most general normalisable solution to the Klein-Gordon equation for a function that only depends on η and ζ. The Poincaré coordinate expressions for the Wightman functions can be obtained by taking the $\omega \to 0$ limit in (4.34) and (4.35).

Now that we have the two Wightman functions at our disposal we are in a position to compute any Green's function that we are interested in. For example the Feynman propagator is given by

$$G_F(x,x') = \theta(T-T')G^+(x,x') + \theta(T'-T)G^-(x,x')\,, \tag{4.36}$$

and the retarded and advanced Green's functions read

$$G_R(x,x') = \theta(T-T')\left(G^+(x,x') - G^-(x,x')\right)\,, \tag{4.37}$$

$$G_A(x,x') = \theta(T'-T)\left(G^+(x,x') - G^-(x,x')\right)\,, \tag{4.38}$$

where $G^+(x,x') - G^-(x,x')$ is called the commutator function.

It is clear, though, that in the Schrödinger case, due to the fact that m is not summed over, there is no mixing between positive and negative frequency Wightman functions. For example, for $m > 0$ the propagator and the retarded Green's functions are the same, while for $m < 0$ the propagator equals the advanced Green's function.

The fact that in the Feynman propagator the step function $\theta(T-T')$ is multiplied by the step function $\theta(m)$ appearing in the Wightman function G^+ and similarly the fact that $\theta(T'-T)$ multiplies $\theta(-m)$ appearing in G^- has the following welcome consequence. Even though T is not a global time function and as such does not allow one to label all causally related events by a different value of T, it is not a problem to define a time ordering since the time ordering in the Feynman propagator is correlated with the sign of m. The failure of T to provide a well-defined global time ordering only applies to events with the same value of T. Propagation between such events with $m > 0$ or $m < 0$ does not occur.

The bulk-to-bulk propagator $G_F(x,x')$, (4.36), satisfies the delta-function sourced Klein-Gordon equation

$$\left(\Box - \tilde{m}_0^2\right) G_F(x,x') = \frac{i}{2\pi} e^{-im(V-V')} R^{d+1} \delta(T-T')\delta(R-R')\delta(\vec{X}-\vec{X}')\,. \tag{4.39}$$

The bulk-to-bulk propagator for the Schrödinger space-time has also been constructed in [34]. Our result agrees with the expression in [34].[9]

We next approximate the bulk-to-bulk propagator for points that are close to being separated by a lightlike line, i.e. for $T-T'$ small, and show how it is related to the Feynman propagator for a massless particle on Minkowski space-time. Using the asymptotic form of the Bessel function we find that for $T-T'$ small the bulk-to-bulk propagator can be approximated by

$$\begin{aligned} G_F(x,x') =\ & \theta(m)\theta(T-T')\frac{1}{2}\frac{i^{-\frac{d+1}{2}} m^{\frac{d-1}{2}}}{(2\pi)^{\frac{d+3}{2}}}\left(\frac{RR'}{T-T'-i\epsilon}\right)^{\frac{d+1}{2}} e^{-im\alpha_-} \\ & +\theta(-m)\theta(T'-T)\frac{1}{2}\frac{i^{\frac{d+1}{2}} (-m)^{\frac{d-1}{2}}}{(2\pi)^{\frac{d+3}{2}}}\left(\frac{RR'}{T-T'+i\epsilon}\right)^{\frac{d+1}{2}} e^{im\alpha_+}\,, \end{aligned} \tag{4.40}$$

where $\alpha_\pm$ is

$$\alpha_\pm = \mp i\left(V - V' - \frac{1}{2}\frac{(\vec{X}-\vec{X}') + (R-R')^2}{T-T' \pm i\epsilon}\right)\,. \tag{4.41}$$

First of all notice that the expression is independent of β. Secondly, the relation with the propagator for a massless particle on Minkowski space-time is obtained by integrating this result over m. Doing so we find

$$\int_{-\infty}^{\infty} dm G_F(x,x') = \frac{1}{\text{Vol}\, S^{d+2}}\frac{1}{d+1}(\sigma + i\epsilon)^{-\frac{d+1}{2}}\,, \tag{4.42}$$

where

$$\text{Vol}\, S^{d+2} = \frac{2\pi^{\frac{d+3}{2}}}{\Gamma(\frac{d+3}{2})}\,. \tag{4.43}$$

In obtaining this expression we used that σ is well approximated by the Minkowski space-time geodesic distance for lightlike separated points. Equation (4.42) is the standard expression for the propagator of a massless particle on Minkowksi space-time. We thus conclude (by inverse Fourier transform) that the behavior of the Schrödinger bulk-to-bulk propagator for points that are close to being separated by a lightlike line is well approximated by the Minkowski space-time propagator for a massless particle with a fixed momentum m in the V direction.

Information about the causal structure probed by scalars can be obtained by looking at the zeros of the commutator function $G^+(x,x') - G^-(x,x')$. By microcausality, the

[9] However, equation (3.27) of [34] contains a misprint. The normalisation constant which they denote by $\tilde{C}_\Delta$ should be the one given in (E.5). This latter normalisation constant agrees with the one in [35].

commutator function must vanish for spacelike separated points x and x'. In a free field theory the commutator function is a classical c-number quantity. Hence, it can only be nonzero whenever two points can be connected by a classical path. The commutator function is therefore sensitive to the possible geodesic non-connectedness. It follows that the commutator function $G^+(x,x') - G^-(x,x')$ must be zero when

1. x and x' are spacelike separated (microcausality),
2. x and x' cannot be connected by a geodesic.

Since G^+ only exists for positive values of m and G^- only for negative values of m, the commutator function vanishes if and only if $G^{\pm}$ vanish separately. Below we will discuss these two types of zeros of $G^{\pm}$. For a recapitulation of the properties of the commutator function in the AdS case which shows similar behaviour we refer to appendix E.

Any two points for which $T - T' \neq 0$ are timelike separated. Hence all spacelike separated points are points for which necessarily $T = T'$ (section 3.2). In appendix D it is shown that points P and $\bar{P}$ for which $T_{\bar{P}} - T_P = \frac{\pi}{\omega}$ that do not satisfy (D.5) are geodesically disconnected. It follows that, by points 1 and 2 above, the commutator function must vanish whenever $\sin\omega(T - T') = 0$. It can be checked that the $i\epsilon$ prescription in the Wightman function $G^{\pm}$ is precisely such that this is the case. Summarising we can say that the commutator function probes the following part of the space-time

$$\bigcup_{n\in\mathbb{Z}} I^+\left(T = T' + (n-1)\frac{\pi}{\omega}\right) \cap I^-\left(T = T' + n\frac{\pi}{\omega}\right) \tag{4.44}$$

which, as we will now discuss, is the scalar field counterpart of the non-distinguishing character of space-time as seen by point particle probes.

The boundary of the region on which the commutator function is nonvanishing is given by $\sin\omega(T - T') = 0$. To contrast this with the AdS case note that there the commutator function is nonvanishing for $|\eta^{\text{AdS}}| < 1$ (see appendix E). In both cases the boundaries are formed by lightlike lines. However, in AdS all lightlike lines are null geodesics and these form a relativistic lightcone structure whereas in the Schrödinger case only null geodesics with $P_- = 0$ (see appendix D) form lightlike lines, and these describe a Galilean lightcone structure. In the case of massive point particles we saw that they probe the entire chronological past and future $I^-(p_0) \cup I^+(p_0)$ of some point p_0 (3.7). The fact that the propagator only probes the horizontal sheets (4.44) rather than $I^-(p_0) \cup I^+(p_0)$ is something that is also observed in the case of the propagator for the non-relativistic harmonic oscillator.

4.4 Bulk-to-boundary propagator

The bulk-to-boundary propagator K_F can be obtained from:

$$K_F(T, V, R, \vec{X}; T', V', \vec{X}') = C \lim_{R'\to 0} R'^{-\Delta_+} G_F(x, x'), \tag{4.45}$$

where $G_F(x,x')$ is the bulk-to-bulk propagator depending on Δ_+. Using the expression for the bulk-to-bulk propagator we find

$$\begin{aligned}
&K_F(T,V,R,\vec{X};T',V',\vec{X}') = \\
&\quad = \theta(m)\theta(T-T')C\frac{i^{-\Delta_+}m^{\Delta_+-1}}{4\pi(2\pi)^{\frac{d}{2}}}\left(\frac{\omega R}{\sin\omega(T-T'-i\epsilon)}\right)^{\Delta_+} e^{im\eta_{-\epsilon}(R'=0)} \qquad (4.46)\\
&\quad\quad +\theta(-m)\theta(T'-T)C\frac{i^{\Delta_+}(-m)^{\Delta_+-1}}{4\pi(2\pi)^{\frac{d}{2}}}\left(\frac{\omega R}{\sin\omega(T-T'+i\epsilon)}\right)^{\Delta_+} e^{im\eta_{+\epsilon}(R'=0)}\,.
\end{aligned}$$

The constant C is determined by requiring (in the sense of distributions):

$$\lim_{\epsilon,R\to 0} R^{\Delta_+-d-2}K_F(T,V,R,\vec{X};T',V',\vec{X}') = \frac{1}{2\pi}e^{-im(V-V')}\delta(T-T')\delta(\vec{X}-\vec{X}')\,. \qquad (4.47)$$

In taking the limit we keep $\frac{T-T'}{R^2}$ and $\frac{\vec{X}-\vec{X}'}{R}$ fixed as R goes to zero and furthermore $\tilde{\epsilon}\equiv\frac{\epsilon}{R^2}$ goes to zero as both ϵ and R go to zero. The result is that the constant C is given by

$$C = \frac{i2^{1-\nu}}{\Gamma(\nu)}\,. \qquad (4.48)$$

The normalisation of the bulk-to-boundary propagator agrees with the corresponding expression in [34, 36]. The Poincaré coordinate expression for the bulk-to-boundary propagator can be obtained by taking the $\omega\to 0$ limit.

When it comes to the bulk-to-boundary propagator there appears an asymmetry in the discussion of the solutions depending on Δ_+ and those in which Δ_+ is replaced by Δ_-. This also happens in AdS and has to do with the fact the bulk-to-boundary propagator with Δ_+ replaced by Δ_- does not approach a boundary delta function in the limit where both points lie on the boundary.

The boundary value of the scalar field $\phi(T,V,R,\vec{X})$ will be denoted by $\phi_0(T,V,\vec{X})$ and is defined by

$$\phi_0(T,V,\vec{X}) = \lim_{R\to 0} R^{\Delta_+-d-2}\phi(T,V,R,\vec{X})\,. \qquad (4.49)$$

A solution to the Klein-Gordon equation for a massive complex scalar on the Schrödinger space-time for a normalisable mode in the background of a non-normalisable mode is given by

$$\begin{aligned}
\phi(T,V,R,\vec{X}) = &\int dT'd^d\vec{X}'dV'K_F(T,V,R,\vec{X};T',V',\vec{X}')\phi_0(T',V',\vec{X}')\\
&+\phi_+(T,V,\vec{X},R)\,, \qquad (4.50)
\end{aligned}$$

where $\phi_+(T,V,\vec{X},R)$ is given in (4.18). The solution $\phi_+(T,V,\vec{X},R)$ corresponds to the normalisable solution (4.18) while the part involving the bulk-to-boundary propagator corresponds to the non-normalisable solution (ϕ_0 is the boundary value of a non-normalisable solution). The non-normalisable solution contains both terms proportional to R^{Δ_-} as well as terms proportional to R^{Δ_+} in the near boundary expansion of the scalar field. The normalisable solution only contributes to the term $\propto R^{\Delta_+}$.

When $\nu > 1$ the term $\propto R^{\Delta_-}$ in the near boundary expansion of the scalar field is dual to a source in the boundary theory. As is well-known when $0 < \nu < 1$ it is possible to instead consider the term $\propto R^{\Delta_+}$ as dual to a source. In the AdS/CFT context the terms proportional to R^{Δ_+} and R^{Δ_-} are conjugate variables in the sense that the generating functional for the theory in which the term $\propto R^{\Delta_+}$ acts as the source can be obtained from the theory in which the on-shell action depends on the term $\propto R^{\Delta_-}$ via a Legendre transformation [37] (see also [38] for the case of Lorentzian AdS/CFT). We expect that a suitably modified version of this statement applies here as well, so that it is sufficiently general to consider only the case where the term proportional to R^{Δ_-} is dual to the source and hence resides in the non-normalisable solution.

5 Discussion

We studied in detail the causal structure of the $z = 2$ Schrödinger space-time from the point of view of both point particle and scalar field probes, emphasising and highlighting those peculiar features of the point particle causal structure that have a counterpart for scalar fields. For scalar fields, it turns out that the restriction to a fixed non-zero lightcone momentum m (as dictated by the representation theory of the Schrödinger group) is sufficient to avoid the occurence of near-to-pathological properties that one does encounter in the case of point particle probes. For example, even though one cannot define a time function and even though the future domain of dependence of slices Σ_T of constant global coordinate time T are empty, one can define a well-posed initial value problem for scalar fields. We have shown that, for a given m this requires specification of the scalar field on Σ_T. This first-order nature of the Klein-Gordon initial value problem is preserved by the so-called double-foliation preserving diffeomorphisms which leave the Galilean-like causal structure (and the lightlike lines) of the $z = 2$ Schrödinger space-time invariant. This Galilean-like causal structure, as defined by the properties of causal curves, is reflected in the properties of the Wightman functions and propagators of the scalar field theory.

One obvious extension of this work is to consider Schrödinger space-times with values of z different from two. The range of z that is interesting from the point of view of non-relativistic physics is $z > 1$. From the study of tidal forces we know that for $1 < z < 2$ the space-times are singular [11]. This leaves us with the range $z > 2$. In this case one would like to construct the counterpart of the $z = 2$ global metric. The construction of such a global metric is hampered by the non-existence of an everywhere timelike Killing vector which means that any global coordinate system is necessarily time-dependent. In order to find an explicit global metric one could try to generalize the isometric embedding presented here to other values of z. This is indeed possible but the result for us was not sufficiently illuminating to derive from it a global metric. What can be stated just from knowing the Poincaré like coordinates for $z > 2$ is that these space-times are non-distinguishing. This can be proven using an appropriately adapted verion of the curve given in [14].

It would also be interesting to study metric perturbations. This is relevant for a number of reasons. First of all, we know that since the $z = 2$ Schrödinger space-time is not stably causal there exist perturbations of the lightcone structure that lead to the existence of closed timelike curves. This raises a number of questions: what kind of metric perturbations produce this kind of behavior? are these physically relevant (e.g. do the perturbations have finite energy in a suitable sense)? and how sensitive would scalar fields be to the presence of such closed timelike curves in the perturbed metric? Secondly, in the analysis of the scalars, representation theory played a dominant role (choosing a fixed nonzero m). It would be nice to understand what this entails for the metric perturbations. Ultimately one would like to understand the precise form of the asymptotic fall-off conditions for the various fields (scalar, gauge, metric, etc.) and the required counterterms that allow one to define a well-defined variational problem and understand the construction of holographic renormalisation (see [3, 10, 43] for a discussion of some of the issues involved). These issues are also relevant for the study of asymptotically Schrödinger black holes. In particular, one might like to understand whether or not black holes in global Schrödinger exhibit any interesting phase transitions [44].

Acknowledgments

This work has been supported by the Swiss National Science Foundation and the "Innovations- und Kooperationsprojekt C-13" of the Schweizerische Universitätskonferenz SUK/CUS.

A Schrödinger algebra and isometric embeddings

We will denote the isometry algebra of the Schrödinger space-time Sch_{d+3} by $\mathfrak{sch}(d)$. It consists of all the elements of the isometry algebra $\mathfrak{so}(2,d{+}2)$ of AdS_{d+3} that commute with the lightcone momentum P_-: d-dimensional spatial rotations M_{ab} and translations P_a, Galilean boosts V_a, time translations H, dilatations D, a special conformal transformation C and, of course, the central element $P_- \equiv N$. The latter four generators $\{H,C,D,N\}$ form the algebra

$$\begin{aligned} &\mathfrak{sch}(d=0) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}_N \cong \mathfrak{so}(2,1) \oplus \mathbb{R}_N \\ &[H,C] = D \qquad [D,C] = 2C \qquad [D,H] = -2H \ . \end{aligned} \tag{A.1}$$

The other non-trivial commutators are

$$\begin{aligned} &[D,P_a] = -P_a \quad [D,V_a] = V_a \quad [P_a,V_b] = \delta_{ab}N \quad [H,V_a] = P_a \quad [C,P_a] = -V_a \\ &[M_{ab},P_c] = \delta_{bc}P_a - \delta_{ac}P_b \qquad [M_{ab},V_c] = \delta_{bc}V_a - \delta_{ac}V_b \\ &[M_{ab},M_{cd}] = \delta_{bc}M_{ad} + \delta_{ad}M_{bc} - \delta_{bd}M_{ac} - \delta_{ac}M_{bd} \end{aligned} \tag{A.2}$$

In Poincaré coordinates a realisation of this algebra is given by

$$\begin{aligned} &H = \partial_t \quad P_a = \partial_a \qquad V_a = x^a\partial_\xi + t\partial_a \quad M_{ab} = x^a\partial_b - x^b\partial_a \\ &N = \partial_\xi \quad D = 2t\partial_t + r\partial_r + x^a\partial_a \qquad C = t^2\partial_t + \frac{1}{2}(r^2+\vec{x}^2)\partial_\xi + tr\partial_r + tx^a\partial_a \end{aligned} \tag{A.3}$$

The embedding of $\mathfrak{sch}(d)$ into the AdS isometry algebra $\mathfrak{so}(2,d+2)$ proceeds principally via the essentially unique embedding of $\mathfrak{sch}(d=0)$ into $\mathfrak{so}(2,2)$ via the double null splitting

$$\mathfrak{so}(2,1) \oplus \mathbb{R}_N \hookrightarrow \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1) \cong \mathfrak{so}(2,2) \ . \tag{A.4}$$

Explicitly, in terms of the generators M_{AB} of $\mathfrak{so}(2,d+2)$ satisfying

$$\begin{aligned}[M_{AB},M_{CD}] &= \eta_{BC}M_{AD}+\eta_{AD}M_{BC}-\eta_{BD}M_{AC}-\eta_{AC}M_{BD} \\ \eta_{AB} &= \mathrm{diag}(-1,+1,\dots,+1,-1) \qquad A,B=0,1,\dots,d+3\end{aligned} \tag{A.5}$$

and null coordinates $(x^0,x^1)\to x^{\pm}$, $(x^{d+2},x^{d+3})\to x^{\hat{\pm}}$ with $\eta_{+-}=\eta_{\hat{+}\hat{-}}=1$, one can choose

$$D=M_{+-}+M_{\hat{+}\hat{-}} \qquad H=M_{-\hat{-}} \qquad C=M_{+\hat{+}} \tag{A.6}$$

and N any element of the other (commuting) $\mathfrak{so}(2,1)\subset\mathfrak{so}(2,2)$ (A.4).

In particular, if one seeks a codimension 1 embedding of Sch_3 into $\mathbb{R}^{2,2}$ (i.e. the Schrödinger analogue of the standard embedding $\mathsf{AdS}_3\hookrightarrow\mathbb{R}^{2,2}$), the equation defining the hypersurface has to be invariant under $\{H,C,D,N\}$ thought of as elements of $\mathfrak{so}(2,2)$ via the above embedding of Lie algebras. Thus let $f=f(x^{\pm},x^{\hat{\pm}})$ be such a function. Invariance under H, say, requires f to satisfy the equation

$$\left(x^{+}\partial_{\hat{-}}-x^{\hat{+}}\partial_{-}\right)f(x^{\pm},x^{\hat{\pm}})=0 \ , \tag{A.7}$$

which is solved by $f=f(x^+,x^{\hat{+}},x^+x^-+x^{\hat{+}}x^{\hat{-}})$. Likewise, invariance under C relates the x^+- and $x^{\hat{+}}$-dependence, and one immediately finds

$$Hf=Cf=0 \quad\Rightarrow\quad f=f(x^+x^-+x^{\hat{+}}x^{\hat{-}}) \ . \tag{A.8}$$

But since

$$x^+x^-+x^{\hat{+}}x^{\hat{-}}=-(x^0)^2+(x^1)^2+(x^{d+2})^2-(x^{d+3})^2 \tag{A.9}$$

this hypersurface describes AdS_3 with the enhanced isometry algebra $\mathfrak{so}(2,2)\supsetneq\mathfrak{sch}(d=0)$. This argument immediately carries over to $d>0$ to preclude the existence of an isometric embedding $\mathsf{Sch}_{d+3}\hookrightarrow\mathbb{R}^{2,d+2}$.

We are thus lead to consider codimension 2 embeddings $\mathsf{Sch}_{d+3}\hookrightarrow\mathbb{R}^{2,d+3}$, with $\eta_{d+4,d+4}=1$, and the corresponding embedding of isometry algebras

$$\mathfrak{sch}(d)\hookrightarrow\mathfrak{so}(2,d+3)\oplus\mathbb{R}^{2,d+3} \ , \tag{A.10}$$

in particular $\mathfrak{so}(2,1)\oplus\mathbb{R}_N\hookrightarrow\mathfrak{so}(2,3)\oplus\mathbb{R}^{2,3}$. A characteristic feature of the Schrödinger algebra and Schrödinger geometry is the existence of the central element N realised as a null Killing vector. Thus N can either arise from a null translation in the translational part of the isometry algebra of the embedding space or from a null rotation. Let us first show that the former is not possible (and that in fact the entire Schrödinger algebra needs to be embedded into the rotational part $\mathfrak{so}(2,d+3)$ of the isometry algebra). The argument is largely insensitive to the dimension and signature of the embedding space, so we can consider a general (semi-direct product) isometry algebra $\mathfrak{so}(p,q)\oplus\mathbb{R}^{p,q}$ with $p+q$ large enough to accommodate the required translations, and we assume that $N=P_-$ is identified with a null translation. Then

* to reproduce $[P_a,V_b]=\delta_{ab}N$, P_a and V_b cannot both be simultaneously translations or rotations, so we choose $P_a\in\mathbb{R}^{p,q}$, $V_a\in\mathfrak{so}(p,q)$ (the opposite choice is related to this via the authomorphism $P_a\leftrightarrow V_a, H\leftrightarrow -C$, $D\to -D$, $N\to -N$);

* since C and D do not commute with P_a, they are elements of $\mathfrak{so}(p,q)$;
* since $[H,C]=D$, one also has $H\in\mathfrak{so}(p,q)$;
* but then $[H,V_a]\in\mathfrak{so}(p,q)$, which contradicts the relation $[H,V_a]=P_a$.

Thus we need to choose $N\in\mathfrak{so}(p,q)$. But then the Schrödinger algebra requires all generators to be elements of $\mathfrak{so}(p,q)$ (by similar reasoning), and we need to consider the embedding of isometry algebras

$$\mathfrak{sch}(d)\hookrightarrow\mathfrak{so}(2,d+3)\ , \tag{A.11}$$

in particular $\mathfrak{so}(2,1)\oplus\mathbb{R}_N\hookrightarrow\mathfrak{so}(2,3)$. In addition to the embedding via $\mathfrak{so}(2,2)\subset\mathfrak{so}(2,3)$ discussed (and dismissed) above, there are the two regular embeddings

$$\begin{aligned}\mathfrak{so}(2,1)\oplus\mathbb{R}_N&\cong\mathfrak{so}(1,2)\oplus\mathfrak{so}(1,1)\subset\mathfrak{so}(2,3)\\ \mathfrak{so}(2,1)\oplus\mathbb{R}_N&\cong\mathfrak{so}(2,1)\oplus\mathfrak{so}(2)\subset\mathfrak{so}(2,3)\end{aligned} \tag{A.12}$$

However, in these cases N is identified either with a timelike boost generator or a spacelike rotation generator and can therefore not possibly be null in the metric induced from the metric on $\mathbb{R}^{2,3}$ (the only embedding that allows a null N is that via $\mathfrak{so}(2,2)$). This argument generalises in an obvious way to $d>0$ (by first embedding the rotations into $\mathfrak{so}(d)\subset\mathfrak{so}(2,d+3)$ and then dealing with the commuting $\mathfrak{so}(2,1)\oplus\mathbb{R}_N$ algebra as above).

We therefore conclude that there are no codimension 2 *equivariant* isometric embeddings of the Schrödinger space-time, i.e. isometric embeddings which are such that all isometries are induced by the linear isometries (pseudo-orthogonal transformations) of the flat embedding space. In this context it is worth noting that there exist G-equivariant versions of the Nash embedding theorem (such as the Moore-Schlafly theorem [39]), but that these do not produce useful upper bounds on the required dimension of the embedding space.

B Schrödinger invariants

In this appendix we briefly discuss the Schrödinger analogue of the AdS chordal distance (or any AdS invariant measure of the distance of two points like the geodesic distance). A characteristic feature of the Schrödinger space-time is that, due to its reduced isometry (and isotropy) algebra, there are two independent invariant building blocks instead of just the one unique chordal distance in the AdS case. To see this, let $\sigma(x,x')$ be any function of two points x and x' which is invariant under the simultaneous action of the isometry group on x and x',

$$\sigma(gx,gx')=\sigma(x,x') \tag{B.1}$$

(geodesic distance is an example of such a function). If we consider $\sigma(x,x')$ as a function of x only, keeping x' fixed, $f_{x'}(x)=\sigma(x,x')$, then this function is invariant under the stabiliser $H_{x'}$ of the point x'.

Concretely in the case of the Schrödinger space-time (2.1) with its Schrödinger isometry group, let us e.g. consider the point $x'=(t',\xi',r',\vec{x}')=(0,\xi',1,\vec{0})$ with ξ'

arbitrary. Its stabiliser is generated by the Killing vectors that vanish at that point. It is easily seen from (A.3) that these are the linear combinations of $\{C - \frac{1}{2}N, V_a, M_{ab}\}$ [40], forming an algebra isomorphic to $\mathfrak{euc}(d) \oplus \mathbb{R}$ (with $\mathfrak{euc}(d)$ the Euclidean algebra). The most general function invariant under these Killing vectors depends on two variables. Indeed, starting with a function of all $d+3$ coordinates, rotation invariance reduces the number to 4 (3+ radial coordinate in the x^a-directions). Then boost invariance reduces this further by one (by correlating the t-dependence with the dependence on this radial coordinate) and finally invariance under $C - \frac{1}{2}N$ reduces this to two.

Since the space-time is homogeneous, this counting argument gives the same number at each point of the space-time. Hence, for each point x' the function $f_{x'}(x)$ depends on two variables. Therefore any Schrödinger invariant function $\sigma(x, x')$ of x and x' is parametrised by two Schrödinger invariant functions that we denote as $\eta(x, x')$ and $\zeta(x, x')$. In Poincaré coordinates they can be choosen to be

$$\eta(x,x') = -(\xi - \xi') + \frac{r^2 + r'^2 + (\vec{x} - \vec{x}')^2}{2(t-t')} \quad , \quad \zeta(x,x') = \frac{rr'}{t-t'} \quad . \tag{B.2}$$

In particular, the standard AdS-invariant chordal distance is

$$\eta^{\text{AdS}} = \frac{\eta}{\zeta} = \frac{-2(\xi-\xi')(t-t') + r^2 + r'^2 + (\vec{x}-\vec{x}')^2}{2rr'} \quad . \tag{B.3}$$

C Chronological future

Here we prove that the chronological future of an arbitrary point p_0 on the constant global time slice Σ_{T_0} consists of all points in the space-time with $T > T_0$. To do so we will show that any two points $(T_0, V_0, R_0, \vec{X}_0)$ and $(T_0 + \varepsilon, V_f, R_f, \vec{X}_f)$ with ε arbitrary can be connected by a timelike curve. These curves can be constructed in strict analogy to the curves that were used in [14] to prove the non-distinguishing character of the $z = 3$ Schrödinger space-time in Poincaré coordinates. First we adapt the curves to the $z = 2$ case, then we simply replace the Poincaré coordinates $(t, \xi, r, \vec{x})$ by the global coordinates $(T, V, R, \vec{X})$. This produces a new curve which is not equivalent to the one used in Poincaré coordinates by a coordinate transformation. Nevertheless, by construction, the new curve has the same nice properties as the one used in [14]: for any two points P_0 and P_f with $T_0 \neq T_f$ (but possibly $T_f - T_0 = \epsilon > 0$ infinitesimal), there exists a causal curve connecting these points.[10] Moreoever, as a consequence of the strictly positive terms proportional to ω^2 appearing in the global metric, the curve produced in this way is now actually everywhere timelike (and not just causal).

For notational simplicity we give the curve in terms of its tangent and its intermediate

[10] Since P_0 and P_f can be spatially arbitrarily close to each other, there exist causal curves that get arbitrarily close to being closed causal curves. This is a violation of strong causality.

JHEP07(2010)069

points:

$$\gamma(\lambda)=\begin{cases}\gamma_1(\lambda) & \text{for } \lambda\in[0,\frac{\varepsilon}{4}]\\ \gamma_2(\lambda) & \text{for } \lambda\in[\frac{\varepsilon}{4},\frac{\varepsilon}{2}]\\ \gamma_3(\lambda) & \text{for } \lambda\in[\frac{\varepsilon}{2},\frac{3\varepsilon}{4}]\\ \gamma_4(\lambda) & \text{for } \lambda\in[\frac{3\varepsilon}{4},\varepsilon]\end{cases}\qquad \begin{aligned}P_0 &= (T_0,V_0,R_0,\vec{X}_0) &&=\gamma(0)\\ P_1 &= (T_0+\tfrac{\varepsilon}{4},V_1,R_1,\vec{X}_0) &&=\gamma(\tfrac{\varepsilon}{4})\\ P_2 &= (T_0+\tfrac{\varepsilon}{2},V_1,R_1,\vec{X}_f) &&=\gamma(\tfrac{\varepsilon}{2})\\ P_3 &= (T_0+\tfrac{3\varepsilon}{4},V_2,R_1,\vec{X}_f) &&=\gamma(\tfrac{3\varepsilon}{4})\\ P_f &= (T_0+\varepsilon,V_f,R_f,\vec{X}_f) &&=\gamma(\varepsilon)\end{aligned} \tag{C.1}$$

$$\dot\gamma(\lambda)=\begin{cases}\dot\gamma_1(\lambda)=\left(1,\frac{8(R_1-R_0)^2}{\varepsilon^2}-\frac{\beta^2}{2R(\lambda)^2},\frac{4(R_1-R_0)}{\varepsilon},0\right) & \text{for } \lambda\in[0,\frac{\varepsilon}{4}]\\ \dot\gamma_2(\lambda)=\left(1,0,0,\frac{4(\vec{X}_f-\vec{X}_0)}{\varepsilon}\right) & \text{for } \lambda\in[\frac{\varepsilon}{4},\frac{\varepsilon}{2}]\\ \dot\gamma_3(\lambda)=\left(1,\frac{4(V_2-V_1)}{\varepsilon},0,0\right) & \text{for } \lambda\in[\frac{\varepsilon}{2},\frac{3\varepsilon}{4}]\\ \dot\gamma_4(\lambda)=\left(1,\frac{8(R_f-R_1)^2}{\varepsilon^2}-\frac{\beta^2}{2R(\lambda)^2},\frac{4(R_f-R_1)}{\varepsilon},0\right) & \text{for } \lambda\in[\frac{3\varepsilon}{4},\varepsilon]\end{cases} \tag{C.2}$$

One sees that γ_1 and γ_4 are timelike by construction without requiring anything else, while in order for the curve to be timelike along the segments γ_2 and γ_3 one needs to satisfy the inequality in (3.2), leading to the conditions

$$\begin{aligned}\tfrac{\beta^2}{R_1^2}+\omega^2\left(\vec{X}(\lambda)^2+R_1^2\right) &> \tfrac{16(\vec{X}_f-\vec{X}_0)^2}{\varepsilon^2} \quad \text{along } \gamma_2\\ \tfrac{\beta^2}{R_1^2}+\omega^2\left(R_1^2+\vec{X}_f^2\right) &> \tfrac{8}{\varepsilon}(V_1-V_2) \quad \text{along } \gamma_3\end{aligned} \tag{C.3}$$

where V_1-V_2 can be expressed in terms of the arbitrary starting and end points as

$$V_1-V_2=V_0-V_f+\frac{2}{\varepsilon}(R_1-R_0)^2+\frac{2}{\varepsilon}(R_f-R_1)^2-\beta^2\frac{\varepsilon}{8R_0R_1}-\beta^2\frac{\varepsilon}{8R_1R_f}\,. \tag{C.4}$$

When $\beta\neq 0$, the conditions (C.3) can be satisfied for any beginning and endpoints P_0 and P_f of the curve, in particular for any given $\epsilon=T_f-T_0\neq 0$, by choosing R_1 small enough (i.e. by taking the path connecting the two points to go sufficiently close to the boundary at $R=0$). We thus find that the chronological future (past) of any point $(T_0,V_0,R_0,\vec{X}_0)$ is the entire set of points with $T>T_0$ ($T<T_0$). In particular, all points on an equal time slice Σ_{T_0} have identical future and past, and in this sense the space-time is *maximally non-distinguishing*. This argument also shows precisely how the construction of this curve, and hence the argument, breaks down for $\beta=0$ (plane wave AdS).

D Geodesics

In this appendix we describe those properties of the solutions to the geodesic equations that are relevant for our purposes. We do not give the explicit solutions to the geodesic equations.

The geodesic equations are

$$\dot T = P_-R^2\,, \tag{D.1}$$

$$\dot V = ER^2-\beta^2P_- -\omega^2P_-R^2(R^2+\vec{X}^2)\,, \tag{D.2}$$

$$\frac{1}{R^2}\frac{d}{d\lambda}\left(\frac{1}{R^2}\dot{\vec{X}}\right) = -\omega^2 P_-^2 \vec{X}\,, \tag{D.3}$$

$$k = \beta^2 P_-^2 + \omega^2 P_-^2 R^4 + (\vec{P}^2 - 2P_- E)R^2 + \frac{\dot{R}^2}{R^2}\,, \tag{D.4}$$

where E, P_- and $\vec{P}$ are integration constants and the dot indicates differentiation with respect to λ which depending on $k = 0, \pm 1$ is either proper time, proper length or some affine parameter. P_- is the lightcone momentum conjugate to V, and solutions to the geodesic equation with $P_- = 0$ either have $k = 0$ (these are lightlike lines - see section 3) or $k = 1$. In this appendix we will always assume that $P_- \neq 0$. There are three families of solutions that depend on whether $\kappa = k - \beta^2 P_-^2$ is negative, zero or positive.

When $\beta = 0$ we have $\kappa = k$ and the three cases split into timelike, null and spacelike geodesics. When $\beta \neq 0$ this does not happen. Both the timelike and the null geodesics are sitting in the $\kappa < 0$ class of solutions, while the spacelike geodesics are divided among all three classes with the $\kappa = 0$ and $\kappa > 0$ classes containing only spacelike geodesics.

Geodesics with $\kappa < 0$ describe bounded motion on $0 < R < \infty$ and never reach the points $R = 0$ and $R = \infty$. Since the $\kappa < 0$ class of solutions also contains spacelike geodesics not all spacelike geodesics go to the boundary. The motion for $\kappa < 0$ is periodic in the R and $\vec{X}$ directions with periods π/ω and $2\pi/\omega$, respectively. The motion in the V direction (for non-compact V) is however not periodic. This is due to the second term (containing β) on the right hand side of (D.2). This term would not be there in plane wave AdS. For compact V the periodicity of the motion in the R and $\vec{X}$ directions does not generically coincide with the periodicity of identifications $V \sim V + 2\pi L$.

All geodesics with $\kappa < 0$ that go through some point P, say, also go through the point $\bar{P}$ with coordinates

$$\left(T_{\bar{P}}, V_{\bar{P}}, R_{\bar{P}}, X^a_{\bar{P}}\right) = \left(T_P + \frac{\pi}{\omega}, V_P - \beta^2 \Delta V, R_P, -X^a_P\right)\ , \tag{D.5}$$

where ΔV is some β independent difference that depends on the locations of P and $\bar{P}$ as well as on the parameters of the geodesic connecting P and $\bar{P}$. In AdS points P and $\bar{P}$ are examples of antipodal points.

It follows from the periodicity of the $\kappa < 0$ class of geodesics that points P and Q with $T_Q - T_P = \pi/\omega$ and with $R_Q \neq R_P$ can never be connected by a $\kappa < 0$ geodesic. Such points P and Q can also not be connected by $\kappa = 0$ or $\kappa > 0$ geodesics because those reach the boundary within a time interval of π/ω or less. This proves that the Schrödinger space-time (just as AdS) is not geodesically connected. In the case of AdS, the geodesic disconnectedness can be compactly described in terms of the invariant distance η^{AdS}: if $\eta^{\mathrm{AdS}}(x, x') \leq -1$ and $x' \neq \bar{x}$, then there is no geodesic connecting x and x'. In particular, for $\beta = 0$ the above pair of points P, Q provides an example of such a pair of points since $\eta^{\mathrm{AdS}}(x_P, x_Q) < -1$.

E AdS commutator function

The AdS Wightman functions, $G^{\pm}_{\mathrm{AdS}}(x, x')$, can be obtained via

$$G^{\pm}_{\mathrm{AdS}}(x, x') = \int_{-\infty}^{\infty} dm G^{\pm}_{\beta=0}(x, x')\,. \tag{E.1}$$

In order to perform the integral over m we allude to the following result taken from [41]

$$\int_0^\infty dx e^{-\alpha x} J_\gamma(\beta x) x^{\mu-1} = \frac{\left(\frac{\beta}{2\alpha}\right)^\gamma \Gamma(\gamma+\mu)}{\alpha^\mu \Gamma(\gamma+1)} F\left(\frac{\gamma+\mu}{2}, \frac{\gamma+\mu+1}{2}; \gamma+1; -\frac{\beta^2}{\alpha^2}\right), \tag{E.2}$$

where F is the hypergeometric function and where we must have

$$\mathrm{Re}\,(\mu+\gamma) > 0 \qquad \text{and} \qquad \mathrm{Re}(\alpha \pm i\beta) > 0\,. \tag{E.3}$$

By taking $x = m$, $\alpha = -i\eta_{-\epsilon}$, $\beta = \zeta_{-\epsilon}$, $\gamma = \pm\nu$, and $\mu = \frac{d+2}{2}$ we obtain for G^+_{AdS}

$$G^+_{\mathrm{AdS}}(x,x') = C_{\Delta_\pm}\left(\eta^{\mathrm{AdS}}_{-\epsilon}\right)^{-\Delta_\pm} F\left(\frac{\Delta_\pm}{2}, \frac{\Delta_\pm+1}{2}; \Delta_\pm - \frac{d}{2}; (\eta^{\mathrm{AdS}}_{-\epsilon})^{-2}\right), \tag{E.4}$$

where $C_{\Delta_\pm}$ is given by

$$C_{\Delta_\pm} = \frac{\Gamma(\Delta_\pm)}{2^{\Delta_\pm}\pi^{\frac{d+2}{2}}(2\Delta_\pm - d - 2)\Gamma(\Delta_\pm - \frac{d}{2} - 1)}, \tag{E.5}$$

and where $\eta^{\mathrm{AdS}}_{-\epsilon}$ is given by

$$\eta^{\mathrm{AdS}}_{-\epsilon} = \frac{\eta_{-\epsilon}}{\zeta_{-\epsilon}}\,. \tag{E.6}$$

Similarly, with $\alpha = i\eta_{+\epsilon}$ and the same choices for β, γ and μ, we obtain for $G^-_{\mathrm{AdS}}(x,x')$ the same expression as we have for G^+_{AdS} but this time as a function of $\eta^{\mathrm{AdS}}_{+\epsilon} = \frac{\eta_{+\epsilon}}{\zeta_{+\epsilon}}$. Note that the $i\epsilon$ prescription is such that the conditions (E.3) for α and β are fulfilled. To see this more explicitly use the fact that to first order in ϵ we have

$$\zeta_{\pm\epsilon} = \zeta \mp i\omega^2\epsilon \frac{RR'\cos\omega(T-T')}{\sin^2\omega(T-T')}, \tag{E.7}$$

$$\eta_{\pm\epsilon} = \eta \mp i\frac{\omega^2\epsilon}{2}\frac{R^2+R'^2+\vec{X}^2+\vec{X}'^2-2\vec{X}\cdot\vec{X}'\cos\omega(T-T')}{\sin^2\omega(T-T')}\,. \tag{E.8}$$

Consider the commutator function $[\phi(x),\phi(x')] = G^+_{\mathrm{AdS}}(x,x') - G^-_{\mathrm{AdS}}(x,x')$. As long as $|\eta^{\mathrm{AdS}}_{\pm\epsilon}| > 1$, where $\eta^{\mathrm{AdS}}_{\pm\epsilon} = \eta_{\pm\epsilon}/\zeta_{\pm\epsilon}$, the hypergeometric function in $G^\mp_{\mathrm{AdS}}$ is defined by its series expansion and is thus single-valued for any ϵ. Since the series converges absolutely for $|\eta^{\mathrm{AdS}}| > 1$, i.e. for $\epsilon = 0$, we can take the limit $\epsilon \to 0$ and we get that for $\epsilon \to 0$ the commutator function $G^+_{\mathrm{AdS}} - G^-_{\mathrm{AdS}}$ vanishes for $|\eta^{\mathrm{AdS}}| > 1$. This result is in agreement with the region where the retarded AdS Green function vanishes[11] [42].

Any two points x and x' in AdS for which $\eta^{\mathrm{AdS}}(x,x') > 1$ are spacelike separated. Any two points x and $x' \neq \bar{x}$ for which $\eta^{\mathrm{AdS}}(x,x') \leq -1$ cannot be connected by any geodesic. The fact that the commutator function vanishes for spacelike separated points is often referred to as microcausality. The fact that the commutator function also vanishes for points x and x' for which $\eta^{\mathrm{AdS}}(x,x') \leq -1$ follows from the fact that the commutator

[11] For values $|\eta^{\mathrm{AdS}}| < 1$ the hypergeometric function in the expression for $G^\pm_{\mathrm{AdS}}$ is defined via its analytic continuation. This analytic continuation does depend on whether or not the function depends on $\eta^{\mathrm{AdS}}_{-\epsilon}$ or on $\eta^{\mathrm{AdS}}_{+\epsilon}$.

function (in the case of a free theory) is a classical object which must vanish for points that cannot be connected by a classical path of propagation. The commutator function is a continuous function of η^{AdS} and since it vanishes for $\eta^{\text{AdS}} > 1$ it also vanishes for $\eta^{\text{AdS}} = 1$. Points x and x' for which $\eta^{\text{AdS}}(x, x') = 1$ are separated by a null geodesic. It turns out that in AdS all null geodesics are also lightlike lines, that is achronal sets.

Summarising, we conclude that we have

$$\lim_{\epsilon \to 0} \left(G^{+}_{\text{AdS}} - G^{-}_{\text{AdS}} \right) = 0 \qquad \text{for} \qquad |\eta^{\text{AdS}}| \geq 1 \,. \tag{E.9}$$

References

[1] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry*, Phys. Rev. **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[2] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, Phys. Rev. Lett. **101** (2008) 061601 [arXiv:0804.4053] [SPIRES].

[3] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, JHEP **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[4] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, JHEP **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[5] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, JHEP **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[6] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, Phys. Rev. **D 78** (2008) 106005 [arXiv:0808.1725] [SPIRES].

[7] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class. Quant. Grav. **26** (2009) 224002 [arXiv:0903.3246] [SPIRES].

[8] J. McGreevy, *Holographic duality with a view toward many-body physics*, arXiv:0909.0518 [SPIRES].

[9] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [SPIRES].

[10] S.F. Ross and O. Saremi, *Holographic stress tensor for non-relativistic theories*, JHEP **09** (2009) 009 [arXiv:0907.1846] [SPIRES].

[11] M. Blau, J. Hartong and B. Rollier, *Geometry of Schrödinger space-times, global coordinates and harmonic trapping*, JHEP **07** (2009) 027 [arXiv:0904.3304] [SPIRES].

[12] G.T. Horowitz and D. Marolf, *Quantum probes of space-time singularities*, Phys. Rev. **D 52** (1995) 5670 [gr-qc/9504028] [SPIRES].

[13] M. Blau, D. Frank and S. Weiss, *Scalar field probes of power-law space-time singularities*, JHEP **08** (2006) 011 [hep-th/0602207] [SPIRES].

[14] V.E. Hubeny, M. Rangamani and S.F. Ross, *Causal structures and holography*, JHEP **07** (2005) 037 [hep-th/0504034] [SPIRES].

[15] S. Hawking and G. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge U.K. (1973) [SPIRES].

[16] E. Minguzzi and M. Sanchez, *The causal hierarchy of spacetimes*, gr-qc/0609119 [SPIRES].

JHEP07(2010)069

[17] Y. Nishida and D.T. Son, *Nonrelativistic conformal field theories*, *Phys. Rev.* **D 76** (2007) 086004 [arXiv:0706.3746] [SPIRES].

[18] D. Brecher, A. Chamblin and H.S. Reall, *AdS/CFT in the infinite momentum frame*, *Nucl. Phys.* **B 607** (2001) 155 [hep-th/0012076] [SPIRES].

[19] J. Rosen, *Embedding of various relativistic Riemannian spaces in pseudo-Euclidean spaces*, *Rev. Mod. Phys.* **37** (1965) 204.

[20] C. Collinson, *Embeddings of the plane-fronted waves and other space-times*, *J. Math. Phys.* **9** (1968) 403.

[21] M. Blau, J.M. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. Quant. Grav.* **19** (2002) 4753 [hep-th/0202111] [SPIRES].

[22] M. Blau and M. O'Loughlin, *Homogeneous plane waves*, *Nucl. Phys.* **B 654** (2003) 135 [hep-th/0212135] [SPIRES].

[23] E. Minguzzi, *Chronological spacetimes without lightlike lines are stably causal*, *Commun. Math. Phys.* **288** (2009) 801 [arXiv:0806.0153] [SPIRES].

[24] V.E. Hubeny, M. Rangamani and S.F. Ross, *Causal inheritance in plane wave quotients*, *Phys. Rev.* **D 69** (2004) 024007 [hep-th/0307257] [SPIRES].

[25] P. Hořava and C.M. Melby-Thompson, *Anisotropic conformal infinity*, arXiv:0909.3841 [SPIRES].

[26] T. Heinzl, *Light-cone quantization: Foundations and applications*, *Lect. Notes Phys.* **572** (2001) 55 [hep-th/0008096] [SPIRES].

[27] S. Hellerman and J. Polchinski, *Compactification in the lightlike limit*, *Phys. Rev.* **D 59** (1999) 125002 [hep-th/9711037] [SPIRES].

[28] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys.* **144** (1982) 249 [SPIRES].

[29] S. Moroz, *Below the Breitenlohner-Freedman bound in the nonrelativistic AdS/CFT correspondence*, *Phys. Rev.* **D 81** (2010) 066002 [arXiv:0911.4060] [SPIRES].

[30] P. Minces and V.O. Rivelles, *Energy and the AdS/CFT correspondence*, *JHEP* **12** (2001) 010 [hep-th/0110189] [SPIRES].

[31] E.D. Rainville, *Special functions*, Chelsea Publ. Co., Bronx New York U.S.A. (1971).

[32] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Third Edition, Springer-Verlag, Berlin Germany (1966).

[33] A. Bezubik and A. Strasburger, *A new form of the spherical expansion of zonal functions and Fourier transforms of* SO(d)*-finite functions*, *SIGMA* **2** (2006) 033 [math-ph/0603011].

[34] A. Volovich and C. Wen, *Correlation functions in non-relativistic holography*, *JHEP* **05** (2009) 087 [arXiv:0903.2455] [SPIRES].

[35] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253 [SPIRES].

[36] R.G. Leigh and N.N. Hoang, *Real-time correlators and non-relativistic holography*, *JHEP* **11** (2009) 010 [arXiv:0904.4270] [SPIRES].

[37] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104] [SPIRES].

[38] D. Marolf and S.F. Ross, *Boundary conditions and new dualities: vector fields in AdS/CFT*, *JHEP* **11** (2006) 085 [hep-th/0606113] [SPIRES].

[39] J. Moore and R. Schlafly, *On equivariant isometric embeddings*, *Math. Z.* **173** (1980) 119.

[40] S. Schäfer-Nameki, M. Yamazaki and K. Yoshida, *Coset construction for duals of non-relativistic CFTs*, *JHEP* **05** (2009) 038 [arXiv:0903.4245] [SPIRES].

[41] I.S. Gradshteyn and I.H. Ryzhik, *Tables of integrals, series and products*, Academic, New York U.S.A. (1980).

[42] U.H. Danielsson, E. Keski-Vakkuri and M. Kruczenski, *Vacua, propagators and holographic probes in AdS/CFT*, *JHEP* **01** (1999) 002 [hep-th/9812007] [SPIRES].

[43] G. Compere, S. de Buyl, S. Detournay and K. Yoshida, *Asymptotic symmetries of Schrödinger spacetimes*, *JHEP* **10** (2009) 032 [arXiv:0908.1402] [SPIRES].

[44] J. Hartong, E. Imeroni and B. Rollier, *Thermodynamic properties of TsT transformed AdS black holes*, work in progress.

JHEP07(2010)069