

## On holographic superconductors with DC current

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**ABSTRACT:** We study direct currents in a simple holographic realization of a superconducting film. We investigate how the presence of a DC current affects the superconducting phase transition, which becomes first order for any non-vanishing value of the current, as well as several other properties of the superconductor such as the AC conductivity. Near the critical temperature we find a quantitative agreement with several properties of Ginzburg-Landau superconducting films, for example the squared ratio of the maximal and minimal condensate is equal to two thirds. We also comment on the extension of our construction to holographic Josephson junctions.

**KEYWORDS:** AdS-CFT Correspondence, Spontaneous Symmetry Breaking, Black Holes

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## 1 Introduction

In [1] it was shown that a black hole solution in a theory with a charged scalar field coupled to Maxwell-Einstein gravity may become classically unstable below some critical temperature  $T_c$ . This instability induces charged scalar hair for the black hole for  $T < T_c$ . According to the AdS/CFT duality, the holographic dual of such a system is a thermal quantum field theory in flat Minkowski space with a global U(1) symmetry, which is spontaneously broken below  $T_c$  by the condensation of the operator dual to the bulk complex scalar. In this sense, the boundary theory has many of the necessary ingredients to describe a superconductor, or a superfluid [2]. This result was first exploited in [3] to assemble a gravity dual of a system undergoing a superconducting phase transition. This construction has been widely studied and generalized, and the different gravity duals sharing these same basic features go under the name of holographic superconductors (see [4–6] for reviews and references).

In a holographic superconductor, a background magnetic field induces a current. However, because of the absence of a dynamical gauge field, this current does not expel the magnetic field, unlike the current induced in ordinary superconductors. In this sense holographic superconductors more closely resemble thin superconducting films or wires. Motivated by this analogy, in the present note we will compare and contrast holographic superconductor phenomenology with that of superconducting films. More precisely, we study a holographic superconductor in two spatial dimensions, i.e. a (extremely) thin superconducting film, with a DC current, analyzing its phase diagram and the behavior of some interesting thermodynamic quantities.

In fact, such a system is interesting to study for a second, more ambitious, reason. It is believed that holographic superconductors may give an understanding of some basic features of high temperature superconductors (HTS). HTS typically enjoy a layered structure and, according to the Lawrence-Doniach model [7], may be approximated by films of superconductors separated by Josephson junctions. Therefore, a holographic realization of a Josephson junction would be desirable. The latter is based on the Josephson effect [8], the phenomenon of current flow across two weakly coupled superconductors separated by a very thin insulating barrier (the Josephson junction). Our model can then be seen as a first, necessary step towards the realization of a Josephson junction.

We pursue the phenomenological approach of [3] and therefore the gravitational system we consider is Einstein-Maxwell theory in four dimensions minimally coupled to a charged massive scalar field. In [3] the following basic set up was considered: the condensation of the charged scalar in a black hole metric at finite charge density. The black hole introduces a temperature  $T$ . The finite charge density, which is taken care of by allowing for a non trivial profile for the temporal component  $A_0$  of the gauge field, provides an independent scale needed to get a critical temperature  $T_c$ . One then finds that for temperatures below  $T_c$  the charged scalar condenses. Under the AdS/CFT map gauge symmetries on the gravity side are dual to global ones on the field theory. Then the condensation of the scalar nicely realizes the spontaneous breaking of a global U(1) symmetry.

We want to modify this basic scenario and allow for the presence of a DC current. To this end, we consider solutions where also a spatial component of the gauge field has a non trivial profile, this providing, via the AdS/CFT map, a current in the dual theory. Such solutions can be easily found in the superconducting phase, where the scalar is non-zero. However, as pointed out in [9], in the normal state (where the symmetry is not broken and the scalar is hence vanishing) the only allowed solutions for the spatial components of the gauge field are the trivial ones. For this reason, within the minimal Einstein-Maxwell framework, we cannot construct a model describing the normal state with DC current. However, as we will discuss in detail, this inconvenience will not impede us to obtain some robust results characterizing the behavior of holographic superconductors at fixed DC current.

An additional limitation of our approach comes from the fact that, as in [3], we work in the probe approximation. This is the limit where the backreaction of the gauge and scalar fields on the metric is neglected. Hence our results are reliable only in the regime where the backreaction can be effectively neglected. Luckily, while the probe approximation breaks down in the zero temperature limit, for temperatures significantly different from zero the results obtained in the probe limit are not substantially modified by the backreaction [10]. There is therefore a large region where our results should not be sensibly different from a honest fully back-reacted model.<sup>1</sup>

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<sup>1</sup>In [11, 12], it has been shown that a phenomenological model of the likes of [3] can be consistently embedded in M-theory or type IIB String Theory. Such embeddings constitute an important advance towards the understanding of the underlying microscopic theory of the holographic superconductors. Unfortunately, sticking to the probe approximation prevents us from using the models of [11, 12] since the charge there is fixed to a finite value, while the probe approximation holds in the large charge limit.

The same model we are going to study here was already considered in [9, 13]. Differently from those analyses we study the system at fixed current. This choice, beside being closer in spirit to real-life experiments, allows us to obtain new results about the phase diagram of this system together with interesting checks and predictions for the behavior of holographic superconductors with DC current, as we now summarize:

- At any finite DC current the transition between the superconducting and the normal state is a first order phase transition. We study the temperature dependence of the condensate and compute the free energy in the superconducting state, concluding that at the phase transition the condensate always jumps a finite distance to zero. This is a clear indication of a first order phase transition.
- We determine the relation between the current and the superfluid velocity. We largely find nice agreement with expectations for physical superconducting films, both for temperatures appreciably lower than, and close to,  $T_c$ . Moreover, it turns out that the form of these curves further justifies the assertion in the previous point, namely that the phase transition is first order. Interestingly, at low temperatures we find that, in contrast with BCS superconducting films, for each value of the superfluid velocity there are two possible values of the current. A free energy computation then shows that only one value, in fact the highest one, is thermodynamically stable.
- We study the temperature dependence of the critical current and of the ratio given by the value of the condensate at zero current over the value at the critical current. For temperatures close to  $T_c$  we find that the holographic superconductors reproduce the universal results predicted by the Ginzburg-Landau (GL) model for superconducting films. On the other hand, at lower temperatures our results deviate significantly from the ones of GL.
- Finally, we study the dependence of the AC conductivity on the DC current. We present results for the conductivity in the direction transverse to the current. At low temperatures we analyze the dependence of the frequency gap on the DC current. As expected on physical grounds, as we increase the current the frequency gap diminishes, and it does so down to a minimal (but not vanishing) value where the first order phase transition occurs.

The rest of this note is organized as follows. In section 2 we present the bulk Lagrangian, the equations of motion which we have solved numerically, and motivate our ansatz and boundary conditions. In section 3 we present the physical output of our numerical studies, namely the checks and predictions mentioned above. Section 4 contains our conclusions as well as a possible strategy for constructing a holographic dual of (an array of) Josephson junctions.

## 2 The gravity dual of a DC superconductor

As advertised, we pursue a bottom-up approach to holographic superconductivity, and consider as a starting point the model originally presented in [3], Einstein-Maxwell theory

in 4-dimensions minimally coupled to a charged, massive scalar field. We stick to the probe approximation and in this case the action of the scalar-Einstein-Maxwell theory reduces to

$$\mathcal{S} = \int dx^4 \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu - i A_\mu) \Psi|^2 - m^2 \Psi^* \Psi \right], \quad (2.1)$$

where the Einstein-Hilbert term has been suppressed, since the backreaction of the fields on the metric can be ignored in the probe limit (the Einstein equations decouple).<sup>2</sup>  $F_{\mu\nu}$  is the U(1) field strength,  $\Psi$  is the complex scalar with charge 1 and mass  $m$ , and  $g$  is the determinant of the metric  $g_{\mu\nu}$ , which we take to be the asymptotically AdS planar black hole metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2} (dx^2 + dy^2) \quad \text{where} \quad f(r) = \frac{r^2}{L^2} - \frac{M}{r}. \quad (2.2)$$

The radial direction extends from the black hole horizon at  $r = r_0 = (ML^2)^{1/3}$  to the boundary of AdS at  $r \rightarrow \infty$ ,  $L$  is the radius of AdS and  $M$  the mass of the black hole. Beside the holographic coordinate  $r$ , we have three others  $(t, x, y)$ , which parametrize the AdS boundary and hence the (2+1)-dimensional dual field theory space-time.

The temperature of the black hole (and hence of the dual field theory) is given by

$$T = \frac{3}{4\pi L^2} r_0. \quad (2.3)$$

As in [3] we will take the scalar mass to be  $m^2 = -2/L^2$  which is above the Breitenlohner-Friedman bound.

## 2.1 The ansatz

According to the AdS/CFT map, the VEV of the U(1) current in the dual field theory is identified with the subleading boundary asymptotics of the bulk gauge field. Hence, to describe holographically a superconductor with a DC current, we need to look for bulk solutions where the black hole develops charged scalar hair in the presence of a non-trivial profile for a spatial component of the gauge field. More precisely, we are interested in getting a current in the  $x$  direction, therefore we will look for solutions which are independent of the time coordinate  $t$  and of  $y$ , but with a non-trivial dependence on both  $r$  and  $x$ . We choose the gauge  $A_r = 0$  (this leaves the freedom to perform  $r$ -independent gauge transformations, as we will do later). As our solutions are  $y$ -independent, we set  $A_y = 0$ . Thus we must determine  $A_x$ ,  $A_t$ , and  $\Psi$  as functions of  $x$  and  $r$ .

We choose the modulus of the scalar to be independent of  $x$  and similarly for  $A_x$  and  $A_t$ . However, having a current then requires that the phase of  $\Psi$  be  $x$ -dependent. The simplest such ansatz reads

$$\Psi(r, x) = \psi(r) e^{i\theta x}, \quad (2.4)$$

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<sup>2</sup>Let us start from the standard Lagrangian  $\sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu - i q A_\mu) \Psi|^2 - m^2 |\Psi|^2 \right] + \text{Einstein}$ , rewrite it in terms of the rescaled fields  $\tilde{\Psi} = q \Psi$  and  $\tilde{A}_\mu = q A_\mu$  and take the limit  $q \rightarrow \infty$  while keeping  $\tilde{\Psi}$ ,  $\tilde{A}_\mu$  fixed. Then, the Lagrangian becomes  $\frac{1}{q^2} \sqrt{-g} \left[ -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - |(\partial_\mu - i \tilde{A}_\mu) \tilde{\Psi}|^2 - m^2 |\tilde{\Psi}|^2 \right] + \text{Einstein}$ . Due to the  $1/q^2$  factor the matter sources decouple from the Einstein equations and the dynamics of the vector and the scalar field are described by the action (2.1) in a vacuum solution of the Einstein equations.

which automatically satisfies the equations of motion for  $A_r$  and for the phase of  $\Psi$ , the latter imposing that  $\theta$  is indeed a constant. Summarizing,  $\psi$ ,  $A_x$  and  $A_t$  are functions of  $r$  while  $\theta$  is a constant. Notice that in a superconductor the spatial derivative of the phase of the condensate is the superfluid velocity [13, 14]. Therefore, in our case we are describing a superconductor with a constant superfluid velocity, since by the AdS/CFT map the latter is given by  $\theta$ .

Notice that a non-zero  $A_x$  contributes positively to the effective scalar mass, hence one expects that a sufficiently large  $A_x$  will win against the negative contribution coming from the time-component of the gauge potential, eventually destroying black hole superconductivity [9]. This corresponds to a critical maximal current in the dual field theory, above which the system enters the normal phase, which is indeed what is expected for physical superconductors.

The equations of motion following from the action (2.1) are

$$\partial_r (r^2 \partial_r A_t) - 2 \frac{r^2 \psi^2}{f} A_t = 0, \tag{2.5}$$

$$\partial_r (f \partial_r A_x) - 2\psi^2 (A_x - \theta) = 0, \tag{2.6}$$

$$\partial_r (r^2 f \partial_r \psi) - L^2 (A_x - \theta)^2 \psi + \frac{r^2 A_t^2}{f} \psi + \frac{2r^2}{L^2} \psi = 0. \tag{2.7}$$

These equations are invariant under two independent scaling symmetries

$$\begin{aligned} r \rightarrow \lambda r, \quad r_0 \rightarrow \lambda r_0, \quad L \rightarrow L, \quad \psi \rightarrow \psi, \quad A_t \rightarrow \lambda A_t, \quad A_x \rightarrow \lambda A_x, \quad \theta \rightarrow \lambda \theta, \\ r \rightarrow r, \quad r_0 \rightarrow r_0, \quad L \rightarrow \nu L, \quad \psi \rightarrow \nu^{-1} \psi, \quad A_t \rightarrow \nu^{-2} A_t, \quad A_x \rightarrow \nu^{-2} A_x, \quad \theta \rightarrow \nu^{-2} \theta. \end{aligned} \tag{2.8}$$

When performing numeric computations we find it convenient to work with variables and coordinates which are invariant with respect to these rescalings (and hence dimensionless). In our case they are

$$\frac{r}{r_0}, \quad \frac{L^2}{r_0} A_\mu, \quad \frac{L^2}{r_0} \theta, \quad L \psi. \tag{2.9}$$

The equations of motion written in terms of these rescaled and dimensionless quantities are the same as before with  $r_0$  and  $L$  set equal to one.

## 2.2 Asymptotics and their dual interpretation

The equations of motion (2.5)–(2.7) are second order, and so we expect six constants of integration, which together with  $\theta$  imply seven parameters. Regularity at the horizon sets  $A_t = 0$  at  $r = 1$  and this, via the equations of motion, imposes two more constraints on  $\psi$  and  $A_x$ . This leaves four parameters. In addition, there are boundary conditions at the boundary of AdS. The leading asymptotics of the fields at large  $r$  read

$$A_x = A_x^{(0)} - \frac{A_x^{(1)}}{r} + O(r^{-2}), \quad A_t = A_t^{(0)} - \frac{A_t^{(1)}}{r} + O(r^{-2}), \quad \psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + O(r^{-3}), \tag{2.10}$$

while  $\theta$  is constant everywhere. The leading contribution of the time and space components of the gauge field correspond, via the AdS/CFT map, to a chemical potential  $\mu$  and a source

for the  $x$ -component of the dual current, respectively. This source for the current will be nothing else than the superfluid velocity  $\nu_x$ . Since the gauge field is covariantly coupled to the scalar, on the boundary we get a term of the form  $\partial_x \varphi - A_x^{(0)} = \theta - A_x^{(0)}$  and we see that one can, through a gauge transformation, trade  $\theta$  for  $A_x^{(0)}$  [13, 14]. In fact, from now on we will choose to work in the gauge  $\theta = 0$  and identify the superfluid velocity with  $A_x^{(0)}$ . The subleading asymptotics correspond instead to the charge density  $\rho$  and the VEV of the current density  $J_x$ .

With our choice of scalar mass term we have two options for the corresponding dual operator. This is because both asymptotic behaviors of the scalar are normalizable at the boundary, so both of them can correspond to a VEV of a dual operator [15]. We can choose the leading asymptotic coefficient  $\psi^{(1)}$  to be a source of a scaling dimension 2 operator  $O_2$ . In this case the expectation value of  $O_2$  will be proportional to the subleading coefficient  $\psi^{(2)}$ . Or, we can choose  $\psi^{(2)}$  to be a source of an operator  $O_1$  with scaling dimension 1. In this case the VEV of  $O_1$  will be proportional to  $\psi^{(1)}$ . For definiteness, in what follows we will work with the operator  $O_2$  (the basic results are not qualitatively different for the opposite choice). To have spontaneous breaking of the U(1) symmetry, we want the source to vanish and hence we will impose  $\psi^{(1)} = 0$ .

In summary, we have seven parameters, three regularity conditions at the horizon plus the two conditions  $\psi^{(1)} = 0$ ,  $\theta = 0$ . Hence, we expect a two-parameter family of solutions, which may be parametrized, for example, by the temperature and current of the superconducting film. Given the temperature and current one may then calculate the value of the order parameter  $\langle O_2 \rangle$ .

Undoing the rescaling described above, we can rewrite the field theory quantities in terms of the asymptotic coefficients of the dimensionless fields. We find

$$\begin{aligned} \mu &= \frac{4\pi}{3} T A_t^{(0)}, & \rho = \langle J_t \rangle &= \frac{16\pi^2}{9} T^2 A_t^{(1)}, \\ \nu_x &= \frac{4\pi}{3} T A_x^{(0)}, & j_x \equiv \langle J_x \rangle &= \frac{16\pi^2}{9} T^2 A_x^{(1)}, & \langle O_2 \rangle &= 2 \frac{16\pi^2}{9} T^2 \psi^{(2)}, \end{aligned} \quad (2.11)$$

where we have used the field and coordinate redefinitions given by eq. (2.9), and written  $r_0$  in terms of the temperature via eq. (2.3).

As can be seen from the expressions for the chemical potential and charge density in eq. (2.11), the asymptotic behavior of  $A_t$  only determines the dimensionless ratios  $\mu/T$  and  $\rho/T^2$ . In other words, the gravity dual only gives us information about the dimensionless ratio of the two scales in the theory: the temperature and the charge density (or the chemical potential). One can then decide to use either the chemical potential or the charge density to fix a scale. We will use the former and study the evolution of  $\langle O_2 \rangle / \mu^2$  as a function of  $T/\mu$  and  $j_x/\mu^2$ . Accordingly, using eq. (2.11), we define  $T_c$ , the critical temperature at zero current, as

$$\frac{T_c}{\mu} = \frac{3}{4\pi} \frac{1}{A_t^{(0)}|_c}, \quad (2.12)$$

where  $A_t^{(0)}|_c$  is the critical value of  $A_t^{(0)}$  for which the condensate turns on at  $j_x = 0$ . When studying the thermodynamics of the system we will work in the grand canonical ensemble, which corresponds to a system at fixed chemical potential.

### 3 Holographic predictions

We have numerically solved the system of coupled differential equations (2.5)–(2.7) and determined the condensate  $\langle O_2 \rangle$  as a function of the current and the temperature. Using a shooting technique we have integrated the equations from the horizon up to the boundary, with the boundary conditions discussed before. Via the AdS/CFT maps detailed in eqs. (2.11), we have then determined the surface of solutions for the condensate as a function of the current and the temperature.

#### 3.1 Phase transition with DC current

The first important thing we want to analyze is how the presence of the current modifies the temperature dependence of the condensate. This is shown in figure 1 for different values of the current and compared with the result at zero current obtained in [3]. Two significant modifications occur. First, at any finite value of the current one can see that the curve  $\langle O_2 \rangle$  vs  $T$  becomes bivaluated. There appears a new branch (the dotted line) corresponding to states where the value of the condensate is much lower. In the following, by computing the free energy we will see that the states with lower value of the condensate have a larger free energy than their counterparts with larger  $\langle O_2 \rangle$  at the same temperature. Therefore, this new branch corresponds to thermodynamically disfavored states.<sup>3</sup> Second and more importantly, we observe that the superconducting state exists up to a maximum value of the temperature (where the plot turns back). Crucially, at that point the value of the condensate is larger than zero. Therefore, at the phase transition the condensate must jump a finite distance to zero. Unless one fine tunes the parameters, such a jump will almost certainly change the energy and so require some latent heat, implying that the phase transition is first order. Moreover, as is expected on physical grounds, the temperature at which the phase transition occurs is always lower than  $T_c$ , the critical temperature at zero current, and its value decreases with increasing current.

This phase transition pattern is quite different from that of refs. [9, 13]. The analysis performed there corresponds to experiments where instead of the current, the superfluid velocity is kept fixed. There it was found that the superconducting phase is separated from the normal phase by a second order phase transition from zero superfluid velocity up to a tricritical point where the phase transition becomes first order and remains so up to the maximum velocity, where the phase transition would be at zero temperature (similar results were found in [16], in the context of superconducting D-brane models). In fact, as we will see in section 3.2, the different phase transition pattern one finds when working at finite current agrees with what is known about the relation between the current  $j_x$  and the superfluid velocity  $v_x$  in superconducting films.

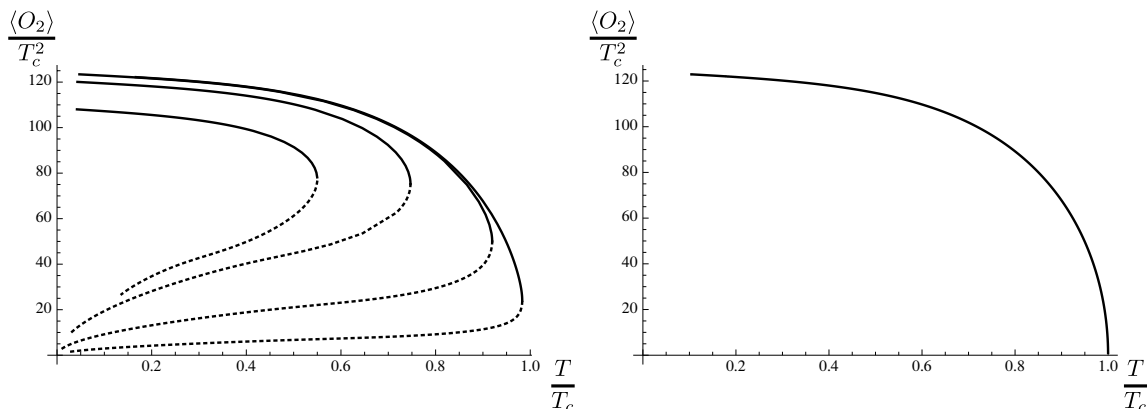
##### 3.1.1 The free energy

In order to confirm our previous claim, namely that the states with lower value of the condensate are metastable, we shall now compute the free energy of the superconducting phase and show that it is larger for the metastable branch (dotted line in figure 1).

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<sup>3</sup>Notice that this branch smoothly joins the (unstable) normal phase branch of [3] in the  $J_x \rightarrow 0$  limit.





**Figure 1.** On the left we plot  $\langle O_2 \rangle$  versus the temperature for several values of the current: from the innermost to the outermost  $j_x/T_c^2 = 28.98, 14.49, 2.90, 0.290$ . The dotted lines correspond to the states with larger free energy than their counterparts at the same temperature. On the right we show for comparison the result at zero current. Notice that at the critical temperature  $\langle O_2 \rangle$  vanishes in this case.

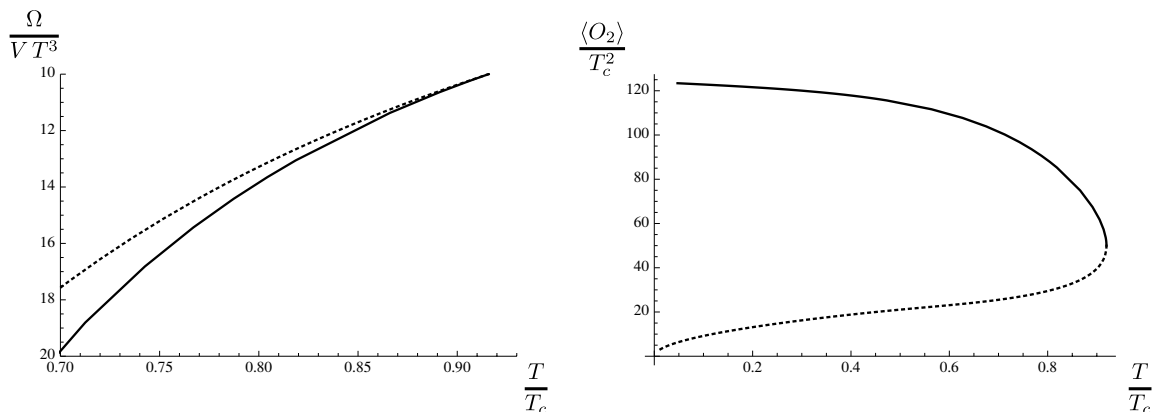
The free energy of the system is determined by the action (2.1) evaluated on-shell  $\Omega = -T S_{\text{os}}$  plus possible boundary counterterms [17]. For the present case the regularized action was presented in [13]. We shall proceed along those lines to compute the free energy of the physical configuration we are interested in. Substituting the equations of motion (2.5)–(2.7) into the action (2.1) one finds

$$S_0 = \int d^3x \left( \frac{r^2}{2} A_t A_t' - \frac{f}{2} A_x A_x' - r^2 f \psi \psi' \right) \Big|_{r=\infty} + \int d^4x \left( \psi^2 A_x^2 - \frac{r^2}{f} \psi^2 A_t^2 \right), \quad (3.1)$$

which is the unregularized on-shell action (the prime means derivative with respect to  $r$ ). This action consists of three boundary terms resulting from the kinetic terms of the temporal and spatial components of the gauge field, and the scalar, respectively; plus a bulk contribution coming from the interaction terms. From the asymptotic behavior of the fields (2.10) it follows that only the boundary term corresponding to the scalar field  $\psi$  is divergent, and thus we need to add the corresponding counterterm. Moreover, one must specify the boundary conditions which are imposed at infinity on the various fields. In our case, one should add boundary terms which take us to an ensemble where  $\psi^{(2)}, A_t^{(0)}$  and  $A_x^{(1)}$  are held fixed, corresponding via eq. (2.11) to  $\langle O^{(2)} \rangle$ , the chemical potential  $\mu$  and the current  $j_x$ . All in all, the boundary term that does the whole job reads (see [18] for a rigorous analysis)<sup>4</sup>

$$\int d^3x \left( r^3 \psi^2 + 2r^4 \psi \psi' + r^2 A_x A_x' \right) \Big|_{r=\infty}. \quad (3.2)$$

<sup>4</sup>Varying  $A_\mu$  in the bulk yields a boundary term  $\sim A_\mu' \delta A_\mu$ , which provides a good variational principle for a boundary condition  $\delta A_\mu|_{r=\infty} = 0$ . This corresponds to an ensemble where we are keeping fixed the asymptotic value of  $A_\mu$ , which in our case means fixed chemical potential ( $\sim A_t|_{r=\infty}$ ) and fixed source of the current ( $\sim A_x|_{r=\infty}$ ). One can go to an ensemble where  $\delta S \sim A_\mu \delta A_\mu'$  by adding a boundary term  $\sim r^2 A_\mu A_\mu'|_{r=\infty}$ .



**Figure 2.** For a value of the current  $j_x/T_c^2 = 2.9$ , we plot the free energy of the superconducting phase zooming in on the region in which  $T$  is closer to its maximum value (where the plot turns back). The dashed line corresponds to the points with a lower value of the condensate at a given temperature. We show on the right the corresponding plot of  $\langle O_2 \rangle$  versus  $T$ . One can see that the lower branch (dotted line) corresponds indeed to states with larger free energy and thus metastable.

Substituting the behavior of the fields written in eq. (2.10) into the regularized on-shell action given by the sum of the contributions (3.1) and (3.2) yields the following expression for the free energy

$$\frac{\Omega(\mu, j_x, \langle O_2 \rangle)}{T^3 V} = - \left[ \frac{1}{2} \left( A_t^{(0)} A_t^{(1)} + A_x^{(0)} A_x^{(1)} \right) - \psi^{(1)} \psi^{(2)} \right] - \int dr \left( \psi^2 A_x^2 - \frac{r^2}{f} \psi^2 A_t^2 \right), \quad (3.3)$$

where  $V$  stands for the volume of the system.

We can now compute the free energy of the superconducting states making up the plot in figure 1 and confirm that the lower branch (dotted line) is metastable. This is shown in figure 2. Notice, however, that we cannot determine precisely at which value of the temperature the phase transition occurs. In order to know this, we would need to compare the free energy of the superconducting state with the free energy of the normal state at the same value of the current. Unfortunately, within the minimal framework we are using, it does not seem possible to describe such a normal state. Let us elaborate a bit more on this. Naively, the first thing one could try to do is to look for a solution with non-trivial  $A_t$  and  $A_x$  but vanishing scalar ( $\psi = 0$ ). However, as noticed in [9], the only such solution satisfying regularity conditions at the horizon has  $A_x = 0$  identically. This result should be expected on physical grounds. In the normal state the dual system is no longer superconducting and thus one expects that in the absence of an electric field the current must vanish (the DC conductivity is now finite). One could then try to switch on a background electric field in the  $x$  direction through the addition of a contribution of the form  $-E \cdot t$  to  $A_x$ . However, in the Maxwell action having  $F_{tx} = -E$  does not modify the equation of motion for  $A_x$  which is then still divergent. One would expect that using a richer model describing non-linear interactions between the bulk fields could solve the problem. Indeed, as shown in [19] and more recently in [20], a DBI action does provide

a solution with non-vanishing current in the normal state. There the conductivity was computed and found to depend both on the electric field and the charge density. Yet it is not clear how to implement the scalar condensation corresponding to the superconducting phase in this scenario.<sup>5</sup> A different approach, closer in spirit to the model we are dealing with here, consists in going beyond the probe approximation, thus looking for a solution which in the normal state would correspond to an asymptotically AdS charged black hole with vector hair (with  $A_t(r)$  and  $A_x = -E \cdot t + h(r)$ ).

Let us emphasize that although within the probe approximation regime we cannot compute the free energy of a normal state with current, the conclusion about the phase transition being first order is robust. It is clear from our computations (figure 1) that at the maximum temperature the value of the condensate is different from zero and then it must jump during the phase transition. If we were able to compute the free energy of the normal state it may be that the phase transition would occur at a value of the temperature somewhat lower than the maximum value. However, as we see from the plot, the condensate would still be different from zero at that point. Simply, the superconducting state would be metastable from the actual temperature of the phase transition up to the maximum temperature (as it happens for instance in [9, 13]).

### 3.2 Current and velocity

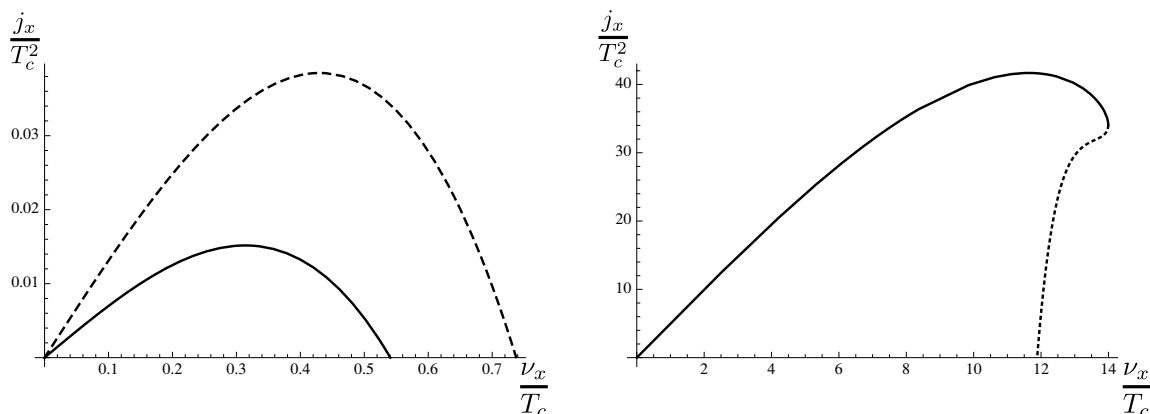
In this section we study the relation between the current  $j_x$  and the superfluid velocity  $\nu_x$ . As we explained at the beginning of section 3, the integration of the equations (2.5)–(2.7) results in a two-parameter family of solutions, which we chose to parametrize in terms of  $T$  and  $j_x$ . This means that once  $T$  and  $j_x$  are fixed all other physical quantities of interest are determined up to a discrete choice, in particular also the superfluid velocity  $\nu_x$ . We will now fix  $T$  and obtain a one-parameter curve of solutions relating  $j_x$  and  $\nu_x$ . The result is presented in figure 3 for several values of  $T$ . Close to the critical temperature (left panel) the relation is an upside down paraboloid which becomes smaller as the temperature approaches  $T_c$  (eventually shrinking to a point for  $T = T_c$ ). On the other hand, for low temperatures (right panel) the relation between  $j_x$  and  $\nu_x$  is linear almost all the way up to a given maximum velocity above which the superconducting state exists no more.

In general our results match nicely with what is known about the relation between the current and the superfluid velocity in thin superconducting films [14]. As we comment below, though, one qualitative difference with respect to BCS superconducting films is that at low temperatures at the maximum superfluid velocity the current is non-zero.

For thin films at temperatures close to  $T_c$ , where the GL model is reliable, the  $j_x$  versus  $\nu_x$  curve has exactly the same features as the one in the left panel of figure 3. As we will now explain this is responsible for the different phase transition pattern one finds when working at fixed current or at fixed superfluid velocity, respectively. For every value of the current there are two values of the superfluid velocity. The current is clearly zero at zero velocity, but also at the maximum value of the superfluid velocity the current falls to zero because the condensate vanishes at this velocity (this holds also in the present holographic

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<sup>5</sup>For a  $p$ -wave superconductor such a DBI construction is possible, see for instance [21, 22].



**Figure 3.** Plots of the current  $j_x$  versus the superfluid velocity  $\nu_x$  at fixed temperature. On the left panel we show the results for two temperatures close to the critical temperature:  $T = 0.998 T_c$  (solid line) and  $T = 0.996 T_c$  (dashed line). On the right we present the curve we find for  $T = 0.244 T_c$ , the dashed line corresponds to metastable states since they are solutions with higher free energy than their counterparts with the same value of the current or the velocity.

model, see [9, 13]). Therefore, at the maximum value of the current the superfluid velocity is not at its maximum and the condensate has a finite non-zero value. This means that if one increases the current a bit further there is no corresponding value of the superfluid velocity in the superconducting phase. Then, the system passes into the normal phase with the condensate jumping a finite distance to zero, which, as already explained, is a signal of a first order phase transition. At the critical temperature this argument breaks down, as no current is possible in the superconducting phase and there is no discontinuity. There the phase transition is second order.

At low temperatures the relation between  $j_x$  and  $\nu_x$  for thin films is linear from zero up to a maximum velocity, the deparing velocity, at which the current falls steeply to zero [14]. As we see on the right plot of figure 3 we find this linear behavior for a long range of currents. However, unlike BCS films, we also find that at the maximum value of the superfluid velocity the current is non-vanishing. According to the analysis in the previous section, for that value of the current the condensate is non-zero and hence if one increases the superfluid velocity the condensate will jump to zero. Hence the phase transition is first order. This agrees with the result of [9, 13] where it is found that the phase transition at low temperatures is indeed first order.

As already noticed, for each value of the current  $j_x$  one finds two solutions with different values of the velocity  $\nu_x$  and so two values of the condensate  $\langle O_2 \rangle$ . As shown in figure 2, we have found that the free energy calculated from the gravity solution using eq. (3.3) is always lower for the configuration in which the magnitude of  $\langle O_2 \rangle$  is higher.

### 3.3 Critical current and critical condensate

In this section we will discuss two further results of our holographic analysis. Recall that our model aims to describe a thin superconducting film and that it is expected to be reliable for the whole range of temperatures (except for very low temperatures, where the backreaction

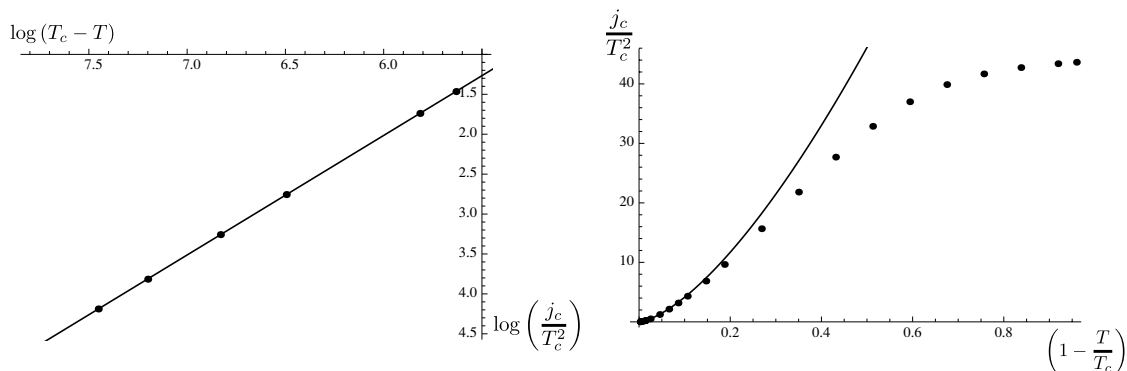
needs to be taken into account). For temperatures near  $T_c$  the GL theory is expected to give an accurate description of such physical system and therefore we have to compare with GL in this regime. It is important in the GL derivation that the film is thin, as this allows one to ignore the free energy contribution of the magnetic field generated by the current. As our magnetic field is non-dynamical, it will not be generated by a current, and so it will not contribute to the free energy. This makes the comparison between holographic superconductors, which are inherently ungauged, with GL model particularly sound for thin films. On the other hand, for temperatures far below the critical temperature our results give new insights on the phase diagram of holographic superconductors.

GL theory predicts that near  $T_c$  the critical current  $j_c$  is proportional to  $(T_c - T)^{3/2}$ . As illustrated in figure 4 we find that this scaling is indeed obeyed by holographic superconductors for temperatures close to  $T_c$ . On the other hand, at low temperatures our results differ appreciably from GL scaling. This is to be expected. For large currents the temperature at which the phase transition occurs is appreciably lower than  $T_c$ , and hence far from the regime where the GL effective description is valid. Moreover, being the phase transition first order, one does not expect any power-law scaling. Notice that we are defining the critical current to be the highest current at which the superconducting solution exists, at a given temperature. As we discussed in section 3.1.1, it might be that the phase transition happens for a lower value of the current and thus the value we are considering would correspond to a metastable state. Nevertheless, at temperatures close to  $T_c$  one can reasonably expect that the corrections to the free energy coming from the very small value of the current are almost negligible and the critical current agrees with the maximum allowed value. The fact that under this assumption the result got for the holographic superconductors agrees with GL is an a posteriori reassurance. Being more conservative, one should take our result as an upper bound, especially for the large current (corresponding to low temperature) regime: in other words, at a given temperature, the corresponding critical current would be at most equal to the one predicted by the plot in figure 4.

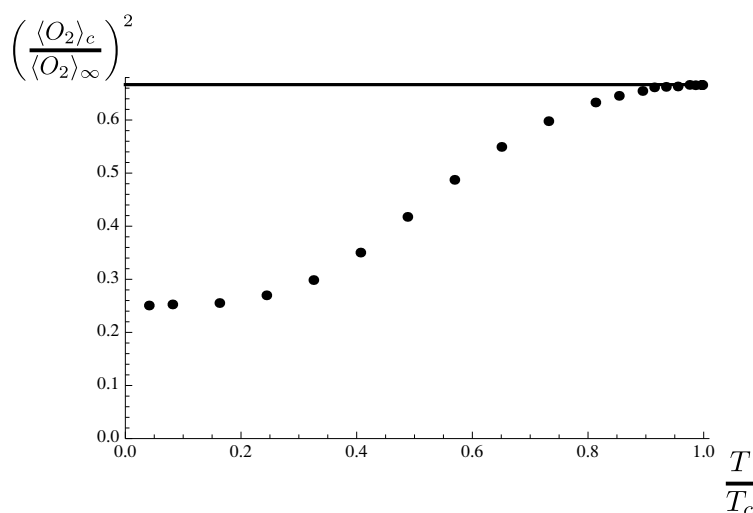
A second prediction of the GL theory is that, at any fixed temperature, the norm of the condensate monotonically decreases with respect to the velocity from its maximum value  $\langle O_2 \rangle_\infty$ . The critical current is reached before the maximum velocity, when the norm of the condensate has an intermediate value  $\langle O_2 \rangle_c$ . More precisely one has

$$\left( \frac{\langle O_2 \rangle_c}{\langle O_2 \rangle_\infty} \right)^2 = \frac{2}{3}. \tag{3.4}$$

We have found numerically that this relation is also satisfied for holographic superconductors at temperatures near the critical temperature  $T_c$ . This can be seen in figure 5 where we plot the ratio (3.4) versus the temperature. Again, away from  $T_c$  the behavior changes sensibly. Notice that the same warning about our inability to determine exactly the critical current applies here and therefore, especially in the region  $T \ll T_c$ , the points in figure 5 should be considered as a lower bound for the ratio  $(\langle O_2 \rangle_c / \langle O_2 \rangle_\infty)^2$ . Notice that the ratio goes to a constant at zero temperature.



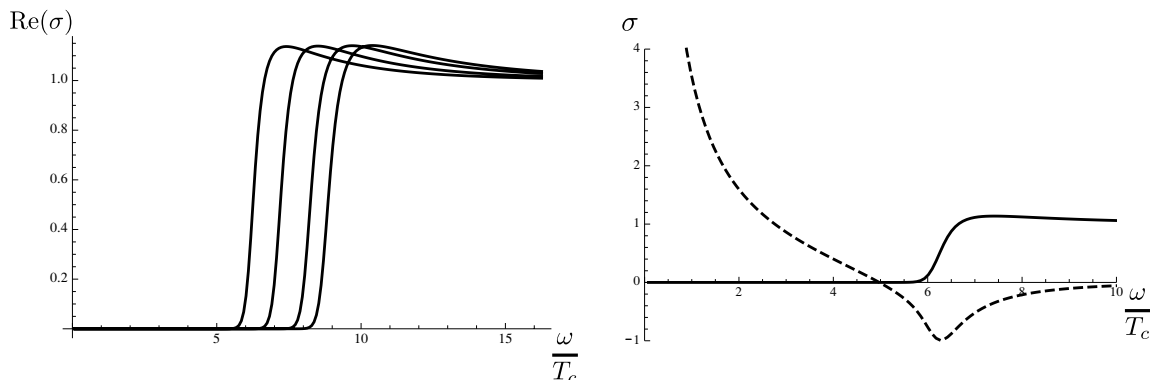
**Figure 4.** Plot of the critical current versus the temperature. The left panel shows a log-log plot from which we can read-off the critical exponent, getting 1.497, which agrees with the expected GL scaling of  $3/2$  within our numerical precision. The right panel shows the departure from GL scaling (solid line) at low temperatures.



**Figure 5.** Plot of the ratio  $(\langle O_2 \rangle_c / \langle O_2 \rangle_\infty)^2$  versus the temperature. The solid line corresponds to the value of  $2/3$  predicted by the GL theory.

### 3.4 Conductivity

In this section we will study the AC conductivity of the system and characterize its dependence on the DC current. To compute the conductivity one must consider an electromagnetic perturbation on top of the hairy black hole solution. This is easy for a perturbation along the direction orthogonal to the current (i.e. a perturbation of  $A_y$ ), since it decouples from other perturbations of the gauge vector or the scalar field. Conversely, a perturbation of  $A_x$  couples to perturbations of  $A_r$  and  $\psi$  and hence the computation of the conductivity along the direction parallel to the current becomes more involved and we will not attempt to do that here. The equation of motion for a zero-momentum perturbation



**Figure 6.** On the left we plot the real part of the conductivity versus the frequency for several values of the current at  $T = 0.04 T_c$ . From left to right:  $j_x/T_c^2 = 43.6, 41.3, 29.6, 2 \cdot 10^{-6}$ . The leftmost curve corresponds to the maximum current at this temperature. On the right we show both the real (solid line) and imaginary (dashed) part of the conductivity as a function of the frequency again at  $T = 0.04 T_c$  and with  $j_x/T_c^2 = 43.6$ , the largest allowed current at that temperature.

$\delta A_y = e^{-i\omega t} A_y(r)$  takes the form

$$\partial_r (f \partial_r A_y) + \left( \frac{\tilde{\omega}^2}{f} - 2\psi^2 \right) A_y = 0, \tag{3.5}$$

where we have applied again the rescalings given in eq. (2.9) and defined  $\tilde{\omega} = 3/(4\pi) \omega/T$ . Notice that this is the same equation as considered in [3], but the background solution for the scalar  $\psi$  is different now, in particular it depends on the current. The boundary asymptotics ( $r \rightarrow \infty$ ) of  $A_y$  take the form

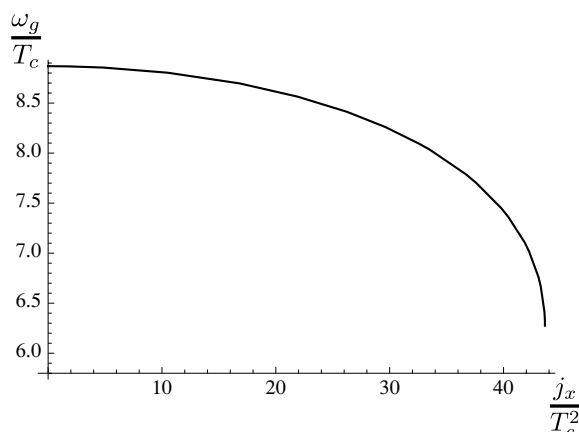
$$A_y = A_y^{(0)} - \frac{A_y^{(1)}}{r} + O(r^{-2}). \tag{3.6}$$

The conductivity is given by the zero-momentum retarded current-current correlator which by using the AdS/CFT dictionary can be calculated in terms of solutions satisfying ingoing wave boundary conditions at the horizon [23]. In fact we recover Ohm’s law on the boundary

$$\sigma_y(\omega) = \frac{\langle J_y \rangle}{E_y} = i \frac{A_y^{(1)}}{\omega A_y^{(0)}}, \tag{3.7}$$

where we have taken into account that  $A_y^{(0)}$  is introducing a background potential on the boundary and thus an electric field  $E_y = -\partial_t A_y^{(0)}$ .

By solving numerically eq. (3.5) with infalling boundary conditions at the horizon ( $r = 1$ ) we can compute the conductivity as a function of the frequency  $\omega$  at given values of temperature and current. In figure 6 we show the results obtained at a low temperature ( $T = 0.04 \cdot T_c$ ) for different values of the current. The AC conductivity displays the features already observed in [3]. At large frequencies it approaches a constant, a characteristic of theories with  $AdS_4$  duals [27]. On the other hand, at  $\omega = 0$  we expect a delta function in  $\text{Re}(\sigma)$ , a fact confirmed, via the Kramers-Kronig relations, through the presence of a pole



**Figure 7.** Plot of  $\omega_g$  as a function of the current. At zero current we find  $\omega_g/T_c = 8.87$ , while at the maximum current  $\omega_g/T_c = 6.28$ .

in the imaginary part of  $\sigma$  at  $\omega = 0$ . Finally, we see that for small enough frequencies, within our numerical precision,  $\text{Re}(\sigma)$  vanishes. This gap can be parametrized in terms of a critical frequency  $\omega_g$ . As we show in figure 6 there is a minimum of  $\text{Im}(\sigma)$  around the point where  $\text{Re}(\sigma)$  becomes non-zero. Then, following [24], we define  $\omega_g$  as the frequency minimizing the imaginary part of the conductivity.

For weakly coupled superconductors the gap is predicted to be  $\omega_g/T_c = 3.5$  at  $T = 0$  [14]. In the original model of [3] the gap was found to be  $\omega_g/T_c \approx 8$  and it seems that such a high value holds quite generically for holographic superconductors. This has been seen as an indication that holographic superconductors are indeed strongly coupled. In figure 7 we plot  $\omega_g$  as a function of the current. In the region of very low current we recover the result  $\omega_g/T_c \approx 8$ . As we increase the current,  $\omega_g$  decreases continuously until we reach the maximum current where it has a finite value. This is consistent with the phase transition being first order at that point. The condensate is non-vanishing and thus we expect  $\omega_g$  to be also different from zero.

In weakly coupled superconductors a definite relation exists between  $\omega_g$  and the energy gap  $\Delta$  at zero temperature,  $\omega_g = 2\Delta$ ,  $\Delta(T)$  being the minimum energy required for charged excitations at a given temperature  $T$ . In strongly coupled superconductors one does not expect the gap to necessarily satisfy this relation so one could have wondered what the relation is for holographic superconductors. However, as noticed in [10, 25, 26] and recently reviewed in [6], holographic superconductors are not hard-gapped, in general, and a non-zero conductivity is present even at small frequencies, though exponentially suppressed (for recent work on hard-gapped holographic superconductors see for instance [28], and [29] for a recent discussion on this point). Indeed, computations of the temperature dependence of the specific heat [25] showed that holographic superconductors behave similarly to some strongly coupled superconductors as heavy fermion compounds: the specific heat does not vanish exponentially at low temperature (this being a consequence of, and hence an indication for, the existence of an energy gap), but as a power law.



## 4 Conclusions and outlook

In this paper we have considered a simple holographic model of a thin superconducting film with DC current. We focused on the modifications that the presence of a DC current induces on the thermodynamics as compared to the same holographic model with vanishing current, originally studied in [3]. Most notably, the phase transition becomes first order for any finite value of the current. Moreover, the conductivity gap becomes a function of the current, too: the frequency gap diminishes as one increases the current but never reaches zero before the phase transition occurs, in agreement with the phase transition being first order. Other results we obtain nicely agree with expectations for thin superconducting films, such as the relation between the current and the superfluid velocity, both at low and high (that is near to  $T_c$ ) temperatures. The only qualitative difference is that at sufficiently low temperatures and high superfluid velocities, the velocity no longer uniquely determines the current.

Ideally one would like to go beyond the probe approximation. Besides leading to better control over the very low temperature regime, this is necessary in order to obtain a holographic description of the normal phase. To describe the superconductor in the normal phase with a DC current one should switch on an external electric field, which is needed to keep a constant current in the normal phase, since there we have a non-vanishing resistivity. However, as we already noticed, only the full system of coupled scalar Maxwell-Einstein gravity equations could in principle allow for a non-trivial but meaningful (that is non-singular) solution. This implies that one should go for a fully back-reacted analysis.

### 4.1 Towards holographic Josephson junctions

Our primary motivation for this work was the observation that HTS's have typically a layered structure and may, according to the highly successful Lawrence-Doniach model [7], be approximated by superconducting films separated by Josephson junctions. While the present model can be seen as a first step in this direction, one would like to find a complete holographic description of a Josephson junction. Let us elaborate a bit on this.

One kind of Josephson junction that appears particularly amenable to a holographic construction is the S-c-S junction. Such a junction is composed entirely of the same superconducting material, but the superconductor is thinner at the junction, and so for example will have a lower critical current. Imposing a space-dependent metric in the boundary theory, one might be able to cook-up a dual model with varying critical current. This might seem quite ad hoc in a bottom-up context but it might possibly arise quite naturally in a string theory context. For example, one may consider M-theory compactified on a 7-dimensional Sasaki-Einstein manifold fibered over  $AdS_4$ . The critical current, like all quantities in a holographic superconductor, depends on the compactification geometry. Then, to construct a junction, one may merely need to vary the moduli of the 7-manifold in the region corresponding to the junction, that is, over a finite interval in one of the field theory directions. An array of junctions would then correspond to moduli that vary periodically in one field theory direction.

Ideally such a spatial dependence of the moduli will be a solution of the supergravity equations of motion. One hope of realizing such a solution is as follows. Each Josephson function may correspond to a brane extending from the boundary to the horizon and along the field theory directions parallel to the Josephson junction, and also potentially wrapping some cycle of the compactification manifold. One may then hope that by correctly choosing this cycle, one may engineer a geometry in which the backreaction of the brane causes the desired deformation of the compactification manifold, reducing the critical current and therefore forming an S-c-S junction. A Lawrence-Doniach HTS may then correspond to an array of such branes.

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