

The tree formula for MHV graviton amplitudes

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ABSTRACT: We present and prove a formula for the MHV scattering amplitude of n gravitons at tree level. Some of the more interesting features of the formula, which set it apart as being significantly different from many more familiar formulas, include the absence of any vestigial reference to a cyclic ordering of the gravitons — making it in a sense a truly gravitational formula, rather than a recycled Yang-Mills result, and the fact that it simultaneously manifests both S_{n-2} symmetry as well as large- z behavior that is $\mathcal{O}(1/z^2)$ term-by-term, without relying on delicate cancellations. The formula is seemingly related to others by an enormous simplification provided by $\mathcal{O}(n^n)$ iterated Schouten identities, but our proof relies on a complex analysis argument rather than such a brute force manipulation. We find that the formula has a very simple link representation in twistor space, where cancellations that are non-obvious in physical space become manifest.

KEYWORDS: Supergravity Models, String Duality

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1 Introduction

The past several years have witnessed tremendous progress in our understanding of the mathematical structure of scattering amplitudes, particularly in maximally supersymmetric theories. It is easy to argue that the seeds of this progress were sown over two decades ago by the discovery [1, 2] of the stunningly simple formula¹

$$\mathcal{A}^{\text{MHV}}(1, \dots, n) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \quad (1.1)$$

for the maximally helicity violating (MHV) color-ordered subamplitude for n -gluon scattering. The importance of this formula goes far beyond simply knowing the answer for a certain scattering amplitude, which one may or may not be particularly interested in. Rather, the mere existence of such a simple formula for something which would normally require enormously tedious calculations using traditional Feynman diagram techniques suggests firstly that the theory must possess some remarkable and deeply hidden mathematical structure, and secondly that if one actually is interested in knowing the answer for a certain amplitude it behooves one to discover and understand this structure. In other words, the formula (1.1) is as important psychologically as it is physically, since it provides strong motivation for digging more deeply into scattering amplitudes.

Much of the progress on gluon amplitudes can be easily recycled and applied to graviton amplitudes due ultimately to the KLT relations [3] which roughly speaking state that

¹Here and throughout the paper we use calligraphic letters \mathcal{A} , \mathcal{M} to denote superspace amplitudes with the overall delta-function of supermomentum conservation suppressed.

“gravity is Yang-Mills squared”. Slightly more precisely, the KLT relations express an n -graviton amplitude as a sum over permutations of the square of the color-ordered n -gluon subamplitude times some simple extra factors (see [4] for a review). There are several indications that maximal supergravity may be an extraordinarily remarkable theory [5–14], and possibly even ultraviolet finite [15–20], but our feeling is that even at tree level we are still far from fully unlocking the structure of graviton amplitudes.

To illustrate this disparity we need look no further than the simplest graviton amplitudes. The original BGK (Berends, Giele and Kuijf) formula for the n -graviton MHV amplitude [21] is now over 20 years old. For later convenience we review here a different form due to Mason and Skinner [22], who proved the equivalence of the original BGK formula to the expression

$$\mathcal{M}_n^{\text{MHV}} = \sum_{P(1,\dots,n-3)} \frac{1}{\langle n\ n-2 \rangle \langle n-2\ n-1 \rangle \langle n-1\ n \rangle \langle 1\ 2 \rangle \cdots \langle n\ 1 \rangle} \frac{1}{\prod_{k=1}^{n-3} \frac{[k|p_{k+1} + \cdots + p_{n-2}|n-1\rangle}{\langle k\ n-1 \rangle}}, \tag{1.2}$$

where the sum indicates a sum over all $(n-3)!$ permutations of the labels $1, \dots, n-3$ and we use the convention

$$[a|p_i + p_j + \cdots |b\rangle = [a\ i] \langle i\ b \rangle + [a\ j] \langle j\ b \rangle + \cdots . \tag{1.3}$$

The fact that any closed form expression exists at all for this quantity, the calculation of which would otherwise be vastly more complicated even than the corresponding one for n gluons, is an amazing achievement. Nevertheless the formula has some features which strongly suggest that it is not the end of the story.

First of all, the formula (1.2) does not manifest the requisite permutation symmetry of an n -graviton superamplitude. Specifically, any superamplitude \mathcal{M}_n must be fully symmetric under all $n!$ permutations of the labels $1, \dots, n$ of the external particles, but only an S_{n-3} subgroup of this symmetry is manifest in (1.2) (several formulas which manifest a slightly larger S_{n-2} subgroup are known [23, 24]). Of course one can check, numerically if necessary, that (1.2) does in fact have this symmetry, but it is far from obvious. Moreover, even the S_{n-3} symmetry arises in a somewhat contrived way, via an explicit sum over permutations. Undoubtedly the summand in (1.2) contains redundant information which is washed out by taking the sum. This situation should be contrasted with that of Yang-Mills theory, where (1.1) is manifestly invariant under the appropriate dihedral symmetry group (not the full permutation group, due to the color ordering of gluons).

Secondly, one slightly disappointing feature of all previously known MHV formulas including (1.2) is the appearance of “ \cdots ”, which indicates that a particular cyclic ordering of the particles must be chosen in order to write the formula, even though a graviton amplitude ultimately cannot depend on any such ordering since gravitons do not carry any color labels. This vestigial feature usually traces back to the use of the KLT relations to calculate graviton amplitudes by recycling gluon amplitudes.

An important feature of graviton amplitudes is that they fall off like $1/z^2$ as the supermomenta of any two particles are taking to infinity in a particular complex direction (see [23, 25–30] and [10] for the most complete treatment), unlike in Yang-Mills theory

where the falloff is only $1/z$ [31]. It has been argued [28] that this exceptionally soft behavior of graviton tree amplitudes is of direct importance for the remarkable ultraviolet cancellations in supergravity loop amplitudes [15, 32–35].

The $1/z^2$ falloff of (1.2) is manifest for each term separately inside the sum over permutations. Two classes of previously known formulas for the n -graviton MHV amplitude are: those like (1.2) which manifest the $1/z^2$ falloff but only S_{n-3} symmetry, and others (see for example [23, 24]) which have a larger S_{n-2} symmetry but only manifest falloff like $1/z$. In the latter class of formulas the stronger $1/z^2$ behavior arises from delicate and non-obvious cancellations between various terms in the sum. This is both a feature and a bug. It is a feature because it implies the existence of linear identities (which have been called bonus relations in [36]) between individual terms in the sum which have proven useful, for example, in establishing the equality of various previously known but not obviously equivalent formulas [36]. But it is a bug because it indicates that the S_{n-2} -invariant formulas contain redundant information distributed amongst the various terms in the sum. The bonus relations allow one to squeeze this redundant information out of any S_{n-2} -invariant formula at the cost of reducing the manifest symmetry to S_{n-3} .

It is difficult to imagine that it might be possible to improve upon the Parke-Taylor formula (1.1) for the n -gluon MHV amplitude. However, for the reasons just reviewed, we feel that (1.2) cannot be the end of the story for gravity. Ideally one would like to have a formula for n -graviton scattering that (1) is manifestly S_n symmetric without the need for introducing an explicit sum over permutations to impose the symmetry *vi et armis*; (2) makes no vestigial reference to any cyclic ordering of the n gravitons, and (3) manifests $1/z^2$ falloff term by term, making it unsqueezable by the bonus relations.

In this paper we present and prove the “tree formula” (2.1) for the MHV scattering amplitude which addresses the second and third points but only manifests S_{n-2} symmetry. In section 2 we introduce the tree formula and discuss several special cases as well as the general soft limit. In section 3 we work out the simple link representation of the amplitude in twistor space, from which new physical space formula follows. Finally the proof is in section 4.

Note Added. After this paper appeared we learned of an ansatz for the MHV graviton amplitude presented in section 6 of [66] which upon inspection is immediately seen to share the nice features of the tree formula. In fact, although terms in the two formulas are arranged in different ways (labeled tree diagrams versus Young tableaux), it is not difficult to check that their content is actually identical. Interestingly the formula of [66] was constructed with the help of “half-soft factors” similar in idea to the “inverse soft limits” which appeared much more recently in [38]. Our work establishes the validity of the ansatz conjectured in [66] and demonstrates that it arises naturally in twistor space.

2 The MHV tree formula

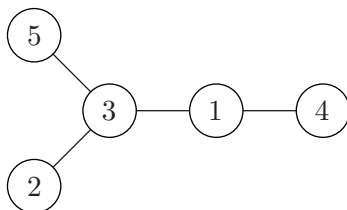
2.1 Statement of the tree formula

Here we introduce a formula for the n -graviton MHV scattering amplitude which we call the “tree formula” since it consists of a sum of terms, each of which is conveniently represented

by a tree diagram. The tree formula manifests an S_{n-2} subgroup of the full permutation group. For the moment we choose to treat particles $n - 1$ and n as special. With this arbitrary choice the formula is:

$$\mathcal{M}_n^{\text{MHV}} = \frac{1}{\langle n-1 n \rangle^2} \sum_{\text{trees}} \left(\prod_{\text{edges } ab} \frac{[ab]}{\langle ab \rangle} \right) \left(\prod_{\text{vertices } a} (\langle a n-1 \rangle \langle a n \rangle)^{\text{deg}(a)-2} \right). \quad (2.1)$$

To write down an expression for the n -point amplitude one draws all inequivalent connected tree graphs with vertices labeled $1, 2, \dots, n - 2$. (It was proven by Cayley that there are precisely $(n - 2)^{n-4}$ such diagrams.) For example, one of the 125 labeled tree graphs contributing to the $n = 7$ graviton amplitude is



According to (2.1) the value of a diagram is then the product of three factors:

1. an overall factor of $1/\langle n-1 n \rangle^2$,
2. a factor of $[ab]/\langle ab \rangle$ for each propagator connecting vertices a and b , and
3. a factor of $(\langle a n-1 \rangle \langle a n \rangle)^{\text{deg}(a)-2}$ for each vertex a , where $\text{deg}(a)$ is the degree of the vertex (the number of edges attached to it).

An alternate description of the formula may be given by noting that a vertex factor of $\langle a n - 1 \rangle \langle a n \rangle$ may be absorbed into each propagator connected to that vertex. This leads to the equivalent formula

$$\mathcal{M}_n^{\text{MHV}} = \frac{1}{\langle n-1 n \rangle^2} \left(\prod_{a=1}^{n-2} \frac{1}{(\langle a n-1 \rangle \langle a n \rangle)^2} \right) \sum_{\text{trees}} \prod_{\text{edges } ab} \frac{[ab]}{\langle ab \rangle} \langle a n-1 \rangle \langle b n-1 \rangle \langle a n \rangle \langle b n \rangle. \quad (2.2)$$

2.2 Examples

We defer to section 4 a formal proof of the tree formula as the impatient reader may be sufficiently convinced by seeing the formula in action here for small n and by noting that it has the correct soft limits for all n , as we discuss shortly.

For each of the trivial cases $n = 3, 4$ there is only a single tree diagram,

$$\mathcal{M}_3^{\text{MHV}} = \textcircled{1} = \frac{1}{(\langle 1 2 \rangle \langle 1 3 \rangle \langle 2 3 \rangle)^2} \quad (2.3)$$

and

$$\mathcal{M}_4^{\text{MHV}} = \textcircled{1} \text{---} \textcircled{2} = \frac{[1 2]}{\langle 1 2 \rangle \langle 1 3 \rangle \langle 1 4 \rangle \langle 2 3 \rangle \langle 2 4 \rangle \langle 3 4 \rangle^2} \quad (2.4)$$

respectively, which immediately reproduce the correct expressions.

For $n = 5$ there are three tree diagrams

$$\begin{aligned}
 \textcircled{1} - \textcircled{2} - \textcircled{3} &= \frac{[12][23]}{\langle 12 \rangle \langle 14 \rangle \langle 15 \rangle \langle 23 \rangle \langle 34 \rangle \langle 35 \rangle \langle 45 \rangle^2} \\
 \textcircled{1} - \textcircled{3} - \textcircled{2} &= \frac{[13][23]}{\langle 13 \rangle \langle 14 \rangle \langle 15 \rangle \langle 23 \rangle \langle 24 \rangle \langle 25 \rangle \langle 45 \rangle^2} \\
 \textcircled{2} - \textcircled{1} - \textcircled{3} &= \frac{[12][13]}{\langle 12 \rangle \langle 13 \rangle \langle 24 \rangle \langle 25 \rangle \langle 34 \rangle \langle 35 \rangle \langle 45 \rangle^2}
 \end{aligned} \tag{2.5}$$

which can easily be verified by hand to sum to the correct expression. Agreement between the tree formula and other known formulas such as (1.2) may be checked numerically for slightly larger values of n by assigning random values to all of the spinor helicity variables. A simple implementation of the tree formula in the Mathematica symbolic computation language is presented in the appendix.

2.3 Relation to other known formulas

The MHV tree formula is evidently quite different in form from most other expressions in the literature. In particular, no reference at all is made to any particular ordering of the particles (there is no vestigial “...”), and the manifest S_{n-2} arises not because of any explicit sum over $P(1, \dots, n-2)$ but rather from the simple fact that the collection of labeled tree diagrams has a manifest S_{n-2} symmetry. In our view these facts serve to highlight the essential “gravitiness” of the formula, in contrast to expressions such as (1.2) which are ultimately recycled from Yang-Mills theory.

One interesting feature of the MHV tree formula is that it is, in a sense, minimally non-holomorphic. Graviton MHV amplitudes, unlike their Yang-Mills counterparts, do not depend only the holomorphic spinor helicity variables λ_i . The tree formula packages all of the non-holomorphicity into the $[ab]$ factors associated with propagators in the tree diagrams. Each diagram has a unique collection of propagators and a correspondingly unique signature of $[\]$'s, which only involve $n-2$ of the n labels.

Like the MHV tree formula, the Mason-Skinner formula (1.2) (unlike most other formulas in the literature, including the original BGK formula) has non-holomorphic dependence on only $n-2$ variables. In our labeling of (1.2) we see that $\tilde{\lambda}_{n-1}$ and $\tilde{\lambda}_n$ do not appear at all. Of course we do not mean to say that \mathcal{M} is “independent” of these two variables since there is a suppressed overall delta function of momentum conservation $\delta^4(\sum_i \lambda_i \tilde{\lambda}_i)$ which one could use to shuffle some $\tilde{\lambda}$'s into others. Rather we mean that the tree and MS formulas have the property that all appearance of two of the $\tilde{\lambda}$'s has already been completely shuffled out.

It is an illuminating exercise to attempt a direct term-by-term comparison of the MHV tree formula with the MS formula (1.2). For the first non-trivial case $n = 5$ the MS formula provides the two terms

$$\frac{[23][1|p_2 + p_3|4\rangle}{\langle 12 \rangle \langle 14 \rangle \langle 15 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle \langle 35 \rangle \langle 45 \rangle^2} - \frac{[13][2|p_1 + p_3|4\rangle}{\langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 24 \rangle \langle 25 \rangle \langle 34 \rangle \langle 35 \rangle \langle 45 \rangle^2}. \tag{2.6}$$

If we now expand out the bracket $[a|p_i + p_j|b\rangle = [a|i\rangle\langle i b\rangle + [a|j\rangle\langle j b\rangle$ then we find four terms: one of them is proportional to $[12][23]$ and is identical to the first line in (2.5), another

proportional to $[12][13]$ is identical to the last line in (2.5). The remaining two terms are both proportional to $[13][23]$ and may be combined as

$$\frac{[13][23](\langle 13\rangle\langle 25\rangle - \langle 15\rangle\langle 23\rangle)}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle\langle 35\rangle\langle 45\rangle^2} \tag{2.7}$$

which with the help of a Schouten identity we recognize as precisely the second line in (2.5).

We beg the reader's pardon for allowing us to indulge in one final example. Expanding the MS formula for $n = 6$ into $\langle \rangle$'s and $[\]$'s yields a total of 36 terms. For example there are 6 terms proportional to the antiholomorphic structure $[14][24][34]$, totalling

$$\frac{[14][24][34]\langle 45\rangle}{\langle 15\rangle\langle 25\rangle\langle 35\rangle\langle 46\rangle\langle 56\rangle^2} \left[\frac{1}{\langle 12\rangle\langle 16\rangle\langle 23\rangle\langle 34\rangle} + \frac{1}{\langle 12\rangle\langle 14\rangle\langle 23\rangle\langle 36\rangle} - \frac{1}{\langle 13\rangle\langle 16\rangle\langle 23\rangle\langle 24\rangle} - \frac{1}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 26\rangle} - \frac{1}{\langle 12\rangle\langle 13\rangle\langle 26\rangle\langle 34\rangle} - \frac{1}{\langle 12\rangle\langle 13\rangle\langle 24\rangle\langle 36\rangle} \right]. \tag{2.8}$$

After repeated use of Schouten identities this amazingly collapses to the single term

$$\frac{[14][24][34]\langle 45\rangle\langle 46\rangle}{\langle 14\rangle\langle 15\rangle\langle 16\rangle\langle 24\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 35\rangle\langle 36\rangle\langle 56\rangle^2} = \begin{array}{c} \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \\ \diagdown \quad | \quad \diagup \\ \textcircled{4} \end{array} \tag{2.9}$$

We believe that these examples are representative of the general case. Expanding out all of the brackets in the n -graviton MS formula generates a total of $[(n - 3)!]^2$ terms, but there are only $(n - 2)^{n-4}$ possible distinct antiholomorphic signatures. Collecting terms with the same signature and repeatedly applying Schouten identities should collapse everything into the terms generated by the MHV tree formula. Note that this is a huge simplification: $(n - 2)^{n-4}$ is smaller than $[(n - 3)!]^2$ by a factor that is asymptotically n^n . We certainly do not have an explicit proof of this cancellation; instead we are relying on fact that the MS formula and the tree formula are separately proven to be correct in order to infer how the story should go.

To conclude this discussion we should note that we are exploring here only the structure of the various formulas, not making any claims about the computational complexity of the MHV tree formula as compared to (1.2) or any other known formula. No practical implementation of the MS formula would proceed by first splitting all of the brackets as we have outlined. Indeed a naive counting of the number of terms, $(n - 3)!$ in (1.2) versus $(n - 2)^{n-4}$ for the tree formula, suggests that for computational purposes the former is almost certainly the clear winner despite the conceptual strengths of the latter.

2.4 Soft limit of the tree formula

Let us consider for a moment the component amplitude

$$M(1^+, \dots, (n - 2)^+, (n - 1)^-, n^-) = \langle n - 1 n \rangle^8 \mathcal{M}_n^{\text{MHV}} \tag{2.10}$$

with particles $n - 1$ and n having negative helicity. The universal soft factor for gravitons is [21, 37]

$$\lim_{p_1 \rightarrow 0} \frac{M(1^+, \dots, (n-2)^+, (n-1)^-, n^-)}{M(2^+, \dots, (n-2)^+, (n-1)^-, n^-)} = \sum_{i=2}^{n-2} g(i^+), \quad g(i^+) = \frac{\langle i n - 1 \rangle \langle i n \rangle [1 i]}{\langle 1 n - 1 \rangle \langle 1 n \rangle \langle 1 i \rangle}. \quad (2.11)$$

It is simple to see that the MHV tree formula satisfies this property: the tree diagrams which do not vanish in the limit $p_1 \rightarrow 0$ are those in which vertex 1 is connected by a propagator to a single other vertex i . Such diagrams remain connected when vertex 1 is chopped off, leaving a contribution to the $n - 1$ -graviton amplitude times the indicated factor $g(i^+)$.

Thinking about this process in reverse therefore suggests a simple interpretation of (2.11) in terms of tree diagrams — it is a sum over all possible places i where the vertex 1 may be attached to the $n - 1$ -graviton amplitude. This structure is exactly that of the “inverse soft factors” suggested recently in [38], and we have checked that the MHV tree formula may be built up by recursively applying the rule proposed there.

3 The MHV tree formula in twistor space

Before turning to the formal proof of the tree formula in the next section, here we work out the link representation of the MHV graviton amplitude in twistor space, which was one of the steps which led to the discovery of the tree formula. Two papers [39, 40] have recently constructed versions of the BCF on-shell recursion relation directly in twistor space variables. We follow the standard notation where $\mu, \tilde{\mu}$ are respectively Fourier transform conjugate to the spinor helicity variables $\lambda, \tilde{\lambda}$, and assemble these together with a four-component Grassmann variable η and its conjugate $\tilde{\eta}$ into the 4|8-component supertwistor variables

$$\mathcal{Z} = \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} \tilde{\mu} \\ \tilde{\lambda} \\ \tilde{\eta} \end{pmatrix}. \quad (3.1)$$

In the approach of [40], in which variables of both chiralities \mathcal{Z} and \mathcal{W} are used simultaneously, an apparently important role is played by the link representation which expresses an amplitude \mathcal{M} in the form

$$\mathcal{M}(\mathcal{Z}_i, \mathcal{W}_J) = \int dc U(c_{iJ}, \lambda_i, \tilde{\lambda}_J) \exp \left[i \sum_{i,J} c_{iJ} \mathcal{Z}_i \cdot \mathcal{W}_J \right]. \quad (3.2)$$

Here one splits the n particles into two groups, one of which (labeled by i) one chooses to represent in \mathcal{Z} space and the other of which (labeled by J) one chooses to represent in \mathcal{W} space. The integral runs over all of the aptly-named link variables c_{iJ} and we refer to the integrand $U(c_{iJ}, \lambda_i, \tilde{\lambda}_J)$ as the link representation of \mathcal{M} . It was shown in [40] that the BCF on-shell recursion in twistor space involves nothing more than a simple integral over \mathcal{Z}, \mathcal{W} variables with a simple (and essentially unique) measure factor.

The original motivation for our investigation was to explore the structure of link representations for graviton amplitudes. We will always adopt the convenient convention of expressing an N^k MHV amplitude in terms of $k + 2$ \mathcal{Z} variables and $n - k - 2$ \mathcal{W} variables. The three-particle MHV and $\overline{\text{MHV}}$ amplitudes

$$U_3^{\text{MHV}} = \frac{|\langle 12 \rangle|}{c_{13}^2 c_{23}^2}, \quad U_3^{\overline{\text{MHV}}} = \frac{|[12]|}{c_{31}^2 c_{32}^2} \quad (3.3)$$

seed the on-shell recursion, which is then sufficient (in principle) to determine the link representation for any desired amplitude.

For example, the four-particle amplitude is the sum of two contributing BCF diagrams

$$U_4^{\text{MHV}} = \frac{\langle 12 \rangle [34]}{c_{13}^2 c_{24}^2 c_{12:34}} + \frac{\langle 12 \rangle [34]}{c_{13}^2 c_{24}^2 c_{14} c_{23}} \quad (3.4)$$

where we use the notation

$$c_{i_1 i_2 : J_1 J_2} = c_{i_1 J_1} c_{i_2 J_2} - c_{i_1 J_2} c_{i_2 J_1}. \quad (3.5)$$

Remarkably the two terms in (3.4) combine nicely into the simple result presented already in [40]:

$$U_4^{\text{MHV}} = \frac{\langle 12 \rangle [34]}{c_{13} c_{14} c_{23} c_{24} c_{12:34}}. \quad (3.6)$$

This simplification seems trivial at the moment but it is just the tip of an iceberg. For larger n the enormous simplifications discussed in the previous section, which are apparently non-trivial in physical space, occur automatically in the link representation.

For example the five particle MHV amplitude is the sum of three BCF diagrams,

$$U_5^{\text{MHV}} = \left\{ \frac{|\langle 12 \rangle| [45] (c_{24} [34] + c_{25} [35])}{c_{13} c_{23} c_{14} c_{25}^2 c_{12:34} c_{12:45}} + (3 \leftrightarrow 4) \right\} + \frac{|\langle 12 \rangle| [34] (c_{24} [45] + c_{23} [35])}{c_{13} c_{14} c_{15} c_{23} c_{24} c_{25}^2 c_{12:34}} \quad (3.7)$$

which nicely simplifies to

$$\frac{1}{|\langle 12 \rangle|} U_5^{\text{MHV}} = \frac{[34][45]}{c_{13} c_{15} c_{23} c_{25} c_{12:34} c_{12:45}} + \frac{[35][45]}{c_{13} c_{14} c_{23} c_{24} c_{12:35} c_{12:45}} + \frac{[34][35]}{c_{14} c_{15} c_{24} c_{25} c_{12:34} c_{12:35}}. \quad (3.8)$$

This expression already exhibits the structure of the MHV tree formula (except that here particles 1 and 2 are singled out, and the vertices of the trees are labeled by $\{3, 4, 5\}$).

Subsequent investigations for higher n reveal the general pattern which is as follows. Returning to the convention where particles $n - 1$ and n are treated as special, the link representation for any desired MHV amplitude may be written down by drawing all tree diagrams with vertices labeled by $\{1, \dots, n - 2\}$ and then assigning

1. an overall factor of $\langle n - 1 n \rangle \text{sign}(\langle n - 1 n \rangle)^n$,
2. for each propagator connecting nodes a and b , a factor of $[ab]/c_{n-1,n;a,b}$,
3. for each vertex a , a factor of $(c_{n-1,a} c_{n,a})^{\text{deg}(a)-2}$, where $\text{deg}(a)$ is the degree of the vertex labeled a .

It is readily verified by direct integration over the link variables that these rules are precisely the link-space representation of the physical space rules for the MHV tree formula given in the previous section.

4 Proof of the MHV tree formula

Here we present a proof of the MHV tree formula. One way one might attempt to prove the formula would be to show directly that it satisfies the BCF on-shell recursion relation [31, 44] for gravity [23, 25, 27], but the structure of the formula is poorly suited for this task. Instead we proceed by considering the usual BCF deformation of the formula M_n^{MHV} by a complex parameter z and demonstrating that $M_n^{\text{MHV}}(z)$ has the same residue at every pole (and behavior at infinity) as the similarly deformed graviton amplitude, thereby establishing equality of the two for all z .

In this section we return to singling out particles 1 and 2, letting the vertices in the tree diagrams carry the labels $\{3, \dots, n\}$. Then the MHV tree formula (2.1) can be written as

$$M_n^{\text{MHV}} = \langle 1 2 \rangle^6 \sum_{\text{trees}} \frac{[\dots][\dots]}{\langle \dots \rangle} \prod_{a=3}^n (\langle 1 a \rangle \langle 2 a \rangle)^{\text{deg}(a)-2} \quad (4.1)$$

(note that we continue to work with the component amplitude (2.10)) where the factors $[\dots][\dots]/\langle \dots \rangle$ associated with the propagators of a diagram are independent of 1 and 2. Let us now make the familiar BCF shift [31]

$$\lambda_1 \rightarrow \lambda_1(z) = \lambda_1 - z\lambda_2, \quad \tilde{\lambda}_2 \rightarrow \tilde{\lambda}_2(z) = \tilde{\lambda}_2 + z\tilde{\lambda}_1 \quad (4.2)$$

which leads to the z -deformed MHV tree formula

$$M_n^{\text{MHV}}(z) = \langle 1 2 \rangle^6 \sum_{\text{trees}} \frac{[\dots][\dots]}{\langle \dots \rangle} \prod_{a=3}^n [(\langle 1 a \rangle - z\langle 2 a \rangle) \langle 2 a \rangle]^{\text{deg}(a)-2}. \quad (4.3)$$

Here we are in a position to observe a nice fact: since each tree diagram is connected, the degrees satisfy the sum rule

$$\sum_{a=3}^n (\text{deg}(a) - 2) = -2, \quad (4.4)$$

which guarantees that each individual term in (4.3) manifestly behaves like $1/z^2$ at large z . This exceptionally soft behavior of graviton amplitudes is completely hidden in the usual Feynman diagram expansion.

A complex function of a single variable which vanishes at infinity is uniquely determined by the locations of its poles as well as its residues. Having noted that (4.3) has the correct behavior at large z , we can conclude the proof of the MHV tree formula by demonstrating that (4.3) has precisely the expected residues at all of its poles. In order to say what the expected residues are we shall use induction on n . As discussed above the tree formula is readily verified for sufficiently small n , so let us assume that it has been established up through $n - 1$. We can then use BCF on-shell recursion (whose terms are displayed graphically in figure 1) to determine what the residues in the deformed n -point amplitude ought to be.

Without loss of generality let us consider just the pole at $z = z_3 \equiv \langle 1 3 \rangle / \langle 2 3 \rangle$. The only tree diagrams which contribute to the residue at this pole are those with $\text{deg}(3) = 1$, meaning that the vertex labeled 3 is connected to the rest of the diagram by a single

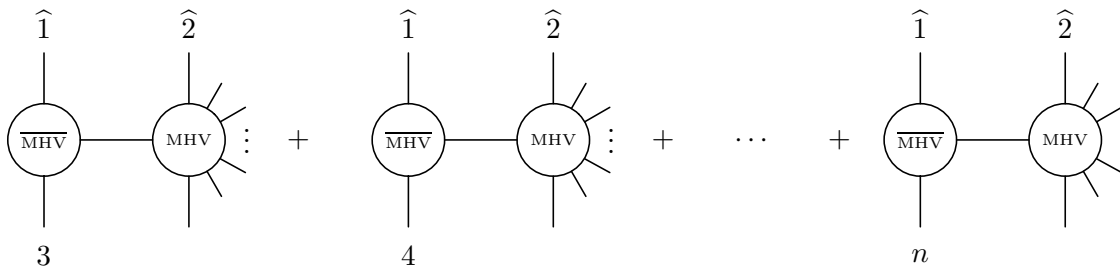


Figure 1. All factorizations contributing to the on-shell recursion relation for the n -point MHV amplitude. Only the first diagram contributes to the residue at $z = \langle 13 \rangle / \langle 23 \rangle$.

propagator. Chopping off vertex 3 gives a subdiagram with vertices labeled $\{4, \dots, n\}$. Clearly all diagrams which contribute to this residue can be generated by first considering the collection of tree diagrams with vertices labeled $\{4, \dots, n\}$ and then attaching vertex 3 in all possible ways to the $n - 3$ vertices of the subdiagram. We therefore have

$$M_n^{\text{MHV}}(z) \sim \langle 12 \rangle^6 \sum_{\text{subdiagrams}} \frac{[\dots][\dots]}{\langle \dots \rangle \dots \langle \dots \rangle} \left(\sum_{b=4}^n \frac{[3b]}{\langle 3b \rangle} \langle \hat{1}b \rangle \langle 2b \rangle \right) \frac{1}{\langle \hat{1}3 \rangle \langle 23 \rangle} \prod_{a=4}^n \left(\langle \hat{1}a \rangle \langle 2a \rangle \right)^{\text{deg}(a)-2} \quad (4.5)$$

where \sim denotes that we have dropped terms which are nonsingular at $z = z_3$, the sum over b runs over all the places where vertex 3 can be attached to the subdiagram, and $[\dots][\dots]/\langle \dots \rangle \dots \langle \dots \rangle$ indicates all edge factors associated the subdiagram, necessarily independent of 3. Using the Schouten identity we find that $\langle \hat{1}b \rangle = \langle 12 \rangle \langle b3 \rangle / \langle 23 \rangle$ so we have after a couple of simple steps (and using (4.4))

$$M_n^{\text{MHV}}(z) \sim \langle 12 \rangle^6 \frac{[13]}{\langle 13 \rangle - z \langle 23 \rangle} \sum_{\text{subdiagrams}} \frac{[\dots][\dots]}{\langle \dots \rangle \dots \langle \dots \rangle} \prod_{a=4}^n \left(\langle 2a \rangle \langle 3a \rangle \right)^{\text{deg}(a)-2}. \quad (4.6)$$

On the other hand we know from the on-shell recursion for the n -point amplitude that the residue at $z = z_3$ comes entirely from the first BCFW diagram in Fig. 1, whose value is

$$M_3^{\overline{\text{MHV}}}(z_3) \times \frac{1}{P^2(z)} \times M_{n-1}^{\text{MHV}}(z_3) \quad (4.7)$$

where

$$P(z) = p_1 + p_3 - z \lambda_2 \tilde{\lambda}_1. \quad (4.8)$$

Assuming the validity of the MHV tree formula for the $n - 1$ -point amplitude on the right, the expression (4.7) evaluates to

$$\frac{[\hat{P}3]^6}{[31]^2 [1\hat{P}]^2} \times \frac{1}{[13](\langle 13 \rangle - z \langle 23 \rangle)} \times \langle \hat{P}2 \rangle^6 \sum_{\text{subdiagrams}} \frac{[\dots][\dots]}{\langle \dots \rangle \dots \langle \dots \rangle} \prod_{a=4}^n \left(\langle \hat{P}a \rangle \langle 2a \rangle \right)^{\text{deg}(a)-2} \quad (4.9)$$

where $\hat{P} = P(z_3)$. After simplifying this result with the help of (4.8) we find precise agreement with (4.5), thereby completing the proof of the MHV tree formula.

5 Discussion and open questions

The tree formula introduced in this paper has several conceptually satisfying features and almost completely fulfills the wish-list outlined in the introduction. It appears to be a genuinely gravitational formula, rather than a recycled Yang-Mills result. Is it, finally, the end of the story for the the MHV amplitude, as the Parke-Taylor formula (1.1) surely is for the n -gluon MHV amplitude?

Among the wish-list items the MHV tree formula fails only in manifesting the full S_n symmetry. Of course it is possible that there simply does not exist any natural more primitive formula which manifests the full symmetry. It is not obvious how one could go about constructing such a formula, but we can draw some encouragement and inspiration from the recent paper [45] which demonstrates how to write manifestly dihedral symmetric formulas for NMHV amplitudes in Yang-Mills theory as certain volume integrals in twistor space. Different ways of dividing the volume into tetrahedra give rise to apparently different but equivalent formulas for NMHV amplitudes. The same goal can apparently also be achieved by writing the amplitude as a certain contour integral where different choices of contour produce different looking but actually equivalent formulas [46, 47]. Perhaps in gravity even the MHV amplitude needs to be formulated in a way which is fundamentally symmetric but which nevertheless requires choosing two of the n gravitons for special treatment.

In Yang-Mills theory the only formula we know of which manifests the full dihedral symmetry for all superamplitudes is the connected prescription [48–52] which follows from Witten’s formulation of Yang-Mills theory as a twistor string theory [53]. Perhaps finding fully S_n symmetric formulas for graviton superamplitudes requires the construction of an appropriate twistor string theory for supergravity, an important question in its own right which has attracted some attention [22, 54–57]. An important motivation for Witten’s twistor string theory was provided by Nair’s observation [58] that the Parke-Taylor formula (1.1) could be computed as a current algebra correlator in a WZW model. The BGK formula (essentially (1.2)) can similarly be related to current correlators and vertex operators in twistor space [59], but we hope that the new MHV tree formula might provide a more appropriate starting point for this purpose and perhaps shed some more light on a twistor-string-like description for supergravity.

Another obvious avenue for future research is to investigate whether any of the advances made here can be usefully applied to non-MHV amplitudes. Unfortunately we have not yet found any very nice structure in the link representation for non-MHV graviton amplitudes. Recently in [61] it was demonstrated how to solve the on-shell recursion for all tree-level supergraviton amplitudes, following steps very similar to those which were used to solve the recursion for supersymmetric Yang-Mills [60]. In [61] a crucial role was played by what was called the graviton subamplitude, which is the summand of an n -particle graviton amplitude inside a sum over $(n - 2)!$ permutations. The decomposition of every amplitude into its subamplitudes allowed for a very efficient application of the on-shell recursion since the same two legs could be singled out and shifted at each step in the recursion. Unfortunately there is no natural notion of a subamplitude for the MHV tree formula, making it very poorly suited as a starting point for attempting to solve the on-shell recursion. In our

view the fact that the tree formula apparently can neither be easily derived from BCF, nor usefully used as an input to BCF, suggests the possible existence of some kind of new rules for the efficient calculation of more general gravity amplitudes.

The arrangement of supergravity amplitudes into ordered subamplitudes also proved very useful in [62, 63] for the purpose of expressing the coefficients of one-loop supergravity amplitudes in terms of one-loop Yang-Mills coefficients. It would certainly be very interesting to see if any of aspects of the MHV tree formula could be useful for loop amplitudes in supergravity, if at least as input for unitarity sums [64, 65].

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A The MHV tree formula in Mathematica

Here we present for the reader's benefit a simple command implementing the MHV tree formula in Mathematica:

```
Needs["Combinatorica`"];
MHV[n_Integer]/;n>4 := 1/ket[n-1,n]^2 1/(Times @@ ((ket[n-1,#] ket[n,#])^2
& /@ Range[n-2])) ((Times @@ (Transpose[#]/.{a___,1,b___,1,c___} :>
prop[Length[{a}]+1,Length[{a,b}]+2])) & /@ IncidenceMatrix /@
CodeToLabeledTree /@ Flatten[Outer[List,Sequence @@
Table[Range[n-2],{n-4}],n-5]) /. prop[a_,b_] ->
bra[a,b]/ket[a,b] ket[n-1,a] ket[n-1,b] ket[n,a] ket[n,b];
```

Here we use the notation $\text{ket}[a, b] = \langle a b \rangle$ and $\text{bra}[a, b] = [a b]$. The (trivial) cases $n = 3, 4$ must be handled separately.

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