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Modularity in Argyres-Douglas theories with $a = c$

Hongliang Jiang

Blackett Laboratory, Imperial College,
London SW7 2AZ, U.K.

E-mail: h.jiang@imperial.ac.uk

Abstract: We consider a family of Argyres-Douglas theories, which are 4D $\mathcal{N} = 2$ strongly coupled superconformal field theories (SCFTs) but share many features with 4D $\mathcal{N} = 4$ super-Yang-Mills theories. In particular, the two central charges of these theories are the same, namely $a = c$. We derive a simple and illuminating formula for the Schur index of these theories, which factorizes into the product of a Casimir term and a term referred to as the Schur partition function. While the former is controlled by the anomaly, the latter is identified with the vacuum character of the corresponding chiral algebra and is expected to satisfy the modular linear differential equation. Our simple expression for the Schur partition function, which can be regarded as the generalization of MacMahon's generalized sum-of-divisor function, allows one to numerically compute the series expansions efficiently, and furthermore find the corresponding modular linear differential equation. In a special case where the chiral algebra is known, we are able to derive the corresponding modular linear differential equation using Zhu's recursion relation. We further study the solutions to the modular linear differential equations and discuss their modular transformations. As an application, we study the high temperature limit or the Cardy-like limit of the Schur index using its simple expression and modular properties, thus shedding light on the 1/4-BPS microstates of genuine $\mathcal{N} = 2$ SCFTs with $a = c$ and their dual quantum gravity via the AdS/CFT correspondence.

Keywords: Conformal and W Symmetry, Supersymmetric Gauge Theory, Extended Supersymmetry

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Contents

1 Introduction

Supersymmetric and superconformal field theories have been attracting intensive interest due to the feasibility of performing exact computations. Many protected sectors have also been discovered in supersymmetric and superconformal field theories, which turn out to enjoy

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rich mathematical structures. In the zoo of supersymmetric field theories, the 4D $\mathcal{N} = 2$ superconformal field theories are of particular interest. On the one hand, the low energy limit on the Coulomb branch of 4D $\mathcal{N} = 2$ superconformal field theories (SCFTs), which is a special Kähler geometry, can be effectively described using the famous Seiberg-Witten theory. On the other hand, the 4D $\mathcal{N} = 2$ superconformal field theories also possess an interesting protected sector, known as the Schur sector, which encodes the information about Higgs branch. The Schur sector is particularly interesting because there is a nice correspondence between the Schur sector and the mathematical notion of vertex operator algebra (VOA), dubbed SCFT/VOA correspondence [1]. One important item in validating this correspondence is given by the so-called Schur index, which just counts the Schur operators in the SCFTs. In general, the Schur index has the following structures:

$$\mathcal{I} = q^{c_{2d}/24}\mathcal{Z} = 1 + \cdots , \qquad \mathcal{Z} = q^{-c_{2d}/24}\mathcal{I} = q^{-c_{2d}/24}(1 + \cdots) = \chi_{\text{vac}} . \tag{1.1}$$

To distinguish the two quantities, we will refer to $\mathcal{I}$ as Schur index, and $\mathcal{Z}$ as Schur partition function. While the Schur index $\mathcal{I}$ is more meaningful for counting and starts with 1 for the unique vacuum state, it is the Schur partition function $\mathcal{Z}$ that is identified with the vacuum character χ_{vac} of the corresponding VOA following the SCFT/VOA correspondence [1]. The Casimir factor $e^{-c_{2d}/24}$ in $\mathcal{Z}$ is crucial to ensure the nice modular behavior in the chiral algebra.[1] See [2, 3] for more discussions on this point.

The superconformal index can be computed in many theories using various techniques. Once the Schur index is known, the Schur partition function can be obtained trivially by multiplying the Casimir factor. The multiplication is straightforward but appears somewhat artificial. It is natural to inquire whether the factorization of the Schur index (1.1) can be achieved in a more natural manner. We will demonstrate that the factorization of Schur index indeed naturally emerges in a family of Argyres-Douglas (AD) theories, which will be denoted as $\mathcal{T}_{(p,N)}$ with two coprime integers $p = 2, 3, 4, 6$ and $N = 2, 3, 4, \cdots$.

The Argyres-Douglas theories $\mathcal{T}_{(p,N)}$ can be obtained by conformally gauging several copies of AD theories $D_{p_i}(\mathrm{SU}(N))$ [4, 5] which have flavor symmetry $\mathrm{SU}(N)$, where the set of p_i depends on the value of p. The Argyres-Douglas theories $\mathcal{T}_{(p,N)}$ share many features with $\mathcal{N} = 4$ $\mathrm{SU}(N)$ super-Yang-Mills (SYM) theories [6, 7]. In particular, the two central charges are exactly the same $a = c$. More surprisingly, the Schur index of $\mathcal{T}_{(p,N)}$ can be obtained from the Schur index of $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM theory by specializing its fugacities to particular values.[2] Meanwhile, the Schur index of $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM admits a closed formula when the flavor fugacities are turned off [9]. The generalization to flavored Schur index was established in [10]. With the closed formula for the flavored Schur index of $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM, we derived the Schur index of $\mathcal{T}_{(p,N)}$ in eq. (2.62) (2.63). Surprisingly, the Schur index derived in this way is very simple and enjoys the obvious and natural factorization of the form in (1.1). We are then led to a remarkably simple formula for the Schur partition function of $\mathcal{T}_{(p,N)}$ AD theory.

Following the SCFT/VOA correspondence, the Schur partition function is identified with the vacuum character of the corresponding chiral algebra. In particular, this implies

[1] We will use vertex operator algebra and chiral algebra interchangeably.

[2] The similarity between $\mathcal{N} = 4$ $\mathrm{SU}(N)$ SYM theory and $\mathcal{T}_{(p,N)}$ AD theory also persists in the non-invertible symmetry, as discussed in [8].

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that Schur partition function satisfies the modular linear differential equation (MLDE) [11], $\mathcal{D}_q^{(k)}\mathcal{Z} = 0$ where $\mathcal{D}_q^{(k)}$ is the modular linear differential operator of weight $2k$ (see appendix A for details). The modular linear differential equations transform covariantly with specific weight under modular transformations, significantly constraining the structure of modular linear differential equations. At a given weight, the modular linear differential equations are almost fully determined, up to several constants. Such a kind of simplicity allows one to numerically search for the modular linear differential equations which are satisfied by the Schur partition function. The numerical search becomes even simpler and tangible thanks to the simple closed form of the Schur partition function of $\mathcal{T}_{(p,N)}$ AD theory. To illustrate, we will find the MLDEs in several examples and study their solutions.

At a fundamental level, the modular linear differential equation is rooted in the existence of a specific kind of null state in the corresponding VOA. Once the null state is known, one can derive the modular linear differential equations systematically using Zhu's recursion relation [12] and the commutation relations in the chiral algebra. We will review the details in appendix B. Unfortunately, the chiral algebra of $\mathcal{T}_{(p,N)}$ AD theory is generally complicated and not known explicitly, except for the simplest case of $\mathcal{T}_{(3,2)}$ AD theory. The VOA of $\mathcal{T}_{(3,2)}$ AD theory is known [13, 14]. With the explicit OPEs at hand, we manage to find the desired null state, and derive the corresponding MLDE of weight 10. Such a MLDE can be verified numerically to very high order.

As a physical application of MLDE, we further study the high temperature limit of the Schur index / Schur partition function. The high temperature limit of the Schur index of 4D $\mathcal{N} \geq 2$ SCFTs has been studied previously in [15, 16] for $\mathcal{N} = 4$ SYM theories and some other $\mathcal{N} = 2$ SCFTs. The $\mathcal{T}_{(p,N)}$ AD theories of interest in this paper have central charges $a = c$ and are honest 4D $\mathcal{N} = 2$ SCFTs without enhanced supersymmetry. So understanding the high temperature limit of the Schur index of $\mathcal{T}_{(p,N)}$ AD theories is important and complementary to the examples studied before. The modularity of solutions to MLDE enables one to derive the high temperature limit systematically. Based on examples studied in this paper and some results in the literature, we are motivated to propose some conjectures on MLDE and particularly a power law asymptotic behavior for the Schur partition function in (4.29), which in the special case of $p = 2$ can be proved using our simple formula for Schur partition functions.

The rest of the paper is organized as follows. In section 2, we will first introduce the $\mathcal{T}_{(p,N)}$ AD theories that will be studied in this paper. Then we will derive a simple and illuminating formula for the Schur index of $\mathcal{T}_{(p,N)}$ which admits an obvious factorization (1.1). In section 3, we will study the modular linear differential equations in $\mathcal{T}_{(p,N)}$. We will also discuss the solutions to the MLDEs and their modular transformation behavior. In section 4, we will study the high temperature limit of the Schur index / partition function based on the MLDEs and modular properties. In section 5, we will summarize the main results of the paper and discuss several open questions for future explorations. We also include a few technical appendices. In appendix A, we will review the concepts of Eisenstein series and modular forms. We will also discuss the general structures of MLDEs and their solutions. In appendix B, we review the basic concepts of VOA, Zhu's recursion relation for torus one-point function, and discuss in detail how to derive MLDE from the null state of the VOA. In appendix C, we will present the explicit MLDEs in several families of AD theories, which are either found numerically or derived rigorously from the null state of the corresponding VOAs.

2 Schur partition function of $\mathcal{T}_{(p,N)}$ Argyres-Douglas theory

In this section, we will first review the properties of $D_p(\mathrm{SU}(N))$ AD theories, which are the building blocks of $\mathcal{T}_{(p,N)}$ AD theories. After discussing the construction and properties of $\mathcal{T}_{(p,N)}$ AD theories, we will then derive a simple formula for the Schur index of $\mathcal{T}_{(p,N)}$ AD theory.

2.1 $D_p(\mathrm{SU}(N))$ AD theory

We will start with a specific type of 4D $\mathcal{N}=2$ SCFTs denoted by $D_p(G)$, which was introduced in [4, 5]. In particular, in this paper we will be mainly focusing on the case of $G=\mathrm{SU}(N)$, namely $D_p(\mathrm{SU}(N))$ where $p, N>1$ are positive integers. Moreover, we will always impose the constraint that p and N are coprime. In this case, the flavor symmetry of $D_p(\mathrm{SU}(N))$ is $\mathrm{SU}(N)$.[3]

The $D_p(\mathrm{SU}(N))$ theories are generalized AD theories [17, 18] and have no direct $\mathcal{N}=2$ Lagrangian description. But they have class $\mathcal{S}$ realization: one can compactify the 6D (0,2) SCFT of type A_{N-1} on a Riemann sphere with an irregular puncture and a full regular puncture. To manifest this construction, one can equivalently use the following notation [18, 19]

$$D_p(\mathrm{SU}(N)) = (A_{N-1}^N[p-N], F) = (I_{N,p-N}, F)\,, \tag{2.1}$$

where F means full puncture which is responsible for the $\mathrm{SU}(N)$ flavor symmetry. The condition $\gcd(p,N)=1$ ensures that the irregular puncture has no further contribution to flavor symmetry.

The flavor central charge k_F and c central charge are [5]

$$k_F = \frac{2N(p-1)}{p}\,, \qquad c = \frac{(p-1)(N^2-1)}{12}\,, \tag{2.2}$$

while a central charge can be computed via

$$8a-4c = \sum_{j=1}^{p-1} \left\lfloor \frac{N}{p} j \right\rfloor \left(2\frac{N}{p} j - \left\lfloor \frac{N}{p} j \right\rfloor \right). \tag{2.3}$$

It turns out for our theories with $\gcd(p,N)=1$, there is a simple formula for a central charge:

$$\frac{a}{c} = 1 - \frac{1}{4p}\,. \tag{2.4}$$

2.2 $\mathcal{T}_{(p,N)}$ AD theory

Suppose we take a collection of $D_{p_i}(\mathrm{SU}(N)))$ theory and gauge the maximal diagonal flavor symmetry, the total one-loop β-function is

$$b = 2\times N - \sum_{i=1}^{\ell} b_i = \left(2 - \sum_{i=1}^{\ell} \frac{p_i-1}{p_i}\right) N\,, \tag{2.5}$$

[3] More precisely, the flavor symmetry is $P\mathrm{SU}(N)$. However, in this paper, we will not be concerned about the global form of the flavor symmetry, so we will just loosely say that the flavor symmetry is $\mathrm{SU}(N)$ for simplicity.

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where the first contribution comes from the vector multiplet and the second contribution comes from the individual $D_{p_i}(\mathrm{SU}(N))$ theory [4].

In order to get a superconformal theory after gauging, we require that the beta function of gauge coupling should vanish, $b = 0$. This is equivalent to the requirement that $\sum_i k_F^{D_{p_i}(\mathrm{SU}(N))} = 4h^\vee(\mathrm{SU}(N)) = 4N$. Then one can easily show that the only possibilities are

$$(p_1, p_2, \cdots, p_\ell) \quad = \quad (2,2,2,2)\,, \qquad (3,3,3)\,, \qquad (2,4,4)\,, \qquad (2,3,6)\,. \tag{2.6}$$

These conformally gauged AD theories are the four series of theories that we will consider in this paper. We will label them respectively as

$$\mathcal{T}_{(2,N)}\,, \qquad \mathcal{T}_{(3,N)}\,, \qquad \mathcal{T}_{(4,N)}\,, \qquad \mathcal{T}_{(6,N)}\,. \tag{2.7}$$

See figure 1 for the four series of theories. Therefore, the theories that we shall study is generally denoted as $\mathcal{T}_{(p,N)}$ where $p = 2,3,4,6$ and $N = 2,3,4\cdots$ subject to the condition that $\gcd(p,N) = 1$. See [7] for further generalizations and an alternative notation for this family of theories that we show in table 1.

A remarkable feature of these theories is that the a and c central charges are exactly the same.

$$c = \sum_i c_{D_{p_i}(\mathrm{SU}(N))} + c_{\mathrm{vec}} \times \dim \mathrm{SU}(N) = \sum_i \frac{(p_i-1)(N^2-1)}{12} + \frac{1}{6}(N^2-1)\,, \tag{2.8}$$

$$a = \sum_i a_{D_{p_i}(\mathrm{SU}(N))} + a_{\mathrm{vec}} \times \dim \mathrm{SU}(N) = \sum_i \left(1 - \frac{1}{4p_i}\right)\frac{(p_i-1)(N^2-1)}{12} + \frac{5}{24}(N^2-1)\,. \tag{2.9}$$

Then it is easy to see that $a - c = b(N^2-1)/48 = 0$. This property is reminiscent of the $\mathcal{N} = 4$ SYM theories which have $a = c$ due to maximally superconformal symmetry.

When $p = 3,4,6$, we can also realize $\mathcal{T}_{(p,N)}$ AD theory by considering IIB string compactified on isolated hypersurface singularity (IHS), which is a Calabi-Yau three-fold defined in terms of a quasi-homogeneous polynomial in $\mathbb{C}^4$ [20]. This geometric engineering way of constructing $\mathcal{T}_{(p,N)}$ AD theories is very useful. For example, one can easily compute the Coulomb branch spectrum from the deformation of singularity. The Higgs branch dimension can also be obtained from the resolution of the singularity [21]. By considering small values of N, one can compute the Higgs branch dimension explicitly.[4] The dependence on N turns out to be very simple and we find that the Higgs branch dimensions of $\mathcal{T}_{(p,N)}$ AD theories fit into the following simple formula

$$\dim_{\mathbb{C}} \mathrm{HB}\left(\mathcal{T}_{(p,N)}\right) = 2 \dim_{\mathbb{H}} \mathrm{HB}\left(\mathcal{T}_{(p,N)}\right) = 2\left\lfloor \frac{N}{p} \right\rfloor\,, \tag{2.10}$$

where $\lfloor \cdot \rfloor$ is the floor function. Note that for $p = 2$, there is no known type IIB realization. However, it was argued in [22] that the $\mathcal{T}_{(2,2k+1)}$ AD theories have Higgs branches of quaternionic dimension k, which is consistent with (2.10).

[4] This Higgs branch dimension is bit tricky, as the naive counting from F-term and hyperkahler quotient gives a zero dimensional Higgs branch. This turns out to be not true due to incomplete Higgsing for these $a = c$ theories: on a generic point on the Higgs branch the gauge group is not completely broken and there are still Coulomb directions in the full moduli space.

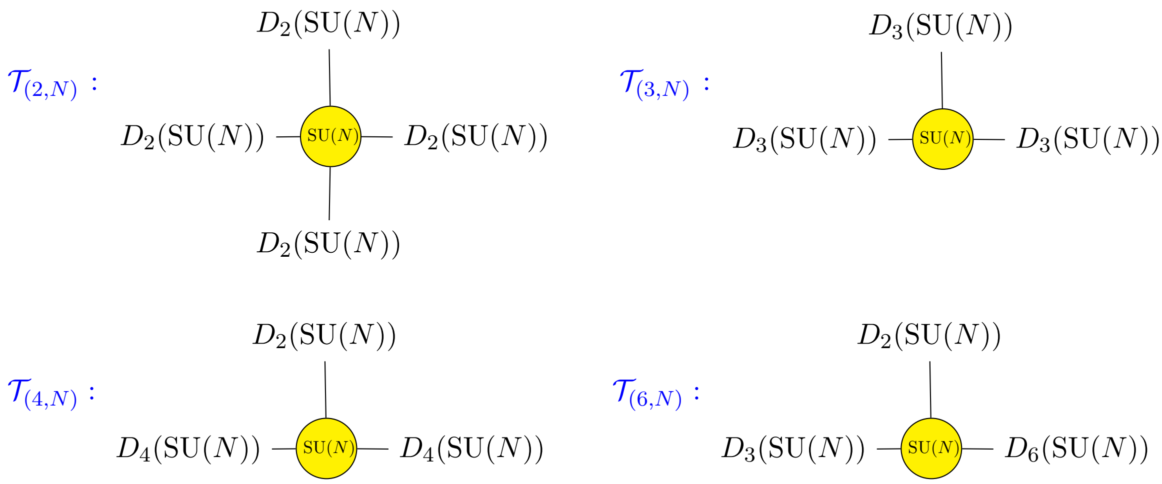

Figure 1. $\mathcal{T}_{(p,N)}$ from gauging of several copies of $D_{p_i}(\mathrm{SU}(N))$. Yellow circle denotes $\mathrm{SU}(N)$ gauge node.

theory	N	IIB string on IHS	$a = c$	$\dim_{\mathbb{H}}$ HB
$\mathcal{T}_{(2,N)} = \widehat{D}_4(\mathrm{SU}(N))$	$2k+1$	-	$\frac{1}{2}(N^2-1)$	k
$\mathcal{T}_{(3,N)} = \widehat{E}_6(\mathrm{SU}(N))$	$3k+1, 3k+2$	$x^3+y^3+z^3+w^N=0$	$\frac{2}{3}(N^2-1)$	k
$\mathcal{T}_{(4,N)} = \widehat{E}_7(\mathrm{SU}(N))$	$2k+1$	$x^2+y^4+z^4+w^N=0$	$\frac{3}{4}(N^2-1)$	$\left\lfloor \frac{k}{2} \right\rfloor$
$\mathcal{T}_{(6,N)} = \widehat{E}_8(\mathrm{SU}(N))$	$6k+1, 6k+5$	$x^2+y^3+z^6+w^N=0$	$\frac{5}{6}(N^2-1)$	k

Table 1. The notations and properties of $\mathcal{T}_{(p,N)}$ theories. Here $N > 1$ and $k, N \in \mathbb{N}$.

We list some properties of $\mathcal{T}_{(p,N)}$ SCFTs in table 1. In particular, we find that the central charges are uniformlly given by

$$a = c = \left(1 - \frac{1}{p}\right)(N^2-1)\,. \tag{2.11}$$

Among the four infinite series of theories, there are several special theories which admit other constructions. For example, we have

$$\mathcal{T}_{(3,2)} = \widehat{E}_6(\mathrm{SU}(2)) = (A_2, D_4) = D_4^6[3] = D_4^4[2]\,, \tag{2.12}$$

where the last notation comes from the class $\mathcal{S}$ realization by compactifying 6d (0,2) SCFT of type D_4 on a Riemann sphere with just one irregular puncture (and without any regular puncture) [18, 23]. The notation (A_2, D_4) also comes from the type IIB realization, which is different but equivalent to the type IIB realization in table 1. Similarly, the other two special theories are

$$\mathcal{T}_{(6,5)} = \widehat{E}_8(\mathrm{SU}(5)) = (A_5, E_8) = E_8^{30}[6]\,, \quad \mathcal{T}_{(4,3)} = \widehat{E}_7(\mathrm{SU}(3)) = (A_3, E_6) = E_6^{12}[4]\,. \tag{2.13}$$

2.3 Schur partition function

In 4D $\mathcal{N}=2$ SCFTs, the Schur index is a useful quantity to count the Schur operators which are 1/4-BPS

$$\mathcal{I}(q,\boldsymbol{x}) = \mathrm{Tr}(-1)^F q^{E-R} \prod_j (x_j)^{F_j}\,, \tag{2.14}$$

where the E is the energy, R is the $\mathrm{SU}(2)_R$ weight, F is the fermion number, x_j and F_j are the fugacity and generators for flavor symmetries.

The Schur index of $D_p(\mathrm{SU}(N))$ takes a very simple form [19]:

$$\mathcal{I}_{D_p(\mathrm{SU}(N))}(q,\boldsymbol{x}) = \mathrm{PE}\left[\frac{q-q^p}{(1-q)(1-q^p)}\chi_{\mathrm{adj}}^{\mathrm{SU}(N)}(\boldsymbol{x})\right], \tag{2.15}$$

where $\boldsymbol{x}$ is the fugacity for $\mathrm{SU}(N)$ flavor symmetry, and $\chi_{\mathrm{adj}}^{\mathrm{SU}(N)}$ is the character of $\mathrm{SU}(N)$ in the adjoint representation.

The index of gauged theory can be obtained by taking the product of the individual matter components and vector multiplet contributions, and then projecting to gauge invariant sector. For our theory $\mathcal{T}_{(p,N)}$, the index is given by

$$\mathcal{I}_{\mathcal{T}(p,N)}(q) = \oint [d\boldsymbol{x}] \prod_{j=1}^{\ell} \mathcal{I}_{D_{p_j}(\mathrm{SU}(N))}(q,\boldsymbol{x}) \times \mathcal{I}_{\mathrm{vec}}(q,\boldsymbol{x})\,, \tag{2.16}$$

where the Haar measure for $\mathrm{SU}(N)$ is

$$[d\boldsymbol{x}] = \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{\substack{j,k=1 \\ j\neq k}}^{N} \left(1-\frac{x_j}{x_k}\right), \qquad x_1 x_2 \cdots x_N = 1\,, \tag{2.17}$$

and the vector multiplet contribution is

$$\mathcal{I}_{\mathrm{vec}}(q,\boldsymbol{x}) = \mathrm{PE}\left[-\frac{2q}{1-q}\chi_{\mathrm{adj}}^{\mathrm{SU}(N)}(\boldsymbol{x})\right]. \tag{2.18}$$

The notation PE stands for plethystic exponential defined by

$$\mathrm{PE}[f(x_1,x_2,\cdots)] = \exp\left(\sum_{k=1}^{\infty} \frac{f(x_1^k,x_2^k,\cdots)}{k}\right). \tag{2.19}$$

After combining all the contributions, the Schur index of $\mathcal{T}_{(p,N)}$ AD theory then takes the following explicit form

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(q) = \oint [d\boldsymbol{x}]\ \mathrm{PE}\left[\left(-\frac{2q}{1-q} + \sum_i \frac{q-q^{p_i}}{(1-q)(1-q^{p_i})}\right)\chi_{\mathrm{adj}}^{\mathrm{SU}(N)}(\boldsymbol{x})\right]. \tag{2.20}$$

On the other hand, the Schur index of $\mathcal{N}=4$ SYM, which can be thought as an $\mathcal{N}=2$ theory with one vector and one hyper transforming in the adjoint representation, is:

$$\mathcal{I}_{\mathcal{N}=4\ \mathrm{SU}(N)\ \mathrm{SYM}}(q,y) = \oint [d\boldsymbol{x}]\, \mathcal{I}_{\mathrm{hyper}}(q,y,\boldsymbol{x}) \mathcal{I}_{\mathrm{vec}}(q,\boldsymbol{x}) \tag{2.21}$$

$$= \oint [d\boldsymbol{x}]\ \mathrm{PE}\left[\left(-\frac{2q}{1-q} + \frac{q^{\frac{1}{2}}}{1-q}(y+y^{-1})\right)\chi_{\mathrm{adj}}^{\mathrm{SU}(N)}(\boldsymbol{x})\right], \tag{2.22}$$

where the hypermultiplet contribution is

$$\mathcal{I}_{\text{hyper}}(q,y,\boldsymbol{x}) = \text{PE}\left[\frac{q^{\frac{1}{2}}}{1-q}\left(y+\frac{1}{y}\right)\chi_{\text{adj}}^{\text{SU}(N)}(\boldsymbol{x})\right], \tag{2.23}$$

and y is the fugacity of the $\text{SU}(2)_F$ symmetry arising from the $\text{SU}(4)_R$ symmetry of $\mathcal{N}=4$ SYM.

It is not obvious, but easy to verify that the indices of $\mathcal{T}_{(p,N)}$ and $\mathcal{N}=4$ SYM theories are related in a simple way

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(q) = \mathcal{I}_{\mathcal{N}=4\ \text{SU}(N)\ \text{SYM}}(q^p, q^{\frac{p}{2}-1}) = \oint [d\boldsymbol{x}]\ \text{PE}\left[\left(\frac{q+q^{p-1}-2q^p}{1-q^p}\right)\chi_{\text{adj}}^{\text{SU}(N)}(\boldsymbol{x})\right]. \tag{2.24}$$

This index relation between AD theories and $\mathcal{N}=4$ SYM theories was first observed in [6]. In the special case of $p=3, N=2$, it has been understood as a consequence of the operator map in VOAs of two theories.

We will use this index relation to derive a simple formula for the Schur index of $\mathcal{T}_{(p,N)}$ theory. For this purpose, we need to first review the known closed formula for the Schur index of $\mathcal{N}=4$ SYM theory, which turns out to be simpler for $\text{U}(N)$ gauge group.

In [9], a closed form expression for the unflavored Schur index of $\mathcal{N}=4$ $\text{U}(N)$ SYM theory was given

$$\mathcal{I}_{\mathcal{N}=4\ \text{U}(N)\ \text{SYM}}(q) = \frac{1}{\theta(q)}\sum_{n=0}^{\infty}(-1)^n\left[\binom{N+n}{N}+\binom{N+n-1}{N}\right]q^{\frac{n^2+Nn}{2}}, \tag{2.25}$$

where

$$\theta(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}} = (q;q)_\infty (q^{\frac{1}{2}};q)_\infty^2, \tag{2.26}$$

and the q-Pochhammer symbol

$$(x;q)_\infty = \prod_{j=0}^{\infty}(1-xq^j). \tag{2.27}$$

One useful property that we will use later is that in the limit $x\to 1$,

$$\lim_{x\to 1}\frac{(x;q)_\infty}{1-x} = \prod_{j=1}^{\infty}(1-q^j) = (q,q)_\infty. \tag{2.28}$$

In [10], the generalization to the case of Schur index with flavor fugacity turned on was achieved using the fermi gas approach. For convenience, we introduce ξ via $y=\sqrt{q}/\xi$, then the flavored Schur index of $\mathcal{N}=4$ $\text{U}(N)$ SYM theory can be written as

$$\mathcal{I}_{\mathcal{N}=4\ \text{U}(N)\ \text{SYM}}(q, y=\sqrt{q}/\xi) = \Lambda_{\text{U}(N)}(q,u,\xi) Z_N(q,u,\xi), \tag{2.29}$$

where

$$\Lambda_{\text{U}(N)}(q,u,\xi) = (-1)^N \xi^{-\frac{1}{2}(N-1)N}\frac{(\frac{q}{u};q)_\infty (u;q)_\infty}{(\frac{q}{u}\xi^{-N};q)_\infty (u\xi^N;q)_\infty}, \tag{2.30}$$

and Z_N is defined via the generating function

$$\Xi(\mu, q, u, \xi) = \sum_{N=0}^{\infty} \mu^N Z_N(q, u, \xi) = \prod_{j=-\infty}^{\infty} \left(1 - \frac{\mu \xi^j}{1 - u q^j}\right). \tag{2.31}$$

Here u is an arbitrary parameter whose contribution to index cancels out finally. We will specialize u to different values in order to simplify the formula.

Let us first consider the limit $u \to 1$, but keep q, ξ generic. Using (2.28), we find (2.30) simplifies

$$\Lambda_{\mathrm{U}(N)}(q, u, \xi) = (1-u)\tilde{\Lambda}_{\mathrm{U}(N)}(q, \xi) + O((1-u)^2), \tag{2.32}$$

where

$$\tilde{\Lambda}_{\mathrm{U}(N)}(q, \xi) = (-1)^N \xi^{-\frac{1}{2}(N-1)N} \frac{(q;q)_\infty^2}{(q\xi^{-N};q)_\infty (\xi^N;q)_\infty}. \tag{2.33}$$

Meanwhile, (2.31) becomes

$$\begin{aligned}\Xi(\mu, q, u, \xi) &= \sum_{N=0}^{\infty} \mu^N Z_N(q, u, \xi) = \left(1 - \frac{\mu}{1-u}\right) \prod_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \left(1 - \frac{\mu \xi^j}{1 - u q^j}\right) \\ &= \frac{1}{1-u}\tilde{\Xi}(\mu, q, \xi) + O((1-u)^0), \end{aligned} \tag{2.34}$$

where

$$\tilde{\Xi}(\mu, q, \xi) = \sum_{N=0}^{\infty} \mu^N \tilde{Z}_N(q, \xi) = -\mu \prod_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \left(1 - \frac{\mu \xi^j}{1 - q^j}\right), \tag{2.35}$$

and we have

$$Z_N(q, u, \xi) = \frac{1}{1-u}\tilde{Z}_N(q, \xi) + O((1-u)^0), \tag{2.36}$$

Combining (2.32) and (2.36), the $\mathcal{N} = 4$ U(N)) SYM index (2.29) can be written as

$$\mathcal{I}_{\mathcal{N}=4\ \mathrm{U}(N)\ \mathrm{SYM}}\left(q, y = \sqrt{q}/\xi\right) = \lim_{u\to 1} \Lambda_{\mathrm{U}(N)}(q, u, \xi) Z_N(q, u, \xi) = \tilde{Z}_N(q, \xi)\tilde{\Lambda}_{\mathrm{U}(N)}(q, \xi), \tag{2.37}$$

Taking $N = 1$, we get

$$\mathcal{I}_{\mathcal{N}=4\ \mathrm{U}(1)\ \mathrm{SYM}}(q, y = \sqrt{q}/\xi) = \tilde{Z}_1(q, \xi)\tilde{\Lambda}_{\mathrm{U}(1)}(q, \xi) = \frac{(q;q)_\infty^2}{(q/\xi;q)_\infty (\xi;q)_\infty}, \tag{2.38}$$

where we used (2.33) and (2.35).

With this U(1) index, we can get the Schur index of $\mathcal{N} = 4$ SU(N) SYM theory:

$$\mathcal{I}_{\mathcal{N}=4\ \mathrm{SU}(N)\ \mathrm{SYM}}\left(q, y = \sqrt{q}/\xi\right) = \frac{\mathcal{I}_{\mathcal{N}=4\ \mathrm{U}(N)\ \mathrm{SYM}}\left(q, y = \sqrt{q}/\xi\right)}{\mathcal{I}_{\mathcal{N}=4\ \mathrm{U}(1)\ \mathrm{SYM}}\left(q, y = \sqrt{q}/\xi\right)} = \Lambda_{\mathrm{SU}(N)}(q, u, \xi) Z_N(q, u, \xi), \tag{2.39}$$

where

$$\Lambda_{\mathrm{SU}(N)}(q, u, \xi) = \frac{\Lambda_{\mathrm{U}(N)}(q, u, \xi)}{\mathcal{I}_{\mathcal{N}=4\ \mathrm{U}(1)\ \mathrm{SYM}}\left(q, y = \sqrt{q}/\xi\right)} \tag{2.40}$$

$$= (-1)^N \xi^{-\frac{1}{2}(N-1)N} \frac{\left(\frac{q}{u};q\right)_\infty (u;q)_\infty}{\left(\frac{q}{u}\xi^{-N};q\right)_\infty (u\xi^N;q)_\infty} \frac{(q/\xi;q)_\infty (\xi;q)_\infty}{(q;q)_\infty^2}. \tag{2.41}$$

Now we can use the index relation (2.24) to get the Schur index of $\mathcal{T}_{(p,N)}$ theory. To distinguish, we will use Q in place of q as the argument for the Schur index of $\mathcal{T}_{(p,N)}$. Then (2.24) implies

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(Q)=\mathcal{I}_{\mathcal{N}=4\,\mathrm{SU}(N)\,\mathrm{SYM}}(q=Q^p,y=\sqrt{q}/\xi=Q^{p/2-1})=\Lambda_{\mathrm{SU}(N)}(Q^p,u,Q)Z_N(Q^p,u,Q)\,. \tag{2.42}$$

Therefore we must set $\xi = Q$ and

$$\begin{aligned}\Lambda_{\mathrm{SU}(N)}(Q^p,u,Q) &= (-1)^N Q^{-\frac{1}{2}(N-1)N}\frac{\left(\frac{Q^p}{u};Q^p\right)_\infty (u;Q^p)_\infty}{\left(\frac{Q^{p-N}}{u};Q^p\right)_\infty (uQ^N;Q^p)_\infty}\\ &\quad\times\frac{(Q^{p-1};Q^p)_\infty (Q;Q^p)_\infty}{(Q^p;Q^p)^2_\infty}\,,\end{aligned} \tag{2.43}$$

$$\Xi(\mu,Q^p,u,Q)=\sum_{N=0}^{\infty}\mu^N Z_N(Q^p,u,Q)=\left(1-\frac{\mu/Q}{1-u/Q^p}\right)\prod_{\substack{j=-\infty\\ j\neq-1}}^{\infty}\left(1-\frac{\mu Q^j}{1-uQ^{jp}}\right). \tag{2.44}$$

To proceed further, we take an alternative limit $u\to Q^p$. Using (2.28), we get

$$\Lambda_{\mathrm{SU}(N)}(\mu,Q^p,u,Q)=(1-Q^p/u)\frac{(-1)^N Q^{-\frac{1}{2}(N-1)N}(Q;Q^p)_\infty (Q^{p-1};Q^p)_\infty}{(Q^{-N};Q^p)_\infty (Q^{N+p};Q^p)_\infty}+O\Big((1-Q^p/u)^2\Big), \tag{2.45}$$

and

$$\Xi(\mu,Q^p,u,Q)=\sum_{N=0}^{\infty}\mu^N Z_N(Q^p,u,Q)=-\frac{\mu/Q}{1-u/Q^p}\Omega(\mu,Q^p,Q)+O\Big((1-Q^p/u)^0\Big), \tag{2.46}$$

where

$$\Omega(\mu,Q^p,Q)=\sum_{N=0}^{\infty}\mu^{N-1}J_N(Q^p,Q)=\prod_{\substack{j=-\infty\\ j\neq-1}}^{\infty}\left(1-\frac{\mu Q^j}{1-Q^{(j+1)p}}\right)=\prod_{\substack{j=-\infty\\ j\neq0}}^{\infty}\left(1-\frac{\mu Q^{j-1}}{1-Q^{jp}}\right). \tag{2.47}$$

This implies

$$Z_N(Q^p,u,Q)=-\frac{1/Q}{1-u/Q^p}J_N(\mu,Q^p,Q)+O\Big((1-Q^p/u)^0\Big)\,. \tag{2.48}$$

Combining (2.45) and (2.48) together, we get

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(Q)=\frac{(-1)^N Q^{-\frac{1}{2}(N-1)N}(Q;Q^p)_\infty (Q^{p-1};Q^p)_\infty}{(Q^{-N};Q^p)_\infty (Q^{N+p};Q^p)_\infty}J_N/Q\,. \tag{2.49}$$

To further simplify the expression, we introduce $\lambda=-\mu/Q$ and I_N via

$$\Omega=\sum_{N=0}^{\infty}\lambda^{N-1}I_N(Q)=\sum_{N=0}^{\infty}(-\lambda Q)^{N-1}J_N=\prod_{\substack{j=-\infty\\ j\neq0}}^{\infty}\left(1+\frac{\lambda Q^j}{1-Q^{jp}}\right). \tag{2.50}$$

It is easy to see that $I_N=(-Q)^{N-1}J_N$. Substituting it to (2.49), we get

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{(p,N)}}(Q) &= \frac{(-1)^N Q^{-\frac{1}{2}(N-1)N}(Q;Q^p)_\infty (Q^{p-1};Q^p)_\infty}{(Q^{-N};Q^p)_\infty (Q^{N+p};Q^p)_\infty}I_N/(-Q)^{N-1}/Q && (2.51)\\ &= -\frac{Q^{-\frac{1}{2}(N+1)N}(Q;Q^p)_\infty (Q^{p-1};Q^p)_\infty}{(Q^{-N};Q^p)_\infty (Q^{N+p};Q^p)_\infty}I_N\,. && (2.52)\end{aligned}$$

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For convenience of notation, we change variables back to q and μ and get

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(q) = C(q) Z_{\mathcal{T}_{(p,N)}}(q)\,, \tag{2.53}$$

where

$$C(q) = -\frac{q^{-\frac{1}{2}N(N+1)}\,(q;q^p)_\infty\,(q^{p-1};q^p)_\infty}{(q^{-N};q^p)_\infty\,(q^{N+p};q^p)_\infty}\,, \qquad \sum_{N=1}^{\infty}\mu^{N-1} Z_{\mathcal{T}_{(p,N)}}(q) = \prod_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left(1+\frac{\mu q^j}{1-q^{jp}}\right). \tag{2.54}$$

Surprisingly, the formula can be further simplified for the $\mathcal{T}_{(p,N)}$ AD theories studied here. In particular, the values of p, N are restricted such that $p = 2,3,4,6$ and $\gcd(p,N) = 1$. So in all cases we always have $N = rp \pm 1$ for $r \in \mathbb{Z}$. If $N = rp+1$, we have

$$\begin{aligned}
\frac{(q;q^p)_\infty\,(q^{p-1};q^p)_\infty}{(q^{-N};q^p)_\infty\,(q^{N+p};q^p)_\infty} &= \frac{(q;q^p)_\infty\,(q^{p-1};q^p)_\infty}{(q^{(r+1)p+1};q^p)_\infty\,(q^{-rp-1};q^p)_\infty} && (2.55)\\
&= \prod_{j=0}^{r}\frac{1-q^{jp+1}}{1-q^{-jp-1}} = \prod_{j=0}^{r}(-q^{jp+1}) && (2.56)\\
&= (-1)^{r+1} q^{\sum_{j=0}^{r} jp+1} && (2.57)\\
&= (-1)^{r+1} q^{(rp+2)(r+1)/2}\,, && (2.58)
\end{aligned}$$

therefore

$$C(q) = -q^{-\frac{1}{2}N(N+1)}(-1)^{r+1} q^{(rp+2)(r+1)/2} = (-1)^r q^{-\frac{1}{2}(p-1)r(pr+2)} = (-1)^r q^{-\frac{1}{2}\left(N^2-1\right)\left(1-\frac{1}{p}\right)}\,. \tag{2.59}$$

Similarly, if $N = rp-1$, we find

$$C(q) = -(-1)^r q^{-\frac{1}{2}(p-1)r(pr-2)} = -(-1)^r q^{-\frac{1}{2}\left(N^2-1\right)\left(1-\frac{1}{p}\right)}\,. \tag{2.60}$$

In both cases, we have

$$\begin{aligned}
C(q) &= (-1)^r s q^{-\frac{1}{2}\left(N^2-1\right)\left(1-\frac{1}{p}\right)} = (-1)^r s q^{-\frac{1}{2}c^{\mathcal{T}_{(p,N)}}} = (-1)^{\left\lfloor \frac{N}{p}\right\rfloor} q^{-\frac{1}{2}c^{\mathcal{T}_{(p,N)}}}\,,\\
N &= rp+s\,, \quad s = \pm 1\,.
\end{aligned} \tag{2.61}$$

where we used (2.11).

We can then redefine $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = (-1)^{\left\lfloor \frac{N}{p}\right\rfloor} Z_{\mathcal{T}_{(p,N)}}(q)$ and get the final formula for the Schur index

$$\mathcal{I}_{\mathcal{T}_{(p,N)}}(q) = q^{-\frac{1}{2}c^{\mathcal{T}_{(p,N)}}}\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = q^{\frac{1}{24}c_{2d}^{\mathcal{T}_{(p,N)}}}\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)\,, \tag{2.62}$$

where $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)$ is given by the generating function

$$\sum_{N=1}^{\infty}\mu^{N-1}(-1)^{\left\lfloor \frac{N}{p}\right\rfloor}\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = \prod_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left(1+\frac{\mu q^j}{1-q^{jp}}\right). \tag{2.63}$$

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Equivalently, we can write the generating function as

$$\sum_{N=1}^{\infty} \mu^{N-1}(-1)^{\left\lfloor \frac{N}{p} \right\rfloor} \mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = \prod_{j=1}^{\infty} \left(1 + \frac{\mu q^j \left(1 - q^{j(p-2)}\right)}{1 - q^{jp}} - \frac{\mu^2 q^{jp}}{(1 - q^{jp})^2}\right). \tag{2.64}$$

Obviously, (2.62) has exactly the factorized form in (1.1). The factorization emerges naturally from our derivation.[5] As we discussed in the introduction, the Schur partition function $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)$ is identified with the vacuum character of the corresponding chiral algebra, and is expected to have nice modular properties. The $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)$ defined via (2.63) is very simple, suggesting that it defines an elementary function with nice modular properties and deserves further studies.

The Schur partition function $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)$ can be more explicitly written as

$$\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = (-1)^{\left\lfloor \frac{N}{p} \right\rfloor} \sum_{\substack{m_1 < m_2 < \cdots < m_{N-1} \\ m_i \in \mathbb{Z}, m_i \neq 0}} \prod_{j=1}^{N-1} \frac{q^{m_j}}{1 - q^{m_j p}}. \tag{2.65}$$

When $p = 2$, it is easy to see that m_i and $m_j = -m_i$ must appear in pair in each summand, otherwise, the two summands with m_i and $-m_i$ would cancel. One can also see this point from the expression in (2.64). Therefore, we should have $N = 2k + 1$, and

$$\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) = (-1)^k \sum_{0 < m_1 < m_2 < \cdots < m_k < \infty} \prod_{j=1}^{k} \frac{-q^{2m_j}}{(1 - q^{2m_j})^2} = \sum_{0 < m_1 < m_2 < \cdots < m_k < \infty} \prod_{j=1}^{k} \frac{q^{2m_j}}{(1 - q^{2m_j})^2}. \tag{2.66}$$

Note that the sign factor $(-1)^{\left\lfloor \frac{N}{p} \right\rfloor} = (-1)^k$ precisely cancels the sign from the product, rendering a function with positive coefficients in q. In this case, $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q)$ turns out to be related to the known function via $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) = A_k(q^2)$,[6] where

$$A_k(q) = \sum_{0 < m_1 < m_2 \cdots < m_k < \infty} \frac{q^{m_1 + \cdots m_k}}{(1 - q^{m_1})^2 \cdots (1 - q^{m_k})^2} \tag{2.67}$$

is known as the MacMahon's generalized 'sum-of-divisor' function [24]. It satisfies the recursion relation [25]

$$A_k(q) = \frac{1}{2k(2k+1)} \left[(6A_1(q) + k(k-1)) A_{k-1}(q) - 2q \frac{d}{dq} A_{k-1}(q) \right], \tag{2.68}$$

and

$$A_1(q) = \frac{1}{2} \mathbb{E}_2(q) + \frac{1}{24}. \tag{2.69}$$

[5]One may complain that $(-1)^{\left\lfloor \frac{N}{p} \right\rfloor}$ is not natural. But this is mild as it is just a sign. Furthermore, this sign is actually important for $\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q)$ in (2.63) to have positive coefficients. This point is particularly obvious for $p = 2$, see (2.66).

[6]This special case of $p = 2$ was noticed before in [7].

Here $\mathbb{E}_{2k}$ are Eisenstein series (A.6). This implies $A_k(q)$ is a quasi-modular form, and can be written as the polynomial of Eisenstein series $\mathbb{E}_2, \mathbb{E}_4, \mathbb{E}_6$. See appendix A for definitions of various concepts.[7]

For $p = 3, N = 2$, we have

$$\mathcal{Z}_{\mathcal{T}_{(3,2)}}(q) = \sum_{m\in\mathbb{Z}, m\neq 0} \frac{q^m}{1-q^{3m}} = \sum_{m\in\mathbb{Z}_+} \frac{q^m - q^{2m}}{1-q^{3m}} = \sum_{m=1}^{\infty} \frac{q^m}{1+q^m+q^{2m}} \tag{2.70}$$

$$= q\Big(1 + q^2 + q^3 + 2q^6 + q^8 + q^{11} + 2q^{12} + q^{15} + \cdots\Big). \tag{2.71}$$

For $p = 4, N = 3$, we have

$$\mathcal{Z}_{\mathcal{T}_{(4,3)}}(q) = \sum_{m>n\geq 1} \frac{q^{m+n} + q^{3(m+n)}}{(1-q^{4m})(1-q^{4n})} - \sum_{m,n\geq 1} \frac{q^{3m+n}}{(1-q^{4m})(1-q^{4n})} \tag{2.72}$$

$$= q^3\Big(1+q^2+q^3+2q^4+3q^6+q^7+3q^8+q^9+3q^{10}+2q^{11}+6q^{12}+4q^{14}+3q^{15}+\cdots\Big). \tag{2.73}$$

Note that (2.63) is derived from physics and the values of p and N are subject to various conditions which are crucial in (2.15) an (2.55)–(2.58). But at the level of mathematics, (2.63) applies to all p and N. Some may be trivial. For example, one can check that

$$\mathcal{Z}_{\mathcal{T}_{(3,3)}}(q) = \sum_{m>n\geq 1} \frac{q^{m+n} + q^{2(m+n)}}{(1-q^{3m})(1-q^{3n})} - \sum_{m,n\geq 1} \frac{q^{2m+n}}{(1-q^{3m})(1-q^{3n})} = 0\,, \tag{2.74}$$

and more generally we expect $\mathcal{Z}_{\mathcal{T}_{(p,jp)}} = 0$ for $j \in \mathbb{N}$. On the other hand for $p = 4, N = 2$, $\mathcal{Z}_{\mathcal{T}_{(4,2)}}(q)$ is not zero and furthermore satisfies the MLDE of weight 18. They do not correspond to the indices of any SCFT constructed here, but it would be interesting to study them further from the mathematical perspective.

3 Modular linear differential equation

In this section, we will study the modular linear differential equations in $\mathcal{T}_{(p,N)}$ AD theories. In the simplest case of $\mathcal{T}_{(3,2)}$ whose VOA is known, we will derive the corresponding MLDE using the formalism in appendix A and B. In other cases, we will present the modular linear differential equations satisfied by Schur partition function, which are found through numerics. We will also discuss the solutions to the MLDEs and their modular transformation properties.

3.1 $\mathcal{T}_{(3,2)}$

The chiral algebra of $\mathcal{T}_{(3,2)}$ AD theory is discussed in [6] and given by the $\mathcal{A}(6)$ algebra [26, 27]. It contains 3 generators, denoted by $T, \Psi, \widetilde{\Psi}$, whose conformal dimensions are 2, 4, 4,

[7]However, $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q)$ seems not to be a quasi-modular form. One can numerically check that $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q)$ can not be written as the polynomial of $\mathbb{E}_2, \mathbb{E}_4, \mathbb{E}_6$ with finite degree. Nevertheless, $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q)$ still enjoys nice modular properties based on the physical expectation of SCFT/VOA correspondence, and satisfies MLDE as we will discuss later.

respectively. While the first generator T is the stress tensor, the latter two $\Psi, \widetilde{\Psi}$ are fermionic Virasoro primary operators. The OPEs among them are given by

$$T(z)T(0) \sim \frac{-12}{z^4}+\frac{2T}{z^2}+\frac{T'}{z}, \tag{3.1}$$

$$T(z)\Psi(0) \sim \frac{4\Psi}{z^2}+\frac{\Psi'}{z}, \tag{3.2}$$

$$T(z)\widetilde{\Psi}(0) \sim \frac{4\widetilde{\Psi}}{z^2}+\frac{\widetilde{\Psi}'}{z}, \tag{3.3}$$

$$\begin{aligned}\Psi(z)\widetilde{\Psi}(0) \sim & -\frac{6}{z^8}+\frac{2T}{z^6}+\frac{T'}{z^5}+\frac{3\,(T''-T^2)}{7z^4}+\frac{2T^{(3)}-9T'T}{21z^3} \\ & +\frac{-48\,(T')^2-84T''T+36T^3+7T^{(4)}}{420z^2}+\frac{60\left(-5T''T'+6T'T^2-2T^{(3)}T\right)+7T^{(5)}}{2800z},\end{aligned} \tag{3.4}$$

where we ignore the coordinate of operators on the r.h.s., which is 0. Actually the full OPE can be easily bootstrapped using the associativity of OPEs and the information about the conformal dimensions of these operators.

The mode expansion of these operators are

$$T(z)=\sum_{n\in\mathbb{Z}} L_n z^{-2-n}\,, \qquad \Psi(z)=\sum_{n\in\mathbb{Z}} \Psi_n z^{-4-n}\,, \qquad \widetilde{\Psi}(z)=\sum_{n\in\mathbb{Z}} \widetilde{\Psi}_n z^{-4-n}\,. \tag{3.5}$$

Using the previous OPEs, we manage to derive the commutation relations of these modes

$$[L_m, L_n] = (m-n)L_{m+n}-2(m^3-m)\delta_{m+n,0}\,, \tag{3.6}$$

$$[L_m, \Psi_n] = (3m-n)\Psi_{m+n}\,, \tag{3.7}$$

$$[L_m, \widetilde{\Psi}_n] = (3m-n)\widetilde{\Psi}_{m+n}\,, \tag{3.8}$$

$$\begin{aligned}\{\Psi_m, \widetilde{\Psi}_n\} = & \frac{1}{840}n\left(n^2\left(n^2-7\right)^2-36\right)\delta_{0,m+n}-\frac{1}{84}(m-n)\left(m^2-mn+n^2-7\right)\Lambda_{m+n} \\ & +\frac{1}{1680}(m-n)\Big(3(m+n)^4-14mn\left(m^2+mn+n^2\right)-39(m+n)^2+98mn+108\Big)L_{m+n} \\ & -\frac{5}{112}(m-n)\widetilde{\Lambda}_{m+n}+\frac{7}{80}(m-n)\Upsilon_{m+n}\,,\end{aligned} \tag{3.9}$$

where[8]

$$\Lambda = T^2-\frac{3}{10}T''\,, \tag{3.10}$$

$$\widetilde{\Lambda} = T^3-\frac{1}{3}(T')^2-\frac{19}{30}T''T-\frac{1}{36}T''''\,, \qquad \Upsilon = T^3-\frac{1}{7}(T')^2-\frac{11}{14}T''T-\frac{19}{588}T''''\,. \tag{3.11}$$

With the explicit OPEs, we can try to find the null operators. See appendix C.3 for detailed discussions. In particular, we find a null operator at dimension 10:

$$\begin{aligned}N_{10} = & T^5+\frac{4}{3}T^{(3)}T'T-\frac{10}{3}\left(T'\right)^2T^2-\frac{10}{3}T''T^3+2T''\left(T'\right)^2+\frac{9}{4}\left(T''\right)^2T-\frac{1}{9}\left(T^{(3)}\right)^2 \\ & -\frac{1}{9}T^{(4)}T^2-\frac{1}{12}T^{(4)}T''-\frac{1}{30}T^{(5)}T'-\frac{13}{360}T^{(6)}T+\frac{1}{5040}T^{(8)}+\frac{70}{9}\Psi''\tilde{\Psi}+\frac{140}{3}\tilde{\Psi}''\Psi\,.\end{aligned} \tag{3.12}$$

[8]The $\Lambda, \widetilde{\Lambda}, \Upsilon$ constructed in this way have simple commutation relations with stress tensor. E.g. $[L_m, \Lambda_n] = \frac{1}{30}(5c+22)m\left(m^2-1\right)L_{m+n}+(3m-n)\Lambda_{m+n}$.

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In terms of modes, the corresponding null state is

$$\mathcal{N}_{10} = \Big(L_{-2}^5 - \frac{20}{3}L_{-4}L_{-2}^3 - \frac{10}{3}L_{-3}^2 L_{-2}^2 - \frac{8}{3}L_{-6}L_{-2}^2 + 9L_{-4}^2 L_{-2} - 26L_{-8}L_{-2} + 8L_{-5}L_{-3}L_{-2}$$
$$-4L_{-5}^2 + 4L_{-4}L_{-3}^2 + 8L_{-10} - 4L_{-6}L_{-4} - 4L_{-7}L_{-3} + \frac{280}{3}\tilde{\Psi}_{-6}\Psi_{-4} + \frac{140}{9}\Psi_{-6}\tilde{\Psi}_{-4}\Big)\Omega\,. \quad (3.13)$$

This null state has the form of (B.13), namely $(L_{-2})^5\Omega \in C_2(\mathcal{V})$ (B.11). The presence of such a kind of null operator enables us to derive the MLDE using Zhu's recursion relation (B.9)(B.10). We defer the detailed discussions and derivations to the appendix C.3, and only provide the final result here. The resulting MLDE we find takes the following simple form

$$\Big[D_q^{(5)} - 140\mathbb{E}_4 D_q^{(3)} - 700\mathbb{E}_6 D_q^{(2)} - 2000\mathbb{E}_4^2 D_q^{(1)}\Big]\mathcal{Z}_{\mathcal{T}_{(3,2)}} = 0\,, \quad (3.14)$$

where $\mathbb{E}_{2k}$ are Eisenstein series (A.6), and $D_q^{(k)}$ are modular covariant differential operators (A.20). See appendix A for review and discussion on the notations and properties.

The explicit and simple expression of $\mathcal{Z}_{\mathcal{T}_{(3,2)}}$ is given in (2.70). One can then numerically verify that the above MLDE is indeed true.

We would like to find the full set of solutions to MLDE, which correspond to the characters of some modules in the corresponding $\mathcal{A}(6)$ chiral algebra. We can use the following ansatz

$$\chi_b = q^b(1 + a_1 q + a_2 q^2 + \cdots)\,, \qquad b = -\frac{c_{2d}}{24} + h\,. \quad (3.15)$$

Then we get the following indicial equation

$$(1-3b)^2(b-1)b^2 = 0\,, \qquad \rightarrow \qquad b = 0,\ 0,\ \frac{1}{3},\ \frac{1}{3},\ 1\,. \quad (3.16)$$

Due to the degeneracy and integral spacing of the roots, the ansatz (3.15) is not valid in general. Instead we should use the following ansatz:

$$\chi_b = q^b \sum_{j=0}^{N_b}\sum_{i=0}^{\infty} a_{ij}(\log q)^j q^i\,, \quad (3.17)$$

where N_b depends on the structure of roots to the indicial equation. See appendix A.3 for more discussions.

We then find the following set of solution to MLDE (3.14)

$$b=0: \quad \chi_0 = 1\,, \quad (3.18)$$
$$b=0: \quad \chi_0^{\log} = \log q\big(1+6q+6q^3+6q^4+12q^7+6q^9+6q^{12}+12q^{13}+6q^{16}+12q^{19}+\cdots\big)\,, \quad (3.19)$$
$$b=\frac{1}{3}: \quad \chi_{\frac{1}{3}} = q^{\frac{1}{3}}\big(1+q+2q^2+2q^4+q^5+2q^6+q^8+2q^9+2q^{10}+2q^{12}+2q^{14}+3q^{16}+\cdots\big)\,, \quad (3.20)$$
$$b=\frac{1}{3}: \quad \chi_{\frac{1}{3}}^{\log} = \chi_{\frac{1}{3}}\log q\,, \quad (3.21)$$
$$b=1: \quad \chi_1 = \mathcal{Z}_{\mathcal{T}_{(3,2)}} = q+q^3+q^4+2q^7+q^9+q^{12}+2q^{13}+q^{16}+2q^{19}+2q^{21}+q^{25}+\cdots\,. \quad (3.22)$$

Note that since there is no constant term in MLDE (3.14), namely the coefficient of $D_q^{(0)}$ is zero, $\chi_0 = 1$ is also a solution. One can also see that

$$\chi_0^{\log}(\tau) = \log q(1+6\chi_1) = 2\pi i\tau\Big(1+6\chi_1(\tau)\Big)\,. \tag{3.23}$$

The five linearly independent solutions above are the full solutions to the MLDE. They are also the characters of some modules in the corresponding chiral algebra. And we expect that they form a (weakly holomorphic logarithmic) vector-valued modular under modular transformation. More precisely, we can consider the vector of solutions

$$\boldsymbol{\chi} = \Big(\chi_0, \quad \chi_0^{\log}, \quad \chi_1, \quad \chi_{\frac{1}{3}}, \quad \chi_{\frac{1}{3}}^{\log}\Big)^T\,. \tag{3.24}$$

Then, under modular S- and T-transformation, we expect

$$\boldsymbol{\chi}\left(-\frac{1}{\tau}\right) = S\boldsymbol{\chi}(\tau)\,, \qquad \boldsymbol{\chi}(\tau+1) = T\boldsymbol{\chi}(\tau)\,, \tag{3.25}$$

where S, T are modular S and T matrices.

The modular T matrix can be easily derived. In the absence of log term, χ_b transforms to $e^{2\pi i b}\chi_b$. In the presence of log term, χ_b gets mixed with other items in $\boldsymbol{\chi}$, and the modular T matrix is not diagonal anymore. For example, it is easy to see that

$$\chi_0^{\log}(\tau+1) = \chi_0^{\log}(\tau)+2\pi i(1+6\chi_1(\tau))\,, \qquad \chi_{\frac{1}{3}}^{\log}(\tau+1) = 2\pi i e^{\frac{2i\pi}{3}}\chi_{\frac{1}{3}}(\tau)+e^{\frac{2i\pi}{3}}\chi_{\frac{1}{3}}^{\log}(\tau)\,. \tag{3.26}$$

As a result, we find the modular T matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2\pi i & 1 & 12\pi i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2i\pi}{3}} & 0 \\ 0 & 0 & 0 & 2\pi i e^{\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}\,. \tag{3.27}$$

The modular S matrix is generally more complicated to compute. Fortunately, we find that some entries in $\boldsymbol{\chi}$ reduce to known functions whose modular transformation is understood. By numerically computing the solution $\chi_{\frac{1}{3}}$ to very high order,[9] we find that it actually can be written as

$$\chi_{\frac{1}{3}}(\tau) = q^{\frac{1}{3}}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \frac{\eta(3\tau)^3}{\eta(\tau)}\,, \qquad q \equiv e^{2\pi i}\,, \tag{3.28}$$

where $\eta(\tau) = q^{\frac{1}{24}}(q,q)_\infty = q^{\frac{1}{24}}\prod_{j=1}^{\infty}(1-q^j)$ is the Dedekind eta function. Under modular transformation, the Dedekind eta function transforms as

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)\,, \qquad \eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau)\,. \tag{3.29}$$

With the help of these formulae, we can easily find that under S-transformation, the solution (3.28) becomes

$$\chi_{\frac{1}{3}}(-\frac{1}{\tau}) = -\frac{i\tau}{3\sqrt{3}}\frac{\eta\left(\frac{\tau}{3}\right)^3}{\eta(\tau)} = -\frac{\log(q)}{6\sqrt{3}\pi}\frac{\eta\left(\frac{\tau}{3}\right)^3}{\eta(\tau)} = \frac{1}{2\sqrt{3}\pi}\chi_{\frac{1}{3}}^{\log}(\tau) - \frac{1}{6\sqrt{3}\pi}\chi_0^{\log}(\tau)\,. \tag{3.30}$$

[9]We have verified it up to the order q^{140}.

where we write the S-transformed character as the linear combination of the original un-transformed characters in light of (3.25); this can be easily achieved by comparing the coefficients of their q-expansions.

Similarly, $\chi_{\frac{1}{3}}^{\log}$ transforms as

$$\chi_{\frac{1}{3}}^{\log}\left(-\frac{1}{\tau}\right) = -\frac{2\pi}{3\sqrt{3}}\frac{\eta\left(\frac{\tau}{3}\right)^3}{\eta(\tau)} = -\frac{4\pi}{\sqrt{3}}\chi_1(\tau) + \frac{2\pi}{\sqrt{3}}\chi_{\frac{1}{3}}(\tau) - \frac{2\pi}{3\sqrt{3}}\chi_0(\tau)\,. \tag{3.31}$$

The modular S matrix has to satisfy the condition $S^2 = 1$. Imposing such a constraint, we are able to determine S matrix completely:[10]

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{2\pi}{\sqrt{3}} & 0 & -4\sqrt{3}\pi & -4\sqrt{3}\pi & 0 \\ -\frac{1}{6} & -\frac{1}{12\sqrt{3}\pi} & 0 & 0 & -\frac{1}{2\sqrt{3}\pi} \\ 0 & -\frac{1}{6\sqrt{3}\pi} & 0 & 0 & \frac{1}{2\sqrt{3}\pi} \\ -\frac{2\pi}{3\sqrt{3}} & 0 & -\frac{4\pi}{\sqrt{3}} & \frac{2\pi}{\sqrt{3}} & 0 \end{pmatrix}. \tag{3.32}$$

Note that the S matrix is not symmetric. It is easy to verify that $S^2 = (ST)^3 = 1$, as required by modular S and T matrices.[11]

As a result, we find

$$\chi_1\left(-\frac{1}{\tau}\right) = -\frac{1}{6}\chi_0(\tau) - \frac{1}{12\sqrt{3}\pi}\chi_0^{\log}(\tau) - \frac{1}{2\sqrt{3}\pi}\chi_{\frac{1}{3}}^{\log}(\tau)\,. \tag{3.33}$$

Using (3.23), we then find

$$\begin{aligned} \chi_0^{\log}\left(-\frac{1}{\tau}\right) &= \frac{-2\pi i}{\tau}\left(1 + 6\chi_1\left(-\frac{1}{\tau}\right)\right) = \frac{i}{\sqrt{3}\tau}\chi_0^{\log}(\tau) + \frac{2\sqrt{3}i}{\tau}\chi_{\frac{1}{3}}^{\log}(\tau) \\ &= -\frac{2\pi}{\sqrt{3}}\chi_0(\tau) - 4\sqrt{3}\chi_1(\tau) - 4\sqrt{3}\chi_{\frac{1}{3}}(\tau)\,, \end{aligned} \tag{3.34}$$

which agrees with (3.32). This thus provides a strong consistency check of our results.

3.2 $\mathcal{T}_{(2,3)}$

For the $\mathcal{T}_{(2,3)}$ AD theory, the corresponding VOA is not known. So we will resort to numerics to find the corresponding MLDE. The Schur partition function is given by (2.66):

$$\mathcal{Z}_{\mathcal{T}_{(2,3)}} = \sum_{m=1}^{\infty} \frac{q^{2m}}{(1-q^{2m})^2}\,. \tag{3.35}$$

This simple expression allows one to numerically compute the series expansion to very high order efficiently. Furthermore, the SCFT/VOA correspondence indicates that $\mathcal{Z}_{\mathcal{T}_{(2,3)}}$ satisfies the MLDE of specific weight, say $2k$:

$$\mathcal{D}_q^{(k)}\mathcal{Z}_{\mathcal{T}_{(2,3)}} = 0\,, \tag{3.36}$$

[10] One can rescale the character by replacing $\log q$ with $\log q/(6\sqrt{3}\pi)$ and get the modular S matrix with rational coefficients. But this would introduce the factor $\sqrt{3}$ to T matrix.

[11] More generally, the condition is $S^2 = (ST)^3 = C$ where C is the charge conjugation matrix.

where $\mathcal{D}_q^{(k)}$ is the modular linear differential operator and transforms covariantly under modular transformations. More explicitly, it takes the following form (A.22)

$$\mathcal{D}_q^{(k)} = D_q^{(k)} + \sum_{r=1}^{k} f_r(\tau) D_q^{(k-r)}\,, \qquad f_r(\tau) \in \mathcal{M}_{2r}(\Gamma)\,. \tag{3.37}$$

Here $\mathcal{M}_{2r}(\Gamma)$ denotes the modular form of weight $2r$, and is freely generated by $\mathbb{E}_4$ and $\mathbb{E}_6$. So f_r can be written as the polynomial of $\mathbb{E}_4$ and $\mathbb{E}_6$ with finitely many undetermined coefficients. Therefore, the MLDO $\mathcal{D}_q^{(k)}$ is almost fixed completely by modular covariance, up to a finite number of coefficients. To fix these coefficients, we can consider the series expansion of both $\mathcal{D}_q^{(k)}$ and $\mathcal{Z}_{\mathcal{T}_{(2,3)}}$ in q, and check MLDE in (3.36). Starting with $k=1$ in (3.36), one can check whether it has is a solution. If yes, we then find the desired MLDE, otherwise we increase k and repeat the same procedure. This then offers an efficient way to find MLDE numerically.

After implementing the algorithm above, we find that up to the order q^{100}, there is indeed a MLDO of weight 24 which annihilates the Schur partition function $\mathcal{Z}_{\mathcal{T}_{(2,3)}}$, namely

$$\mathcal{D}_q^{(12)} \mathcal{Z}_{\mathcal{T}_{(2,3)}} = 0\,, \tag{3.38}$$

where

$$\begin{aligned}
\mathcal{D}_q^{(12)} \;=\; & D_q^{(12)} - 1510\mathbb{E}_4 D_q^{(10)} - 55440\mathbb{E}_6 D_q^{(9)} - 233400\mathbb{E}_4^2 D_q^{(8)} + 2364600\mathbb{E}_4\mathbb{E}_6 D_q^{(7)} \\
& + 2000\left(31228\mathbb{E}_4^3 - 41013\mathbb{E}_6^2\right) D_q^{(6)} + 1422624000\mathbb{E}_4^2\mathbb{E}_6 D_q^{(5)} \\
& + \left(3925360000\mathbb{E}_4^4 + 40438916000\mathbb{E}_4\mathbb{E}_6^2\right) D_q^{(4)} \\
& + \left(420470400000\mathbb{E}_4^3\mathbb{E}_6 + 344509200000\mathbb{E}_6^3\right) D_q^{(3)} \\
& + \left(1168824000000\mathbb{E}_4^5 + 7510426000000\mathbb{E}_4^2\mathbb{E}_6^2\right) D_q^{(2)} \\
& + \left(23905224000000\mathbb{E}_4^4\mathbb{E}_6 + 31682361200000\mathbb{E}_4\mathbb{E}_6^3\right) D_q^{(1)}\,.
\end{aligned} \tag{3.39}$$

Note that as in the case of $\mathcal{T}_{(3,2)}$ AD theory, there is also no constant term in $\mathcal{D}_q^{(12)}$ here.

The MLDE above is a differential equation of order 12, so it should have other 11 solutions corresponding to other modules of the VOA, in addition to the vacuum character $\mathcal{Z}_{\mathcal{T}_{(2,3)}}$. To find the rest of solutions, we can use the ansatz $\chi_i = q^b(1 + a_1 q + \cdots)$ and substitute it into $\mathcal{D}_q^{(12)}\chi_i = 0$. At leading order in q, this gives the indicial equation

$$(b-2)(b-1)^3 b^3 (2b-1)^3 \left(8b^2 - 36b + 39\right) = 0\,. \tag{3.40}$$

The solutions can be easily found to be (ignoring multiplicity)

$$b = 0,\ 1/2,\ 1,\ 2,\ \frac{1}{4}(9-\sqrt{3}),\ \frac{1}{4}(9+\sqrt{3})\,. \tag{3.41}$$

In particular, note that $b=0$ is the minimal root and has degeneracy 3.

In principle, one can proceed further and find the full set of solutions order by order in q. In practice, this computation is tedious as the MLDO is a differential operator of very high order being 12. The computation is further complicated by the degeneracies of the roots which means logarithmic term in the solutions. Given these complications, we will not study the explicit solutions here, and just be content with the indicial equation. As we will discuss later, the indicial equation is already useful enough and can be used to understand the high temperature limit of the Schur index / partition function.

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3.3 $\mathcal{T}_{(4,3)}$

We can use the same numerical algorithm as before to find the MLDE in $\mathcal{T}_{(4,3)}$ AD theory. It turns out that up to the order of q^{100}, the Schur partition function (2.72) satisfies the following MLDE of weight 34

$$\mathcal{D}_q^{(17)} \mathcal{Z}_{\mathcal{T}_{(4,3)}} = 0\,, \tag{3.42}$$

where the explicit form of MLDO $\mathcal{D}_q^{(17)}$ is given in (C.73).

Similarly, we find the indicial equation

$$(b-3)(b-1)^2 b^3 (2b-3)^2 (2b-1)^3 (4b-1)^2 g(b) = 0\,, \tag{3.43}$$

where

$$g(b) = b^4 - \frac{38b^3}{3} + \frac{21578712128131b^2}{344459812152} - \frac{245015329302 47b}{172229906076} + \frac{167773381022507}{1377839248608}\,. \tag{3.44}$$

The set of solutions is given by (ignoring multiplicity)

$$b = 0,\ \frac{1}{4},\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 3,\ \beta_1,\ \cdots,\ \beta_4\,, \tag{3.45}$$

where β_s are the four roots of $g(b) = 0$. In particular, note that $b = 0$ is the minimal root and has degeneracy 3.

4 High temperature limit

One virtue of modularity is that it relates states in the UV to that in the IR, which allows one to infer the high energy or high temperature behavior. For example, modular invariance of 2D CFT gives rise to the Cardy formula which characterises the high energy density of states universally in terms of central charge [28]. The same philosophy applies here for the Schur partition function of 4D $\mathcal{N} = 2$ SCFTs, as we will discuss below.

We would like to understand the behavior of the Schur index $\mathcal{I}(q)$, or equivalently the Schur partition function $\mathcal{Z}(q)$, in the limit $\tau \to 0$, which will be referred to as the high temperature limit or Cardy limit.

In many cases, the leading asymptotic of the Schur index / partition function is [29, 30]

$$\mathcal{I}(q) \simeq \mathcal{Z}(q) \simeq e^{-\frac{2\pi i}{\tau} 2(a-c)}\,, \qquad \tau \to 0\,. \tag{4.1}$$

It has been observed that this is valid in many example where $a - c < 0$, but violated in a few examples when $a - c > 0$. See [16] for more discussion on this point. The theories studied here have exactly $a = c$, so the formula (4.1), if correct, would predict a finite constant leading term in the limit $\tau \to 0$ or equivalently $q \to 1$. For $\mathcal{N} = 4$ SYM theories which have $a = c$, the Cardy limit of the index has been studied a lot. In particular, the Schur partition function of $\mathcal{N} = 4$ SU(2) SYM with central charge $a = c = 3/4$ has the leading asymptotic [31]

$$\mathcal{Z}^{\mathcal{N}=4\ \mathrm{SU}(2)\ \mathrm{SYM}}(q) \simeq \frac{1}{-4i\tau} - \frac{1}{2\pi} + \cdots\,, \qquad q \equiv e^{2\pi i \tau}\,, \tag{4.2}$$

which is obviously different from (4.1). This indicates that (4.1) may be not valid in theories with equal central charges $a = c$. However, the $\mathcal{N} = 4$ SYM theories are special as they have enhanced SUSY.

Instead the $\mathcal{T}_{(p,N)}$ AD theories studied here are genuine 4D $\mathcal{N} = 2$ SCFTs with $a = c$. Of course, one can try to infer the high temperature behavior of Schur index of $\mathcal{T}_{(p,N)}$ theories based on their index relation with $\mathcal{N} = 4$ SYM. However, except for the special case of $p = 2$, this requires the knowledge of high temperature limit of flavored Schur index of $\mathcal{N} = 4$ SYM, which is generally not known.[12] Given this fact, we will study the high temperature asymptotic behavior of Schur index of $\mathcal{T}_{(p,N)}$ theories directly using MLDE and modular property. In the case of $p = 2$, it turns out that we can derive the high temperature limit for all $\mathcal{T}_{(2,2k+1)}$ using recursion relation and the defining generating function. These results motivate us to make some conjectures about MLDE and high temperature limit of Schur partition function.

4.1 $\mathcal{T}_{(3,2)}$

Let us first consider the $\mathcal{T}_{(3,2)}$ AD theory, whose modular properties have been discussed extensively in subsection 3.1. We wan to understand the high temperature limit $\tau \to 0$ of Schur partition function $\mathcal{Z}_{\mathcal{T}_{(3,2)}}(\tau)$, which is identified with the vacuum character χ_1. For our purpose of application, we rewrite (3.33) as

$$\chi_1(\tau) = \frac{i}{\sqrt{3}\tau}\left(\chi_1\left(-\frac{1}{\tau}\right) + \chi_{\frac{1}{3}}\left(-\frac{1}{\tau}\right)\right) + \frac{i}{6\sqrt{3}\tau} - \frac{1}{6}\,. \tag{4.3}$$

This can be used to study the behavior of the Schur partition function in the high temperature limit, namely $\tau \to 0, \tilde{\tau} = -1/\tau \to \infty$, $q \to 1$, $\tilde{q} = e^{2\pi i\tilde{\tau}} = e^{-2\pi i/\tau} \to 0$. In this limit, it is easy to see that $\chi_1(\tilde{\tau}), \chi_{\frac{1}{3}}(\tilde{\tau}) \to 0$ up to exponentially small corrections. Therefore in the high temperature limit $\tau \to 0$, we have

$$\mathcal{Z}_{\mathcal{T}_{(3,2)}}(\tau) = \chi_1(\tau) = \frac{i}{6\sqrt{3}\tau} - \frac{1}{6} + O(e^{-2\pi i/3/\tau})\,, \qquad \tau \to 0\,, \tag{4.4}$$

up to exponentially suppressed corrections.

4.2 $\mathcal{T}_{(2,2k+1)}$

We now derive the high temperature asymptotic behavior of the Schur partition function of $\mathcal{T}_{(2,2k+1)}$ AD theories. We will provide two ways to derive it.

The first way to derive is to use $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) = A_k(q^2)$, where $A_k(q)$ is MacMahon's generalized 'sum-of-divisor' function defined in (2.67) and satisfies the recursion relation (2.68) and (2.69). It turns out these formulae are useful enough to derive the asymptotic behavior of $A_k(q)$.

Let us first derive the asymptotic behavior of $A_1(q)$. Using the modular behavior of $\mathbb{E}_2$ in (A.7), we find

$$\mathbb{E}_2(\tau) = \frac{1}{\tau^2}\mathbb{E}_2\left(-\frac{1}{\tau}\right) + \frac{1}{2\pi i\tau}\,. \tag{4.5}$$

[12] It would be interesting to use flavored MLDE to understand the high temperature limit of flavored Schur index of $\mathcal{N} = 4$ SYM. The flavored MLDE has been studied e.g. in [32].

Meanwhile, we have

$$\mathbb{E}_2(\tilde{\tau}) = -\frac{1}{12} + 2\tilde{q} + 6\tilde{q}^2 + 8\tilde{q}^3 + \cdots\,, \qquad \tilde{q} = e^{2\pi i\tilde{\tau}} \to 0\,, \tag{4.6}$$

where $\tilde{\tau} = -\frac{1}{\tau} \to \infty$. Combining them together, we get

$$\mathbb{E}_2(\tau) = \frac{1}{\tau^2}\mathbb{E}_2(\tilde{\tau}) + \frac{1}{2\pi i\tau} = -\frac{1}{12\tau^2} + \frac{1}{2\pi i\tau} + O(e^{\#/\tau})\,, \qquad \tau \to 0\,, \tag{4.7}$$

up to exponential corrections. Further using (2.69), we get

$$A_1(q) = \frac{1}{2}\mathbb{E}_2(q) + \frac{1}{24} = -\frac{1}{24\tau^2} + \frac{1}{4\pi i\tau} + \frac{1}{24} + O(e^{\#/\tau})\,, \tag{4.8}$$

which gives the asymptotic behavior of $A_1(q)$. We can find similar formula for other $A_k(q)$ by using the recursion relation (2.68)

$$A_k(q) = \frac{1}{2k(2k+1)}\left[(6A_1(q) + k(k-1))A_{k-1}(q) - 2q\frac{d}{dq}A_{k-1}(q)\right]. \tag{4.9}$$

Obviously, we can insert (4.8) into this recursion relation and get the asymptotic behavior of $A_2(q)$. Repeating the procedure in a recursive way, we can find the asymptotic behavior of all $A_k(q)$. For simplicity, let us focus on the most singular terms of $A_k(q)$ in the limit $\tau \to 0$. By noticing that $A_1 \sim -\frac{1}{24\tau^2}$ and $q\frac{d}{dq} = \tau\frac{d}{d\tau}$, we easily see that the most singular term in (4.9) is

$$A_k(q) \sim \frac{3}{k(2k+1)}A_1(q)A_{k-1}(q) \;\sim -\frac{1}{24\tau^2}\frac{3}{k(2k+1)}A_{k-1}(q)\,. \tag{4.10}$$

With this relation, we can easily derive the following asymptotic behavior

$$A_k(q) \sim \frac{(-1)^k}{8^k k!(2k+1)!!}\frac{1}{\tau^{2k}} = \frac{(-1)^k}{4^k(2k+1)!}\frac{1}{\tau^{2k}} \xrightarrow{k\to\infty} \frac{(-1)^k}{\sqrt{\pi}4^{2k+1}e^{-2k}k^{2k+\frac{3}{2}}}\frac{1}{\tau^{2k}}\,, \qquad \tau \to 0\,, \tag{4.11}$$

where we also show the large k limit of the coefficients using Stirling's formula.

This also gives the high temperature asymptotic behavior of the Schur partition function

$$\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q = e^{2\pi i\tau}) = A_k(q^2) = \frac{(-1)^k}{4^{2k}(2k+1)!}\frac{1}{\tau^{2k}} \xrightarrow{k\to\infty} \frac{(-1)^k}{\sqrt{\pi}4^{3k+1}e^{-2k}k^{2k+\frac{3}{2}}}\frac{1}{\tau^{2k}}\,, \qquad \tau \to 0. \tag{4.12}$$

It is easy to verify that this asymptotic behavior of $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}$ is consistent with that of $\mathcal{N} = 4$ SU$(N = 2k+1)$ SYM theory in [31], once we use the index relation (2.24).

It is very straightforward to generalize the above derivation and computing all the subleading corrections. For example, at sub-leading order, we have

$$\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}} \sim \frac{(-1)^k}{4^{2k}(2k+1)!}\frac{1}{\tau^{2k}} + \frac{12i(-1)^k k}{\pi 4^{2k}(2k+1)!}\frac{1}{\tau^{2k-1}} + \cdots\,, \qquad \tau \to 0\,. \tag{4.13}$$

We now give another derivation of the above asymptotic formula based on the defining generating function (2.63), which will be denoted as $\Omega_p(\mu,\tau)$:

$$\begin{aligned}\Omega_p(\mu,\tau) &= \sum_{N=1}^{\infty}\mu^{N-1}(-1)^{\left\lfloor\frac{N}{p}\right\rfloor}\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) = \prod_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left(1 + \frac{\mu q^j}{1-q^{jp}}\right)\\ &= \prod_{j=1}^{\infty}\left(1 + \frac{\mu q^j\left(1-q^{j(p-2)}\right)}{1-q^{jp}} - \frac{\mu^2 q^{jp}}{(q^{jp}-1)^2}\right).\end{aligned} \tag{4.14}$$

In the particular case of $p = 2$, we have

$$\Omega_2(\mu,\tau) = \sum_{N=1}^{\infty} \mu^{N-1}(-1)^{\left\lfloor \frac{N}{p} \right\rfloor} \mathcal{Z}_{\mathcal{T}_{(2,N)}}(q) = 1+\sum_{k=1}^{\infty} \mu^{2k}(-1)^k \mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) = \prod_{j=1}^{\infty}\left(1-\frac{\mu^2 q^{2j}}{(q^{2j}-1)^2}\right), \tag{4.15}$$

where we used $\mathcal{Z}_{\mathcal{T}_{(2,2k)}} = 0$ and $\mathcal{Z}_{\mathcal{T}_{(2,1)}} = 1$. We would like to use this formula to derive the high temperature limit $\tau \to 0$ of $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}$. It turns out that in the generating function Ω_2, we need to consider the double scaling limit by taking $\mu, \tau \to 0$ but keeping the ratio μ/τ fixed. In such a limit, the multiplicand in the infinite product of (4.15) reduces to

$$1 - \frac{\mu^2 q^{2j}}{(q^{2j}-1)^2} = \left(\frac{\mu^2}{12}+1\right) + \frac{\mu^2}{16\pi^2 j^2 \tau^2} + O(\tau^2) \xrightarrow{\mu,\tau\to 0,\ \mu/\tau \text{ fixed}} 1 + \frac{\mu^2}{16\pi^2 j^2 \tau^2}. \tag{4.16}$$

Substituting it back to the infinite product, we get[13]

$$\Omega_2(\mu,\tau) = \prod_{j=1}^{\infty}\left(1-\frac{\mu^2 q^{2j}}{(q^{2j}-1)^2}\right) \xrightarrow{\mu,\tau\to 0,\ \mu/\tau \text{ fixed}} \prod_{j=1}^{\infty}\left(1+\frac{\mu^2}{16\pi^2 j^2\tau^2}\right) = \frac{4\tau}{\mu}\sinh\left(\frac{\mu}{4\tau}\right), \tag{4.17}$$

where we used the formula

$$\sinh z = z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{(\pi n)^2}\right). \tag{4.18}$$

One can then perform series expansion in μ in order to get the asymptotic behavior of $\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}$ in (4.12). Equivalently, we will show that (4.12) gives the same generating function (4.17).

Indeed, from (4.12) we have

$$\mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) = \frac{(-1)^k}{4^{2k}(2k+1)!}\frac{1}{\tau^{2k}} + \frac{\#}{\tau^{2k-1}} + \cdots, \tag{4.19}$$

where # and dots represent less singular terms and exponentially suppressed terms. Substituting it back to the generating function (4.15), we get

$$\Omega_2(\mu,\tau) = 1+\sum_{k=1}^{\infty}\mu^{2k}(-1)^k \mathcal{Z}_{\mathcal{T}_{(2,2k+1)}}(q) \simeq 1+\sum_{k=1}^{\infty}\mu^{2k}(-1)^k \frac{(-1)^k}{4^{2k}(2k+1)!}\frac{1}{\tau^{2k}} = \frac{4\tau}{\mu}\sinh\left(\frac{\mu}{4\tau}\right), \tag{4.20}$$

which exactly recovers (4.16). Note that in the above formula, we also need to take the double scaling limit $\mu, \tau \to 0$ with μ/τ fixed, in order to suppress the contribution from the less singular terms in (4.19).

In the case of $k = 1$, (4.12) gives

$$\mathcal{Z}^{\mathcal{T}_{(2,3)}}(q) \sim -\frac{1}{96\tau^2}, \tag{4.21}$$

which has also been verified numerically.

[13]Rigorously speaking, (4.16) would fail if $j\tau \gtrsim 1$. However, for big enough j, both the l.h.s. and r.h.s. of (4.16) approach to 1 due to the suppression of both $q^{2j}, 1/j^2 \to 0$ in the large j limit.

4.3 General case

To study the high temperature limit of Schur partition function, the key point is to use the modular transformation of characters:

$$\mathcal{Z}(q) = \chi_{\mathbf{1}}(\tau) = \sum_j S_{\mathbf{1}j}\chi_j\left(-\frac{1}{\tau}\right), \tag{4.22}$$

where $\mathbf{1}$ corresponds to the vacuum, and χ_j are the characters of some modules of the corresponding VOA, which are also the solutions to MLDE. Once we understand the behavior of $\chi_j(\tilde{\tau})$ in the limit of $\tilde{\tau} \to \infty$ or equivalently $\tilde{q} = e^{2\pi i\tilde{\tau}} \to 0$, and the modular S-matrix, we can establish the high temperature limit of Schur partition function via (4.22).

Using the modular transformation properties, the authors in [11] proposed the following high temperature limit

$$\mathcal{Z} \sim e^{\frac{\pi i c_{\text{eff}}}{12\tau}}, \qquad c_{\text{eff}} = c_{2d} - 24\min_i h_i = -24\min_i b_i \tag{4.23}$$

where $b = -\frac{c_{2d}}{24} + h$. By comparing (4.1) and (4.23), it was further proposed in [11] that

$$c_{\text{eff}} = -24\min_i b_i = 48(c-a)\,. \tag{4.24}$$

From our previous examples of $\mathcal{T}_{(3,2)}$ and $\mathcal{T}_{(2,2k+1)}$, we know that the asymptotic behaviors in both (4.1) and (4.23) are not valid. However, (4.24) seems to be still valid.

In AD $\mathcal{T}_{(p,N)}$ theories considered here, the two central charges are the same, so $c - a = 0$. On the other hand, in the case of $\mathcal{T}_{(3,2)}, \mathcal{T}_{(2,3)}, \mathcal{T}_{(4,3)}$, by analyzing the indicial equation of MLDE, we do find that $\min_i b_i = 0$.

In [16], a more careful analysis was done and found that the following high temperature limit for Schur partition function

$$\mathcal{Z}(q) \sim \sum_i \tilde{A}_i\,\tilde{q}^{b_i}(\log\tilde{q})^{d_i-1} + \cdots \sim \sum_i A_i\tau^{1-d_i}e^{-2\pi i b_i/\tau} + \cdots \tag{4.25}$$

where d_i is the degeneracy of the root b_i (counting also all the roots that are less than b_i by an integer), and $\tilde{q} = e^{2\pi i\tilde{\tau}}, \tilde{\tau} = -1/\tau$.

Focusing on the most singular term which gives the leading contribution, we get

$$\mathcal{Z}(q) \sim A_j\tau^{1-d_j}e^{-2\pi i b_j/\tau}, \qquad b_j = \min_i b_i\,, \tag{4.26}$$

where b_j is the minimal b_i. Depending on the sign of $\min_i b_i$, we can have either exponentially enhanced / suppressed leading contribution, or power law leading asymptotic if $\min_i b_i = 0$.

If $\min_i b_i > 0$, we see $\mathcal{Z}(q) \to 0$ in the limit $\tau \to 0$, which looks very unlikely as it means an almost perfect cancellation between bosonic and fermionic states. If $\min_i b_i = -\alpha < 0, \alpha > 0, \tau = i\beta$, then we have exponentially growing contribution of the form $e^{2\pi\alpha/\beta}$, namely

$$\log\mathcal{Z}(q) \sim \frac{2\pi\alpha}{\beta} \to +\infty\,, \qquad \beta \to 0\,. \tag{4.27}$$

This is reminiscent of the index for counting black hole entropy.[14] Since our theories have $a = c$ which means that they are supposed to have nice holographic dual description, such a

[14]Indeed, the 1/16-BPS black hole entropy was reproduced from the exponentially large term of $\mathcal{N} = 4$ SYM index in the Cardy-like limit [33]. Note that to rigorously reproduce the black hole entropy from the index or partition function, one needs to consider the large-N limit.

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kind of exponential contribution to Schur partition function with exponent $\alpha \sim N^2$ would indicate the presence of 1/4-BPS black holes.[15] However, in all our explicit examples, we did not see any case with $\min_i b_i < 0$.

On the other hand, if $\min_i b_i = 0$, we have

$$\mathcal{Z}(q) \sim \frac{A_0}{\tau^{d_0-1}}\,, \qquad \tau \to 0 \tag{4.28}$$

where d_0 is the degeneracy of the root $b = 0$, namely in the indicial equation we have the factor b^{d_0}. Interestingly, we find that all the examples we studied fall into this class. For $\mathcal{T}_{(3,2)}$, the asymptotic growth is given by (4.4), while for $\mathcal{T}_{(2,2k+1)}$, the asymptotic growth (4.12). For $\mathcal{T}_{(2,3)}$ and $\mathcal{T}_{(4,3)}$, their MLDEs are shown explicitly in the previous section, and their indicial relation are given by (3.40) (3.43). As a result, we do see that the $\min_i b_i = 0$ and the degeneracy is 3, namely $d_0 = 3$ in (4.28).

Based on these discussions and the results in the literature, we are then motivated to make the following set of conjectures:

1) There is no constant term in MLDE, so $\chi = 1$ is a solution to MLDE and $b = 0$ is a root to the indicial equation;

2) $b = 0$ is the minimal root to indicial equation;

3) The degeneracy of root $b = 0$ is N, so the high temperature asymptotic behavior of the Schur partition function of $\mathcal{T}_{(p,N)}$ with $\gcd(p, N) = 1$ is

$$\mathcal{Z}_{\mathcal{T}_{(p,N)}}(q) \sim \frac{\#}{\tau^{N-1}}\,, \qquad \tau \to 0\,. \tag{4.29}$$

Note that this kind of asymptotic behavior has been proved for $p = 2$. Moreover conjecture 1) and 2) are consequences of (4.24).

To further understand the case of other p, let us attempt to generalize the previous generating function techniques. As in (4.16), we can take the double scaling limit of the multiplicands in (4.14)

$$\left(1 + \frac{\mu q^j \left(1 - q^{j(p-2)}\right)}{1 - q^{jp}} - \frac{\mu^2 q^{jp}}{\left(q^{jp} - 1\right)^2}\right) \xrightarrow{\mu,\tau\to 0,\ \mu/\tau \text{ fixed}} 1 + \frac{\mu^2}{4\pi^2 j^2 p^2 \tau^2}\,, \tag{4.30}$$

where the linear term in μ is absent due to the double scaling limit. Taking infinite product, we get

$$\Omega_p = \prod_{j=1}^{\infty} \left(1 + \frac{\mu^2}{4\pi^2 j^2 p^2 \tau^2}\right) = \frac{2p\tau}{\mu} \sinh\left(\frac{\mu}{2p\tau}\right)\,, \tag{4.31}$$

[15] By 1/4-BPS black holes, we mean black hole solutions that preserve 4 out of 16 supersymmetries in 5d $\mathcal{N} = 4$ gauged supergravity. The minimally supersymmetric black holes are 1/8-BPS. To the best of our knowledge, no such 1/4-BPS black holes have been constructed in supergravity. The microstates of a 1/4-BPS black hole, if it exists, are supposed to be in one-to-one correspondence with the Schur operators, which are also 1/4-BPS in 4d $\mathcal{N} = 2$ superconformal field theory. The absence of exponential growth in our Schur indices is consistent with the absence of 1/4-BPS black holes. Note that in the context of $\mathcal{N} = 4$ SYM theory, [34] also argued for the absence of 1/8-BPS black holes in the dual string theory. (Note that the 1/8-BPS black hole in that context corresponds to the 1/4-BPS black hole in this paper as they are both counted by Schur operators).

where we used (4.18) again. This is an even function in μ, so it would give zero to $\mathcal{Z}_{\mathcal{T}_{(p,N)}}$ for even N according to (4.14). But (4.4) has shown that $\mathcal{Z}_{\mathcal{T}_{(3,2)}} \sim \frac{i}{6\sqrt{3}\tau}$. This discrepancy is supposed to arise from the order of taking the double scaling limits and infinite product. To exemplify this point, let us consider the μ-linear term in (4.14):

$$\Omega_p = 1 + \sum_{j=1}^{\infty} \frac{\mu q^j \left(1 - q^{j(p-2)}\right)}{1 - q^{jp}} + O(\mu^2) \tag{4.32}$$

$$\sim 1 + \sum_{j=1}^{\infty} \frac{p-2}{p} \mu q^j + O(\mu^2) \tag{4.33}$$

$$= 1 + \mu \frac{p-2}{p} \frac{1}{1-q} + O(\mu^2) \tag{4.34}$$

$$= 1 + \frac{2-p}{p} \frac{\mu}{2\pi i \tau} + O(\mu^2)\,. \tag{4.35}$$

This would give $\mathcal{Z}_{\mathcal{T}_{(3,2)}} \sim \frac{i}{6\pi\tau}$, which is slightly different from but close to (4.4); the difference in the coefficients is due to our approximation in (4.33). Nevertheless, we get the right scaling behavior consistent with conjecture (4.29) for $N = 2$. A very careful and systematic analysis is needed to study the general case and get the exact coefficient, and we leave this important question to the future.

5 Conclusion

In this paper, we studied the Schur sector of a family of AD theories denoted as $\mathcal{T}_{(p,N)}$. The theories we have studied are interesting as they share many features with $\mathcal{N} = 4$ SYM theory. In particular, the two central charges are the same $a = c$, indicating that they have holographic dual descriptions in terms of supergravity in AdS with some special features.

We derived an enlightening formula (2.62) for the Schur index of this family of AD theories, which naturally factors out the Casimir term. The remaining Schur partition function takes a simple form (2.63) and is expected to satisfy the modular linear differential equation. We study the MLDEs numerically based on our simple formula. We also derive the MLDE analytically and investigate the modular properties of its solution for the theory $\mathcal{T}_{(3,2)}$, whose VOA is known. Combining the modularity of MLDE with the explicit simple formula for Schur partition function, we discuss their high temperature limits. All of our explicit results suggest that the high temperature limit of Schur index / partition function of $\mathcal{T}_{(p,N)}$ AD theories diverges following a power law, rather than exponentially. This motivates us to propose a set of conjectures about MLDEs and the high temperature behavior (4.29) for general $\mathcal{T}_{(p,N)}$ AD theories. In the case of $p = 2$, we prove the conjecture and show the asymptotic growth explicitly (4.29). In general, the exponential growth of index in the high temperature limit is closely related to the black hole entropy in the dual AdS quantum gravity.[16] Our conjecture on power law divergence (4.29) indicates the absence of 1/4-BPS black hole.

[16] More precisely, one needs to consider the large N limit which is different from the high temperature limit or Cardy limit. The presence of black holes would give rise to an exponential growth of partition function with proper exponent in the large N limit.

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In the appendix, we also review many important concepts and useful techniques which are used for developing the results in the main body. In addition, we also present some new results there, including computing the torus one-point function to higher weight (B.27)(B.28), and deriving the explicit MLDEs for some AD theories in the family of (A_{k-1}, A_{n-1}) and $D_p(\mathrm{SU}(N))$ in appendix C.1 and C.2, whose chiral algebras are W-algebras and Kac-Moody algebras.

There are some open questions. First of all, it would be interesting to study the properties of the functions (2.63) from a mathematical perspective. In particular, these functions are well-defined for general p and N (some may be trivial), rather than just for coprime integers $p = 2, 3, 4, 6$ and $N = 2, 3, 4\cdots$. We expect they all enjoy some nice modular properties. For example, one can take $p = 4$ and $N = 2$ which does not correspond to the SCFT studied here, and check that $\mathcal{Z}_{\mathcal{T}_{(4,2)}}$ defined by (2.63) satisfies a MLDE of weight 18. It is also important to study the asymptotic behavior of these functions in the limit $\tau \to 0$. This characterises the high energy/temperature growth of 1/4-BPS states in SCFTs. We have made the conjecture in subsection 4.3, and it would be interesting to prove or disprove it. Another interesting and related limit is $\tau \to \mathbb{Q}$.

Secondly, it would be fascinating to find the chiral algebra of $\mathcal{T}_{(p,N)}$ AD theory, and use it to derive the MLDE analytically. So far, only the chiral algebra of $\mathcal{T}_{(3,2)}$ AD theory is known. The intriguing operator map between $\mathcal{T}_{(p,N)}$ AD theory and $\mathcal{N} = 4$ SYM theory proposed in [6] may offer some insights and help to find or bootstrap the chiral algebra.

Thirdly, our Schur partition function is derived from the index of $\mathcal{N} = 4$ SYM theory based on the index relation (2.24). Then a natural question is whether there is a direct connection at the level of MLDE. It would be amazing if one could establish an explicit map between the (flavored) MLDE of $\mathcal{N} = 4$ SYM theory and the MLDE of $\mathcal{T}_{(p,N)}$ AD theory. For $\mathcal{N} = 4$ SU(N) SYM with odd $N \leq 7$, it has been observed in [11] that Schur partition functions satisfy monic MLDEs of weight $(N+1)^2/2$. It is natural to ask about the dependence of the weights of MLDEs on the parameters p and N for $\mathcal{T}_{(p,N)}$ AD theory.

Finally, it would be interesting to find the holographic dual of $\mathcal{T}_{(p,N)}$ AD theories and study various properties using supergravity techniques. The important feature of $a = c$ implies some remarkable cancellations of higher derivative corrections in the supergravity Lagrangian. Once the holographic dual is known, one could then study the "giant graviton expansion" of the Schur index of $\mathcal{T}_{(p,N)}$ AD theories and try to reproduce it from supergravity.[17]

We leave these interesting questions to the future.

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[17] In particular, it is easy to show that, in the supergravity limit $N \to \infty$, the Schur index is given by $\mathcal{I}_{\mathcal{T}_{(p,\infty)}} = \prod_{k=1}^{\infty} \frac{1-q^{kp}}{(1-q^k)(1-q^{k(p-1)})} / \, \mathrm{PE}\left[\frac{q^{p-1}-2q^p+q}{1-q^p}\right] = \frac{(q;q^p)_\infty (q^{p-1};q^p)_\infty}{(q;q)_\infty (q^{p-1};q^{p-1})_\infty (q^p;q^p)_\infty}$.

A Modular form and modular linear differential equation

We review some basics of modular forms and modular linear differential equations. See e.g. [35] for more details.

A.1 Modular form and Eisenstein series

Let $\tau \in \mathbb{H}$ be the modular parameter taking value in the upper half place. The modular group of interest here is $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. It acts on the modular parameter as

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \quad a,b,c,d \in \mathbb{Z}\,, \quad ad-bc=1\,. \tag{A.1}$$

The modular group is generated by two elements S and T

$$S : \tau \to -\frac{1}{\tau}, \qquad T : \tau \to \tau+1 \tag{A.2}$$

subject to the relations $S^2 = (ST)^3 = 1$.

For convenience, we define $q \equiv e^{2\pi i\tau}$. The modular group action on q is naturally given by

$$\gamma \circ q = e^{2\pi i\tau \frac{a\tau+b}{c\tau+d}}\,. \tag{A.3}$$

A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that transforms according to

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{A.4}$$

and remains finite in the limit $\mathrm{Im}\,\tau \to +\infty$. Due to the invariance under T transformation, any modular form has a convergent Fourier expansion in q and is finite in the limit $q \to 0$:

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \qquad q = e^{2\pi i\tau}\,. \tag{A.5}$$

One can relax the finiteness condition at infinity and allow a finite number of terms with negative q exponents in the above Fourier expansion, which defines a weakly holomorphic modular form.

A particular set of modular forms is given by the Eisenstein series which is defined by[18]

$$\mathbb{E}_{2k}(\tau) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}\,, \qquad q \equiv e^{2\pi i\tau}\,. \tag{A.6}$$

where B_{2k} is the $2k$-th Bernoulli number. When the integer $k > 1$, $\mathbb{E}_{2k}(\tau)$ is a modular form for Γ of weight $2k$. The case of $\mathbb{E}_2(\tau)$ is special as it is not a modular form but a quasi-modular form since it transforms anomalously under modular transformations:

$$\mathbb{E}_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\mathbb{E}_2(\tau) - \frac{c(c\tau+d)}{2\pi i}\,. \tag{A.7}$$

[18]We will be sloppy about the notation, and use both $\mathbb{E}_{2k}(q)$ and $\mathbb{E}_{2k}(\tau)$ to denote the same Eisenstein series. More generally, we will use $f(q)$ and $f(\tau)$ to denote the same function.

The space of modular forms of weight k is denoted by $\mathcal{M}_k(\Gamma)$. The ring of the modular forms for modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ is freely generated by $\mathbb{E}_4(\tau)$ and $\mathbb{E}_6(\tau)$:

$$\bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma) = \mathbb{C}[E_4(\tau), E_6(\tau)]\,. \tag{A.8}$$

In particular, this implies that $\mathbb{E}_{2k}$ with $k \geq 4$ can be written as the polynomial of $\mathbb{E}_4$ and $\mathbb{E}_6$. For example,

$$\mathbb{E}_8 = \frac{3}{7}\mathbb{E}_4^2\,, \qquad \mathbb{E}_{10} = \frac{5}{11}\mathbb{E}_4\mathbb{E}_6\,, \qquad \cdots\,. \tag{A.9}$$

The space $\mathcal{M}_k(\Gamma)$ is finite dimensional:

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} 0\,, & k<0 \text{ or } k \text{ is odd} \\ \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1\,, & \text{otherwise} \end{cases}\,. \tag{A.10}$$

We can further genearlize and define the vector-valued modular form similarly. Consider a homomorphism

$$\rho : \Gamma \to \mathrm{GL}(n,\mathbb{C})\,, \tag{A.11}$$

which gives an n-dim representation of Γ. A vector-valued modular form of weight $k \in \mathbb{Z}$ and multiplier system ρ is a function $\boldsymbol{\chi} = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix} : \mathbb{H} \to \mathbb{C}^n$ which transforms as

$$\boldsymbol{\chi}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \rho(\gamma) \boldsymbol{\chi}(\tau)\,, \qquad \gamma \in \Gamma\,, \tag{A.12}$$

and all the component functions are finite in the limit $\operatorname{Im}\tau \to +\infty$. If the component functions have exponential growth in this limit, then $\chi(\tau)$ is referred to as a weakly holomorphic vector-valued modular form.

A weakly holomorphic logarithmic vector-valued modular form of weight $k \in \mathbb{Z}$ with multiplier system ρ is a function $\chi : \mathbb{H} \to \mathbb{C}^n$ which transforms as (A.12) and such that its q expansion contains logarithms of q.

Let us also introduce the notion of quasi-modular form. The function $f : \mathbb{H} \to \mathbb{C}$ is a quasi-modular form of weight k and depth s if there exist holomorphic functions $f_0, \cdots, f_s$ with $f_s \neq 0$, such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \sum_{j=0}^{s} f_j(\tau) \left(\frac{c}{c\tau+d}\right)^j, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,, \qquad \tau \in \mathbb{H}\,. \tag{A.13}$$

We use $\mathcal{M}_k^s$ to denote the set of quasi-modular forms of weight k and depth s, and $\mathcal{M}_k^{\leq s}$ to denote that of weight k and depth non-greater than s. Given $f \in \mathcal{M}_k^s$, we can always write

$$f(\tau) = \sum_{i=0}^{s} g_i(\tau)\, \mathbb{E}_2^i(\tau)\,, \qquad \exists\, g_i(\tau) \in \mathcal{M}_{k-2i}\,. \tag{A.14}$$

This means

$$\mathcal{M}_k^{\leq s} = \bigoplus_{i=0}^{s} \mathcal{M}_{k-2i}\, \mathbb{E}_2^i\,. \tag{A.15}$$

So the ring of quasi-modular forms is $\mathbb{C}[\mathbb{E}_2, \mathbb{E}_4, \mathbb{E}_6]$ and the maximal power of $\mathbb{E}_2$ gives the depth.

A.2 Modular covariant derivative and MLDE

The modular forms also transform nicely under the differential operations. For this purpose, we need to introduce the Serre derivative $\partial_{(k)}$ which maps modular forms of a fixed weight to modular forms of higher weight:

$$\partial_{(k)} : \mathcal{M}_k(\Gamma) \to \mathcal{M}_{k+2}(\Gamma)\,. \tag{A.16}$$

More explicitly, Serre derivatives are defined as:

$$\partial_{(k)} f(q) = (q\partial_q + k\mathbb{E}_2(\tau))f(q)\,. \tag{A.17}$$

When acting on the Eisenstein series of low-weight, we have

$$\partial_{(4)}\mathbb{E}_4 = q\partial_q\mathbb{E}_4 + 4\mathbb{E}_2\mathbb{E}_4 = 14\mathbb{E}_6\,, \qquad \partial_{(6)}\mathbb{E}_6 = q\partial_q\mathbb{E}_6 + 6\mathbb{E}_2\mathbb{E}_6 = \frac{60}{7}\mathbb{E}_4^2\,, \qquad \cdots\,, \tag{A.18}$$

and

$$\partial_{(2)}\mathbb{E}_2 = q\partial_q\mathbb{E}_2 + 2\mathbb{E}_2^2 = 5\mathbb{E}_4 + \mathbb{E}_2^2\,, \tag{A.19}$$

although $\mathbb{E}_2$ is not a modular form.

Using Serre derivatives we can define k-th order modular differential operators

$$D_q^{(k)} = \partial_{(2k-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)}\,, \tag{A.20}$$

which naturally act on objects of modular weight zero, as one can see from (A.16). We also define $D_q^{(0)} f(q) := f(q)$. The modular differential operators transform with weight $2k$ under the modular group action,

$$D_{\gamma\circ q}^{(k)} = (c\tau + d)^{2k} D_q^{(k)}\,, \qquad \gamma \in \Gamma\,. \tag{A.21}$$

This enables us to construct a large class of modular linear differential operators (MLDO) of a fixed weight $2k$ as sums of modular differential of weight $2k - 2r$ multiplied by modular forms of weight $2r$. We are particularly interested in those that are holomorphic and monic

$$\mathcal{D}_q^{(k)} = D_q^{(k)} + \sum_{r=1}^{k} f_r(\tau) D_q^{(k-r)}\,, \qquad f_r(\tau) \in \mathcal{M}_{2r}(\Gamma)\,. \tag{A.22}$$

Here monic means the coefficient of $D_q^{(k)}$ is one, and by holomorphic we mean that the modular forms $f_r(\tau)$ defining the coefficients of the MLDO are modular forms and hence finite in the limit $q \to 0$.

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The elements of the kernel of a MLDO define a modular linear differential equation (MLDE):

$$\mathcal{D}_q^{(k)}\chi(\tau) = \left(D_q^{(k)} + \sum_{r=1}^{k} f_r(\tau) D_q^{(k-r)}\right)\chi(\tau) = 0\,. \tag{A.23}$$

From (A.10), one can show that the total number of undetermined coefficients in f_r is given by

$$\dim \sum_{r=1}^{k} f_r(\tau) D_q^{(k-r)} = \sum_{r=1}^{k} \dim \mathcal{M}_{2r}(\Gamma) = \left\lfloor \frac{k^2}{12} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor\,. \tag{A.24}$$

This growth is mild, and allows one to find the MLDE satisfied by χ numerically with low computational cost. For example, if we want to find a MLDE of weight 60 with $k = 30$ above, we need to fix less than 100 coefficients (A.23), which means we can determine such a MLDE by just expanding χ up to the order around q^{100+b} where b is the smallest power of q.

A.3 Solution to MLDE

Given the MLDE, we would like to find the full set of the solutions, which are supposed to transform as a vector-valued modular form.

It is easy to see that the MLDO takes the following form

$$\mathcal{D}_q^{(k)} = \sum_{i=0}^{k} P_{2i}(\tau) q^{k-i} \frac{\partial^{k-i}}{\partial q^{k-i}}\,, \tag{A.25}$$

where $P_{2i}(\tau)$ are polynomials in Einstein series $\mathbb{E}_2, \mathbb{E}_4, \mathbb{E}_6$. So it is a Fuchsian ordinary differential equations with only a regular singularity at $q = 0$ inside the unit disk.

To solve MLDE (A.23), one can make the following ansatz

$$\chi(\tau) = q^{\alpha} \sum_{n=0}^{\infty} a_n q^n\,, \tag{A.26}$$

and substitute it back to (A.23), then we get

$$q^{\alpha-k}\Big(\alpha(\alpha-1)\cdots(\alpha-k+1)a_0 + \cdots\Big) + q^{\alpha-k+1}\Big(\cdots\Big) + \cdots = 0\,, \tag{A.27}$$

where the coefficient of $q^{\alpha-k}$ is the indicial polynomial with degree k. So there are k roots to the indicial polynomial, denoted by $\alpha_1, \cdots, \alpha_k$.

If all the roots are different and do not differ by an integer, namely $e^{2\pi\alpha_i}$ are all different, then the set of series

$$\chi_i(\tau) = q^{\alpha_i}\phi_i(q) = q^{\alpha_i} \sum_{n=0}^{\infty} a_{i,n} q^n\,, \qquad a_{i,0} \neq 0\,, \tag{A.28}$$

is the full set of linearly independent solutions to the MLDE, where the coefficients $a_{i,n}$ are fixed recursively from (A.27) by requiring that the coefficients of all $q^{\#}$ should vanish. They would transform as a vector-valued modular form under the modular group Γ (A.12), and $\rho(T)$ is a diagonal matrix with diagonal elements $e^{2\pi i\alpha_i}$.

On the other hand, if some roots are the same or have an integer difference, then the solution of MLDE may contain logarithmic terms. In this case, the general solution of a monic holomorphic MLDE of degree k is a k-dimensional vector with components

$$\chi_i(\tau) = q^{\alpha_i} \sum_{j=0}^{N_i-1} (\log q)^j \phi_{i,j}(q)\,, \qquad \phi_{i,j}(q) = \sum_{l=0}^{\infty} a_{i,j,l} q^l\,, \tag{A.29}$$

where N_i depends on the structure of the roots to indicial equation. See e.g. [36] for detailed discussions. In particular $N_i \leq k$.

The set of linearly independent solutions of MLDE transform as a weakly holomorphic logarithmic vector-valued modular form under the modular group Γ, and $\rho(T)$ is not a diagonal matrix any more; the off-diagonal entries arise from the logarithmic term due to the fact that $\log q \to \log q + 2\pi i$ when we perform T-transformation $\tau \to \tau + 1$.

B Zhu's recursion relation

We will review Zhu's recursion relation [12] and its application in deriving modular linear differential equation. A detailed discussion can be found in the original reference [12] and [11, 37].

B.1 Vertex operator algebra

We now introduce the vertex operator algebras. We will only introduce the basic concepts and refer the reader to [11, 37] for details. For simplicity, we will only consider $\mathbb{Z}_{\geq 0}$-graded conformal vertex operator algebra, meaning that the vertex operator algebra has a subalgebra which is isomorphic to Virasoro vertex operator algebra, and the conformal dimension of all operators are non-negative integers. As a result, the vertex operator algebra $\mathcal{V}$ can be decomposed as

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n\,, \qquad \dim \mathcal{V}_n < \infty\,, \tag{B.1}$$

and L_0 operator from Virasoro algebra acts as $L_0 a = na$ for $a \in \mathcal{V}_n$.

The operator-state correspondence dictates that for each state a with integer weight h_a, the corresponding mode expansion of the corresponding vertex operator is given by

$$a(z) = Y(a,z) = \sum_{n\in\mathbb{Z}} a_{-h_a-n} z^n = \sum_{n\in\mathbb{Z}} a_n z^{-h_a-n}\,, \tag{B.2}$$

where the modes act as endomorphism $a_n : \mathcal{V}_k \to \mathcal{V}_{k+n}$.

We further introduce the following notation for the zero mode of a

$$o(a) := a_0\,. \tag{B.3}$$

For the convenience of formulating recursion relations on the torus, we consider an alternative expansion of the same vertex operator

$$Y[a,z] = e^{zh_a} Y(a, e^z - 1) = \sum_{n\in\mathbb{Z}} a_{[-h_a-n]} z^n\,, \tag{B.4}$$

where we have introduced the "square-bracket" modes. They are related to the usual modes in (B.2) via

$$a_{[n]} = \sum_{j\geq n} c(j+h_a-1, n+h_a-1; h_a) a_j \,, \tag{B.5}$$

where the coefficients are defined via

$$(1+z)^{h-1}\Big(\log(1+z)\Big)^n = \sum_{j\geq n} c(j,n;h) z^j \,. \tag{B.6}$$

For Virasoro primary state a, the commutation relations for $a_{[n]}$ are identical to those of a_n. For stress tensor, which is not Virasoro primary, we instead define

$$L_{[n]} = \sum_{j\geq n} c(j+1, n+1; h_a) L_j - \frac{c}{24}\delta_{n+2,0} \,. \tag{B.7}$$

These square-bracket Virasoro modes then satisfy the same commutation relation as the original Virasoro modes L_n.

We also need the vacuum state which is defined such that[19]

$$a_n \Omega = a_{[n]}\Omega = 0 \,, \qquad n \geq -h_a \,. \tag{B.8}$$

The two definitions of vacuum are consistent as $a_{[n]} = a_n + c(n+1, n; h_a) a_{n+1} + \cdots$ which can be shown from (B.6).

Since the vacuum state and the commutation relations defined via the square-bracket modes are the same as those of usual modes, a null vector in the vacuum Verma module of a VOA formulated in terms of the usual mode will still be null after replacing the usual modes with square bracket modes. In another word, once we have a null state acting on vacuum, we can replace all the a_n in the null state with $a_{[n]}$, and the resulting state is still null.

B.2 Zhu's recursion relation for torus one-point function

The virtue of the square-bracket modes is that one can elegantly formulate the recursion relation for torus one-point functions in the VOA. We will call such kinds of recursion relation as Zhu's recursion relation, as they were originally derived by Zhu [12]. The simplest Zhu's recursion relation is[20]

$$\mathrm{STr}_{\mathcal{V}}\Big(o(a_{[-h_a]}b)q^{L_0-\frac{c}{24}}\Big) = \mathrm{Tr}\Big(o(a)o(b)q^{L_0-\frac{c}{24}}\Big) + \sum_{k\geq 1}\mathbb{E}_{2k}(\tau)\,\mathrm{Tr}\Big(o(a_{[-h_a+2k]}b)q^{L_0-\frac{c}{24}}\Big) \,. \tag{B.9}$$

More generally, for $n \geq 1$ we have

$$\mathrm{STr}_{\mathcal{V}}\Big(o(a_{[-h_a-n]}b)q^{L_0-\frac{c}{24}}\Big) = (-1)^n \sideset{}{'}\sum_{2k\geq n+1}\binom{2k-1}{n}\mathbb{E}_{2k}(\tau)\,\mathrm{Tr}\Big(o(a_{[-h_a-n+2k]}b)q^{L_0-\frac{c}{24}}\Big) \,, \tag{B.10}$$

[19] With this property of vacuum, we can define a state $|a\rangle = \lim_{z\to 0} a(z)\,|\Omega\rangle = a_{-h_a}\,|\Omega\rangle$.

[20] Here we consider the general case of vertex operator superalgebras, which have even and odd parts $\mathcal{V} = \mathcal{V}_{\mathrm{even}} \oplus \mathcal{V}_{\mathrm{odd}}$. The supertrace of an endomorphsim $\mathcal{O}: \mathcal{V}\to\mathcal{V}$ is defined as $\mathrm{STr}_{\mathcal{V}} \equiv \mathrm{Tr}_{\mathcal{V}_{\mathrm{even}}}\mathcal{O} - \mathrm{Tr}_{\mathcal{V}_{\mathrm{odd}}}\mathcal{O}$.

where the prime indicates that the term with $2k = n+1$ should be removed because $o(a_{[-h_a+1]}b)$ is a commutator and hence $\mathrm{STr}\Big(o(a_{[-h_a+1]}b)q^{L_0-\frac{c}{24}}\Big) = 0$. Note that since on the r.h.s., we have $-h_a - n + 2k \geq -h_a + 1$, so $a_{[-h_a-n+2k]}$ annihilates the vacuum (B.8). After using the commutation relation of the algebra, we can move the left-most oscillator $a_{[-h_a-n+2k]}$ to the right inside $o(a_{[-h_a-n+2k]}b)$ and obtain the zero mode of state with strictly smaller conformal dimension. This then gives a way to compute the torus one-point function recursively.

B.3 MLDE from null state

Given a VOA $\mathcal{V}$, one can define a subspace $C_2(\mathcal{V}) \subset \mathcal{V}$ as

$$C_2(\mathcal{V}) := \mathrm{Span}\Big\{a_{-h_a-1}b\Big|a, b \in \mathcal{V}\Big\}\,. \tag{B.11}$$

By operator-state correspondence, the state $a_{-h_a-1}\Omega$ is associated to the vertex operator $\partial^n a$. The vector space $C_2(\mathcal{V})$ can be understood as the space of those normally-ordered composite operators that contain at least one derivative, including those operators without explicit derivatives apparently but can be rewritten in terms of operators with derivatives after using appropriate null relations in the VOA.

We can further define the quotient space

$$\mathcal{R}_{\mathcal{V}} := \mathcal{V}/C_2(\mathcal{V})\,. \tag{B.12}$$

The corresponding algebra is known as the C_2-algebra of $\mathcal{V}$, which is a commutative, associative Poisson algebra.

In many interesting cases, the VOA contains some null state of the form

$$\mathcal{N}_T = (L_{-2})^k\Omega + \sum_i a^i_{-h_i-1}\varphi_i\,. \tag{B.13}$$

In other words, we have $(L_{-2})^k\Omega \in C_2(\mathcal{V})$ for $k \in \mathbb{Z}_+$. The existence of such a null state is closely related to the existence of MLDE for the vacuum character of $\mathcal{V}$.

Due to the isomorphism between square modes and usual modes that we discussed before, a null state of the form in (B.13) indicates another null state in the Verma module of VOA

$$\mathcal{N}_{[T]} = (L_{[-2]})^k\Omega + \varphi = (L_{[-2]})^k\Omega + \sum_i a^i_{[-h_i-1]}b_i\,, \qquad \varphi \in C_{[2]}(\mathcal{V})\,, \tag{B.14}$$

where $C_{[2]}(\mathcal{V})$ is the square mode analogue of (B.11). Because $\mathcal{N}_{[T]}$ is a null state, correlation functions with insertions of $\mathcal{N}_{[T]}$ must vanish. In particular, the torus one-point function of $\mathcal{N}_{[T]}$ must be zero

$$\mathrm{STr}_{\mathcal{V}}\Big(o(\mathcal{N}_{[T]})q^{L_0-\frac{c}{24}}\Big) = \mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^k\Omega)q^{L_0-\frac{c}{24}}\Big) + \sum_i \mathrm{STr}_{\mathcal{V}}\Big(o(a^i_{[-h_i-1]}b_i)q^{L_0-\frac{c}{24}}\Big) = 0\,. \tag{B.15}$$

The torus one-point function with insertions above can be evaluated using Zhu's recursion relation in (B.9) and (B.10). Consequently and in ideal cases,[21] we get a modular covariant

[21] After using the recursion relation, one may encounter the zero mode of the form $o(a^1_{-h_1}\cdots a^j_{-h_j})$, which generally gives an obstruction for further evaluation. See [11] for discussions. We don't encounter such kinds of obstructions in this paper.

differential operator acting on the vacuum character, namely we obtain the MLDE for the vacuum character of VOA (A.23)

$$\mathcal{D}_q^{(k)}\chi(\tau)=\Big(D_q^{(k)}+\sum_{r=1}^{k}f_r(\tau)D_q^{(k-r)}\Big)\chi(\tau)=0\,,\qquad \chi(\tau)=\mathrm{STr}_{\mathcal{V}}\Big(q^{L_0-\frac{c}{24}}\Big)\,. \tag{B.16}$$

From these discussions, we see that the MLDE for the vacuum character arises as the consequence of a null state of the form (B.13).

B.4 Stress tensor one-point function

To derive the MLDE, one important ingredient is to derive the torus one-point function with insertion $o((L_{[-2]})^k\Omega)$. Let us start with the simplest case of $k=1$. Using Zhu's recursion relation (B.9), we get

$$\begin{aligned}\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-2]}\Omega)q^{L_0-\frac{c}{24}}\Big)&=\mathrm{STr}_{\mathcal{V}}\Big(o(L)o(\Omega)q^{L_0-\frac{c}{24}}\Big)=\mathrm{STr}_{\mathcal{V}}\Big((L_0-c/24)q^{L_0-\frac{c}{24}}\Big)\\&=q\frac{d}{dq}\mathrm{STr}_{\mathcal{V}}\Big(q^{L_0-\frac{c}{24}}\Big)\,,\end{aligned} \tag{B.17}$$

where we used that $L_{[i]}\Omega=0$ for $i>-2$ and $o(L_{[-2]}\Omega)=o(L)=L_0-\frac{c}{24}$. We also note that $o(L_{[-1]}a)=o(L_{-1}a+L_0a)=(L_{-1}a+L_0a)_0=-(h_a+0)a_0+h_aa_0=0$ where $L_{[-1]}=L_{-1}+L_0$.

For higher k, we can similarly derive the torus one-point function using (B.9) and (B.10).

$$\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-2]}(L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big) \tag{B.18}$$

$$\begin{aligned}&=\mathrm{STr}_{\mathcal{V}}\Big(o(L)o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)+\mathbb{E}_2(q)\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[0]}(L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)\\&\quad+\sum_{k\geq2}\mathbb{E}_{2k}(q)\;\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-2+2k]}(L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)\\&=\mathrm{STr}_{\mathcal{V}}\Big(\Big(L_0-\frac{c}{24}\Big)o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)+2r\mathbb{E}_2\;\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)\\&\quad+\sum_{k\geq2}\mathbb{E}_{2k}\;\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-2+2k]}(L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)\\&=\Big(q\frac{\partial}{\partial q}+2r\mathbb{E}_2\Big)\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)+\sum_{k\geq2}\mathbb{E}_{2k}\;\mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-2+2k]}(L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)\\&=\partial^{(2r)}\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)+\sum_{k\geq2}A_{k,r}\mathbb{E}_{2k}\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^{r-k+1}\Omega)q^{L_0-\frac{c}{24}}\Big)\,,\end{aligned} \tag{B.19}$$

where $A_{k,r}$ is defined by

$$L_{[-2+2k]}(L_{[-2]})^r\Omega=A_{k,r}(L_{[-2]})^{r-k+1}\Omega\,. \tag{B.20}$$

Explicitly, one can compute $A_{k,r}$ using Virasoro commutation relations repeatedly and the property of vacuum $L_{[m]}\Omega=0$ for $m>-2$. In particular, one can easily show that $A_{1,r}=2r$, namely $L_{[0]}(L_{[-2]})^r\Omega=2r(L_{[-2]})^r\Omega$.

As a result, we find the following trace formula

$$\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^r\Omega)q^{L_0-\frac{c}{24}}\Big)=\mathcal{P}_{2r}\circ\mathrm{STr}_{\mathcal{V}}\Big(o(q^{L_0-\frac{c}{24}}\Big)\,, \tag{B.21}$$

where

$$\mathcal{P}_2 = D_q^{(1)}, \tag{B.22}$$

$$\mathcal{P}_4 = D_q^{(2)} + \frac{c}{2}\mathbb{E}_4, \tag{B.23}$$

$$\mathcal{P}_6 = D_q^{(3)} + 2\left(4+\frac{3c}{4}\right)\mathbb{E}_4 D_q^{(1)} + 10c\mathbb{E}_6, \tag{B.24}$$

$$\mathcal{P}_8 = D_q^{(4)} + (32+3c)\mathbb{E}_4 D_q^{(2)} + 40(4+c)\mathbb{E}_6\, D_q^{(1)} + \frac{3}{4}c(144+c)\mathbb{E}_4^2, \tag{B.25}$$

$$\begin{aligned}\mathcal{P}_{10} = {} & D_q^{(5)} + 5\left((16+c)\mathbb{E}_4\right) D_q^{(3)} + 100(8+c)\mathbb{E}_6\, D_q^{(2)} + \frac{5}{4}(1536+c(464+3c))\mathbb{E}_4^2\, D_q^{(1)} \\ & + \frac{50}{11}c(816+11c)\mathbb{E}_4\mathbb{E}_6,\end{aligned} \tag{B.26}$$

$$\begin{aligned}\mathcal{P}_{12} = {} & D_q^{(6)} + \frac{5}{2}(64+3c)\mathbb{E}_4\, D_q^{(4)} + 200\left((12+c)\mathbb{E}_6\right) D_q^{(3)} + 20\left(608+93c+\frac{9c^2}{16}\right)\mathbb{E}_4^2 D_q^{(2)} \\ & + \frac{100}{11}(7936+c(2756+33c))\mathbb{E}_4\mathbb{E}_6\, D_q^{(1)} + \frac{15}{104}c(285696+13c(432+c))\mathbb{E}_4^3 \\ & + \frac{1000}{13}c(744+13c)\mathbb{E}_6^2,\end{aligned} \tag{B.27}$$

$$\begin{aligned}\mathcal{P}_{14} = {} & D_q^{(7)} + \frac{7}{2}(80+3c)\mathbb{E}_4\, D_q^{(5)} + 350(16+c)\mathbb{E}_6\, D_q^{(4)} + \frac{35}{4}(5120+3c(176+c))\mathbb{E}_4^2\, D_q^{(3)} \\ & + \frac{350}{11}(17280+c(3064+33c))\mathbb{E}_4\mathbb{E}_6\, D_q^{(2)} + \left(\frac{15}{104}(6062080+7c(335616+13c(448+c)))\mathbb{E}_4^3\right. \\ & \left. + \frac{7000}{13}(2176+c(848+13c))\mathbb{E}_6^2\right) D_q^{(1)} + \frac{525}{22}c(240+c)(576+11c)\mathbb{E}_4^2\mathbb{E}_6,\end{aligned} \tag{B.28}$$

$$\begin{aligned}\mathcal{P}_{16} = {} & D_q^{(8)} + 14\left((32+c)\mathbb{E}_4\right) D_q^{(6)} + 560\left((20+c)\mathbb{E}_6\right) D_q^{(5)} + \frac{35}{2}(7168+c(560+3c))\mathbb{E}_4^2\, D_q^{(4)} \\ & + \frac{2800}{11}(9344+c(1124+11c))\mathbb{E}_4\mathbb{E}_6\, D_q^{(3)} + \left(\frac{5}{26}(40845312+7c(1163264+39c(464+c)))\mathbb{E}_4^3\right. \\ & \left. + \frac{28000}{13}(4768+c(952+13c))\mathbb{E}_6^2\right) D_q^{(2)} + \frac{700}{11}(1155072+c(492608+c(10132+33c)))\mathbb{E}_4^2\mathbb{E}_6\, D_q^{(1)} \\ & + \frac{105c(1207885824+17c(1951488+13c(864+c)))}{3536}\mathbb{E}_4^4 \\ & + \frac{14000c(18873216+17c(29400+143c))}{2431}\mathbb{E}_4\mathbb{E}_6^2,\end{aligned} \tag{B.29}$$

where c is the central charge of Virasoro algebra (not the 4d central charge). The expressions of $\mathcal{P}_{2k}$ for larger k can also be obtained easily, but we don't write them down explicitly here as they are becoming more and more complicated. Note that $\mathcal{P}_{2k}$'s up to $k = 5$ were presented in [37].

The same method can be used to derive the general torus one-point function with any Virasoro insertion $o(L_{-n_1}\cdots L_{-n_s}\Omega)$, based on Zhu's recursion relation (B.9) (B.10) and the Virasoro commutation relation. A particularly simple case is

$$\mathrm{STr}_{\mathcal{V}}\left(o(L_{[-n]}\Omega)q^{L_0-\frac{c}{24}}\right) = 0\,, \qquad n > 2\,, \tag{B.30}$$

which can be understood from the fact that the resulting modes $L_{[-h_a-n+2k]}\Omega$, arising from applying the recursion relation (B.10), annihilate the vacuum.

Before closing this subsection, we would like to recall some useful formulae for operator modes and OPEs in order to facilitate the actual derivation of MLDE in general VOAs with generators besides the Virasoro ones.

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Given the mode expansion in (B.2), one can take derivatives and find that modes for the derivative of vertex operator is related to the modes of original vertex operator via

$$(\partial a)_m = -(h_a+m)a_m\,, \qquad (\partial^k a)_m = (-1)^k(h_a+m+k-1)\cdots(h_a+m)a_m\,. \tag{B.31}$$

In physics, one can concretely specify the VOA via the OPEs of the generators

$$A(z)B(0) \sim \sum_n \frac{[AB]_n}{z^n}\,. \tag{B.32}$$

In particular, the $n=0$ term defines the normal order product $:AB:\equiv [AB]_0$ which will also be abbreviated as AB for simplicity. The modes of the normal order product are given by

$$(AB)_n = \sum_{k\le -h_A} A_k B_{n-k} + (-1)^{|A||B|}\sum_{k>-h_A} B_{n-k}A_k\,, \tag{B.33}$$

where $(-1)^{|A||B|}$ takes into account the statistics of two operators. We then have $|(AB)\rangle = (AB)_{-h_A-h_B}|0\rangle = A_{-h_A}B_{-h_B}|0\rangle$. Furthermore, we have $A_nB = [AB]_{n+h_A}$.

From OPE in (B.32), one can then obtain the commutation relation of their modes [38]

$$[A_m, B_n] = \sum_{l>0}\binom{m+h_A-1}{l-1}([AB]_l)_{m+n}\,. \tag{B.34}$$

Finally, the commutation relation between Virasoro modes and modes of primary operator is given by

$$[L_m, O_n] = ((h-1)m-n)O_{m+n}\,. \tag{B.35}$$

C MLDE for families of AD theories

C.1 (A_{k-1}, A_{n-1})

Let us consider the AD theory (A_{k-1}, A_{n-1}) subject to the coprime condition that $\gcd(k,n)=1$. This family of theory is simple as the chiral algebra is the W-algebra minimal model $\mathcal{W}(k,k+n)$ [39].

The central charges of the corresponding 2d VOAs are

$$c_{2d}^{(A_{k-1},A_{n-1})} = -12c_{4d}^{(A_{k-1},A_{n-1})} = -\frac{(k-1)(n-1)(k+n+nk)}{n+k}\,, \tag{C.1}$$

The Schur index is given by [19, 40]

$$\mathcal{I}_{(A_{k-1},A_{n-1})} = \mathrm{PE}\left[\frac{q^2(1-q^{k-1})(1-q^{n-1})}{(1-q)^2(1-q^{k+n})}\right]. \tag{C.2}$$

In the special case of $k=2, n=2r+1$, we get (A_1, A_{2r}) AD theory, whose chiral algebra is simply given by $(2,2r+3)$ minimal model VOA, whose only strong generator is the stress tensor. The SCFT/VOA correspondence then implies the vacuum character of $(2,2r+3)$ VOA coincides with the Schur index of (A_1, A_{2r}) up to the Casimir factor, namely

$$\chi_{(2,2r+3)} \equiv \mathrm{STr}_{\mathcal{V}_{(2,2r+3)}} q^{L_0-\frac{c}{24}} \tag{C.3}$$

$$= \mathcal{Z}_{(A_1,A_{2r})} = q^{-\frac{c_{2d}^{(A_1,A_{2r})}}{24}}\mathcal{I}_{(A_1,A_{2r})} = q^{\frac{r(6r+5)}{12(2r+3)}}\,\mathrm{PE}\left[\frac{q^2\left(1-q^{2r}\right)}{(1-q)\left(1-q^{2r+3}\right)}\right]. \tag{C.4}$$

(A_1, A_2) AD theory. The VOA of (A_1, A_2) AD theory is given by the (2,5) Lee-Yang VOA. It has a null operator of the form

$$T^2 - \frac{3}{10}\partial^2 T = 0\,. \tag{C.5}$$

Equivalently, there is a null state of the form

$$\mathcal{N}_4 \equiv (L_{-4} - \frac{5}{3}L_{-2}^2)\Omega = 0\,. \tag{C.6}$$

Following the previous general discussions, the torus one-point function with the insertion of the zero mode this null state is

$$\mathrm{STr}_{\mathcal{V}}\Big(o(\mathcal{N}_4)q^{L_0-\frac{c_{2d}}{24}}\Big) = \mathrm{STr}_{\mathcal{V}}\Big(o(L_{[-4]}\Omega)q^{L_0-\frac{c_{2d}}{24}}\Big) - \frac{5}{3}\mathrm{STr}_{\mathcal{V}}\Big(o((L_{[-2]})^2\Omega)q^{L_0-\frac{c_{2d}}{24}}\Big) \tag{C.7}$$

$$= -\frac{5}{3}\mathcal{P}_4 \circ \mathrm{STr}_{\mathcal{V}}\Big(q^{L_0-\frac{c}{24}}\Big) = 0\,, \tag{C.8}$$

where we have used (B.30) and (B.21). Using (B.27), we then find the following MLDE

$$\Big(D_q^{(2)} - \frac{11}{5}\mathbb{E}_4\Big)\chi_{(2,5)} = 0\,. \tag{C.9}$$

(A_1, A_4) AD theory. The VOA of (A_1, A_4) AD theory is given by the (2,7) minimal model VOA. It has a null operator of the form

$$T^3 - \frac{11}{14}T''T - \frac{1}{7}T'T' - \frac{19}{588}T^{(4)} = 0\,. \tag{C.10}$$

Equivalently, there is a null state of the form

$$\mathcal{N}_6 = \Big(L_{-6} + \frac{77}{38}L_{-4}L_{-2} + \frac{7}{38}L_{-3}L_{-3} - \frac{49}{38}L_{-2}L_{-2}L_{-2}\Big)\Omega = 0\,. \tag{C.11}$$

Following the general discussions in the previous subsections, the insertion of zero mode of this null state in the torus partition function leads to a MLDE. In particular, the insertion of the first and last mode in the bracket of (C.11) haven been computed in (B.30) and (B.21), and the insertion of the rest of modes can be computed using Zhu's recursion relation (B.9) (B.10). Consequently, we find the following MLDE

$$\Big(D_q^{(3)} - \frac{100}{7}\mathbb{E}_4 D_q^{(1)} - \frac{1700}{49}\mathbb{E}_6\Big)\chi_{(2,7)} = 0\,. \tag{C.12}$$

(A_1, A_6) AD theory. The VOA of (A_1, A_6) AD theory is given by the (2,9) minimal model VOA, which has a null state of the form

$$\mathcal{N}_8 = \Big(L_{-2}^4 - \frac{26}{9}L_{-4}L_{-2}^2 - \frac{4}{9}L_{-3}^2L_{-2} - \frac{88}{27}L_{-6}L_{-2} + \frac{7L_{-4}^2}{9} + \frac{4}{27}L_{-5}L_{-3} - \frac{278L_{-8}}{81}\Big)\Omega = 0\,. \tag{C.13}$$

As before, this null state gives rise to the following MLDE

$$\Big(D_q^{(4)} - \frac{130}{3}\mathbb{E}_4 D_q^{(2)} - \frac{7420}{27}\mathbb{E}_6 D_q^{(1)} - \frac{6325}{27}\mathbb{E}_4^2\Big)\chi_{(2,9)} = 0\,. \tag{C.14}$$

(A_1, A_8) AD theory. The VOA of (A_1, A_8) AD theory is given by the (2,11) minimal model VOA, which has a null state of the form

$$\mathcal{N}_{10} = \Big(L_{-2}^5 - \frac{50}{11}L_{-4}L_{-2}^3 - \frac{10}{11}L_{-3}^2L_{-2}^2 - \frac{1004}{121}L_{-6}L_{-2}^2 + \frac{411}{121}L_{-4}^2L_{-2} \tag{C.15}$$

$$+ \frac{52}{121}L_{-5}L_{-3}L_{-2} - \frac{22914L_{-8}L_{-2}}{1331} + \frac{6}{11}L_{-4}L_{-3}^2 + \frac{5052L_{-6}L_{-4}}{1331} \tag{C.16}$$

$$+ \frac{984L_{-7}L_{-3}}{1331} - \frac{164L_{-5}^2}{1331} - \frac{205200L_{-10}}{14641}\Big)\Omega = 0\,. \tag{C.17}$$

As before, this null state gives rise to the following MLDE

$$\left(D_q^{(5)} - \frac{1060}{11}\mathbb{E}_4 D_q^{(3)} - \frac{123900}{121}\mathbb{E}_6 D_q^{(2)} - \frac{2460400}{1331}\mathbb{E}_4^2 D_q^{(1)} - \frac{706764800}{161051}\mathbb{E}_4\mathbb{E}_6\right)\chi_{(2,11)} = 0\,. \tag{C.18}$$

One can also verify this equation numerically as the character is known (C.3).

The case of $k = 3$ is also simple, and gives (A_2, A_{n-1}) AD theories. If $\gcd(n, 3) = 1$, we have $(A_2, A_{3r}), (A_2, A_{3r+1})$ AD theories, whose VOAs are W_3-algebra, with strong generators T and W.

The OPEs are given by

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T}{z^2} + \frac{T'}{z}\,, \tag{C.19}$$

$$T(z)W(0) \sim \frac{3W}{z^2} + \frac{W'}{z}\,, \tag{C.20}$$

$$W(z)W(0) \sim \frac{c/3}{z^6} + \frac{2T}{z^4} + \frac{T'}{z^3} + \frac{\frac{3}{10}T'' + \frac{32}{22+5c}\Lambda}{z^2} + \frac{\frac{1}{15}T''' + \frac{16}{22+5c}\Lambda'}{z}\,, \tag{C.21}$$

where Λ is given by (3.10).

In terms of modes, the commutation relations are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,, \tag{C.22}$$

$$[L_m, W_n] = (2m-n)W_{m+n}\,, \tag{C.23}$$

$$[W_m, W_n] = (m-n)\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right]L_{m+n} \tag{C.24}$$

$$+\frac{16}{22+5c}(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}\,. \tag{C.25}$$

(A_2, A_3) AD theory. The VOA is W_3 algebra with $c = -114/7$. At dimension 5, we find a null operator

$$N_5 = W'' - \frac{14}{3}TW = 0\,. \tag{C.26}$$

At dimension 6, we find a null operator

$$N_6 = 168\,(T')^2 + 336T''T - 196T^3 - 17T^{(4)} + 1092W^2 = 0\,. \tag{C.27}$$

Then we can consider the normal order product TN_6, where TW^2 becomes $W''W$ using the null operator N_5. Consequently, we find a null operator at dimension 8:

$$T^4 - \frac{6}{7}(T')^2 T - \frac{12}{7}T''T^2 + \frac{9}{28}(T'')^2 + \frac{78}{49}W''W + \frac{1}{7}T^{(3)}T' - \frac{9}{98}T^{(4)}T - \frac{3}{490}T^{(6)} . \tag{C.28}$$

The corresponding state is

$$\Big(L_{-2}^4 - \frac{24}{7}L_{-4}L_{-2}^2 - \frac{6}{7}L_{-3}^2 L_{-2} - \frac{108}{49}L_{-6}L_{-2} + \frac{9L_{-4}^2}{7} + \frac{6}{7}L_{-5}L_{-3} - \frac{216L_{-8}}{49} + \frac{156}{49}W_{-5}W_{-3}\Big)\Omega = 0 . \tag{C.29}$$

Obviously, we have $L_{-2}^4 \in C_2(\mathcal{V})$. We can then use Zhu's recursion relation to derive the MLDE. The only non-trivial computation is for W generators, which can be computed using (C.24):

$$\begin{aligned} &W_m W_{-3}\Omega = [W_m, W_{-3}]\Omega \qquad (C.30)\\ &= \left[\frac{1}{156}(m+3)\Big(\left(23m^2+3m+52\right)L_{m-3} - 42(T^2)_{m-3}\Big) - \frac{19}{420}m\left(m^4-5m^2+4\right)\delta_{0,m-3}\right]\Omega , \end{aligned}$$

which is valid for $m > -3$, as $W_m\Omega = 0$. Here $(T^2)_{-4}\Omega = L_{-2}^2\Omega$ and $(T^2)_{m-3}\Omega = 0$ for $m > -1$.

As a result, we find the following MLDE

$$\left(D_q^{(4)} - \frac{370}{7}\mathbb{E}_4 D_q^{(2)} - \frac{10340}{49}\mathbb{E}_6 D_q^{(1)} - \frac{115425}{343}\mathbb{E}_4^2\right)\mathcal{Z}_{(A_2,A_3)} = 0 . \tag{C.31}$$

(A_2, A_4) AD theory. The VOA is W_3 algebra with $c = -23$. The MLDE can be derived similarly and turns out to be given by

$$\left(D_q^{(5)} - 125\mathbb{E}_4 D_q^{(3)} - \frac{24125}{16}\mathbb{E}_4^2 D_q^{(1)} - \frac{6825}{8}\mathbb{E}_6 D_q^{(2)} - \frac{221375\mathbb{E}_4\mathbb{E}_6}{32}\right)\mathcal{Z}_{(A_2,A_4)} = 0 . \tag{C.32}$$

(A_2, A_6) AD theory. A monic MLDE has been found to be

$$\begin{aligned} &\Bigg[D_q^{(10)} - \frac{6928}{5}\mathbb{E}_4 D_q^{(8)} + 115066\mathbb{E}_4^2 D_q^{(6)} - \frac{65828}{5}\mathbb{E}_6 D_q^{(7)} + \frac{27134436}{25}\left(\mathbb{E}_4\mathbb{E}_6\right) D_q^{(5)} - \frac{5064230556}{125}\left(\mathbb{E}_4^2\mathbb{E}_6\right) D_q^{(3)} \\ &+ \left(\frac{4157344488\mathbb{E}_4^3}{125} - \frac{1904916944\mathbb{E}_6^2}{25}\right) D_q^{(4)} + \left(-\frac{153425310291\mathbb{E}_4^4}{625} - \frac{48280541232}{125}\mathbb{E}_4\mathbb{E}_6^2\right) D_q^{(2)} \\ &+ \left(-\frac{223567701588}{125}\mathbb{E}_4^3\mathbb{E}_6 - \frac{1342610519712\mathbb{E}_6^3}{125}\right) D_q^{(1)} - \frac{6298771946544}{125}\mathbb{E}_4^2\mathbb{E}_6^2\Bigg]\mathcal{Z}_{(A_2,A_6)} = 0 . \end{aligned} \tag{C.33}$$

Meanwhile, a non-monic MLDE is also found

$$\begin{aligned} &\Bigg[\mathbb{E}_4 D_q^{(8)} - \frac{2128}{5}\mathbb{E}_4^2 D_q^{(6)} - 28\mathbb{E}_6 D_q^{(7)} + 588\left(\mathbb{E}_4\mathbb{E}_6\right) D_q^{(5)} - \frac{472164}{25}\left(\mathbb{E}_4^2\mathbb{E}_6\right) D_q^{(3)} \\ &+ \left(-48230\mathbb{E}_4^3 + \frac{751856\mathbb{E}_6^2}{5}\right) D_q^{(4)} + \left(-\frac{11004312\mathbb{E}_4^4}{125} - \frac{1880032}{5}\mathbb{E}_4\mathbb{E}_6^2\right) D_q^{(2)} \\ &+ \left(-\frac{1370396412}{125}\mathbb{E}_4^3\mathbb{E}_6 + \frac{448292768}{25}\mathbb{E}_6^3\right) D_q^{(1)} - \frac{8034147891\mathbb{E}_4^5}{625}\Bigg]\mathcal{Z}_{(A_2,A_6)} = 0 . \end{aligned} \tag{C.34}$$

C.2 $D_p(\mathrm{SU}(N))$

We now consider the family of $D_p(\mathrm{SU}(N))$ AD theory reviewed in (2.1). Here we again impose the condition $\gcd(p,N) = 1$.

The central charge of the corresponding chiral algebra is

$$c_{2d}^{D_p(\mathrm{SU}(N))} = -12c_{4d}^{D_p(\mathrm{SU}(N))} = -(p-1)(N^2-1) \tag{C.35}$$

The Schur index is given in (2.15). For simplicity, we turn off all the flavor fugacities. Then the corresponding Schur partition function is

$$\mathcal{Z}_{D_p(\mathrm{SU}(N))}(q) = q^{-\frac{c_{2d}^{D_p(\mathrm{SU}(N))}}{24}} \mathcal{I}_{D_p(\mathrm{SU}(N))}(q, \boldsymbol{x} = 1) = q^{(p-1)(N^2-1)/24}\, \mathrm{PE}\left[\frac{(q-q^p)(N^2-1)}{(1-q)(1-q^p)}\right]. \tag{C.36}$$

For $D_p(\mathrm{SU}(N))$ theory, the chiral algebra is given by the Kac-Moody algebra $\widehat{\mathfrak{su}(N)}_{-\frac{N(p-1)}{p}}$, and the only strong generators are Kac-Moody currents J^a with $a = 1, 2, \cdots, N^2-1$. In particular, the stress tensor is given via Sugawara construction:

$$T \sim \sum_a J^a J^a\,, \qquad L_{-2}\Omega + \#\sum_a J^a_{-1}J^a_{-1}\Omega = 0\,. \tag{C.37}$$

But this kind of null relation does not lead to a MLDE as $L_{-2}\Omega \notin C_2(\mathcal{V})$.

In case $N = 2$, we have AD theories which can be denoted alternatively as $D_{2k-1}(\mathrm{SU}(2)) = (I_{2,2k-3}, F) = (A_1, D_{2k-1})$. In [11], the authors found the null state of the form (for small k explicitly)

$$J^A_{-1}(J^1_{-1}J^1_{-1} + J^2_{-1}J^2_{-1} + J^3_{-1}J^3_{-1})^{k-1}\Omega \in C_2(\mathcal{V})\,. \tag{C.38}$$

As a result $(L_{-2})^k\Omega \in C_2(\mathcal{V})$, which leads the MLDE of weight $2k$.[22]

$D_3(\mathrm{SU}(2))$ AD theory.

$$\left(D_q^{(2)} - 15\mathbb{E}_4\right)\mathcal{Z}_{D_3(\mathrm{SU}(2))} = 0\,. \tag{C.39}$$

$D_5(\mathrm{SU}(2))$ AD theory.

$$\left(D_q^{(3)} - \frac{236}{5}\mathbb{E}_4 D_q^{(1)} - \frac{756\mathbb{E}_6}{5}\right)\mathcal{Z}_{D_5(\mathrm{SU}(2))} = 0\,. \tag{C.40}$$

$D_7(\mathrm{SU}(2))$ AD theory.

$$\left(D_q^{(4)} - \frac{730}{7}\mathbb{E}_4 D_q^{(2)} - \frac{36980}{49}\mathbb{E}_6 D_q^{(1)} - \frac{164025\mathbb{E}_4^2}{343}\right)\mathcal{Z}_{D_7(\mathrm{SU}(2))} = 0\,. \tag{C.41}$$

[22] However, this seems to be not true for $k = 8$. In this case, we find a MLDE of weight 12 satisfied by the Schur partition function $\Big[D_q^{(6)} - \frac{3493}{5}\mathbb{E}_4 D_q^{(4)} + \frac{7028}{5}\mathbb{E}_6 D_q^{(3)} + \frac{354331}{25}\mathbb{E}_4^2 D_q^{(2)} + \frac{323204}{5}\mathbb{E}_4\mathbb{E}_6\, D_q^{(1)} + \frac{1}{25}\left(6596205\mathbb{E}_4^3 - 13130040\mathbb{E}_6^2\right)\Big]\mathcal{Z}_{D_{15}(\mathrm{SU}(2))} = 0\,.$

$D_2(\mathrm{SU}(3))$ AD theory.

$$\left(D_q^{(2)} - 40\mathbb{E}_4\right)\mathcal{Z}_{D_2(\mathrm{SU}(3))} = 0\,. \tag{C.42}$$

$D_2(\mathrm{SU}(5))$ AD theory.

$$\left(D_q^{(3)} - 400\mathbb{E}_4 D_q^{(1)}\right)\mathcal{Z}_{D_2(\mathrm{SU}(5))} = 0\,. \tag{C.43}$$

$D_2(\mathrm{SU}(7))$ AD theory.

$$\left(D_q^{(3)} - 1840\mathbb{E}_4 D_q^{(1)} + 30240\mathbb{E}_6\right)\mathcal{Z}_{D_2(\mathrm{SU}(7))} = 0\,. \tag{C.44}$$

$D_2(\mathrm{SU}(9))$ AD theory.

$$\left(D_q^{(3)} - 5440\mathbb{E}_4 D_q^{(1)} + 196000\mathbb{E}_6\right)\mathcal{Z}_{D_2(\mathrm{SU}(9))} = 0\,. \tag{C.45}$$

$D_2(\mathrm{SU}(2k+1))$ AD theory. For $p = 2$ and $N = 2k+1 > 3$, the above examples suggest that the corresponding MLDEs always have weight 6. Assuming this and using $\mathcal{I}_{D_2(\mathrm{SU}(2k+1))} = 1 + 4k(1+k)q + \cdots$, we can determine the general form of MLDE completely

$$\begin{aligned}\left(D_q^{(3)} - 5\left(3k^4+6k^3-3k^2-6k+8\right)\mathbb{E}_4 D_q^{(1)} + 35\left(k^6+3k^5-3k^4-11k^3-6k^2\right)\mathbb{E}_6\right)& \\ \times\mathcal{Z}_{D_2(\mathrm{SU}(9))} = 0\,.&\end{aligned} \tag{C.46}$$

This suggests that $(L_{-2})^3\Omega \in C_2(\mathcal{V})$.

C.3 $\mathcal{T}_{(3,2)}$

The chiral algebra of $\mathcal{T}_{(3,2)}$ AD theory is given by the $\mathcal{A}(6)$ algebra [6, 26, 27]. It contains 3 strong generators, denoted by $T, \Psi, \widetilde{\Psi}$, whose conformal dimensions are 2, 4, 4, respectively. While the first generator T is the stress tensor, the latter two $\Psi, \widetilde{\Psi}$ are fermionic Virasoro primary operators. The OPEs among them are given by

$$T(z)T(0) \sim \frac{-12}{z^4} + \frac{2T}{z^2} + \frac{T'}{z}\,, \tag{C.47}$$

$$T(z)\Psi(0) \sim \frac{4\Psi}{z^2} + \frac{\Psi'}{z}\,, \tag{C.48}$$

$$T(z)\tilde{\Psi}(0) \sim \frac{4\tilde{\Psi}}{z^2} + \frac{\tilde{\Psi}'}{z}\,, \tag{C.49}$$

$$\begin{aligned}\Psi(z)\tilde{\Psi}(0) \sim& -\frac{6}{z^8} + \frac{2T}{z^6} + \frac{T'}{z^5} + \frac{3\left(T''-T^2\right)}{7z^4} + \frac{2T^{(3)} - 9T'T}{21z^3} \\ &+\frac{-48\left(T'\right)^2 - 84T''T + 36T^3 + 7T^{(4)}}{420z^2} \\ &+\frac{60\left(-5T''T' + 6T'T^2 - 2T^{(3)}T\right) + 7T^{(5)}}{2800z}\,,\end{aligned} \tag{C.50}$$

where we ignore the argument of operators on the r.h.s., which is 0. Actually the full OPE can be easily bootstrapped using the associativity of OPEs and the information of conformal dimensions of these operators.[23]

This algebra admits a free field realization in terms of chiral boson φ satisfying the OPE

$$\varphi(z)\varphi(0) \sim \log z \,. \tag{C.51}$$

We can write the generator of the chiral algebra using φ [26, 27]

$$\Psi(w) = e^{-\sqrt{3}\varphi(w)}\,, \qquad T = \frac{1}{2}(\partial\varphi)^2 + \frac{5}{2\sqrt{3}}\partial^2\varphi\,, \tag{C.52}$$

$$\widetilde{\Psi}(w) = \frac{1}{2\pi i}\oint \frac{dz}{z-w} e^{2\sqrt{3}\varphi(z)}\Psi(w) = P_5(\partial\varphi, \partial^2\varphi, \cdots \partial^5\varphi)e^{\sqrt{3}\varphi(z)}\,, \tag{C.53}$$

where P_5 is an operator with conformal dimension 5 built as a polynomial in $\partial^j\varphi$. Since it is quite complicated, we don't write it down explicitly here. One can verify that the free field realization is consistent with the OPEs above.

The mode expansion of these operators are

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-2-n}\,, \qquad \Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{-4-n}\,, \qquad \widetilde{\Psi}(z) = \sum_{n\in\mathbb{Z}} \widetilde{\Psi}_n z^{-4-n}\,. \tag{C.54}$$

Using the previous OPEs, we can derive the commutation relation of these modes[24]

$$[L_m, L_n] = (m-n)L_{m+n} - 2(m^3-m)\delta_{m+n,0}, \tag{C.55}$$

$$[L_m, \Psi_n] = (3m-n)\Psi_{m+n}, \tag{C.56}$$

$$[L_m, \widetilde{\Psi}_n] = (3m-n)\widetilde{\Psi}_{m+n}, \tag{C.57}$$

$$\begin{aligned}\{\Psi_m, \widetilde{\Psi}_n\} = {} & \frac{1}{840}n\left(n^2\left(n^2-7\right)^2-36\right)\delta_{0,m+n} - \frac{1}{84}(m-n)\left(m^2-mn+n^2-7\right)\Lambda_{m+n} \\ & + \frac{1}{1680}(m-n)\left(3(m+n)^4-14mn\left(m^2+mn+n^2\right)-39(m+n)^2+98mn+108\right)L_{m+n} \\ & - \frac{5}{112}(m-n)\widetilde{\Lambda}_{m+n} + \frac{7}{80}(m-n)\Upsilon_{m+n},\end{aligned} \tag{C.58}$$

where[25]

$$\Lambda = T^2 - \frac{3}{10}T''\,, \tag{C.59}$$

$$\widetilde{\Lambda} = T^3 - \frac{1}{3}(T')^2 - \frac{19}{30}T''T - \frac{1}{36}T''''\,, \qquad \Upsilon = T^3 - \frac{1}{7}(T')^2 - \frac{11}{14}T''T - \frac{19}{588}T''''\,. \tag{C.60}$$

We need to understand the null operators of this VOA. At dimension 6 we find 2 null operators

$$N_6 = \Psi'' - 6T\Psi = 0\,, \qquad \widetilde{N}_6 = \widetilde{\Psi}'' - 6T\widetilde{\Psi} = 0\,. \tag{C.61}$$

[23]When computing the OPEs, we make heavy use of the Mathematica package `OPEdefs` [41].

[24]In deriving the mode algebra, we use the formulae in (B.31)–(B.35).

[25]The $\Lambda, \widetilde{\Lambda}, \Upsilon$ constructed in this way have simple commutation relations with stress tensor. E.g. $[L_m, \Lambda_n] = \frac{1}{30}(5c+22)m\left(m^2-1\right)L_{m+n} + (3m-n)\Lambda_{m+n}$. They become the null operators of Virasoro algebra for specific values of central charge.

Besides, we also have the null operators $7T\Psi - \Psi T = 0$ and $7T\widetilde{\Psi} - \widetilde{\Psi}T = 0$, which arise from exchanging the order of two operators.

At dimension 8 we find a non-trivial null operator[26]

$$N_8 = T^4 + \frac{140}{3}\Psi\widetilde{\Psi} - \frac{10}{3}\left(T'\right)^2 T - \frac{10}{3}T''T^2 + \frac{5}{4}\left(T''\right)^2 + \frac{10}{9}T^{(3)}T' + \frac{5}{18}T^{(4)}T - \frac{5}{216}T^{(6)}\,. \quad \text{(C.62)}$$

Due to the presence of $\Psi\widetilde{\Psi}$, the corresponding state does not fit into the form of (B.13), namely $(L_{-2})^4\Omega \notin C_2(\mathcal{V})$. However, we can consider the normal order product TN_8, then the fermion bilinear terms can be rewritten $T\Psi\tilde{\Psi} \to \Psi''\tilde{\Psi}$ using the null operator (C.61). Consequently, we get a null operator at level 10:

$$\begin{aligned} N_{10} = {}& T^5 + \frac{4}{3}T^{(3)}T'T - \frac{10}{3}\left(T'\right)^2 T^2 - \frac{10}{3}T''T^3 + 2T''\left(T'\right)^2 + \frac{9}{4}\left(T''\right)^2 T - \frac{1}{9}\left(T^{(3)}\right)^2 \\ & - \frac{1}{9}T^{(4)}T^2 - \frac{1}{12}T^{(4)}T'' - \frac{1}{30}T^{(5)}T' - \frac{13}{360}T^{(6)}T + \frac{1}{5040}T^{(8)} + \frac{70}{9}\Psi''\tilde{\Psi} + \frac{140}{3}\tilde{\Psi}''\Psi\,. \end{aligned} \quad \text{(C.63)}$$

In terms of modes, the corresponding null state is

$$\begin{aligned} \mathcal{N}_{10} = \Big(& L_{-2}^5 - \frac{20}{3}L_{-4}L_{-2}^3 - \frac{10}{3}L_{-3}^2 L_{-2}^2 - \frac{8}{3}L_{-6}L_{-2}^2 + 9L_{-4}^2 L_{-2} - 26L_{-8}L_{-2} + 8L_{-5}L_{-3}L_{-2} \\ & - 4L_{-5}^2 + 4L_{-4}L_{-3}^2 + 8L_{-10} - 4L_{-6}L_{-4} - 4L_{-7}L_{-3} + \frac{280}{3}\widetilde{\Psi}_{-6}\Psi_{-4} + \frac{140}{9}\Psi_{-6}\widetilde{\Psi}_{-4}\Big)\Omega. \end{aligned} \quad \text{(C.64)}$$

This null state has the form of (B.13), namely $(L_{-2})^5\Omega \in C_2(\mathcal{V})$. The presence of such a null operator enables us to derive the MLDE. The derivation of Virasoro mode contribution is the same as before, so we only show the nontrivial contribution from fermionic modes.

Using Zhu's recursion relation (B.10), we find the insertion of fermion bilinear zero mode reduces to

$$\Psi_{-6}\widetilde{\Psi}_{-4}\Omega \quad \to \quad \Psi_{-2+2r}\widetilde{\Psi}_{-4}\Omega = \{\Psi_{-2+2r}, \widetilde{\Psi}_{-4}\}\Omega\,, \qquad r = 0,1,2,3,\cdots\,, \quad \text{(C.65)}$$

where we used the property of vacuum that $\Psi_n\Omega = \widetilde{\Psi}_n\Omega = 0$ for $n > -4$. The anti-commutators of fermions are given by Virasoro modes in (C.58). Explicitly, we find

$$\Psi_{-2}\widetilde{\Psi}_{-4}\Omega = \left(\frac{3}{35}L_{-2}^3 - \frac{2}{5}L_{-4}L_{-2} + \frac{2}{5}L_{-6} - \frac{4}{35}L_{-3}^2\right)\Omega\,, \quad \text{(C.66)}$$

$$\Psi_0\widetilde{\Psi}_{-4}\Omega = \left(\frac{6}{7}L_{-4} - \frac{3}{7}L_{-2}^2\right)\Omega\,, \quad \text{(C.67)}$$

$$\Psi_2\widetilde{\Psi}_{-4}\Omega = 2L_{-2}\Omega\,, \quad \text{(C.68)}$$

$$\Psi_4\widetilde{\Psi}_{-4}\Omega = -6\Omega\,, \quad \text{(C.69)}$$

$$\Psi_n\widetilde{\Psi}_{-4}\Omega = 0\,, \qquad n > 4\,. \quad \text{(C.70)}$$

Therefore, everything reduces to the stress tensor insertion again, which can be computed easily. The contribution from inserting the zero mode of $\widetilde{\Psi}_{-6}\Psi_{-4}\Omega$ can computed in exactly

[26] There are of course other null operators which are obtained from the one of lower dimension by taking derivative and multiplying with other operators. They can be obtained straightforwardly, so we will not show them explicitly.

the same way; actually their contribution is just the opposite of inserting $\Psi_{-6}\widetilde{\Psi}_{-4}\Omega$, as one can see from the commutation relation (C.58). After combining all contributions together, we arrive at the following simple MLDE

$$\left[D_q^{(5)} - 140\mathbb{E}_4 D_q^{(3)} - 700\mathbb{E}_6 D_q^{(2)} - 2000\mathbb{E}_4^2 D_q^{(1)}\right]\mathcal{Z}_{\mathcal{T}_{(3,2)}} = 0\,, \tag{C.71}$$

where the Schur partition function

$$\mathcal{Z}_{\mathcal{T}_{(3,2)}} = q^{-c_{2d}/24}\mathcal{I} = q^{c_{4d}/2}\mathcal{I}\,, \qquad \mathcal{I} = 1 + \cdots\,. \tag{C.72}$$

Explicitly, it is given in (2.70). One can then numerically verify that the MLDE (C.71) is indeed satisfied.

C.4 $\mathcal{T}_{(4,3)}$

The Schur partition function of $\mathcal{T}_{(4,3)}$ AD theory is annihilated by the following MLDO of weight 34:

$$\begin{aligned}
\mathcal{D}_q^{(17)} = {} & D_q^{(17)} - \frac{51533242520305}{14352492173}\mathbb{E}_4 D_q^{(15)} - \frac{6973916257268590}{14352492173}\mathbb{E}_6 D_q^{(14)} - \frac{85177879156262050}{14352492173}\mathbb{E}_4^2 D_q^{(13)} \\
& + \frac{14443540566243470150}{14352492173}\mathbb{E}_4\mathbb{E}_6 D_q^{(12)} \\
& + \left(\frac{1816106608843919305 00\mathbb{E}_4^3}{14352492173} + \frac{5090954917066018442 75\mathbb{E}_6^2}{14352492173}\right) D_q^{(11)} \\
& + \frac{1659222240987437280000}{14352492173}\mathbb{E}_4^2\mathbb{E}_6 D_q^{(10)} \\
& + \left(\frac{15968624650880199775000\mathbb{E}_4^4}{14352492173} - \frac{38160696348954260340 8750\mathbb{E}_4\mathbb{E}_6^2}{14352492173}\right) D_q^{(9)} \\
& + \left(-\frac{230351539542689805537 5000\mathbb{E}_4^3\mathbb{E}_6}{14352492173} - \frac{350881620853176693069 1250\mathbb{E}_6^3}{14352492173}\right) D_q^{(8)} \\
& + \left(-\frac{204303702383003898350 00000\mathbb{E}_4^5}{14352492173} - \frac{2292747657932555425 0000\mathbb{E}_4^2\mathbb{E}_6^2}{161263957}\right) D_q^{(7)} \\
& + \left(\frac{334591267928888659150 00000\mathbb{E}_4^4\mathbb{E}_6}{14352492173} - \frac{333254744268135740656 550000\mathbb{E}_4\mathbb{E}_6^3}{14352492173}\right) D_q^{(6)} \\
& + \left(\frac{350540943451154796600 000000\mathbb{E}_4^6}{14352492173} + \frac{260643405940116687647 6000000\mathbb{E}_4^3\mathbb{E}_6^2}{14352492173}\right. \\
& \qquad \left. - \frac{559300998615517929348 4100000\mathbb{E}_6^4}{14352492173}\right) D_q^{(5)} \\
& + \left(-\frac{221128011675518101425 60000000\mathbb{E}_4^5\mathbb{E}_6}{14352492173} + \frac{866820551626002839502 00000000\mathbb{E}_4^2\mathbb{E}_6^3}{14352492173}\right) D_q^{(4)} \\
& + \left(\frac{424411033216159954980 0000000\mathbb{E}_4^7}{14352492173} + \frac{221096430874162940607 00000000\mathbb{E}_4^4\mathbb{E}_6^2}{14352492173}\right. \\
& \qquad \left. + \frac{163123204241712899740 202500000\mathbb{E}_4\mathbb{E}_6^4}{14352492173}\right) D_q^{(3)} \\
& + \left(\frac{191602959513327117390 600000000\mathbb{E}_4^6\mathbb{E}_6}{14352492173} + \frac{173472811242429480314 1900000000\mathbb{E}_4^3\mathbb{E}_6^3}{14352492173}\right. \\
& \qquad \left. - \frac{986866725033974720394 987500000\mathbb{E}_6^5}{14352492173}\right) D_q^{(2)}
\end{aligned}$$

$$+\left(-\frac{846400892357004288480000000000\mathbb{E}_4^8}{14352492173}+\frac{866037293897187651694560000000\mathbb{E}_4^5\mathbb{E}_6^2}{14352492173}\right.$$
$$\left.-\frac{52016089913494409621646000000000\mathbb{E}_4^2\mathbb{E}_6^4}{14352492173}\right)D_q^{(1)}. \tag{C.73}$$

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