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Black hole horizon edge partition functions

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ABSTRACT: We extend a formula for 1-loop black hole determinants by Denef, Hartnoll, and Sachdev (DHS) to spinning fields on any $(d + 1)$ -dimensional static spherically symmetric black hole. By carefully analyzing the regularity condition imposed on the Euclidean eigenfunctions, we reveal an unambiguous bulk-edge split in the 1-loop Euclidean partition function for tensor fields of arbitrary integer spin: the bulk part captures the “renormalized” thermal canonical partition function recently discussed in [1]; the edge part is related to quasinormal modes (QNMs) that fail to analytically continue to a subset of Euclidean modes with enhanced fall-offs near the origin. Since the edge part takes the form of a path integral on S^{d-1} , this suggests that these are associated with degrees of freedom living on the bifurcation surface in the Lorentzian two-sided black hole geometry. For massive higher spin on static BTZ and massive vector on Nariai black holes, we find that the edge partition function is related to the QNMs with lowest overtone numbers.

KEYWORDS: Black Holes, Models of Quantum Gravity, Thermal Field Theory

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1 Introduction

Ever since the seminal work [2], the Euclidean gravitational path integral has been a prominent tool that has led to tremendous progress in thermodynamic and entanglement aspects of quantum black holes. In some cases, one finds exact agreement with microscopic calculations in string theory or holographic CFTs, even beyond the leading order in G_N [3–7].

Operationally, one starts with a formal path integral integrating over all metrics and matter fields, and then expands $g = g^* + \delta g$, $\Phi = \Phi^* + \delta\phi$ around the saddle points (g^*, Φ^*)

$$Z = \int \mathcal{D}g \mathcal{D}\Phi e^{-S[g, \Phi]} \approx \sum_{g^*, \Phi^*} e^{-S[g^*, \Phi^*]} Z_{1\text{-loop}}[g^*, \Phi^*] (1 + \dots). \quad (1.1)$$

Many aspects of such a formal object remain to be understood. For example, what exactly should we sum over in \sum_{g^*, Φ^*} ? Another well-known confusion is that in the gravity sector¹ there is a conformal mode that renders the gravitational action unbounded from below [9].

This paper is a continuation of [1], concerning the 1-loop contributions from matter fields and the graviton around a $(d+1)$ -dimensional static spherically symmetric black hole background. In Euclidean signature, this means we always have $U(1) \times SO(d)$ symmetry, associated with the thermal circle and the codimension-2 sphere, as part of the isometries. In [1], we considered the 1-loop Euclidean path integral for a real scalar on such a background, which takes the form of a functional determinant

$$Z_{\text{PI}}(m^2) = \int \mathcal{D}\phi e^{-\frac{1}{2} \int (\nabla\phi)^2 + m^2\phi^2} = \frac{1}{\det(-\nabla^2 + m^2)^{1/2}}. \quad (1.2)$$

Our key result in [1] is that (1.2) has a canonical interpretation through the relation

$$Z_{\text{PI}} = \tilde{Z}_{\text{bulk}}, \quad \tilde{Z}_{\text{bulk}} \equiv \frac{Z_{\text{bulk}}}{Z_{\text{bulk}}^{\text{Rin}}} \quad (\text{scalar}). \quad (1.3)$$

Here $Z_{\text{bulk}} \equiv \text{Tr} e^{-\beta_H \hat{H}}$ is the formal thermal canonical partition function at the inverse black hole temperature β_H for the scalar living outside the horizon, while $Z_{\text{bulk}}^{\text{Rin}}$ is analogously defined but on a Rindler-like wedge at the inverse temperature β_H . As explained

¹In general, for any massless fields with spin $s \geq 2$ there are finite number of modes with a wrong sign of kinetic term, which has been demonstrated explicitly in [8] for the case of massless higher spin fields on a sphere.

in [1] and briefly reviewed in section 2.2, while these traces Tr are ill-defined, their ratios can be unambiguously defined. Explicitly, the “renormalized” partition function is given by the formula

$$\log \tilde{Z}_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-2\pi t/\beta_H}}{1 - e^{-2\pi t/\beta_H}} \chi_{\text{QNM}}(t), \quad \chi_{\text{QNM}}(t) \equiv \sum_z N_z e^{-izt}. \quad (1.4)$$

Here $\chi_{\text{QNM}}(t)$ is a “quasinormal mode (QNM) character” defined as a sum over the QNM spectrum, with z the frequencies of the QNMs and N_z their degeneracies. The relation (1.3) has been verified in [1] for the case of scalars on static BTZ, Nariai, and the de Sitter static patch.

In this work, we extend these considerations to arbitrary spinning fields. While we will focus on massive fields, since the 1-loop path integrals for massless gauge fields are given by ratios of determinants of differential operators, we can simply put together the massive results taking the masses to specific values in order to obtain the massless results.²

As demonstrated by our explicit examples of massive higher spin (HS) fields on static BTZ and massive vectors on Nariai, one can still define \tilde{Z}_{bulk} as a formal ratio like (1.3), which continues to be given explicitly by the formula (1.4). However, it turns out that $Z_{\text{PI}} \neq \tilde{Z}_{\text{bulk}}$ for any spin $s \geq 1$. In fact, for massive symmetric tensor [11, 12] and p -form fields [13] with arbitrary spins on a round sphere S^{d+1} or $EAdS_{d+1}$, it was observed that their Euclidean path integrals could be brought into the form

$$Z_{\text{PI}} = \frac{\tilde{Z}_{\text{bulk}}}{Z_{\text{edge}}}. \quad (1.5)$$

Our goal is to provide an explanation for this bulk-edge split and systematically characterize the edge part Z_{edge} for general higher spin fields on any static black hole background.

To arrive at (1.5), we first note that the formula (1.4) is equivalent to a formula derived by Denef, Hartnoll, and Sachdev (DHS) [14] for the scalar Euclidean path integral (1.2). The DHS derivation was based on the analytic properties of $Z_{\text{PI}}(m^2)$ as a function on the complex m^2 -plane, and the fact that *any* QNM would Wick-rotate to a *regular* Euclidean mode at the correct (complex) value of m^2 .

As explained in section 3, the key subtlety for spinning fields is associated with the regularity condition (i.e. smoothness and single-valuedness around the Euclidean time direction) imposed on the field configurations included in the Euclidean path integration. While this condition seems innocuous (and is naturally assumed in any calculation of 1-loop determinants in the literature), a careful analysis reveals that it has non-trivial consequences in the DHS derivation, as already pointed out in the context of spin-2 fields on a BTZ background in [15].³ As we will see, the regularity condition for spinning fields creates

²For compact spaces there could be new subtleties for massless fields, such as residual group volume or Polchinski’s phase coming from Wick-rotating the conformal modes [8–10]. These are contributions from a finite number of modes and do not affect the general consideration of this paper.

³In [15], the analysis was phrased in terms of (local) square integrability at the origin, which is implied if the functions are regular at the origin.

an obstruction for some QNMs to Wick-rotate to a subset of regular Euclidean modes and eventually leads to the form (1.5), with the edge part explicitly given by

$$\log Z_{\text{edge}} = \int_0^\infty \frac{dt}{2t} \sum_{k=-(s-1)}^{s-1} \sum_{z_{e,k}} e^{-\left(\frac{2\pi}{\beta_H} |k| + iz_{e,k}\right)t} \quad (1.6)$$

for a massive spin- s field. Here, $z_{e,k} = z_{e,k}(m^2)$ are the frequencies of those QNMs which fail to Wick-rotate to a regular Euclidean mode with $U(1)$ quantum number $|k|$ for any complex value of mass m^2 . The sum $\sum_{z_{e,k}}$ receives contributions from $SO(d)$ representations of spin $0, 1, \dots, s-1$. Since Z_{edge} is characterized based on the regularity condition near the origin, it is natural to associate these $SO(d)$ degrees of freedom as living on the bifurcation surface S^{d-1} in the Lorentzian signature, thus justifying the terminology ‘‘edge’’. In sections 4 and 5, we work out the explicit form of (1.6) for massive HS on static BTZ and massive vector on Nariai. In combination with (1.4), we then find exact agreement with Z_{PI} as in (1.5).

While in this work we do not have a canonical interpretation for Z_{edge} , the structure (1.5) is generally expected from studies of entanglement entropy in gauge theories and gravity. We will comment more on this as we conclude in section 6.

As mentioned earlier on, the general form (1.5) for higher spin fields on a sphere was first observed in [11], where, however, the precise $SO(d)$ contents for Z_{edge} were somewhat obscure. To clarify those, one could follow the same procedure of checking the Euclidean continuation of QNMs demonstrated in our explicit examples in sections 4 and 5, and put Z_{edge} into the form (1.6). However, it turns out that there exists yet another way to work out the precise $SO(d)$ contents for Z_{edge} , by exploiting powerful methods from representation theory. This will be explained in an upcoming work [16].

Plan of the paper. We review the DHS formula for scalars and its Lorentzian interpretation in section 2. In section 3, we examine the Euclidean regularity condition and generalize the DHS arguments to arbitrary spinning fields. In sections 4 and 5 we work out the explicit examples of massive HS on static BTZ and massive vector on Nariai respectively. We collect some helpful basic facts for scalar and vector spherical harmonics in appendix A. Appendices B–D contain technical calculations that are useful in our analysis.

2 Comments on the Denef-Hartnoll-Sachdev formula

2.1 Review of the Denef-Hartnoll-Sachdev argument for scalars

The following discussion applies to arbitrary $(d+1)$ -dimensional static spherically symmetric backgrounds:

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_{d-1}^2. \quad (2.1)$$

Here $d\Omega_{d-1}^2$ is the metric on the unit S^{d-1} . There is a horizon at $r = r_H$ if $F(r_H) = 0$, with inverse Hawking temperature $\beta_H = \frac{1}{T_H} = \frac{4\pi}{F'(r_H)}$. Wick-rotating $t = -it_E$ in (2.1)

and making it periodic

$$ds^2 \rightarrow ds_E^2 = F(r)dt_E^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_{d-1}^2, \quad t_E \simeq t_E + \beta_H, \quad (2.2)$$

we obtain a smooth geometry that arises as a saddle point in the Euclidean gravitational path integral. The above analytic continuation maps the horizon at $r = r_H$ to the origin, near which we can make a change of variables

$$\rho^2 = \frac{4}{F'(r_H)}(r - r_H), \quad \varphi = \frac{2\pi}{\beta_H} t_E, \quad (2.3)$$

so that the near-horizon spacetime takes the product form

$$ds^2 \approx d\rho^2 + \rho^2 d\varphi^2 + r_H^2 d\Omega_{d-1}^2 = dud\bar{u} + r_H^2 d\Omega_{d-1}^2. \quad (2.4)$$

In the last equality we have introduced the complex coordinates

$$u = \rho e^{-i\varphi}, \quad \bar{u} = \rho e^{i\varphi}. \quad (2.5)$$

At 1-loop, corrections to the gravitational path integral are given by integrating quadratic fluctuations of matter fields (including the graviton) living on (2.2). For instance, the 1-loop contribution of a real scalar ϕ with mass m^2 is given by

$$Z_{\text{PI}}(m^2) = \int \mathcal{D}\phi e^{-\frac{1}{2} \int (\nabla\phi)^2 + m^2\phi^2} = \frac{1}{\det(-\nabla^2 + m^2)^{1/2}}. \quad (2.6)$$

Regularity condition. We demand the functions in the functional integration (2.6) to be smooth at the origin $\rho = 0$ and single-valued in the Euclidean time direction, i.e. they should be regular functions. This means that ϕ has a Taylor expansion in the complex coordinates u, \bar{u} near the origin. More precisely, a mode with thermal frequency k has the following $\rho \rightarrow 0$ behavior:

$$\phi_k \sim \rho^{|k|} e^{-ik\varphi} = \begin{cases} u^k, & k \geq 0 \\ \bar{u}^{-k}, & k \leq 0 \end{cases}. \quad (2.7)$$

As part of the definition of the path integral, ϕ is typically required to satisfy other boundary conditions (e.g. standard or alternate boundary condition in asymptotically AdS black holes).

The idea of [14] is that we assume the functional determinant (2.6) to be a meromorphic function on the complex m^2 -plane, and try to match its poles and zeros. $Z_{\text{PI}}(m^2)$ has no zero, and hits a pole whenever

$$(-\nabla^2 + m^2)\phi = 0. \quad (2.8)$$

Solving this equation near $\rho = 0$, we deduce the near-origin behavior

$$\phi \sim \rho^{\mp i \frac{z}{2\pi T_H}} e^{-\frac{z}{2\pi T_H} \varphi} = \rho^{\mp i \frac{z}{2\pi T_H}} e^{-zt_E}, \quad (2.9)$$

where $z = z(m^2)$ is a function of m^2 . If we Wick rotate (2.9) back to real time, the $\rho \rightarrow 0$ behavior becomes the near-horizon behavior

$$\phi \sim \rho^{\mp i \frac{z}{2\pi T_H}} e^{-izt} = e^{-iz(t \pm x)}, \quad x \equiv \frac{\ln \rho}{2\pi T_H}. \quad (2.10)$$

This is the boundary condition satisfied by (anti-)QNMs purely approaching (leaving) the horizon. Therefore, for *physical* m^2 , z is a QNM or anti-QNM frequency.

Now, for m^2 to be a pole of $Z_{\text{PI}}(m^2)$, we need the Euclidean solution ϕ to be regular at the origin. We can see from (2.9) that for *generic* m^2 this will not be the case. However, by varying m^2 (and thus $z(m^2)$) over the complex plane, we encounter a regular solution of (2.8) every time that (2.9) matches onto either branch of (2.7). The \mp branch in (2.9) can only be matched onto the $k > 0$ ($k < 0$) branch in (2.7), while either can match onto the $k = 0$ mode. Therefore, we conclude that⁴

$$\frac{1}{\det(-\nabla^2 + m^2)} = \prod_z \prod_{k=-\infty}^{\infty} \left(|k| + \frac{iz}{2\pi T_H} \right)^{-N_z/2} \left(|k| - \frac{i\bar{z}}{2\pi T_H} \right)^{-N_{\bar{z}}/2}. \quad (2.11)$$

Here \bar{z} are anti-QNM frequencies. When the theory is PT-symmetric, \bar{z} can be taken to be the complex conjugate of z . Alternatively, we observe that since the Lorentzian equation of motion is invariant under $t \rightarrow -t$, for a QNM with frequency z , there is an anti-QNM with frequency $-z$. Therefore, we can replace $\bar{z} \rightarrow -z$ in (2.11), and we have simply

$$Z_{\text{PI}}(m^2) = \prod_z \prod_{k=-\infty}^{\infty} \left(|k| + \frac{iz}{2\pi T_H} \right)^{-N_z/2}. \quad (2.12)$$

We will focus on this case from now on. Using $\log x = \int_0^\infty \frac{dt}{t} e^{-xt}$ (ignoring the issue of UV-divergence), we can formally write [1]

$$\log Z_{\text{PI}} = \int_0^\infty \frac{dt}{2t} \sum_z \sum_{k=-\infty}^{\infty} N_z e^{-\left(|k| + \frac{iz}{2\pi T_H} \right) t} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-2\pi t/\beta_H}}{1 - e^{-2\pi t/\beta_H}} \chi_{\text{QNM}}(t). \quad (2.13)$$

In the second equality we performed the sum over k , scaled $t \rightarrow 2\pi t/\beta_H$, and expressed in terms of the ‘‘QNM character’’

$$\chi_{\text{QNM}}(t) \equiv \sum_z N_z e^{-izt}. \quad (2.14)$$

2.2 Black hole scattering and the renormalized partition function

The main result of [1] is a Lorentzian calculation that reproduces the 1-loop Euclidean path integral as computed by the DHS formula (2.13), which we review in this section. We refer the reader to [1] for a more detailed discussion. For concreteness we focus on the case of asymptotically AdS black holes for $d \geq 3$.

⁴Generally there is a holomorphic function $e^{P(m^2)}$ multiplying (2.11), which is related to the UV-divergences (including the logarithmic divergence $d+1$ is even) of (2.11) and can be determined by comparing $m^2 \rightarrow \infty$ asymptotics of (2.11) and the heat kernel coefficients [14]. We proceed formally neglecting this issue, and will provide a rigorous regularization when we discuss explicit examples.

Black hole scattering. To start with, we separate

$$\phi_{\omega l}(t, r, \Omega) = e^{-i\omega t} \frac{\psi_l(r)}{r^{\frac{d-1}{2}}} Y_l(\Omega). \quad (2.15)$$

For every integer $l \geq 0$, Y_l are the $(d-1)$ -dimensional spherical harmonics. Making use of this ansatz and the tortoise coordinate $x \equiv \int_{\infty}^r \frac{dr'}{F(r')}$, the Klein-Gordon equation $(-\nabla^2 + m^2)\phi = 0$ on the background (2.1) is recast into a 1D Schrödinger form for each l :

$$\left(-\partial_x^2 + V_l(x)\right) \psi_l(x) = \omega^2 \psi_l(x), \quad (2.16)$$

with the effective potential

$$V_l(x) = F(r) \left[\frac{d-1}{2r^{\frac{d-1}{2}}} \partial_r \left(r^{\frac{d-3}{2}} F(r) \right) + \left(\frac{l(l+d-2)}{r^2} + m^2 \right) \right]. \quad (2.17)$$

In the near-horizon regime ($x \rightarrow -\infty$), the normalizable solution to (2.16) satisfying the standard boundary condition at infinity ($x = 0$) takes the asymptotic form

$$\psi_l(x \rightarrow -\infty) \sim A_l^{\text{out}}(\omega) e^{-i\omega x} + A_l^{\text{in}}(\omega) e^{i\omega x}. \quad (2.18)$$

Here by “in” (“out”) we mean the waves travel away from (towards) the horizon, as opposed to the common terminology in studies of QNMs. For real ω , $A_l^{\text{in}}(\omega) = A_l^{\text{out}*}(\omega)$, and the ratio

$$\mathcal{S}_l(\omega) = \frac{A_l^{\text{out}}(\omega)}{A_l^{\text{in}}(\omega)} \equiv e^{2i\theta_l(\omega)} \quad (2.19)$$

is a pure phase, or a rank-1 unitary S-matrix.

The renormalized partition function. A naive Lorentzian calculation to be compared with the 1-loop Euclidean path integral (2.6), would be that of the ideal gas canonical partition function for the scalar field living on the background (2.1):

$$\log Z_{\text{bulk}} \equiv \log \text{Tr} e^{-\beta_H \hat{H}} = \int_0^{\infty} d\omega \rho(\omega) \log \left(e^{\beta_H \omega/2} - e^{-\beta_H \omega/2} \right). \quad (2.20)$$

Here $\rho(\omega) = \sum_l D_l^d \rho_l(\omega)$ is the total single-particle density of states (DOS). As it is, (2.20) is pathological: for every $\text{SO}(d)$ angular momenta $l \geq 0$, there is a continuum of normal modes in any small interval $\Delta\omega$, and thus $\rho_l(\omega)$ is strictly infinite. This infinity is distinct from the usual UV-divergences coming from integrating over all $\omega > 0$ and summing over all $l \geq 0$.

The key realization of [1] is that the non-trivial information about the spacetime and the scalar field encoded in the potential (2.17) can be extracted by comparing the scattering problem (2.16) to a reference problem with potential $\bar{V}_l(x)$. The difference of $\rho_l(\omega)$ from the reference $\bar{\rho}_l(\omega)$ is a completely finite quantity, related to the scattering matrices (2.19):

$$\Delta\rho_l(\omega) = \rho_l(\omega) - \bar{\rho}_l(\omega) = \frac{1}{2\pi i} \partial_{\omega} \left(\log \mathcal{S}_l(\omega) - \log \bar{\mathcal{S}}_l(\omega) \right). \quad (2.21)$$

Here the difference in the first equality is understood in a limiting sense explained in [1]. Therefore, instead of (2.20), a class of better defined objects are given by *differences* of free energies:

$$\log Z_{\text{bulk}} - \log \bar{Z}_{\text{bulk}} = \int_0^\infty d\omega \Delta\rho(\omega) \log \left(e^{\beta_H \omega/2} - e^{-\beta_H \omega/2} \right) \quad (2.22)$$

where

$$\Delta\rho(\omega) = \sum_{l=0}^\infty D_l^d \Delta\rho_l(\omega) = \frac{1}{2\pi i} \partial_\omega \sum_{l=0}^\infty D_l^d \left(\log \mathcal{S}_l(\omega) - \log \bar{\mathcal{S}}_l(\omega) \right). \quad (2.23)$$

Quantities like (2.22) are still UV-divergent due to the integration over all ω and the sum over all $l \geq 0$, but these are the usual divergences that are absorbed into the renormalization of the cosmological constant, Newton’s constant and curvature couplings once we couple our theory to gravity.

A priori, there is no canonical choice of the reference scattering problem. For example, one could consider the reference with the minimal potential $\bar{V}_l(x) = 0$. Any choice of $\bar{V}_l(x)$ would lead (after UV-regularization) to a finite “renormalized” free energy (2.22). By working out the examples for scalars on static BTZ, Nariai and static patch in de Sitter, [1] observes that choosing \bar{Z}_{bulk} to be that on a Rindler-like wedge at the inverse black hole temperature β_H , with the associated scattering problem

$$\left[-\partial_x^2 + V^{\text{Rin}}(\beta_H, x) \right] \psi(x) = \omega^2 \psi(x), \quad V^{\text{Rin}}(\beta, x) \equiv \left(\frac{4\pi}{\beta} \right)^2 e^{\frac{4\pi}{\beta} x}, \quad (2.24)$$

the renormalized free energy equals the 1-loop Euclidean partition function:

$$\tilde{Z}_{\text{bulk}} = Z_{\text{PI}}, \quad \tilde{Z}_{\text{bulk}} \equiv \frac{Z_{\text{bulk}}}{Z_{\text{bulk}}^{\text{Rin}}(\beta_H)} \quad (\text{scalar}). \quad (2.25)$$

In particular, from all the examples, one finds that the S-matrix for the original problem always takes the form $\mathcal{S}_l(\omega) = \mathcal{S}_l^{\text{QNM}}(\omega) \mathcal{S}^{\text{Rin}}(\beta_H, \omega)$, where $\mathcal{S}_l^{\text{QNM}}(\omega)$ contains QNM frequencies as poles and anti-QNM frequencies as zeros, and

$$\mathcal{S}^{\text{Rin}}(\beta, \omega) = \frac{\Gamma\left(\frac{i\beta\omega}{2\pi}\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi}\right)}. \quad (2.26)$$

is the scattering matrix for the Rindler problem (2.24), which has the Matsubara frequencies as zeros and poles. Therefore, choosing the Rindler problem (2.24) as the reference, the renormalized DOS is the Fourier transform of the QNM character

$$\Delta\rho(\omega) = \frac{1}{2\pi i} \sum_z N_z \left(\frac{1}{\omega + z} - \frac{1}{\omega - z} \right) = \int_0^\infty \frac{dt}{2\pi} \left(e^{i\omega t} + e^{-i\omega t} \right) \chi_{\text{QNM}}(t). \quad (2.27)$$

While in principle there could be a holomorphic part contributing to $\Delta\rho(\omega)$, in all the explicit examples $\Delta\rho(\omega)$ does not receive such a contribution and (2.27) gives the complete answer. Plugging (2.27) and performing the ω -integral gives the DHS formula (2.13).

Generalization to higher spins. To conclude this section, we note that the Lorentzian considerations above readily generalize to spinning fields. In appendix B, we explicitly solve the scattering problem for a massive higher spin (HS) field on the Rindler-like wedge and obtain the S-matrices.

In section 4, we study the example of massive higher spin on static BTZ. For any spin $s \geq 1$, it remains true that for each angular momentum $l \in \mathbb{Z}$, the S-matrix for the associated problem takes the product form $\mathcal{S}_l(\omega) = \mathcal{S}_l^{\text{QNM}}(\omega) \mathcal{S}^{\text{Rin},(s)}(\beta_H, \omega)$, where $\mathcal{S}_l^{\text{QNM}}(\omega)$ contains QNM frequencies as poles and anti-QNM frequencies as zeros, and $\mathcal{S}^{\text{Rin},(s)}(\beta_H, \omega)$ is the Rindler S-matrix (B.60) generalized to a spin- s field. Therefore, choosing the reference to be the Rindler problem, the renormalized free energy is still given by the formula

$$\log \tilde{Z}_{\text{bulk}} = \int_0^\infty dt \frac{1 + e^{-2\pi t/\beta_H}}{2t(1 - e^{-2\pi t/\beta_H})} \chi_{\text{QNM}}(t), \quad (2.28)$$

where the QNM character $\chi_{\text{QNM}}(t)$ is analogously defined as (2.14). However, in contrast to the scalar case, this turns out *not* to be equal to the 1-loop Euclidean path integral. The latter needs to be modified by “edge” corrections. We turn to this next.

3 Edge partition functions for spinning fields

In this section we extend the DHS formula to spinning fields. As we will see, the regularity condition in Euclidean signature is more subtle than its scalar counterpart; in certain sectors some components are required to have enhanced fall-offs near the origin. This eventually leads to a natural bulk-edge split for the Euclidean path integral.

3.1 Spin-1

As an illustration of the idea, we first consider a massive vector A^μ living on the background (2.1), with the 1-loop path integral

$$Z_{\text{PI}}(m^2) = \int \mathcal{D}A e^{-\int \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \right)} = \det \left(-\nabla_{(1)}^2 + m^2 \right)^{-1/2}. \quad (3.1)$$

We denote $-\nabla_{(1)}^2$ as the Laplacian acting on transverse vector fields. On compact spaces such as a sphere, there will be an extra correction due to a normalizable constant scalar mode [8, 11]. The inclusion of this mode is essential for consistency with locality and unitarity [17]. Here we neglect such a contribution to keep the argument as simple as possible. Such a subtlety will matter when we study the example of Nariai spacetime in section 5.

Regularity condition and the analyticity argument. Similar to the scalar case, we demand the vector fields A_μ in the functional integration (3.1) to be smooth at the origin $\rho = 0$ and single-valued in the Euclidean time direction, i.e. they are regular vector fields. Once again, the most convenient way to assess regularity is to work with the complex coordinates $u = \rho e^{-i\varphi}$ and $\bar{u} = \rho e^{i\varphi}$. At the origin these are well-defined, unlike the polar coordinates (ρ, φ) . The components $(A_u, A_{\bar{u}})$ are related to (A_ρ, A_φ) through

$$A_\rho = e^{-i\varphi} A_u + e^{i\varphi} A_{\bar{u}} \quad , \quad A_\varphi = -i\rho \left(e^{-i\varphi} A_u - e^{i\varphi} A_{\bar{u}} \right). \quad (3.2)$$

A mode with U(1) quantum number $k \in \mathbb{Z}$, takes the form

$$(A_\rho, A_\varphi, A_i) \propto e^{-ik\varphi}, \quad (3.3)$$

which from (3.2) implies that A_u , $A_{\bar{u}}$ and A_i contain the factors

$$A_u \propto e^{-i(k-1)\varphi}, \quad A_{\bar{u}} \propto e^{-i(k+1)\varphi}, \quad A_i \propto e^{-ik\varphi}. \quad (3.4)$$

The regularity condition boils down to requiring $(A_u, A_{\bar{u}}, A_i)$ to have a Taylor series expansion in u and \bar{u} near the origin, which means that the leading term of the ρ -expansions of A_u , $A_{\bar{u}}$ and A_i must combine with (3.4) to form non-negative powers of u or \bar{u} . Explicitly, the result is

- for $k \geq 1$,

$$A_u \sim u^{k-1}, \quad A_{\bar{u}} \sim u^{k+1}, \quad A_i \sim u^k; \quad (3.5)$$

- for $k \leq -1$,

$$A_u \sim \bar{u}^{-k+1}, \quad A_{\bar{u}} \sim \bar{u}^{-k-1}, \quad A_i \sim \bar{u}^{-k}; \quad (3.6)$$

- for $k = 0$,

$$A_u \sim \bar{u}, \quad A_{\bar{u}} \sim u, \quad A_i \sim u^0. \quad (3.7)$$

Observe that when $k = 0$, the fall-offs of A_u and $A_{\bar{u}}$ do not follow the same pattern as the generic $|k| \geq 1$ sectors. Indeed, naively putting $k = 0$ in (3.5) or (3.6) would lead to a mode that diverges at the origin. This highlights the qualitative difference between spinning fields and scalars.

Now we repeat the DHS analyticity argument for the path integral (3.1) as a function on the complex m^2 -plane. The functional determinant hits a zero whenever

$$(-\nabla_{(1)}^2 + m^2)A_\mu = 0 \quad (3.8)$$

has a solution on the space of smooth vector fields. Here comes an important difference compared to the scalar case. Recall that in the latter case, *any* QNM would Wick rotate to a regular Euclidean mode with U(1) quantum number $k \in \mathbb{Z}_{\geq 0}$ as we vary m^2 so that

$$\frac{iz}{2\pi T_H} = -k. \quad (3.9)$$

This is not true for the massive vector: because of the enhanced fall-off (3.7), a *subset* of QNMs cannot be Wick-rotated to the $k = 0$ sector. Similar comments apply to anti-QNMs. We will focus on PT-symmetric theories, where \bar{z} can be taken to be $-z$.

Edge partition function. Denoting by z_e the QNMs that *cannot* be Wick-rotated to the $k = 0$ Euclidean modes due to the fall-off condition (3.7), we have a modified DHS formula:

$$Z_{\text{PI}} = \frac{\tilde{Z}_{\text{bulk}}}{Z_{\text{edge}}} \quad (3.10)$$

where

$$\log \tilde{Z}_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-2\pi t/\beta}}{1 - e^{-2\pi t/\beta}} \chi_{\text{QNM}}(t), \quad \log Z_{\text{edge}} = \int_0^\infty \frac{dt}{2t} \sum_{z_e} e^{-iz_e t}. \quad (3.11)$$

Here $\chi_{\text{QNM}}(t)$ is the QNM character defined analogously as (2.14). As discussed at the end of section 2, the bulk part \tilde{Z}_{bulk} has an unambiguous meaning of a Rindler-renormalized thermal canonical partition function. Our analyticity argument reveals that the Euclidean path integral demands a division by the edge partition function Z_{edge} , which accounts for the fact that some QNMs cannot be Wick-rotated to $k = 0$ Euclidean modes due to the fall-off condition (3.7).

Note that the modes with frequency z_e must have non-zero A_u and $A_{\bar{u}}$ components, which must be constructed from $\text{SO}(d)$ scalars; therefore, Z_{edge} can be thought of as a path integral of a scalar on S^{d-1} . Since our argument is based on the behavior of the vector field near the origin, it is natural to identify this S^{d-1} with the bifurcation surface in the Lorentzian signature, thus justifying the terminology ‘‘edge’’.

3.2 Spin-2 and beyond

The argument above readily generalizes to higher spin fields. For instance, for a symmetric spin-2 field $h_{\mu\nu}$, requiring $h_{uu}, h_{u\bar{u}}, h_{\bar{u}\bar{u}}, h_{ui}, h_{\bar{u}i}, h_{ij}$ to have a Taylor expansion in u, \bar{u} leads to the fall-offs near the origin:

- for $k \geq 2$,

$$h_{uu} \propto u^{k-2}, \quad h_{u\bar{u}} \propto u^k, \quad h_{\bar{u}\bar{u}} \propto u^{k+2}, \quad h_{ui} \propto u^{k-1}, \quad h_{\bar{u}i} \propto u^{k+1}, \quad h_{ij} \propto u^k; \quad (3.12)$$

- for $k = 1$,

$$h_{uu} \propto \bar{u}, \quad h_{u\bar{u}} \propto u, \quad h_{\bar{u}\bar{u}} \propto u^3, \quad h_{ui} \propto u^0, \quad h_{\bar{u}i} \propto u^2, \quad h_{ij} \propto u; \quad (3.13)$$

- for $k = 0$,

$$h_{uu} \propto \bar{u}^2, \quad h_{u\bar{u}} \propto u^0, \quad h_{\bar{u}\bar{u}} \propto u^2, \quad h_{ui} \propto \bar{u}, \quad h_{\bar{u}i} \propto u, \quad h_{ij} \propto u^0; \quad (3.14)$$

- for $k = -1$,

$$h_{uu} \propto \bar{u}^3, \quad h_{u\bar{u}} \propto \bar{u}, \quad h_{\bar{u}\bar{u}} \propto u, \quad h_{ui} \propto \bar{u}^2, \quad h_{\bar{u}i} \propto \bar{u}^0, \quad h_{ij} \propto \bar{u}; \quad (3.15)$$

- for $k \leq -2$,

$$h_{uu} \propto \bar{u}^{-k+2}, \quad h_{u\bar{u}} \propto \bar{u}^{-k}, \quad h_{\bar{u}\bar{u}} \propto \bar{u}^{-k-2}, \quad h_{ui} \propto \bar{u}^{-k+1}, \quad h_{\bar{u}i} \propto \bar{u}^{-k-1}, \quad h_{ij} \propto \bar{u}^{-k}. \quad (3.16)$$

Observe the enhanced fall-off of h_{uu} in the $k = 0, 1$ sector, that of $h_{\bar{u}\bar{u}}$ for $k = 0, -1$, and those of $h_{ui}, h_{\bar{u}\bar{i}}$ for $k = 0$. Note that $h_{uu}, h_{\bar{u}\bar{u}}$ can only be constructed from $\text{SO}(d)$ scalars, while $h_{ui}, h_{\bar{u}\bar{i}}$ can be constructed from either $\text{SO}(d)$ scalars or vectors.

The pattern goes on for tensor fields of arbitrary rank $s \geq 1$. The component

$$\phi_{\underbrace{u \cdots u}_a \underbrace{\bar{u} \cdots \bar{u}}_b \underbrace{i_1 \cdots i_c}_c}, \quad a + b + c = s, \quad (3.17)$$

would have enhanced fall-offs for $k = 0, 1, \dots, a - b - 1$ if $a > b$, or $k = 0, -1, \dots, a - b + 1$ if $a < b$. Such a component can be constructed from $\text{SO}(d)$ representations of spin $0, 1, \dots, c$. Repeating the analyticity argument, we then expect that for each fixed $|k| = 0, 1, \dots, s - 1$, there will be a subset of (anti-)QNMs with frequencies $z_{e,k}$ ($\bar{z}_{e,k}$) that cannot be Wick-rotated to regular Euclidean modes of $\text{U}(1)$ quantum number $|k|$ ($-|k|$). For PT-symmetric theories where $\bar{z}_{e,k}$ can be taken to be $-z_{e,k}$, the edge partition function will take the general form

$$\log Z_{\text{edge}} = \int_0^\infty \frac{dt}{2t} \sum_{k=-(s-1)}^{s-1} \sum_{z_{e,k}} e^{-\left(\frac{2\pi}{\beta_H} |k| + iz_{e,k}\right)t}. \quad (3.18)$$

In sections 4 and 5, we will work out the explicit expressions of Z_{edge} for massive HS fields on static BTZ and massive vector on Nariai, and check (3.10) against the full Euclidean path integrals obtained by direct derivations.

4 Example: massive higher spin on static BTZ

As our prime example, we consider massive higher spin (HS) fields living on the static BTZ background (setting $\ell_{\text{AdS}} = 1$):

$$ds^2 = -\left(r^2 - r_H^2\right) dt^2 + \frac{dr^2}{r^2 - r_H^2} + r^2 d\phi^2 = \frac{r_H^2}{\sinh^2(r_H x)} \left(-dt^2 + dx^2 + \cosh^2(r_H x) d\vartheta^2\right). \quad (4.1)$$

In the second equality we have written in terms of the tortoise coordinate $r(x) = -r_H \coth(r_H x)$. We recall that $r_H \equiv M_{\text{BH}} = 2\pi T_H$.

A spin- s ($s \geq 1$) field of mass $m^2 = (\Delta - s)(\Delta + s - 2)$ living on such a background is described by either (\mp) set of first-order equations [18]

$$\epsilon_{\mu_1}^{\alpha\beta} \nabla_\alpha \phi_{\beta\mu_2 \cdots \mu_s} = \mp M \phi_{\mu_1 \mu_2 \cdots \mu_s}, \quad M = \Delta - 1. \quad (4.2)$$

In the current setting we are interested in the parity-invariant theory that includes both \pm solutions.

It turns out to be natural to study components with respect to the coordinates

$$y_\pm = e^{\mp r_H t} \text{sech}(r_H x), \quad (4.3)$$

in terms of which the metric becomes

$$ds^2 = \frac{1}{4(1 - y_+ y_-)^2} \left(y_-^2 dy_+^2 + 2(2 - y_+ y_-) dy_+ dy_- + y_+^2 dy_-^2 \right) + \frac{r_H^2}{1 - y_+ y_-} d\vartheta^2. \quad (4.4)$$

Notice that near horizon $x \rightarrow -\infty$, $y_{\pm} \rightarrow 2 e^{r_H(x \mp t)}$ and

$$ds^2 \approx dy_+ dy_- + r_H^2 d\vartheta^2. \tag{4.5}$$

Comparing this to (2.4), we see that y_+ and y_- are essentially the Lorentzian analogs for the complex coordinates \bar{u} and u in (2.5) respectively. Thus, working with the components with respect to these coordinates make the comparison with section 3 more direct.

Upon Wick-rotating $t \rightarrow -it_E$ and identifying $t_E \sim t_E + \beta$ in (4.1), the resulting Euclidean BTZ (EBTZ) geometry is related to thermal AdS_3 ($TAdS_3$) by a large diffeomorphism. As a result, their path integrals are equal upon the modular transformation

$$\tau \rightarrow -\frac{1}{\tau} \quad \tau = 2\pi iT_H. \tag{4.6}$$

This is expected to be true for any theories. Reproducing the $TAdS_3$ results (reviewed in appendix C) hence serves as a consistency check for our method after we obtain Z_{PI}^{BTZ} in section 4.3.

4.1 Explicit solutions, scattering matrices, and quasinormal modes

While the system (4.2) for massive HS fields on BTZ has been solved in for example [19], we present a simpler version of this computation in appendix D, where we work with components with respect to the coordinates (4.4).

4.1.1 Massive scalar

We start with the simplest case of a massive scalar, whose normal mode functions will serve as the seed solutions for constructing those for general massive HS fields.

For a scalar with mass $m^2 = \Delta(\Delta - 2)$, the normal mode solutions to the Klein-Gordon equation $(-\nabla^2 + m^2)\phi = 0$ are solved with the ansatz

$$\phi(t, x, \vartheta) = e^{-i\omega t + il\vartheta} \sqrt{-\tanh(r_H x)} \psi_{\omega l}^{\text{Scalar}}(x). \tag{4.7}$$

The normalizable solution satisfying the standard boundary condition is

$$\psi_{\omega l}^{\text{Scalar}}(x) = \frac{(\cosh(r_H x))^{\frac{i l}{r_H}} (-\sinh(r_H x))^{\Delta}}{\sqrt{-\tanh(r_H x)}} {}_2F_1(a_{\omega l}, a_{-\omega l}; \Delta; -\sinh^2(r_H x)), \tag{4.8}$$

where

$$a_{\omega l} = \frac{\Delta}{2} + \frac{i(-\omega + l)}{2r_H}. \tag{4.9}$$

This solution has the near-horizon behavior

$$\psi_{\omega l}^{\text{Scalar}}(x \rightarrow -\infty) \propto \frac{\Gamma\left(\frac{i\omega}{r_H}\right)}{\Gamma(a_{-\omega, l}) \Gamma(a_{-\omega, -l})} e^{-i\omega x} + \frac{\Gamma\left(-\frac{i\omega}{r_H}\right)}{\Gamma(a_{\omega l}) \Gamma(a_{\omega, -l})} e^{i\omega x}. \tag{4.10}$$

The ratio of the incoming and outgoing coefficients defines a unitary S-matrix:

$$\begin{aligned} \mathcal{S}_l(\omega) &= \mathcal{S}_l^{\text{BTZ}}(\omega) \mathcal{S}^{\text{Rin}}\left(\frac{2\pi}{r_H}, \omega\right) \\ \mathcal{S}_l^{\text{BTZ}}(\omega) &= \mathcal{S}_l^{\text{BTZ}, L}(\omega) \mathcal{S}_l^{\text{BTZ}, R}(\omega), \quad \mathcal{S}_l^{\text{BTZ}, L}(\omega) = \mathcal{S}_{-l}^{\text{BTZ}, R}(\omega) \equiv \frac{\Gamma(a_{\omega l})}{\Gamma(a_{-\omega, -l})}. \end{aligned} \tag{4.11}$$

Here $\mathcal{S}^{\text{Rin}}(\beta, \omega)$ is the Rindler S-matrix (2.26). The poles of $\mathcal{S}_l^{\text{BTZ}}(\omega)$ are the QNM frequencies

$$z_{nl}^L = l - 2\pi T_H i(\Delta + 2n), \quad z_{nl}^R = -l - 2\pi T_H i(\Delta + 2n), \quad (4.12)$$

while its zeros are the anti-QNM frequencies $-z_{nl}^{L/R}$ or $(z_{nl}^{L/R})^*$.

4.1.2 Massive higher spin

As explained in appendix D, the incoming and outgoing behaviors for a normal mode solution to (4.2) are dominated by the components with all +- and --indices respectively. We will focus on these and use the shorthand notation⁵

$$\phi_{(\pm)} \equiv \underbrace{\phi_{\pm \dots \pm}}_s. \quad (4.13)$$

With our explicit calculations in appendix D, we find the normal mode solutions to be

$$\phi_{(\pm)}^{(\mp)} = C_{\omega l, (\pm)}^{(\mp)} e^{\pm sr_H t - i\omega t + il\vartheta} (-\tanh(r_H x))^{\frac{1}{2} - s} \psi_{\omega \pm isr_H, l}^{\text{Scalar}}(x), \quad (4.14)$$

with $\psi_{\omega l}^{\text{Scalar}}(x)$ defined in (4.8). Here the superscript (\mp) corresponds to the \mp -equations (4.2). In (4.14) we have the relative polarization constants fixed by (4.2) to be

$$C_{\omega l, (+)}^{(\mp)} \frac{\Gamma(a_{-\omega + isr_H, \mp l})}{\Gamma(a_{-\omega - isr_H, \mp l})} = (-)^s C_{\omega l, (-)}^{(\mp)} \frac{\Gamma(a_{\omega + isr_H, \pm l})}{\Gamma(a_{\omega - isr_H, \pm l})}. \quad (4.15)$$

Combining (4.10), (4.14) and (4.15), we can deduce

$$\left(\phi_{(+)}, \phi_{(-)}\right)_{\omega l}^{(\mp)}(x \rightarrow -\infty) \propto B_{\omega l}^{(\mp), \text{out}}(0, (-)^s) e^{(-i\omega - sr_H)(t+x)} + B_{\omega l}^{(\mp), \text{in}}(1, 0) e^{(-i\omega + sr_H)(t-x)} \quad (4.16)$$

where $(0, (-)^s) e^{(-i\omega - sr_H)(t+x)}$ and $(1, 0) e^{(-i\omega + sr_H)(t-x)}$ are the outgoing and incoming waves respectively, with coefficients

$$B_{\omega l}^{(\mp), \text{out}} = \frac{\Gamma\left(\frac{i\omega}{r_H} + s\right)}{\Gamma(a_{-\omega - isr_H, \mp l}) \Gamma(a_{-\omega + isr_H, \pm l})}, \quad B_{\omega l}^{(\mp), \text{in}} = \frac{\Gamma\left(-\frac{i\omega}{r_H} + s\right)}{\Gamma(a_{\omega - isr_H, \pm l}) \Gamma(a_{\omega + isr_H, \mp l})}. \quad (4.17)$$

The ratio between the coefficients is again a pure phase and takes the form

$$\mathcal{S}_l^{(\mp)}(\omega) \equiv \frac{B_{\omega l}^{(\mp), \text{out}}}{B_{\omega l}^{(\mp), \text{in}}} = \mathcal{S}_{s, l}^{\text{BTZ}, (\mp)}(\omega) \mathcal{S}_l^{\text{Rin}, (s)}(\beta_H, \omega) \quad (4.18)$$

with $\mathcal{S}_l^{\text{Rin}, (s)}(\beta, \omega)$ defined in (B.60) and

$$\begin{aligned} \mathcal{S}_{s, l}^{\text{BTZ}, (\mp)}(\omega) &= \mathcal{S}_{s, l}^{\text{BTZ}, (\mp, L)}(\omega) \mathcal{S}_{s, l}^{\text{BTZ}, (\mp, R)}(\omega), \\ \mathcal{S}_{s, l}^{\text{BTZ}, (\mp, L)}(\omega) &\equiv \frac{\Gamma(a_{\omega \mp isr_H, l})}{\Gamma(a_{-\omega \mp isr_H, -l})}, \quad \mathcal{S}_{s, l}^{\text{BTZ}, (\mp, R)}(\omega) \equiv \frac{\Gamma(a_{\omega \pm isr_H, -l})}{\Gamma(a_{-\omega \pm isr_H, l})}, \end{aligned} \quad (4.19)$$

⁵In terms of the notation (B.51) in appendix D, $\phi_{(+)} = \phi_{(s)(0)(0)}$ and $\phi_{(-)} = \phi_{(0)(s)(0)}$.

in accordance with the discussion at the end of section 2.2. QNM frequencies are the poles of the S-matrices $\mathcal{S}_{s,l}^{\text{BTZ},(\mp)}(\omega)$, which can be summarized by

$$z_{nl}^{(\mp),L} = l - 2\pi T_H i(\Delta \mp s + 2n) \quad z_{nl}^{(\mp),R} = -l - 2\pi T_H i(\Delta \pm s + 2n), \quad (4.20)$$

while the anti-QNM frequencies are given by $(z_{nl}^{(\mp),L/R})^*$, the zeros of $\mathcal{S}_{s,l}^{\text{BTZ},(\mp)}(\omega)$. In a parity invariant theory where both (\mp) -QNMs (4.20) are included, the set of anti-QNMs is also spanned by $-z_{nl}^{(\mp),L/R}$, the zeros of $\mathcal{S}_{s,l}^{\text{BTZ},(\pm)}(\omega)$.

4.2 Euclidean continuation of the quasinormal modes

In this part we examine the Euclidean continuation of the QNMs. The analysis for anti-QNMs is analogous.

4.2.1 Massive scalar

As a warm-up illustration, we first look at the case of the massive scalar. At the QNM frequencies (4.12), with (4.7), (4.8) we can write down the full explicit mode functions

$$\begin{aligned} \phi_{nl}^L &\propto e^{-iz_{nl}^L t} (\cosh(r_H x))^{\frac{i}{r_H}} (-\sinh(r_H x))^\Delta {}_2F_1\left(-n, \frac{iz_{nl}^L}{r_H} - n; \Delta; -\sinh^2(r_H x)\right), \\ \phi_{nl}^R &\propto e^{-iz_{nl}^R t} (\cosh(r_H x))^{-\frac{i}{r_H}} (-\sinh(r_H x))^\Delta {}_2F_1\left(-n, \frac{iz_{nl}^R}{r_H} - n; \Delta; -\sinh^2(r_H x)\right). \end{aligned} \quad (4.21)$$

We have suppressed the $e^{i\theta}$ dependence which is unimportant for the following. In deriving $\phi_{nl}^R(x)$ we have used

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \quad (4.22)$$

Following the DHS analyticity argument, we vary m^2 or Δ such that

$$\frac{iz_{nl}^L}{r_H} = -|k| \quad \text{or} \quad \frac{iz_{nl}^R}{r_H} = -|k|, \quad k \in \mathbb{Z}. \quad (4.23)$$

At these (complex) mass values, upon Wick rotating $t \rightarrow -it_E = -i\frac{1}{r_H}\varphi$, the mode functions (4.21) behave near the origin like

$$\begin{aligned} \phi_{nl}^L(-it_E, x \rightarrow -\infty) &\propto e^{-i|k|\varphi} e^{(|k|+2n)r_H x} {}_2F_1\left(-n, -|k| - n; \Delta; -\frac{1}{4}e^{-2r_H x}\right) \\ \phi_{nl}^R(-it_E, x \rightarrow -\infty) &\propto e^{-i|k|\varphi} e^{(|k|+2n)r_H x} {}_2F_1\left(-n, -|k| - n; \Delta; -\frac{1}{4}e^{-2r_H x}\right). \end{aligned} \quad (4.24)$$

To proceed, we will make use of another identity

$$F(-m, b; c; z) = \frac{\Gamma(b+m)\Gamma(c)}{\Gamma(b)\Gamma(c+m)} (-z)^m F\left(-m, 1-c-m; 1-b-m; \frac{1}{z}\right), \quad (4.25)$$

where m is a non-negative integer. It is important to note that when $b = -j$ for j a non-negative integer, (4.25) holds if $m \leq j$. Using this, one can show that for any $n = 0, 1, 2, \dots$ and $k \in \mathbb{Z}$, (4.24) is equivalent to

$$\phi_{nl}^L(-it_E, x \rightarrow -\infty) \propto u^{|k|} \quad \text{or} \quad \phi_{nl}^R(-it_E, x \rightarrow -\infty) \propto e^{|k|(r_H x - i\varphi)} = u^{|k|}. \quad (4.26)$$

We thus conclude that these modes obey the regularity condition (2.7).

4.2.2 Massive higher spin

Now, let us study the Wick-rotation of QNMs for a massive spin- s field, for general $s \geq 1$. From (4.16), we observe that at the QNM frequencies (4.20), the incoming piece vanishes and we are left with the purely outgoing piece determined by $\phi_{(-)}$. Using (4.14), we first write down the explicit normal modes for $\phi_{(-)}$:

$$\phi_{\omega l(-)} \propto e^{-sr_H t - i\omega t} \frac{(\cosh(r_H x))^{\frac{i l}{r_H}} (-\sinh(r_H x))^\Delta}{(-\tanh(r_H x))^s} {}_2F_1(a_{\omega - isr_H, l}, a_{-\omega + isr_H, l}; \Delta; -\sinh^2(r_H x)). \quad (4.27)$$

We have suppressed the $e^{i l \theta}$ dependence which is unimportant for the following. The QNMs at frequencies (4.20) read explicitly

$$\begin{aligned} \phi_{nl,(-)}^{(-),L} &\propto e^{-i(z_{nl}^{(-),L} - isr_H)t} \frac{(\cosh(r_H x))^{\frac{i l}{r_H}} (-\sinh(r_H x))^\Delta}{(-\tanh(r_H x))^s} {}_2F_1\left(-n, \frac{iz_{nl}^{(-),L}}{r_H} - n + s; \Delta; -\sinh^2(r_H x)\right), \\ \phi_{nl,(-)}^{(-),R} &\propto e^{-i(z_{nl}^{(-),R} - isr_H)t} \frac{(\cosh(r_H x))^{-\frac{i l}{r_H}} (-\sinh(r_H x))^\Delta}{(-\tanh(r_H x))^s} {}_2F_1\left(-n - s, \frac{iz_{nl}^{(-),R}}{r_H} - n; \Delta; -\sinh^2(r_H x)\right), \\ \phi_{nl,(-)}^{(+),L} &\propto e^{-i(z_{nl}^{(+),L} - isr_H)t} \frac{(\cosh(r_H x))^{\frac{i l}{r_H}} (-\sinh(r_H x))^\Delta}{(-\tanh(r_H x))^s} {}_2F_1\left(-n - s, \frac{iz_{nl}^{(+),L}}{r_H} - n; \Delta; -\sinh^2(r_H x)\right), \\ \phi_{nl,(-)}^{(+),R} &\propto e^{-i(z_{nl}^{(+),R} - isr_H)t} \frac{(\cosh(r_H x))^{-\frac{i l}{r_H}} (-\sinh(r_H x))^\Delta}{(-\tanh(r_H x))^s} {}_2F_1\left(-n, \frac{iz_{nl}^{(+),R}}{r_H} - n + s; \Delta; -\sinh^2(r_H x)\right), \end{aligned} \quad (4.28)$$

Again, we vary m^2 or Δ such that

$$\frac{iz_{nl}^{(\mp),L}}{r_H} = -|k| \quad \text{or} \quad \frac{iz_{nl}^{(\mp),R}}{r_H} = -|k|, \quad k \in \mathbb{Z}. \quad (4.29)$$

At these (complex) values of masses, upon Wick rotating $t \rightarrow -it_E = -i\frac{1}{r_H}\varphi$, the mode functions (4.28) behave near the origin like

$$\begin{aligned} \phi_{nl,(u)}^{(-),L}(-it_E, x \rightarrow -\infty) &\propto e^{-i(|k|-s)\varphi} e^{(|k|+2n-s)r_H x} {}_2F_1\left(-n, -|k| - n + s; \Delta; -\frac{1}{4}e^{-2r_H x}\right) \\ \phi_{nl,(u)}^{(-),R}(-it_E, x \rightarrow -\infty) &\propto e^{-i(|k|-s)\varphi} e^{(|k|+2n-s)r_H x} {}_2F_1\left(-n - s, -|k| - n; \Delta; -\frac{1}{4}e^{-2r_H x}\right), \\ \phi_{nl,(u)}^{(+),L}(-it_E, x \rightarrow -\infty) &\propto e^{-i(|k|-s)\varphi} e^{(|k|+2n-s)r_H x} {}_2F_1\left(-n - s, -|k| - n; \Delta; -\frac{1}{4}e^{-2r_H x}\right) \\ \phi_{nl,(u)}^{(+),R}(-it_E, x \rightarrow -\infty) &\propto e^{-i(|k|-s)\varphi} e^{(|k|+2n-s)r_H x} {}_2F_1\left(-n, -|k| - n + s; \Delta; -\frac{1}{4}e^{-2r_H x}\right). \end{aligned} \quad (4.30)$$

Here we recall that under the Wick rotation, the $--$ -component becomes the u -component. Using (4.25), one can show that for any $|k| \geq s$, all four mode functions (4.30) are regular near the origin

$$\phi_{nl,(u)}^{(\mp),L}(-it_E, x \rightarrow -\infty) \propto \phi_{nl,(u)}^{(\mp),R}(-it_E, x \rightarrow -\infty) \propto e^{(|k|-s)r_H x} e^{-i(|k|-s)\varphi} \propto u^{|k|-s}. \quad (4.31)$$

The case for $|k| < s$ is more intricate. On the one hand, we always have

$$\phi_{nl,(u)}^{(+),L}(-it_E, x \rightarrow -\infty) \propto \phi_{nl,(u)}^{(-),R}(-it_E, x \rightarrow -\infty) \propto e^{(s-|k|)r_H x} e^{-i(|k|-s)\varphi} \propto \bar{u}^{s-|k|}, \quad (4.32)$$

so that these are regular at the origin. On the other, when $|k| + n \geq s > |k|$,

$$\phi_{nl,(u)}^{(-),L}(-it_E, x \rightarrow -\infty) \propto \phi_{nl,(u)}^{(+),R}(-it_E, x \rightarrow -\infty) \propto e^{(s-|k|)r_H x} e^{-i(|k|-s)\varphi} = \bar{u}^{s-|k|}, \quad (4.33)$$

while for $|k| + n < s$,

$$\phi_{nl,(u)}^{(-),L}(-it_E, x \rightarrow -\infty) \propto \phi_{nl,(u)}^{(+),R}(-it_E, x \rightarrow -\infty) \propto e^{|k|-s|r_H x} e^{-i(|k|-s)\varphi} = u^{-(s-|k|)}. \quad (4.34)$$

We can see that (4.33) is a regular behavior while (4.34) is not.

To summarize, for a fixed $k \in \mathbb{Z}$, any QNM can Wick-rotate to a regular Euclidean mode with U(1) quantum number $|k|$ at complex masses (4.29), except for those with frequencies

$$z_{nl}^{(-),L} = l - 2\pi T_H i(\Delta - s + 2n) \quad z_{nl}^{(+),R} = -l - 2\pi T_H i(\Delta - s + 2n), \quad n < s - |k|. \quad (4.35)$$

The irregularity of such modes agrees with the case of $s = 2$ first pointed out in [15].

4.3 Euclidean path integral

4.3.1 Quasinormal mode character and renormalized bulk partition function

With the QNM spectrum (4.20) we can compute the QNM character

$$\chi_{[\Delta,s]}^{\text{BTZ}}(t) = \sum_{l \in \mathbb{Z}} \sum_{n=0}^{\infty} \sum_{\pm} \left(e^{-iz_{n,l,\pm}^L t} + e^{-iz_{n,l,\pm}^R t} \right) = \frac{4\pi e^{-2\pi T_H \Delta t}}{1 - e^{-4\pi T_H t}} 2 \cosh(2\pi T_H s t) \sum_{k \in \mathbb{Z}} \delta(t - 2\pi k). \quad (4.36)$$

Substituting this into (2.28) yields the renormalized bulk partition function

$$\log \tilde{Z}_{\text{bulk}}^{\text{BTZ}}[\Delta, s] = \sum_{k=1}^{\infty} \frac{q_k^s + q_k^{-s}}{k} \frac{q_k^\Delta}{(1 - q_k)^2}, \quad q_k = e^{-(2\pi)^2 T_H k}. \quad (4.37)$$

Note that this is not related to the $TAdS_3$ expression (C.4) through the modular transformation (4.6) for any $s \geq 1$.

4.3.2 Edge partition function

As we checked explicitly in section 4.2, for a fixed $k \in \mathbb{Z}$, QNMs with frequencies (4.35) cannot Wick-rotate to a regular Euclidean mode with U(1) quantum number $|k|$ at complex masses (4.29). They contribute to the integrand (3.18) as

$$\begin{aligned} & \frac{1}{2t} \sum_{k=-(s-1)}^{s-1} \sum_{l \in \mathbb{Z}} \sum_{n=0}^{s-1-|k|} \left(e^{-\left(2\pi T_H |k| + iz_{nl}^{(-),L}\right)t} + e^{-\left(2\pi T_H |k| + iz_{nl}^{(+),R}\right)t} \right) \\ &= \frac{2\pi e^{-2\pi T_H \Delta t}}{t} \sum_{j \in \mathbb{Z}} \delta(t - 2\pi j) \frac{e^{-2\pi s T_H t} + e^{2\pi s T_H t} - 2}{(1 - e^{-2\pi t T_H})^2} \end{aligned} \quad (4.38)$$

Substituting this into (3.18) yields the BTZ edge partition function

$$\log Z_{\text{edge}} = \sum_j \frac{q_j^s + q_j^{-s} - 2}{j} \frac{q_j^\Delta}{(1 - q_j)^2}. \quad (4.39)$$

4.3.3 The full Euclidean path integral

Taking the difference between the renormalized bulk partition function (4.37) and the edge partition function (4.39), we obtain the full path integral

$$\log Z_{\text{PI}}^{\text{BTZ}} = \log \tilde{Z}_{\text{bulk}}^{\text{BTZ}} - \log Z_{\text{edge}}^{\text{BTZ}} = \sum_{k=1}^{\infty} \frac{2}{k} \frac{q_k^{\Delta}}{(1 - q_k)^2}, \quad (4.40)$$

which precisely equals the $TAdS_3$ result (C.4) after the modular transformation (4.6). In [19] the authors found a prescription to modify the DHS formula so that the result (4.40) was reproduced. Our discussion in section 3 gave a justification for their prescription.

5 Example: massive vector on Nariai

In this section we study a free massive vector A_{μ} on Nariai spacetime ($d \geq 3$):

$$ds^2 = - (1 - y^2) dt^2 + \frac{\ell_N^2}{1 - y^2} dy^2 + r_N^2 d\Omega_{d-1}^2, \quad -1 < y < 1. \quad (5.1)$$

Here ℓ_N and r_N are related to the dS length ℓ_{dS} through

$$\ell_N \equiv \frac{\ell_{\text{dS}}}{\sqrt{d}}, \quad r_N \equiv \sqrt{\frac{d-2}{d}} \ell_{\text{dS}}, \quad \ell_{\text{dS}} \equiv \sqrt{\frac{d(d-1)}{2\Lambda}}. \quad (5.2)$$

This geometry is locally $dS_2 \times S^{d-1}$, with isometry group $\text{SO}(1, 2) \times \text{SO}(d)$. There are two horizons (cosmological and black hole) at $y = \pm 1$ with the same Hawking temperatures $T_N = \frac{1}{2\pi\ell_N}$. Note that this temperature is higher than the temperature $T_{\text{dS}} = \frac{1}{2\pi\ell_{\text{dS}}}$ for pure de Sitter.

Upon Wick-rotating $t \rightarrow -it_E$ and identifying $t_E \sim t_E + 2\pi\ell_N$, the geometry (5.1) becomes $S^2 \times S^{d-1}$. The 1-loop path integral for a massive vector on such a geometry is

$$Z_{\text{PI}} = \int \mathcal{D}A e^{-S[A]}, \quad S[A] = \int_{S^2 \times S^{d-1}} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_{\mu} A^{\mu} \right). \quad (5.3)$$

The derivation in [8] is readily carried over to this case:

$$Z_{\text{PI}} = Z_{\text{PI}}^T Z_{\text{PI}}^L \quad (5.4)$$

with

$$Z_{\text{PI}}^T = \det \left(-\nabla_{(1)}^2 + m^2 + \frac{1}{\ell_N^2} \right)^{-1/2}, \quad Z_{\text{PI}}^L = (m^2)^{1/2}. \quad (5.5)$$

Here $-\nabla_{(1)}^2$ is the Laplacian acting on transverse vector fields on $S^2 \times S^{d-1}$. We have used the fact that the S^2 and S^{d-1} factors in the Euclidean Nariai geometry have respective radii ℓ_N and r_N defined in (5.2). The factor Z_{PI}^L comes from the integration over the off-shell longitudinal modes and corresponds to the normalizable constant scalar mode [8, 11]. Our general discussions in section 3 apply to Z_{PI}^T . We will comment on Z_{PI}^L when we directly compute Z_{PI} in section 5.3.

5.1 Explicit solutions and quasinormal modes

The Proca equation of motion $\nabla^\mu F_{\mu\nu} = m^2 A_\nu$ on (5.1) is equivalent to

$$\left(-\frac{1}{\ell_N^2} \nabla_{dS_2}^2 - \frac{1}{r_N^2} \nabla_{S^{d-1}}^2 + m^2 + \frac{1}{\ell_N^2} \right) A_\mu = 0. \quad (5.6)$$

In deriving this, it is important to use the relation (5.2) between ℓ_N and r_N . In these expressions, $\nabla_{dS_2}^2$ and $\nabla_{S^{d-1}}^2$ are the Laplacians on the *unit* dS_2 and S^{d-1} respectively; the former acts on (A_t, A_y) as a vector and A_i as a scalar, while the latter acts on (A_t, A_y) as scalars and A_i as a vector. All components are related through the transversality condition

$$\nabla^\mu A_\mu = -\frac{1}{1-y^2} \partial_t A_t + \frac{1-y^2}{\ell_N^2} \partial_y A_y + \frac{1}{r_N^2} \nabla_{S^{d-1}}^i A_i = 0. \quad (5.7)$$

There are three types of solutions according to their $SO(d)$ transformation properties. In appendix A, we collect some basic facts about scalar and vector spherical harmonics that are useful for our analysis.

Vector mode. This tower of solutions take the form

$$A_t = A_y = 0, \quad A_i = e^{-i\omega t} R^V(y) Y_{l,i}^{d-1}(\Omega), \quad l \geq 1, \quad (5.8)$$

where $Y_{l,i}^{d-1}(\Omega)$ are vector spherical harmonics on S^{d-1} . The transversality condition as well as the y, t -components of the equation of motion (5.6) are automatically satisfied. The i -component of (5.6) implies that

$$(1-y^2) \partial_y^2 R - 2y \partial_y R + \left(\frac{\omega^2 \ell_N^2}{1-y^2} - \Delta_{V,l} \bar{\Delta}_{V,l} \right) R = 0, \quad (5.9)$$

where

$$\Delta_{V,l} = \frac{1}{2} + i \nu_{V,l}, \quad \nu_{V,l} = \sqrt{\ell_N^2 m_{V,l}^2 + \frac{3}{4}}, \quad m_{V,l}^2 \equiv m^2 + \frac{l(l+d-2)-1}{r_N^2}, \quad (5.10)$$

and $\bar{\Delta}_{V,l} \equiv 1 - \Delta_{V,l}$. There are two linearly independent solutions to (5.9):

$$R_{\omega l}^{V,\text{even}} = (1-y^2)^{-\frac{i\omega \ell_N}{2}} {}_2F_1 \left(\frac{\Delta_{V,l} - i\omega \ell_N}{2}, \frac{\bar{\Delta}_{V,l} - i\omega \ell_N}{2}; \frac{1}{2}, y^2 \right) \quad (5.11)$$

and

$$R_{\omega l}^{V,\text{odd}} = y (1-y^2)^{-\frac{i\omega \ell_N}{2}} {}_2F_1 \left(\frac{1 + \Delta_{V,l} - i\omega \ell_N}{2}, \frac{1 + \bar{\Delta}_{V,l} - i\omega \ell_N}{2}; \frac{3}{2}, y^2 \right). \quad (5.12)$$

Both modes are regular at the location of the observer $y = 0$ and thus should be included as solutions. The label even/odd denotes the parity under $y \rightarrow -y$. Expanding near the horizons in terms of the tortoise coordinate $x = \frac{\ell_N}{2} \log \frac{1+y}{1-y}$ we find respectively

$$R_{\omega l}^{V,\text{even}}(|x| \rightarrow \infty) \propto \frac{\Gamma(i\omega \ell_N)}{\Gamma\left(\frac{\Delta_{V,l} + i\omega \ell_N}{2}\right) \Gamma\left(\frac{\bar{\Delta}_{V,l} + i\omega \ell_N}{2}\right)} e^{i\omega|x|} + (\omega \rightarrow -\omega), \quad (5.13)$$

and

$$R^{V,\text{odd}}(|x| \rightarrow \infty) \propto \frac{\Gamma(i\omega\ell_N)}{\Gamma\left(\frac{1+\Delta_{V,l}+i\omega\ell_N}{2}\right)\Gamma\left(\frac{1+\bar{\Delta}_{V,l}+i\omega\ell_N}{2}\right)} e^{i\omega|x|} + (\omega \rightarrow -\omega). \quad (5.14)$$

Therefore, at QNM frequencies

$$iz_{\Delta_{nl}}^V \ell_N = \Delta_{V,l} + n, \quad iz_{\bar{\Delta}_{nl}}^V \ell_N = \bar{\Delta}_{V,l} + n, \quad n = 0, 1, 2, \dots, \quad (5.15)$$

the even (odd) modes (5.13) are purely outgoing at both horizons when n is even (odd). At these frequencies, one can solve (5.9) to get

$$R_{\Delta_{nl}}^V(y) = P_{-\Delta_{V,l}}^{n+\Delta_{V,l}}(y) \quad \text{and} \quad R_{\bar{\Delta}_{nl}}^V(y) = P_{-\bar{\Delta}_{V,l}}^{n+\bar{\Delta}_{V,l}}(y) \quad (5.16)$$

where $n = 0, 1, 2, \dots$. Alternatively, (5.16) can be obtained (up to an overall normalization) by substituting (5.16) into (5.11) ((5.12)) when n is even (odd) together with the relation between hypergeometric and associated Legendre functions (See for instance [20, §14.3(iii)]). The spectrum of anti-QNM can be solved in a similar way, and the explicit mode functions can be obtained by flipping $t \rightarrow -t$ in the QNM ones.

Scalar mode I. This tower of scalar solutions takes the form

$$A_t = i \frac{1-y^2}{\ell_N} R'(y) e^{-i\omega t} Y_l^{d-1}(\Omega), \quad A_y = \frac{\omega\ell_N}{1-y^2} R(y) e^{-i\omega t} Y_l^{d-1}(\Omega), \quad A_i = 0, \quad l \geq 0, \quad (5.17)$$

where $Y_l^{d-1}(\Omega)$ are scalar spherical harmonics on S^{d-1} . Plugging this ansatz into (5.6) and (5.7), one finds that R satisfies (5.9), with $\Delta_{V,l}$ replaced by $\Delta_{S,l}$ defined as

$$\Delta_{S,l} = \frac{1}{2} + i\nu_{S,l}, \quad \nu_{S,l} = \sqrt{m_{S,l}^2 \ell_N^2 - \frac{1}{4}} = \sqrt{m^2 \ell_N^2 + l(l+d-2) - \frac{1}{4}}, \quad \bar{\Delta}_l \equiv 1 - \Delta_l. \quad (5.18)$$

We essentially get a KK-tower of dS_2 vectors with masses $m_{S,l}^2$. The subsequent analysis is the same as before but with $\Delta_{V,l}$ replaced by $\Delta_{S,l}$. In particular, we can immediately write down the QNM frequencies

$$iz_{\Delta_{nl}}^{(S,1)} \ell_N = \Delta_{S,l} + n, \quad iz_{\bar{\Delta}_{nl}}^{(S,1)} \ell_N = \bar{\Delta}_{S,l} + n, \quad n = 0, 1, 2, \dots, \quad (5.19)$$

and the mode functions

$$R_{\Delta_{nl}}^{(S,1)}(y) = P_{-\Delta_{S,l}}^{n+\Delta_{S,l}}(y) \quad \text{and} \quad R_{\bar{\Delta}_{nl}}^{(S,1)}(y) = P_{-\bar{\Delta}_{S,l}}^{n+\bar{\Delta}_{S,l}}(y). \quad (5.20)$$

The mode functions of the anti-QNMs can be obtained by flipping $t \rightarrow -t$ in the QNM ones.

Scalar mode II. Another tower of scalar solutions are obtained with the ansatz

$$\begin{aligned} A_t &= -i\omega R(y) e^{-i\omega t} Y_l^{d-1}(\Omega), \quad A_y = R'(y) e^{-i\omega t} Y_l^{d-1}(\Omega), \\ A_i &= C_l R(y) e^{-i\omega t} \frac{\partial_i Y_l^{d-1}(\Omega)}{l(l+d-2)}, \quad l \geq 1. \end{aligned} \quad (5.21)$$

The equations of motion again are satisfied when R solves (5.9) with $\Delta_{V,l}$ replaced by $\Delta_{S,l}$ defined in (5.18). For this set of solutions, the transversality constraint (5.7) implies that the respective divergences along the dS_2 and S^{d-1} directions cancel each other, with the polarization constant \mathcal{C}_l fixed to be

$$\Delta_{S,l}\bar{\Delta}_{S,l} = \frac{\mathcal{C}_l}{d-2}. \quad (5.22)$$

Again the QNM frequencies are given by replacing $\Delta_{V,l} \rightarrow \Delta_{S,l}$ in (5.15):

$$iz_{\Delta_{nl}}^{(S,2)}\ell_N = \Delta_{S,l} + n, \quad iz_{\bar{\Delta}_{nl}}^{(S,2)}\ell_N = \bar{\Delta}_{S,l} + n, \quad n = 0, 1, 2, \dots \quad (5.23)$$

These are identical to (5.19) but we stress that here $l \geq 1$. The mode functions are

$$R_{\Delta_{nl}}^{(S,2)}(y) = P_{-\Delta_{S,l}}^{n+\Delta_{S,l}}(y) \quad \text{and} \quad R_{\bar{\Delta}_{nl}}^{(S,2)}(y) = P_{-\bar{\Delta}_{S,l}}^{n+\bar{\Delta}_{S,l}}(y). \quad (5.24)$$

Again, the anti-QNM functions can be obtained by flipping $t \rightarrow -t$ in the QNM ones.

5.1.1 Renormalized bulk partition function

With the QNM spectra (5.15), (5.19) and (5.23), we can immediately write down the QNM character

$$\chi(t) = \sum_{l \geq 1} \left(D_{l,1}^d \frac{q^{\Delta_{V,l}} + q^{\bar{\Delta}_{V,l}}}{1-q} + D_l^d \frac{q^{\Delta_{S,l}} + q^{\bar{\Delta}_{S,l}}}{1-q} \right) + \sum_{l \geq 0} D_l^d \frac{q^{\Delta_{S,l}} + q^{\bar{\Delta}_{S,l}}}{1-q}. \quad (5.25)$$

Here we have defined $q \equiv e^{-t/\ell_N}$. Plugging this into (2.28) then gives

$$\log \tilde{Z}_{\text{bulk}}^T = \int_0^\infty \frac{dt}{2t} \frac{1+q}{1-q} \left[\sum_{l \geq 1} \left(D_{l,1}^d \frac{q^{\Delta_{V,l}} + q^{\bar{\Delta}_{V,l}}}{1-q} + D_l^d \frac{q^{\Delta_{S,l}} + q^{\bar{\Delta}_{S,l}}}{1-q} \right) + \sum_{l \geq 0} D_l^d \frac{q^{\Delta_{S,l}} + q^{\bar{\Delta}_{S,l}}}{1-q} \right]. \quad (5.26)$$

5.2 Quasinormal modes and eigenfunctions on Euclidean Nariai

5.2.1 Spectrum for the vector Laplacian on $S^2 \times S^{d-1}$

The eigenfunctions of the Laplace operator $-\nabla_{(1)}^2$ in (5.4) can be easily obtained by combining together the spherical harmonics on S^2 and S^{d-1} . We summarize this below, where we use i, j, k and I, J, K to denote indices on the S^{d-1} and S^2 factors respectively.

Vector type. These eigenfunctions take the form

$$A_\varphi = A_\theta = 0, \quad A_i = Y_{Lp}^2(\theta, \varphi) Y_{l,i}^{d-1}(\Omega), \quad L \geq 0, \quad -L \leq p \leq L, \quad l \geq 1 \quad (5.27)$$

with eigenvalues

$$\lambda_{L,l}^{(V)} = \frac{L(L+1)}{\ell_N^2} + \frac{l(l+d-2)-1}{r_N^2} \quad (5.28)$$

and degeneracy $D_L^3 D_{l,1}^d$.

Scalar type I. The first type of scalar eigenfunctions take the form

$$(A_\varphi, A_\theta) = Y_{Lp,I}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = 0, \quad L \geq 1, \quad -L \leq p \leq L, \quad l \geq 0, \quad (5.29)$$

with eigenvalues

$$\lambda_{L,l}^{(S,1)} = \frac{L(L+1) - 1}{\ell_N^2} + \frac{l(l+d-2)}{r_N^2} \quad (5.30)$$

and degeneracy $D_L^3 D_l^d$.

Scalar type II. Another type of scalar eigenfunctions take the form

$$(A_\varphi, A_\theta) = \partial_I Y_{Lp}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = C_{Ll} Y_{Lp}^2(\theta, \varphi) \partial_i Y_l^{d-1}(\Omega), \quad (5.31)$$

$$L \geq 1, \quad -L \leq p \leq L, \quad l \geq 1,$$

with eigenvalues

$$\lambda_{L,l}^{(S,2)} = \frac{L(L+1) - 1}{\ell_N^2} + \frac{l(l+d-2)}{r_N^2} = \frac{L(L+1)}{\ell_N^2} + \frac{l(l+d-2) - (d-2)}{r_N^2} \quad (5.32)$$

and degeneracy $D_L^3 D_l^d$. Here C_{Ll} is a relative constant fixed by the Casimir equation.

5.2.2 Euclidean continuation of the quasinormal modes

Since we have the exact expressions for the QNMs discussed in section 5.1 and the Euclidean eigenfunctions summarized above, we can directly compare them after the Wick rotation $t \rightarrow -it_E$. We will ignore overall normalization constants unimportant for this analysis. Also, for a better comparison we change to the variables $y = \cos \theta$ and $\varphi = \frac{1}{\ell_N} t_E$.

Vector type. For every $l \geq 1$, whenever the mass m^2 is varied such that

$$iz_{\Delta nl}^V \ell_N = n + \Delta_{V,l} = -|k| \quad \text{or} \quad iz_{\bar{\Delta} nl}^V \ell_N = n + \bar{\Delta}_{V,l} = -|k|, \quad k \in \mathbb{Z}, \quad (5.33)$$

upon Wick rotation $t \rightarrow -it_E$, the QNM with frequency (5.15) becomes

$$A_\varphi = A_\theta = 0, \quad A_i = Y_{n+|k|, -|k|}^2(\theta, \varphi) Y_{l,i}^{d-1}(\Omega) \quad (5.34)$$

while the anti-QNM given by flipping $t \rightarrow -t$ becomes

$$A_\varphi = A_\theta = 0, \quad A_i = Y_{n+|k|, |k|}^2(\theta, \varphi) Y_{l,i}^{d-1}(\Omega). \quad (5.35)$$

It is clear that running over $n = 0, 1, 2, \dots$ and $k \in \mathbb{Z}$, (5.34) and (5.35) span the set of eigenfunctions of the vector type (5.27).

Scalar type I. For every $l \geq 0$, whenever the mass m^2 is varied such that

$$iz_{\Delta nl}^{(S,1)} \ell_N = n + \Delta_{S,l} = -|k| \quad \text{or} \quad iz_{\bar{\Delta} nl}^{(S,1)} \ell_N = n + \bar{\Delta}_{S,l} = -|k|, \quad k \in \mathbb{Z}, \quad (5.36)$$

the QNM with frequency (5.19) upon Wick rotation $t \rightarrow -it_E$ becomes

$$(A_\varphi, A_\theta) = Y_{n+|k|, -|k|, I}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = 0, \quad (5.37)$$

while the anti-QNM given by flipping $t \rightarrow -t$ becomes

$$(A_\varphi, A_\theta) = Y_{n+|k|,|k|,I}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = 0. \quad (5.38)$$

Running over $n = 1, 2, \dots$ and $k \in \mathbb{Z}$, (5.34) and (5.35) span the set of eigenfunctions of the scalar type I (5.29). Notice that when $n = k = 0$, the Wick-rotated mode does not belong to this set.

Scalar type II. For every $l \geq 1$, whenever the mass m^2 is varied such that

$$iz_{\Delta_{nl}}^{(S,2)} \ell_N = n + \Delta_{S,l} = -|k| \quad \text{or} \quad iz_{\bar{\Delta}_{nl}}^{(S,2)} \ell_N = n + \bar{\Delta}_{S,l} = -|k|, \quad k \in \mathbb{Z}, \quad (5.39)$$

the QNM with frequency (5.23) upon Wick rotation $t \rightarrow -it_E$ becomes

$$(A_\varphi, A_\theta) = \partial_I Y_{n+|k|,-|k|}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = C_{n+|k|,l} Y_{n+|k|,-|k|}^2(\theta, \varphi) \partial_i Y_l^{d-1}(\Omega), \quad (5.40)$$

while the anti-QNM given by flipping $t \rightarrow -t$ becomes

$$(A_\varphi, A_\theta) = \partial_I Y_{n+|k|,|k|}^2(\theta, \varphi) Y_l^{d-1}(\Omega), \quad A_i = C_{n+|k|,l} Y_{n+|k|,|k|}^2(\theta, \varphi) \partial_i Y_l^{d-1}(\Omega), \quad (5.41)$$

In these expressions, C_{Li} is as defined in (5.31). Running over $n = 1, 2, \dots$ and $k \in \mathbb{Z}$, (5.40) and (5.41) span the set of eigenfunctions of the scalar type II (5.31). Notice that when $n = k = 0$, the Wick-rotated mode does not belong to this set.

5.2.3 Edge partition function

From the last section, we see that all QNMs Wick-rotate to Euclidean modes for the correct value of masses, except the $n = 0$ modes of both scalar type I and II with frequencies

$$iz_{\Delta_{nl}}^{(S,1)} \ell_N = \Delta_{S,l}, \quad iz_{\bar{\Delta}_{nl}}^{(S,1)} \ell_N = \bar{\Delta}_{S,l}, \quad l \geq 0, \quad (5.42)$$

and

$$iz_{\Delta_{\Delta,n=0,l}}^{(S,2)} \ell_N = \Delta_{S,l}, \quad iz_{\bar{\Delta}_{\Delta,n=0,l}}^{(S,2)} \ell_N = \bar{\Delta}_{S,l}, \quad l \geq 1 \quad (5.43)$$

respectively. As we saw these modes do not Wick-rotate to the $k = 0$ Euclidean modes for any value of masses, and contribute to the edge partition function as

$$\log Z_{\text{edge}}^T = \int_0^\infty \frac{dt}{2t} \left(\sum_{l \geq 1} + \sum_{l \geq 0} \right) D_l^d \left(q^{\Delta_{S,l}} + q^{\bar{\Delta}_{S,l}} \right). \quad (5.44)$$

5.3 Euclidean path integral

With the eigenvalues and degeneracies of the spin-1 Laplacian on $S^2 \times S^{d-1}$, we write down the 1-loop path integral

$$\begin{aligned} \log Z_{\text{PI}} = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left[\sum_{l \geq 1} \sum_{L \geq 0} D_{l,1}^d D_L^3 e^{-\left(\lambda_{L,l}^{(V)} + m^2 + \frac{1}{\ell_N^2} \right) \tau} \right. \\ \left. + \left(\sum_{l \geq 1} + \sum_{l \geq 0} \right) \sum_{L \geq 1} D_l^d D_L^3 e^{-\left(\lambda_{L,l}^{(S)} + m^2 + \frac{1}{\ell_N^2} \right) \tau} - e^{-m^2 \tau} \right], \quad (5.45) \end{aligned}$$

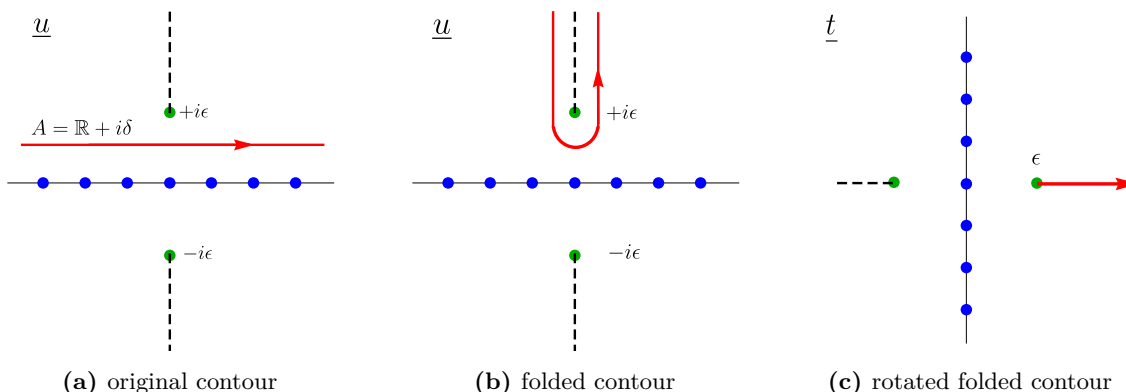


Figure 1. We fold the contour A (red) along the branch cut around the branch point $+i\epsilon$ (green dot), and then rotate $u = it$. The blue dots represent the poles of $f(u)$.

where we have abbreviated $\lambda_{L,l}^{(S,1)} = \lambda_{L,l}^{(S,2)} = \lambda_{L,l}^{(S)}$. The last term in the bracket comes from the factor $Z_{\text{PI}}^L = (m^2)^{1/2}$ in (5.4). Here we have inserted the UV regulator $e^{-\frac{\epsilon^2}{4\tau}}$ so that this integral is convergent for $\epsilon > 0$.

To proceed, we substitute (5.28), (5.30), (5.32) and use the Hubbard-Stratonovich trick, following the approach in [1, 11]. For the sum over L in the first line, we can write

$$\sum_{L \geq 0} D_L^3 e^{-\left(\lambda_{L,l}^{(V)} + m^2 + \frac{1}{\ell_N^2}\right)\tau} = e^{-\frac{\nu_{V,l}^2}{\ell_N^2}\tau} \sum_{L=0}^{\infty} D_L^3 e^{-\tau(L+\frac{1}{2})^2/\ell_N^2} = e^{-\frac{\nu_{V,l}^2}{\ell_N^2}\tau} \int_A du \frac{e^{-u^2/4\tau}}{\sqrt{4\pi\tau}} f(u), \quad (5.46)$$

with the integration contour $A = \mathbb{R} + i\delta$, $\delta > 0$ (see figure 1). Here we have defined

$$f(u) \equiv \sum_{L=0}^{\infty} D_L^3 e^{iu(L+\frac{1}{2})/\ell_N} = \left(\frac{1 + e^{iu/\ell_N}}{1 - e^{iu/\ell_N}} \right) \frac{e^{i\frac{u}{2}/\ell_N}}{1 - e^{iu/\ell_N}}. \quad (5.47)$$

For the sum over L and the last term in the second line of (5.45), we can write similarly

$$\sum_{L \geq 1} D_L^3 e^{-\left(\lambda_{L,l}^{(S)} + m^2 + \frac{1}{\ell_N^2}\right)\tau} = e^{-\frac{\nu_{S,l}^2}{\ell_N^2}\tau} \int_A du \frac{e^{-u^2/4\tau}}{\sqrt{4\pi\tau}} \left(f(u) - e^{i\frac{u}{2\ell_N}} \right) \quad (5.48)$$

and

$$-e^{-m^2\tau} = -e^{-\tau\nu_{S,0}^2/\ell_N^2} e^{-\tau\left(\frac{1}{2\ell_N}\right)^2} = -e^{-\tau\nu_{S,0}^2/\ell_N^2} \int_A du \frac{e^{-u^2/4\tau}}{\sqrt{4\pi\tau}} e^{i\frac{u}{2\ell_N}}, \quad (5.49)$$

with the same contour A . We can then perform the τ -integral in (5.45) (keeping $\text{Im } u = \delta < \epsilon$). Finally, after deforming the contour A as in figure 1 and changing variables to $u = it$, we arrive at the regularized formula

$$\log Z_{\text{PI}} = \log \tilde{Z}_{\text{bulk}} - \log Z_{\text{edge}} \quad (5.50)$$

where

$$\log \tilde{Z}_{\text{bulk}} = \int_{\epsilon}^{\infty} \frac{dt}{2\sqrt{t^2 - \epsilon^2}} \frac{1+q}{1-q} \left[\sum_{l \geq 1} D_l^d \frac{e^{-\frac{t}{2\ell_N} - i\nu_{V,l}\sqrt{t^2 - \epsilon^2}} + e^{-\frac{t}{2\ell_N} + i\nu_{V,l}\sqrt{t^2 - \epsilon^2}}}{1-q} + \left(\sum_{l \geq 1} + \sum_{l \geq 0} \right) D_l^d \frac{e^{-\frac{t}{2\ell_N} - i\nu_{S,l}\sqrt{t^2 - \epsilon^2}} + e^{-\frac{t}{2\ell_N} + i\nu_{S,l}\sqrt{t^2 - \epsilon^2}}}{1-q} \right] \quad (5.51)$$

and

$$\log Z_{\text{edge}} = 2 \int_{\epsilon}^{\infty} \frac{dt}{2\sqrt{t^2 - \epsilon^2}} \sum_{l \geq 0} D_l^d \left(e^{-\frac{t}{2\ell_N} - i\nu_{S,l}\sqrt{t^2 - \epsilon^2}} + e^{-\frac{t}{2\ell_N} + i\nu_{S,l}\sqrt{t^2 - \epsilon^2}} \right). \quad (5.52)$$

Putting $\epsilon = 0$, we see that (5.51) recovers (5.26), while (5.52) recovers (5.44) plus the term $\log Z_{\text{PI}}^L$. Notice the curious overall factor of 2 in (5.52), which might be associated with the fact that we have two horizons in the Lorentzian geometry.

6 Discussion and outlook

We have revealed a natural structure for 1-loop Euclidean path integrals of general spinning fields through the relation

$$Z_{\text{PI}} = \frac{\tilde{Z}_{\text{bulk}}}{Z_{\text{edge}}}. \quad (6.1)$$

While we have explained in [1] and generalized in this work the unambiguous canonical meaning of $\tilde{Z}_{\text{bulk}} \equiv Z_{\text{bulk}}/Z_{\text{bulk}}^{\text{Rin}}$ as a ratio of thermal canonical partition functions, we have not given the Hilbert space interpretation of Z_{edge} .

The presence of the “edge” contributions is not surprising in view of past studies of entanglement entropy in gauge theories and gravity. In the early work [21], a “contact term” was found in the entanglement entropy for Maxwell theory on black holes computed as a conical entropy. A considerable number of works have been devoted to the proper interpretation for such a contact term as “edge” degrees of freedom living on the entanglement surface (the bifurcation surface S^{d-1} in the case of black holes). See [22–40] for a partial list. While the vast majority of these studies focus on gauge theories and gravity, these edge degrees of freedom are expected to be present for massive spinning fields as well. For one thing, the origin for the contact term in [21] is the linear curvature coupling R present in the kinetic term $S \sim \int A(-\nabla^2 + R)A + \dots$, which is also present for massive fields. For another, from the Lorentzian two-sided geometry point of view, the object $Z_{\text{bulk}} \equiv \text{Tr} e^{-\beta_H \hat{H}}$ can be thought of as computing the normalization of the reduced density matrix after tracing out one side. This assumes the global Hilbert space factorizes. For all spinning fields, there are obstructions to this factorization of Hilbert space due to the presence of constraints. In gauge theories and gravity, we have gauge constraints such as the Gauss law constraint $\nabla \cdot \mathbf{E} = 0$ for Maxwell theory; for massive spinning fields, we have for instance the transversality condition $\nabla^\lambda A_\lambda = 0$ for a Proca field. The edge degrees of freedom account for the non-factorization of the global Hilbert space due to such constraints. This point has been discussed in [37] for the Proca field on Rindler space. It is

possible that by employing the Stückelberg trick, one could understand the massive theory as a gauge theory, so that their edge modes can be understood in the same formalism as in gauge theories and gravity.

In any case, with our purely Euclidean characterization described in section 3, it would be very interesting to connect our work to existing approaches to edge modes and understand their canonical pictures. To that end, we note the crucial role played by the regularity condition imposed on the Euclidean eigenfunctions in the path integral. It is very plausible that the regularity condition is closely related to “shrinkable boundary condition” recently discussed by several authors [41, 42].

Finally, in the context of holography, our results capture $O(G_N^0)$ -effects in the bulk low-energy effective field theory. It would be extremely interesting to investigate the boundary interpretations of the results in [1] and the current paper. In particular, the boundary counterpart of the edge modes uncovered in this paper might serve as a boundary signature of the bulk black hole horizon. Our results for massive HS on static BTZ could provide a set of concrete data for exploring this direction.

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A Scalar and vector spherical harmonics on S^{d-1}

Throughout this paper we use Latin letters such as i, j, k to denote components on S^{d-1} . We also use a tilde to denote quantities living intrinsically on S^{d-1} ; for example, $\tilde{\nabla}_i$ acts as a covariant derivative with the standard round metric and Levi-Civita connection on S^{d-1} .

When $d \geq 4$, we denote the $(d - 1)$ -dimensional spherical harmonics by $Y_l(\Omega)$, which satisfies

$$-\tilde{\nabla}^2 Y_l = l(l + d - 2)Y_l, \quad l \geq 0, \tag{A.1}$$

with degeneracy

$$D_l^d = \frac{2l + d - 2}{d - 2} \binom{l + d - 3}{d - 3}. \tag{A.2}$$

Vector spherical harmonics are denoted by $Y_{l,i}(\Omega)$, satisfying

$$-\tilde{\nabla}^2 Y_{l,i} = (l(l + d - 2) - 1) Y_{l,i}, \quad l \geq 1, \tag{A.3}$$

and the transversality condition

$$\tilde{\nabla}^i Y_{l,i} = 0, \tag{A.4}$$

with degeneracy

$$D_{l,1}^d = \frac{l(d+l-2)(d+2l-2)}{(d-3)(l+1)} \binom{d+l-4}{d-4}. \quad (\text{A.5})$$

When $d = 3$, i.e. on S^2 with standard metric $ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, the scalar spherical harmonics are the familiar ones (restoring the magnetic quantum number m)

$$Y_{lm}(\theta, \varphi) \propto e^{im\varphi} P_l^m(\cos \theta), \quad -l \leq m \leq l, \quad (\text{A.6})$$

where $P_l^m(x)$ is the associated Legendre polynomial. The proportionality constant is not important to us. Vector spherical harmonics are related to these by

$$Y_{lm,i}(\theta, \varphi) = \frac{1}{\sqrt{l(l+1)}} \epsilon_{ij} \partial^j Y_{lm}(\theta, \varphi) \quad (l \geq 1) \quad (\text{A.7})$$

where $\epsilon_{\theta\phi} = \sin \theta$. Explicitly,

$$\begin{aligned} Y_{lm,\theta}(\theta, \varphi) &\propto \sin \theta \partial^\varphi Y_{lm}(\theta, \varphi) = \frac{im}{\sin \theta} Y_{lm}(\theta, \varphi) \\ Y_{lm,\varphi}(\theta, \varphi) &\propto \sin \theta \partial^\theta Y_{lm}(\theta, \varphi) \propto \sin^2 \theta e^{im\varphi} \partial_x P_l^m(x) \Big|_{x=\cos \theta}. \end{aligned} \quad (\text{A.8})$$

B Scattering in the Rindler-like region

In this appendix we study massive spinning fields on the Rindler-like wedge:

$$ds^2 = e^{\frac{4\pi}{\beta}x} \left(-dt^2 + dx^2 \right) + r_H^2 d\Omega_{d-1}^2, \quad -\infty < x < \infty. \quad (\text{B.1})$$

This is nothing but a product of a 2D Rindler space and a transverse sphere with constant radius r_H . Another coordinate system that turns out to be useful is

$$y_{\pm} = \frac{\beta}{2\pi} e^{\mp \frac{2\pi}{\beta}t + \frac{2\pi}{\beta}x}, \quad (\text{B.2})$$

in terms of which the metric becomes

$$ds^2 = dy_+ dy_- + r_H^2 d\Omega_{d-1}^2. \quad (\text{B.3})$$

B.1 Massive scalar

This case has been studied in the appendix A of [1]. The Klein-Gordon equation $(-\nabla^2 + m^2)\phi = 0$ is equivalent to

$$\left[-\partial_t^2 + \partial_x^2 + e^{\frac{4\pi}{\beta}x} \left(\frac{\tilde{\nabla}^2}{r_H^2} - m^2 \right) \right] \phi = 0. \quad (\text{B.4})$$

Solving with the ansatz

$$\phi(t, x, \Omega) = e^{-i\omega t} \psi(x) Y_l(\Omega), \quad (\text{B.5})$$

the normalizable solution is

$$\psi_{\omega l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi}} \left(2e^{\frac{2\pi x'}{\beta}} \right), \quad (\text{B.6})$$

where

$$x' = x + \frac{\beta}{2\pi} \log \frac{\beta M_l}{4\pi}, \quad M_l \equiv \sqrt{\frac{l(l+d-2)}{r_H^2} + m^2}. \quad (\text{B.7})$$

Near the horizon (B.6) takes the asymptotic form

$$\psi_{\omega l}^{\text{Scalar}}(x' \rightarrow -\infty) \propto \Gamma\left(\frac{i\beta\omega}{2\pi}\right) e^{-i\omega x'} + \Gamma\left(-\frac{i\beta\omega}{2\pi}\right) e^{i\omega x'}. \quad (\text{B.8})$$

The ratio between the coefficients of the outgoing and incoming waves

$$\mathcal{S}^{\text{Rin}}(\beta, \omega) = \frac{\Gamma\left(\frac{i\beta\omega}{2\pi}\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi}\right)} \quad (\text{B.9})$$

is a pure phase, or rank-1 unitary S-matrix.

B.2 Massive vector in any $d \geq 3$

In this section we consider a vector field of mass m^2 on the wedge (B.1). In the coordinates (B.3), all Christoffel symbols except Γ_{jk}^i are trivial, and one immediately concludes that the Proca equation of motion $\nabla^\mu F_{\mu\nu} = m^2 A_\nu$ is equivalent to

$$\left[-\partial_t^2 + \partial_x^2 + e^{\frac{4\pi}{\beta}x} \left(\frac{\tilde{\nabla}^2}{r_H^2} - m^2 \right) \right] A_\pm = 0 = \left[-\partial_t^2 + \partial_x^2 + e^{\frac{4\pi}{\beta}x} \left(\frac{\tilde{\nabla}^2 - (d-2)}{r_H^2} - m^2 \right) \right] A_i, \quad (\text{B.10})$$

together with the transversality condition

$$2(\partial_- A_+ + \partial_+ A_-) + \frac{1}{r_H^2} \tilde{\nabla}^i A_i = 0, \quad (\text{B.11})$$

Here we have abbreviated $\partial_\pm \equiv \partial_{y_\pm}$. Note A_\pm are related to A_t, A_x through

$$A_\pm = e^{\pm \frac{2\pi}{\beta}t - \frac{2\pi}{\beta}x} (\mp A_t + A_x). \quad (\text{B.12})$$

Since all equations in (B.10) take the same form as the scalar case (B.4), we can immediately employ the results from the last section, except that there can be different types of solutions according to $\text{SO}(d)$ irreducible representations.

Vector type. First we have the vector-type solutions of the form

$$A_\pm = 0, \quad A_i = e^{-i\omega t} \psi_l^{(V)}(x) Y_{l,i}(\Omega), \quad (\text{B.13})$$

where the angular dependence of A_i are taken to be vector spherical harmonics $Y_{l,i}(\Omega)$. For this type of solutions, the transversality condition (B.11) is trivially satisfied. The only non-trivial equation in (B.10) with ansatz (B.13) is

$$\left[-\partial_{x'}^2 + \left(\frac{4\pi}{\beta} \right)^2 e^{\frac{4\pi}{\beta}x'} \right] \psi_l^{(V)}(x') = \omega^2 \psi_l^{(V)}(x'), \quad (\text{B.14})$$

Here we have defined

$$x' = x + \frac{\beta}{2\pi} \log \frac{\beta M_{1,l}}{4\pi}, \quad M_{1,l} \equiv \sqrt{\frac{(l+1)(l+d-3)}{r_H^2} + m^2}. \quad (\text{B.15})$$

The normalizable solution to (B.14) is

$$\psi_{\omega l}^{(V)}(x') = \psi_{\omega l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi}} \left(2 e^{\frac{2\pi x'}{\beta}} \right), \quad (\text{B.16})$$

and the S-matrix is simply

$$\mathcal{S}^{(\text{Rin},V)}(\beta, \omega) = \mathcal{S}^{\text{Rin}}(\beta, \omega) = \frac{\Gamma\left(\frac{i\beta\omega}{2\pi}\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi}\right)}. \quad (\text{B.17})$$

Scalar type. Next we have the scalar-type solutions, with ansatz

$$A_{\pm} = e^{\pm \frac{2\pi}{\beta} t - i\omega t} \psi_{\pm, l}^S(x) Y_l(\Omega), \quad A_i = e^{-i\omega t} \psi_{i, l}^S(x) \frac{\tilde{\nabla}_i Y_l(\Omega)}{l(l+d-2)}. \quad (\text{B.18})$$

Here we have inserted a factor $e^{\pm \frac{2\pi}{\beta} t}$ to compensate for the corresponding factors in (B.12) to get a normal mode with time dependence $e^{-i\omega t}$. We will focus on solving for $\psi_{\pm, l}^S$, with which the angular solution $\psi_{i, l}^S$ is completely determined through (B.11). For each $l \geq 1$, there are two linearly independent solutions, which can be taken to be

$$\text{Scalar } \pm\text{-type:} \quad A_{\pm} \neq 0, \quad A_{\mp} = 0, \quad A_i \neq 0. \quad (\text{B.19})$$

For the \pm -solution with $A_{\mp} = 0$, plugging the ansatz (B.18) into (B.10) leads to

$$\begin{aligned} \left[-\partial_{x'}^2 + \left(\frac{4\pi}{\beta}\right)^2 e^{\frac{4\pi}{\beta} x'} \right] \psi_{\pm, l}^{S, \pm}(x') &= \left(\omega \pm i \frac{2\pi}{\beta}\right)^2 \psi_{\pm, l}^{S, \pm}(x') \\ \left[-\partial_{x'}^2 + \left(\frac{4\pi}{\beta}\right)^2 e^{\frac{4\pi}{\beta} x'} \right] \psi_{i, l}^{S, \pm}(x') &= \omega^2 \psi_{i, l}^{S, \pm}(x'). \end{aligned} \quad (\text{B.20})$$

Here we have defined

$$x' = x + \frac{\beta}{2\pi} \log \frac{\beta M_{0,l}}{4\pi}, \quad M_{0,l} \equiv \sqrt{\frac{l(l+d-2)}{r_H^2} + m^2}. \quad (\text{B.21})$$

The normalizable solutions are

$$\psi_{\pm, l}^{S, \pm}(x') = \psi_{\omega \pm i \frac{2\pi}{\beta}, l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi} \mp 1} \left(2 e^{\frac{2\pi x'}{\beta}} \right), \quad (l \geq 1). \quad (\text{B.22})$$

Using

$$\frac{d}{dz} K_{\alpha}(z) = \pm \frac{\alpha}{z} K_{\alpha}(z) - K_{\alpha \pm 1}(z), \quad (\text{B.23})$$

one can check that the angular solutions $\psi_{i, l}^S$ obtained through (B.11) automatically solve the angular equation (B.20). Since the \pm -modes (B.22) are same as (B.6) but with $\omega \rightarrow \omega \pm i \frac{2\pi}{\beta}$, we can immediately write down the S-matrices

$$\mathcal{S}_{l \geq 1}^{\text{Rin}, \pm}(\beta, \omega) = \mathcal{S}^{\text{Rin}}\left(\beta, \omega \pm i \frac{2\pi}{\beta}\right), \quad (\text{B.24})$$

with $\mathcal{S}^{\text{Rin}}(\beta, \omega)$ defined in (B.9). Since the equations (B.20) for $\psi_{\pm, l}^{S, \pm}$ are related through $\omega \rightarrow -\omega$, they can be thought of as a time-reversal dual pair of scattering problems, and their S-matrices (B.24) satisfy the unitary condition

$$\mathcal{S}^{\text{Rin}, +}(\beta, -\omega) \mathcal{S}^{\text{Rin}, -}(\beta, \omega) = 1. \quad (\text{B.25})$$

When $l = 0$, the ansatz for A_i in (B.18) breaks down, and we have instead

$$A_{\pm} = e^{\pm \frac{2\pi}{\beta} t - i\omega t} \psi_{\pm, l=0}^S, \quad A_i = 0, \quad (\text{B.26})$$

where both A_{\pm} must be non-zero. We still have the first line of (B.20), with normalizable solutions

$$\psi_{\pm, l=0}^S(x') = \psi_{\omega \pm i \frac{2\pi}{\beta}, l=0}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi} \mp 1} \left(2 e^{\frac{2\pi x'}{\beta}} \right), \quad (l = 0). \quad (\text{B.27})$$

The transversality condition (B.12) fixes the relative coefficients of these two solutions, so that the full solution is

$$(A_+, A_-)_{l=0} = e^{-i\omega t} \left(e^{\frac{2\pi}{\beta} t} \psi_{+, l=0}^S, -e^{-\frac{2\pi}{\beta} t} \psi_{-, l=0}^S \right). \quad (\text{B.28})$$

Near horizon, this behaves as

$$\begin{aligned} (A_+, A_-)_{l=0}(x' \rightarrow -\infty) &\propto \Gamma\left(\frac{i\beta\omega}{2\pi} + 1\right) (0, 1) e^{(-i\omega - \frac{2\pi}{\beta})(t+x')} \\ &+ \Gamma\left(-\frac{i\beta\omega}{2\pi} + 1\right) (-1, 0) e^{(-i\omega + \frac{2\pi}{\beta})(t-x')}. \end{aligned} \quad (\text{B.29})$$

Here the second and first terms correspond to waves incoming from and outgoing to the horizon respectively, the ratio of their coefficients again defines a unitary S-matrix:

$$\mathcal{S}_{l=0}^{\text{Rin}}(\beta, \omega) = \frac{\Gamma\left(\frac{i\beta\omega}{2\pi} + 1\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi} + 1\right)}. \quad (\text{B.30})$$

B.3 Massive higher spin in $d = 2$

When $d = 2$, the transverse sphere becomes a circle S^1 with radius r_H , and the metric is simply

$$ds^2 = e^{\frac{4\pi}{\beta} x} \left(-dt^2 + dx^2 \right) + r_H^2 d\vartheta^2, \quad -\infty < x < \infty, \quad \vartheta \simeq \vartheta + 2\pi. \quad (\text{B.31})$$

B.3.1 Massive vector

Now let us study a massive vector on (B.31). A special feature for the case of $d = 2$ is that (B.10) and (B.11) can be equivalently described by either (\mp) set of first-order equations

$$\epsilon_{\alpha\mu\nu} \nabla^\mu A^\nu = \mp m A_\alpha. \quad (\text{B.32})$$

Here we take the following convention for the Levi-Civita symbol

$$\epsilon^{\mu\nu\lambda} \equiv \frac{\tilde{\epsilon}^{\mu\nu\lambda}}{\sqrt{-g}}, \quad \tilde{\epsilon}^{-+\vartheta} = -\tilde{\epsilon}_{-\vartheta} = 1. \quad (\text{B.33})$$

It is straightforward to check that solutions to (B.32) satisfy (B.10) and (B.11) at $d = 2$. In a parity-invariant theory, both sets of solutions should be included. To obtain the explicit solutions, note that equations (B.10) and (B.11) remain valid. The only difference is that the isometry group for S^1 is $U(1)$, and we do not have to separate the solutions into vector or scalar type. The full set of normal mode solutions is obtained using the ansatz

$$A_{\pm} = C_{\pm} e^{\pm \frac{2\pi}{\beta} t - i\omega t} \psi_{\pm,l}(x) e^{il\vartheta}, \quad A_{\vartheta} = C_{\vartheta} e^{-i\omega t} \psi_{\vartheta,l}(x) e^{il\vartheta}, \quad l = 0, \pm 1, \pm 2, \dots \quad (\text{B.34})$$

Here we have inserted overall constants C_{\pm} and C_{ϑ} to be determined below. Notice that the $U(1)$ angular momentum l takes values over all integers. For each $l \in \mathbb{Z}$, we obtain

$$\psi_{\pm,l}(x') = \psi_{\omega \pm i \frac{2\pi}{\beta}, l}^{\text{Scalar}}(x'), \quad \psi_{\vartheta,l}(x') = \psi_{\omega, l}^{\text{Scalar}}(x') \quad l \in \mathbb{Z}, \quad (\text{B.35})$$

where

$$x' = x + \frac{\beta}{2\pi} \log \frac{\beta M_{0,l}}{4\pi}, \quad M_{0,l} \equiv \sqrt{\frac{l^2}{r_H^2} + m^2}. \quad (\text{B.36})$$

Polarization vectors. In the first-order formulation (B.32), all components (B.35) are coupled. To find their relative coefficients, we note that the ϑ -component of (B.32) reads explicitly

$$2r_H (\partial_+ A_- - \partial_- A_+) = \pm m A_{\vartheta}. \quad (\text{B.37})$$

On the other hand, the transversality condition (B.11) at $d = 2$ implies for the ansatz (B.34)

$$il A_{\vartheta} = -2r_H^2 (\partial_+ A_- + \partial_- A_+). \quad (\text{B.38})$$

These then lead to the relation

$$(il \pm mr_H) \partial_+ A_- = (il \mp mr_H) \partial_- A_+, \quad (\text{B.39})$$

where the upper (lower) signs correspond to the $- (+)$ -branch (B.32). Plugging in (B.34) and (B.35), one finds

$$C_{l,-}^{(\mp)} (il \pm mr_H) = C_{l,+}^{(\mp)} (il \mp mr_H). \quad (\text{B.40})$$

To summarize, we have determined the two sets of normal modes

$$(A_+, A_-)_{\omega l}^{(\mp)} = e^{-i\omega t + il\vartheta} \left(C_{l,+}^{(\mp)} e^{\frac{2\pi}{\beta} t} \psi_{\pm,l}(x'), C_{l,-}^{(\mp)} e^{-\frac{2\pi}{\beta} t} \psi_{-,l}(x') \right) \quad (\text{B.41})$$

satisfying the \mp equations (B.32) respectively, with A_{ϑ} uniquely determined by (B.37) or (B.38).

S-matrices. Comparing (B.35) and (B.8), we see that near horizon the incoming and outgoing waves are dominated by the components A_+ and A_- respectively. Explicitly,

$$(A_+, A_-)_{\omega l}^{(\mp)} (x' \rightarrow -\infty) \propto B_{\omega l}^{(\mp), \text{out}}(0, 1) e^{-i(\omega - i \frac{2\pi}{\beta})(t+x')} + B_{\omega l}^{(\mp), \text{in}}(1, 0) e^{-i(\omega + i \frac{2\pi}{\beta})(t-x')}. \quad (\text{B.42})$$

We have suppressed the $e^{il\vartheta}$ dependence. The outgoing and incoming coefficients are

$$B_{\omega l}^{(\mp), \text{out}} = (il \mp mr_H) \Gamma\left(\frac{i\beta\omega}{2\pi} + 1\right), \quad B_{\omega l}^{(\mp), \text{in}} = (il \pm mr_H) \Gamma\left(-\frac{i\beta\omega}{2\pi} + 1\right), \quad (\text{B.43})$$

whose ratio

$$\mathcal{S}_l^{\text{Rin},(\pm,s=1)}(\beta,\omega) = \frac{B_{\omega l}^{(\mp),\text{out}}}{B_{\omega l}^{(\mp),\text{in}}} = \frac{il \mp mr_H}{il \pm mr_H} \frac{\Gamma\left(\frac{i\beta\omega}{2\pi} + 1\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi} + 1\right)} \equiv \frac{il \mp mr_H}{il \pm mr_H} \mathcal{S}^{\text{Rin},(s=1)}(\omega) \quad (\text{B.44})$$

defines a unitary S-matrix.

B.3.2 Massive higher spin

We now study a general spin- s symmetric tensor field $\phi_{\mu_1\mu_2\cdots\mu_s}$ with mass m^2 , described by either (\mp) set of first-order equations

$$\epsilon_{\mu_1}^{\alpha\beta} \nabla_\alpha \phi_{\beta\mu_2\cdots\mu_s} = \mp m \phi_{\mu_1\mu_2\cdots\mu_s}. \quad (\text{B.45})$$

One can show that the solutions to these equations solve the Fierz-Pauli system

$$\left(-\nabla^2 + m^2\right) \phi_{\mu_1\mu_2\cdots\mu_s} = 0, \quad \nabla^\lambda \phi_{\lambda\mu_1\mu_2\cdots\mu_{s-1}} = 0, \quad \phi^\lambda_{\lambda\mu_1\mu_2\cdots\mu_{s-2}} = 0. \quad (\text{B.46})$$

In the coordinates (B.31), all Christoffel symbols are trivial, and one immediately concludes that the components with respect to (B.3) all satisfy the scalar equation

$$\left[-\partial_t^2 + \partial_x^2 + e^{\frac{4\pi}{\beta}x} \left(\frac{\tilde{\nabla}^2}{r_H^2} - m^2\right)\right] \phi_{A_1A_2\cdots A_s} = 0, \quad (\text{B.47})$$

where $A_I \in (\pm, \vartheta)$. Because of this, the normal mode functions are all given by the scalar mode function (B.6) (with x' defined in (B.36)), with appropriate (complex) shifts in ω dictated by the relation between the $(+, -, \vartheta)$ -components and the original (t, x, ϑ) -components. For example, for a spin-2 field $h_{\mu\nu}$,

$$\begin{aligned} h_{tt} &= e^{-\frac{4\pi}{\beta}(t-x)} h_{++} - 2e^{\frac{4\pi}{\beta}x} h_{+-} + e^{\frac{4\pi}{\beta}(t+x)} h_{--} \\ h_{tx} &= -2e^{-\frac{4\pi}{\beta}(t-x)} h_{++} + 2e^{\frac{4\pi}{\beta}(t+x)} h_{--} \\ h_{xx} &= e^{-\frac{4\pi}{\beta}(t-x)} h_{++} + 2e^{\frac{4\pi}{\beta}x} h_{+-} + e^{\frac{4\pi}{\beta}(t+x)} h_{--} \\ h_{t\vartheta} &= -e^{-\frac{2\pi}{\beta}(t-x)} h_{+\vartheta} + e^{\frac{2\pi}{\beta}(t+x)} h_{-\vartheta} \\ h_{x\vartheta} &= e^{-\frac{2\pi}{\beta}(t-x)} h_{+\vartheta} + e^{\frac{2\pi}{\beta}(t+x)} h_{-\vartheta}, \end{aligned} \quad (\text{B.48})$$

which imply that the normal modes are solved with the ansatz

$$\begin{aligned} h_{\pm\pm} &= e^{\pm\frac{4\pi}{\beta}t - i\omega t + il\vartheta} \psi_{\pm\pm,l}(x), & h_{+-} &= e^{-i\omega t + il\vartheta} \psi_{+-,l}(x), \\ h_{\pm\vartheta} &= e^{\pm\frac{2\pi}{\beta}t - i\omega t + il\vartheta} \psi_{\pm\vartheta,l}(x), & h_{\vartheta\vartheta} &= e^{-i\omega t + il\vartheta} \psi_{\vartheta\vartheta,l}(x), \end{aligned} \quad (\text{B.49})$$

where $l \in \mathbb{Z}$. We can immediately write down the explicit solutions:

$$\begin{aligned} \psi_{\pm\pm,\omega l}(x) &= \psi_{\omega \pm 2i\frac{2\pi}{\beta},l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi} \mp 2} \left(2e^{\frac{2\pi x'}{\beta}}\right), \\ \psi_{\pm\vartheta,\omega l}(x) &= \psi_{\omega \pm i\frac{2\pi}{\beta},l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi} \mp 1} \left(2e^{\frac{2\pi x'}{\beta}}\right), \\ \psi_{+-,\omega l}(x) &= \psi_{\vartheta\vartheta,l}(x) = \psi_{\omega l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi}} \left(2e^{\frac{2\pi x'}{\beta}}\right). \end{aligned} \quad (\text{B.50})$$

It is straightforward to generalize to arbitrary spin $s \geq 2$. We use the notation $\phi_{(a)(b)(c)}$ to denote the component of a spin- s field with a +-, b --, and $c = s - a - b$ ϑ -indices, i.e.

$$\phi_{(a)(b)(c)} \equiv \underbrace{\phi_{+\dots+}}_a \underbrace{-\dots-}_{b} \underbrace{\vartheta \dots \vartheta}_c. \quad (\text{B.51})$$

Normal modes are solved with the ansatz

$$\phi_{(a)(b)(c)} = C_{(a)(b)(c)} e^{(a-b)\frac{2\pi}{\beta}t - i\omega t} \psi_{(a)(b)(c),\omega l}(x) e^{il\vartheta}, \quad l \in \mathbb{Z}, \quad (\text{B.52})$$

where we have inserted the relative coefficients $C_{(a)(b)(c)}$ to be determined. The solutions are then

$$\psi_{(a)(b)(c),\omega l}(x) = \psi_{\omega+(a-b)i\frac{2\pi}{\beta},l}^{\text{Scalar}}(x') = K_{\frac{i\beta\omega}{2\pi}+(b-a)} \left(2e^{\frac{2\pi x'}{\beta}} \right). \quad (\text{B.53})$$

Polarization tensors. In the first-order formulation (B.45), all components (B.52) are coupled. To find their relative coefficients, we note that the ϑ -component of (B.45) reads explicitly

$$2r_H \left(\partial_+ \phi_{(a-1)(b+1)(c)} - \partial_- \phi_{(a)(b)(c)} \right) = \pm m \phi_{(a-1)(b)(c+1)}. \quad (\text{B.54})$$

On the other hand, the transversality condition (B.11) at $d = 2$ implies for the ansatz (B.34)

$$il \phi_{(a-1)(b)(c+1)} = -2r_H^2 \left(\partial_+ \phi_{(a-1)(b+1)(c)} + \partial_- \phi_{(a)(b)(c)} \right). \quad (\text{B.55})$$

These then lead to the relation

$$(il \pm mr_H) \partial_+ \phi_{(a-1)(b+1)(c)} = (il \mp mr_H) \partial_- \phi_{(a)(b)(c)}, \quad (\text{B.56})$$

where the upper (lower) signs correspond to the $-$ ($+$)-branch (B.45). Plugging in (B.52) and (B.53), one then finds

$$C_{l,(a-1)(b+1)(c)}^{(\mp)} (il \pm mr_H) = C_{l,(a)(b)(c)}^{(\mp)} (il \mp mr_H). \quad (\text{B.57})$$

S-matrices. Analogous to the vector case, near horizon the incoming and outgoing waves are dominated by the components $\phi_{(s)(0)(0)}$ and $\phi_{(0)(s)(0)}$ respectively, and we have

$$\begin{aligned} & \left(\phi_{(s)(0)(0)}, \phi_{(0)(s)(0)} \right)_{\omega l}^{(\mp)} (x \rightarrow -\infty) \\ & \propto B_{\omega l}^{(\mp),\text{out}}(0,1) e^{-i(\omega - is\frac{2\pi}{\beta})(t+x')} + B_{\omega l}^{(\mp),\text{in}}(1,0) e^{-i(\omega + is\frac{2\pi}{\beta})(t-x')}, \end{aligned} \quad (\text{B.58})$$

where we have suppressed the $e^{il\vartheta}$ dependence, and the outgoing and incoming coefficients are

$$B_{\omega l}^{(\mp),\text{out}} = (il \mp mr_H)^s \Gamma\left(\frac{i\beta\omega}{2\pi} + s\right), \quad B_{\omega l}^{(\mp),\text{in}} = (il \pm mr_H)^s \Gamma\left(-\frac{i\beta\omega}{2\pi} + s\right), \quad (\text{B.59})$$

whose ratio

$$\mathcal{S}_l^{\text{Rin},(\pm,s)}(\beta,\omega) = \frac{B_{\omega l}^{(\mp),\text{out}}}{B_{\omega l}^{(\mp),\text{in}}} = \left(\frac{il \mp mr_H}{il \pm mr_H} \right)^s \frac{\Gamma\left(\frac{i\beta\omega}{2\pi} + s\right)}{\Gamma\left(-\frac{i\beta\omega}{2\pi} + s\right)} = \left(\frac{il \mp mr_H}{il \pm mr_H} \right)^s \mathcal{S}^{\text{Rin},(s)}(\beta,\omega) \quad (\text{B.60})$$

defines a unitary S-matrix. Notice that the overall factor is independent of ω and drops out in relation (2.21).

C Massive higher spin on global AdS_3

Even though global AdS_3 (setting $\ell_{AdS} = 1$)

$$ds^2 = -\left(1 + r^2\right) dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\vartheta^2 \tag{C.1}$$

does not have a horizon and the considerations in the main text do not apply, we include this example due to its relation with the BTZ case. Also, it is instructive to highlight the difference between the two computations.

Thermal canonical partition function. The *normal* mode spectrum for a field with spin $s \geq 1$ and generic mass $m^2 = (\Delta - s)(\Delta + s - 2)$ on global AdS_3 is well-known:

$$\omega_{nl} = 2n + |l| + \Delta \tag{C.2}$$

where $n = 0, 1, 2, \dots$ and $l = 0, \pm 1, \pm 2, \dots$ labels the $U(1)$ angular momentum quantum number. In this case the density of state is simply a sum of delta functions over the discrete spectrum (C.2). The thermal canonical partition function is

$$\log Z_{\text{bulk}}^{AdS_3} \equiv \log \text{Tr} e^{-\beta \hat{H}} = -2 \sum_{n,l} \log\left(1 - e^{-\omega_{nl}\beta}\right). \tag{C.3}$$

Here we have dropped an infinite contribution from zero point energies that renormalizes the cosmological constant. Expanding the logarithm and performing the sums over n, l , we have

$$\log Z_{\text{bulk}}^{AdS_3} = \sum_{k=1}^{\infty} \frac{2}{k} \frac{e^{-\Delta k\beta}}{(1 - e^{-k\beta})^2} = \sum_{k=1}^{\infty} \frac{\chi_{[\Delta,s]}^{AdS_3}(k\beta)}{k}, \quad \chi_{[\Delta,s]}^{AdS_3}(t) = 2 \frac{e^{-\Delta t}}{(1 - e^{-t})^2}. \tag{C.4}$$

In the last equality we have expressed the result in terms of the $SO(2, 2)$ group character $\chi_{[\Delta,s]}^{AdS_3}(t)$.

Path integral on thermal AdS ($TAdS$). The same result can be obtained by computing the Euclidean path integral on $TAdS_3$:

$$\log Z_{\text{PI}}^{TAdS_3} = \int_0^{\infty} \frac{d\tau}{2\tau} e^{-\epsilon^2/4\tau} \text{Tr} e^{-(\nabla_s^2 + M_s^2)\tau} \tag{C.5}$$

where $e^{-\epsilon^2/4\tau}$ is a UV regulator. Here the trace Tr is over the spectrum of the Laplace operator $-\nabla_s^2 + M_s^2$ on $TAdS_3$. This has been computed by the image method in [43]:⁶

$$\text{Tr} e^{-(\nabla_s^2 + M_s^2)\tau} = \sum_{k=1}^{\infty} \frac{\beta}{\sqrt{4\pi\tau} \sinh^2 \frac{k\beta}{2}} e^{-\frac{k\beta^2}{4\tau}} e^{-(\Delta-1)^2\tau}. \tag{C.6}$$

Performing the τ -integral in (C.5) and putting $\epsilon = 0$, we recover the canonical result (C.4).

⁶The heat kernel for $s \leq 2$ on $TAdS_3$ was first computed in [44].

D Massive higher spin on BTZ: normal modes

In this appendix we find explicitly the normal mode solutions for massive higher spin (HS) fields on a static BTZ black hole (setting $\ell_{\text{AdS}} = 1$):

$$ds^2 = \frac{r_H^2}{\sinh^2(r_H x)} \left(-dt^2 + dx^2 + \cosh^2(r_H x) d\vartheta^2 \right), \quad -\infty < x < 0. \quad (\text{D.1})$$

Another coordinate system that turns out to be useful is given by

$$y_{\pm} = e^{\mp r_H t} \operatorname{sech}(r_H x), \quad (\text{D.2})$$

in terms of which the metric becomes

$$ds^2 = \frac{1}{4(1 - y_+ y_-)^2} \left(y_-^2 dy_+^2 + 2(2 - y_+ y_-) dy_+ dy_- + y_+^2 dy_-^2 \right) + \frac{r_H^2}{1 - y_+ y_-} d\vartheta^2. \quad (\text{D.3})$$

When expressing quantities in these coordinates, we use the shorthand notations \pm to denote the y_{\pm} -components. The non-zero Christoffel symbols associated with the metric (D.3) are

$$\Gamma_{\pm\pm}^{\pm} = \frac{y_{\mp}}{1 - y_+ y_-}, \quad \Gamma_{\pm\mp}^{\pm} = \frac{1}{2} \frac{y_{\pm}}{1 - y_+ y_-}, \quad \Gamma_{\vartheta\vartheta}^{\pm} = -r_H^2 y_{\pm}, \quad \Gamma_{\pm\vartheta}^{\vartheta} = \frac{1}{2} \frac{y_{\mp}}{1 - y_+ y_-}, \quad (\text{D.4})$$

which satisfy

$$\Gamma_{\pm\pm}^{\pm} = 2\Gamma_{\pm\vartheta}^{\vartheta} = 2\Gamma_{\pm\mp}^{\mp}. \quad (\text{D.5})$$

D.1 Massive scalars

For a scalar with mass $m^2 = \Delta(\Delta - 2)$, rescaling

$$\phi(t, x, \vartheta) = \sqrt{-\tanh(r_H x)} \psi(t, x, \vartheta), \quad (\text{D.6})$$

the Klein-Gordon equation $(-\nabla^2 + m^2)\phi = 0$ becomes

$$\left[-\partial_t^2 + \partial_x^2 + \frac{r_H^2}{\sinh^2(2r_H x)} - \frac{(\Delta - 1)^2 r_H^2}{\sinh^2(r_H x)} + \frac{\partial_{\vartheta}^2}{\cosh^2(r_H x)} \right] \psi(x) = 0. \quad (\text{D.7})$$

Solving with the ansatz

$$\psi(t, x, \vartheta) = e^{-i\omega t + i l \vartheta} \psi_{\omega l}(x), \quad (\text{D.8})$$

the normalizable solution satisfying the standard boundary condition is

$$\psi_{\omega l}^{\text{Scalar}}(x) = \frac{(\cosh(r_H x))^{r_H} (-\sinh(r_H x))^{\Delta}}{\sqrt{-\tanh(r_H x)}} {}_2F_1(a_{\omega l}, a_{-\omega l}; \Delta; -\sinh^2(r_H x)), \quad (\text{D.9})$$

with

$$a_{\omega l} = \frac{\Delta}{2} + \frac{i(-\omega + l)}{2r_H}. \quad (\text{D.10})$$

D.2 Massive vector

A massive vector on static BTZ is described by the first-order equations

$$\epsilon_{\alpha\mu\nu}\nabla^\mu A^\nu = \mp m A_\alpha. \quad (\text{D.11})$$

Here we take the following convention for the Levi-Civita symbol

$$\epsilon^{\mu\nu\lambda} \equiv \frac{\tilde{\epsilon}^{\mu\nu\lambda}}{\sqrt{-g}}, \quad \tilde{\epsilon}^{-+\vartheta} = -\tilde{\epsilon}_{-\vartheta} = 1. \quad (\text{D.12})$$

The solutions to each of the \pm -equations (D.11) furnish an irreducible representation of $\text{SO}(2,2)$. In a parity-invariant theory, both sets of solutions should be included. It is straightforward to show that the solutions to each equation satisfy the Fierz-Pauli system

$$\left(-\nabla^2 + \Delta(\Delta - 2) - 1\right) A_\mu = 0, \quad \nabla^\lambda A_\lambda = 0, \quad (\text{D.13})$$

where $\Delta = 1 + m$. We will focus on the components A_\pm , which uniquely determine A_ϑ through the ϑ -component of (D.11). Working out the \pm -components of (D.11) and using the transversality condition (D.13), one finds that A_\pm satisfy

$$\left[-\nabla_S^2 + (\Delta - 1)^2 + \tanh^2(r_H x) - \frac{2}{r_H} \tanh(r_H x) \partial_x\right] A_\pm = 0, \quad (\text{D.14})$$

where ∇_S^2 is the scalar Laplacian:

$$\nabla_S^2 \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu) = \frac{\sinh^2(r_H x)}{r_H^2} (-\partial_t^2 + \partial_x^2) - \frac{\tanh(r_H x)}{r_H} \partial_x + \frac{\tanh^2(r_H x)}{r_H^2} \partial_\vartheta^2. \quad (\text{D.15})$$

If we further define

$$A_\pm = \frac{\bar{A}_\pm}{\sqrt{-\tanh(r_H x)}}, \quad (\text{D.16})$$

we have

$$\left[-\partial_t^2 + \partial_x^2 + \frac{r_H^2}{\sinh^2(2r_H x)} - \frac{(\Delta - 1)^2 r_H^2}{\sinh^2(r_H x)} + \frac{\partial_\vartheta^2}{\cosh^2(r_H x)}\right] \bar{A}_\pm = 0. \quad (\text{D.17})$$

These take exactly the same form as the scalar equation (D.7). Note that in the near-horizon limit $x \rightarrow -\infty$, (D.17) reduces to the Rindler-like form (B.4). Since A_\pm are related to A_t, A_x through

$$A_\pm = \frac{e^{\pm r_H t}}{\tanh(r_H x)} (\pm \sinh(r_H x) A_t + \cosh(r_H x) A_x), \quad (\text{D.18})$$

normal modes correspond to the ansatz

$$\bar{A}_\pm = C_{\omega l, \pm} e^{\pm r_H t - i\omega t + i l \vartheta} \psi_{\omega l, \pm}(x), \quad A_\vartheta = e^{-i\omega t + i l \vartheta} \psi_{\omega l, \vartheta}(x). \quad (\text{D.19})$$

Here we have inserted the relative constants $C_{\omega l, \pm}$ to be determined below. Since (D.17) has exactly the same form as the scalar case (D.7), $\psi_{\omega l, \pm}(x)$ are simply given by shifting the scalar solution (D.9) by $\omega \rightarrow \omega \pm i r_H$, that is

$$\psi_{\omega l, \pm}(x) = \psi_{\omega \pm i r_H, l}^{\text{Scalar}}(x), \quad (\text{D.20})$$

where $\psi_{\omega l}^{\text{Scalar}}(x)$ is defined in (D.9).

Polarization vectors. In the first-order formulation (D.11), the solutions (D.19) are not independent. We first consider the $(-)$ -branch in (D.11), which explicitly reads

$$\begin{aligned} mA_{\pm} &= \pm \frac{1}{2r_H} \left[y_{\pm}^2 (\partial_{\mp} A_{\vartheta} - \partial_{\vartheta} A_{\mp}) + (2 - y_+ y_-) (\partial_{\vartheta} A_{\pm} - \partial_{\pm} A_{\vartheta}) \right], \\ mA_{\vartheta} &= 2r_H (1 - y_+ y_-) (\partial_+ A_- - \partial_- A_+). \end{aligned} \quad (\text{D.21})$$

Multiplying the first equation by y_{\pm} and taking the sum, we have

$$+ mr_H (y_+ A_+ + y_- A_-) = \frac{1}{r_H} \partial_t A_{\vartheta} + \partial_{\vartheta} (y_+ A_+ - y_- A_-). \quad (\text{D.22})$$

For the normal mode ansatz (D.19), we can replace

$$\partial_t \rightarrow -i\omega, \quad \partial_{\vartheta} \rightarrow il. \quad (\text{D.23})$$

Using the ϑ equation (D.21) in (D.22), we then arrive at

$$\frac{2i\omega}{m} (1 - y_+ y_-) \partial_- A_+ - (mr_H - il) y_+ A_+ = \frac{2i\omega}{m} (1 - y_+ y_-) \partial_+ A_- + (mr_H + il) y_- A_- . \quad (\text{D.24})$$

Plugging in (D.19) and (D.20), this implies the relation

$$C_{\omega l, +}^{(-)} (a_{-\omega + ir_H, -l} - 1) = -C_{\omega l, -}^{(-)} (a_{\omega + ir_H, +l} - 1) \quad (\text{D.25})$$

with $a_{\omega l}$ defined in (D.10). Here the superscript $(-)$ means that this is associated with the $(-)$ -branch (D.11). Similarly, for the $(+)$ -branch we have

$$C_{\omega l, +}^{(+)} (a_{-\omega + ir_H, +l} - 1) = -C_{\omega l, -}^{(+)} (a_{\omega + ir_H, -l} - 1). \quad (\text{D.26})$$

To summarize, we have determined the two sets of normal modes

$$(A_+, A_-)_{\omega l}^{(\mp)} = e^{-i\omega t + il\vartheta} \left(C_{\omega l, +}^{(\mp)} e^{r_H t} \psi_{\omega + ir_H, l}^{\text{Scalar}}(x), C_{\omega l, -}^{(\mp)} e^{-r_H t} \psi_{\omega - ir_H, l}^{\text{Scalar}}(x) \right) \quad (\text{D.27})$$

satisfying the \pm equations (D.11) respectively.

D.3 Massive higher spin

Having worked out the warm-up cases of massive scalar and vector, we now study a general spin- s symmetric tensor field $\phi_{\mu_1 \mu_2 \dots \mu_s}$ with mass $m^2 = (\Delta + s - 2)(\Delta - s)$, described by either (\mp) set of first-order equations

$$\epsilon_{\mu_1}^{\alpha\beta} \nabla_{\alpha} \phi_{\beta \mu_2 \dots \mu_s} = \mp M \phi_{\mu_1 \mu_2 \dots \mu_s}, \quad M \equiv \Delta - 1. \quad (\text{D.28})$$

The solutions to each of the first-order equations (D.28) satisfy the Fierz-Pauli system [19]

$$\left(-\nabla^2 + \Delta(\Delta - 2) - s \right) \phi_{\mu_1 \mu_2 \dots \mu_s} = 0, \quad \nabla^{\lambda} \phi_{\lambda \mu_1 \mu_2 \dots \mu_{s-1}} = 0, \quad \phi^{\lambda}_{\lambda \mu_1 \mu_2 \dots \mu_{s-2}} = 0. \quad (\text{D.29})$$

Solutions to both sets of equations (D.28) should be included for a parity-invariant theory.

To proceed, we first note that the action of the Laplacian takes the general form

$$\begin{aligned} \nabla^2 \phi_{\mu_1 \mu_2 \dots \mu_s} = & \nabla_S^2 \phi_{\mu_1 \mu_2 \dots \mu_s} - \frac{1}{\sqrt{g}} \partial_\alpha \left(\sqrt{g} g^{\alpha\lambda} \Gamma_{\lambda(\mu_1)}^\beta \right) \phi_{\mu_2 \dots \mu_s) \beta} - 2 \Gamma_{\lambda(\mu_1}^\alpha \nabla^\lambda \phi_{\mu_2 \dots \mu_s) \alpha} \\ & - g^{\alpha\lambda} \Gamma_{\alpha\beta}^\rho \Gamma_{\lambda(\mu_1}^\beta \phi_{\mu_2 \dots \mu_s) \rho} - \sum_{i \neq j} g^{\alpha\lambda} \Gamma_{\alpha\mu_i}^\beta \Gamma_{\lambda\mu_j}^\rho \phi_{\mu_1 \dots \hat{\mu}_i \hat{\mu}_j \dots \mu_s \beta \rho}. \end{aligned} \quad (\text{D.30})$$

In this expression, the symmetrization convention is simply to sum over permutations without extra factors. The summation in the last line has $s(s-1)$ terms. ∇_S^2 is the scalar Laplacian (D.15).

From now on, we use the notation $\phi_{(a)(b)(c)}$ to denote the component of a spin- s symmetric field with a (+)-, b (-)-, and $c = s - a - b$ (ϑ)-indices, analogous to (B.51). For the most part, we will focus on the components with only \pm -indices (i.e. those with $c = 0$). Solving for these will then uniquely determine the other components through (D.28). For these components, after a lengthy calculation we find explicitly (suppressing the ($c = 0$) subscript)

$$\nabla^2 \phi_{(a)(b)} = \nabla_S^2 \phi_{(a)(b)} - 2s(1 - y_+ y_-) (y_+ \partial_+ + y_- \partial_-) \phi_{(a)(b)} - 3s \phi_{(a)(b)} + s^2 y_+ y_- \phi_{(a)(b)}. \quad (\text{D.31})$$

In deriving this, we have simplified in (D.30) the second term using

$$\frac{1}{\sqrt{g}} \partial_\alpha \left(\sqrt{g} g^{\alpha\lambda} \Gamma_{\lambda\mu}^\beta \right) = \begin{cases} 3, & \beta = \mu = \pm \\ 2, & \beta = \mu = \vartheta, \\ 0, & \text{otherwise} \end{cases}, \quad (\text{D.32})$$

the third term using the transversality condition (D.29), the fourth term using

$$g^{\alpha\lambda} \Gamma_{\alpha\beta}^\rho \Gamma_{\lambda\mu}^\beta = \begin{cases} 2y_+ y_-, & \rho = \mu = \pm \\ y_\pm^2, & \rho = \pm, \mu = \mp, \\ 0, & \text{otherwise} \end{cases}, \quad (\text{D.33})$$

and the last term using

$$\begin{aligned} g^{\alpha\lambda} \Gamma_{\alpha\pm}^\beta \Gamma_{\lambda\pm}^\rho &= \left(\frac{1}{2} \Gamma_{\pm\pm}^\pm \right)^2 g^{\beta\rho} + \delta_\pm^\beta \delta_\pm^\rho 2y_+ y_- + \left(\delta_+^\beta \delta_-^\rho + \delta_-^\beta \delta_+^\rho \right) \frac{y_\pm^2}{2}, \\ g^{\alpha\lambda} \Gamma_{\alpha\pm}^\beta \Gamma_{\lambda\mp}^\rho &= \frac{1}{4} \Gamma_{++}^+ \Gamma_{--}^- g^{\beta\rho} + \delta_+^\beta \delta_+^\rho \frac{y_+^2}{2} + \delta_-^\beta \delta_-^\rho \frac{y_-^2}{2} + \delta_\pm^\beta \delta_\mp^\rho 2y_+ y_-, \end{aligned} \quad (\text{D.34})$$

together with the tracelessness condition (D.29).

In terms of the variables t, x , we can write (D.31) as

$$\begin{aligned} \nabla^2 \phi_{(a)(b)} = & \left(\frac{\sinh^2(r_H x)}{r_H^2} (-\partial_t^2 + \partial_x^2) - \frac{\tanh(r_H x)}{r_H} \partial_x + \frac{\tanh^2(r_H x)}{r_H^2} \partial_\vartheta^2 \right) \phi_{(a)(b)} \\ & + 2s \frac{\tanh(r_H x)}{r_H} \partial_x \phi_{(a)(b)} - 3s \phi_{(a)(b)} + s^2 \text{sech}^2(r_H x) \phi_{(a)(b)}. \end{aligned} \quad (\text{D.35})$$

Finally, rescaling

$$\phi_{(a)(b)} = (-\tanh(r_H x))^{\frac{1}{2}-s} \bar{\phi}_{(a)(b)}, \quad (\text{D.36})$$

we find that the second-order equations (D.29) for these components are reduced to

$$\left[-\partial_t^2 + \partial_x^2 + \frac{r_H^2}{\sinh^2(2r_H x)} - \frac{(\Delta - 1)^2 r_H^2}{\sinh^2(r_H x)} + \frac{\partial_\vartheta^2}{\cosh^2(r_H x)} \right] \bar{\phi}_{(a)(b)} = 0. \quad (\text{D.37})$$

Therefore, we have a set of decoupled equations that take the scalar form (D.7). Dictated by the relations between \pm - and t, x -components, normal modes correspond to the ansatz

$$\bar{\phi}_{(a)(b)(c)} = C_{\omega l, (a)(b)(c)} e^{(a-b)r_H t - i\omega t + il\vartheta} \psi_{\omega l, (a)(b)(c)}(x). \quad (\text{D.38})$$

This is true even for $c \neq 0$. Here $C_{\omega l, (a)(b)(c)}$ are polarization constants to be determined.

Since (D.37) has exactly the same form as the scalar case (D.7), $\psi_{\omega l, (a)(b)}(x)$ are simply given by shifting the scalar solution (D.9) by $\omega \rightarrow \omega + i(a-b)r_H$, that is

$$\psi_{\omega l, (a)(b)}(x) = \psi_{\omega + i(a-b)r_H, l}^{\text{Scalar}}(x), \quad (\text{D.39})$$

with $\psi_{\omega l}^{\text{Scalar}}(x)$ defined in (D.9).

Polarization tensors. In the first-order formulation (D.28), the solutions (D.39) are not independent. Analogous to the vector case, we multiply the $(a)(b)(c)$ - and $(a-1)(b+1)(c)$ -components of (D.28) by y_+ and y_- respectively and take their sum, which leads to

$$\begin{aligned} & \pm Mr_H \left(y_+ \phi_{(a)(b)(c)} + y_- \phi_{(a-1)(b+1)(c)} \right) \\ &= \frac{1}{r_H} \partial_t \phi_{(a-1)(b)(c+1)} + \partial_\vartheta \left(y_+ \phi_{(a)(b)(c)} - y_- \phi_{(a-1)(b+1)(c)} \right) \\ & \quad + cr_H^2 \left(y_+^2 \phi_{(a+1)(b)(c-1)} - y_-^2 \phi_{(a-1)(b+2)(c-1)} \right). \end{aligned} \quad (\text{D.40})$$

Here the upper (lower) sign corresponds to the $- (+)$ -branch (D.28). For the normal mode ansatz (D.38), we can replace

$$\partial_t \rightarrow -i\omega + (a-1-b)r_H, \quad \partial_\vartheta \rightarrow il. \quad (\text{D.41})$$

For $c = 0$, using also the $(a-1)(b)(c+1)$ -component of (D.28), (D.40) leads to the relation

$$\begin{aligned} & \frac{2i(\omega + i(a-1-b)r_H)}{M} \left((1 - y_+ y_-) \partial_- \phi_{(a)(b)} - \frac{s-1}{2} y_+ \phi_{(a)(b)} \right) - (Mr_H \mp il) y_+ \phi_{(a)(b)} \\ &= \frac{2i(\omega + i(a-1-b)r_H)}{M} \left((1 - y_+ y_-) \partial_+ \phi_{(a-1)(b+1)} - \frac{s-1}{2} y_- \phi_{(a-1)(b+1)} \right) \\ & \quad + (Mr_H \pm il) y_- \phi_{(a-1)(b+1)}. \end{aligned} \quad (\text{D.42})$$

Substituting (D.38) and (D.39), we arrive at the recursion relation

$$C_{\omega l, (a)(b)}^{(\mp)} (a_{-\omega - i(a-b-2)r_H, \mp l} - 1) = -C_{\omega l, (a-1)(b+1)}^{(\mp)} (a_{\omega + i(a-b)r_H, \pm l} - 1), \quad (\text{D.43})$$

with $a_{\omega l}$ defined in (D.10). Here (\mp) means that this is associated with the \mp -branch (D.28). Using (D.43), it is straightforward to derive the relation (4.15).

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