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Functional reduction of one-loop Feynman integrals with arbitrary masses

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Abstract: A method of functional reduction for the dimensionally regularized one-loop Feynman integrals with massive propagators is described in detail.

The method is based on a repeated application of the functional relations proposed by the author. Explicit formulae are given for reducing one-loop scalar integrals to a simpler ones, the arguments of which are the ratios of polynomials in the masses and kinematic invariants. We show that a general scalar n-point integral, depending on $n(n+1)/2$ generic masses and kinematic variables, can be expressed as a linear combination of integrals depending only on n variables. The latter integrals are given explicitly in terms of hypergeometric functions of $(n-1)$ dimensionless variables. Analytic expressions for the 2-, 3- and 4-point integrals, that depend on the minimal number of variables, were also obtained by solving the dimensional recurrence relations. The resulting expressions for these integrals are given in terms of Gauss' hypergeometric function ${}_2F_1$, the Appell function F_1 and the hypergeometric Lauricella — Saran function F_S. A modification of the functional reduction procedure for some special values of kinematic variables is considered.

Keywords: Electroweak Precision Physics, Higher Order Electroweak Calculations, Higher-Order Perturbative Calculations

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Contents

1 Introduction

Feynman integrals play an important role in making precise perturbative predictions in quantum field theory and statistical physics. Theoretical predictions for experiments at the LHC [1, 2] as well as at future colliders such as the FCC [3] demand knowledge of precise radiative corrections. Precise experimental measurements are to be interpreted with sufficient precision of theoretical predictions [4]. The complexity of the evaluation of such radiative corrections is related, in particular, to the difficulties in calculating integrals corresponding to Feynman diagrams with many external legs depending on many kinematic variables. Purely numerical evaluation of such integrals sometimes cannot provide sufficiently high precision within the reasonable computer time. Problems of numerical evaluation of the one-loop integrals were considered, for example, in refs. [5–8]. Numerical instability in evaluating one-loop scalar integrals near exceptional momentum configurations was addressed in refs. [9–11].

At present, there are many various methods of evaluating Feynman integrals. These integrals depend on two significantly different sets of variables. They are functions of continuous variables — scalar products of external momenta and masses, as well as functions of discrete parameters — powers of propagators and space — time dimension parameter d. External kinematic invariants and squared masses were used to derive differential equations [12] (see also reviews [13, 14]). The space-time dimension d and powers of propagators were used to derive difference equations [15–17] for these integrals. Then the results for integrals are obtained by solving these equations. Practical application of the method of differential equation and methods based on recurrence relations to evaluating high-order, multi-leg Feynman diagrams clearly demonstrates the need for further improvements and development of methods for solving differential and recursion relations.

On the other hand, it is possible to extend the applicability of these methods by combining them with other approaches. For instance, these methods can be used in combination with the approach proposed in refs. [18–20]. In ref. [18], a new type of relations among Feynman integrals, namely functional relations was discovered. In ref. [19], a simple method was proposed for deriving functional relations applicable to integrals corresponding to Feynman diagrams with any number of loops and legs. Using these relations, a method of functional reduction was formulated and applied to several massless integrals in ref. [20]. This method allows one to express the integral of interest in terms of integrals with fewer variables. In general, the latter integrals will be easier to evaluate by the above mentioned methods than the original integral.

Integrals appearing in the final results of functional reduction have two important features. Firstly, they depend on the minimal number of variables (MNV) and, secondly, these variables are the ratios of Gram determinants.

As for our representation of integrals in terms of functions that depend explicitly on the ratios of the Gram determinants, we would like to mention refs. [6, 11] where the importance of representing the kinematic dependence of integrals in terms of the ratios of the Gram determinants was demonstrated. As the authors have shown, such a representation turns out to be useful for the stability of numerical calculation of integrals.

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The primary purpose of the paper is to apply the method of functional reduction to scalar one-loop integrals that depend on arbitrary kinematic variables and masses.

The article is organized as follows. In section 2, we briefly describe the method for deriving functional relations given in ref. [19]. In section 3, we describe the method of functional reduction proposed in ref. [20]. In section 4, the functional reduction of the 2-point integral is considered. In section 5, a two-step functional reduction of the integral corresponding to a 3-point Feynman diagram is described. The Feynman parameter representation and dimensional recurrence relations for the integrals arising at the final stage of the functional reduction are given. We present the analytic result derived by using the dimensional recurrence relation and the result in terms of the double hypergeometric series obtained by expanding the Feynman parameter integral.

In section 6, we propose the three-step functional reduction procedure for the 4-point integral. Solving the dimensional recurrence relation, we obtained an analytic result for the integral depending on the MNV. A representation of this integral in terms of the triple hypergeometric series is also given.

In section 7, we describe the derivation of the functional relations of a four-step reduction procedure for a 5-point integral. We also give here the Feynman parameter representation of the 5-point integral depending on the MNV and the dimensional recurrence relation for this integral. Using the parametric representation of the integral, we express it as a fourfold hypergeometric series.

In section 8, we describe 5 steps of the functional reduction of a 6-point integral. The Feynman parameter representation and the dimensional recurrence relation for the integral with the MNV are given. Using the parametric representation, we express the integral as a multiple hypergeometric series.

In section 9, we describe a modification of the functional reduction method for integrals depending on special values of kinematic variables and present analytic results for these integrals.

In section 10, a general method is proposed for obtaining the final formula of the functional reduction for an arbitrary one-loop n-point integral. The parametric representation is also given for the n-point integrals depending on the MNV. Using this parametric representation, we obtain a representation of the integral in terms of multiple hypergeometric series.

We offer some concluding remarks in section 11.

Finally, in the appendix, we give the definition of kinematic determinants, provide useful formulae for hypergeometric functions and describe a method of deriving Feynman parameter representations of the integrals depending on the MNV.

2 Algebraic relations among propagators

We consider the one-loop scalar integral in the general dimension d corresponding to the Feynman diagram with n external lines and n internal propagators with arbitrary masses m_i and external momenta

$$I_n^{(d)}(\{m_j^2\};\{s_{ik}\}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{D_1 \ldots D_n} \tag{2.1}$$

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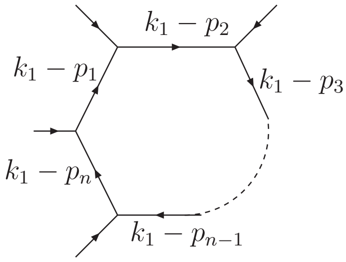

Figure 1. The generic n-point one-loop graph.

where the inverse massive propagators have the form

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta. \tag{2.2}$$

In what follows, we omit the $i\eta$ term assuming that all masses have such a correction. The propagators and momenta are labeled as in figure 1. As was shown in ref. [19], functional relations for these integrals can be derived from the following algebraic relations among the products of propagators:

$$\prod_{r=1}^{n} \frac{1}{D_r} = \frac{1}{D_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{D_j}. \tag{2.3}$$

We assume that k_1 is an arbitrary momentum and p_j correspond to external momenta. The proceeding equation is satisfied if p_0, m_0^2 and x_j, $(j = 1, \dots n)$ are chosen to satisfy the system of equations. In order to obtain such a system, we proceed as follows. We multiply both sides of eq. (2.3) by the product $\prod_{j=0}^{n} D_j$ and get

$$D_0 = \sum_{r=1}^{n} x_r D_r, \tag{2.4}$$

or

$$k_1^2 - 2k_1 p_0 + p_0^2 - m_0^2 = \sum_{r=1}^{n} x_r (k_1^2 - 2k_1 p_r + p_r^2 - m_r^2). \tag{2.5}$$

It is assumed that k_1 will be an integration momentum and p_j, x_r do not depend on it. Differentiating both sides of eq. (2.5) with respect to $k_{1\mu}$, one gets a linear equation in k_1 from which two equations follow:

$$1 = \sum_{r=1}^{n} x_r, \tag{2.6}$$

$$p_0 = \sum_{j=1}^{n} x_j p_j. \tag{2.7}$$

Substituting eqs. (2.6), (2.7) into eq. (2.5) yields the following equation:

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=2}^{n} \sum_{l=1}^{j-1} x_j x_l s_{lj} = 0, \tag{2.8}$$

where the kinematic invariants s_{ij} are defined as

$$s_{ij} = s_{i,j} = (p_i - p_j)^2. \tag{2.9}$$

Solving eq. (2.6) for one of the parameters x_j and then substituting this solution into eq. (2.8) gives a quadratic equation for the remaining parameters x_i. This quadratic equation can be solved with respect to one of the parameters x_j. Thus, the solution of the system of equations (2.6), (2.8) depends on $(n-2)$ of the remaining arbitrary parameters x_i and one arbitrary mass m_0.

Integrating algebraic relation (2.3) over momentum k_1 yields a functional equation for a general one-loop n-point integral

$$I_n^{(d)}(\{m_r^2\};\{s_{ik}\}) = \sum_{j=1}^{n} x_j \, I_n^{(d)}(\{m_r^2\};\{s_{ik}\})\Big|_{m_j^2 \to m_0^2, s_{jk} \to s_{0k}}. \tag{2.10}$$

This equation will be our starting equation for deriving relations for the functional reduction of integrals $I_n^{(d)}(\{m_r^2\};\{s_{ik}\})$.

Notice that linear relations among inverse propagators were also derived in ref. [21] and used for finding relationships among one-loop Feynman integrals. These relationships allow one to reduce n-point integrals to a combination of $(n-1)$-point integrals. Relations among propagators were obtained as a result of vanishing of the Gram determinants for the set of n vectors considered in d-dimensional space with $n > d$ and d being integer. In our approach, we assume that all vectors are d dimensional with d being noninteger. Our original algorithm [18] for finding functional relations among n- point integrals was based on vanishing the Gram determinants made of a set of momenta for the $(n+1)$ point integrals. However, for obtaining functional relations, we find it more convenient to introduce an additional propagator depending on an auxiliary external vector and an arbitrary mass [19]. Arbitrariness of these parameters is easier to use for reducing the number of variables in integrals, as compared to our original algorithm [18].

In the next sections, we will consider in detail the derivation of functional relations for reducing integrals $I_2^{(d)}$,…,$I_6^{(d)}$.

3 Method of functional reduction

By choosing arbitrary parameters x_j, m_0^2, we can try to express the integral of interest in terms of integrals with fewer variables. If we manage to find these parameters, we will actually solve the functional equation for the integral.

The systematic method for solving functional equations for Feynman integrals was presented in ref. [20]. In a sense, this is a generalization of the method that is used to solve the usual Sincov functional equation [22–24]

$$f(x,y) = f(x,z) - f(y,z). \tag{3.1}$$

By setting $z = 0$ in this equation, we get a general solution

$$f(x,y) = g(x) - g(y), \tag{3.2}$$

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where

$$g(x) = f(x, 0), \tag{3.3}$$

i.e. the function $f(x, y)$ is a combination of its 'boundary values', which may be completely arbitrary.

As for solving functional equations for Feynman integrals, the situation is much more complicated here — too many variables are involved, too many functions. For this reason, we used a computer to systematically search for possible relationships among the arguments of integrals leading to a decrease in the number of variables of these integrals. To reduce the number of variables, we impose the following simple conditions on the new variables s_{j0}, m_0^2

$$\begin{aligned} &s_{0j} = 0, \qquad s_{0j} - s_{0i} = 0, \quad s_{0j} \pm s_{ik} = 0, \quad s_{0j} \pm m_0^2 = 0, \quad m_j^2 \pm m_0^2 = 0, \\ &m_0^2 = 0, \quad s_{0j} \pm m_0^2 \pm m_k^2 = 0, \qquad (i, j, k = 1 \ldots n). \end{aligned} \tag{3.4}$$

From the set of equations obtained by combining eqs. (3.4), (2.6) and (2.8), we have formed various systems of equations with $2, 3, 4$, etc. equations in each system. Solutions of these systems of equations and analysis of these solutions were performed using the computer algebra system MAPLE. The number of these systems depends on n and varies from 10^3 to 10^6. CPU execution time ranged from a few minutes to several hours. Many solutions of these equations have been found. Some of them lead to a simultaneous decrease in the number of variables in all integrals on the right-hand side of the functional equation (2.10). In the following sections, we will describe in detail how this method works.

4 Functional reduction of the 2-point integral $I_2^{(d)}$

We start by considering a simple one-loop integral depending on arbitrary masses and external momentum

$$I_2^{(d)}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1 - p_1)^2 - m_1^2][(k_1 - p_2)^2 - m_2^2]}. \tag{4.1}$$

Setting $n = 2$ in eq. (2.3), leads to an algebraic relation between the products of two propagators

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}. \tag{4.2}$$

At $n = 2$, according to (2.6)–(2.8), the parameters x_j, m_0^2 and momentum p_0 in this equation must obey the following conditions:

$$\begin{aligned} &x_1 + x_2 = 1, \qquad p_0 = x_1 p_1 + x_2 p_2, \\ &m_0^2 - x_1 m_1^2 - x_2 m_2^2 + x_1 x_2 s_{12} = 0. \end{aligned} \tag{4.3}$$

Integrating algebraic relation (4.2) over momentum k_1 yields

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2; s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2; s_{10}). \tag{4.4}$$

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By solving the system of equations (4.3) for x_1, x_2, we get

$$x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{4s_{12}(m_0^2 - r_{12})}}{2s_{12}}, \qquad x_2 = 1 - x_1. \tag{4.5}$$

Both kinematic invariants s_{10}, s_{20} can be expressed in terms of x_1 as

$$\begin{aligned} s_{10} &= (p_1 - p_0)^2 = (1 - x_1)^2 s_{12} = m_1^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})}, \\ s_{20} &= (p_2 - p_0)^2 = x_1^2 s_{12} = m_2^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})}, \end{aligned} \tag{4.6}$$

where

$$r_{12} = -\frac{\lambda_{12}}{g_{12}} = \frac{2m_1^2 m_2^2 + 2s_{12}m_1^2 + 2s_{12}m_2^2 - m_1^4 - m_2^4 - s_{12}^2}{4s_{12}}. \tag{4.7}$$

The definitions of the determinants λ_{12}, g_{12} are given in the appendix.

Equation (4.4) strongly resembles Sincov's equation (3.1). By setting the only remaining arbitrary parameter m_0^2 to some special value, one can try to reduce the number of variables simultaneously for both integrals on the right-hand side of eq. (4.4). We will consider three different cases.

Case 1. $\boldsymbol{m_0^2 = 0}$. The most obvious choice is to take $m_0^2 = 0$. Substituting this value into eq. (4.4), we obtain

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = \overline{x}_1 I_2^{(d)}(m_2^2, 0; \overline{s}_{20}) + \overline{x}_2 I_2^{(d)}(m_1^2, 0; \overline{s}_{10}), \tag{4.8}$$

where

$$\overline{x}_{1,2} = x_{1,2}\big|_{m_0^2=0}, \qquad \overline{s}_{01} = s_{01}\big|_{m_0^2=0}, \qquad \overline{s}_{02} = s_{02}\big|_{m_0^2=0}. \tag{4.9}$$

The analytic expression for the integral $I_2^{(d)}(m^2, 0;\ p^2)$ is well known (see refs. [25, 26])

$$I_2^{(d)}(m^2, 0;\ p^2) = -\Gamma\left(1 - \frac{d}{2}\right) m^{d-4}\, {}_2F_1\!\left[\begin{matrix} 1, 2 - \frac{d}{2}\, ; \\ \frac{d}{2}\, ; \end{matrix} \frac{p^2}{m^2}\right]. \tag{4.10}$$

Note that the $\varepsilon = (4-d)/2$ expansion of the hypergeometric function ${}_2F_1$ in eq. (4.10) is known to all orders in ε [27–29]. Using eq. (4.8), one can easily obtain ε expansion of the original integral $I_2^{(d)}(m_1^2, m_2^2; s_{12})$.

Case 2. $\boldsymbol{m_0^2 = r_{12}}$. The second special value of m_0^2, which leads to a simultaneous decrease in the number of variables in both integrals on the right-hand side of eq. (4.4), is $m_0^2 = r_{12}$. In this case, the square roots in eqs. (4.5), (4.6) vanish, and we get

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = \kappa_{12} I_2^{(d)}(r_{12}, r_2; r_2 - r_{12}) + \kappa_{21} I_2^{(d)}(r_{12}, r_1; r_1 - r_{12}), \tag{4.11}$$

where

$$\kappa_{12} = \frac{\partial r_{12}}{\partial m_1^2}, \qquad \kappa_{21} = \frac{\partial r_{12}}{\partial m_2^2}, \qquad r_i = m_i^2. \tag{4.12}$$

The analytic result for integrals $I_2^{(d)}(r_{12}, r_j, r_j - r_{12})$, $(j = 1, 2)$ can be obtained either from the Feynman parameter representation

$$I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) = \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 h_2^{\frac{d}{2}-2} dx_1, \tag{4.13}$$

where

$$h_2 = r_{12} - (r_{12} - r_j)x_1^2, \tag{4.14}$$

or by solving the dimensional recurrence relation

$$(d-1)I_2^{(d+2)}(r_{12}, r_j; r_j - r_{12}) = -2r_{12}I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) - I_1^{(d)}(r_j). \tag{4.15}$$

In the latter case, the result reads

$$\begin{aligned} I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) = & \frac{-\pi^{\frac{3}{2}} r_{12}^{\frac{d}{2}-2}}{2\sin\frac{\pi d}{2}\Gamma\left(\frac{d-1}{2}\right)} \sqrt{\frac{r_{12}}{r_{12} - r_j}} \\ & + \frac{\pi}{2r_{12}} \frac{r_j^{\frac{d}{2}-1}}{\sin\frac{\pi d}{2}\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left[\begin{matrix} 1, \frac{d-1}{2}\,; \\ \frac{d}{2}\,; \end{matrix} \frac{r_j}{r_{12}}\right]. \end{aligned} \tag{4.16}$$

It is valid for $|r_j/r_{12}| < 1$. In order to solve the dimensional recurrence relation (4.15), we used the method described in ref. [30].

By changing the variable in the integral (4.13) and comparing the result with the integral representation of the ${}_2F_1$ function (A.20), we find

$$I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) = r_{12}^{\frac{d}{2}-2}\Gamma\left(2 - \frac{d}{2}\right) {}_2F_1\left[\begin{matrix} 2 - \frac{d}{2}, \frac{1}{2}\,; \\ \frac{3}{2}\,; \end{matrix} 1 - \frac{r_j}{r_{12}}\right]. \tag{4.17}$$

This result may also be obtained by expanding the integrand in eq. (4.13) in powers of $z_j = 1 - r_j/r_{12}$, assuming $|z_j| < 1$, and then integrating with respect to x_1 term by term. Formula (4.16) can be obtained from eq. (4.17) by performing an analytic continuation of the hypergeometric function ${}_2F_1$.

Case 3. Combination of two equations. The third reduction of integrals can be achieved in a slightly more complicated way. First, we set $m_0^2 = m_2^2$ in eq. (4.4) and obtain

$$\begin{aligned} I_2(m_1^2, m_2^2; s_{12}) = & \frac{m_1^2 - m_2^2}{s_{12}} I_2^{(d)}\left(m_1^2, m_2^2;\ \frac{(m_1^2 - m_2^2)^2}{s_{12}}\right) \\ & + \frac{s_{12} - m_1^2 + m_2^2}{s_{12}} I_2^{(d)}\left(m_2^2, m_2^2;\ \frac{(s_{12} - m_1^2 + m_2^2)^2}{s_{12}}\right). \end{aligned} \tag{4.18}$$

Then we interchange masses $m_1^2 \leftrightarrow m_2^2$ in this formula and add the result to (4.18). Due to the invariance of $I_2^{(d)}$ under this permutation, two terms in this sum having different signs cancel out, so we get

$$\begin{aligned} I_2(m_1^2, m_2^2;\ s_{12}) = & \frac{s_{12} + m_1^2 - m_2^2}{2s_{12}} I_2^{(d)}\left(m_1^2, m_1^2;\ \frac{(s_{12} + m_1^2 - m_2^2)^2}{s_{12}}\right) \\ & + \frac{s_{12} - m_1^2 + m_2^2}{2s_{12}} I_2^{(d)}\left(m_2^2, m_2^2;\ \frac{(s_{12} - m_1^2 + m_2^2)^2}{s_{12}}\right). \end{aligned} \tag{4.19}$$

The same result can be derived by setting $m_0^2 = m_1^2$ in eq. (4.4) and then adding the obtained result to the result obtained by setting $m_0^2 = m_2^2$ in eq. (4.4). The analytic expression for the integral with equal masses is well known (see refs. [25, 26]):

$$I_2^{(d)}(m^2, m^2; p^2) = m^{d-4}\,\Gamma\left(2 - \frac{d}{2}\right)\,{}_2F_1\left[\begin{matrix}1, 2-\frac{d}{2}; \\ \frac{3}{2};\end{matrix}\ \frac{p^2}{4m^2}\right]. \tag{4.20}$$

Thus, we have presented three different possibilities of reducing the integral $I_2^{(d)}$ to a sum of integrals with fewer variables. Different reduction formulae can be used in different kinematic domains.

Using eqs. (4.8), (4.11) and (4.19), one can easily find relations among integrals that appeared in the right-hand sides of these equations. For example, setting $m_1^2 = r_{12}$, $m_2^2 = r_j$, $s_{12} = r_j - r_{12}$ in eq. (4.19), we get

$$I_2(r_{12}, r_j; r_j - r_{12}) = I_2(r_j, r_j; 4(r_j - r_{12})) \qquad (j = 1, 2). \tag{4.21}$$

We conclude this section with a remark about the differences between integrals found in the three reduction procedures. In eq. (4.8), the arguments of integrals on the right-hand side depend on the square roots of the ratios of polynomials, while in eqs. (4.11), (4.19) the arguments of integrals on the right-hand side are just the ratios of polynomials. In all these cases, the integrals were expressed in terms of the hypergeometric function ${}_2F_1$. However, the $\varepsilon = (4-d)/2$ expansion of the functions in eq. (4.8) is technically slightly simpler than the expansion of the ${}_2F_1$ functions in eqs. (4.11), (4.19). The reason is that in the first case, the parameters of the ${}_2F_1$ functions are integers plus terms proportional to the ε while in the ${}_2F_1$ functions from eqs. (4.11), (4.19) some parameters are half integers. The ε expansion of the ${}_2F_1$ functions with half integer parameters contains logarithms and polylogarithms depending on the square roots of the argument of the ${}_2F_1$ function [27–29], while expansion of the ${}_2F_1$ functions with integer parameters do not have such square roots.

Our preliminary study shows that a similar situation takes place with the integrals $I_3^{(d)}$ and $I_4^{(d)}$. Analytical results for these integrals involve the function ${}_2F_1$ as well as more complicated hypergeometric functions. We expect that finding relationships among those functions with integer and half integer parameters will be helpful in performing the ε expansion of the integrals $I_3^{(d)}$ and $I_4^{(d)}$.

5 Functional reduction of the 3-point integral $I_3^{(d)}$

Now we turn to a 3-point integral with arbitrary internal mass scales and arbitrary external momenta

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = \frac{1}{i\pi^{d/2}} \int \frac{d^dk_1}{D_1D_2D_3}. \tag{5.1}$$

Setting $n = 3$ in equation (2.3) leads to an algebraic relation for the products of three propagators [19]:

$$\frac{1}{D_1D_2D_3} = \frac{x_1}{D_0D_2D_3} + \frac{x_2}{D_1D_0D_3} + \frac{x_3}{D_1D_2D_0}. \tag{5.2}$$

Equation (5.2) holds if

$$p_0 = x_1 p_1 + x_2 p_2 + x_3 p_3, \tag{5.3}$$

and the parameters m_0^2, x_j obey the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 x_2 s_{12} + x_1 x_3 s_{13} + x_2 x_3 s_{23} - x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2 + m_0^2 &= 0. \end{aligned} \tag{5.4}$$

Integrating (5.2) over momentum k_1 gives a functional relation for the one-loop integral $I_3^{(d)}$ with arbitrary masses and kinematic variables:

$$\begin{aligned} I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}&(m_0^2, m_2^2, m_3^2; s_{23}, s_{03}, s_{02}) \\ + x_2 I_3^{(d)}&(m_1^2, m_0^2, m_3^2; s_{03}, s_{13}, s_{01}) \\ + x_3 I_3^{(d)}&(m_1^2, m_2^2, m_0^2; s_{02}, s_{01}, s_{12}). \end{aligned} \tag{5.5}$$

Now our aim is to find the values of m_0^2, x_j $(j = 1, 2, 3)$, leading to a simultaneous reduction in the number of variables in all integrals on the right-hand side of eq. (5.5). Equation (5.5) will be our starting point at all steps of the functional reduction.

5.1 Functional reduction procedure

Functional reduction of the 3-point integral is not so straightforward as compared to the integral $I_2^{(d)}$. We will work out a two-step procedure of functional reduction allowing one to express an integral $I_3^{(d)}$ that depends on 6 variables in terms of integrals depending on 3 variables.

Reduction of the integral $I_3^{(d)}$, step 1. One of the solutions of the systems of equations (3.4), taken at $n = 3$ and combined with equations (5.3), (5.4), leads to the desired relation

$$\begin{aligned} I_3\left(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}\right) = &\,\kappa_{123} I_3\left(r_{123}, r_2, r_3; s_{23}, r_3 - r_{123}, r_2 - r_{123}\right) \\ &+ \kappa_{213} I_3\left(r_{123}, r_1, r_3; s_{13}, r_3 - r_{123}, r_1 - r_{123}\right) \\ &+ \kappa_{312} I_3\left(r_{123}, r_2, r_1; s_{12}, r_1 - r_{123}, r_2 - r_{123}\right), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} r_{123} &= -\frac{\lambda_{123}}{g_{123}}, \qquad r_i = m_i^2, \\ \kappa_{123} &= \frac{\partial r_{123}}{\partial m_1^2}, \quad \kappa_{213} = \frac{\partial r_{123}}{\partial m_2^2}, \quad \kappa_{312} = \frac{\partial r_{123}}{\partial m_3^2}, \end{aligned} \tag{5.7}$$

and the determinants λ_{123}, g_{123} are defined in the appendix. Note that all integrals on the right-hand side of equation (5.6) depend only on 4 variables.

This is not the only functional relation that reduces the number of variables of the integral $I_3^{(d)}$. We have discovered another functional relationship that reduces the number of variables by one

$$I_3\left(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}\right) = \frac{s_{23}(s_{23}-s_{12}-s_{13})}{g_{123}} I_3\left(m_0^2, m_2^2, m_3^2; s_{23}, s_{123}, s_{123}\right) + \frac{s_{13}(s_{13}-s_{12}-s_{23})}{g_{123}} I_3\left(m_1^2, m_0^2, m_3^2; s_{123}, s_{13}, s_{123}\right) + \frac{s_{12}(s_{12}-s_{13}-s_{23})}{g_{123}} I_3\left(m_1^2, m_2^2, m_0^2; s_{123}, s_{123}, s_{12}\right), \quad (5.8)$$

where

$$m_0^2 = -\frac{2\delta_3}{g_{123}}, \qquad s_{123} = -\frac{2s_{12}s_{13}s_{23}}{g_{123}}, \quad (5.9)$$

$$\delta_3 = -s_{12}s_{13}s_{23} + s_{23}(s_{12}+s_{13}-s_{23})m_1^2 + s_{13}(s_{12}-s_{13}+s_{23})m_2^2 - s_{12}(s_{12}-s_{13}-s_{23})m_3^2, \quad (5.10)$$

and g_{123} is defined in the appendix. We have found many other functional relations that reduce the number of variables, although not in all integrals at once. However, as was shown in section 4, integrals without reducing the number of variables can be eliminated by combining various functional relations (see eq. (4.19)). Derivation of this kind of functional relations will be studied in more detail in a forthcoming publication.

Reduction of the integral $I_3^{(d)}$, step 2 Now we proceed to the next step of the functional reduction. Applying relation (5.5) to the first integral on the right-hand side of eq. (5.6) and solving for the new variables m_0^2, s_{0j}, x_k the corresponding system of equations from (3.4), combined with eqs. (5.3), (5.4), leads to the equation

$$I_3^{(d)}(r_{123}, r_2, r_3; s_{23}, r_3 - r_{123}, r_2 - r_{123}) = \kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) + \kappa_{32} I_3^{(d)}(r_{123}, r_{23}, r_2; r_2 - r_{23}, r_2 - r_{123}, r_{23} - r_{123}), \quad (5.11)$$

where

$$\kappa_{23} = \frac{\partial r_{23}}{\partial m_2^2}, \qquad \kappa_{32} = \frac{\partial r_{23}}{\partial m_3^2}.$$

By an appropriate change of variables, two more equations for reducing other integrals in the right-hand side of eq. (5.6) can be obtained from eq. (5.11).

Combining eq. (5.6), eq. (5.11) and two equations that follow from eq. (5.11) by changing variables, we get the final reduction formula for the integral $I_3^{(d)}$:

$$\begin{aligned} I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = {} & \kappa_{123}\kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\ & + \kappa_{123}\kappa_{32} I_3^{(d)}(r_{123}, r_{23}, r_2; r_2 - r_{23}, r_2 - r_{123}, r_{23} - r_{123}) \\ & + \kappa_{213}\kappa_{31} I_3^{(d)}(r_{123}, r_{13}, r_1; r_1 - r_{13}, r_1 - r_{123}, r_{13} - r_{123}) \\ & + \kappa_{213}\kappa_{13} I_3^{(d)}(r_{123}, r_{13}, r_3; r_3 - r_{13}, r_3 - r_{123}, r_{13} - r_{123}) \\ & + \kappa_{312}\kappa_{12} I_3^{(d)}(r_{123}, r_{12}, r_2; r_2 - r_{12}, r_2 - r_{123}, r_{12} - r_{123}) \\ & + \kappa_{312}\kappa_{21} I_3^{(d)}(r_{123}, r_{12}, r_1; r_1 - r_{12}, r_1 - r_{123}, r_{12} - r_{123}). \end{aligned} \quad (5.12)$$

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This formula allows one to express the integral $I_3^{(d)}$, which depends on 6 variables in terms of integrals depending only on 3 variables.

It is interesting to note that the replacement of masses and kinematic invariants on both sides of eq. (5.12) with the arguments of the first integral $I_3^{(d)}$ on the right-hand side of this equation, i.e.

$$\begin{aligned} &m_1^2 \to r_{123}, \qquad &&m_2 \to r_{23}, \qquad &&m_3^2 \to r_3, \\ &s_{23} \to r_3 - r_{23}, \qquad &&s_{13} \to r_3 - r_{123}, \qquad &&s_{12} \to r_{23} - r_{123}, \end{aligned} \tag{5.13}$$

leads to the following transformations of the coefficients and arguments of integrals on the right-hand side of eq. (5.12)

$$\begin{aligned} &r_{123} \to r_{123}, \qquad &&\kappa_{123} \to 1, \qquad \kappa_{213} \to 0, \qquad \kappa_{312} \to 0, \\ &r_{23} \to r_{23}, \qquad &&r_{13} \to r_{123}, \qquad r_{12} \to r_{123}, \\ &\kappa_{23}, \kappa_{13}, \kappa_{12} \to 1, \qquad &&\kappa_{32}, \kappa_{31}, \kappa_{21} \to 0. \end{aligned} \tag{5.14}$$

As expected, in eq. (5.12), after these substitutions only the first term remains. Change of variables (5.13) leads to a factorization of the determinants λ and g

$$\begin{aligned} \lambda_{123} &= 8 r_{123}(r_{23} - r_3)(r_{123} - r_{23}), \qquad & g_{123} &= -8(r_{23} - r_3)(r_{123} - r_{23}), \\ \lambda_{23} &= -4 r_{23}(r_{23} - r_3), & g_{23} &= -4(r_3 - r_{23}), \end{aligned} \tag{5.15}$$

and as follows from these relations, r_{123}, r_{23} remain invariant under substitutions (5.13).

5.2 Analytic results for integrals depending on the MNV

An analytic result for the integral $I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123})$ can be obtained, for example, either by solving the dimensional recurrence relation or by calculating the Feynman parameter integral.

The dimensional recurrence relation for this integral reads

$$\begin{aligned} &(d-2) I_3^{(d+2)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\ &\quad = -2 r_{123} I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\ &\qquad - I_2^{(d)}(r_{23}, r_3; r_3 - r_{23}). \end{aligned} \tag{5.16}$$

The solution of the dimensional recurrence relation (5.16) was obtained by the method described in ref. [30]. Assuming that $|r_3/r_{123}| < 1$, $|r_3/r_{23}| < 1$, $|r_{23}/r_{123}| < 1$, we found the following result:

$$\begin{aligned} &I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\ &= \frac{1}{\sin\frac{\pi d}{2}} \left\{ \frac{r_{123}^{\frac{d-6}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} C_3(x,y) \right. \\ &\qquad + \frac{\pi^{\frac{3}{2}} r_{23}^{\frac{d-4}{2}}}{4 r_{123} \Gamma\left(\frac{d-1}{2}\right)} \sqrt{\frac{r_{23}}{r_{23} - r_3}}\ {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}; \\ \frac{d-1}{2}; \end{matrix} \frac{r_{23}}{r_{123}}\right] \\ &\qquad \left. - \frac{\pi r_3^{\frac{d-2}{2}}}{4\Gamma\left(\frac{d}{2}\right)(r_{23} - r_3) r_{123}} \sqrt{1 - \frac{r_3}{r_{23}}}\ F_1\left(\frac{d-2}{2}, 1, \frac{1}{2}, \frac{d}{2}; \frac{r_3}{r_{123}}, \frac{r_3}{r_{23}}\right) \right\}, \end{aligned} \tag{5.17}$$

where

$$C_3(x,y) = \frac{\pi x y^2}{4(x^2-y^2)^{\frac{1}{2}}} \ln\left(\frac{x-(x^2-y^2)^{\frac{1}{2}}}{x+(x^2-y^2)^{\frac{1}{2}}}\right), \tag{5.18}$$

and the variables x, y are defined as

$$x = \sqrt{\frac{r_{123}}{r_{123}-r_3}}, \qquad y = \sqrt{\frac{r_{123}}{r_{123}-r_{23}}}. \tag{5.19}$$

The function $C_3(x,y)$ was derived from the system of differential equations

$$\begin{aligned} x\frac{\partial C_3(x,y)}{\partial x} + y\frac{\partial C_3(x,y)}{\partial y} &= 2C_3(x,y), \\ (x^2-y^2)x\frac{\partial C_3(x,y)}{\partial x} &= -y^2 C_3(x,y) - \frac{1}{2}\pi x^2 y^2. \end{aligned} \tag{5.20}$$

This system was obtained from the system of differential equations for the integral $I_3^{(d)}$. We would like to notice the coefficient $1/\sin(\pi d/2)$ in front of the braces which is singular at $d=4$. Since the integral $I_3^{(4)}$ is finite, the terms in the braces at $d=4$ must cancel. This fact makes it possible to easily obtain the hypergeometric Appell function F_1 at $d=4$ as a combination of logarithms

$$F_1\left(1,1,\frac{1}{2},2;\frac{r_3}{r_{123}},\frac{r_3}{r_{23}}\right) = \frac{x^2\sqrt{1-y^2}}{1-x^2}\left[\ln\left(\frac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}}\right)+\ln\left(\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}\right)\right], \tag{5.21}$$

where x,y are defined in (5.19). This expression has been checked numerically to a precision of at least 200 decimal digits.

Another hypergeometric representation of the integral $I_3^{(d)}$ was derived directly from the Feynman parameter integral

$$I_3^{(d)}(r_{123},r_{23},r_3;r_3-r_{23},r_3-r_{123},r_{23}-r_{123}) = -\Gamma\left(3-\frac{d}{2}\right)\int_0^1\int_0^1 x_1 h_3^{\frac{d}{2}-3}dx_1dx_2, \tag{5.22}$$

where

$$h_3 = r_{123} - (r_{123}-r_{23})x_1^2 - (r_{23}-r_3)x_1^2x_2^2. \tag{5.23}$$

The method to derive the Feynman parameter representation of the integrals $I_n^{(d)}$ depending on the MNV is described in the appendix. Expanding the integrand in powers of variables

$$z_1 = \frac{r_{123}-r_{23}}{r_{123}}, \quad z_2 = \frac{r_{23}-r_3}{r_{123}}, \tag{5.24}$$

assuming that $|z_1|<1$, $|z_2|<1$ and integrating over x_1, x_2, we then get

$$\begin{aligned} &I_3^{(d)}(r_{123},r_{23},r_3;r_3-r_{23},r_3-r_{123},r_{23}-r_{123}) \\ &= -\frac{1}{2}\Gamma\left(3-\frac{d}{2}\right)\sum_{n_1,n_2=0}^{\infty}\left(3-\frac{d}{2}\right)_{n_1+n_2}\frac{(1)_{n_1+n_2}}{(2)_{n_1+n_2}}\frac{\left(\frac{1}{2}\right)_{n_2}}{\left(\frac{3}{2}\right)_{n_2}}\frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}. \end{aligned} \tag{5.25}$$

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Here $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the so-called Pochhammer symbol. The double series in eq. (5.25) can be written [31] as the hypergeometric function S_1

$$
\begin{aligned}
&I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\
&\quad = -\frac{1}{2}\Gamma\left(3-\frac{d}{2}\right) r_{123}^{\frac{d}{2}-3}\; S_1\left(3-\frac{d}{2}, 1, \frac{1}{2}, 2, \frac{3}{2}, \frac{r_{23}-r_3}{r_{123}}, 1-\frac{r_{23}}{r_{123}}\right). \qquad (5.26)
\end{aligned}
$$

The definition of the hypergeometric function S_1 is given in the appendix. Using the formula for the analytic continuation of the function S_1 presented in ref. [31], the integral $I_3^{(d)}$ can be written in terms of the hypergeometric functions ${}_2F_1$ and F_1:

$$
\begin{aligned}
&I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) = \frac{\Gamma\left(2-\frac{d}{2}\right)}{2(r_{123}-r_{23})} r_{123}^{\frac{d}{2}-2} \\
&\quad \times \left\{ {}_2F_1\left[\begin{matrix}1, \frac{1}{2}; \\ \frac{3}{2};\end{matrix} \frac{r_3 - r_{23}}{r_{123}-r_{23}}\right] - \frac{r_{23}^{\frac{d}{2}-2}}{r_{123}^{\frac{d}{2}-2}} F_1\left(\frac{1}{2}, 1, 2-\frac{d}{2}, \frac{3}{2}; \frac{r_{23}-r_3}{r_{23}-r_{123}}, 1-\frac{r_3}{r_{23}}\right)\right\}. \qquad (5.27)
\end{aligned}
$$

The formula for the analytic continuation of the function S_1 is given in the appendix (see eq. (A.17)). Note that the results in terms of the hypergeometric function S_1 for some $I_3^{(d)}$ integrals were presented in ref. [31].

6 Functional reduction of the 4-point integral $I_4^{(d)}$

Now we proceed to formulate a functional reduction procedure for the 4-point integral. For $n = 4$, the algebraic relation (2.3) reads

$$
\frac{1}{D_1D_2D_3D_4} = \frac{x_1}{D_0D_2D_3D_4} + \frac{x_2}{D_1D_0D_3D_4} + \frac{x_3}{D_1D_2D_0D_4} + \frac{x_4}{D_1D_2D_3D_0}. \qquad (6.1)
$$

Equation (6.1) holds if

$$
p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4, \qquad (6.2)
$$

and the parameters m_0^2, x_j obey the following system of equations:

$$
\begin{aligned}
&x_1 + x_2 + x_3 + x_4 = 1, \\
&x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_2x_3s_{23} + x_2x_4s_{24} + x_3x_4s_{34} \\
&\qquad -x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 + m_0^2 = 0. \qquad (6.3)
\end{aligned}
$$

Integrating equation (6.1) over momentum k_1 yields

$$
\begin{aligned}
&I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
&\quad = x_1 I_4^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2; s_{02}, s_{23}, s_{34}, s_{04}, s_{24}, s_{03}) \\
&\qquad + x_2 I_4^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2; s_{01}, s_{03}, s_{34}, s_{14}, s_{04}, s_{13}) \\
&\qquad + x_3 I_4^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2; s_{12}, s_{02}, s_{04}, s_{14}, s_{24}, s_{01}) \\
&\qquad + x_4 I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2; s_{12}, s_{23}, s_{03}, s_{01}, s_{02}, s_{13}). \qquad (6.4)
\end{aligned}
$$

This will be our initial equation for deriving functional relations in all three steps of the reduction procedure.

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6.1 Functional reduction procedure

In this subsection, we provide functional relations for expressing the integral $I_4^{(d)}$ that depends on 10 variables in terms of the integrals $I_4^{(d)}$ depending on 4 variables.

Reduction of the integral $I_4^{(d)}$, step 1. Solving various systems of equations, formed from equations (6.3) combined with equations (3.4), taken at $n = 4$, we obtained one solution which leads to the functional relation that reduces the number of variables by 3 in all integrals on the right-hand side of eq. (6.4). The functional relation corresponding to this solution reads

$$\begin{aligned} &I_4(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\ &\quad= \kappa_{1234} I_4(r_{1234}, r_2, r_3, r_4; r_2 - r_{1234}, s_{23}, s_{34}, r_4 - r_{1234}, s_{24}, r_3 - r_{1234}) \\ &\qquad+ \kappa_{2134} I_4(r_{1234}, r_1, r_3, r_4; r_1 - r_{1234}, s_{13}, s_{34}, r_4 - r_{1234}, s_{14}, r_3 - r_{1234}) \\ &\qquad+ \kappa_{3124} I_4(r_{1234}, r_2, r_1, r_4; r_2 - r_{1234}, s_{12}, s_{14}, r_4 - r_{1234}, s_{24}, r_1 - r_{1234}) \\ &\qquad+ \kappa_{4123} I_4(r_{1234}, r_2, r_3, r_1; r_2 - r_{1234}, s_{23}, s_{13}, r_1 - r_{1234}, s_{12}, r_3 - r_{1234}), \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} &r_{1234} = -\frac{\lambda_{1234}}{g_{1234}}, \qquad r_i = m_i^2, \\ &\kappa_{1234} = \frac{\partial r_{1234}}{\partial m_1^2}, \qquad \kappa_{2134} = \frac{\partial r_{1234}}{\partial m_2^2}, \qquad \kappa_{3124} = \frac{\partial r_{1234}}{\partial m_3^2}, \qquad \kappa_{4123} = \frac{\partial r_{1234}}{\partial m_4^2}. \end{aligned} \tag{6.6}$$

At the next step, the integrals on the right-hand side of equation (6.5) depending on 7 variables will be expressed in terms of integrals depending on 5 variables.

Reduction of the integral $I_4^{(d)}$, step 2. Applying formula (6.4) to the first integral on the right-hand side of eq. (6.5) and solving the systems of equations formed from equations (6.3) combined with equations (3.4) given for the kinematics of this integral, we found the following relation:

$$\begin{aligned} &I_4(r_{1234}, r_2, r_3, r_4; r_2 - r_{1234}, s_{23}, s_{34}, r_4 - r_{1234}, s_{24}, r_3 - r_{1234}) \\ &\quad= \kappa_{234} I_4(r_{1234}, r_{234}, r_3, r_4; r_{234} - r_{1234}, r_3 - r_{234}, s_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_3 - r_{1234}) \\ &\qquad+ \kappa_{324} I_4(r_{1234}, r_{234}, r_2, r_4; r_{234} - r_{1234}, r_2 - r_{234}, s_{24}, r_4 - r_{1234}, r_4 - r_{234}, r_2 - r_{1234}) \\ &\qquad+ \kappa_{423} I_4(r_{1234}, r_{234}, r_3, r_2; r_{234} - r_{1234}, r_3 - r_{234}, s_{23}, r_2 - r_{1234}, r_2 - r_{234}, r_3 - r_{1234}), \end{aligned} \tag{6.7}$$

where

$$r_{234} = -\frac{\lambda_{234}}{g_{234}}, \qquad \kappa_{234} = \frac{\partial r_{234}}{\partial m_2^2}, \qquad \kappa_{324} = \frac{\partial r_{234}}{\partial m_3^2}, \qquad \kappa_{423} = \frac{\partial r_{234}}{\partial m_4^2}. \tag{6.8}$$

Similar expressions for all other integrals on the right-hand side of eq. (6.5) can be obtained from eq. (6.7) by changing variables and coefficients appropriately.

After reducing integrals depending on 7 variables to integrals depending on 5 variables, the next step is to reduce the latter integrals to integrals depending on 4 variables.

Reduction of the integral $I_4^{(d)}$, step 3. Applying our initial functional relation (6.4) to the first integral on the right-hand side of eq. (6.7) and solving the systems of equations corresponding to this case, we obtained several solutions. One of these solutions leads to the two-term functional relation

$$\begin{aligned}
&I_4(r_{1234},r_{234},r_3,r_4;r_{234}-r_{1234},r_3-r_{234},s_{34},r_4-r_{1234},r_4-r_{234},r_3-r_{1234})\\
&\quad=\kappa_{34}I_4(r_{1234},r_{234},r_{34},r_4;\\
&\qquad\qquad r_{234}-r_{1234},r_{34}-r_{234},r_4-r_{34},r_4-r_{1234},r_4-r_{234},r_{34}-r_{1234})\\
&\qquad+\kappa_{43}I_4(r_{1234},r_{234},r_{34},r_3;\\
&\qquad\qquad r_{234}-r_{1234},r_{34}-r_{234},r_3-r_{34},r_3-r_{1234},r_3-r_{234},r_{34}-r_{1234}),
\end{aligned} \tag{6.9}$$

where

$$r_{34}=-\frac{\lambda_{34}}{g_{34}},\qquad \kappa_{34}=\frac{\partial r_{34}}{\partial m_3^2},\qquad \kappa_{43}=\frac{\partial r_{34}}{\partial m_4^2}. \tag{6.10}$$

Note that both integrals on the right-hand side of eq. (6.9) depend only on 4 variables. Combining eqs. (6.5), (6.7), (6.9) and all required relations that follow from these equations by changing variables as mentioned previously, we obtain the final functional reduction formula for the integral $I_4^{(d)}$

$$\begin{aligned}
&I_4(m_1^2,m_2^2,m_3^2,m_4^2;s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})\\
&\quad=\kappa_{1234}\kappa_{234}\kappa_{34}I_4^{(d)}(r_{1234},r_{234},r_{34},r_4;\\
&\qquad\qquad r_{234}-r_{1234},r_{34}-r_{234},r_4-r_{34},r_4-r_{1234},r_4-r_{234},r_{34}-r_{1234})\\
&\qquad+\kappa_{1234}\kappa_{234}\kappa_{43}I_4^{(d)}(r_{1234},r_{234},r_{34},r_3;\\
&\qquad\qquad r_{234}-r_{1234},r_{34}-r_{234},r_3-r_{34},r_3-r_{1234},r_3-r_{234},r_{34}-r_{1234})\\
&\qquad+\kappa_{1234}\kappa_{324}\kappa_{24}I_4^{(d)}(r_{1234},r_{234},r_{24},r_4;\\
&\qquad\qquad r_{234}-r_{1234},r_{24}-r_{234},r_4-r_{24},r_4-r_{1234},r_4-r_{234},r_{24}-r_{1234})\\
&\qquad+\kappa_{1234}\kappa_{324}\kappa_{42}I_4^{(d)}(r_{1234},r_{234},r_{24},r_2;\\
&\qquad\qquad r_{234}-r_{1234},r_{24}-r_{234},r_2-r_{24},r_2-r_{1234},r_2-r_{234},r_{24}-r_{1234})\\
&\qquad+\kappa_{1234}\kappa_{423}\kappa_{23}I_4^{(d)}(r_{1234},r_{234},r_{23},r_3;\\
&\qquad\qquad r_{234}-r_{1234},r_{23}-r_{234},r_3-r_{23},r_3-r_{1234},r_3-r_{234},r_{23}-r_{1234})\\
&\qquad+\kappa_{1234}\kappa_{423}\kappa_{32}I_4^{(d)}(r_{1234},r_{234},r_{23},r_2;\\
&\qquad\qquad r_{234}-r_{1234},r_{23}-r_{234},r_2-r_{23},r_2-r_{1234},r_2-r_{234},r_{23}-r_{1234})\\
&\qquad+\kappa_{2134}\kappa_{134}\kappa_{34}I_4^{(d)}(r_{1234},r_{134},r_{34},r_4;\\
&\qquad\qquad r_{134}-r_{1234},r_{34}-r_{134},r_4-r_{34},r_4-r_{1234},r_4-r_{134},r_{34}-r_{1234})\\
&\qquad+\kappa_{2134}\kappa_{134}\kappa_{43}I_4^{(d)}(r_{1234},r_{134},r_{34},r_3;\\
&\qquad\qquad r_{134}-r_{1234},r_{34}-r_{134},r_3-r_{34},r_3-r_{1234},r_3-r_{134},r_{34}-r_{1234})\\
&\qquad+\kappa_{2134}\kappa_{314}\kappa_{14}I_4^{(d)}(r_{1234},r_{134},r_{14},r_4;\\
&\qquad\qquad r_{134}-r_{1234},r_{14}-r_{134},r_4-r_{14},r_4-r_{1234},r_4-r_{134},r_{14}-r_{1234})
\end{aligned}$$

$$
\begin{aligned}
&+\kappa_{2134}\kappa_{314}\kappa_{41}I_4^{(d)}(r_{1234},r_{134},r_{14},r_1;\\
&\qquad r_{134}-r_{1234},r_{14}-r_{134},r_1-r_{14},r_1-r_{1234},r_1-r_{134},r_{14}-r_{1234})\\
&+\kappa_{2134}\kappa_{413}\kappa_{13}I_4^{(d)}(r_{1234},r_{134},r_{13},r_3;\\
&\qquad r_{134}-r_{1234},r_{13}-r_{134},r_3-r_{13},r_3-r_{1234},r_3-r_{134},r_{13}-r_{1234})\\
&+\kappa_{2134}\kappa_{413}\kappa_{31}I_4^{(d)}(r_{1234},r_{134},r_{13},r_1;\\
&\qquad r_{134}-r_{1234},r_{13}-r_{134},r_1-r_{13},r_1-r_{1234},r_1-r_{134},r_{13}-r_{1234})\\
&+\kappa_{3124}\kappa_{124}\kappa_{24}I_4^{(d)}(r_{1234},r_{124},r_{24},r_4;\\
&\qquad r_{124}-r_{1234},r_{24}-r_{124},r_4-r_{24},r_4-r_{1234},r_4-r_{124},r_{24}-r_{1234})\\
&+\kappa_{3124}\kappa_{124}\kappa_{42}I_4^{(d)}(r_{1234},r_{124},r_{24},r_2;\\
&\qquad r_{124}-r_{1234},r_{24}-r_{124},r_2-r_{24},r_2-r_{1234},r_2-r_{124},r_{24}-r_{1234})\\
&+\kappa_{3124}\kappa_{214}\kappa_{14}I_4^{(d)}(r_{1234},r_{124},r_{14},r_4;\\
&\qquad r_{124}-r_{1234},r_{14}-r_{124},r_4-r_{14},r_4-r_{1234},r_4-r_{124},r_{14}-r_{1234})\\
&+\kappa_{3124}\kappa_{214}\kappa_{41}I_4^{(d)}(r_{1234},r_{124},r_{14},r_1;\\
&\qquad r_{124}-r_{1234},r_{14}-r_{124},r_1-r_{14},r_1-r_{1234},r_1-r_{124},r_{14}-r_{1234})\\
&+\kappa_{3124}\kappa_{412}\kappa_{12}I_4^{(d)}(r_{1234},r_{124},r_{12},r_2;\\
&\qquad r_{124}-r_{1234},r_{12}-r_{124},r_2-r_{12},r_2-r_{1234},r_2-r_{124},r_{12}-r_{1234})\\
&+\kappa_{3124}\kappa_{412}\kappa_{21}I_4^{(d)}(r_{1234},r_{124},r_{12},r_1;\\
&\qquad r_{124}-r_{1234},r_{12}-r_{124},r_1-r_{12},r_1-r_{1234},r_1-r_{124},r_{12}-r_{1234})\\
&+\kappa_{4123}\kappa_{123}\kappa_{23}I_4^{(d)}(r_{1234},r_{123},r_{23},r_3;\\
&\qquad r_{123}-r_{1234},r_{23}-r_{123},r_3-r_{23},r_3-r_{1234},r_3-r_{123},r_{23}-r_{1234})\\
&+\kappa_{4123}\kappa_{123}\kappa_{32}I_4^{(d)}(r_{1234},r_{123},r_{23},r_2;\\
&\qquad r_{123}-r_{1234},r_{23}-r_{123},r_2-r_{23},r_2-r_{1234},r_2-r_{123},r_{23}-r_{1234})\\
&+\kappa_{4123}\kappa_{213}\kappa_{13}I_4^{(d)}(r_{1234},r_{123},r_{13},r_3;\\
&\qquad r_{123}-r_{1234},r_{13}-r_{123},r_3-r_{13},r_3-r_{1234},r_3-r_{123},r_{13}-r_{1234})\\
&+\kappa_{4123}\kappa_{213}\kappa_{31}I_4^{(d)}(r_{1234},r_{123},r_{13},r_1;\\
&\qquad r_{123}-r_{1234},r_{13}-r_{123},r_1-r_{13},r_1-r_{1234},r_1-r_{123},r_{13}-r_{1234})\\
&+\kappa_{4123}\kappa_{312}\kappa_{12}I_4^{(d)}(r_{1234},r_{123},r_{12},r_2;\\
&\qquad r_{123}-r_{1234},r_{12}-r_{123},r_2-r_{12},r_2-r_{1234},r_2-r_{123},r_{12}-r_{1234})\\
&+\kappa_{4123}\kappa_{312}\kappa_{21}I_4^{(d)}(r_{1234},r_{123},r_{12},r_1;\\
&\quad r_{123}-r_{1234},r_{12}-r_{123},r_1-r_{12},r_1-r_{1234},r_1-r_{123},r_{12}-r_{1234}). \qquad (6.11)
\end{aligned}
$$

This formula allows us to express an integral $I_4^{(d)}$, that depends on 15 variables, as a linear combination of 24 integrals, each depending only on 4 variables.

We conclude this subsection by noting an interesting property of this equation. By replacing the masses and kinematic invariants in eq. (6.11) with the corresponding arguments of the first integral on the right-hand side of this equation

$$\begin{aligned}
&m_1^2 \to r_{1234}, && m_2^2 \to r_{234}, && m_3^2 \to r_{34}, && m_4^2 \to r_4, \\
&s_{12} \to r_{234} - r_{1234}, && s_{23} \to r_{34} - r_{234}, && s_{34} \to r_4 - r_{34}, && \\
&s_{14} \to r_4 - r_{1234}, && s_{24} \to r_4 - r_{234}, && s_{13} \to r_{34} - r_{1234}, &&
\end{aligned} \tag{6.12}$$

we get the following transformations of the coefficients and arguments of integrals in this equation

$$\begin{aligned}
r_{1234} \to r_{1234}, \quad r_{234} \to r_{234}, \quad r_{34} \to r_{34}, \\
\kappa_{1234}, \kappa_{234}, \kappa_{34} \to 1, \qquad \kappa_{2134}, \kappa_{3124}, \kappa_{4123} \to 0.
\end{aligned} \tag{6.13}$$

As a result of these substitutions, only the first integral remains on the right-hand side of the equation. Note that the transformations (6.12) lead to a factorization of the determinants λ and g such as

$$\begin{aligned}
\lambda_{1234} &= 16 r_{1234}(r_{34} - r_4)(r_{34} - r_{234})(r_{1234} - r_{234}), \\
g_{1234} &= -16(r_{34} - r_4)(r_{34} - r_{234})(r_{1234} - r_{234}).
\end{aligned} \tag{6.14}$$

Similar factorization holds for lower order determinants. We observed analogous factorization of determinants appearing in the final functional relations for the integrals $I_5^{(d)}$ and $I_6^{(d)}$.

6.2 Analytic results for integrals depending on the MNV

An analytic expression for the integral

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; m_2^2 - m_1^2, m_3^2 - m_2^2, m_4^2 - m_3^2, m_4^2 - m_1^2, m_4^2 - m_2^2, m_3^2 - m_1^2), \tag{6.15}$$

which depends on the MNV, can be obtained, for example, by solving the dimensional recurrence relation or by evaluating the Feynman parameter integral.

The dimensional recurrence relation for this integral reads

$$\begin{aligned}
&(d-3) I_4^{(d+2)}(r_{1234}, r_{234}, r_{34}, r_4; \\
&\qquad\qquad r_{234} - r_{1234}, r_{34} - r_{234}, r_4 - r_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_{34} - r_{1234}) \\
&\quad = -2 r_{1234} I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4; \\
&\qquad\qquad\qquad r_{234} - r_{1234}, r_{34} - r_{234}, r_4 - r_{34}, r_4 - r_{1234}, r_4 - r_{234}, r_{34} - r_{1234}) \\
&\qquad - I_3^{(d)}(r_{234}, r_{34}, r_4; r_4 - r_{34}, r_4 - r_{234}, r_{34} - r_{234}).
\end{aligned} \tag{6.16}$$

To solve this equation, we used for the integral $I_3^{(d)}$ the analytic result given in eq. (5.17).

Applying the method described in ref. [30], we get

$$\begin{aligned}
&I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4;\\
&\qquad r_{234}-r_{1234}, r_{34}-r_{234}, r_4-r_{34}, r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234})\\
&= \frac{1}{\sin\frac{\pi d}{2}}\left\{\frac{r_{1234}^{\frac{d}{2}-4}}{\Gamma\left(\frac{d-3}{2}\right)}C_4(x,y,z)+\frac{\pi r_{234}^{\frac{d}{2}-2}\arctan\frac{(r_{34}-r_4)^{\frac{1}{2}}}{(r_{234}-r_{34})^{\frac{1}{2}}}}{4r_{1234}(r_{34}-r_4)^{\frac{1}{2}}(r_{234}-r_{34})^{\frac{1}{2}}\Gamma\left(\frac{d-2}{2}\right)}\right.\\
&\qquad \times {}_2F_1\left[\begin{matrix}1, \frac{d-3}{2};\\ \frac{d-2}{2};\end{matrix}\ \frac{r_{234}}{r_{1234}}\right]-\frac{\pi^{\frac{3}{2}}r_{34}^{\frac{d}{2}-2}}{8r_{1234}r_{234}\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{r_{34}}{r_{34}-r_4}\right)^{\frac{1}{2}}\left(\frac{r_{234}}{r_{234}-r_{34}}\right)^{\frac{1}{2}}\\
&\qquad \times F_1\left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; \frac{r_{34}}{r_{1234}}, \frac{r_{34}}{r_{234}}\right)\\
&\qquad + \frac{\pi r_4^{\frac{d}{2}-1}}{8r_{1234}(r_{34}-r_4)(r_{234}-r_4)\Gamma\left(\frac{d}{2}\right)}\\
&\qquad \left.\times F_S\left(\frac{d-3}{2}, 1, 1, 1, 1, \frac{1}{2}, \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{r_4}{r_{1234}}, \frac{r_4}{r_4-r_{234}}, \frac{r_4}{r_4-r_{34}}\right)\right\},
\end{aligned} \tag{6.17}$$

where

$$C_4(x,y,z) = \frac{\pi^{\frac{3}{2}}\ xy^2z^2}{8(x^2-y^2)^{\frac{1}{2}}(z^2-y^2)^{\frac{1}{2}}}\ln\left[\frac{xz+y^2-(z^2-y^2)^{\frac{1}{2}}(x^2-y^2)^{\frac{1}{2}}}{xz+y^2+(z^2-y^2)^{\frac{1}{2}}(x^2-y^2)^{\frac{1}{2}}}\right], \tag{6.18}$$

and the definition of the hypergeometric Lauricella-Saran function F_S is given in the appendix (see eq. (A.13)). Here the variables x, y, z are defined as

$$x = \sqrt{\frac{r_{1234}}{r_{1234}-r_4}}, \qquad y = \sqrt{\frac{r_{1234}}{r_{1234}-r_{34}}}, \qquad z = \sqrt{\frac{r_{1234}}{r_{1234}-r_{234}}}. \tag{6.19}$$

The function $C_4(x,y,z)$ was obtained by solving the system of differential equations

$$\begin{aligned}
&x\frac{\partial C_4(x,y,z)}{\partial x}+y\frac{\partial C_4(x,y,z)}{\partial y}+z\frac{\partial C_4(x,y,z)}{\partial z}=3C_4(x,y,z),\\
&\frac{\partial C_4(x,y,z)}{\partial x}=-\frac{y^2}{x(x^2-y^2)}C_4(x,y,z)-\frac{\pi^{\frac{3}{2}}xy^2z^2}{4(x^2-y^2)(x+z)}\\
&\frac{\partial C_4(x,y,z)}{\partial z}=\frac{(2y^2-z^2)}{z(y^2-z^2)}C_4(x,y,z)+\frac{\pi^{\frac{3}{2}}\ xy^2z^2}{4(y^2-z^2)(x+z)}.
\end{aligned} \tag{6.20}$$

This system was derived from the system of differential equations for the integral $I_4^{(d)}$. The constant of integration that occurs in solving the system of differential equations (6.20) was determined by comparing the asymptotic behavior of eq. (6.17) with the asymptotic behavior of the integral $I_4^{(d)}$ as $r_{1234} \to \infty$.

Note that in calculations of Feynman integrals, the Lauricella-Saran function F_S was first discovered when calculating the one-loop box integral [32]. In ref. [32], it was shown

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that this function can be represented by the one-fold integral

$$F_S\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};x,y,z\right)$$
$$=\frac{\Gamma\left(\frac{d}{2}\right)\ (y-z)^{-\frac{1}{2}}}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{3}{2}\right)}\int_0^1\frac{\arcsin\sqrt{\frac{(y-z)t}{1-tz}}(1-t)^{\frac{d-5}{2}}}{(1-x+tx)\sqrt{1-ty}}\,dt. \tag{6.21}$$

We note that similarly to the integral $I_3^{(d)}$, there is a factor $1/\sin(\pi d/2)$ in front of the braces in eq. (6.17) which is singular at $d=4$. Since the integral $I_4^{(4)}$ is finite, the terms in the braces must cancel at $d=4$. This fact makes it possible to easily obtain the hypergeometric function F_S at $d=4$ as a combination of logarithms

$$F_S\left(\frac{1}{2},1,1,1,1,\frac{1}{2},2,2,2;\frac{r_4}{r_{1234}},\frac{r_4}{r_4-r_{234}},\frac{r_4}{r_4-r_{34}}\right)$$
$$=\frac{(x^2-y^2)^{\frac{1}{2}}(x^2-z^2)}{(z^2-y^2)^{\frac{1}{2}}(1-x^2)x}\left\{\ln\left[\frac{xz+y^2-(z^2-y^2)^{\frac{1}{2}}(x^2-y^2)^{\frac{1}{2}}}{xz+y^2+(z^2-y^2)^{\frac{1}{2}}(x^2-y^2)^{\frac{1}{2}}}\right]\right.$$
$$\left.-\ln\left[\frac{x(y^2-z^2)^{\frac{1}{2}}-z(y^2-x^2)^{\frac{1}{2}}}{x(y^2-z^2)^{\frac{1}{2}}+z(y^2-x^2)^{\frac{1}{2}}}\right]+\ln\left[\frac{(z^2-y^2)^{\frac{1}{2}}-(1-y^2)^{\frac{1}{2}}}{(z^2-y^2)^{\frac{1}{2}}+(1-y^2)^{\frac{1}{2}}}\right]\right\}, \tag{6.22}$$

where x,y and z are defined in eq. (6.19). This expression has been checked numerically to a precision of at least 200 decimal digits. Complications in analytic calculation of a periodic function that appears in solving the recurrence relation for the one-loop box integral were discussed in ref. [33].

The integral $I_4^{(d)}$ depending on the MNV can be represented as a triple hypergeometric series. Such a representation can be derived from the Feynman parameter integral

$$I_4^{(d)}(r_{1234},r_{234},r_{34},r_4;r_{234}-r_{1234},r_{34}-r_{234},r_3-r_{34},r_4-r_{234},r_{34}-r_{1234})$$
$$=\Gamma\left(4-\frac{d}{2}\right)\int_0^1\int_0^1\int_0^1 x_1^2x_2\ h_4^{\frac{d}{2}-4}dx_1dx_2dx_3, \tag{6.23}$$

where

$$h_4=r_{1234}-(r_{1234}-r_{234})x_1^2-(r_{234}-r_{34})x_1^2x_2^2-(r_{34}-r_4)x_1^2x_2^2x_3^2. \tag{6.24}$$

Expanding the integrand of (6.23) with respect to three variables

$$z_1=1-\frac{r_{234}}{r_{1234}},\qquad z_2=\frac{r_{234}-r_{34}}{r_{1234}},\qquad z_3=\frac{r_{34}-r_4}{r_{1234}}, \tag{6.25}$$

assuming that $|z_1|<1$, $|z_2|<1$, $|z_3|<1$ and integrating over x_1, x_2, x_3 term by term, we obtain the series representation

$$I_4^{(d)}(r_{1234},r_{234},r_{34},r_4;r_{234}-r_{1234},r_{34}-r_{234},r_3-r_{34},r_4-r_{234},r_{34}-r_{1234})$$
$$=\frac{1}{6}\Gamma\left(4-\frac{d}{2}\right)r_{1234}^{\frac{d}{2}-4}\sum_{n_1,n_2,n_3=0}^{\infty}\left(4-\frac{d}{2}\right)_{n_1+n_2+n_3}$$
$$\times\frac{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{5}{2}\right)_{n_1+n_2+n_3}}\frac{(1)_{n_2+n_3}}{(2)_{n_2+n_3}}\frac{\left(\frac{1}{2}\right)_{n_3}}{\left(\frac{3}{2}\right)_{n_3}}\frac{z_1^{n_1}z_2^{n_2}z_3^{n_3}}{n_1!n_2!n_3!}. \tag{6.26}$$

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The analytic continuation of this series can be expressed in terms of the hypergeometric function ${}_2F_1$, the Appell functions F_1, F_3 and various hypergeometric Lauricella-Saran functions. Formula (6.17) is an example of such a representation. Other examples can be found in refs. [34, 35]. The relationship between the hypergeometric Lauricella-Saran functions F_S and F_N is given in the appendix (see eq. (A.14)).

7 Functional reduction of the 5-point integral $I_5^{(d)}$

In this section, we describe the functional reduction of the 5-point integral. At $n = 5$, the algebraic relation that follows from eq. (2.3) reads

$$\begin{aligned}\frac{1}{D_1D_2D_3D_4D_5} &= \frac{x_1}{D_0D_2D_3D_4D_5} + \frac{x_2}{D_1D_0D_3D_4D_5} \\ &\quad + \frac{x_3}{D_1D_2D_0D_4D_5} + \frac{x_4}{D_1D_2D_3D_0D_5} + \frac{x_5}{D_1D_2D_3D_4D_0}. \end{aligned} \tag{7.1}$$

Equation (7.1) is valid if

$$p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5, \tag{7.2}$$

and the parameters m_0^2, x_j obey the following system of equations:

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 + x_5 = 1, \\ &x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_1x_5s_{15} + x_2x_3s_{23} + x_2x_4s_{24} \\ &\quad + x_2x_5s_{25} + x_3x_4s_{34} + x_3x_5s_{35} + x_4x_5s_{45} \\ &\quad - x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 - x_5m_5^2 + m_0^2 = 0. \end{aligned} \tag{7.3}$$

Integrating equation (7.1) over k_1 yields the functional relation

$$\begin{aligned}&I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\ &\quad = x_1 I_5^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{02}, s_{23}, s_{34}, s_{45}, s_{05}, s_{03}, s_{04}, s_{24}, s_{25}, s_{35}) \\ &\qquad + x_2 I_5^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2, m_5^2; s_{01}, s_{03}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{04}, s_{05}, s_{35}) \\ &\qquad + x_3 I_5^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2, m_5^2; s_{12}, s_{02}, s_{04}, s_{45}, s_{15}, s_{01}, s_{14}, s_{24}, s_{25}, s_{05}) \\ &\qquad + x_4 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2, m_5^2; s_{12}, s_{23}, s_{03}, s_{05}, s_{15}, s_{13}, s_{01}, s_{02}, s_{25}, s_{35}) \\ &\qquad + x_5 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_0^2; s_{12}, s_{23}, s_{34}, s_{04}, s_{01}, s_{13}, s_{14}, s_{24}, s_{02}, s_{03}). \end{aligned} \tag{7.4}$$

This equation will be our initial equation in deriving functional relations at all stages of the reduction procedure.

7.1 Functional reduction procedure

In this subsection, we describe four steps of the functional reduction procedure, which will allow us to represent the integral $I_5^{(d)}$ that depends on 15 variables in terms of integrals depending on 5 variables.

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Reduction of the integral $I_5^{(d)}$, step 1. The systems of equations that were formed from a set of equations (3.4) and eqs. (7.2), (7.3) have many solutions. One of these solutions leads to the following relation:

$$\begin{aligned}
&I_5^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2;s_{12},s_{23},s_{34},s_{45},s_{15},s_{13},s_{14},s_{24},s_{25},s_{35})\\
&\quad=\kappa_{12345}I_5^{(d)}(r_{12345},r_2,r_3,r_4,r_5;\\
&\qquad\qquad r_2-r_{12345},s_{23},s_{34},s_{45},r_5-r_{12345},r_3-r_{12345},r_4-r_{12345},s_{24},s_{25},s_{35})\\
&\qquad+\kappa_{21345}I_5^{(d)}(r_1,r_{12345},r_3,r_4,r_5;\\
&\qquad\qquad r_1-r_{12345},r_3-r_{12345},s_{34},s_{45},s_{15},s_{13},s_{14},r_4-r_{12345},r_5-r_{12345},s_{35})\\
&\qquad+\kappa_{31245}I_5^{(d)}(r_1,r_2,r_{12345},r_4,r_5;\\
&\qquad\qquad s_{12},r_2-r_{12345},r_4-r_{12345},s_{45},s_{15},r_1-r_{12345},s_{14},s_{24},s_{25},r_5-r_{12345})\\
&\qquad+\kappa_{41235}I_5^{(d)}(r_1,r_2,r_3,r_{12345},r_5;\\
&\qquad\qquad s_{12},s_{23},r_3-r_{12345},r_5-r_{12345},s_{15},s_{13},r_1-r_{12345},r_2-r_{12345},s_{25},s_{35})\\
&\qquad+\kappa_{51234}I_5^{(d)}(r_1,r_2,r_3,r_4,r_{12345};\\
&\qquad\qquad s_{12},s_{23},s_{34},r_4-r_{12345},r_1-r_{12345},s_{13},s_{14},s_{24},r_2-r_{12345},r_3-r_{12345}),
\end{aligned}\tag{7.5}$$

where

$$r_{12345}=-\frac{\lambda_{12345}}{g_{12345}},\qquad \kappa_{12345}=\frac{\partial r_{12345}}{\partial m_1^2},\qquad \kappa_{21345}=\frac{\partial r_{12345}}{\partial m_2^2},\qquad r_i=m_i^2,$$
$$\kappa_{31245}=\frac{\partial r_{12345}}{\partial m_3^2},\qquad \kappa_{41235}=\frac{\partial r_{12345}}{\partial m_4^2},\qquad \kappa_{51234}=\frac{\partial r_{12345}}{\partial m_5^2}.$$

This equation is the first step in the functional reduction procedure expressing an integral depending on 15 variables as a linear combination of integrals depending on 11 variables.

Reduction of the integral $I_5^{(d)}$, step 2. By setting the masses and kinematic variables in equation (7.4) to be equal to the arguments of the first integral in the right-hand side of equation (7.5) and solving the corresponding sets of systems of equations, we found

$$\begin{aligned}
&I_5^{(d)}(r_{12345},r_2,r_3,r_4,r_5;\\
&\qquad r_2-r_{12345},s_{23},s_{34},s_{45},r_5-r_{12345},r_3-r_{12345},r_4-r_{12345},s_{24},s_{25},s_{35})\\
&\quad=\kappa_{2345}I_5^{(d)}(r_{12345},r_{2345},r_3,r_4,r_5;r_{2345}-r_{12345},r_3-r_{2345},s_{34},\\
&\qquad\qquad s_{45},r_5-r_{12345},r_3-r_{12345},r_4-r_{12345},r_4-r_{2345},r_5-r_{2345},s_{35})\\
&\qquad+\kappa_{3245}I_5^{(d)}(r_{12345},r_2,r_{2345},r_4,r_5;r_2-r_{12345},r_2-r_{2345},r_4-r_{2345},\\
&\qquad\qquad s_{45},r_5-r_{12345},r_{2345}-r_{12345},r_4-r_{12345},s_{24},s_{25},r_5-r_{2345})\\
&\qquad+\kappa_{4235}I_5^{(d)}(r_{12345},r_2,r_3,r_{2345},r_5;r_2-r_{12345},s_{23},r_3-r_{2345},r_5-r_{2345},\\
&\qquad\qquad r_5-r_{12345},r_3-r_{12345},r_{2345}-r_{12345},r_2-r_{2345},s_{25},s_{35})\\
&\qquad+\kappa_{5234}I_5^{(d)}(r_{12345},r_2,r_3,r_4,r_{2345};r_2-r_{12345},s_{23},s_{34},r_4-r_{2345},\\
&\qquad\qquad r_{2345}-r_{12345},r_3-r_{12345},r_4-r_{12345},s_{24},r_2-r_{2345},r_3-r_{2345}),
\end{aligned}\tag{7.6}$$

where

$$r_{2345} = -\frac{\lambda_{2345}}{g_{2345}},$$
$$\kappa_{2345} = \frac{\partial r_{2345}}{\partial m_2^2}, \quad \kappa_{3245} = \frac{\partial r_{2345}}{\partial m_3^2}, \quad \kappa_{4235} = \frac{\partial r_{2345}}{\partial m_4^2}, \quad \kappa_{5234} = \frac{\partial r_{2345}}{\partial m_5^2}. \tag{7.7}$$

Equation (7.6) allows one to express the first integral on the right-hand side of equation (7.5), which depends on 11 variables, in terms of integrals depending on 8 variables. Similar equations that reduce the number of variables by 3 for other integrals in the right-hand side of equation (7.5) can be obtained by a proper change of arguments and coefficients in equation (7.6).

Reduction of the integral $I_5^{(d)}$, step 3. Integrals depending on 8 variables can be expressed in terms of integrals depending on 6 variables. Similar to the previous steps, we apply relation (7.4) to the first integral on the right-hand side of equation (7.6), and solving the corresponding sets of systems of equations, we get

$$\begin{aligned}
&I_5^{(d)}(r_{12345}, r_{2345}, r_3, r_4, r_5; r_{2345}-r_{12345}, r_3-r_{2345},\\
&\qquad s_{34}, s_{45}, r_5-r_{12345}, r_3-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_5-r_{2345}, s_{35})\\
&\quad = \kappa_{345} I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_4, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_4-r_{345},\\
&\qquad s_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\qquad + \kappa_{435} I_5^{(d)}(r_{12345}, r_{2345}, r_3, r_{345}, r_5; r_{2345}-r_{12345}, r_3-r_{2345}, r_3-r_{345},\\
&\qquad r_5-r_{345}, r_5-r_{12345}, r_3-r_{12345}, r_{345}-r_{12345}, r_{345}-r_{2345}, r_5-r_{2345}, s_{35})\\
&\qquad + \kappa_{534} I_5^{(d)}(r_{12345}, r_{2345}, r_3, r_4, r_{345}; r_{2345}-r_{12345}, r_3-r_{2345}, s_{34}, r_4-r_{345},\\
&\qquad r_{345}-r_{12345}, r_3-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_{345}-r_{2345}, r_3-r_{345}),
\end{aligned} \tag{7.8}$$

where

$$r_{345} = -\frac{\lambda_{345}}{g_{345}}, \quad \kappa_{345} = \frac{\partial r_{345}}{\partial m_3^2}, \quad \kappa_{435} = \frac{\partial r_{345}}{\partial m_4^2}, \quad \kappa_{534} = \frac{\partial r_{345}}{\partial m_5^2}.$$

The functional relation (7.8) reduces an integral depending on 8 variables to a linear combination of integrals depending on 6 variables. From equation (7.8) one can obtain similar equations for reducing all the other integrals in the right-hand side of eq. (7.6).

Reduction of the integral $I_5^{(d)}$, step 4. Now we proceed to derive the last set of reduction equations. To do this, we apply relation (7.4) to the first integral on the right-hand side of (7.8), solve the systems of equations consisting of equations (3.4), (7.2), (7.3) and find

$$
\begin{aligned}
&I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_4, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_4-r_{345}, s_{45},\\
&\qquad r_5-r_{12345}, r_{345}-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad= \kappa_{45} I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345}, r_5-r_{45},\\
&\qquad\quad r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad\ + \kappa_{54} I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_4, r_{45}; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_4-r_{345}, r_4-r_{45},\\
&\qquad\quad r_{45}-r_{12345}, r_{345}-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_{45}-r_{2345}, r_{45}-r_{345}), \qquad (7.9)
\end{aligned}
$$

where

$$
r_{45} = -\frac{\lambda_{45}}{g_{45}}, \qquad \kappa_{45} = \frac{\partial r_{45}}{\partial m_4^2}, \qquad \kappa_{54} = \frac{\partial r_{45}}{\partial m_5^2}.
$$

Again, functional relations for reducing all other integrals on the right-hand side of eq. (7.8) can be obtained from (7.9) by a proper change of variables.

By combining equations (7.5), (7.6), (7.8), (7.9), and all required relations that follow from these equations by changing variables and coefficients, as mentioned previously, we obtained a formula that will allow us to express the integral $I_5^{(d)}$, depending on 15 variables, as a linear combination of 120 integrals, each depending only on 5 variables. All resulting integrals in this combination depend on the MNV and have the form

$$
\begin{aligned}
I_5^{(d)}(m_i^2, m_j^2, m_k^2, m_l^2, m_r^2;\ & m_j^2-m_i^2, m_k^2-m_j^2, m_l^2-m_k^2, m_r^2-m_l^2, m_r^2-m_i^2,\\
& m_k^2-m_i^2, m_l^2-m_i^2, m_l^2-m_j^2, m_r^2-m_j^2, m_r^2-m_k^2), \qquad (7.10)
\end{aligned}
$$

where m_i^2, m_j^2,m_k^2,m_l^2,m_r^2 are the ratios of polynomials in masses and kinematic invariants. The coefficients in front of these integrals are also the ratios of polynomials in masses and kinematic invariants.

The final reduction formula for the integral $I_5^{(d)}$ is too lengthy to present in the manuscript. Instead, we provide these formulae in electronic form in the supplementary material distributed with this article.

7.2 Dimensional recurrence relation and series representation

The dimensional recurrence relation for the integral $I_5^{(d)}$ depending on the MNV reads

$$
\begin{aligned}
&(d-4) I_5^{(d+2)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345},\\
&\qquad r_5-r_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad= -2r_{12345} I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_{45}, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_{45}-r_{345},\\
&\qquad r_5-r_{45}, r_5-r_{12345}, r_{345}-r_{12345}, r_{45}-r_{12345}, r_{45}-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad\ - I_4^{(d)}(r_{2345}, r_{345}, r_{45}, r_5;\\
&\qquad r_{345}-r_{2345}, r_{45}-r_{345}, r_5-r_{45}, r_5-r_{2345}, r_5-r_{345}, r_{45}-r_{2345}). \qquad (7.11)
\end{aligned}
$$

Notice that the inhomogeneous part of this equation consists of only one term — the integral $I_4^{(d)}$ which also depends on the MNV. The solution of the dimensional recurrence relation for the integral $I_5^{(d)}$ is a bit cumbersome but straightforward. We will present the result and details of the derivation in a separate publication.

At $d = 4$ the term with the integral $I_5^{(d+2)}$ in eq. (7.11) drops out, and we get a simple relation

$$\begin{aligned}
&2r_{12345}I_5^{(4)}(r_{12345},r_{2345},r_{345},r_{45},r_5;r_{2345}-r_{12345},r_{345}-r_{2345},r_{45}-r_{345},\\
&\qquad r_5-r_{45},r_5-r_{12345},r_{345}-r_{12345},r_{45}-r_{12345},r_{45}-r_{2345},r_5-r_{2345},r_5-r_{345})\\
&\quad=-I_4^{(4)}(r_{2345},r_{345},r_{45},r_5;\\
&\qquad\qquad r_{345}-r_{2345},r_{45}-r_{345},r_5-r_{45},r_5-r_{2345},r_5-r_{345},r_{45}-r_{2345}).
\end{aligned} \tag{7.12}$$

Now we proceed to derive a multiple series representation for the integral $I_5^{(d)}$ depending on the MNV. To do this, we will use the following Feynman parameter integral representation

$$\begin{aligned}
&I_5^{(d)}(r_{12345},r_{2345},r_{345},r_{45},r_5;r_{2345}-r_{12345},r_{345}-r_{2345},r_{45}-r_{345},r_5-r_{45},r_5-r_{12345},\\
&\qquad r_{345}-r_{12345},r_{45}-r_{12345},r_{45}-r_{2345},r_5-r_{2345},r_5-r_{345})\\
&\quad=-\Gamma\left(5-\frac{d}{2}\right)\int_0^1\int_0^1\int_0^1\int_0^1 x_1^3x_2^2x_3h_5^{\frac{d}{2}-5}dx_1dx_2dx_3dx_4,
\end{aligned} \tag{7.13}$$

where

$$\begin{aligned}
h_5&=r_{12345}-(r_{12345}-r_{2345})x_1^2\\
&\quad-(r_{2345}-r_{345})x_1^2x_2^2-(r_{345}-r_{45})x_1^2x_2^2x_3^2-(r_{45}-r_5)x_1^2x_2^2x_3^2x_4^2.
\end{aligned} \tag{7.14}$$

A multiple series representation of the integral $I_5^{(d)}$ can be derived in a similar manner, as it was done for the integrals $I_3^{(d)}$, $I_4^{(d)}$. Expanding the integrand in eq. (7.13) with respect to four variables

$$z_1=\frac{r_{12345}-r_{2345}}{r_{12345}},\qquad z_2=\frac{r_{2345}-r_{345}}{r_{12345}},\qquad z_3=\frac{r_{345}-r_{45}}{r_{12345}},\qquad z_4=\frac{r_{45}-r_5}{r_{12345}}, \tag{7.15}$$

assuming that all $|z_i|<1$, and then integrating term by term over $x_1,\dots,x_4$, we get

$$\begin{aligned}
&I_5^{(d)}(r_{12345},r_{2345},r_{345},r_{45},r_5;r_{2345}-r_{12345},r_{345}-r_{2345},r_{45}-r_{345},r_5-r_{45},\\
&\qquad r_5-r_{12345},r_{345}-r_{12345},r_{45}-r_{12345},r_{45}-r_{2345},r_5-r_{2345},r_5-r_{345})\\
&\quad=-\frac{1}{24}\Gamma\left(5-\frac{d}{2}\right)\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\left(5-\frac{d}{2}\right)_{n_1+n_2+n_3+n_4}\\
&\qquad\times\frac{(2)_{n_1+n_2+n_3+n_4}}{(3)_{n_1+n_2+n_3+n_4}}\frac{\left(\frac{3}{2}\right)_{n_2+n_3+n_4}}{\left(\frac{5}{2}\right)_{n_2+n_3+n_4}}\frac{(1)_{n_3+n_4}}{(2)_{n_3+n_4}}\frac{\left(\frac{1}{2}\right)_{n_4}}{\left(\frac{3}{2}\right)_{n_4}}\frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}\frac{z_3^{n_3}}{n_3!}\frac{z_4^{n_4}}{n_4!}.
\end{aligned} \tag{7.16}$$

Note the similarity of the summand of this series to summands of the series given in eqs. (5.25), (6.26).

8 Functional reduction of the 6-point integral $I_6^{(d)}$

Now we proceed to derive functional relations for reducing the 6-point integral. At $n = 6$ the algebraic relation (2.3) reads

$$\frac{1}{D_1D_2D_3D_4D_5D_6} = \frac{x_1}{D_0D_2D_3D_4D_5D_6} + \frac{x_2}{D_1D_0D_3D_4D_5D_6} + \frac{x_3}{D_1D_2D_0D_4D_5D_6} + \frac{x_4}{D_1D_2D_3D_0D_5D_6} + \frac{x_5}{D_1D_2D_3D_4D_0D_6} + \frac{x_6}{D_1D_2D_3D_4D_5D_0}. \tag{8.1}$$

Equation (8.1) is valid if

$$p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6, \tag{8.2}$$

and the parameters m_0^2, x_j obey the following system of equations:

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1, \\
&x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_1x_5s_{15} + x_1x_6s_{16} + x_2x_3s_{23} + x_2x_4s_{24} \\
&\quad + x_2x_5s_{25} + x_2x_6s_{26} + x_3x_4s_{34} + x_3x_5s_{35} + x_3x_6s_{36} + x_4x_5s_{45} + x_4x_6s_{46} \\
&\quad + x_5x_6s_{56} - x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 - x_5m_5^2 - x_6m_6^2 + m_0^2 = 0.
\end{aligned} \tag{8.3}$$

Integrating both sides of eq. (8.1) over k_1 yields the functional relation

$$\begin{aligned}
&I_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, \\
&\qquad s_{13}, s_{14}, s_{15}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46}) \\
&= x_1 I_6^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; s_{02}, s_{23}, s_{34}, s_{45}, s_{56}, s_{06}, \\
&\qquad s_{03}, s_{04}, s_{05}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46}) \\
&\quad + x_2 I_6^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2, m_5^2, m_6^2; s_{01}, s_{03}, s_{34}, s_{45}, s_{56}, s_{16}, \\
&\qquad s_{13}, s_{14}, s_{15}, s_{04}, s_{05}, s_{06}, s_{35}, s_{36}, s_{46}) \\
&\quad + x_3 I_6^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2, m_5^2, m_6^2; s_{12}, s_{02}, s_{04}, s_{45}, s_{56}, s_{16}, \\
&\qquad s_{01}, s_{14}, s_{15}, s_{24}, s_{25}, s_{26}, s_{05}, s_{06}, s_{46}) \\
&\quad + x_4 I_6^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2, m_5^2, m_6^2; s_{12}, s_{23}, s_{03}, s_{05}, s_{56}, s_{16}, \\
&\qquad s_{13}, s_{01}, s_{15}, s_{02}, s_{25}, s_{26}, s_{35}, s_{36}, s_{06}) \\
&\quad + x_5 I_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_0^2, m_6^2; s_{12}, s_{23}, s_{34}, s_{04}, s_{06}, s_{16}, \\
&\qquad s_{13}, s_{14}, s_{01}, s_{24}, s_{02}, s_{26}, s_{03}, s_{36}, s_{46}) \\
&\quad + x_6 I_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_0^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{05}, s_{01}, \\
&\qquad s_{13}, s_{14}, s_{15}, s_{24}, s_{25}, s_{02}, s_{35}, s_{03}, s_{04}).
\end{aligned} \tag{8.4}$$

This equation will be used at all steps of the functional reduction of the integral $I_6^{(d)}$. Derivation of the reduction formulae is completely analogous to that of the integrals $I_2^{(d)}, \ldots, I_5^{(d)}$.

8.1 Functional reduction procedure

In this subsection, we will describe five steps of the functional reduction procedure allowing us to represent the integral $I_6^{(d)}$ depending on 21 variables in terms of integrals depending on 6 variables.

Reduction of the integral $I_6^{(d)}$, step 1. At the first step, we formed various systems of equations by combining eqs. (3.4) taken at $n = 6$ and eqs. (8.2), (8.3), and solved these systems for x_j, m_0^2. Many solutions have been found. One of these solutions leads to a functional relation reducing 5 variables simultaneously for all integrals in the right-hand side of eq. (8.4). The functional relation corresponding to this solution reads

$$\begin{aligned}
&I_6^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2;s_{12},s_{23},s_{34},s_{45},s_{56},s_{16},\\
&\qquad s_{13},s_{14},s_{15},s_{24},s_{25},s_{26},s_{35},s_{36},s_{46})\\
&\quad=\kappa_{123456}I_6^{(d)}(r_{123456},r_2,r_3,r_4,r_5,r_6;r_2-r_{123456},s_{23},s_{34},s_{45},s_{56},r_6-r_{123456},\\
&\qquad\quad r_3-r_{123456},r_4-r_{123456},r_5-r_{123456},s_{24},s_{25},s_{26},s_{35},s_{36},s_{46})\\
&\qquad+\kappa_{213456}I_6^{(d)}(r_1,r_{123456},r_3,r_4,r_5,r_6;r_1-r_{123456},r_3-r_{123456},s_{34},s_{45},s_{56},s_{16},\\
&\qquad\quad s_{13},s_{14},s_{15},r_4-r_{123456},r_5-r_{123456},r_6-r_{123456},s_{35},s_{36},s_{46})\\
&\qquad+\kappa_{312456}I_6^{(d)}(r_1,r_2,r_{123456},r_4,r_5,r_6;s_{12},r_2-r_{123456},r_4-r_{123456},s_{45},s_{56},s_{16},\\
&\qquad\quad r_1-r_{123456},s_{14},s_{15},s_{24},s_{25},s_{26},r_5-r_{123456},r_6-r_{123456},s_{46})\\
&\qquad+\kappa_{412356}I_6^{(d)}(r_1,r_2,r_3,r_{123456},r_5,r_6;s_{12},s_{23},r_3-r_{123456},r_5-r_{123456},s_{56},\\
&\qquad\quad s_{16},s_{13},r_1-r_{123456},s_{15},r_2-r_{123456},s_{25},s_{26},s_{35},s_{36},r_6-r_{123456})\\
&\qquad+\kappa_{512346}I_6^{(d)}(r_1,r_2,r_3,r_4,r_{123456},r_6;s_{12},s_{23},s_{34},r_4-r_{123456},r_6-r_{123456},\\
&\qquad\quad s_{16},s_{13},s_{14},r_1-r_{123456},s_{24},r_2-r_{123456},s_{26},r_3-r_{123456},s_{36},s_{46})\\
&\qquad+\kappa_{612345}I_6^{(d)}(r_1,r_2,r_3,r_4,r_5,r_{123456};s_{12},s_{23},s_{34},s_{45},r_5-r_{123456},r_1-r_{123456},\\
&\qquad\quad s_{13},s_{14},s_{15},s_{24},s_{25},r_2-r_{123456},s_{35},r_3-r_{123456},r_4-r_{123456}),
\end{aligned}\tag{8.5}$$

where

$$\begin{aligned}
&r_{123456}=-\frac{\lambda_{123456}}{g_{123456}},\quad \kappa_{123456}=\frac{\partial r_{123456}}{\partial m_1^2},\quad \kappa_{213456}=\frac{\partial r_{123456}}{\partial m_2^2},\quad \kappa_{213456}=\frac{\partial r_{123456}}{\partial m_3^2},\\
&\kappa_{213456}=\frac{\partial r_{123456}}{\partial m_4^2},\quad \kappa_{213456}=\frac{\partial r_{123456}}{\partial m_5^2},\quad \kappa_{612345}=\frac{\partial r_{123456}}{\partial m_6^2},\qquad r_i=m_i^2.
\end{aligned}\tag{8.6}$$

There are several other solutions of the systems of equations (3.4), (8.2), (8.3) allowing us to reduce the number of variables simultaneously for all integrals in the right-hand side of eq. (8.4), but the number of variables reducible by the functional relations corresponding to these solutions were less than 5. We obtained also many solutions leading to the functional relations reducing the number of variables but not for all integrals simultaneously. Some of these solutions depend on the square roots of the polynomials in kinematic variables and masses.

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Reduction of the integral $I_6^{(d)}$, step 2. At the second step of the reduction we take the arguments of the first integral in the right-hand side of equation (8.5) and substitute them into the initial functional equation (8.4). By solving the systems of equations composed of equations (3.4) and equations (8.2), (8.3), for the new unknowns m_0^2, s_{0j}, x_j we found a solution allowing us to reduce four variables simultaneously for all integrals. This solution leads the following reduction formula:

$$
\begin{aligned}
&I_6^{(d)}(r_{123456}, r_2, r_3, r_4, r_5, r_6; r_2 - r_{123456}, s_{23}, s_{34}, s_{45}, s_{56}, r_6 - r_{123456},\\
&\qquad r_3 - r_{123456}, r_4 - r_{123456}, r_5 - r_{123456}, s_{24}, s_{25}, s_{26}, s_{35}, s_{36}, s_{46})\\
&\quad = \kappa_{23456} I_6^{(d)}(r_{123456}, r_{23456}, r_3, r_4, r_5, r_6; r_{23456} - r_{123456}, r_3 - r_{23456}, s_{34}, s_{45}, s_{56},\\
&\qquad r_6 - r_{123456}, r_3 - r_{123456}, r_4 - r_{123456}, r_5 - r_{123456}, r_4 - r_{23456},\\
&\qquad r_5 - r_{23456}, r_6 - r_{23456}, s_{35}, s_{36}, s_{46})\\
&\quad + \kappa_{32456} I_6^{(d)}(r_{123456}, r_2, r_{23456}, r_4, r_5, r_6; r_2 - r_{123456}, r_2 - r_{23456}, r_4 - r_{23456}, s_{45}, s_{56},\\
&\qquad r_6 - r_{123456}, r_{23456} - r_{123456}, r_4 - r_{123456}, r_5 - r_{123456}, s_{24}, s_{25}, s_{26},\\
&\qquad r_5 - r_{23456}, r_6 - r_{23456}, s_{46})\\
&\quad + \kappa_{42356} I_6^{(d)}(r_{123456}, r_2, r_3, r_{23456}, r_5, r_6; r_2 - r_{123456}, s_{23}, r_3 - r_{23456}, r_5 - r_{23456}, s_{56},\\
&\qquad r_6 - r_{123456}, r_3 - r_{123456}, r_{23456} - r_{123456}, r_5 - r_{123456}, r_2 - r_{23456}, s_{25}, s_{26},\\
&\qquad s_{35}, s_{36}, r_6 - r_{23456})\\
&\quad + \kappa_{52346} I_6^{(d)}(r_{123456}, r_2, r_3, r_4, r_{23456}, r_6; r_2 - r_{123456}, s_{23}, s_{34}, r_4 - r_{23456}, r_6 - r_{23456},\\
&\qquad r_6 - r_{123456}, r_3 - r_{123456}, r_4 - r_{123456}, r_{23456} - r_{123456}, s_{24}, r_2 - r_{23456}, s_{26},\\
&\qquad r_3 - r_{23456}, s_{36}, s_{46})\\
&\quad + \kappa_{62345} I_6^{(d)}(r_{123456}, r_2, r_3, r_4, r_5, r_{23456}; r_2 - r_{123456}, s_{23}, s_{34}, s_{45}, r_5 - r_{23456},\\
&\qquad r_{23456} - r_{123456}, r_3 - r_{123456}, r_4 - r_{123456}, r_5 - r_{123456}, s_{24}, s_{25}, r_2 - r_{23456}, s_{35},\\
&\qquad r_3 - r_{23456}, r_4 - r_{23456}),
\end{aligned}
\tag{8.7}
$$

where

$$
\begin{aligned}
r_{23456} &= -\frac{\lambda_{23456}}{g_{23456}}, \quad \kappa_{23456} = \frac{\partial r_{23456}}{\partial m_2^2}, \quad \kappa_{32456} = \frac{\partial r_{23456}}{\partial m_3^2},\\
\kappa_{42356} &= \frac{\partial r_{23456}}{\partial m_4^2}, \quad \kappa_{52346} = \frac{\partial r_{23456}}{\partial m_5^2}, \quad \kappa_{62345} = \frac{\partial r_{23456}}{\partial m_6^2}.
\end{aligned}
\tag{8.8}
$$

Similar considerations apply to other integrals from the right-hand side of equation (8.5). Hence with the aid of the functional relation (8.7), the original integral depending on 16 variables will be reduced to a combination of integrals depending on 12 variables.

Reduction of the integral $I_6^{(d)}$, step 3. At the third step, integrals depending on 12 variables were reduced to integrals depending on 9 variables. We substitute arguments of the first integral in the right-hand side of equation (8.7) into our initial equation (8.4) and solve the systems of equations composed of equations (3.4) and (8.2), (8.3) for the new unknowns x_j, m_0^2.

One of the obtained solutions yields the required reduction formula

$$
\begin{aligned}
&I_6^{(d)}(r_{123456}, r_{23456}, r_3, r_4, r_5, r_6; r_{23456}-r_{123456}, r_3-r_{23456}, s_{34}, s_{45}, s_{56}, r_6-r_{123456},\\
&\qquad r_3-r_{123456}, r_4-r_{123456}, r_5-r_{123456}, r_4-r_{23456}, r_5-r_{23456}, r_6-r_{23456}, s_{35}, s_{36}, s_{46})\\
&\ = \kappa_{3456} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_4, r_5, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_4-r_{3456},\\
&\qquad s_{45}, s_{56}, r_6-r_{123456}, r_{3456}-r_{123456}, r_4-r_{123456}, r_5-r_{123456}, r_4-r_{23456},\\
&\qquad r_5-r_{23456}, r_6-r_{23456}, r_5-r_{3456}, r_6-r_{3456}, s_{46})\\
&\quad + \kappa_{4356} I_6^{(d)}(r_{123456}, r_{23456}, r_3, r_{3456}, r_5, r_6; r_{23456}-r_{123456}, r_3-r_{23456}, r_3-r_{3456},\\
&\qquad r_5-r_{3456}, s_{56}, r_6-r_{123456}, r_3-r_{123456}, r_{3456}-r_{123456},\\
&\qquad r_5-r_{123456}, r_{3456}-r_{23456}, r_5-r_{23456}, r_6-r_{23456}, s_{35}, s_{36}, r_6-r_{3456})\\
&\quad + \kappa_{5346} I_6^{(d)}(r_{123456}, r_{23456}, r_3, r_4, r_{3456}, r_6; r_{23456}-r_{123456}, r_3-r_{23456}, s_{34},\\
&\qquad r_4-r_{3456}, r_6-r_{3456}, r_6-r_{123456}, r_3-r_{123456}, r_4-r_{123456}, r_{3456}-r_{123456},\\
&\qquad r_4-r_{23456}, r_{3456}-r_{23456}, r_6-r_{23456}, r_3-r_{3456}, s_{36}, s_{46})\\
&\quad + \kappa_{6345} I_6^{(d)}(r_{123456}, r_{23456}, r_3, r_4, r_5, r_{3456}; r_{23456}-r_{123456}, r_3-r_{23456}, s_{34},\\
&\qquad s_{45}, r_5-r_{3456}, r_{3456}-r_{123456}, r_3-r_{123456}, r_4-r_{123456}, r_5-r_{123456},\\
&\qquad r_4-r_{23456}, r_5-r_{23456}, r_{3456}-r_{23456}, s_{35}, r_3-r_{3456}, r_4-r_{3456}),
\end{aligned}
\tag{8.9}
$$

where

$$
\begin{aligned}
&r_{3456} = -\frac{\lambda_{3456}}{g_{3456}}, \quad \kappa_{3456} = \frac{\partial r_{3456}}{\partial m_3^2}, \quad \kappa_{4356} = \frac{\partial r_{3456}}{\partial m_4^2},\\
&\kappa_{5346} = \frac{\partial r_{3456}}{\partial m_5^2}, \quad \kappa_{6345} = \frac{\partial r_{3456}}{\partial m_6^2}.
\end{aligned}
\tag{8.10}
$$

Functional relations for reducing other integrals from the right-hand side of eq. (8.7) can be obtained from eq. (8.9) by an appropriate change of variables.

Reduction of the integral $I_6^{(d)}$, step 4. At the next step, we derive a formula for expressing integrals depending on 9 variables in terms of integrals depending on 7 variables. Again, as it was done in the previous step, we substitute the arguments of the first integral on the right-hand side of eq. (8.9) into eq. (8.4), solve appropriate systems of equations for the new unknowns and get

$$
\begin{aligned}
&I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_4, r_5, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_4-r_{3456},\\
&\qquad s_{45}, s_{56}, r_6-r_{123456}, r_{3456}-r_{123456}, r_4-r_{123456}, r_5-r_{123456}, r_4-r_{23456},\\
&\qquad r_5-r_{23456}, r_6-r_{23456}, r_5-r_{3456}, r_6-r_{3456}, s_{46})\\
&\ = \kappa_{456} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_5, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_{456}-r_{3456},\\
&\qquad r_5-r_{456}, s_{56}, r_6-r_{123456}, r_{3456}-r_{123456}, r_{456}-r_{123456}, r_5-r_{123456}, r_{456}-r_{23456},\\
&\qquad r_5-r_{23456}, r_6-r_{23456}, r_5-r_{3456}, r_6-r_{3456}, r_6-r_{456})\\
&\quad + \kappa_{546} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_4, r_{456}, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_4-r_{3456},\\
&\qquad r_4-r_{456}, r_6-r_{456}, r_6-r_{123456}, r_{3456}-r_{123456}, r_4-r_{123456}, r_4-r_{23456},\\
&\qquad r_{456}-r_{123456}, r_{456}-r_{23456}, r_6-r_{23456}, r_{456}-r_{3456}, r_6-r_{3456}, s_{46})
\end{aligned}
$$

$$+\,\kappa_{645} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_4, r_5, r_{456}; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_4-r_{3456},$$
$$s_{45}, r_5-r_{456}, r_{456}-r_{123456}, r_{3456}-r_{123456}, r_4-r_{123456}, r_5-r_{123456}, r_4-r_{23456},$$
$$r_5-r_{23456}, r_{456}-r_{23456}, r_5-r_{3456}, r_{456}-r_{3456}, r_4-r_{456}), \tag{8.11}$$

where

$$r_{456} = -\frac{\lambda_{456}}{g_{456}}, \quad \kappa_{456} = \frac{\partial r_{456}}{\partial m_4^2}, \quad \kappa_{546} = \frac{\partial r_{456}}{\partial m_5^2}, \quad \kappa_{645} = \frac{\partial r_{456}}{\partial m_6^2}. \tag{8.12}$$

Notice that all integrals in the right-hand side of equation (8.11) depend on 7 variables. These integrals may be expressed in terms of integrals depending on 6 variables.

Reduction of the integral $I_6^{(d)}$, step 5. The final formula for the first integral on the right-hand side of eq. (8.11) was derived by the same method which was used in the previous steps and reads

$$I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_5, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_{456}-r_{3456},$$
$$r_5-r_{456}, s_{56}, r_6-r_{123456}, r_{3456}-r_{123456}, r_{456}-r_{123456}, r_5-r_{123456}, r_{456}-r_{23456},$$
$$r_5-r_{23456}, r_6-r_{23456}, r_5-r_{3456}, r_6-r_{3456}, r_6-r_{456})$$
$$=\kappa_{56} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_{56}, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_{456}-r_{3456},$$
$$r_{56}-r_{456}, r_6-r_{56}, r_6-r_{123456}, r_{3456}-r_{123456}, r_{456}-r_{123456}, r_{56}-r_{123456},$$
$$r_{456}-r_{23456}, r_{56}-r_{23456}, r_6-r_{23456}, r_{56}-r_{3456}, r_6-r_{3456}, r_6-r_{456})$$
$$+\,\kappa_{65} I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_5, r_{56}; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_{456}-r_{3456},$$
$$r_5-r_{456}, r_6-r_{56}, r_{56}-r_{123456}, r_{3456}-r_{123456}, r_{456}-r_{123456}, r_5-r_{123456},$$
$$r_{456}-r_{23456}, r_5-r_{23456}, r_{56}-r_{23456}, r_5-r_{3456}, r_{56}-r_{3456}, r_{56}-r_{456}), \tag{8.13}$$

where

$$r_{56} = -\frac{\lambda_{56}}{g_{56}}, \quad \kappa_{56} = \frac{\partial r_{56}}{\partial m_5^2}, \quad \kappa_{65} = \frac{\partial r_{56}}{\partial m_6^2}. \tag{8.14}$$

Analogous formulae for the reduction of other integrals on the right-hand side of eq. (8.11) can be obtained form eq. (8.13) by changing variables appropriately. This completes the derivation of the reduction formulae for the integral $I_6^{(d)}$.

Composition of equations (8.5), (8.7), (8.9), (8.11), (8.13) and all required relations that follow from these equations by changing variables as mentioned previously, gives a formula for the complete functional reduction of the integral $I_6^{(d)}$. This formula represents the integral depending on 21 variables as a sum of 720 integrals, each depending only on 6 variables. All the resulting integrals in this sum have the form

$$I_6^{(d)}(m_i^2, m_j^2, m_k^2, m_l^2, m_r^2, m_s^2; \;\; m_j^2-m_i^2, m_k^2-m_j^2, m_l^2-m_k^2, m_r^2-m_l^2,$$
$$m_s^2-m_r^2, m_s^2-m_i^2, m_k^2-m_i^2, m_l^2-m_i^2, m_r^2-m_i^2, m_l^2-m_j^2,$$
$$m_r^2-m_j^2, m_s^2-m_j^2, m_r^2-m_k^2, m_s^2-m_k^2, m_s^2-m_l^2), \tag{8.15}$$

where m_i^2, m_j^2, m_k^2, m_l^2, m_r^2, m_s^2 are the ratios of polynomials in masses and kinematic invariants. The coefficients in front of these integrals are also the ratios of polynomials in masses and kinematic invariants.

The final formula of the reduction is too lengthy to display here, but we provide it< in a computer-readable ancillary file attached to this article.

8.2 Dimensional recurrence relation and series representation

An analytic result for the integral $I_6^{(d)}$ can be obtained, for example, by solving the dimensional recurrence relation or by evaluating the Feynman parameter integral.

The dimensional recurrence relation for the integral $I_6^{(d)}$ depending on the MNV reads

$$\begin{aligned}(d-5)I_6^{(d+2)}&(r_{123456},r_{23456},r_{3456},r_{456},r_{56},r_6;r_{23456}-r_{123456},r_{3456}-r_{23456},r_{456}-r_{3456},\\ &r_{56}-r_{456},r_6-r_{56},r_6-r_{123456},r_{3456}-r_{123456},r_{456}-r_{123456},r_{56}-r_{123456},\\ &r_{456}-r_{23456},r_{56}-r_{23456},r_6-r_{23456},r_{56}-r_{3456},r_6-r_{3456},r_6-r_{456})\\ =-2r_{123456}I_6^{(d)}&(r_{123456},r_{23456},r_{3456},r_{456},r_{56},r_6;r_{23456}-r_{123456},r_{3456}-r_{23456},\\ &r_{456}-r_{3456},r_{56}-r_{456},r_6-r_{56},r_6-r_{123456},r_{3456}-r_{123456},r_{456}-r_{123456},r_{56}-r_{123456},\\ &r_{456}-r_{23456},r_{56}-r_{23456},r_6-r_{23456},r_{56}-r_{3456},r_6-r_{3456},r_6-r_{456})\\ -I_5^{(d)}&(r_{23456},r_{3456},r_{456},r_{56},r_6;r_{3456}-r_{23456},r_{456}-r_{3456},r_{56}-r_{456},r_6-r_{56},r_6-r_{23456},\\ &r_{456}-r_{23456},r_{56}-r_{23456},r_{56}-r_{3456},r_6-r_{3456},r_6-r_{456}). \qquad (8.16)\end{aligned}$$

Notice that the inhomogeneous term in this equation is an integral depending on the MNV. The solution of this recurrence relation is straightforward but cumbersome. The result is a bit lengthy and for this reason it will not be presented in this article.

We obtained the following Feynman parameter representation of this integral:

$$\begin{aligned}&I_6^{(d)}(r_{123456},r_{23456},r_{3456},r_{456},r_{56},r_6;\{s_{ij}\})\\ &\quad=\Gamma\left(6-\frac{d}{2}\right)\int_0^1\dots\int_0^1 \; x_1^4x_2^3x_3^2x_4\; h_6^{\frac{d}{2}-6}dx_1\dots dx_5, \qquad (8.17)\end{aligned}$$

where

$$\begin{aligned}h_6=r_{123456}&-(r_{123456}-r_{23456})x_1^2-(r_{23456}-r_{3456})x_1^2x_2^2-(r_{3456}-r_{456})x_1^2x_2^2x_3^2\\ &-(r_{456}-r_{56})x_1^2x_2^2x_3^2x_4^2-(r_{56}-r_6)x_1^2x_2^2x_3^2x_4^2x_5^2, \qquad (8.18)\end{aligned}$$

and

$$\begin{aligned}&s_{ij}=m_j^2-m_i^2,(j>i),\\ &m_1^2=r_{123456},\;m_2^2=r_{23456},\;m_3^2=r_{3456},\;m_4^2=r_{456},\;m_5^2=r_{56},\;m_6^2=r_6. \qquad (8.19)\end{aligned}$$

A series representation of the integral $I_6^{(d)}$ can be obtained by the same method which was used in deriving the series representation of integrals $I_3^{(d)}$, $I_4^{(d)}$, $I_5^{(d)}$. Expanding the integrand of (8.17) in terms of the variables

$$\begin{aligned}&z_1=\frac{r_{123456}-r_{23456}}{r_{123456}}, \qquad z_2=\frac{r_{123456}-r_{23456}}{r_{123456}}, \qquad z_3=\frac{r_{123456}-r_{23456}}{r_{123456}},\\ &z_4=\frac{r_{123456}-r_{23456}}{r_{123456}}, \qquad z_5=\frac{r_{123456}-r_{23456}}{r_{123456}}, \qquad (8.20)\end{aligned}$$

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assuming that all $|z_j| < 1$ and integrating over $x_1,\ldots,x_5$ term by term, we get

$$
\begin{aligned}
&I_6^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2;\{s_{ij}\})\\
&\quad = \frac{r_{123456}^{\frac{d}{2}-6}}{120}\Gamma\left(6-\frac{d}{2}\right)\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\left(6-\frac{d}{2}\right)_{n_1+n_2+n_3+n_4+n_5}\frac{\left(\frac{5}{2}\right)_{n_1+n_2+n_3+n_4+n_5}}{\left(\frac{7}{2}\right)_{n_1+n_2+n_3+n_4+n_5}}\\
&\quad\quad\times\frac{(2)_{n_2+n_3+n_4+n_5}}{(3)_{n_2+n_3+n_4+n_5}}\frac{\left(\frac{3}{2}\right)_{n_3+n_4+n_5}}{\left(\frac{5}{2}\right)_{n_3+n_4+n_5}}\frac{(1)_{n_4+n_5}}{(2)_{n_4+n_5}}\frac{\left(\frac{1}{2}\right)_{n_5}}{\left(\frac{3}{2}\right)_{n_5}}\frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}\frac{z_3^{n_3}}{n_3!}\frac{z_4^{n_4}}{n_4!}\frac{z_5^{n_5}}{n_5!}. \qquad (8.21)
\end{aligned}
$$

One can see that the summand of this series is very similar to that of the series given in eqs. (5.25), (6.26), (7.16).

9 Functional reduction of integrals with special kinematics

The procedure of functional reduction must be modified if the Gram determinant $g_{12\ldots n} = 0$. In this case, the integral $I_n^{(d)}$ can be reduced [36] to a combination of integrals $I_{n-1}^{(d)}$. The functional reduction can be applied to integrals obtained after such a reduction. Notice that if the lower order Gram determinant $g_{ij..k}$ vanishes, then it means that the Gram determinant $g_{12\ldots n}$ also vanishes [37]. A modification of the functional reduction is needed in the case when some kinematic invariants s_{ij} vanish.

If some $s_{ij} = 0$ ($r_{ij} \to \infty$), then the corresponding last step of the functional reduction must be skipped. There is no further functional reduction of integrals with such values of kinematic invariants. Analytic results for these integrals are simpler than those for integrals depending on general kinematics. We will consider derivation of these results integral by integral.

The integral $I_3^{(d)}$ at $s_{23} = 0$. If, at the last step of the functional reduction, the kinematic invariant of the integral $I_3^{(d)}$, say $s_{23} = 0$, then the application of formula (5.11) should be skipped.

The Feynman parameter representation of this integral reads

$$
\begin{aligned}
&I_3(r_{123},r_2,r_3;\ 0,r_3-r_{123},r_2-r_{123})\\
&\quad = -\Gamma\left(3-\frac{d}{2}\right)\int_0^1\int_0^1 x_1[r_{123}-(r_{123}-r_3)x_1^2-(r_3-r_2)x_1^2x_2]^{\frac{d}{2}-3}dx_1dx_2. \qquad (9.1)
\end{aligned}
$$

Notice a small difference between this expression and the Feynman parameter representation of the integral $I_3^{(d)}$ given in eq. (5.22). The integral (9.1) can be easily evaluated. First, integrating with respect to x_1 and then with respect to x_2 yields

$$
\begin{aligned}
&I_3^{(d)}(r_{123},r_2,r_3;\ 0,r_3-r_{123},r_2-r_{123}) = \frac{-\pi}{2\sin\frac{\pi d}{2}\Gamma\left(\frac{d}{2}-1\right)}\\
&\quad\times\left\{\frac{r_3^{\frac{d}{2}-2}}{r_3-r_{123}}F_1\left(1,1,2-\frac{d}{2},2;\frac{r_3-r_2}{r_3-r_{123}},\frac{r_3-r_2}{r_3}\right)-\frac{r_{123}^{\frac{d}{2}-2}}{r_2-r_3}\ln\frac{r_{123}-r_2}{r_{123}-r_3}\right\}. \qquad (9.2)
\end{aligned}
$$

The result for this integral may be obtained in a slightly different form. Expanding the integrand of (9.1) with respect to the variables

$$z_1 = \frac{r_{123} - r_3}{r_{123}}, \qquad z_2 = \frac{r_3 - r_2}{r_{123}}, \tag{9.3}$$

assuming $|z_1| < 1$, $|z_2| < 1$ and integrating with respect to x_1 and x_2 term by term, we get the series representation

$$\begin{aligned} &I_3^{(d)}(r_{123}, r_2, r_3;\ 0, r_3 - r_{123}, r_2 - r_{123}) \\ &\quad = -\frac{r_{123}^{\frac{d}{2}-3}}{2}\Gamma\left(3 - \frac{d}{2}\right) \sum_{n_1,n_2=0}^{\infty} \left(3 - \frac{d}{2}\right)_{n_1+n_2} \frac{(1)_{n_1+n_2}}{(2)_{n_1+n_2}} \frac{(1)_{n_2}}{(2)_{n_2}} \frac{z_1^{n_1}}{n_1!} \frac{z_2^{n_2}}{n_2!}. \end{aligned} \tag{9.4}$$

Another series representation can be obtained by a slight modification of the above derivation. We expand the integrand with respect to z_1 and integrate over x_1 first. Finally, performing the integration with respect to x_2, we get the result in terms of two generalized hypergeometric series ${}_3F_2$

$$\begin{aligned} &I_3^{(d)}(r_{123}, r_2, r_3;\ 0, r_3 - r_{123}, r_2 - r_{123}) = -\frac{r_{123}^{\frac{d}{2}-3}}{2}\Gamma\left(3 - \frac{d}{2}\right) \\ &\quad \times \left\{ \frac{z_1 + z_2}{z_2}\, {}_3F_2\left[\begin{matrix} 3 - \frac{d}{2}, 1, 1\,; \\ 2, 2\,; \end{matrix}\ z_1 + z_2\right] - \frac{z_1}{z_2}\, {}_3F_2\left[\begin{matrix} 3 - \frac{d}{2}, 1, 1\,; \\ 2, 2\,; \end{matrix}\ z_1\right]\right\}. \end{aligned} \tag{9.5}$$

The analytical expression for the integral can also be obtained by solving dimensional recurrence relation

$$\begin{aligned} &(d-2) I_3^{(d+2)}(r_{123}, r_2, r_3;\ 0, r_3 - r_{123}, r_2 - r_{123}) \\ &\quad = -2r_{123} I_3^{(d)}(r_{123}, r_2, r_3;\ 0, r_3 - r_{123}, r_2 - r_{123}) - I_2^{(d)}(r_2, r_3; 0). \end{aligned} \tag{9.6}$$

Here the integral $I_2^{(d)}$ is a combination of two tadpole integrals. The solution of this dimensional recurrence relation reads

$$\begin{aligned} &I_3^{(d)}(r_{123}, r_2, r_3;\ 0, r_3 - r_{123}, r_2 - r_{123}) \\ &\quad = \frac{\pi r_{123}^{\frac{d}{2}-2}}{(r_2 - r_3)\sin\frac{\pi d}{2}\Gamma\left(\frac{d}{2}\right)} \left\{ \frac{d-2}{4}\ln(r_{123} - r_2) - \frac{d-2}{4}\ln(r_{123} - r_3) \right. \\ &\quad \left. + \frac{1}{2}\left(\frac{r_2}{r_{123}}\right)^{\frac{d}{2}-1} {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}\,; \\ \frac{d}{2}\,; \end{matrix}\ \frac{r_2}{r_{123}}\right] - \frac{1}{2}\left(\frac{r_3}{r_{123}}\right)^{\frac{d}{2}-1} {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}\,; \\ \frac{d}{2}\,; \end{matrix}\ \frac{r_3}{r_{123}}\right] \right\}. \end{aligned} \tag{9.7}$$

An arbitrary function, invariant under $d \to d + 2$, appearing in the solution of eq. (9.6), was obtained by solving the system of differential equations with respect to kinematic variables.

As was shown in ref. [38], different representations of the Feynman integrals can be used to find new relations among hypergeometric functions. In particular, a comparison of eq. (9.5) with eq. (9.7) yields the following relationship:

$${}_3F_2\left[\begin{matrix} 3 - \frac{d}{2}, 1, 1\,; \\ 2, 2\,; \end{matrix}\ z\right] = \frac{2}{z(d-4)} \left\{ \frac{2(1-z)^{\frac{d-2}{2}}}{d-2}\, {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}\,; \\ \frac{d}{2}\,; \end{matrix}\ 1 - z\right] + \ln z + \psi\left(\frac{d}{2} - 1\right) + \gamma \right\}, \tag{9.8}$$

where the function $\psi(x)$ is the logarithmic derivative of the Euler's Γ function, $\psi(x) = d\ \ln\Gamma(x)/dx$, and $\gamma = 0.57721566\ldots$ denotes Euler's or Mascheroni's constant [39].

By comparing eq. (9.2) with eq. (9.7), we get the following relationship:

$$F_1\left(1,1,2-\frac{d}{2},2;x,y\right) = \frac{2}{(d-2)y}\left\{\frac{(1-y)^{\frac{d}{2}-1}}{x-1}\,{}_2F_1\left[\begin{matrix}1,1\,;\, \frac{x(1-y)}{y(1-x)}\\ \frac{d}{2}\,;\end{matrix}\right] + {}_2F_1\left[\begin{matrix}1,1\,;\,\frac{x}{y}\\ \frac{d}{2}\,;\end{matrix}\right]\right\}. \tag{9.9}$$

This formula can be used to evaluate a high-order series expansion in $\varepsilon = (4-d)/2$ of the hypergeometric Appell function F_1. Such an expansion can easily be derived, as expansion of the hypergeometric functions ${}_2F_1$ from eq. (9.9) is known to any order in ε.

The integral $I_4^{(d)}$ at $s_{34} = 0$. The situation concerning the integral $I_4^{(d)}$ is very similar to the case of the integral $I_3^{(d)}$. If, for example, a kinematic variable of the integral $I_4^{(d)}$ in equation (6.9), say $s_{34} = 0$, then the application of relation (6.9) must be skipped.

An analytic result for the integral $I_4^{(d)}$ in this case can also be obtained by solving the dimensional recurrence relation

$$\begin{aligned}
&(d-3)I_4^{(d+2)}(r_{1234}, r_{234}, r_{34}, r_4;\\
&\qquad\qquad r_{234}-r_{1234}, r_{34}-r_{234}, 0, r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234})\\
&\quad = -2r_{1234}I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4;\\
&\qquad\qquad\qquad r_{234}-r_{1234}, r_{34}-r_{234}, 0, r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234})\\
&\qquad - I_3^{(d)}(r_{234}, r_{34}, r_4; 0, r_4-r_{234}, r_{34}-r_{234}).
\end{aligned} \tag{9.10}$$

In order to solve this equation, we used the analytic result (9.7) for the integral $I_3^{(d)}$ and obtained

$$\begin{aligned}
&I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4;\\
&\quad r_{234}-r_{1234}, r_{34}-r_{234}, 0, r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234})\\
&\quad = \frac{r_{1234}^{\frac{d}{2}}\ c_4(r_{1234}, r_{234}, r_{34}, r_4)}{\Gamma\left(\frac{d-3}{2}\right)\sin\frac{\pi d}{2}} - \frac{\pi}{8\sin\frac{\pi d}{2}\Gamma\left(\frac{d}{2}\right)\ r_{1234}(r_{34}-r_4)}\\
&\qquad \times\left\{(d-2)r_{234}^{\frac{d}{2}-2}\left[\ln(r_{234}-r_{34}) - \ln(r_{234}-r_4)\right]{}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}\,;\, \frac{r_{234}}{r_{1234}}\\ \frac{d-2}{2}\,;\end{matrix}\right]\right.\\
&\qquad\quad + \frac{2r_{34}^{\frac{d}{2}-1}}{r_{234}-r_{34}}F_3\left(1,1,1,\frac{d-3}{2},\frac{d}{2};\frac{r_{34}}{r_{34}-r_{234}},\frac{r_{34}}{r_{1234}}\right)\\
&\qquad\quad \left.- \frac{2r_4^{\frac{d}{2}-1}}{r_{234}-r_4}F_3\left(1,1,1,\frac{d-3}{2},\frac{d}{2};\frac{r_4}{r_4-r_{234}},\frac{r_4}{r_{1234}}\right)\right\},
\end{aligned} \tag{9.11}$$

where

$$
\begin{aligned}
&c_4(r_{1234}, r_{234}, r_{34}, r_4) \\
&\quad = \frac{\pi^{\frac{3}{2}}}{4r_{1234}^{\frac{5}{2}}(r_{34}-r_4)(r_{1234}-r_{234})^{\frac{1}{2}}} \Bigg\{ \ln(r_{234}-r_{34}) - \ln(r_{234}-r_4) \\
&\qquad - \ln \frac{\sqrt{r_{1234}-r_{34}} - \sqrt{r_{1234}-r_{234}}}{\sqrt{r_{1234}-r_{34}} + \sqrt{r_{1234}-r_{234}}} + \ln \frac{\sqrt{r_{1234}-r_4} - \sqrt{r_{1234}-r_{234}}}{\sqrt{r_{1234}-r_4} + \sqrt{r_{1234}-r_{234}}} \Bigg\}. \qquad (9.12)
\end{aligned}
$$

The definition of the hypergeometric Appell function F_3 is given in the appendix. The function $c_4(r_{1234}, r_{234}, r_{34}, r_4)$ appeared as an arbitrary periodic function in the solution of the dimensional recurrence relation for the integral $I_4^{(d)}$. This function was obtained by solving the system of differential equations which was derived from the system of differential equations for the integral $I_4^{(d)}$.

The integral $I_4^{(d)}$ can also be evaluated using the Feynman parameter representation. In the case under consideration, the integral representation is slightly different from the representation given in (6.23)

$$
\begin{aligned}
&I_4^{(d)}(r_{1234}, r_{234}, r_3, r_4; r_{234}-r_{1234}, r_3-r_{234}, 0, r_4-r_{1234}, r_4-r_{234}, r_3-r_{1234}) \\
&\quad = \Gamma\left(4-\frac{d}{2}\right) \int_0^1 \int_0^1 \int_0^1 x_1^2 x_2 h_4^{\frac{d}{2}-4} dx_1 dx_2 dx_3, \qquad (9.13)
\end{aligned}
$$

where

$$
h_4 = r_{1234} - (r_{1234}-r_{234})x_1^2 - (r_{234}-r_3)x_1^2x_2^2 - (r_3-r_4)x_1^2x_2^2x_3. \qquad (9.14)
$$

The difference between this h_4 and h_4 from eq. (6.24) is only in the last term. Expanding the integrand of (9.13) with respect to three variables

$$
z_1 = \frac{r_{1234}-r_{234}}{r_{1234}}, \quad z_2 = \frac{r_{234}-r_3}{r_{1234}}, \quad z_3 = \frac{r_3-r_4}{r_{1234}}, \qquad (9.15)
$$

assuming that $|z_j| < 1$, $(j = 1, 2, 3)$, and integrating with respect to x_1, x_2, x_3 term by term, we then obtain

$$
\begin{aligned}
&I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4; r_{234}-r_{1234}, r_{34}-r_{234}, 0, \\
&\quad r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234}) \\
&= \frac{1}{6}\Gamma\left(4-\frac{d}{2}\right) r_{1234}^{\frac{d}{2}-4} \sum_{n_1,n_2,n_3=0}^{\infty} \left(4-\frac{d}{2}\right)_{n_1+n_2+n_3} \frac{\left(\frac{3}{2}\right)_{n_1+n_2+n_3}}{\left(\frac{5}{2}\right)_{n_1+n_2+n_3}} \frac{(1)_{n_2+n_3}}{(2)_{n_2+n_3}} \frac{(1)_{n_3}}{(2)_{n_3}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3!}.
\end{aligned}
$$

Note that the summand of this series is slightly different from the summand of the integral $I_4^{(d)}$ for general kinematics in eq. (6.26).

The integral $I_5^{(d)}$ at $s_{45} = 0$. If one of the kinematic variables of the integral $I_5^{(d)}$, say $s_{45} = 0$, then the application of the reduction relation (7.9) must be skipped.

An analytic result for such an integral can be obtained either by solving the dimensional recurrence relation or by calculating the Feynman parameter integral. Solving the dimensional recurrence relation for the integral $I_5^{(d)}$ is somewhat cumbersome and the result is relatively long. For these reasons, we will not present it here. Instead, we have derived the expression for the integral $I_5^{(d)}$ in terms of multiple hypergeometric series. The Feynman parameter representation of this integral reads

$$\begin{aligned}
&I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_4, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_4-r_{345}, 0, r_5-r_{12345},\\
&\quad r_{345}-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad = -\Gamma\left(5-\frac{d}{2}\right)\int_0^1\int_0^1\int_0^1\int_0^1 x_1^3x_2^2x_3h_5^{\frac{d}{2}-5}dx_1dx_2dx_3dx_4, \qquad (9.16)
\end{aligned}$$

where

$$\begin{aligned}
h_5 &= r_{12345}-(r_{12345}-r_{2345})x_1^2\\
&\quad -(r_{2345}-r_{345})x_1^2x_2^2-(r_{345}-r_4)x_1^2x_2^2x_3^2-(r_4-r_5)x_1^2x_2^2x_3^2x_4. \qquad (9.17)
\end{aligned}$$

Expanding the Feynman parameter integrand with respect to the four variables

$$z_1=\frac{r_{12345}-r_{2345}}{r_{12345}},\quad z_2=\frac{r_{2345}-r_{345}}{r_{12345}},\quad z_3=\frac{r_{345}-r_4}{r_{12345}},\quad z_4=\frac{r_4-r_5}{r_{12345}}, \qquad (9.18)$$

and integrating over $x_1,\ldots,x_4$ term by term then yields

$$\begin{aligned}
&I_5^{(d)}(r_{12345}, r_{2345}, r_{345}, r_4, r_5; r_{2345}-r_{12345}, r_{345}-r_{2345}, r_4-r_{345}, 0, r_5-r_{12345},\\
&\quad r_{345}-r_{12345}, r_4-r_{12345}, r_4-r_{2345}, r_5-r_{2345}, r_5-r_{345})\\
&\quad = -\frac{1}{24}\Gamma\left(5-\frac{d}{2}\right)\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\left(5-\frac{d}{2}\right)_{n_1+n_2+n_3+n_4}\\
&\qquad \times\frac{(2)_{n_1+n_2+n_3+n_4}}{(3)_{n_1+n_2+n_3+n_4}}\frac{\left(\frac{3}{2}\right)_{n_2+n_3+n_4}}{\left(\frac{5}{2}\right)_{n_2+n_3+n_4}}\frac{(1)_{n_3+n_4}}{(2)_{n_3+n_4}}\frac{(1)_{n_4}}{(2)_{n_4}}\frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}\frac{z_3^{n_3}}{n_3!}\frac{z_4^{n_4}}{n_4!}. \qquad (9.19)
\end{aligned}$$

Note a slight difference between this representation and series representation (7.16) of the integral $I_5^{(d)}$ for general kinematics.

The integral $I_6^{(d)}$ at $s_{56}=0$. The integral $I_6^{(d)}$ for the case when one of the kinematic variables, say $s_{56}=0$, should be considered in the same way as the integral $I_5^{(d)}$. The application of the reduction formula (8.13) in the last step must be skipped.

The analytic calculation of the integral can be performed either by solving the dimensional recurrence relation or by evaluating the Feynman parameter integral. The solution of the dimensional recurrence relation is straightforward but cumbersome and the result is relatively long. For this reason the derivation of the solution and the result will not be considered in the present paper.

The parametric representation of the integral reads

$$\begin{aligned}
&I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_5, r_6; r_{23456}-r_{123456}, r_{3456}-r_{23456}, r_{456}-r_{3456},\\
&\quad r_5-r_{456}, 0, r_6-r_{123456}, r_{3456}-r_{123456}, r_5-r_{123456}, r_{456}-r_{23456},\\
&\quad r_{456}-r_{123456}, r_5-r_{23456}, r_6-r_{23456}, r_5-r_{3456}, r_6-r_{3456}, r_6-r_{456})\\
&\quad = \Gamma\left(6-\frac{d}{2}\right)\int_0^1\ldots\int_0^1 x_1^4x_2^3x_3^2x_4h_6^{\frac{d}{2}-6}dx_1dx_2dx_3dx_4dx_5, \qquad (9.20)
\end{aligned}$$

where

$$h_6 = r_{123456} - (r_{123456} - r_{23456})x_1^2 - (r_{23456} - r_{3456})x_1^2x_2^2 \\ - (r_{3456} - r_{456})x_1^2x_2^2x_3^2 - (r_{456} - r_5)x_1^2x_2^2x_3^2x_4^2 - (r_5 - r_6)x_1^2x_2^2x_3^2x_4^2x_5. \tag{9.21}$$

The last term of h_6 here differs from that given in eq. (8.18). Expanding the integrand with respect to the five variables

$$z_1 = \frac{r_{123456} - r_{23456}}{r_{123456}}, \qquad z_2 = \frac{r_{23456} - r_{3456}}{r_{123456}}, \\ z_3 = \frac{r_{3456} - r_{456}}{r_{123456}}, \qquad z_4 = \frac{r_{456} - r_5}{r_{123456}}, \qquad z_5 = \frac{r_5 - r_6}{r_{123456}}, \tag{9.22}$$

and integrating over $x_1,\ldots,x_5$ term by term, we get the multiple series representation

$$I_6^{(d)}(r_{123456}, r_{23456}, r_{3456}, r_{456}, r_5, r_6; r_{23456} - r_{123456}, r_{3456} - r_{23456}, r_{456} - r_{3456}, \\ r_5 - r_{456}, 0, r_6 - r_{123456}, r_{3456} - r_{123456}, r_{456} - r_{123456}, r_5 - r_{123456}, r_{456} - r_{23456}, \\ r_5 - r_{23456}, r_6 - r_{23456}, r_5 - r_{3456}, r_6 - r_{3456}, r_6 - r_{456}) \\ = \frac{r_{123456}^{\frac{d}{2}-6}}{120}\Gamma\left(6 - \frac{d}{2}\right) \sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty} \left(6 - \frac{d}{2}\right)_{n_1+n_2+n_3+n_4+n_5} \frac{\left(\frac{5}{2}\right)_{n_1+n_2+n_3+n_4+n_5}}{\left(\frac{7}{2}\right)_{n_1+n_2+n_3+n_4+n_5}} \\ \times \frac{(2)_{n_2+n_3+n_4+n_5}}{(3)_{n_2+n_3+n_4+n_5}} \frac{\left(\frac{3}{2}\right)_{n_3+n_4+n_5}}{\left(\frac{5}{2}\right)_{n_3+n_4+n_5}} \frac{(1)_{n_4+n_5}}{(2)_{n_4+n_5}} \frac{(1)_{n_5}}{(2)_{n_5}} \frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}\frac{z_3^{n_3}}{n_3!}\frac{z_4^{n_4}}{n_4!}\frac{z_5^{n_5}}{n_5!}. \tag{9.23}$$

Note similarities between summand of this multiple series and the summands of (9.4), (9.16), (9.19).

This concludes our consideration of integrals for special values of kinematic variables.

10 General algorithm of the functional reduction

Comparing expressions (4.11), (5.12), (6.11), (7.5), (7.6), (7.8), (7.9), (8.5), (8.7), (8.9), (8.11), (8.13), it is not hard to see common features and similarities between them. Based on these observations, we have developed a regular algorithm for obtaining final reduction formulae, which is valid for the integrals considered in the article. We assume that the algorithm can be applied to integrals $I_n^{(d)}$ with $n > 6$ as well.

Final functional reduction formulae for the integrals $I_2^{(d)}, \ldots, I_6^{(d)}$ can be obtained by exploiting the following algorithm:

- write down the term

$$\kappa_{1\ldots n}\kappa_{2\ldots n}\cdots\kappa_{n-1\ n} \quad I_n^{(d)}(m_1^2, m_2^2, \ldots m_n^2; s_{12}, s_{23}, \ldots) \tag{10.1}$$

- replace in the integral $s_{ij} \to m_j^2 - m_i^2 \ \ (j > i)$
- replace in the integral $m_1^2 \to r_{1\ldots n}$, $m_2^2 \to r_{2\ldots n}, \ldots, m_n^2 \to r_n$
- replace $\kappa_{ij\ldots} \to \frac{\partial r_{ij\ldots}}{\partial m_i^2}$

- generate $n!-1$ terms by symmetrizing the term (10.1) with respect to the indices $1,2,\ldots n$ and add all these terms to (10.1).

All steps are very straightforward and easily achieved with a computer program. This algorithm works perfectly for the integrals $I_2^{(d)},\ldots,I_6^{(d)}$. We have verified numerically that it is also valid for the integrals $I_7^{(d)}$, $I_8^{(d)}$. Notice that the number of terms in the final reduction formula for massless integrals is $n!/2$.

We found that the parametric representation of the integral $I_n^{(d)}$ depending on the MNV can be written as

$$I_n^{(d)}(m_1^2,\ldots,m_n^2;\{s_{ik}=m_k^2-m_i^2|k>i\})$$
$$=\frac{(-1)^n r_{1\ldots n}^{\frac{d}{2}-n}}{2^{n-1}}\Gamma\left(n-\frac{d}{2}\right)\int_0^1\frac{dt_1}{\sqrt{t_1}}\int_0^{t_1}\frac{dt_2}{\sqrt{t_2}}\cdots\int_0^{t_{n-2}}\frac{h_n^{\frac{d}{2}-n}}{\sqrt{t_{n-1}}}\,dt_{n-1}, \tag{10.2}$$

where h_n is a polynomial linear in the integration variables

$$h_n=1-z_1t_1-z_2t_2-\ldots-z_{n-1}t_{n-1}, \qquad z_i=\frac{m_i^2-m_{i+1}^2}{m_1^2}. \tag{10.3}$$

The parametric representation of integrals depending on the special kinematics considered in section 9 differs from that of (10.2) and reads

$$I_n^{(d)}(m_1^2,\ldots,m_n^2;\{s_{n-1,n}=0;s_{ik}=m_k^2-m_i^2|i<k\})$$
$$=\frac{(-1)^n r_{1\ldots n}^{\frac{d}{2}-n}}{2^{n-2}}\Gamma\left(n-\frac{d}{2}\right)$$
$$\times\int_0^1\frac{dt_1}{\sqrt{t_1}}\int_0^{t_1}\frac{dt_2}{\sqrt{t_2}}\cdots\int_0^{t_{n-4}}\frac{dt_{n-3}}{\sqrt{t_{n-3}}}\int_0^{t_{n-3}}\frac{dt_{n-2}}{t_{n-2}}\int_0^{t_{n-2}}h_n^{\frac{d}{2}-n}\,dt_{n-1}, \tag{10.4}$$

where h_n is given in (10.3). The integration with respect to t_{n-1} can be performed explicitly. As a result of this integration, the integral $I_n^{(d)}$ that depends on $n-1$ variables will be expressed as a difference of two functions, each depending on $n-2$ variables. In section 9 such a representation was derived for the integrals $I_3^{(d)}$, $I_4^{(d)}$.

The multiple series representation of the integral $I_n^{(d)}$ for $n=2,\ldots,6$ was given in the previous sections. The generic form of all these series is

$$T_k(a,\{\beta_j\};\{\gamma_i\};z_1,\ldots,z_k)$$
$$=\sum_{n_1,\ldots,n_k=0}^{\infty}(a)_{n_1+n_2\ldots+n_k}\frac{(\beta_1)_{n_1+n_2\ldots+n_k}}{(\gamma_1)_{n_1+n_2\ldots+n_k}}\frac{(\beta_2)_{n_2+\ldots+n_k}}{(\gamma_2)_{n_2+\ldots+n_k}}\cdots\frac{(\beta_k)_{n_k}}{(\gamma_k)_{n_k}}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_k^{n_k}}{n_k!}. \tag{10.5}$$

This representation holds for integrals depending on general kinematics as well as for integrals depending on the special kinematics considered in section 9. As one can see from the considered examples, the integral $I_n^{(d)}$ depending on general kinematics can be written in terms of the function T_{n-1} with the parameters

$$a=n-\frac{d}{2},\quad \beta_k=\frac{n-k}{2},\quad \gamma_k=\beta_k+1,\quad 1\le k\le n-1. \tag{10.6}$$

The integral $I_n^{(d)}$ depending on the special kinematics can be written in terms of the function T_{n-1} with the parameters

$$
\begin{aligned}
&a = n - \frac{d}{2}, \qquad \beta_{n-1} = 1, \qquad \beta_k = \frac{n-k}{2}, \qquad 1 \le k \le n-2; \\
&\gamma_j = \beta_j + 1, \qquad 1 \le j \le n-1.
\end{aligned} \tag{10.7}
$$

We assume that for $n > 6$ the integrals $I_n^{(d)}$ depending on generic as well as special kinematics can also be expressed in terms of the hypergeometric series given in (10.5) with the parameters β_k, γ_k defined in (10.6), (10.7).

Note that the functions T_1 and T_2 can be identified with the already known hypergeometric functions ${}_2F_1$ and S_1:

$$
\begin{aligned}
T_1(a,\beta_1;\gamma_1;z_1) &= {}_2F_1\left[\begin{matrix} a,\beta_1\,; \\ \gamma_1\,; \end{matrix} z_1\right], \\
T_2(a,\beta_1,\beta_2;\gamma_1,\gamma_2;z_1,z_2) &= S_1(a,\beta_1,\beta_2,\gamma_1,\gamma_2;z_2,z_1).
\end{aligned} \tag{10.8}
$$

The function $T_k(a,\{\beta_i\};\{\gamma_j\};\{z_n\})$ can be considered as a generalization of the hypergeometric functions S_1 and ${}_2F_1$.

At present, there are several publications where the series representations of one-loop integrals were considered. In ref. [40], it was shown that the n-point one-loop integral can be represented by a generalized hypergeometric power series depending on $n(n-1)/2$ variables. In refs. [41, 42], the representation of the general scalar n-point one-loop Feynman integral in terms of the $n(n+1)/2$-fold multiple hypergeometric series was derived by using Mellin-Barnes technique.

We expect that our representation of one-loop integrals in terms of the $(n-1)$-fold hypergeometric series will be useful for the analytic continuations as well as for the ε expansion of one-loop integrals. We also hope that the parametric representations (10.2), (10.4), (A.33) can be of interest in other approaches for evaluating Feynman integrals, for example, for methods based on the intersection theory proposed in ref. [43] or just for direct evaluation of integrals at fixed integer values of the dimension d.

11 Conclusions and outlook

In this paper, we provided a systematic approach for reducing a generic n-point one-loop integral with arbitrary masses and kinematic invariants to a linear combination of integrals that depend on n variables. The integrals depending on the MNV encountered at the last stage of the reduction were expressed in terms of the multiple hypergeometric series depending on $n-1$ dimensionless variables. We have not found functional relations allowing for a further reduction in the number of variables. Probably, some additional relations among integrals depending on the MNV can be obtained for integer values of the space-time dimension d. A new class of identities for Feynman integrals valid at fixed integer value of the dimension d was discovered in ref. [44]. Such identities were derived by using Schouten identities which are valid only for integer d. It will be interesting to investigate the applicability of the method of ref. [44] to our integrals depending on the MNV.

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We have shown that analytic results for integrals with the MNV can be derived by solving dimensional recurrence relations. The explicit expressions for the integrals $I_2^{(d)}$, $I_3^{(d)}$, $I_4^{(d)}$ as solutions of the dimensional recurrence relations were given. Arbitrary periodic functions appearing in the solutions of the dimensional recurrence relations were found by solving systems of differential equations.

The choice of integrals depending on the MNV is not unique. One can find relationships among integrals depending on different minimal sets of variables using our functional relations and rewrite the results in the most preferable set of functions. In section 4, such a relationship was given for the integral $I_2^{(d)}$. Relevant relationship for the integral $I_3^{(d)}$ was presented in ref. [38] and analogous relationships will be given for other integrals in a forthcoming publication.

We expect that our representation of one-loop integrals can be helpful for deriving $\varepsilon = (4-d)/2$ expansion of these integrals. For instance, multiple series (10.5) can be expanded in ε by exploiting the methods proposed in refs. [45, 46] or by solving the system of differential equations for this series. In the latter case, to effectively solve the problem, one should construct an appropriate alphabet. As shown in ref. [47], the alphabet for the one-loop integrals can be expressed in terms of the Gram determinants. We expect that our representation of integrals in terms of multiple hypergeometric series with arguments depending explicitly on the Gram determinants, can be useful for finding a canonical basis used to solve a system of differential equations as well as for finding an alphabet of these integrals.

The new set of hypergeometric series T_k, encountered in computation of integrals depending on the MNV, will be studied in detail in our future publications.

We plan to formulate a systematic procedure based on functional relations that would allow analytic continuation of Feynman integrals to different kinematic domains. As it was discovered in the course of our preliminary investigation (see also [38]), the functional relations can help to find still unknown relationships among hypergeometric functions.

One of our next directions of research will be the derivation and investigation of functional relations and functional reduction of multi-loop integrals. Integrating algebraic relations for products of propagators with loop integrals, one can easily get functional equations for multi-loop integrals. For example, integrating the three term relation (5.2) multiplied by the one-loop vertex type integral depending on k_1, one can get a functional equation for the two-loop pentagon integral. Certainly, the functional reduction of multi-loop integrals will be more complicated. It will include integrals corresponding to diagrams with a different topology but with the same leading Landau singularity.

We also plan to apply the functional reduction method for evaluating the Feynman diagrams required for computing radiative corrections for modern experiments.

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A Useful formulae for kinematic determinants, hypergeometric functions and parametric representation of integrals

A.1 Kinematic determinants

The modified Cayley and the Gram determinants occurring in many formulae of the paper are defined as

$$\Delta_n \equiv \Delta_n(\{p_1, m_1\}, \dots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \tag{A.1}$$

$$Y_{ij} = m_i^2 + m_j^2 - s_{ij}, \tag{A.2}$$

$$G_{n-1} \equiv G_{n-1}(p_1, \dots, p_n) = -2 \begin{vmatrix} S_{11} & S_{12} & \dots & S_{1\ n-1} \\ S_{21} & S_{22} & \dots & S_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1\ 1} & S_{n-1\ 2} & \dots & S_{n-1\ n-1} \end{vmatrix}, \tag{A.3}$$

$$S_{ij} = s_{in} + s_{jn} - s_{ij}, \tag{A.4}$$

where $s_{ij}^2 = (p_i - p_j)^2$, p_i are the combinations of external momenta flowing through the i-th lines, respectively, and m_i is the mass of the i-th line. We will use throughout the article indexed notation for Δ_n and G_{n-1}

$$\begin{aligned} \lambda_{i_1 i_2 \dots i_n} &= \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \dots, \{p_{i_n}, m_{i_n}\}), \\ g_{i_1 i_2 \dots i_n} &= G_{n-1}(p_{i_1}, p_{i_2}, \dots, p_{i_n}). \end{aligned} \tag{A.5}$$

Our results depend on the ratios of $\lambda_{i_1 i_2 \dots i_n}$ and $g_{i_1 i_2 \dots i_n}$ and, therefore, it is convenient to introduce the notation

$$r_{ij\dots k} = -\frac{\lambda_{ij\dots k}}{g_{ij\dots k}}. \tag{A.6}$$

Coefficients in front of the integrals in reduction formulae are expressed in terms of derivatives of $r_{i\dots k}$ with respect to masses. For convenience we use the following shorthand notation:

$$\kappa_{j_r j_1 \dots j_{r-1} j_{r+1} \dots j_n} = \frac{\partial r_{j_1 \dots j_r \dots j_n}}{\partial m_{j_r}^2}. \tag{A.7}$$

The imaginary part of r is rather simple. Using

$$\sum_{j=1}^{n} \partial_j \lambda_{i_1 \dots i_n} = -g_{i_1 \dots i_n} = -G_{n-1}(p_{i_1}, p_{i_2}, \dots, p_{i_n}), \tag{A.8}$$

one shows that to all orders in η

$$\lambda_{i_1 i_2 \dots i_n}(\{m_r^2 - i\eta\}) = \lambda_{i_1 i_2 \dots i_n}(\{m_r^2\}) + i g_{i_1 i_2 \dots i_n}\ \eta, \tag{A.9}$$

and, therefore, the causal η prescription for r is (with the same η for all masses)

$$r_{ij\dots k}\big|_{m_j^2 - i\eta} = r_{ij\dots k}\big|_{m_j^2} - i\eta. \tag{A.10}$$

A.2 Hypergeometric functions for the integrals $I_2^{(d)}$, $I_3^{(d)}$, $I_4^{(d)}$

In this subsection, we provide a collection of formulae related to different hypergeometric functions which were encountered in the derivation of some results of the paper.

A.2.1 Series representation

Series representation of the Appell function F_1 [48]

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{A.11}$$

The Appell function F_3 is defined by [48]

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m y^n}{m!\;n!}. \tag{A.12}$$

The Lauricella-Saran function F_S was introduced in [49, 50] and it is defined by a triple hypergeometric series

$$\begin{aligned}&F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z)\\&\quad=\sum_{r,m,n=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_{m+n}(\beta_1)_r(\beta_2)_m(\beta_3)_n}{(\gamma_1)_{r+m+n}}\frac{x^r y^m z^n}{r!\;m!\;n!}.\end{aligned} \tag{A.13}$$

A relation between the hypergeometric Lauricella-Saran functions F_S and yet another Lauricella-Saran function F_N [51]

$$\begin{aligned}&F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z)\\&\quad=\frac{z^{\beta_2}}{y^{\beta_2}}F_N\left(\beta_2,\alpha_1,\alpha_2,\beta_2+\beta_3,\beta_1,\beta_2+\beta_3;\beta_2+\beta_3,\gamma_1,\gamma_1;1-\frac{z}{y},x,z\right),\end{aligned} \tag{A.14}$$

where

$$\begin{aligned}&F_N(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)\\&\quad=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_n(\alpha_3)_p(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.\end{aligned} \tag{A.15}$$

More relations between the F_S and F_N functions can be found in ref. [51].

The generalized Kampé de Fériet hypergeometric function S_1 in equation (5.27) is defined by a double series

$$S_1\left(\alpha,\alpha',\beta,\gamma,\delta;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\alpha')_{m+n}(\beta)_m}{(\gamma)_{m+n}(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{A.16}$$

The domain of convergence of this series $|x|+|y|<1$. The analytic continuation formula for the function S_1, which was used in the derivation of eq. (5.27) reads [31]

$$\begin{aligned}&S_1(\alpha,\alpha',\beta,\gamma,\delta;x,y)\\&\quad=\frac{\Gamma\left(\alpha'-\alpha\right)\Gamma\left(\gamma\right)}{\Gamma\left(\gamma-\alpha\right)\Gamma\left(\alpha'\right)}(-y)^{-\alpha}F_2\Big(\alpha,\beta,\alpha+1-\gamma,\delta,\alpha+1-\alpha';-\frac{x}{y},\frac{1}{y}\Big)\\&\qquad+\frac{\Gamma\left(\alpha-\alpha'\right)\Gamma\left(\gamma\right)}{\Gamma\left(\gamma-\alpha'\right)\Gamma\left(\alpha\right)}(-y)^{-\alpha'}F_2\Big(\alpha',\beta,\alpha'+1-\gamma,\delta,\alpha'+1-\alpha;-\frac{x}{y},\frac{1}{y}\Big).\end{aligned} \tag{A.17}$$

The function F_3 can be reduced to the function F_1 by means of [39]

$$F_3(\alpha, \alpha', \beta, \beta', \alpha + \alpha'; x, y) = (1-y)^{-\beta'} F_1\left(\alpha, \beta, \beta', \alpha + \alpha'; x, \frac{y}{y-1}\right). \tag{A.18}$$

A similar reduction formula takes place for the Appell function F_1

$$F_1\left(a, b, b', b + b'; w, z\right) = (1-z)^{-a}\, {}_2F_1\left[\begin{matrix} a, b\, ; & \frac{w-z}{1-z} \\ b+b'\, ; & \end{matrix}\right]. \tag{A.19}$$

A.2.2 Integral representations

Euler's integral representation of the hypergeometric Gauss function ${}_2F_1$

$${}_2F_1\left(\alpha, \beta, \gamma, x\right) = \frac{\Gamma\left(\gamma\right)}{\Gamma\left(\beta\right)\Gamma\left(\gamma-\beta\right)} \int_0^1 du\, u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha}. \tag{A.20}$$

$$\beta > 0, \quad \gamma - \beta > 0. \tag{A.21}$$

Euler's integral representation of the Appell function F_1

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma\left(\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma-\alpha\right)} \int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}. \tag{A.22}$$

Euler's integral representation of the Appell function F_3

$$\begin{aligned} F_3\left(\alpha, \alpha', \beta, \beta', \gamma; x, y\right) = &\frac{\Gamma\left(\gamma\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)\Gamma\left(\gamma-\beta-\beta'\right)} \\ &\times \iint_{u\geq 0,\, v\geq 0}^{u+v\leq 1} \frac{du\, dv\, u^{\beta-1} v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}}{(1-ux)^{\alpha}(1-vy)^{\alpha'}}, \end{aligned} \tag{A.23}$$

$$\mathrm{Re}(\beta) > 0, \quad \mathrm{Re}(\beta') > 0, \quad \mathrm{Re}(\gamma - \beta - \beta') > 0. \tag{A.24}$$

An integral representation of the function S_1 [31]

$$\begin{aligned} &S_1\left(\alpha, \alpha', \beta, \gamma, \delta; x, y\right) \\ &= \frac{\Gamma\left(\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma-\alpha\right)} \int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1} F_2(\alpha', \beta, 1, \delta, 1; ux, uy) \\ &= \frac{\Gamma\left(\gamma\right)\Gamma\left(\delta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma-\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\delta-\beta\right)} \\ &\quad \times \int_0^1 du \int_0^1 dv\, u^{\alpha-1} v^{\beta-1}(1-u)^{\gamma-\alpha-1}(1-v)^{\delta-\beta-1}(1-uvx-uy)^{-\alpha'} \end{aligned} \tag{A.25}$$

In the derivation of eq. (6.21) the following integral representation of the Lauricella-Saran function F_S was used [51]:

$$\begin{aligned} &\frac{\Gamma(\alpha_1)\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1)} F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) \\ &\quad = \int_0^1 \frac{t^{\gamma_1-\alpha_1-1}(1-t)^{\alpha_1-1}}{(1-x+tx)^{\beta_1}} F_1(\alpha_2, \beta_2, \beta_3, \gamma_1-\alpha_1; ty, tz)dt. \end{aligned} \tag{A.26}$$

A.3 Differential relations for the Lauricella-Saran function F_S

In order to obtain a system of differential equations for the boundary function $C_4(x,y,z)$ in equation (6.17), we used the following differential relations for the Lauricella-Saran function F_S:

$$
\begin{aligned}
&2(1-x)x(x+y-xy)(x+z-xz)\frac{\partial}{\partial x}F_s^{(d)}(x,y,z)\\
&\quad= x^2(d-2) - x^2\,{}_2F_1\!\left[\begin{matrix}1,\frac{d-3}{2};\\ \frac{d}{2};\end{matrix}\,x\right] + (1-x)xz(d-3)\,{}_2F_1\!\left[\begin{matrix}\frac{1}{2},1;\\ \frac{d}{2};\end{matrix}\,z\right]\\
&\qquad + y(d-4)(xz-x-z)(x-1)F_1\left(1,1,\frac{1}{2},\frac{d}{2};y,z\right)\\
&\qquad + (dxy-dx-dy-3xy+3x+4y)(xz-x-z)(x-1)F_s^{(d)}(x,y,z),\\
&2(1-y)(y-z)(x+y-xy)\frac{\partial}{\partial y}F_s^{(d)}(x,y,z)\\
&\quad= x(1-y)\,{}_2F_1\!\left[\begin{matrix}1,\frac{d-3}{2};\\ \frac{d}{2};\end{matrix}\,x\right] - z(d-3)\,{}_2F_1\!\left[\begin{matrix}\frac{1}{2},1;\\ \frac{d}{2};\end{matrix}\,z\right]\\
&\qquad - (y-1)(2xy-xz-x-2y+z)F_s^{(d)}(x,y,z)\\
&\qquad - (y-z)(d-4)F_1\left(1,1,\frac{1}{2},\frac{d}{2};y,z\right) + y(d-2),\\
&2(1-z)(y-z)(x+z-xz)\frac{\partial}{\partial z}F_s^{(d)}(x,y,z)\\
&\quad= x(z-1)\,{}_2F_1\!\left[\begin{matrix}1,\frac{d-3}{2};\\ \frac{d}{2};\end{matrix}\,x\right] + z(d-3)\,{}_2F_1\!\left[\begin{matrix}1,\frac{1}{2};\\ \frac{d}{2};\end{matrix}\,z\right]\\
&\qquad + (z-1)(xz-x-z)F_s^{(d)}(x,y,z) - z(d-2),
\end{aligned}
\tag{A.27}
$$

where

$$
F_s^{(d)}(x,y,z) = F_S\left(\frac{d-3}{2},1,1,1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};x,y,z\right). \tag{A.28}
$$

These differential relations were derived by using series representation (A.13).

Solving the dimensional recurrence relation for the integral $I_4^{(d)}$, we used the following recurrence relation for the hypergeometric function F_S:

$$
(d-3)xF_s^{(d+2)}(x,y,z) = dF_s^{(d)}(x,y,z) - dF_1\left(1,1,\frac{1}{2},\frac{d}{2};y,z\right). \tag{A.29}
$$

Additionally, we provide here differential relations for the Appell function F_1, which were used to find a system of differential equations for the function $C_4(x,y,z)$

$$
\begin{aligned}
\frac{\partial}{\partial x}F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) &= -\frac{(dx-dy-3x+4y)}{2x(x-y)}F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right)\\
&\quad - \frac{y(d-4)}{2x(x-y)}\,{}_2F_1\!\left[\begin{matrix}\frac{1}{2},\frac{d-3}{2};\\ \frac{d-1}{2};\end{matrix}\,y\right] + \frac{(d-3)\sqrt{1-y}}{2(x-y)(1-x)},\\
\frac{\partial}{\partial y}F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) &= \frac{1}{2(x-y)}F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right)\\
&\quad + \frac{(d-4)}{2(x-y)}\,{}_2F_1\!\left[\begin{matrix}\frac{1}{2},\frac{d-3}{2};\\ \frac{d-1}{2};\end{matrix}\,y\right] - \frac{d-3}{2(x-y)\sqrt{1-y}}
\end{aligned}
\tag{A.30}
$$

In order to obtain these relations, we used the series representation (A.11).

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A.4 Derivation of parametric representation of integrals with MNV

In order to derive a Feynman parametric representation of the integrals depending on the MNV, we used the following parametric formula (see, for example, [52], p.632):

$$\begin{aligned}\frac{1}{D_1 D_2 \ldots D_n} &= \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_{n-1} \\ &\times \frac{\Gamma(n)\;\; x_1^{n-2} x_2^{n-3} \ldots x_{n-2}}{[D_n x_1 \ldots x_{n-1} + D_{n-1} x_1 \ldots x_{n-2}(1-x_{n-1}) + \ldots + D_1(1-x_1)]^n},\end{aligned} \tag{A.31}$$

where D_j are defined in (2.2). Shifting k_1 in order to remove the linear term and integrating over k_1 by means of

$$\int \frac{d^d k_1}{[i\pi^{d/2}]} \frac{1}{(k_1^2 - m_i^2)^\alpha} = (-1)^\alpha \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)(m_i^2)^{\alpha - \frac{d}{2}}}, \tag{A.32}$$

we obtain

$$\begin{aligned}&I_n^{(d)}(m_1^2, \ldots, m_n^2; \{s_{ik} = m_k^2 - m_i^2 | k > i\}) \\ &\quad = (-1)^n \Gamma\left(n - \frac{d}{2}\right) \int_0^1 dx_1 \ldots \int_0^1 dx_{n-1} x_1^{n-2} x_2^{n-3} \ldots x_{n-2}\; h_n^{\frac{d}{2}-n},\end{aligned} \tag{A.33}$$

where h_n are the polynomials of form

$$h_n = m_1^2 - \sum_{j=1}^{n-1} (m_j^2 - m_{j+1}^2) \prod_{k=1}^{j} x_k^2. \tag{A.34}$$

The integral representation (10.2) can be obtained from (A.33) by changing variables. To our knowledge, such representation has not been found so far in the literature.

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