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Coadjoint representation of the BMS group on celestial Riemann surfaces

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Abstract: The coadjoint representation of the BMS group in four dimensions is constructed in a formulation that covers both the sphere and the punctured plane. The structure constants are worked out for different choices of bases. The conserved current algebra of non-radiative asymptotically flat spacetimes is explicitly interpreted in these terms.

Keywords: Classical Theories of Gravity, Differential and Algebraic Geometry, Gauge-gravity correspondence, Space-Time Symmetries

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1 Introduction

The BMS group [1–5] is the symmetry group of four-dimensional asymptotically flat spacetimes at null infinity. The precise form of this group changes when one replaces the celestial sphere by a different Riemann surface [6, 7]. Whereas unitary irreducible representations of the BMS group are directly relevant for the quantum theory [8–10], the coadjoint representation is intimately connected to classical solution space through the momentum map. Unitary irreducible representations come later, after the classification of coadjoint orbits, via geometric quantization.

In this paper, we provide the detailed construction of the coadjoint representation of the BMS group and of the algebra on the celestial sphere and the punctured plane. In the case of the sphere, we explicitly identify the coadjoint representation in the gravitational data of non-radiative spacetimes.

In order to set the stage, we start in section 2 by providing the (corrected) commutation relations of the $\mathfrak{bms}_4$ algebra on the celestial sphere. Since the structure of the BMS group is the same as that of the Poincaré group in the sense that both are semi-direct product groups with an abelian ideal, we recall in section 3 the structure of the coadjoint representation of such groups and algebras [11].

In a next step in section 4, we provide a description of the coadjoint representation of the BMS group in four dimensions in terms of suitably weighted functions on a two-dimensional surface by focussing on local aspects. For the presentation, after some generic considerations based on [12, 13], we will use a Weyl covariant derivative instead of the standard "eth" operator [5, 14, 15]. Note that our conventions for these derivatives differ somewhat from those originally introduced in [16–18] for related reasons. The description applies both to the "global" and "local" versions of the algebra [19–22], which are studied explicitly in sections 5 and 6, respectively. In the former section, we use extensively results of sections 4.14 and 4.15 of [17]. Besides the standard choice of rotation and boost generators as used originally in [3], we also provide explicit commutation relations adapted to the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ decomposition of the Lorentz algebra. In the latter section, our expansions follow the standard conventions used in the context of two-dimensional conformal field theories.

In section 7, we briefly comment the case of the cylinder. Finally, in the case of the sphere and for non-radiative asymptotically flat spacetimes, we explicitly construct the equivariant map from the free gravitational data at $\mathscr{I}^+$ to the coadjoint representation in section 8.

The coadjoint representation of the BMS group in three dimensions [23] (see also [24, 25]) has been investigated in [26]. In that case, the abelian factor can be identified with the Lie algebra of the non abelian factor acted upon by the adjoint representation, which simplifies the classification of coadjoint orbits considerably. Furthermore, central extensions are the familiar ones directly related to the Virasoro group and algebra. Neither of these simplifications occur in four dimensions. As a consequence, we will not discuss the classification of coadjoint orbits in this paper. Also, central extensions that are relevant in the gravitational context are of a different nature [27], and will not be considered here.

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2 Poincaré and BMS algebras on the celestial sphere

The structure constants of the $\mathfrak{bms}_4$ algebra on the celestial sphere have been worked out in [3] (see also [28] for corrections, and [29, 30] for reviews). More details on the geometric interpretation can be found in [5, 15] (see also [17, 18, 31–33]). In this section, we start by providing the standard commutation relations of the $\mathfrak{bms}_4$ algebra in terms of rotation and boost generators.

Let x^a, $a = 0, \dots 3$, be Cartesian coordinates on Minkowski spacetime where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ and its inverse η^{ab} are used to lower and raise indices. The starting point is the Poincaré algebra with generators

$$L^{ab} = L^{[ab]} = -\left(x^a \frac{\partial}{\partial x_b} - x^b \frac{\partial}{\partial x_a}\right), \quad P^a = \frac{\partial}{\partial x_a}, \tag{2.1}$$

satisfying

$$\begin{aligned} [L^{ab}, L^{cd}] &= -(\eta^{bc} L^{ad} - \eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc}), \\ [P^a, L^{bc}] &= -(\eta^{ab} P^c - \eta^{ac} P^b). \end{aligned} \tag{2.2}$$

When splitting into suitable combinations of rotation and boost generators and of translation generators,

$$\begin{aligned} &L_z = L^{12}, \quad L^{\pm} = \pm i L^{23} + L^{13}, \quad K_z = L^{30}, \qquad K^{\pm} = \mp i L^{20} - L^{10}, \\ &H = P^0, \quad P_z = -\frac{1}{2} P^3, \qquad P^{\pm} = \frac{1}{2}(i P^2 \pm P^1), \end{aligned} \tag{2.3}$$

the non-vanishing commutation relations of the Poincaré algebra become

$$\begin{aligned} &[L^+, L^-] = 2iL_z, && [L_z, L^{\pm}] = \pm i L^{\pm}, \quad [K^+, K^-] = -2iL_z, \\ &[K_z, K^{\pm}] = L^{\pm}, && [L^+, K^-] = 2K_z, \quad [L^-, K^+] = 2K_z, \\ &[L_z, K^{\pm}] = \pm i K^{\pm}, && [L^{\pm}, K_z] = -K^{\pm}, \\ &[K_z, H] = 2P_z, && [K^{\pm}, H] = \pm 2P^{\pm}, \quad [L^{\pm}, P_z] = \mp P^{\pm}, \quad [K_z, P_z] = \frac{1}{2} H, \\ &[L_z, P^{\pm}] = \pm i P^{\pm}, && [L^+, P^-] = -2P_z = -[L^-, P^+], \\ &[K^+, P^-] = -H = -[K^-, P^+]. \end{aligned} \tag{2.4}$$

In terms of spherical coordinates and a retarded time coordinate,

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \quad u = x^0 - r, \quad r\cos\theta = x^3, \quad r\sin\theta e^{i\phi} = x^1 + i x^2, \tag{2.5}$$

the Poincaré generators read

$$
\begin{aligned}
L_z &= \partial_\phi, \\
L^\pm &= -e^{\pm i\phi}[\partial_\theta \pm i\cot\theta\partial_\phi], \\
K_z &= -\left(1+\frac{u}{r}\right)\cos\theta(r\partial_r) + \cos\theta(u\partial_u) + \left(1+\frac{u}{r}\right)\sin\theta\partial_\theta, \\
K^\pm &= e^{\pm i\phi}\left[\left(1+\frac{u}{r}\right)\sin\theta(r\partial_r) - \sin\theta(u\partial_u) + \left(1+\frac{u}{r}\right)\cos\theta\partial_\theta \pm \left(1+\frac{u}{r}\right)\frac{i}{\sin\theta}\partial_\phi\right], \\
H &= \partial_u, \\
-2P_z &= \cos\theta(-\partial_r+\partial_u) + \frac{1}{r}\sin\theta\partial_\theta, \\
\pm 2P^\pm &= e^{\pm i\phi}\left[\sin\theta(\partial_r-\partial_u) + \frac{1}{r}\cos\theta\partial_\theta \pm \frac{1}{r\sin\theta}\partial_\phi\right].
\end{aligned}
\tag{2.6}
$$

As may be shown on general grounds or explicitly checked, the Poincaré algebra in the form (2.4) may also be represented in terms of these generators restricted to the surface $r = cte \to \infty$,

$$
\begin{aligned}
&L_z = \partial_\phi, \qquad && L^\pm = -e^{\pm i\phi}[\partial_\theta \pm i\cot\theta\partial_\phi], \\
&K_z = \cos\theta(u\partial_u) + \sin\theta\partial_\theta, \qquad && K^\pm = e^{\pm i\phi}\left[-\sin\theta(u\partial_u) + \cos\theta\partial_\theta \pm \frac{i}{\sin\theta}\partial_\phi\right], \\
&H = \partial_u, \quad -2P_z = \cos\theta\partial_u, \quad \pm 2P^\pm = -e^{\pm i\phi}\sin\theta\partial_u.
\end{aligned}
\tag{2.7}
$$

The next step is to represent the Poincaré algebra at $u = 0$. This can be done by simply restricting the Lorentz generators to that surface, and by representing the translation generators by suitable functions on that surface,

$$
\begin{aligned}
&L_z = \partial_\phi, \qquad L^\pm = -e^{\pm i\phi}[\partial_\theta \pm i\cot\theta\partial_\phi], \\
&K_z = \sin\theta\partial_\theta, \qquad K^\pm = e^{\pm i\phi}\left[\cos\theta\partial_\theta \pm \frac{i}{\sin\theta}\partial_\phi\right], \\
&H = 1 = {}_0Z_{0,0}, \quad P_z = -\frac{1}{2}\cos\theta = {}_0Z_{1,0}, \quad P^\pm = \mp\frac{1}{2}e^{\pm i\phi}\sin\theta = {}_0Z_{1,\pm 1},
\end{aligned}
\tag{2.8}
$$

while in addition, defining for any function f on the sphere,

$$
\begin{aligned}
&[L_z, f] = L_z(f), \qquad && [L^\pm, f] = L^\pm(f), \\
&[K_z, f] = K_z(f) - \cos\theta f, \quad && [K^\pm, f] = K^\pm(f) + e^{\pm i\phi}\sin\theta f.
\end{aligned}
\tag{2.9}
$$

When applied to the four functions in the last line of (2.8), this reproduces the commutation relations of the Lorentz with the translation generators in the Poincaré algebra, i.e., the last two lines of (2.4). The general expression for the unnormalized spherical harmonics ${}_sZ_{j,m}$ are explicitly given in appendix A.

How the Poincaré algebra is enhanced to the $\mathfrak{bms}_4$ algebra in the context of asymptotically flat spacetimes at null infinity is discussed in the references at the beginning of this section. Besides the original reference [3], we also refer to the re-derivation in [20] for more details.

In the $\mathfrak{bms}_4$ algebra, the commutation relations for the Lorentz sub-algebra are unchanged and given in the first three lines of (2.4). The commutation relations for the $\mathfrak{bms}_4$ algebra are then completed by choosing a basis for functions on the sphere. Whatever basis is chosen, the supertranslation generators $\mathcal{T}^G_A$ commute,

$$[\mathcal{T}^G_A, \mathcal{T}^G_{A'}] = 0. \tag{2.10}$$

To get explicit structure constants for the commutators involving Lorentz and supertranslation generators, one may start with an unnormalized basis involving associated Legendre functions,

$$\mathcal{T}^U_{j,m} = P^m_j(\cos\theta)e^{im\phi}, \quad P^m_j(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_j(x). \tag{2.11}$$

The action of the Lorentz generators on the $\mathcal{T}^U_{j,m}$ is then worked out according to (2.9) by using suitable properties of associated Legendre functions (see e.g. 8.733 and 8.735 of [34]),

$$[L_z, \mathcal{T}^U_{j,m}] = im\mathcal{T}^U_{j,m}, \tag{2.12}$$

$$[L^+, \mathcal{T}^U_{j,m}] = -\mathcal{T}^U_{j,m+1}, \tag{2.13}$$

$$[L^-, \mathcal{T}^U_{j,m}] = (j-m+1)(j+m)\mathcal{T}^U_{j,m-1}, \tag{2.14}$$

$$[K_z, \mathcal{T}^U_{j,m}] = \frac{(j-1)(j-m+1)}{2j+1}\mathcal{T}^U_{j+1,m} - \frac{(j+2)(j+m)}{2j+1}\mathcal{T}^U_{j-1,m}, \tag{2.15}$$

$$[K^+, \mathcal{T}^U_{j,m}] = \frac{j-1}{2j+1}\mathcal{T}^U_{j+1,m+1} + \frac{j+2}{2j+1}\mathcal{T}^U_{j-1,m+1}, \tag{2.16}$$

$$\begin{aligned}[K^-, \mathcal{T}^U_{j,m}] = &-\frac{(j-1)(j-m+1)(j-m+2)}{2j+1}\mathcal{T}^U_{j+1,m-1} \\ &- \frac{(j+2)(j+m)(j+m-1)}{2j+1}\mathcal{T}^U_{j-1,m-1}.\end{aligned} \tag{2.17}$$

For a normalized basis in terms of standard spherical harmonics,

$$\mathcal{T}^S_{j,m} = {}_0Y_{j,m} = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P^m_j(\cos\theta)e^{im\phi}, \tag{2.18}$$

one finds instead

$$[L_z, \mathcal{T}^S_{j,m}] = im\mathcal{T}^S_{j,m}, \tag{2.19}$$

$$[L^\pm, \mathcal{T}^S_{j,m}] = \mp\sqrt{(j\mp m)(j\pm m+1)}\mathcal{T}^S_{j,m\pm1}, \tag{2.20}$$

$$[K_z, \mathcal{T}^S_{j,m}] = (j-1)\sqrt{\frac{(j+m+1)(j-m+1)}{(2j+1)(2j+3)}}\mathcal{T}^S_{j+1,m} \tag{2.21}$$

$$- (j+2)\sqrt{\frac{(j+m)(j-m)}{(2j-1)(2j+1)}}\mathcal{T}^S_{j-1,m}, \tag{2.22}$$

$$\begin{aligned}[K^\pm, \mathcal{T}^S_{j,m}] = &\pm(j-1)\sqrt{\frac{(j\pm m+2)(j\pm m+1)}{(2j+1)(2j+3)}}\mathcal{T}^S_{j+1,m\pm1} \\ &\pm (j+2)\sqrt{\frac{(j\mp m)(j\mp m-1)}{(2j-1)(2j+1)}}\mathcal{T}^S_{j-1,m\pm1}.\end{aligned} \tag{2.23}$$

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This is the form under which the commutation relations between Lorentz and supertranslation generators usually appear in the literature (with due care devoted to various conventions and correction of misprints).

3 Coadjoint representations of semi-direct product groups and algebras

For a semi-direct product group of the form $G \ltimes_\sigma A$, with G a Lie group and A an abelian Lie group seen as a vector space with the addition, the group law is given by

$$(f,\alpha)\cdot(g,\beta) = (f\cdot g, \alpha + \sigma_f(\beta)), \tag{3.1}$$

while σ is a representation of G on A. The associated Lie algebra is $\mathfrak{g} \oplus_\Sigma A$, the Lie algebra of A being identified with A itself, and

$$[(X,\alpha),(Y,\beta)] = ([X,Y], \Sigma_X\beta - \Sigma_Y\alpha), \tag{3.2}$$

where Σ is the differential of σ. The adjoint actions of the group and the algebra are then given by

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X, \sigma_f\beta - \Sigma_{\mathrm{Ad}_f X}\alpha), \tag{3.3}$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y], \Sigma_X\beta - \Sigma_Y\alpha). \tag{3.4}$$

The dual space to the Lie algebra is given by $\mathfrak{g}^* \oplus A^*$, with non-degenerate pairing denoted by

$$\langle (j,p),(X,\alpha)\rangle = \langle j, X\rangle + \langle p,\alpha\rangle, \tag{3.5}$$

and coadjoint actions

$$\langle \mathrm{Ad}^*_{(f,\alpha)}(j,p),(Y,\beta)\rangle = \langle (j,p), \mathrm{Ad}_{(f,\alpha)^{-1}}(Y,\beta)\rangle, \tag{3.6}$$

$$\langle \mathrm{ad}^*_{(X,\alpha)}(j,p),(Y,\beta)\rangle = \langle (j,p), -\mathrm{ad}_{(X,\alpha)}(Y,\beta)\rangle. \tag{3.7}$$

Defining $\times : A \oplus A^* \to \mathfrak{g}^*$ by

$$\langle \alpha \times p, X\rangle = \langle p, \Sigma_X \alpha\rangle, \tag{3.8}$$

and σ^* to be the dual representation associated with σ, $\sigma^* : G \times A^* \to A^*$,

$$\langle \sigma^*_f p, \alpha\rangle = \langle p, \sigma_{f^{-1}}\alpha\rangle, \tag{3.9}$$

the coadjoint representations are given by

$$\mathrm{Ad}^*_{(f,\alpha)}(j,p) = (\mathrm{Ad}^*_f j + \alpha \times \sigma^*_f p, \sigma^*_f p), \tag{3.10}$$

$$\mathrm{ad}^*_{(X,\alpha)}(j,p) = (\mathrm{ad}^*_X j + \alpha \times p, \Sigma^*_X p). \tag{3.11}$$

In terms of generators, (e_A, e_α) of $\mathfrak{g} \oplus_\Sigma A$, with (e^A_*, e^α_*) the associated dual basis of $\mathfrak{g}^* \oplus A^*$,

$$[e_A, e_B] = f^C_{AB} e_C, \quad [e_A, e_\alpha] = f^\beta_{A\alpha} e_\beta, \quad [e_\alpha, e_\beta] = 0, \tag{3.12}$$

the coadjoint representation of the algebra (3.11) becomes

$$\mathrm{ad}^*_{e_A} e^B_* = -f^B_{AC} e^C_*, \quad \mathrm{ad}^*_{e_\alpha} e^B_* = 0, \quad \mathrm{ad}^*_{e_A} e^\beta_* = -f^\beta_{A\gamma} e^\gamma_*, \quad \mathrm{ad}^*_{e_\alpha} e^\beta_* = -f^\beta_{\alpha C} e^C_*. \tag{3.13}$$

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4 General structure of the coadjoint representation of BMS4

4.1 Background structure

4.1.1 Extended conformal transformations

Consider an n-dimensional Riemannian manifold with coordinates x^α and metric $g_{\alpha\beta}(x)$ which transforms under invertible coordinate transformations $x'^\alpha = x'^\alpha(x)$ as

$$g'_{\gamma\delta}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta}. \tag{4.1}$$

Conformal coordinate transformations are such invertible coordinate transformations for which

$$g'_{\gamma\delta}(x') = g_{\gamma\delta}(x')\Omega^2(x'). \tag{4.2}$$

Consider then a two-dimensional surface $\mathcal{S}$ with coordinates $x^\alpha = (\xi, \bar{\xi})$ and a conformally flat metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -2(P\bar{P})^{-1}d\xi d\bar{\xi}, \tag{4.3}$$

for some nowhere vanishing $P(x)$. In this case, coordinate transformations of the form

$$\xi' = \xi'(\xi), \quad \bar{\xi}' = \bar{\xi}'(\bar{\xi}) \tag{4.4}$$

are conformal coordinate transformations with

$$\Omega(x') = \left[\frac{(P\bar{P})(x')}{(P\bar{P})(x)}J\right]^{\frac{1}{2}}, \quad J = \frac{\partial\xi}{\partial\xi'}\frac{\partial\bar{\xi}}{\partial\bar{\xi}'}. \tag{4.5}$$

For such conformal transformations, the transformation law

$$P'(x') = P(x)\frac{\partial\xi'}{\partial\xi}, \quad \bar{P}'(x') = \bar{P}(x)\frac{\partial\bar{\xi}'}{\partial\bar{\xi}}, \tag{4.6}$$

induces the transformation (4.2) of the metric components. The more general transformation law

$$P'(x') = P(x)\frac{\partial\xi'}{\partial\xi}e^{-E(x')}, \quad \bar{P}'(x') = \bar{P}(x)\frac{\partial\bar{\xi}'}{\partial\bar{\xi}}e^{-\bar{E}(x')}, \tag{4.7}$$

with E a complex scalar field can be understood as follows. Writing the metric with suitable zweibeins as,

$$ds^2 = e^A{}_\alpha dx^\alpha \eta_{AB} e^B{}_\beta dx^\beta, \quad \eta_{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad e_1{}^\alpha\partial_\alpha = P\partial, \quad e_2{}^\alpha\partial_\alpha = \bar{P}\bar{\partial}, \tag{4.8}$$

they correspond to the transformations of the zweibeins under conformal coordinate transformations. At the same time, the imaginary part iE_I produces a local rotation of the zweibeins while the real part of E_R generates the Weyl rescaling of the metric. The associated conformal factor is

$$\Omega(x') = \left[\frac{(P\bar{P})(x')}{(P\bar{P})(x)}J\right]^{\frac{1}{2}} e^{E_R(x')}. \tag{4.9}$$

Three relevant subclasses of the extended transformations (4.7) are

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(i) Conformal coordinate transformation. taking $E = 0 = \bar{E}$ in (4.7) leads back to (4.6), and thus, when applied to the metric components, to the same transformations (4.2) as those coming from the conformal coordinate transformation applied to the metric tensor.

(ii) Complex Weyl rescaling. taking $\xi' = \xi$, $\bar{\xi}' = \bar{\xi}$ in (4.7) gives

$$P'(x) = P(x)e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x)e^{-\bar{E}(x)}. \tag{4.10}$$

This induces a real local Weyl rescaling on the metric,

$$g'_{\alpha\beta}(x) = e^{2E_R} g_{\alpha\beta}(x). \tag{4.11}$$

When the real part $E_R = 0$, the metric is unchanged.

(iii) Fixed conformal factor. fixing as in [5, 15] the conformal factor to be a prescribed function of its arguments, $P(x) = P_F(x)$, $\bar{P}(x) = \bar{P}_F(x)$ and $P'(x') = P_F(x')$, $\bar{P}'(x') = \bar{P}_F(x')$ in (4.7), implies that complex Weyl rescalings are frozen to

$$e^{E(x')} = \frac{P_F(x)}{P_F(x')}\frac{\partial\xi'}{\partial\xi} \Longleftrightarrow \begin{cases} e^{E_R(x')} = J^{-\frac{1}{2}}\left[\frac{(P_F\bar{P}_F)(x)}{(P_F\bar{P}_F)(x')}\right]^{\frac{1}{2}}, \\ e^{iE_I(x')} = \left[\frac{(P_F/\bar{P}_F)(x)}{(P_F/\bar{P}_F)(x')}\left(\frac{\partial\xi'/\partial\xi}{\partial\bar{\xi}'/\partial\bar{\xi}}\right)\right]^{\frac{1}{2}}, \end{cases} \tag{4.12}$$

where $E = E_R + iE_I$. We will mostly be interested in 2 particular cases below. The first is when $\mathcal{S}$ is a 2-sphere of radius R with metric $ds^2 = -R^2(d\theta^2 + \sin^2\theta d\phi^2)$ and $\xi = \bar{\zeta} = \cot\frac{\theta}{2}e^{-i\phi}$, so that

$$P_S = \frac{1+\xi\bar{\xi}}{R\sqrt{2}} = \bar{P}_S, \tag{4.13}$$

see [17] section 4.15 for details.

The second is the punctured complex plane, the complex plane with the origin removed $\mathbb{C}_0 = \mathbb{C} - \{0\}$, with standard metric $ds^2 = -2dzd\bar{z}$ so that

$$P = 1 = \bar{P}. \tag{4.14}$$

4.1.2 Conformal fields and weighted scalars

Under conformal coordinate transformations and complex Weyl rescalings, fields $\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}$ of conformal dimensions $(h, \bar{h})$ and Weyl weights $(\lambda, \bar{\lambda})$ transform as

$$\phi'^{\lambda,\bar{\lambda}}_{h,\bar{h}}(x') = e^{\lambda E(x')}e^{\bar{\lambda}\bar{E}(x')}\left(\frac{\partial\xi}{\partial\xi'}\right)^h\left(\frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\right)^{\bar{h}}\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}(x). \tag{4.15}$$

It follows from (4.7) that the conformal dimensions and Weyl weights of P are both $(-1, 0)$ whereas those of $\bar{P}$ are both $(0, -1)$. These quantities can be used to map the fields $\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}$ into scalars $\eta^{s,w}$

$$\eta^{s,w} = P^h\bar{P}^{\bar{h}}\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}, \tag{4.16}$$

of spin and conformal weights $[s, w]$,

$$s = (h - \bar{h}) - (\lambda - \bar{\lambda}), \quad w = -(h + \bar{h}) + (\lambda + \bar{\lambda}), \tag{4.17}$$

which transform under conformal coordinate transformations and complex Weyl rescalings as

$$\eta'^{s,w}(x') = e^{wE_R(x')} e^{-isE_I(x')} \eta^{s,w}(x). \tag{4.18}$$

4.1.3 Derivative operators

The only non-vanishing components of the Levi-Civita connection associated with (4.3) are

$$\Gamma^{\xi}_{\xi\xi} = -\partial \ln(P\bar{P}), \quad \Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} = -\bar{\partial} \ln(P\bar{P}), \tag{4.19}$$

where $\partial = \partial_\xi, \bar{\partial} = \partial_{\bar{\xi}}$. Equation (4.7) induces their transformation law under conformal coordinate transformations combined with Weyl rescalings,

$$\Gamma'^{\xi'}_{\xi'\xi'}(x') = \Gamma^{\xi}_{\xi\xi}(x) \frac{\partial \xi}{\partial \xi'} + \frac{\partial \xi'}{\partial \xi} \frac{\partial^2 \xi}{\partial \xi' \partial \xi'} + 2\partial' E_R(x'), \tag{4.20}$$

with a similar transformation law for $\Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}}$. In addition to the conformally flat metric (4.3), one supposes that $\mathcal{S}$ is endowed with a Weyl connection $(W, \bar{W})$ (see e.g. [12]) that transforms as

$$W'(x') = \left(\frac{\partial \xi}{\partial \xi'}\right) W(x) + 2\partial' E_R(x'), \quad \bar{W}'(x') = \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right) \bar{W}(x) + 2\bar{\partial}' E_R(x'). \tag{4.21}$$

Using P, $\bar{P}$, W, $\bar{W}$, one can define

$$\begin{aligned} K &= \frac{1}{2}(\partial \ln \mu - \Gamma^{\xi}_{\xi\xi}) + W = \partial \ln \bar{P} + W, \quad O = \frac{1}{2}(\Gamma^{\xi}_{\xi\xi} - \partial \ln \mu) = -\partial \ln \bar{P} \\ \bar{K} &= \frac{1}{2}(\bar{\partial} \ln \bar{\mu} - \Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}}) + \bar{W} = \bar{\partial} \ln P + \bar{W}, \quad \bar{O} = \frac{1}{2}(\Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} - \bar{\partial} \ln \bar{\mu}) = -\bar{\partial} \ln P, \end{aligned} \tag{4.22}$$

where $\mu = \frac{\bar{P}}{P}$ is a Beltrami differential. These objects transform as

$$\begin{aligned} K'(x') &= \left(\frac{\partial \xi}{\partial \xi'}\right) K(x) + \partial' E(x'), \quad O'(x') = \left(\frac{\partial \xi}{\partial \xi'}\right) O(x) + \partial' \bar{E}(x'), \\ \bar{K}'(x') &= \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right) \bar{K}(x) + \bar{\partial}' \bar{E}(x'), \quad \bar{O}'(x') = \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right) \bar{O}(x) + \bar{\partial}' E(x'). \end{aligned} \tag{4.23}$$

The Weyl covariant derivative can then be defined as

$$\begin{aligned} D\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}} &= [\nabla + (h - \lambda)K + (h - \bar{\lambda})O]\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}, \\ \bar{D}\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}} &= [\bar{\nabla} + (\bar{h} - \bar{\lambda})\bar{K} + (\bar{h} - \lambda)\bar{O}]\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}, \end{aligned} \tag{4.24}$$

where $\nabla \equiv \nabla_\xi$ and $\bar{\nabla} \equiv \nabla_{\bar{\xi}}$ are the components of the covariant derivative associated to the Levi-Civita connection (4.19). Notice that the field P $(\bar{P})$ of Weyl weights $(\lambda, \bar{\lambda}) = (-1, 0)$ (resp. $(\lambda, \bar{\lambda}) = (0, -1)$) and conformal dimensions $(h, \bar{h}) = (-1, 0)$ (resp. $(h, \bar{h}) = (0, -1)$)

are holomorphic (resp. anti-holomorphic) with respect to the Weyl covariant derivative, namely $\bar{D}P = 0$ (resp. $D\bar{P} = 0$). Under conformal coordinate transformations and complex Weyl rescalings, we have

$$\begin{aligned}(D\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}})'(x') &= e^{\lambda E(x')}e^{\bar{\lambda}\bar{E}(x')}\left(\frac{\partial\xi}{\partial\xi'}\right)^{h+1}\left(\frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\right)^{\bar{h}}(D\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}})(x),\\ (D\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}})'(x') &= e^{\lambda E(x')}e^{\bar{\lambda}\bar{E}(x')}\left(\frac{\partial\xi}{\partial\xi'}\right)^{h}\left(\frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\right)^{\bar{h}+1}(\bar{D}\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}})(x).\end{aligned} \tag{4.25}$$

Therefore, the operator D ($\bar{D}$) acts on fields of Weyl weights $(\lambda, \bar{\lambda})$ and conformal dimensions $(h, \bar{h})$ to produce fields of Weyl weights $(\lambda, \bar{\lambda})$ and conformal dimensions $(h + 1, \bar{h})$ (resp. $(h, \bar{h} + 1)$).

In the following we will assume that the only fields carrying non-vanishing Weyl weights are $P, \bar{P}$. All other fields are thus of the form $\phi^{0,0}_{h,\bar{h}}$ with associated scalars $\eta^{s,w} = P^h\bar{P}^{\bar{h}}\phi^{0,0}_{h,\bar{h}}$ so that

$$s = h - \bar{h}, \quad w = -(h + \bar{h}), \quad h = \frac{s - w}{2}, \quad \bar{h} = -\frac{s + w}{2}. \tag{4.26}$$

If

$$\eth\eta^{s,w} = P^{h+1}\bar{P}^{\bar{h}}(\nabla\phi^{0,0}_{h,\bar{h}}), \quad \bar{\eth}\eta^{s,w} = P^h\bar{P}^{\bar{h}+1}(\bar{\nabla}\phi^{0,0}_{h,\bar{h}}), \tag{4.27}$$

then

$$\begin{aligned}\eth\eta^{s,w} &= P\bar{P}^{-s}\partial(\bar{P}^s\eta^{s,w}) = P(\partial - sO)\eta^{s,w},\\ \bar{\eth}\eta^{s,w} &= \bar{P}P^s\bar{\partial}(P^{-s}\eta^{s,w}) = \bar{P}(\bar{\partial} + s\bar{O})\eta^{s,w}.\end{aligned} \tag{4.28}$$

in agreement with expressions (4.14.34) and (4.14.33) of [17]. Under conformal coordinate transformations and complex Weyl rescalings,

$$\begin{aligned}(\eth\eta^{s,w})'(x') &= e^{(w-1)E_R(x')}e^{-i(s+1)E_I(x')}[\eth + (w - s)P\partial E_R(x'(x))]\eta^{s,w}(x),\\ (\bar{\eth}\eta^{s,w})'(x') &= e^{(w-1)E_R(x')}e^{-i(s-1)E_I(x')}[\bar{\eth} + (w + s)\bar{P}\bar{\partial}E_R(x'(x))]\eta^{s,w}(x).\end{aligned} \tag{4.29}$$

Hence, the scalars $\eth\eta^{s,w}$ and $\bar{\eth}\eta^{s,w}$ transform as scalars of weights $[s+1, w-1]$ respectively $[s - 1, w - 1]$ only if $w = s \iff h = 0$ respectively $w = -s \iff \bar{h} = 0$. Alternatively, one may limit oneself to complex Weyl rescalings with $E_R = 0$ so that only rotations of the zweibeins are allowed, with no Weyl rescaling of the metric. In this case, only spin weight s is relevant.

When using the Weyl covariant derivative D instead of the covariant derivative ∇ associated to the Christoffel connection, this issue does not arise. Denoting $\mathcal{D}\eta^{s,w}$ and $\bar{\mathcal{D}}\eta^{s,w}$ the images under the mapping (4.16) of $D\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}$ and of $\bar{D}\phi^{\lambda,\bar{\lambda}}_{h,\bar{h}}$, respectively, we have

$$\begin{aligned}\mathcal{D}\eta^{s,w} &= \left[\eth + \left(\frac{s - w}{2}\right)(\mathcal{O} + \mathcal{K})\right]\eta^{s,w} = P\left[\partial - \frac{s}{2}(O - K) - \frac{w}{2}(O + K)\right]\eta^{s,w},\\ \bar{\mathcal{D}}\eta^{s,w} &= \left[\bar{\eth} - \left(\frac{w + s}{2}\right)(\bar{\mathcal{O}} + \bar{\mathcal{K}})\right]\eta^{s,w} = \bar{P}\left[\bar{\partial} + \frac{s}{2}(\bar{O} - \bar{K}) - \frac{w}{2}(\bar{O} + \bar{K})\right]\eta^{s,w},\end{aligned} \tag{4.30}$$

where $\mathcal{O} = PO$, $\mathcal{K} = PK$, $\bar{\mathcal{O}} = \bar{P}\bar{O}$, $\bar{\mathcal{K}} = \bar{P}\bar{K}$. Under conformal coordinate transformations and complex Weyl rescalings, we now have

$$\begin{aligned}(\mathcal{D}\eta^{s,w})'(x') &= e^{(w-1)E_R(x')}e^{-i(s+1)E_I(x')}(\mathcal{D}\eta^{s,w})(x),\\ (\bar{\mathcal{D}}\eta^{s,w})'(x') &= e^{(w-1)E_R(x')}e^{-i(s-1)E_I(x')}(\bar{\mathcal{D}}\eta^{s,w})(x).\end{aligned} \tag{4.31}$$

Therefore, the operator $\mathcal{D}$ ($\bar{\mathcal{D}}$) acts on weighted scalars $[s, w]$ to produce weighted scalars $[s+1, w-1]$ (resp. $[s-1, w-1]$). The following property holds:

$$[\mathcal{D}, \bar{\mathcal{D}}]\eta^{s,w} = -P\bar{P}\Big(s\partial\bar{\partial}\ln(P\bar{P}) + \frac{s-w}{2}\bar{\partial}W + \frac{s+w}{2}\partial\bar{W}\Big)\eta^{s,w}. \tag{4.32}$$

Note that $R = -2P\bar{P}\partial\bar{\partial}\ln(P\bar{P})$ is the scalar curvature of $\mathcal{S}$.

4.1.4 Ingredients

In the considerations below, all fields except for $P, \bar{P}$ have Weyl weights $(0, 0)$. We need the following ingredients:

(i) Supertranslation field. a real conformal field $\tilde{\mathcal{T}}$ of dimensions $(-\frac{1}{2}, -\frac{1}{2})$ and its assocated weighted scalar $\mathcal{T}$ under the map (4.16) of weights $[0, 1]$.

(ii) Superrotation field. a conformal field $\tilde{\mathcal{Y}}$ of dimensions $(-1, 0)$, its associated weighted scalar $\mathcal{Y}$ of weights $[-1, 1]$, and the complex conjugates $\bar{\tilde{\mathcal{Y}}}$ and $\bar{\mathcal{Y}}$. These fields satisfy the conformal Killing equation which becomes

$$\bar{\mathcal{D}}\mathcal{Y} = 0 \iff \bar{D}\tilde{\mathcal{Y}} = 0, \tag{4.33}$$

together with the complex conjugate relations. Locally, the solutions are simply $\tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}(\xi)$ and $\mathcal{Y} = P^{-1}\tilde{\mathcal{Y}}(\xi)$, with $\tilde{\mathcal{Y}}(\xi)$ arbitrary. This will not be the case when taking global restrictions into account. Note also that, because $s = -w$ for $\mathcal{Y}$, it follows from the second of (4.30) that the first of (4.33) can also be written using $\bar{\eth}$ instead of $\bar{\mathcal{D}}$.

(iii) Supermomentum. a real conformal field $\tilde{\mathcal{P}}$ of dimensions $(\frac{3}{2}, \frac{3}{2})$ and its associated weighted scalar $\mathcal{P}$ of weights $[0, -3]$.

(iv) Super angular momentum. a conformal field $\tilde{\mathcal{J}}$ of dimensions $(1, 2)$ and its associated weighted scalar $\mathcal{J}$ of weights $[-1, -3]$, together with the complex conjugates $\bar{\tilde{\mathcal{J}}}$ and $\bar{\mathcal{J}}$. We consider equivalence classes $[\mathcal{J}]$ such that $\mathcal{J} \sim \mathcal{J} + \mathcal{D}\mathcal{L}$ with $\mathcal{L}$ characterized by the weights $[-2, -2]$ and their complex conjugates. In this case, it follows from the first of (4.30) that, since $s = w$ for $\mathcal{L}$, one may also write $\eth\mathcal{L}$ in the equivalence relation. Similarly, we consider equivalence classes $[\tilde{\mathcal{J}}]$ such that $\tilde{\mathcal{J}} \sim \tilde{\mathcal{J}} + D\tilde{\mathcal{L}}$ with $\tilde{\mathcal{L}}$ characterized by the conformal dimensions $(0, 2)$ and their complex conjugates). These equivalence classes may be called super angular momenta.

The conformal dimensions, the Weyl weights, and the spin and conformal weights of the different ingredients are summarized in the tables 1 below. The objects $\tilde{d\mu}$, $d\mu$ represent the integration measure (see below).

Under complex conjugation, $\overline{(h, \bar{h})} = (\bar{h}, h)$, $\overline{(\lambda, \bar{\lambda})} = (\bar{\lambda}, \lambda)$, $\overline{[s, w]} = [-s, w]$.

4.2 Coadjoint representation of the algebra

4.2.1 Weighted scalars

In terms of above ingredients, the $\mathfrak{bms}_4$ algebra may be defined by triplets $s = (\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{T})$ with the commutation relations

$$[(\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{T}_1), (\mathcal{Y}_2, \bar{\mathcal{Y}}_2, \mathcal{T}_2)] = (\hat{\mathcal{Y}}, \hat{\bar{\mathcal{Y}}}, \hat{\mathcal{T}}), \tag{4.34}$$

$\phi_{h,\bar{h}}$	$\tilde{\mathcal{T}}$	$\tilde{\mathcal{Y}}$	$\tilde{\mathcal{P}}$	$\tilde{\mathcal{J}}$	$\tilde{d\mu}$	P	D
h	$-\frac{1}{2}$	-1	$\frac{3}{2}$	1	-1	-1	1
$\bar{h}$	$-\frac{1}{2}$	0	$\frac{3}{2}$	2	-1	0	0
λ	0	0	0	0	0	-1	0
$\bar{\lambda}$	0	0	0	0	0	0	0

$\eta^{s,w}$	$\mathcal{T}$	$\mathcal{Y}$	$\mathcal{P}$	$\mathcal{J}$	$d\mu$	$\mathcal{D}$
s	0	-1	0	-1	0	1
w	1	1	-3	-3	2	-1

Table 1. Dimensions and weights.

where

$$\begin{cases} \hat{\mathcal{Y}} = \mathcal{Y}_1\mathcal{D}\mathcal{Y}_2 - \mathcal{Y}_2\mathcal{D}\mathcal{Y}_1\,, \\ \hat{\mathcal{T}} = \mathcal{Y}_1\mathcal{D}\mathcal{T}_2 - \frac{1}{2}\mathcal{D}\mathcal{Y}_1\mathcal{T}_2 - (1\leftrightarrow 2) + \text{c.c.}\,. \end{cases} \tag{4.35}$$

Elements of the type $(\mathcal{Y},\bar{\mathcal{Y}},0)$ form a sub-algebra $\mathfrak{g}$. As usual, we identify the individual entries of the triplets/doublets with the triplets/doublets where all other entries are zero. For weighted scalar $\eta^{s,w}$, a representation of $\mathfrak{g}$ is defined by

$$\mathcal{Y}\cdot\eta^{s,w} = \mathcal{Y}\mathcal{D}\eta^{s,w} + \frac{s-w}{2}\mathcal{D}\mathcal{Y}\eta^{s,w}, \quad \bar{\mathcal{Y}}\cdot\eta^{s,w} = \bar{\mathcal{Y}}\bar{\mathcal{D}}\eta^{s,w} - \frac{s+w}{2}\bar{\mathcal{D}}\bar{\mathcal{Y}}\eta^{s,w}. \tag{4.36}$$

At this stage, we note that this representation, and also the $\mathfrak{bms}_4$ algebra above and the coadjoint representation below, may also be written with $\eth,\bar{\eth}$ instead of $\mathcal{D},\bar{\mathcal{D}}$ because the additional terms cancel.

In the notation of section 3, we thus have $X=(\mathcal{Y},\bar{\mathcal{Y}})$, $\alpha=\mathcal{T}$, and

$$\Sigma_X\alpha = (\mathcal{Y},\bar{\mathcal{Y}})\cdot\mathcal{T} = \mathcal{Y}\mathcal{D}\mathcal{T} - \frac{1}{2}\mathcal{D}\mathcal{Y}\mathcal{T} + \text{c.c.}\,. \tag{4.37}$$

Elements of $\mathfrak{bms}_4^*$ are denoted by triplets $([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P})$ where the pairing is given by

$$\langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}),(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_{\mathcal{S}} d\mu\,[\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}]. \tag{4.38}$$

The measure

$$d\mu(\xi,\bar{\xi}) = \frac{iC}{P\bar{P}}d\xi\wedge d\bar{\xi}, \tag{4.39}$$

for some normalization constant C, has dimensions $(0,0)$ and weights $[0,2]$. At this stage, we assume that the integral annihilates total $\mathcal{D}$ and $\bar{\mathcal{D}}$ derivatives. Furthermore, we require the pairing to be non-degenerate, which can only be the case when taking quotients with respect to the equivalence relations $\mathcal{J}\sim\mathcal{J}+\mathcal{D}\mathcal{L}$ and $\bar{\mathcal{J}}\sim\bar{\mathcal{J}}+\bar{\mathcal{D}}\bar{\mathcal{L}}$. Concrete realizations where these assumptions hold will be discussed below.

From the definition of the coadjoint representation (3.7), it then follows that

$$\begin{aligned} \mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} &= \bar{\mathcal{Y}}\bar{\mathcal{D}}\mathcal{J} + 2\bar{\mathcal{D}}\bar{\mathcal{Y}}\mathcal{J} + \mathcal{D}(\mathcal{Y}\mathcal{J}) + \frac{1}{2}\mathcal{T}\bar{\mathcal{D}}\mathcal{P} + \frac{3}{2}\bar{\mathcal{D}}\mathcal{T}\mathcal{P}, \\ \mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} &= \mathcal{Y}\mathcal{D}\mathcal{P} + \frac{3}{2}\mathcal{D}\mathcal{Y}\mathcal{P} + \text{c.c.}\,, \end{aligned} \tag{4.40}$$

where the third term in the first of the above equations does not appear but can be added because it is equivalent to zero. This is useful in order to have a transformation law consistent with the conformal dimensions of $\mathcal{J}$.

Remarks.

(i) The definition makes sense on the level of equivalence classes,

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}([0],[0],0)=([0],[0],0), \tag{4.41}$$

because

$$\bar{\mathcal{Y}}\bar{\mathcal{D}}\mathcal{D}\mathcal{L}+2\bar{\mathcal{D}}\bar{\mathcal{Y}}\mathcal{D}\mathcal{L}=\mathcal{D}(\bar{\mathcal{Y}}\bar{\mathcal{D}}\mathcal{L}+2\bar{\mathcal{D}}\bar{\mathcal{Y}}\mathcal{L}). \tag{4.42}$$

(ii) Since the inner product involves complex conjugation, we have

$$\mathrm{ad}^*_{\bar{\mathcal{Y}}}\mathcal{J}=\bar{\mathcal{Y}}\bar{\mathcal{D}}\mathcal{J}+2\bar{\mathcal{D}}\bar{\mathcal{Y}}\mathcal{J},\quad \mathrm{ad}^*_{\bar{\mathcal{Y}}}\bar{\mathcal{J}}=\mathcal{D}(\mathcal{Y}\mathcal{J})\sim 0,\quad \mathrm{ad}^*_{\bar{\mathcal{Y}}}\mathcal{P}=\bar{\mathcal{Y}}\bar{\mathcal{D}}\mathcal{P}+\frac{3}{2}\bar{\mathcal{D}}\bar{\mathcal{Y}}\mathcal{P}, \tag{4.43}$$

together with the complex conjugates of these relations.

(iii) In the notation of section 3, $j=([\mathcal{J}],[\bar{\mathcal{J}}])$, $p=\mathcal{P}$ and

$$\begin{aligned}\Sigma^*_X p&=\mathcal{Y}\mathcal{D}\mathcal{P}+\frac{3}{2}\mathcal{D}\mathcal{Y}\mathcal{P}+\text{c.c.}\,,\\ \alpha\times p&=\left(\left[\frac{1}{2}\mathcal{T}\bar{\mathcal{D}}\mathcal{P}+\frac{3}{2}\bar{\mathcal{D}}\mathcal{T}\mathcal{P}\right],\left[\frac{1}{2}\mathcal{T}\mathcal{D}\mathcal{P}+\frac{3}{2}\mathcal{D}\mathcal{T}\mathcal{P}\right]\right)\,.\end{aligned} \tag{4.44}$$

This relation encodes the change of super angular momentum under an infinitesimal supertranslation, which depends linearly on supermomentum. Note also that when using a vector $\mathcal{T}$ which is not real, the contribution of $\alpha\times p$ to $\mathcal{J}$ is $\frac{1}{2}\bar{\mathcal{T}}\bar{\mathcal{D}}\mathcal{P}+\frac{3}{2}\bar{\mathcal{D}}\bar{\mathcal{T}}\mathcal{P}$.

(iv) On the level of integrands, if we define

$$\mathcal{J}^u_s=\bar{\mathcal{J}}\mathcal{Y}+\mathcal{J}\bar{\mathcal{Y}}+\mathcal{P}\mathcal{T}, \tag{4.45}$$

equation (3.7) reads

$$\mathrm{ad}^*_{s_1}\mathcal{J}^u_{s_2}=-\mathcal{J}^u_{[s_1,s_2]}+\mathcal{D}\mathcal{L}_{s_1,s_2}+\bar{\mathcal{D}}\bar{\mathcal{L}}_{s_1,s_2}, \tag{4.46}$$

where

$$\mathcal{L}_{s_1,s_2}=\bar{\mathcal{J}}\mathcal{Y}_1\mathcal{Y}_2+(\mathcal{J}\mathcal{Y}_1\bar{\mathcal{Y}}_2)+\mathcal{P}\mathcal{Y}_1\mathcal{T}_2+\frac{1}{2}\mathcal{P}\mathcal{T}_1\mathcal{Y}_2. \tag{4.47}$$

(v) In the case when there is no non-degenerate pairing, it is still true that the vector space of elements $([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P})$ forms a representation under $\mathfrak{bms}_4$. This representation $\rho_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}$, which is no longer the coadjoint representation, is defined by replacing $\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}$ by $\rho_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}$ in the left hand side of (4.40).

4.2.2 Conformal fields

The above definitions are expressed in terms of weighted scalars. The analogous definitions in terms of the associated conformal fields are obtained by a direct rewriting that consists in adding tilde's on all the scalars and replacing $\mathcal{D}$ by D and $\bar{\mathcal{D}}$ by $\bar{D}$.

In particular, under the mapping (4.16), the representation (4.36) becomes

$$\tilde{\mathcal{Y}}\cdot\phi_{h,\bar{h}}=\tilde{\mathcal{Y}}D\phi_{h,\bar{h}}+hD\tilde{\mathcal{Y}}\phi_{h,\bar{h}},\quad \tilde{\bar{\mathcal{Y}}}\cdot\phi_{h,\bar{h}}=\tilde{\bar{\mathcal{Y}}}\bar{D}\phi_{h,\bar{h}}+\bar{h}\bar{D}\tilde{\bar{\mathcal{Y}}}\phi_{h,\bar{h}}. \tag{4.48}$$

The Weyl covariant derivatives $D, \bar{D}$ may again be replaced by the ordinary derivatives $\partial, \bar{\partial}$ in these representations because the additional terms cancel.

The pairing is given by

$$\langle\!([\tilde{\mathcal{J}}], [\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}}), (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}, \tilde{\mathcal{T}})\tilde{\rangle} = \int_{\mathcal{S}} \tilde{d\mu}\, [\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}], \tag{4.49}$$

where the measure has conformal dimensions $(-1,-1)$, Weyl weights $(0,0)$ and is associated to the measure (4.39) through $d\mu = (P\bar{P})^{-1}\tilde{d\mu}$, so that

$$\tilde{d\mu} = iC d\xi \wedge d\bar{\xi}. \tag{4.50}$$

4.3 Coadjoint representation of the group

4.3.1 Conformal fields

Consider conformal coordinate transformations, $(\xi'(\xi), \bar{\xi}'(\bar{\xi})) = (g(\xi), \bar{g}(\bar{\xi}))$, such that $\frac{\partial g}{\partial \xi} > 0$, $\frac{\partial \bar{g}}{\partial \bar{\xi}} > 0$. They form a group G under composition. For a conformal field $\phi_{h,\bar{h}}(x)$ of dimensions $(h, \bar{h})$ (and vanishing Weyl weights), a representation of G is defined through

$$((g,\bar{g}) \cdot \phi_{h,\bar{h}})(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^{h} \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{\bar{h}} \phi_{h,\bar{h}}(x). \tag{4.51}$$

The BMS$_4$ group is determined by elements $(g, \bar{g}, \tilde{\mathcal{T}})$ with multiplication

$$(g_1, \bar{g}_1, \tilde{\mathcal{T}}_1) \cdot (g_2, \bar{g}_2, \tilde{\mathcal{T}}_2) = (g_1 \circ g_2, \bar{g}_1 \circ \bar{g}_2, \tilde{\mathcal{T}}_1 + (g_1, \bar{g}_1) \cdot \tilde{\mathcal{T}}_2). \tag{4.52}$$

Elements of the form $(g, \bar{g}, 0)$ form a subgroup isomorphic to G. In the notation of section 3, we thus have $f = (g, \bar{g})$, $X = (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}})$ and $\alpha = \tilde{\mathcal{T}}$ with

$$(\sigma_f(\alpha))(x') = ((g,\bar{g}) \cdot \tilde{\mathcal{T}})(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^{-\frac{1}{2}} \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{-\frac{1}{2}} \tilde{\mathcal{T}}(x). \tag{4.53}$$

For the adjoint action, defined in equation (3.3), we get

$$(\mathrm{Ad}_f X)(x') = ((g,\bar{g}) \cdot (\mathcal{Y}, \bar{\mathcal{Y}}))(x') = \left(\left(\frac{\partial \xi}{\partial \xi'}\right)^{-1} \tilde{\mathcal{Y}}, \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{-1} \tilde{\bar{\mathcal{Y}}}\right)(x), \tag{4.54}$$

$$(\Sigma_{\mathrm{Ad}_f X}\alpha)(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^{-\frac{1}{2}} \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{-\frac{1}{2}} \left(\tilde{\mathcal{Y}} D\tilde{\mathcal{T}} - \frac{1}{2} D\tilde{\mathcal{Y}}\tilde{\mathcal{T}} + \text{c.c.}\right)(x), \tag{4.55}$$

whereas definition (3.10) for the coadjoint representation gives

$$(\mathrm{Ad}^*_f \tilde{\mathcal{J}})(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)\left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{2} \tilde{\mathcal{J}}(x), \tag{4.56}$$

$$(\sigma^*_f \tilde{\mathcal{P}})(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^{\frac{3}{2}} \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{\frac{3}{2}} \tilde{\mathcal{P}}(x), \tag{4.57}$$

$$(\tilde{\mathcal{T}} \times \sigma^*_f \tilde{\mathcal{P}})(x') = \left(\left(\frac{\partial \xi}{\partial \xi'}\right)\left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{2} \left(\frac{1}{2}\tilde{\mathcal{T}}\bar{D}\tilde{\mathcal{P}} + \frac{3}{2}\bar{D}\tilde{\mathcal{T}}\tilde{\mathcal{P}}\right), \text{c.c.}\right)(x), \tag{4.58}$$

where c.c. denotes the complex conjugate of the expression to the left of the comma.

Remarks

(i) As usual for diffeomorphisms, the adjoint and coadjoint representations of the algebra discussed previously are the differentials of those of the group discussed in this section up to an overall minus sign.

(ii) As usual in conformal field theory, on the level of the algebra, we will consider not only the Lie algebra of the globally well-defined conformal transformations but also the algebra of infinitesimal local conformal transformations.

(iii) The formulas for the group can also be used to understand how the coadjoint representation behaves under conformal mappings. We briefly discuss the standard map from the punctured plane to the cylinder below.

4.3.2 Weighted scalars

The description in terms of weighted scalars is very similar. As compared to the previous section, one simply removes the tilde's and replaces D, $\bar{D}$ by $\mathcal{D}$, $\bar{\mathcal{D}}$, while at the same time replacing $(\frac{\partial\xi}{\partial\xi'})^h(\frac{\partial\bar{\xi}}{\partial\bar{\xi}'})^{\bar{h}}$ by $e^{wE_R(x')}e^{-isE_I(x')}$ using table 1.

For future reference, let us nevertheless provide explicit formulas. In this case, the BMS_4 group is determined by elements $(g,\bar{g},\mathcal{T})$ with multiplication

$$(g_1,\bar{g}_1,\mathcal{T}_1)\cdot(g_2,\bar{g}_2,\mathcal{T}_2)=(g_1\circ g_2,\bar{g}_1\circ\bar{g}_2,\mathcal{T}_1+(g_1,\bar{g}_1)\cdot\mathcal{T}_2), \tag{4.59}$$

where the representation of G on a weighted scalar $\eta^{s,w}$ is defined through

$$((g,\bar{g})\cdot\eta^{s,w})(x')=e^{wE_R(x')}e^{-isE_I(x')}\eta^{s,w}(\xi,\bar{\xi}). \tag{4.60}$$

In the notation of section 3, we now have $f=(g,\bar{g})$, $X=(\mathcal{Y},\bar{\mathcal{Y}})$, and $\alpha=\mathcal{T}$ with

$$(\sigma_f(\alpha))(x')=((g,\bar{g})\cdot\mathcal{T})(x')=e^{E_R(x')}\mathcal{T}(x). \tag{4.61}$$

For the adjoint representation, we get

$$(\mathrm{Ad}_f X)(x')=((g,\bar{g})\cdot(\mathcal{Y},\bar{\mathcal{Y}}))(x')=\left(e^{E_R(x')}e^{iE_I(x')}\mathcal{Y}(x),e^{E_R(x')}e^{-iE_I(x')}\bar{\mathcal{Y}}(x)\right), \tag{4.62}$$

$$(\Sigma_{\mathrm{Ad}_f X}\alpha)(x')=e^{E_R(x')}\left(\mathcal{Y}\mathcal{D}\mathcal{T}-\frac{1}{2}\mathcal{D}\mathcal{Y}\mathcal{T}+\text{c.c.}\right)(x), \tag{4.63}$$

whereas for the coadjoint representation, we get

$$(Ad^*_f\mathcal{J})(x')=e^{-3E_R(x')}e^{iE_I(x')}\mathcal{J}(x), \tag{4.64}$$

$$(\sigma^*_f\mathcal{P})(x')=e^{-3E_R(x')}\mathcal{P}(x), \tag{4.65}$$

$$(\mathcal{T}\times\sigma^*_f\mathcal{P})(x')=\left(e^{-3E_R(x')}e^{iE_I(x')}\left(\frac{1}{2}\mathcal{T}\bar{\mathcal{D}}\mathcal{P}+\frac{3}{2}\bar{\mathcal{D}}\mathcal{T}\mathcal{P}\right)(x),\text{c.c.}\right). \tag{4.66}$$

4.4 Weyl invariance

The structure described above is covariant with respect to conformal coordinate transformations combined with complex Weyl rescalings since it is defined in terms of suitable covariant derivatives.

In particular, the above descriptions are valid for all conformal factors P and $\bar{P}$ and all two-dimensional surfaces $\mathcal{S}$ such that total $\mathcal{D}, \bar{\mathcal{D}}$ derivatives are annihilated and the pairing is non-degenerate. In the remainder of the paper, we mainly focus on two particular cases: (i) the sphere S^2 of radius R with $P_S = \frac{1+\xi\bar{\xi}}{R\sqrt{2}} = \bar{P}_S$, and (ii) the punctured complex plane $\mathbb{C} - \{0\} = \mathbb{C}_0$ with $P = 1 = \bar{P}$.

5 Realization on the sphere

5.1 Generalities

If $\xi = \cot\frac{\theta}{2}e^{-i\phi}$, the standard metric on the sphere of radius R is

$$ds^2 = -2(P_S\bar{P}_S)^{-1}d\xi d\bar{\xi}, \quad P_S = \frac{1+\xi\bar{\xi}}{R\sqrt{2}}. \tag{5.1}$$

The globally well-defined conformal coordinate transformations for the sphere are the fractional linear unimodular transformations

$$\xi' = \frac{a\xi + b}{c\xi + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}. \tag{5.2}$$

In particular,

$$\frac{\partial\xi}{\partial\xi'} = (c\xi + d)^2. \tag{5.3}$$

Under combined conformal coordinate transformations and Weyl rescalings, the metric takes the standard form in the new coordinates if one freezes the Weyl transformations as in equation (4.12). For $P_F = P_S$, we have [5, 15]

$$e^{E(x')} = \frac{P_S(x)}{P_S(x')}\frac{\partial\xi'}{\partial\xi} \iff e^{E_R(x')} = \frac{1+\xi\bar{\xi}}{|a\xi+b|^2 + |c\xi+d|^2}, \quad e^{iE_I(x')} = \frac{\bar{c}\bar{\xi}+\bar{d}}{c\xi+d}. \tag{5.4}$$

In this context, w is referred to as the boost weight.

The derivative operators (4.28) now take the explicit form (cf. section 4.15 of [17])

$$\eth\eta^{s,w} = P_S^{1-s}\partial(P_S^s\eta^{s,w}), \quad \bar{\eth}\eta^{s,w} = P_S^{1+s}\bar{\partial}(P_S^{-s}\eta^{s,w}). \tag{5.5}$$

The pairing on the sphere is defined between scalars $\eta^{s,w}$ of weights $[s,w]$ and $\kappa^{s,-w-2}$ of weights $[s,-w-2]$ as follows,

$$\langle\kappa^{s,-w-2}, \eta^{s,w}\rangle = \frac{1}{4\pi R^2}\int_{S^2}\frac{id\xi\wedge d\bar{\xi}}{P_S\bar{P}_S}\,\overline{\kappa^{s,-w-2}}\eta^{s,w}, \tag{5.6}$$

where the normalization $C = (4\pi R^2)^{-1}$ is chosen so that

$$\frac{1}{4\pi R^2}\int_{S^2}\frac{id\xi\wedge d\bar{\xi}}{P_S\bar{P}_S} = \frac{1}{2\pi}\int_{S^2}\frac{id\xi\wedge d\bar{\xi}}{(1+\xi\bar{\xi})^2} = \frac{1}{4\pi}\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi = 1. \tag{5.7}$$

When compared to (4.38), we thus have

$$\langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}),(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle=\langle\mathcal{J},\mathcal{Y}\rangle+\langle\bar{\mathcal{J}},\bar{\mathcal{Y}}\rangle+\langle\mathcal{P},\mathcal{T}\rangle,\quad d\mu(\xi,\bar{\xi})=\frac{id\xi\wedge d\bar{\xi}}{4\pi R^2P_S\bar{P}_S}. \tag{5.8}$$

This pairing has all the required properties.

5.2 Adjoint and coadjoint representations of the group

In terms of spin-weighted scalars, the (co)adjoint representation of the BMS_4 group on the sphere is described by the general formulas established in subsection 4.3.2, where now $f=(g,\bar{g})$ are given by general linear fractional transformations of (5.2) (and the associated transformation of the complex conjugate variable) and the factors $e^{E_R(x')}$ and $e^{iE_I(x')}$ are given in (5.4). Let us write them out explicitly, with $x=\xi,\bar{\xi}$.

For the adjoint representation, under a combined transformation f and a supertranslation α with which one acts and a supertranslation β on which one acts, (where α,β are two different supertranslation fields with the same weights than $\mathcal{T}$),

$$\begin{aligned}
&\mathcal{Y}'(x')=e^{E_R(x')}e^{iE_I(x')}\mathcal{Y}(x),\\
&\bar{\mathcal{Y}}'(x')=e^{E_R(x')}e^{-iE_I(x')}\bar{\mathcal{Y}}(x),\\
&\beta'(x')=e^{E_R(x')}\Big(\beta-\Big(\mathcal{Y}\eth\alpha-\frac{1}{2}\alpha\eth\mathcal{Y}+\text{c.c.}\Big)\Big)(x).
\end{aligned} \tag{5.9}$$

For the coadjoint representation, if we denote by $\mathcal{T}$ instead of α the supertranslation with which one acts,

$$\begin{aligned}
&\mathcal{J}'(x')=e^{-3E_R(x')}e^{iE_I(x')}\Big(\mathcal{J}+\Big(\frac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P}+\frac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}\Big)\Big)(x)\\
&\bar{\mathcal{J}}'(x')=e^{-3E_R(x')}e^{-iE_I(x')}\Big(\bar{\mathcal{J}}+\Big(\frac{1}{2}\mathcal{T}\eth\mathcal{P}+\frac{3}{2}\eth\mathcal{T}\mathcal{P}\Big)\Big)(x)\\
&\mathcal{P}'(x')=e^{-3E_R(x')}\mathcal{P}(x).
\end{aligned} \tag{5.10}$$

Not surprisingly, when using the associated conformal fields, these transformations simplify. The formulas of section 4.3.1 apply. The Jacobians $\partial\xi/\partial\xi',\partial\bar{\xi}/\partial\bar{\xi}'$ are explicitly given by (5.3) and its complex conjugate. In this case, the integration measure is

$$\tilde{d\mu}=\frac{id\xi\wedge d\bar{\xi}}{4\pi R^2}, \tag{5.11}$$

and

$$\begin{aligned}
&\tilde{\mathcal{Y}}'(\xi')=(c\xi+d)^{-2}\tilde{\mathcal{Y}}(\xi),\\
&\tilde{\bar{\mathcal{Y}}}'(\bar{\xi}')=(\bar{c}\bar{\xi}+\bar{d})^{-2}\tilde{\bar{\mathcal{Y}}}(\xi),\\
&\tilde{\beta}'(x')=(c\xi+d)^{-1}(\bar{c}\bar{\xi}+\bar{d})^{-1}\Big(\tilde{\beta}-\Big(\tilde{\mathcal{Y}}\partial\tilde{\alpha}-\frac{1}{2}\tilde{\alpha}\partial\tilde{\mathcal{Y}}+\text{c.c.}\Big)\Big)(x).
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&\tilde{\mathcal{J}}'(x')=(c\xi+d)^2(\bar{c}\bar{\xi}+\bar{d})^4\Big(\tilde{\mathcal{J}}(x)+\Big(\frac{1}{2}\tilde{\mathcal{T}}\bar{\partial}\tilde{\mathcal{P}}+\frac{3}{2}\bar{\partial}\tilde{\mathcal{T}}\tilde{\mathcal{P}}\Big)\Big)(x)\\
&\tilde{\bar{\mathcal{J}}}'(x')=(c\xi+d)^4(\bar{c}\bar{\xi}+\bar{d})^2\Big(\tilde{\bar{\mathcal{J}}}+\Big(\frac{1}{2}\tilde{\mathcal{T}}\partial\tilde{\mathcal{P}}+\frac{3}{2}\partial\tilde{\mathcal{T}}\tilde{\mathcal{P}}\Big)\Big)(x)\\
&\tilde{\mathcal{P}}'(x')=(c\xi+d)^3(\bar{c}\bar{\xi}+\bar{d})^3\tilde{\mathcal{P}}(x).
\end{aligned} \tag{5.13}$$

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5.3 Expansions

5.3.1 Spin-weighted spherical harmonics

We now decompose the relevant spin-weighted scalars in terms of spin-weighted spherical harmonics (see appendix A for conventions).

For the $\mathfrak{bms}_4$ Lie algebra, we have $\bar{\mathcal{D}}\mathcal{Y} = \bar{\eth}\mathcal{Y} = 0$ with $\mathcal{Y}$ of weights $[-1, 1]$. It follows from array (4.15.60) of [17] that $\mathcal{Y}$ belongs to the three-dimensional vector space of spherical harmonics with spin weight $s = -1$ and $j = 1$. Hence, defining

$$\mathcal{Y}_m = -R\sqrt{2}\ {}_{-1}Z_{1,m}, \quad m = -1, 0, 1, \tag{5.14}$$

gives rise to the decomposition

$$\bar{\eth}\mathcal{Y} = 0 \iff \mathcal{Y} = \sum_{m=-1}^{1} y_m \mathcal{Y}_m. \tag{5.15}$$

In the same way,

$$\bar{\mathcal{Y}}_m = (-1)^m R\sqrt{2}\ {}_{1}Z_{1,-m} \Longrightarrow \bar{\mathcal{Y}} = \sum_{m=-1}^{1} \bar{y}_m \bar{\mathcal{Y}}_m, \tag{5.16}$$

while

$$\mathcal{T}_{j,m} = {}_0Z_{j,m} \Longrightarrow \mathcal{T} = \sum_{j,|m|\leq j} t_{j,m}\mathcal{T}_{j,m}, \tag{5.17}$$

where we impose $\bar{t}_{j,m} = (-1)^m t_{j,-m}$ since $\mathcal{T}$ is real.

When taking into account the pairing (5.6) and the normalization of the ${}_sZ_{jm}$ in (A.4), this choice of basis for the algebra implies the following choice for the associated dual basis of the coadjoint representation,

$$\begin{aligned} \mathcal{Y}^m_* &= \frac{-6}{R\sqrt{2}(1+m)!(1-m)!}\ {}_{-1}Z_{1,m}, \\ \bar{\mathcal{Y}}^m_* &= \frac{(-1)^m 6}{R\sqrt{2}(1+m)!(1-m)!}\ {}_{1}Z_{1,-m}, \\ \mathcal{T}^{j,m}_* &= \frac{(2j+1)!(2j)!}{j!j!(j+m)!(j-m)!}\ {}_0Z_{j,m}, \end{aligned} \tag{5.18}$$

and thus also the following expansions,

$$\mathcal{J} = \sum_{m=-1}^{1} j_m \mathcal{Y}^m_*, \quad \bar{\mathcal{J}} = \sum_{m=-1}^{1} \bar{j}_m \bar{\mathcal{Y}}^m_*, \quad \mathcal{P} = \sum_{j,|m|\leq j} p_{j,m}\mathcal{T}^{j,m}_*, \tag{5.19}$$

where $\bar{p}_{j,m} = (-1)^m p_{j,-m}$ since $\mathcal{P}$ is real.

Note that the explicit expressions for ${}_{-1}Z_{1,m}$ in (A.2) gives

$$\tilde{\mathcal{Y}}_m = \mathcal{Y}_m P_S = \xi^{1-m}, \quad \tilde{\bar{\mathcal{Y}}}_m = \bar{\mathcal{Y}}_m \bar{P}_S = \bar{\xi}^{1-m}. \tag{5.20}$$

Remarks.

(i) From the discussion of the behavior of spin-weighted spherical harmonics under Lorentz transformations in section 4.15 of [17], it follows that, if $w \geq |s|$ then $\eth^{w-s+1}\eta^{s,w}$, $\bar{\eth}^{w+s+1}\eta^{s,w}$ have definite spin and boosts weights given by $[w+1, s-1]$ and $[-w-1, -s-1]$ respectively. This is the case for

$$\eth\bar{\mathcal{Y}},\ \bar{\eth}\mathcal{Y},\ \eth^3\mathcal{Y},\ \bar{\eth}^3\bar{\mathcal{Y}}. \tag{5.21}$$

As shown there, the equations $\bar{\eth}\mathcal{Y} = 0$ and $\eth^3\mathcal{Y} = 0$ on the one hand, and $\eth\bar{\mathcal{Y}} = 0$ and $\bar{\eth}^3\bar{\mathcal{Y}} = 0$ on the other, define the same Lorentz invariant three-dimensional subspaces described above.

For the dual situation where $w \leq -|s| - 2$, $\eth^{s-w-1}\kappa^{w+1,s-1}$ and $\bar{\eth}^{-s-w-1}\kappa^{-w-1,-s-1}$ have definite spin and boost weights $[s, w]$, it is shown that equivalence classes $[\eta^{s,w}], \eta^{s,w} \sim \eta^{s,w} + \eth^{s-w-1}\kappa^{w+1,s-1}$ or $\eta^{s,w} \sim \eta^{s,w} + \bar{\eth}^{-s-w-1}\kappa^{-w-1,-s-1}$ define Lorentz invariant subspaces. This is the case for

$$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\eth}\bar{\mathcal{L}}, \quad \bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth^3\mathcal{M}, \tag{5.22}$$

where $\bar{\mathcal{L}} : [2, -2]$ and $\mathcal{M} : [-2, 0]$ and both equivalence classes define the same three-dimensional Lorentz invariant subspaces. Similarly, by complex conjugation

$$\mathcal{J} \sim \mathcal{J} + \eth\mathcal{L}, \quad \mathcal{J} \sim \mathcal{J} + \bar{\eth}^3\bar{\mathcal{M}} \tag{5.23}$$

where $\bar{\mathcal{L}} : [-2, -2]$ and $\bar{\mathcal{M}} : [2, 0]$.

(ii) The (well-known) coadjoint representation of the Poincaré group may be discussed from the perspective developed here by imposing in addition the conditions $\eth^2\mathcal{T} = 0 = \bar{\eth}^2\mathcal{T}$ reducing super to ordinary translations. Again, these equations define a four-dimensional Lorentz invariant subspace because $\mathcal{T}$ has the required weights. At the same time, one should consider equivalence classes $\mathcal{P} \sim \mathcal{P} + \eth^2\mathcal{N} + \bar{\eth}^2\bar{\mathcal{N}}$, where $\mathcal{N} : [-2, -1]$, $\bar{\mathcal{N}} : [2, -1]$ have the required weights and which also defines a Lorentz invariant four-dimensional subspace.

5.3.2 Overcomplete set of functions

The representation of the generators $\mathcal{Y}_m, \bar{\mathcal{Y}}_m$ of the Lorentz algebra on weighted scalars $\eta^{s,w}$ is explicitly given by

$$\begin{aligned}
\mathcal{Y}_m \cdot \eta^{s,w} &= \xi^{-m}\Big(\xi\partial\eta^{s,w} + \Big(\frac{s-w}{2}(1-m) + w\frac{\xi\bar{\xi}}{1+\xi\bar{\xi}}\Big)\eta^{s,w}\Big), \\
\bar{\mathcal{Y}}_m \cdot \eta^{s,w} &= \bar{\xi}^{-m}\Big(\bar{\xi}\bar{\partial}\eta^{s,w} + \Big(-\frac{s+w}{2}(1-m) + w\frac{\xi\bar{\xi}}{1+\xi\bar{\xi}}\Big)\eta^{s,w}\Big).
\end{aligned} \tag{5.24}$$

This follows from using (4.36) written in terms of $\eth$ and $\bar{\eth}$ together with (5.20).

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For the associated conformal field $\phi_{h,\bar{h}} = P_S^w \eta^{s,w}$, this simplifies to

$$\begin{aligned} \tilde{\mathcal{Y}}_m \cdot \phi_{h,\bar{h}} &= \xi^{-m}(\xi\partial\phi_{h,\bar{h}} + h(1-m)\phi_{h,\bar{h}}), \\ \tilde{\bar{\mathcal{Y}}}_m \cdot \phi_{h,\bar{h}} &= \bar{\xi}^{-m}(\bar{\xi}\bar{\partial}\phi_{h,\bar{h}} + \bar{h}(1-m)\phi_{h,\bar{h}}), \end{aligned} \tag{5.25}$$

where $s = h - \bar{h}, w = -h - \bar{h}$.

Rather than expanding the spin-weighted scalar $\eta^{s,w}$ in terms of (unnormalized) spin-weighted spherical harmonics, one may also work with suitable sets of over-complete functions. We follow [14], section 4.C (up to conventions). Let $|s| \leq L$. For a fixed $L \in \mathbb{N}$, there is an invertible matrix that relates the spin-weighted spherical harmonics ${}_sY_{j,m}$ with $j \leq L$ to the functions

$${}_sZ^L_{m_1,m_2} = (1+\xi\bar{\xi})^{-L}\xi^{L-s-m_1}\bar{\xi}^{L+s-m_2}, \quad 0 \leq m_1 \leq L-s, \quad 0 \leq m_2 \leq L+s. \tag{5.26}$$

Depending on the conformal weight w, one may label these same functions as

$${}_{h,\bar{h}}Z^{\tilde{L}}_{k,l} = P_S^{-w}\,{}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l}, \quad {}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l} = (R\sqrt{2})^{h+\bar{h}}(1+\xi\bar{\xi})^{-\tilde{L}}\xi^{\tilde{L}-h-k}\bar{\xi}^{\tilde{L}-\bar{h}-l}, \tag{5.27}$$

where

$$\tilde{L} = L + h + \bar{h}, \quad k = m_1 + h, \quad l = m_2 + \bar{h}, \quad h \leq k \leq \tilde{L} - h, \quad \bar{h} \leq l \leq \tilde{L} - \bar{h}. \tag{5.28}$$

In particular, if $h, \bar{h}$ are half-integer, so are k, l.

When taking for $\eta^{s,w}$ one of the functions ${}_{s,w}Z^{\tilde{L}}_{k,l}$, it follows that

$$\begin{aligned} \mathcal{Y}_m \cdot {}_{h,\bar{h}}Z^{\tilde{L}}_{k,l} &= -(hm+k)\,{}_{h,\bar{h}}Z^{\tilde{L}+1}_{k+m,l} + (\tilde{L} - (hm+k))\,{}_{h,\bar{h}}Z^{\tilde{L}+1}_{k+m+1,l+1}, \\ \bar{\mathcal{Y}}_m \cdot {}_{h,\bar{h}}Z^{\tilde{L}}_{k,l} &= -(\bar{h}m+l)\,{}_{h,\bar{h}}Z^{\tilde{L}+1}_{k,l+m} + (\tilde{L} - (\bar{h}m+l))\,{}_{h,\bar{h}}Z^{\tilde{L}+1}_{k+1,l+m+1}, \end{aligned} \tag{5.29}$$

where the following (elementary) relations have been used,

$${}_{h,\bar{h}}Z^{\tilde{L}}_{k,l} = {}_{h,\bar{h}}Z^{\tilde{L}+1}_{k,l} + {}_{h,\bar{h}}Z^{\tilde{L}+1}_{k+1,l+1}. \tag{5.30}$$

By construction, when taking for the conformal fields $\phi_{h,\bar{h}}$ the functions ${}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l}$, the relations (5.29) hold with the substitutions $\mathcal{Y}_m \to \tilde{\mathcal{Y}}_m$, $\bar{\mathcal{Y}}_m \to \tilde{\bar{\mathcal{Y}}}_m$, ${}_{h,\bar{h}}Z^{\tilde{L}}_{k,l} \to {}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l}$.

When taking into account that

$$\xi = \cot\frac{\theta}{2}e^{-i\phi}, \quad \mu = \cos\theta, \quad \xi\bar{\xi} = \frac{1+\mu}{1-\mu} = \cot^2\frac{\theta}{2}, \quad 1+\xi\bar{\xi} = \frac{2}{1-\mu}, \tag{5.31}$$

it follows that

$$\begin{aligned} \langle {}_sZ^L_{m_1',m_2'}, {}_sZ^L_{m_1,m_2}\rangle &= \delta_m^{m_1+m_2'}\delta_m^{m_1'+m_2}\frac{1}{2}\int_{-1}^{1} d\mu \left(\frac{1-\mu}{2}\right)^{2L}\left(\frac{1+\mu}{1-\mu}\right)^{2L-m} \\ &= \delta_m^{m_1+m_2'}\delta_m^{m_1'+m_2}\frac{m!(2L-m)!}{(2L+1)!} \\ &= \delta_m^{m_1+m_2'}\delta_m^{m_1'+m_2}\beta(m+1, 2L-m+1), \end{aligned} \tag{5.32}$$

where $0 \leq m \leq 2L$. Instead of reverting to angular variables for the integrals, they may also be worked out directly in complex coordinates:

$$\begin{aligned}\langle {}_sZ^L_{m_1',m_2'}, {}_sZ^L_{m_1,m_2}\rangle &= \frac{i}{2\pi}\int d\xi \wedge d\bar{\xi}\ (1+\xi\bar{\xi})^{-2L-2}\xi^{2L-m_1-m_2'}\bar{\xi}^{2L-m_1'-m_2} \\ &= \frac{1}{2L+1}\frac{1}{2\pi i}\int d\xi \wedge d\bar{\xi}\ \partial((1+\xi\bar{\xi})^{-2L-1}\bar{\xi}^{2L-m_1'-m_2-1})\xi^{2L-m_1-m_2'},\end{aligned} \tag{5.33}$$

where $0 \leq m_1' + m_2 \leq 2L$, $0 \leq m_1 + m_2' \leq 2L$. If $m_1 + m_2' = 2L$, one then proceeds by using Stokes' theorem together with a kind of Cauchy residue theorem for the remaining line integral (see for instance [35] in the current context). If $m_1 + m_2' < 2L$, one makes an integration by parts to lower the degree of ξ, and applies the same reasoning for all integrals that involve a total ∂ with expressions that have poles in $\bar{\xi}$.

Because of the weights $[s, w]$ and $[s, -w-2]$ of the spin-weighted spherical harmonics involved in (5.6), the relevant integrals pair ${}_{h,\bar{h}}Z^{\tilde{L}}_{k,l}$ on the right with ${}_{h',\bar{h}'}Z^{\tilde{L}'}_{k',l'}$ on the left, where $h' = -\bar{h}+1$, $\bar{h}' = -h+1$ and $\tilde{L}' = \tilde{L} - 2h - 2\bar{h} + 2$,

$$\langle {}_{-\bar{h}+1,-h+1}Z^{\tilde{L}-2h-2\bar{h}+2}_{k',l'}, {}_{h,\bar{h}}Z^{\tilde{L}}_{k,l}\rangle = \delta^{k+l'-1}_m \delta^{k'+l-1}_m \beta(m+1, 2(\tilde{L}-h-\bar{h})-m+1). \tag{5.34}$$

By construction, the associated conformal fields ${}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l}$ have the same integrals when using the measure $\tilde{d\mu}$ given in (5.11),

$$\langle {}_{-\bar{h}+1,-h+1}\tilde{Z}^{\tilde{L}-2h-2\bar{h}+2}_{k',l'}, {}_{h,\bar{h}}\tilde{Z}^{\tilde{L}}_{k,l}\tilde{\rangle} = \delta^{k+l'-1}_m \delta^{k'+l-1}_m \beta(m+1, 2(\tilde{L}-h-\bar{h})-m+1). \tag{5.35}$$

5.4 Structure constants

When taking as generators for supertranslations the unnormalized spherical harmonics (5.17), one can now work out the structure constants of the $\mathfrak{bms}_4$ algebra (4.34)–(4.35) by using properties (A.7) and (A.10). For the first part of the algebra, they can be read off from the commutation relations

$$[\mathcal{Y}_m, \mathcal{Y}_n] = (m-n)\mathcal{Y}_{m+n}, \quad [\bar{\mathcal{Y}}_m, \bar{\mathcal{Y}}_n] = (m-n)\bar{\mathcal{Y}}_{m+n}, \quad [\mathcal{Y}_m, \bar{\mathcal{Y}}_n] = 0, \tag{5.36}$$

while those involving supertranslation generators can be obtained from

$$\begin{aligned}[\mathcal{Y}_{-1}, \mathcal{T}_{j,m}] &= +\frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m-1} - \frac{(j+m)}{2}\mathcal{T}_{j,m-1} \\ &\quad + (j-1)\mathcal{T}_{j+1,m-1},\end{aligned} \tag{5.37}$$

$$[\mathcal{Y}_0, \mathcal{T}_{j,m}] = -\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m} - \frac{m}{2}\mathcal{T}_{j,m} + (j-1)\mathcal{T}_{j+1,m}, \tag{5.38}$$

$$\begin{aligned}[\mathcal{Y}_1, \mathcal{T}_{j,m}] &= +\frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m+1} + \frac{(j-m)}{2}\mathcal{T}_{j,m+1} \\ &\quad + (j-1)\mathcal{T}_{j+1,m+1},\end{aligned} \tag{5.39}$$

The commutation relations involving $\bar{\mathcal{Y}}_m$ and $\mathcal{T}_{j,m}$ may then be obtained by complex conjugation. They are explicitly given by

$$[\bar{\mathcal{Y}}_{-1}, \mathcal{T}_{j,m}] = -\frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m+1} + \frac{(j-m)}{2}\mathcal{T}_{j,m+1} - (j-1)\mathcal{T}_{j+1,m+1}, \tag{5.40}$$

$$[\bar{\mathcal{Y}}_{0}, \mathcal{T}_{j,m}] = -\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m} + \frac{m}{2}\mathcal{T}_{j,m} + (j-1)\mathcal{T}_{j+1,m}, \tag{5.41}$$

$$[\bar{\mathcal{Y}}_{1}, \mathcal{T}_{j,m}] = -\frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)}\mathcal{T}_{j-1,m-1} - \frac{(j+m)}{2}\mathcal{T}_{j,m-1} - (j-1)\mathcal{T}_{j+1,m-1}. \tag{5.42}$$

Finally, the supertranslations generators commute with each other,

$$[\mathcal{T}_{j,m}, \mathcal{T}_{j',m'}] = 0. \tag{5.43}$$

In order to establish the relation to the commutation relations of section 2, one defines

$$l_m = \tilde{\mathcal{Y}}_m \partial, \quad \bar{l}_m = \bar{\tilde{\mathcal{Y}}}_m \bar{\partial}, \tag{5.44}$$

and takes into account equation (5.20) together with $\xi = \cot\frac{\theta}{2}e^{-i\phi}$. If one makes the identification of the generators as in (2.8) at $r\to\infty$ and $u=0$, it follows that

$$\begin{aligned}
L_z &= -i(l_0 - \bar{l}_0) = -i(\xi\partial - \bar{\xi}\bar{\partial}), & K_z &= -(l_0 + \bar{l}_0) = -(\xi\partial + \bar{\xi}\bar{\partial}),\\
L^+ &= +(l_1 + \bar{l}_{-1}) = \partial + \bar{\xi}^2\bar{\partial}, & L^- &= +(\bar{l}_1 + l_{-1}) = \bar{\partial} + \xi^2\partial,\\
K^+ &= -(\bar{l}_{-1} - l_1) = \partial - \bar{\xi}^2\bar{\partial}, & K^- &= -(l_{-1} - \bar{l}_1) = \bar{\partial} - \xi^2\partial.
\end{aligned} \tag{5.45}$$

This allows one to explicitly relate the commutation relations for the Lorentz algebra in (5.36) to those in (2.2), respectively to the first part of (2.4). The Poincaré generators are represented by

$$H = 1, \quad P_z = \frac{1-\xi\bar{\xi}}{2(1+\xi\bar{\xi})}, \quad P^+ = -\frac{\bar{\xi}}{1+\xi\bar{\xi}}, \quad P^- = \frac{\xi}{1+\xi\bar{\xi}}. \tag{5.46}$$

For functions f on the sphere, we now get instead of (2.9)

$$\begin{aligned}
&[L_z, f] = L_z(f), \quad [L^\pm, f] = L^\pm(f), \quad [K_z, f] = K_z(f) + \frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}f,\\
&[K^+, f] = K^+(f) + \frac{2\bar{\xi}}{1+\xi\bar{\xi}}f, \qquad [K^-, f] = K^-(f) + \frac{2\xi}{1+\xi\bar{\xi}}f.
\end{aligned} \tag{5.47}$$

When applied to the four Poincaré generators in (5.46), this reproduces the second part of (2.4).

In order to relate the action of the Lorentz generators on the supertranslation generators (5.17) given in (5.37)–(5.42) to the more standard form (2.19)–(2.23), we may start from the (first equality in the) relations (5.45) and use (5.37)–(5.42) to show that

$$\begin{aligned}
[L_z, \mathcal{T}_{j,m}] &= im\mathcal{T}_{j,m}, \quad [L^\pm, \mathcal{T}_{j,m}] = \pm(j \mp m)\mathcal{T}_{j,m\pm1}, \\
[K_z, \mathcal{T}_{j,m}] &= -2(j-1)\mathcal{T}_{j+1,m} + \frac{(j+2)(j+m)(j-m)}{2(2j+1)(2j-1)}\mathcal{T}_{j-1,m}, \\
[K^\pm, \mathcal{T}_{j,m}] &= \pm 2(j-1)\mathcal{T}_{j+1,m\pm1} \pm \frac{(j+2)(j\mp m)(j\mp m-1)}{2(2j+1)(2j-1)}\mathcal{T}_{j-1,m\pm1}.
\end{aligned} \tag{5.48}$$

When taking the normalization (A.3) into account, we then recover the commutation relations (2.19)–(2.23). Note that the commutation relations of L_z with the supertranslations generators are particularly simple since the latter are expressed in terms of (unnormalized) spherical harmonics.

The choice of basis for the Lorentz algebra in (5.36) is adapted to the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ decomposition. It is thus useful to organize the supertranslation generators, or more generally, the functions on the sphere, accordingly. This will also allow us to compare directly with the realization on the punctured complex plane to be discussed below. Hence, instead of providing the commutation relation between the Lorentz and supertranslation generators, one may replace the latter by the overcomplete set of functions adapted to $\mathcal{T}$ of weights $[0,1]$,

$$\mathcal{T}^{\tilde{L}}_{k,l} = {}_{-\frac{1}{2},-\frac{1}{2}}Z^{\tilde{L}}_{k,l}. \tag{5.49}$$

One then finds

$$\begin{aligned}
[\mathcal{Y}_m, \mathcal{T}^{\tilde{L}}_{k,l}] &= \left(\frac{m}{2} - k\right)\mathcal{T}^{\tilde{L}+1}_{k+m,l} + \left(\tilde{L} + \frac{m}{2} - k\right)\mathcal{T}^{\tilde{L}+1}_{k+m+1,l+1}, \\
[\bar{\mathcal{Y}}_m, \mathcal{T}^{\tilde{L}}_{k,l}] &= \left(\frac{m}{2} - l\right)\mathcal{T}^{\tilde{L}+1}_{k,l+m} + \left(\tilde{L} + \frac{m}{2} - l\right)\mathcal{T}^{\tilde{L}+1}_{k+1,l+m+1},
\end{aligned} \tag{5.50}$$

$$[\mathcal{T}^{\tilde{L}}_{k,l}, \mathcal{T}^{\tilde{L}}_{k',l'}] = 0. \tag{5.51}$$

5.5 Coadjoint representation of the algebra

The coadjoint representation may now be written explicitly using (3.13). Alternatively, it can be derived using the results of subsection 4.2 together with (A.7) and (A.10). One finds,

$$\mathrm{ad}^*_{\mathcal{Y}_m}\mathcal{Y}^n_* = (-2m+n)\mathcal{Y}^{n-m}_*, \quad \mathrm{ad}^*_{\mathcal{Y}_m}\bar{\mathcal{Y}}^n_* = 0, \tag{5.52}$$

$$\mathrm{ad}^*_{\bar{\mathcal{Y}}_m}\bar{\mathcal{Y}}^n_* = (-2m+n)\bar{\mathcal{Y}}^{n-m}_*, \quad \mathrm{ad}^*_{\bar{\mathcal{Y}}_m}\mathcal{Y}^n_* = 0, \tag{5.53}$$

$$\begin{aligned}\mathrm{ad}^*_{\mathcal{Y}_{-1}}\mathcal{T}^{j,m}_* &= -\frac{(j+3)(j+m+2)(j+m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m+1}_* \\ &\quad +\frac{(j+m+1)}{2}\mathcal{T}^{j,m+1}_* - (j-2)\mathcal{T}^{j-1,m+1}_*,\end{aligned} \tag{5.54}$$

$$\mathrm{ad}^*_{\mathcal{Y}_0}\mathcal{T}^{j,m}_* = \frac{(j+3)(j+m+1)(j-m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m}_* + \frac{m}{2}\mathcal{T}^{j,m}_* - (j-2)\mathcal{T}^{j-1,m}_*, \tag{5.55}$$

$$\begin{aligned}\mathrm{ad}^*_{\mathcal{Y}_1}\mathcal{T}^{j,m}_* &= -\frac{(j+3)(j-m+2)(j-m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m-1}_* \\ &\quad -\frac{(j-m+1)}{2}\mathcal{T}^{j,m-1}_* - (j-2)\mathcal{T}^{j-1,m-1}_*,\end{aligned} \tag{5.56}$$

$$\begin{aligned}\mathrm{ad}^*_{\bar{\mathcal{Y}}_{-1}}\mathcal{T}^{j,m}_* &= +\frac{(j+3)(j-m+2)(j-m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m-1}_* \\ &\quad -\frac{(j-m+1)}{2}\mathcal{T}^{j,m-1}_* + (j-2)\mathcal{T}^{j-1,m-1}_*,\end{aligned} \tag{5.57}$$

$$\mathrm{ad}^*_{\bar{\mathcal{Y}}_0}\mathcal{T}^{j,m}_* = +\frac{(j+3)(j+m+1)(j-m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m}_* - \frac{m}{2}\mathcal{T}^{j,m}_* - (j-2)\mathcal{T}^{j-1,m}_*, \tag{5.58}$$

$$\begin{aligned}\mathrm{ad}^*_{\bar{\mathcal{Y}}_1}\mathcal{T}^{j,m}_* &= +\frac{(j+3)(j+m+2)(j+m+1)}{4(2j+3)(2j+1)}\mathcal{T}^{j+1,m+1}_* \\ &\quad +\frac{(j+m+1)}{2}\mathcal{T}^{j,m+1}_* + (j-2)\mathcal{T}^{j-1,m+1}_*,\end{aligned} \tag{5.59}$$

$$\mathrm{ad}^*_{\mathcal{T}_{j,m}}\mathcal{Y}^p_* = 0 = \mathrm{ad}^*_{\mathcal{T}_{j,m}}\bar{\mathcal{Y}}^p_*, \tag{5.60}$$

$$\begin{aligned}\mathrm{ad}^*_{\mathcal{T}_{j,m}}\mathcal{T}^{j',m'}_* = &\left(-\frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} + \frac{(j+m)}{2}\delta^{j'}_j - (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m-1}\mathcal{Y}^{-1}_* \\ &+\left(-\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} - \frac{m}{2}\delta^{j'}_j + (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m}\mathcal{Y}^{0}_* \\ &+\left(-\frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} - \frac{(j-m)}{2}\delta^{j'}_j - (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m+1}\mathcal{Y}^{1}_* \\ &+\left(+\frac{(j+2)(j-m)(j-m-1)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} - \frac{(j-m)}{2}\delta^{j'}_j + (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m+1}\bar{\mathcal{Y}}^{-1}_* \\ &+\left(-\frac{(j+2)(j+m)(j-m)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} + \frac{m}{2}\delta^{j'}_j + (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m}\bar{\mathcal{Y}}^{0}_* \\ &+\left(+\frac{(j+2)(j+m)(j+m-1)}{4(2j+1)(2j-1)}\delta^{j'}_{j-1} + \frac{(j+m)}{2}\delta^{j'}_j + (j-1)\delta^{j'}_{j+1}\right)\delta^{m'}_{m-1}\bar{\mathcal{Y}}^{1}_*.\end{aligned} \tag{5.61}$$

In terms of the overcomplete sets of functions, if one uses ${}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+4}_{k,l}$ rather than $\mathcal{T}^{j,m}_*$ for the expansion of $\mathcal{P}$, one may use (5.29) to replace equations (5.54)–(5.59) through

$$\begin{aligned}\mathcal{Y}_m \cdot {}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+4}_{k,l} &= -\left(\frac{3}{2}m+k\right){}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+5}_{k+m,l} + \left(\tilde{L}+4-\left(\frac{3}{2}m+k\right)\right){}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+5}_{k+m+1,l+1}, \\ \bar{\mathcal{Y}}_m \cdot {}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+4}_{k,l} &= -\left(\frac{3}{2}m+l\right){}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+5}_{k,l+m} + \left(\tilde{L}+4-\left(\frac{3}{2}m+l\right)\right){}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+5}_{k+1,l+m+1},\end{aligned} \tag{5.62}$$

while equations (5.60) become

$$\mathrm{ad}^*_{\mathcal{T}^{\tilde{L}}_{k,l}}\mathcal{Y}^p_* = 0 = \mathrm{ad}^*_{\mathcal{T}^{\tilde{L}}_{k,l}}\bar{\mathcal{Y}}^p_*. \tag{5.63}$$

Finally, it also follows from

$$\langle{}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+4}_{k',l'}, \mathcal{T}^{\tilde{L}}_{k,l}\rangle = \delta^{k'+l-1}_m \delta^{k+l'-1}_m \beta(m+1, 2\tilde{L}+3-m), \tag{5.64}$$

the commutation relations (5.50), (5.51) and the definition of the coadjoint representation that

$$\begin{aligned}
\mathrm{ad}^*_{\mathcal{T}^{\tilde{L}}_{k,l}}\ {}_{\frac{3}{2},\frac{3}{2}}Z^{\tilde{L}+5}_{k',l'} = &\Bigg(\frac{k'-3k+l-l'}{2}\beta(k'+l, 2\tilde{L}+6-k'-l) \\
&+\left(\tilde{L}+\frac{k'-3k+l-l'}{2}\right)\beta(k'+l+1, 2\tilde{L}+5-k'-l)\Bigg)\mathcal{Y}^{k'-k+l-l'}_* \\
&+\Bigg(\frac{l'-3l+k-k'}{2}\beta(l'+k, 2\tilde{L}+6-l'-k) \\
&+\left(\tilde{L}+\frac{l'-3l+k-k'}{2}\right)\beta(l'+k+1, 2\tilde{L}+5-l'-k)\Bigg)\bar{\mathcal{Y}}^{l'-l+k-k'}_*.
\end{aligned} \tag{5.65}$$

6 Realization on the punctured complex plane

6.1 Generalities

Since the whole structure is Weyl invariant, one may start from the sphere with radius R and perform a Weyl rescaling as in (4.10) with

$$e^{-E(\xi,\bar{\xi})} = \frac{\sqrt{2}}{1+\xi\bar{\xi}}, \tag{6.1}$$

followed by the (conformal) coordinate transformations that consists of a simple rescaling $\xi = R^{-1}z$, $\bar{\xi} = R^{-1}\bar{z}$, so that the metric becomes

$$ds^2 = -2dzd\bar{z}. \tag{6.2}$$

The next step is to remove the points at infinity and at the origin to go to the 1-punctured complex plane $\mathbb{C}_0$. This changes the allowed space of functions. Conformal coordinate transformations are of the form

$$z' = z'(z), \quad \bar{z}' = \bar{z}'(\bar{z}), \tag{6.3}$$

where the globally well-defined ones that are connected to the identity are $z' = az, a \in \mathbb{C}, a \neq 0$. The derivative operators $\eth$ and $\bar{\eth}$ defined in (4.28) simply become ∂ and $\bar{\partial}$, respectively. There is no difference between conformal fields and weighted scalars. Indeed, freezing the conformal factor as in (4.12) with $P_F = 1 = \bar{P}_F$ yields

$$e^{E(x')} = \frac{\partial z'}{\partial z} \iff e^{E_R(x')} = \left(\frac{\partial z'}{\partial z}\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\frac{1}{2}},\ e^{iE_I(x')} = \left(\frac{\partial z'/\partial z}{\partial \bar{z}'/\partial \bar{z}}\right)^{\frac{1}{2}}, \tag{6.4}$$

which implies that conformal fields (of vanishing Weyl weights) and their associated weighted scalars through the map (4.16) are equal and transform in the same way (compare (4.15) and (4.18) by taking (6.4) into account). In the following, we use the notation for conformal fields $\phi_{h,\bar{h}}$.

We assume here that conformal fields on the punctured complex plane may be expanded in series as

$$\phi_{h,\bar{h}}(z,\bar{z}) = \sum_{k,l} a_{k,l}\ {}_{h,\bar{h}}\tilde{Z}_{k,l}, \quad {}_{h,\bar{h}}\tilde{Z}_{k,l} = z^{-h-k}\bar{z}^{-\bar{h}-l}, \tag{6.5}$$

where the coefficients $a_{k,l} \in \mathbb{C}$ and satisfy suitable conditions that we will not discuss in detail here (see e.g. [36] for more details). We also assume that $h, \bar{h}$ are either integer or half-integer. In the former case $k, l \in \mathbb{Z}$, whereas in the latter case $k, l \in \frac{1}{2} + \mathbb{Z}$. Other choices are also possible. The reason we are choosing Neveu-Schwarz conditions here is that, up to factors of $(1 + z\bar{z})$, the functions that appear here then include those that have appeared naturally in the case of the sphere.

Residues with respect to z and $\bar{z}$ are defined as

$$\mathrm{Res}_z[\phi_{h,\bar{h}}](\bar{z}) = \sum_l a_{1-h,l}\bar{z}^{-\bar{h}-l}, \quad \mathrm{Res}_{\bar{z}}[\phi_{h,\bar{h}}](z) = \sum_k a_{k,1-\bar{h}} z^{-h-k}. \tag{6.6}$$

This allows one to define pairing

$$\langle \psi_{-\bar{h}+1,-h+1}, \phi_{h,\bar{h}} \tilde{\rangle} = \mathrm{Res}_z \mathrm{Res}_{\bar{z}}[\overline{\psi_{-\bar{h}+1,-h+1}}\phi_{h,\bar{h}}]. \tag{6.7}$$

This pairing is non-degenerate, and since $\mathrm{Res}_z[\partial\phi] = 0 = \mathrm{Res}_{\bar{z}}[\bar{\partial}\phi]$, it annihilates total derivatives ∂ and $\bar{\partial}$, as it should. The pairing can then be defined as

$$\langle ([\tilde{\mathcal{J}}], [\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}}), (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}, \tilde{\mathcal{T}}) \tilde{\rangle} = \langle \tilde{\mathcal{J}}, \tilde{\mathcal{Y}} \tilde{\rangle} + \langle \tilde{\bar{\mathcal{J}}}, \tilde{\bar{\mathcal{Y}}} \tilde{\rangle} + \langle \tilde{\mathcal{P}}, \tilde{\mathcal{T}} \tilde{\rangle}. \tag{6.8}$$

6.2 Adjoint and coadjoint representations of the group

The formulas for the adjoint and coadjoint representations of the group are the same than those for the conformal fields on the sphere, except for the general Jacobians $\partial z/\partial z', \partial\bar{z}/\partial\bar{z}'$,

$$\begin{aligned}
\tilde{\mathcal{Y}}'(z') &= \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde{\mathcal{Y}}(z), \\
\tilde{\bar{\mathcal{Y}}}'(\bar{z}') &= \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{-1} \tilde{\bar{\mathcal{Y}}}(z), \\
\tilde{\beta}'(x') &= \left(\frac{\partial z}{\partial z'}\right)^{-\frac{1}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{-\frac{1}{2}} \left(\tilde{\beta} - \left(\tilde{\mathcal{Y}}\partial\tilde{\alpha} - \frac{1}{2}\tilde{\alpha}\partial\tilde{\mathcal{Y}} + \text{c.c.}\right)\right)(x),
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
\tilde{\mathcal{J}}'(x') &= \left(\frac{\partial z}{\partial z'}\right)^{1} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{2} \left(\tilde{\mathcal{J}} + \left(\frac{1}{2}\tilde{\mathcal{T}}\bar{\partial}\tilde{\mathcal{P}} + \frac{3}{2}\bar{\partial}\tilde{\mathcal{T}}\tilde{\mathcal{P}}\right)\right)(x) \\
\tilde{\bar{\mathcal{J}}}'(x') &= \left(\frac{\partial z}{\partial z'}\right)^{2} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{1} \left(\tilde{\bar{\mathcal{J}}} + \left(\frac{1}{2}\tilde{\mathcal{T}}\partial\tilde{\mathcal{P}} + \frac{3}{2}\partial\tilde{\mathcal{T}}\tilde{\mathcal{P}}\right)\right)(x) \\
\tilde{\mathcal{P}}'(x') &= \left(\frac{\partial z}{\partial z'}\right)^{\frac{3}{2}} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{\frac{3}{2}} \tilde{\mathcal{P}}(x).
\end{aligned} \tag{6.10}$$

6.3 Expansions

In terms of the basis functions defined in (6.5),

$$\langle {}_{-\bar h+1,-h+1}\tilde Z_{k',l'},\, {}_{h,\bar h}\tilde Z_{k,l}\tilde\rangle = \delta^0_{l'+k}\delta^0_{k'+l}. \tag{6.11}$$

In particular, the dual becomes

$$({}_{h,\bar h}\tilde Z_{k,l})^* = {}_{-\bar h+1,-h+1}\tilde Z_{-l,-k} \iff (z^{-h-k}\bar z^{-\bar h-l})^* = z^{\bar h-1+l}\bar z^{h-1+k}. \tag{6.12}$$

The basis for the conformal fields relevant for the algebra is

$$\tilde{\mathcal{Y}}_m = {}_{-1,0}\tilde Z_{m,0} = z^{1-m}, \quad \bar{\tilde{\mathcal{Y}}}_m = {}_{0,-1}\tilde Z_{0,m} = \bar z^{1-m}, \quad \tilde{\mathfrak{T}}_{k,l} = {}_{-\frac12,-\frac12}\tilde Z_{k,l} = z^{\frac12-k}\bar z^{\frac12-l}, \tag{6.13}$$

where $m, k+\frac12, l+\frac12 \in \mathbb{Z}$. We have

$$\tilde{\mathcal{Y}} = \sum_{m\in\mathbb{Z}} \tilde y_m \tilde{\mathcal{Y}}_m, \quad \bar{\tilde{\mathcal{Y}}} = \sum_{m\in\mathbb{Z}} \bar{\tilde y}_m \bar{\tilde{\mathcal{Y}}}_m, \quad \tilde{\mathcal{T}} = \sum_{k,l\in\frac12+\mathbb{Z}} \tilde t_{k,l}\tilde{\mathfrak{T}}_{k,l}, \tag{6.14}$$

where $\bar{\tilde t}_{k,l} = \tilde t_{l,k}$ since $\tilde{\mathcal{T}}$ is real. For the coadjoint representation, one finds from (6.12) (or from the definition with equivalence classes when taking into account that $z^{-1}, \bar z^{-1}$ are not equivalent to zero because they are not the derivative of a monomial but of the logarithm),

$$\tilde{\mathcal{Y}}^m_* = z^{-1}\bar z^{-2+m}, \quad \bar{\tilde{\mathcal{Y}}}^m_* = z^{-2+m}\bar z^{-1}, \quad \tilde{\mathfrak{T}}^{k,l}_* = z^{-\frac32+l}\bar z^{-\frac32+k}, \tag{6.15}$$

we have

$$\tilde{\mathcal{J}} = \sum_{m\in\mathbb{Z}} \tilde j_m \tilde{\mathcal{Y}}^m_*, \quad \bar{\tilde{\mathcal{J}}} = \sum_{m\in\mathbb{Z}} \bar{\tilde j}_m \bar{\tilde{\mathcal{Y}}}^m_*, \quad \tilde{\mathcal{P}} = \sum_{k,l\in\frac12+\mathbb{Z}} \tilde p_{k,l}\tilde{\mathfrak{T}}^{k,l}_*, \tag{6.16}$$

where $\bar{\tilde p}_{k,l} = p_{l,k}$ since $\tilde{\mathcal{P}}$ is real.

In terms of basis elements, the representation (4.48) becomes

$$\tilde{\mathcal{Y}}_m \cdot {}_{h,\bar h}\tilde Z_{k,l} = -(hm+k)\; {}_{h,\bar h}\tilde Z_{k+m,l}, \quad \bar{\tilde{\mathcal{Y}}}_m \cdot {}_{h,\bar h}\tilde Z_{k,l} = -(\bar hm+l)\; {}_{h,\bar h}\tilde Z_{k,l+m}, \tag{6.17}$$

while

$$\begin{aligned} \tilde{\mathcal{Y}}_m \cdot ({}_{h,\bar h}\tilde Z_{k,l})^* &= [(\bar h-1)m+l]\; ({}_{h,\bar h}\tilde Z_{k,l-m})^*, \\ \bar{\tilde{\mathcal{Y}}}_m \cdot ({}_{h,\bar h}\tilde Z_{k,l})^* &= [(h-1)m+k]\; ({}_{h,\bar h}\tilde Z_{k-m,l})^*. \end{aligned} \tag{6.18}$$

6.4 Structure constants

As discussed in subsection 4.2, all the results stated there can be readily expressed in terms of conformal fields. In particular, the $\mathfrak{bms}_4$ algebra (4.34)–(4.35) simplifies to

$$[(\tilde{\mathcal{Y}}_1, \bar{\tilde{\mathcal{Y}}}_1, \tilde{\mathcal{T}}_1), (\tilde{\mathcal{Y}}_2, \bar{\tilde{\mathcal{Y}}}_2, \tilde{\mathcal{T}}_2)] = (\hat{\tilde{\mathcal{Y}}}, \hat{\bar{\tilde{\mathcal{Y}}}}, \hat{\tilde{\mathcal{T}}}), \tag{6.19}$$

where

$$\begin{cases} \hat{\tilde{\mathcal{Y}}} = \tilde{\mathcal{Y}}_1\partial\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_2\partial\tilde{\mathcal{Y}}_1\,, \\ \hat{\tilde{\mathcal{T}}} = \tilde{\mathcal{Y}}_1\partial\tilde{\mathcal{T}}_2 - \dfrac12\partial\tilde{\mathcal{Y}}_1\tilde{\mathcal{T}}_2 - (1\leftrightarrow 2) + \text{c.c.}\,. \end{cases} \tag{6.20}$$

In the basis (6.13), the commutation relations become

$$\begin{aligned} &[\tilde{\mathcal{Y}}_m, \tilde{\mathcal{Y}}_n] = (m-n)\tilde{\mathcal{Y}}_{m+n}, && [\bar{\tilde{\mathcal{Y}}}_m, \bar{\tilde{\mathcal{Y}}}_n] = (m-n)\bar{\tilde{\mathcal{Y}}}_{m+n}, \\ &[\tilde{\mathcal{Y}}_m, \tilde{\mathfrak{T}}_{k,l}] = \left(\frac12 m - k\right)\tilde{\mathfrak{T}}_{m+k,l}, && [\bar{\tilde{\mathcal{Y}}}_m, \tilde{\mathfrak{T}}_{k,l}] = \left(\frac12 m - l\right)\tilde{\mathfrak{T}}_{k,m+l} \\ &[\tilde{\mathcal{Y}}_m, \bar{\tilde{\mathcal{Y}}}_n] = 0 = [\tilde{\mathfrak{T}}_{k,l}, \tilde{\mathfrak{T}}_{r,s}]. \end{aligned} \tag{6.21}$$

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6.5 Coadjoint representation of the algebra

The coadjoint representation in the basis (6.15) may be obtained from the structure constants of the algebra contained in (6.21) using (3.13). Alternatively, it can be derived using the results of subsection 4.2 together with the explicit expressions of the generators and their duals (6.15), and also from (6.18). Explicitly,

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathcal{Y}}_m}\tilde{\mathcal{Y}}^n_* = (-2m+n)\tilde{\mathcal{Y}}^{n-m}_*, && \mathrm{ad}^*_{\tilde{\tilde{\mathcal{Y}}}_m}\tilde{\tilde{\mathcal{Y}}}^n_* = (-2m+n)\tilde{\tilde{\mathcal{Y}}}^{n-m}_*,\\
&\mathrm{ad}^*_{\tilde{\mathcal{Y}}_m}\tilde{\mathfrak{T}}^{k,l}_* = \left(-\frac{3}{2}m+k\right)\tilde{\mathfrak{T}}^{k-m,l}_*, && \mathrm{ad}^*_{\tilde{\tilde{\mathcal{Y}}}_m}\tilde{\mathfrak{T}}^{k,l}_* = \left(-\frac{3}{2}m+l\right)\tilde{\mathfrak{T}}^{k,l-m}_*,\\
&\mathrm{ad}^*_{\tilde{\mathfrak{T}}_{k,l}}\tilde{\mathfrak{T}}^{r,s}_* = \left(\frac{r-3k}{2}\right)\delta^s_l\tilde{\mathcal{Y}}^{r-k}_* + \left(\frac{s-3l}{2}\right)\delta^r_k\tilde{\tilde{\mathcal{Y}}}^{s-l}_*, &&\\
&\mathrm{ad}^*_{\tilde{\mathcal{Y}}_m}\tilde{\tilde{\mathcal{Y}}}^n_* = 0 = \mathrm{ad}^*_{\tilde{\tilde{\mathcal{Y}}}_m}\tilde{\mathcal{Y}}^n_*, && \mathrm{ad}^*_{\tilde{\mathfrak{T}}_{k,l}}\tilde{\mathcal{Y}}^m_* = 0 = \mathrm{ad}^*_{\tilde{\mathfrak{T}}_{k,l}}\tilde{\tilde{\mathcal{Y}}}^m_*.
\end{aligned} \tag{6.22}$$

7 Comments on the cylinder

The mapping from the punctured plane to the vertical cylinder is standard in the context of conformal field theory. It is defined through

$$z = e^{-i\frac{2\pi}{L_1}w}, \quad w = w_1 + iw_2, \quad w_1 \sim w_1 + L_1. \tag{7.1}$$

According to (4.51), conformal fields on the cylinder are related to those on the punctured plane through

$$\phi^{C_V}_{h,\bar{h}}(w,\bar{w}) = \left(-i\frac{2\pi}{L_1}z\right)^h\left(i\frac{2\pi}{L_1}\bar{z}\right)^{\bar{h}}\phi_{h,\bar{h}}(z,\bar{z}). \tag{7.2}$$

When naively substituting the expansion adapted to the punctured plane (6.5), the associated expansion on the cylinder is

$$\phi^{C_V}_{h,\bar{h}}(w,\bar{w}) = \sum_{k,l} a_{k,l}\ {}_{h,\bar{h}}Z^{C_V}_{k,l}, \quad {}_{h,\bar{h}}Z^{C_V}_{k,l} = i^{\bar{h}-h}\left(\frac{2\pi}{L_1}\right)^{h+\bar{h}} e^{i\frac{2\pi}{L_1}kw}\ e^{-i\frac{2\pi}{L_1}l\bar{w}}, \tag{7.3}$$

with k,l semi-integer when $h,\bar{h}$ are semi-integer. As usual for Neveu-Schwarz boundary conditions, it follows that for half-integer conformal weights, holomorphic or anti-holomorphic fields on the cylinder are anti-periodic.

The generators (6.13) of the $\mathfrak{bms}_4$ algebra become

$$\mathcal{Y}^{C_V}_m = i\left(\frac{2\pi}{L_1}\right)^{-1} e^{i\frac{2\pi}{L_1}mw}, \ \bar{\mathcal{Y}}^{C_V}_m = -i\left(\frac{2\pi}{L_1}\right)^{-1} e^{-i\frac{2\pi}{L_1}m\bar{w}}, \ \mathcal{T}^{C_V}_{k,l} = \left(\frac{2\pi}{L_1}\right)^{-1} e^{i\frac{2\pi}{L_1}kw}e^{-i\frac{2\pi}{L_1}l\bar{w}}, \tag{7.4}$$

while those of the coadjoint representation (6.15) become

$$\mathcal{Y}^m_{C_V*} = i\left(\frac{2\pi}{L_1}\right)^3 e^{i\frac{2\pi}{L_1}m\bar{w}}, \ \bar{\mathcal{Y}}^m_{C_V*} = -i\left(\frac{2\pi}{L_1}\right)^3 e^{-i\frac{2\pi}{L_1}mw}, \ \mathcal{T}^{k,l}_{C_V*} = \left(\frac{2\pi}{L_1}\right)^3 e^{-i\frac{2\pi}{L_1}lw}e^{i\frac{2\pi}{L_1}k\bar{w}}. \tag{7.5}$$

By construction, the commutation relations of the elements in (7.4) are unchanged: they are obtained from (6.21) by adding a superscript C_V to the generators.

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For the coadjoint representation, matters are more subtle. It remains true that the vector space generated by the elements of (7.5) is a representation of $\mathfrak{bms}_4$ in the sense of Remark (v) of section 4.2, which is explicitly given by adding a superscript, respectively subscript, C_V to the generators of (6.22). This is not however the coadjoint representation since there are issues with the pairing on the infinite cylinder. Indeed,

$$\begin{aligned}\langle\psi^{C_V}_{-\bar{h}+1,-h+1},\phi^{C_V}_{h,\bar{h}}\rangle_{C_V} &\equiv \frac{1}{8\pi^2}\int idw\wedge d\bar{w}\,[\overline{\psi^{C_V}_{-\bar{h}+1,-h+1}}\;\phi^{C_V}_{h,\bar{h}}]\\ &= \frac{1}{4\pi^2}\int_0^{L_1} dw_1\int_{-\infty}^{\infty} dw_2\,[\overline{\psi^{C_V}_{-\bar{h}+1,-h+1}}\;\phi^{C_V}_{h,\bar{h}}],\end{aligned}\tag{7.6}$$

and in particular,

$$\begin{aligned}\langle{}_{-\bar{h}+1,-h+1}Z^{C_V}_{k',l'},{}_{h,\bar{h}}Z^{C_V}_{k,l}\rangle_{C_V} &= \frac{1}{L_1}\delta^{k+l'}_m\delta^{k'+l}_m\quad\int_{-\infty}^{+\infty}dw_2\;e^{-\frac{4\pi}{L_1}w_2 m}\\ &= \frac{1}{2}\delta^{k+l'}_m\delta^{k'+l}_m\quad\int_{-\infty}^{+\infty}\frac{d\kappa}{2\pi}\,e^{\kappa m}.\end{aligned}\tag{7.7}$$

The remaining integral does not impose $m=0$, as in (6.11) in the context of the coadjoint representation on the punctured plane. For the infinite vertical cylinder, the functions

$${}_{h,\bar{h}}Z^{C_V}_{k,l} = i^{\bar{h}-h}\left(\frac{2\pi}{L_1}\right)^{h+\bar{h}} e^{i\frac{2\pi}{L_1}(k-l)w_1}\;e^{-\frac{2\pi}{L_1}(k+l)w_2},\tag{7.8}$$

are not appropriate for expansions. One should rather use

$$\phi^{C_V}_{h,\bar{h}}(w,\bar{w}) = \sum_m\int_{-\infty}^{+\infty}d\kappa\;a_m(\kappa)Z^{C_V}_m(\kappa),\quad Z^{C_V}_m(\kappa) = i^{\bar{h}-h}\left(\frac{2\pi}{L_1}\right)^{h+\bar{h}} e^{i\frac{2\pi}{L_1}mw_1}e^{i\frac{2\pi}{L_1}\kappa w_2},\tag{7.9}$$

which satisfy

$$\langle{}_{-\bar{h}+1,-h+1}Z^{C_V}_{m'}(\kappa'),{}_{h,\bar{h}}Z^{C_V}_m(\kappa)\rangle_{C_V} = \delta^{m'}_m\,\delta(\kappa'-\kappa).\tag{7.10}$$

8 Identification in non-radiative asymptotically flat spacetimes

We limit ourselves in this section to the case of the sphere. Asymptotically flat spacetimes in the Newman-Penrose-Unti sense are for instance defined in [18], end of section 9.8. Here we consider the case with the Maxwell field turned off, $\varphi_1 = 0 = \varphi_2$. Non-radiative spacetimes correspond to the subset of solutions with u-independent asymptotic part of the shear,

$$\partial_u\sigma^0 = 0,\tag{8.1}$$

(as well as its complex conjugate and all higher order u derivatives), so that the news and also Ψ^0_3, Ψ^0_4 vanish. It follows that

$$\Psi^0_2 - \bar{\Psi}^0_2 = \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0,\tag{8.2}$$

while the evolution equations imply that

$$\partial_u\Psi^0_2 = 0,\quad \Psi^0_1 = \Psi^0_1(\xi,\bar{\xi}) + u\eth\Psi^0_2.\tag{8.3}$$

Such non-radiative space-times are completely characterized by specifying, at the cut $u = 0$ of $\mathscr{I}^+$, the free data

$$\Psi_2^0 + \bar{\Psi}_2^0, \Psi_1^0, \sigma^0, \tag{8.4}$$

together with the different orders Ψ_0^n in a $1/r$ expansion of Ψ_0,

$$\Psi_0 = \sum_{n\geq 0} \Psi_0^n(\xi, \bar{\xi})\, r^{-5-n} \tag{8.5}$$

Besides the linear dependence u dependence of Ψ_1^0 in (8.3), there are also evolution equations that govern the u-dependence of Ψ_0^n which do not concern us here.

The transformation of this data under BMS symmetries has been worked out in different ways and under various assumptions in [22, 37–39][1] In the case of the sphere, $P = P_S$ and the scalar curvature is $R_S = 2$. Furthermore, the solutions $\mathcal{Y}, \bar{\mathcal{Y}}$ to the conformal Killing equation on the sphere are given by (5.15) and (5.16). This implies in particular that

$$\eth^3\mathcal{Y} = 0 \quad \eth R_S = 0, \tag{8.6}$$

together with the complex conjugate relations.[2] In the non-radiating case and at $u = 0$, the infinitesimal transformations reduce to[3]

$$\begin{aligned}
\delta_s \Psi_2^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}}\right] \Psi_2^0, \\
\delta_s \Psi_1^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}\right] \Psi_1^0 + \mathcal{T}\eth\Psi_2^0 + 3\eth\mathcal{T}\Psi_2^0, \\
\delta_s \sigma^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}}\right] \sigma^0 - \eth^2\mathcal{T}, \\
\delta_s \Psi_0^0 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{5}{2}\eth\mathcal{Y} + \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}}\right] \Psi_0^0 + \mathcal{T}\eth\Psi_1^0 + 3\mathcal{T}\sigma^0\Psi_2^0 + 4\eth\mathcal{T}\Psi_1^0, \\
\delta_s \Psi_0^1 &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 3\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}\right] \Psi_0^1 - \bar{\eth}\left[5\eth\mathcal{T}\Psi_0^0 + \mathcal{T}\eth\Psi_0^0 + 4\mathcal{T}\Psi_1^0\sigma^0\right].
\end{aligned} \tag{8.7}$$

There are increasingly complicated transformations laws for the higher Ψ_0^n, $n \geq 2$, that are not relevant for our purpose here.

When expressing the first two of the equations in (8.7) in terms of the free data by taking the constraint (8.2) into account, one finds (trivially) that

$$\delta_s(\Psi_2^0 + \bar{\Psi}_2^0) = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}}](\Psi_2^0 + \bar{\Psi}_2^0), \tag{8.8}$$

and

$$\begin{aligned}
\delta_s \Psi_1^0 = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}]\Psi_1^0 &+ \frac{1}{2}\mathcal{T}\eth(\Psi_2^0 + \bar{\Psi}_2^0 + \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0) \\
&+ \frac{3}{2}\eth\mathcal{T}(\Psi_2^0 + \bar{\Psi}_2^0 + \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0).
\end{aligned} \tag{8.9}$$

[1]The arxiv version of the last reference is preferable to the published one on account of typesetting issues in the latter.

[2]Note that in the considerations below the value of $R_S = 2$ on the sphere is never needed, only the second of (8.6) is used.

[3]Up to a conventional overall sign that we have changed here.

Following the analysis in three dimensions, one fixes the normalization by computing the surface charge algebra, directly related to linear super momentum and angular momentum of the system. In the non-radiating case, this has been discussed for instance in section 4.2 of [38] (see also [39]). Let us summarize the relevant part of those results in the notation and conventions adopted here. Let

$$f = \mathcal{T} + \frac{1}{2}u(\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}}), \tag{8.10}$$

and consider the 2-form

$$J_s = \frac{i}{R^2}[(P_S\bar{P}_S)^{-1}\mathcal{J}^u_s d\xi\wedge d\bar{\xi} + P_S^{-1}\mathcal{J}^{\bar{\xi}}_s du\wedge d\xi - \bar{P}_S^{-1}\mathcal{J}^{\xi}_s du\wedge d\bar{\xi}], \tag{8.11}$$

with $\mathcal{J}^\xi_s = \mathcal{J}_s$, $\mathcal{J}^{\bar{\xi}} = \bar{\mathcal{J}}$ and

$$\begin{aligned}\mathcal{J}^u_s &= -\frac{1}{8\pi G}\Big[(\Psi^0_2+\bar{\Psi}^0_2)f + \Psi^0_{1\bar{J}}\mathcal{Y} + \bar{\Psi}^0_{1\bar{J}}\bar{\mathcal{Y}}\Big],\\ \mathcal{J}_s &= \frac{1}{8\pi G}\Big[\Psi^0_2\mathcal{Y} + \frac{1}{2}\eth\bar{\sigma}^0(\eth\mathcal{Y}-\bar{\eth}\bar{\mathcal{Y}}) - \frac{1}{2}\bar{\sigma}^0\eth(\eth\mathcal{Y}-\bar{\eth}\bar{\mathcal{Y}})\Big],\\ \Psi^0_{1\bar{J}} &= \Psi^0_1 + \sigma^0\eth\bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0).\end{aligned} \tag{8.12}$$

The transformation law of $\Psi^0_{1\bar{J}}$ turns out to be

$$\begin{aligned}\delta_s\Psi^0_{1\bar{J}} = &[\mathcal{Y}\eth+2\eth\mathcal{Y}]\Psi^0_{1\bar{J}}+\bar{\eth}(\bar{\mathcal{Y}}\Psi^0_{1\bar{J}})+\frac{1}{2}\mathcal{T}\eth(\Psi^0_2+\bar{\Psi}^0_2)+\frac{3}{2}\eth\mathcal{T}(\Psi^0_2+\bar{\Psi}^0_2)\\ &+\frac{1}{2}\bar{\eth}\left(\mathcal{T}\bar{\eth}\eth\sigma^0-\bar{\eth}\mathcal{T}\eth\sigma^0+3\eth\mathcal{T}\bar{\eth}\sigma^0-3\bar{\eth}\eth\mathcal{T}\sigma^0-\frac{3}{2}R_S\mathcal{T}\sigma^0\right)-\frac{1}{2}\eth^3(\mathcal{T}\bar{\sigma}^0),\end{aligned} \tag{8.13}$$

where the terms on the second line are irrelevant when multiplied by $\mathcal{Y}$ and integrated over the sphere (cf. Remark (i) in section 5.3.1).

When taking the retarded time-dependence of Ψ^0_1 in (8.3) and the constraint (8.2) into account, this 2-form is closed,

$$dJ_s = 0 \iff \partial_u\mathcal{J}^u_s + \eth\mathcal{J}_s + \bar{\eth}\bar{\mathcal{J}}_s = 0. \tag{8.14}$$

Furthermore,

$$\delta_{s_1}\mathcal{J}^u_{s_2} = -\mathcal{J}^u_{[s_1,s_2]} + \eth\mathcal{L}_{s_2,s_1} + \bar{\eth}\overline{\mathcal{L}_{s_2,s_1}}, \tag{8.15}$$

where the concrete expression for $\mathcal{L}_{s_2,s_1}$ is not needed here. This transformation law is in line with (4.46). The charges defined by

$$Q_s = \int_{S^2,u=u_0} J^u_s, \tag{8.16}$$

with u_0 constant, are conserved in the sense that they do not depend on u and

$$\delta_{s_1}Q_{s_2} = -Q_{[s_1,s_2]}. \tag{8.17}$$

More precisely, the polynomial algebra $\mathcal{F}$ generated by the free data $\Psi_2^0 + \bar{\Psi}_2^0, \Psi_1^0, \sigma^0, \bar{\Psi}_1^0, \bar{\sigma}^0$ carries a representation δ_s of the BMS_4 algebra. It then follows from the identification at $u = 0$,

$$Q_s = \langle([J], [\bar{J}], \mathcal{P}), (\mathcal{Y}, \bar{\mathcal{Y}}, \mathcal{T})\rangle, \tag{8.18}$$

that the pre-moment map $\mu : \mathcal{F} \to \mathfrak{bms}_4^*$ defined by

$$\mu\left(-\frac{1}{2G}[\Psi_2^0 + \bar{\Psi}_2^0]\right) = \mathcal{P}, \quad \mu\left(-\frac{1}{2G}\Psi_{1\bar{J}}^0\right) = [\bar{\mathcal{J}}], \quad \mu\left(-\frac{1}{2G}\bar{\Psi}_{1}^{0}{}_{\bar{\mathcal{J}}}\right) = [\mathcal{J}], \tag{8.19}$$

is compatible with the representation,

$$\mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu. \tag{8.20}$$

The transformation law of the asymptotic part of the shear implies in particular that

$$\delta_s(\bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0) = \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}}\right](\bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0). \tag{8.21}$$

This means that constraining the asymptotic part of the shear to be electric [5],

$$\bar{\eth}^2\sigma_e^0 = \eth^2\bar{\sigma}_e^0, \tag{8.22}$$

is a BMS invariant condition. In this case, the transformation law (8.9) simplifies and suggests

$$\mu\left(-\frac{1}{2G}\Psi_1^0\right) = \bar{\mathcal{J}} \sim \Psi_1^0, \quad \mu\left(-\frac{1}{2G}\bar{\Psi}_1^0\right) = \mathcal{J}. \tag{8.23}$$

That this is compatible with the previous identification can be seen as follows. The electric condition is solved by a real field $\chi_e = \bar{\chi}_e$ with the same weights $s = 0, w = 1$ than $\mathcal{T}$,

$$\begin{aligned}\sigma_e^0 &= \eth^2\chi_e, \quad \bar{\sigma}^0 = \bar{\eth}^2\chi_e,\\ \delta_s\chi_e &= \left[\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} - \frac{1}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}}\right]\chi_e - \mathcal{T} + \sum_{j\leq 1,m} \lambda^{jm}{}_0 Z_{j,m},\end{aligned} \tag{8.24}$$

where $\lambda^{jm} \in \mathbb{R}$. Inserting this solution into $\Psi_{1\bar{J}}^0$ one finds that it indeed agrees with Ψ_1^0 up to terms that are projected to zero by the map,

$$\Psi_{1\bar{J}}^0 = \Psi_1^0 + \frac{1}{2}\bar{\eth}\left(\eth^3\chi_e\bar{\eth}\chi_e + 3\eth^2\chi_e\eth\bar{\eth}\chi_e - \frac{3}{4}R_S\eth\chi_e\eth\chi_e\right) - \frac{1}{4}\eth^3(\bar{\eth}\chi_e\bar{\eth}\chi_e). \tag{8.25}$$

Relevant formulas for the group can be found in [22] and will not be repeated here. On the punctured plane, in order to have room for the Witt algebra, one cannot limit oneself to non-radiative spacetimes since turning off the news requires $\partial^3\tilde{\mathcal{Y}} = 0 = \bar{\partial}^3\bar{\tilde{\mathcal{Y}}}$. In the presence of news, currents are no longer conserved. Current algebra is broken both by flux terms and by a field dependent central extension discussed in more details in [27]. In this case, the last term in (8.13) is no longer trivial and becomes the associated (field-dependent) Souriau cocycle.

9 Discussion and perspectives

For the generalized BMS group on the sphere introduced in [40] (see also [41–44] for further considerations), the coadjoint representation is obtained from the approach developed here simply by removing the conformal Killing equation on infinitesimal superrotations $\bar{\eth}\mathcal{Y} = 0 = \eth\bar{\mathcal{Y}}$ and the associated equivalence relations on super-angular momentum $\mathcal{J}, \bar{\mathcal{J}}$. All fields should then simply be expanded in terms of spin-weighted spherical harmonics according to their weights.

A detailed recent study of the coadjoint representation of closely related semi-direct product groups involving diffeomeorphisms on the sphere, along the lines of our analysis in three dimensions [25, 26] (see also [45] for a review), has recently appeared in [46].

For the $\mathfrak{bms}_4$ algebra on the punctured plane, a discussion of central extensions can be found in [21], whereas deformations have been studied in detail in [47].

After having set up the basics in this paper, the next steps are to classify the coadjoint orbits, to re-discuss unitary irreducible representations [10, 48–52] from the viewpoint of the orbit method [53] and to construct the associated geometric actions [54, 55], as in the three dimensional case [56] (see [57] in this context). One could also explore whether some aspects of positive energy theorems for the Bondi mass [58–64] might be understood from such a perspective, again as in three dimensions [65].

The most interesting question is to understand in detail how such effective actions for the sector captured by the coadjoint representation interacts with the radiative degrees of freedom, as described in [66, 67] and more recently in [68], see also [69, 70] in this context.

Another more technical question is to extend the considerations in section 8 to a full-fledged momentum map at null infinity, as recently constructed at spatial infinity [71, 72], by starting from [73] and also [74].

In the case of celestial scattering amplitudes and soft theorems, the relevant surface is neither the (Riemann) sphere nor the plane, but rather two Riemann spheres with punctures related by an antipodal map. On each of these surfaces, the superrotation part of the extended algebra is given by the Witt algebra only if there are two particles/punctures. In this context, complementary aspects of the BMS group have been discussed in [75–93].

In this exposition here, we have followed the general relativity route from the sphere to the punctured plane. In conformal field theory, one travels in the opposite direction. For more punctures, the appropriate algebra should presumably be Krichever-Novikov algebras [94, 95].

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A Spin-weighted spherical harmonics

We follow the conventions of [17], section 4.15. Instructive alternative presentations and perspectives can be found in [96–99], section 1.10, [100, 101].

Let

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{i}{\sqrt{1+\xi\bar{\xi}}} \begin{pmatrix} -1 & \xi \\ \bar{\xi} & 1 \end{pmatrix}. \tag{A.1}$$

Let also s be integer or half-integer, $j \pm s \in \mathbb{N}$, $j \pm m \in \mathbb{N}$, $|m| \leq j$, $|s| \leq j$, and consider

$${}_sZ_{j,m} = \sum_r \frac{(j+m)!(j-m)!(j+s)!(j-s)!\alpha^r \beta^{j-m-r}\gamma^{j+s-r}\delta^{r+m-s}}{(2j)!r!(j-m-r)!(j+s-r)!(r+m-s)!}, \tag{A.2}$$

where the summation extends over integer values of r in the range $\max(0, s-m) \leq r \leq \min(j-m, j+s)$.

The spin-weighted spherical harmonics ${}_sY_{j,m}$ are then defined by

$${}_sY_{j,m} = (-1)^{j+m}\ {}_sZ_{j,m}\sqrt{\frac{(2j+1)!(2j)!}{4\pi(j+s)!(j-s)!(j+m)!(j-m)!}}. \tag{A.3}$$

For $s = 0$, one recovers the usual spherical harmonics functions, i.e., ${}_0Y_{j,m}$. The following properties hold:

- For each s, the ${}_sZ_{j,m}$ form an orthogonal basis for the spin-weighted scalars η^s on the sphere with

$$\langle {}_sZ_{j',m'}, {}_sZ_{j,m}\rangle = \frac{(j+s)!(j-s)!(j+m)!(j-m)!}{(2j+1)!(2j)!}\delta_{jj'}\delta_{mm'}, \tag{A.4}$$

  for the pairing (5.6). The ${}_sY_{j,m}$ form an orthonormal basis with

$$4\pi\langle {}_sY_{j',m'}, {}_sY_{j,m}\rangle = \delta_{jj'}\delta_{mm'}. \tag{A.5}$$

- The behavior under complex conjugation is

$$\overline{{}_sZ_{j,m}} = (-1)^{m+s}\ {}_{-s}Z_{j,-m}, \quad \overline{{}_sY_{j,m}} = (-1)^{3m+s}\ {}_{-s}Y_{j,-m}. \tag{A.6}$$

- The action of the operators ð and $\bar{\text{ð}}$ defined in (5.5) is explicitly given by

$$\eth\ {}_sZ_{j,m} = -\left(\frac{j-s}{R\sqrt{2}}\right) {}_{s+1}Z_{j,m}, \qquad \bar{\eth}\ {}_sZ_{j,m} = \left(\frac{j+s}{R\sqrt{2}}\right) {}_{s-1}Z_{j,m}. \tag{A.7}$$

$$\eth\ {}_sY_{j,m} = -\sqrt{\frac{(j+s+1)(j-s)}{2R^2}}\,{}_{s+1}Y_{j,m},$$

$$\bar{\eth}\ {}_sY_{j,m} = \sqrt{\frac{(j-s+1)(j+s)}{2R^2}}\,{}_{s-1}Y_{j,m}. \tag{A.8}$$

- The ${}_sZ_{j,m}$ and ${}_sY_{j,m}$ are eigenfunctions of the operator $\bar{\eth}\eth$:

$$\bar{\eth}\eth {}_sZ_{j,m} = -(j+s+1)(j-s)\frac{1}{2}\ {}_sZ_{j,m}, \quad \bar{\eth}\eth {}_sY_{j,m} = -(j+s+1)(j-s)\frac{1}{2}\ {}_sY_{j,m}. \tag{A.9}$$

- Products of spin-weighted spherical harmonics can be decomposed as

$$
\begin{aligned}
{}_{s_1}Z_{j_1,m_1}\ {}_{s_2}Z_{j_2,m_2} = & \sqrt{\frac{(j_1+s_1)!(j_1-s_1)!(j_1+m_1)!(j_1-m_1)!}{(2j_1)!(2j_1)!}} \\
& \times \sqrt{\frac{(j_2+s_2)!(j_2-s_2)!(j_2+m_2)!(j_2-m_2)!}{(2j_2)!(2j_2)!}} \\
& \times \sum_j (-1)^{j_1+j_2+j}\ {}_{(s_1+s_2)}Z_{j,(m_1+m_2)} \\
& \times \sqrt{\frac{(2j)!(2j)!}{(j+s_1+s_2)!(j-s_1-s_2)!(j+m_1+m_2)!(j-m_1-m_2)!}} \\
& \times \langle j_1,s_1;j_2,s_2|j,(s_1+s_2)\rangle\ \langle j_1,m_1;j_2,m_2|j,(m_1+m_2)\rangle,
\end{aligned}
\tag{A.10}
$$

or

$$
\begin{aligned}
{}_{s_1}Y_{j_1,m_1}\ {}_{s_2}Y_{j_2,m_2} = & \sum_j \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j+1)}}\ {}_{(s_1+s_2)}Y_{j,(m_1+m_2)} \\
& \times \langle j_1,s_1;j_2,s_2|j,(s_1+s_2)\rangle\ \langle j_1,m_1;j_2,m_2|j,(m_1+m_2)\rangle,
\end{aligned}
\tag{A.11}
$$

where the summation extends over integer values of j in the range $\max(|j_1-j_2|, |s_1+s_2|, |m_1+m_2|) \leq j \leq j_1+j_2$, and where $\langle j_1,m_1;j_2,m_2|j,(m_1+m_2)\rangle$ is a Clebsch-Gordan coefficient of the rotation group (see e.g. [15, 100] in this context).

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