

KLT factorization of winding string amplitudes

Jaume Gomis,^a Ziqi Yan^b and Matthew Yu^a

^a*Perimeter Institute for Theoretical Physics,
31 Caroline St N, Waterloo, ON N2L 2Y5, Canada*

^b*Nordita, KTH Royal Institute of Technology and Stockholm University,
Hannes Alfvéns väg 12, SE-106 91 Stockholm, Sweden*

E-mail: jgomis@pitp.ca, ziqi.yan@su.se, myu@pitp.ca

ABSTRACT: We uncover a Kawai-Lewellen-Tye (KLT)-type factorization of closed string amplitudes into open string amplitudes for closed string states carrying winding and momentum in toroidal compactifications. The winding and momentum closed string quantum numbers map respectively to the integer and fractional winding quantum numbers of open strings ending on a D-brane array localized in the compactified directions. The closed string amplitudes factorize into products of open string scattering amplitudes with the open strings ending on a D-brane configuration determined by closed string data.

KEYWORDS: Bosonic Strings, Scattering Amplitudes, D-branes, Differential and Algebraic Geometry

ARXIV EPRINT: [2103.05013](https://arxiv.org/abs/2103.05013)

Contents

1	Introduction	1
2	Elements on winding closed strings	4
2.1	Closed string tachyons and cocycles	5
2.2	Higher-order vertex operators	7
2.3	Four closed string tachyon amplitudes	7
3	Open string amplitude and KLT relation	11
3.1	Winding open strings	11
3.2	Four open string tachyon amplitudes	13
3.3	Four-point KLT relation	16
4	KLT relation for higher-point amplitudes	18
4.1	Higher-point closed string amplitude	18
4.2	Splitting the worldsheet integrals	20
4.3	Higher-point KLT relation	23
5	Conclusions and outlooks	26
A	KLT and twisted period relation	27
A.1	Twisted cocycles and closed string amplitudes	27
A.2	Loaded cycles and open string amplitudes	29
A.3	Intersection number and twisted period relation	30

1 Introduction

The celebrated Kawai-Lewellen-Tye (KLT) relations show that tree-level closed string amplitudes factorize into a sum of quadratic products of open string amplitudes [1]. Intuitively, the KLT relations are a consequence of the factorization of complex integrals describing closed string amplitudes into contour integrals describing open string amplitudes. The simplest incarnation of such a factorization is the classic Riemann bilinear identity.

The KLT relations have revealed elegant structures in scattering amplitudes and led to surprising connections between different theories (see e.g. [2–11]). Originally, the KLT relations were proposed as a powerful method for evaluating tree-level closed string amplitudes [1]. Intriguingly, the discovery of such relations has also led to numerous insights into quantum field theory (QFT) amplitudes by considering the infinite tension limit of string amplitudes. For example, it has been shown that Born-Infeld, non-linear sigma models, and special Galileons fit into the QFT type of KLT relations [12], via the framework of the

CHY formalism [13]. The KLT relations also inspired studies of gravity as a double copy of gauge theory over the past decade [14]. The importance of the double-copy construction is justified by its use in significantly simplifying the calculation of gravity amplitudes at loop level [15, 16]. A deeper understanding of these remarkable relations may hold new surprises in simplifying practical computations of amplitudes in both field and string theory.

Connecting string theory with nature requires compactifying string theory down to four spacetime dimensions. Upon compactification, the extended nature of the string clearly distinguishes string theory from local QFT. The existence of string states winding along a compact direction, which do not have a local field theory description, enriches the quantum numbers carried by string states and the collection of string amplitudes in the theory, and leads to inherently stringy phenomena, such as T-duality.

In this paper we derive a KLT-like relation for \mathcal{N} -point scattering amplitudes of closed strings when a spatial direction is compactified on a circle and string states carry winding (and momentum) on the circle. This allows us to express any closed string amplitude in terms of open string amplitudes. But which open string amplitudes? At first sight there seems to be a puzzle. A closed string state on a circle of radius R can carry both momentum n/R and winding w along that circle, where $n, w \in \mathbb{Z}$. In contrast, an open string state can either carry momentum or winding along the circle, depending upon whether we impose Neumann or Dirichlet boundary conditions along the circle, but *cannot* carry both momentum and winding. The resolution of this conundrum is that, while the winding number w_i of the i -th closed string can be identified with the winding number of the i -th open string obeying Dirichlet boundary conditions, the closed string momentum n_i is encoded in the D-brane configuration where open string amplitudes are defined, and corresponds to the fractional winding number of an open string stretched between D-branes. Figure 1 shows the scattering of \mathcal{N} closed strings with momentum and winding data (n_i, w_i) . Figure 2 illustrates the corresponding D-brane configuration. The D-branes are equally separated by a distance which is T-dual to the circumference of the compactified circle. The associated open string amplitude is depicted in figure 3, and involves open strings defined on an array of $n_+ + 1$ D-branes, with n_+ defined in figure 1. The i -th open string winds w_i times around the full spatial circle; in addition, it traverses $n_i - 1$ D-branes, with n_i corresponding to the “fractional” part of winding number of the open string stretched between two D-branes. The fractional winding number is equal to $n_i \tilde{R}/R = \alpha' n_i/R^2$. The above mapping between the quantum numbers describing closed and open string states in the KLT-like relation is summarized in table 1. The precise meaning of these terms will be further clarified in section 3.2.¹

This paper is organized as follows. In section 2, we give an introduction to essential ingredients of winding closed strings, and evaluate in section 2.3 the four closed string

¹There is also a straightforward T-dual interpretation for open string scattering amplitudes with open strings ending on the D-brane configuration described in figure 2. In the T-dual frame, an open string ends on a stack of spacetime-filling D-branes in the presence of Wilson lines. Such an open string satisfies Neumann boundary conditions in the compactified direction and can carry nonzero momentum number (both integral and fractional, with the latter attributed to the Wilson line) but only zero winding. See section 3.3 for details.

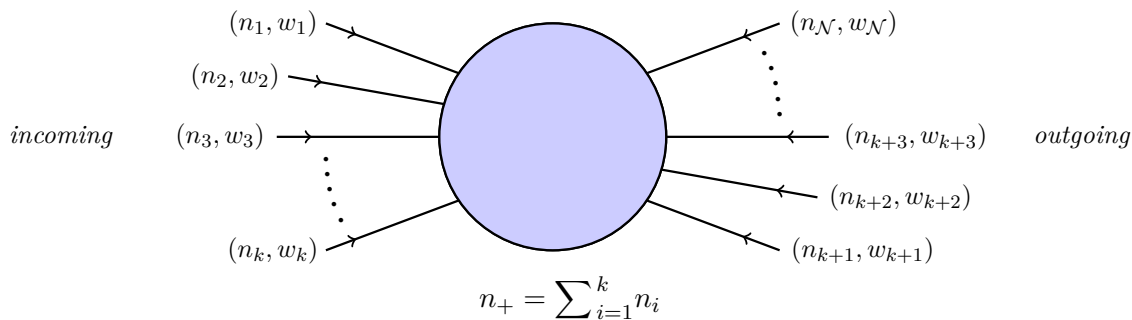


Figure 1. Closed string amplitude with momentum and winding data (n_i, w_i) obeying conservation laws $\sum_{i=1}^{\mathcal{N}} n_i = 0$ and $\sum_{i=1}^{\mathcal{N}} w_i = 0$. We choose $n_1, \dots, n_k > 0$ and $n_{k+1}, \dots, n_{\mathcal{N}} \leq 0$, splitting strings into incoming and outgoing states. The integer n_+ is the total incoming momentum. We show that the number of D-branes in the open string scattering amplitude appearing in the KLT-like factorization is $n_+ + 1$.

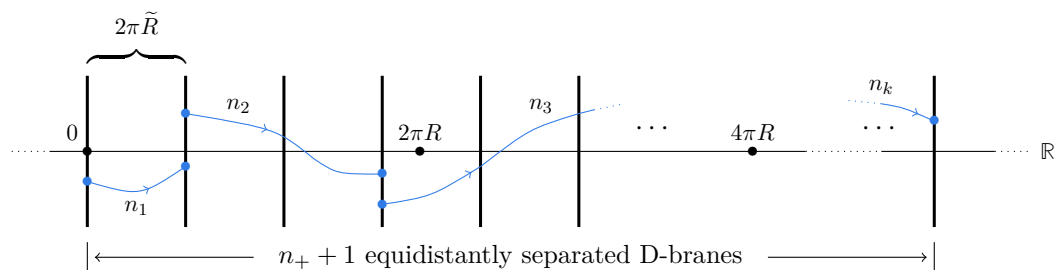


Figure 2. The horizontal line represents the real axis, which under the $x \sim x + 2\pi R$ identification becomes a circle of radius R . The vertical lines represent an array of equidistantly separated D-branes. The distance between consecutive D-branes is $2\pi\tilde{R}$, where $\tilde{R} = \alpha'/R$ is the T-dual radius. Each blue curve represents an incoming open string stretched between different D-branes. The momentum n_i of the i -th closed string is encoded in the number of D-branes traversed by the i -th open string and physically corresponds to fractional winding.

closed string	open string
momentum	fractional winding
winding	integer winding

Table 1. The mapping between quantum numbers of closed and open strings in the KLT relations.

tachyon² amplitude in the presence of windings. In section 3, we discuss winding open strings and derive in section 3.3 the KLT relation between four closed and open string tachyon amplitudes. In section 4, we generalize the KLT relation for winding strings to higher-point amplitudes that involve massless and massive vertex operators, and present the general form of the KLT relation in (4.23). We conclude our paper in section 5. In

²Although a closed string tachyon cannot simultaneously have nonzero momentum and winding along the compactified circle, the four-tachyon amplitude still provides a sufficiently interesting example that captures the salient properties of the KLT relation for winding strings.

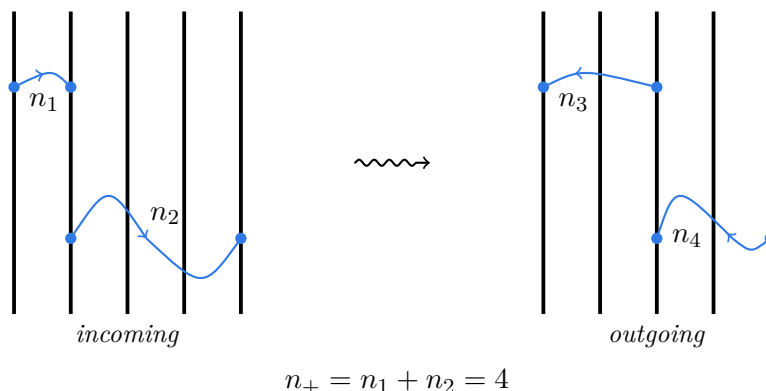


Figure 3. A four-point open string amplitude with $n_1 = 1$, $n_2 = 3$, $n_3 = -2$, and $n_4 = -2$. Open strings can only join or split at the same D-brane. There is a total of $n_+ + 1 = 5$ D-branes for this choice of closed string quantum numbers.

appendix A, we recast our new KLT relation in more modern terms using intersection theory, which gives rise to a compact rewriting of the KLT relation in (A.26).

2 Elements on winding closed strings

We consider string theory described by a sigma model that maps the worldsheet Σ to a d -dimensional target space \mathcal{M} . We take the worldsheet to be Euclidean and denote the worldsheet coordinates by $\sigma^\alpha = (\tau, \sigma)$, with $\tau \in \mathbb{R}$ and $\sigma \in [0, 2\pi]$. In the target space, we denote the coordinates by $\mathbb{X}^M = (X^\mu, X)$, where $M = 0, \dots, d - 1$ and $\mu = 0, 1, \dots, d - 2$. We have defined $X^\mu \equiv \mathbb{X}^\mu$ and $X \equiv \mathbb{X}^{d-1}$. We compactify X over a circle of radius R , such that

$$X \sim X + 2\pi R. \tag{2.1}$$

A closed string that winds w times around the circle along the X -direction satisfies

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w, \quad w \in \mathbb{Z}. \tag{2.2}$$

The mode expansion of X is

$$X(\tau, \sigma) = x - i\alpha' p \tau + R w \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} \left(\alpha_m e^{-im\sigma} + \tilde{\alpha}_m e^{im\sigma} \right) e^{-m\tau}, \tag{2.3}$$

where p denotes the momentum in the X -direction and α' is the Regge slope. The single-valuedness of the operator $\exp(ipX)$ requires the quantization condition

$$p = \frac{n}{R}, \quad n \in \mathbb{Z}. \tag{2.4}$$

Here, n is the Kaluza-Klein (KK) excitation number.

To perform radial quantization, we define

$$z = e^{\tau+i\sigma}, \quad \bar{z} = e^{\tau-i\sigma}, \tag{2.5}$$

in terms of which the mode expansion of X is $X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$, where

$$X_L(z) = x_L - \frac{i}{2} \alpha' p_L \ln z + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m}{m z^m}, \quad (2.6a)$$

$$X_R(\bar{z}) = x_R - \frac{i}{2} \alpha' p_R \ln \bar{z} + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\tilde{\alpha}_m}{m \bar{z}^m}, \quad (2.6b)$$

with $x = x_L + x_R$ and

$$p_L = \frac{n}{R} + \frac{Rw}{\alpha'}, \quad p_R = \frac{n}{R} - \frac{Rw}{\alpha'}. \quad (2.7)$$

Note that p_L and p_R are eigenvalues of the operators

$$\hat{p}_L = \frac{1}{\pi \alpha'} \oint_{\mathcal{C}} dz \partial_z X_L(z), \quad \hat{p}_R = -\frac{1}{\pi \alpha'} \oint_{\mathcal{C}} d\bar{z} \partial_{\bar{z}} X_R(\bar{z}), \quad (2.8)$$

where the counterclockwise oriented contour \mathcal{C} encloses the vertex operator associated with the string.

2.1 Closed string tachyons and cocycles

The closed string tachyon is described by the following vertex operator [17]:

$$\begin{aligned} \mathcal{V}_{\mathcal{C}}(z, \bar{z}) &= g_c \exp\left[\frac{i}{4} \pi \alpha' (p_L - p_R)(\hat{p}_L + \hat{p}_R)\right] : e^{iK_L \cdot \mathbb{X}_L(z) + iK_R \cdot \mathbb{X}_R(\bar{z})} : \\ &= g_c \exp\left[\frac{i}{2} \pi R w (\hat{p}_L + \hat{p}_R)\right] : e^{iK_L \cdot \mathbb{X}_L(z) + iK_R \cdot \mathbb{X}_R(\bar{z})} :, \end{aligned} \quad (2.9)$$

where $K_{L,R} \cdot \mathbb{X}_{L,R} = \eta_{MN} K_{L,R}^M \mathbb{X}_{L,R}^N$, and

$$\mathbb{X}_L^M = (X^\mu, X_L), \quad K_L^M = (k^\mu, p_L), \quad (2.10a)$$

$$\mathbb{X}_R^M = (X^\mu, X_R), \quad K_R^M = (k^\mu, p_R). \quad (2.10b)$$

The extra phase factor in (2.9) is known as the *cocycle*, which is important for removing the phases arising from crossing certain branch cuts when different vertex operators are interchanged with each other.

To demonstrate that (2.9) is the correct vertex operator, we first note the operator product expansions (OPEs),

$$X_L(z_1) X_L(z_2) \sim -\frac{\alpha'}{2} \ln z_{12}, \quad X_L(z_1) X^\mu(z_2, \bar{z}_2) \sim 0, \quad (2.11a)$$

$$X_R(\bar{z}_1) X_R(\bar{z}_2) \sim -\frac{\alpha'}{2} \ln \bar{z}_{12}, \quad X_R(\bar{z}_1) X^\mu(z_2, \bar{z}_2) \sim 0, \quad (2.11b)$$

$$X_L(z_1) X_R(\bar{z}_2) \sim 0, \quad (2.11c)$$

and

$$X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z_{12}|^2. \quad (2.11d)$$

We defined $z_{12} \equiv z_1 - z_2$ and $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$. It follows that

$$\hat{p}_L \mathcal{V}_C(z, \bar{z}) = \frac{1}{\pi \alpha'} \oint_C dw : \partial_w X_L(w) : \mathcal{V}_C(z, \bar{z}) = p_L \mathcal{V}_C(z, \bar{z}). \quad (2.12)$$

Therefore, $\mathcal{V}_C(z, \bar{z})$ corresponds to the eigenstate of $\hat{p}_{L,R}$. Similarly, we have

$$\hat{p}_R \mathcal{V}_C(z, \bar{z}) = p_R \mathcal{V}_C(z, \bar{z}). \quad (2.13)$$

We next note that the BRST invariance of (2.9) requires

$$-K_L^2 = -K_R^2 = -\frac{4}{\alpha'} \quad \implies \quad -k^2 = p_L^2 - \frac{4}{\alpha'} = p_R^2 - \frac{4}{\alpha'}, \quad (2.14)$$

where $k^2 = k_\mu k^\mu$. It then follows that the dispersion relation is

$$-k^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} - \frac{4}{\alpha'}, \quad (2.15)$$

when supplemented with the level matching condition

$$n w = 0. \quad (2.16)$$

This level matching condition implies that an asymptotic tachyonic state cannot have both nonzero Kaluza-Klein and winding numbers.

Finally, we consider the product of two tachyon operators, such as

$$\mathcal{V}_{C_1}(z_1, \bar{z}_1) \mathcal{V}_{C_2}(z_2, \bar{z}_2), \quad (2.17)$$

which is taken to be radially ordered, i.e. C_1 encloses C_2 . It follows that

$$\begin{aligned} & \mathcal{V}_{C_1}(z_1, \bar{z}_1) \mathcal{V}_{C_2}(z_2, \bar{z}_2) \\ & \sim g_c^2 e^{i\pi w_1 n_2} z_{12}^{\frac{1}{2}\alpha' K_{L1} \cdot K_{L2}} \bar{z}_{12}^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} : e^{i(K_{L1} + K_{L2}) \cdot \mathcal{X}_L(z) + i(K_{R1} + K_{R2}) \cdot \mathcal{X}_R(\bar{z})} : \\ & = g_c^2 e^{i\pi w_1 n_2} \left(e^{\frac{1}{2}i\pi\alpha' K_{L1} \cdot K_{L2}} z_{21}^{\frac{1}{2}\alpha' K_{L2} \cdot K_{L1}} \right) \left(e^{-\frac{1}{2}i\pi\alpha' K_{R1} \cdot K_{R2}} \bar{z}_{21}^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R1}} \right) \\ & \quad \times : e^{i(K_{L2} + K_{L1}) \cdot \mathcal{X}_L(z) + i(K_{R2} + K_{R1}) \cdot \mathcal{X}_R(\bar{z})} : . \end{aligned} \quad (2.18)$$

Using

$$\frac{1}{2} \alpha' (K_{L1} \cdot K_{L2} - K_{R1} \cdot K_{R2}) = n_1 w_2 + n_2 w_1, \quad (2.19)$$

we find

$$\begin{aligned} & \mathcal{V}_{C_1}(z_1, \bar{z}_1) \mathcal{V}_{C_2}(z_2, \bar{z}_2) \\ & \sim e^{2i\pi w_1 n_2} \left[g_c^2 e^{i\pi w_2 n_1} z_{21}^{\frac{1}{2}\alpha' K_{L2} \cdot K_{L1}} \bar{z}_{21}^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R1}} : e^{i(K_{L2} + K_{L1}) \cdot \mathcal{X}_L(z) + i(K_{R2} + K_{R1}) \cdot \mathcal{X}_R(\bar{z})} : \right] \\ & \sim e^{2i\pi w_1 n_2} \mathcal{V}_{C_2}(z_2, \bar{z}_2) \mathcal{V}_{C_1}(z_1, \bar{z}_1). \end{aligned} \quad (2.20)$$

Since both w_1 and n_2 are integers, we find that $\mathcal{V}_{C_1}(z_1, \bar{z}_1)$ and $\mathcal{V}_{C_2}(z_2, \bar{z}_2)$ commute.

2.2 Higher-order vertex operators

More general vertex operators can be constructed by multiplying vertex operator with factors of the form $\partial^m \mathbb{X}_L^M$ and $\bar{\partial}^m \mathbb{X}_R^M$, for positive integer m . BRST invariance imposes various conditions on such operators. For example, we have the BRST-invariant (up to total derivatives) and gauge-fixed vertex operators

$$e^{\frac{1}{2} i\pi R w (\hat{p}_L + \hat{p}_R)} \varepsilon_{I_1 \dots I_N J_1 \dots J_{\tilde{N}}} \times : \partial \mathbb{X}_L^{I_1} \dots \partial \mathbb{X}_L^{I_N} \bar{\partial} \mathbb{X}_R^{J_1} \dots \bar{\partial} \mathbb{X}_R^{J_{\tilde{N}}} e^{iK_L \cdot \mathbb{X}_L(z) + iK_R \cdot \mathbb{X}_R(\bar{z})} :, \quad (2.21)$$

where $\varepsilon_{M_1 \dots M_k N_1 \dots N_\ell}$ is symmetric in $\{M_1, \dots, M_k\}$ and in $\{N_1, \dots, N_\ell\}$, respectively, and

$$K_L^{M_i} \varepsilon_{M_1 \dots M_i \dots M_k N_1 \dots N_\ell} = 0, \quad \eta^{M_i N_j} \varepsilon_{M_1 \dots M_i \dots M_k N_1 \dots N_j \dots N_\ell} = 0, \quad (2.22a)$$

$$K_R^{N_i} \varepsilon_{M_1 \dots M_k N_1 \dots N_i \dots N_\ell} = 0. \quad (2.22b)$$

The dispersion relation and the level matching conditions are

$$-k^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} (N + \tilde{N} - 2), \quad n w = \tilde{N} - N. \quad (2.23)$$

Unlike closed string tachyons, the vertex operator (2.21) can simultaneously have nonzero KK and winding quantum numbers.

For an example of a BRST-invariant vertex operator that contains higher derivatives acting on $X_{L,R}$, we consider the gauge-fixed operator

$$e^{\frac{1}{2} i\pi R w (\hat{p}_L + \hat{p}_R)} \varepsilon_{MN} : \left(\frac{i}{2} \alpha' K_L^M \partial^2 \mathbb{X}_L^N + \partial \mathbb{X}_L^M \partial \mathbb{X}_L^N \right) e^{iK_L \cdot \mathbb{X}_L(z) + iK_R \cdot \mathbb{X}_R(\bar{z})} :, \quad (2.24)$$

where ε_{MN} is a symmetric tensor that satisfies

$$\varepsilon^M_M = \alpha' K_L^M K_L^N \varepsilon_{MN}. \quad (2.25)$$

The dispersion relation and the level matching condition in this case are

$$-k^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2}, \quad n w = -2. \quad (2.26)$$

Other higher-derivative vertex operators can be constructed similarly.

2.3 Four closed string tachyon amplitudes

We start with the tree-level four tachyon amplitude, which exhibits the simplest nontrivial KLT relation. Due to the level matching condition (2.16), a tachyonic state can either have a nonzero KK or winding number. Later in section 4, we will consider amplitudes that involve massless and massive vertex operators discussed in section 2.2, in which case

an asymptotic closed string state can have both nonzero KK and winding numbers. The amplitude for four closed string tachyons on a spherical worldsheet is

$$\begin{aligned} \mathcal{A}_c^{(4)} &= e^{-\chi \Phi_0} \int_{\mathbb{C}} d^2 z_2 \left\langle :c(z_1) \tilde{c}(\bar{z}_1) : :c(z_3) \tilde{c}(\bar{z}_3) : :c(z_4) \tilde{c}(\bar{z}_4) : \prod_{i=1}^4 : \mathcal{V}_{\mathcal{C}_i}(z_i, \bar{z}_i) : \right\rangle_{\mathbb{S}^2} \\ &\propto i (2\pi)^{25} \delta^{(25)}(k_1 + \dots + k_4) \delta_{n_1 + \dots + n_4, 0} \delta_{w_1 + \dots + w_4, 0} \mathcal{M}^{(4)}. \end{aligned} \quad (2.27)$$

Here, $\chi = 2$ is the Euler characteristic of the sphere and Φ_0 is the expectation value of the dilaton field. We have dropped a proportionality constant that depends on the coupling g_c , which we do not need in this paper. The vertex operators $\mathcal{V}_{\mathcal{C}_i}$ are ordered radially, with the integration contour \mathcal{C}_i enclosing \mathcal{C}_j , $i < j$. It follows that

$$\mathcal{M}^{(4)} = \frac{1}{\alpha'} |z_{13}|^2 |z_{14}|^2 |z_{34}|^2 \int_{\mathbb{C}} d^2 z_2 \prod_{\substack{i,j=1 \\ i < j}}^4 e^{i\pi w_i n_j} z_{ij}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj}} \bar{z}_{ij}^{\frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}}. \quad (2.28)$$

The vertex operator $\mathcal{V}_{\mathcal{C}_i}(z_i, \bar{z}_i)$ is defined in (2.9). We have used the Möbius transformations to fix $z_1 = 0$, $z_3 = 1$, and $z_4 = \infty$. Using the identity

$$\begin{aligned} z_{ij}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj}} \bar{z}_{ij}^{\frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}} &= \left(e^{\frac{1}{2}i\pi\alpha' K_{Li} \cdot K_{Lj}} z_{ji}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj}} \right) \left(e^{-\frac{1}{2}i\pi\alpha' K_{Ri} \cdot K_{Rj}} \bar{z}_{ji}^{\frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}} \right) \\ &= e^{i\pi(n_i w_j + n_j w_i)} z_{ji}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj}} \bar{z}_{ji}^{\frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}}, \end{aligned} \quad (2.29)$$

we find that (2.27) becomes

$$\mathcal{M}_c^{(4)} = \frac{1}{\alpha'} \exp \left(i\pi \sum_{\substack{i,j=1 \\ i < j}}^4 n_i w_j \right) \mathcal{I}^{(4)}, \quad (2.30)$$

where

$$\mathcal{I}^{(4)} = \int_{\mathbb{C}} d^2 z_2 z_2^{\frac{1}{2}\alpha' K_{L1} \cdot K_{L2}} \bar{z}_2^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (1 - z_2)^{\frac{1}{2}\alpha' K_{L2} \cdot K_{L3}} (1 - \bar{z}_2)^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R3}}. \quad (2.31)$$

This integral can be studied using intersection theory, which gives rise to a geometrical interpretation of the associated KLT relation in terms of the twisted (co)homology and intersection number. We will discuss this modern interpretation later in appendix A. In practice, however, we will perform the integrals following the method used in the original work in [1], without resorting to intersection theory. This allows us to focus on the novelties brought by nonzero windings.

Define $z_2 = x + i\tilde{y}$ (and $\bar{z}_2 = x - i\tilde{y}$) for real variables x and \tilde{y} , and then promote the integral over \tilde{y} to be in the complex plane. Then, the integrand for \tilde{y} has its branch points at $\pm ix$ and $\pm i(1-x)$. We deform the contour for the \tilde{y} -integral that is along the real axis counterclockwise by $\frac{\pi}{2} - \epsilon$, for $\epsilon \rightarrow 0^+$. This deformation is captured by the following change of variables:

$$\tilde{y} = \exp \left[i \left(\frac{\pi}{2} - \epsilon \right) \right] y \approx iy + \epsilon y. \quad (2.32)$$

Therefore,³

$$\begin{aligned} \mathcal{I}^{(4)} = & \frac{i}{2} \int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} d\xi (\zeta + i\delta)^{\frac{1}{2}\alpha' K_{L1} \cdot K_{L2}} (1 - \zeta - i\delta)^{\frac{1}{2}\alpha' K_{L2} \cdot K_{L3}} \\ & \times (\xi - i\delta)^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (1 - \xi + i\delta)^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R3}}, \end{aligned} \quad (2.33)$$

where

$$\xi \equiv x + y, \quad \zeta \equiv x - y, \quad \delta \equiv \epsilon y = \frac{1}{2} \epsilon (\xi - \zeta). \quad (2.34)$$

For a fixed $\zeta \in (-\infty, 0)$, the ξ -integral has branch points at

$$i\delta|_{\xi=0} = \frac{1}{2} i\epsilon(-\zeta), \quad 1 + i\delta|_{\xi=1} = 1 + \frac{1}{2} i\epsilon(1 - \zeta). \quad (2.35)$$

Importantly, both of the branch points reside in the upper half plane. Completing the integration contour for ξ in the lower half plane, the integral over ξ yields zero. A similar argument also shows that $\zeta \in (1, \infty)$ contributes zero to the amplitude. Therefore, only when $\zeta \in (0, 1)$, the amplitude receive nonzero contributions and

$$\mathcal{I}^{(4)} = \frac{1}{2} i \mathcal{I}_L^{(4)} \mathcal{I}_R^{(4)}, \quad (2.36)$$

where

$$\mathcal{I}_L^{(4)} = \int_0^1 d\zeta \zeta^{\frac{1}{2}\alpha' K_{L1} \cdot K_{L2}} (1 - \zeta)^{\frac{1}{2}\alpha' K_{L2} \cdot K_{L3}}, \quad (2.37a)$$

$$\mathcal{I}_R^{(4)} = \int_{\mathbb{R}} d\xi (\xi - i\delta)^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (1 - \xi + i\delta)^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R3}}. \quad (2.37b)$$

In $\mathcal{I}_R^{(4)}$, there are two branch points at the positions given in (2.35), but now with $\zeta \in (0, 1)$. This implies that the branch point at $i\delta|_{\xi=0}$ resides in the lower half plane, and the branch point at $1 + i\delta|_{\xi=1}$ resides in the upper half plane. We then take the part of the ξ -integral contour that lies to the right of $\xi = 0$, and rotate it by 180° across the lower half plane of the complex plane, around the branch point $\xi = i\delta|_{\xi=0}$. Note that this contour deformation does not cross any branch points. See [1] for more details. We find

$$\mathcal{I}_R^{(4)} = 2i \sin\left(\frac{1}{2} \pi \alpha' K_{R1} \cdot K_{R2}\right) \int_{-\infty}^0 d\xi (-\xi)^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (1 - \xi)^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R3}}. \quad (2.38)$$

We can alternatively take the part of the original ξ -integral contour that lies to the left of $\xi = 1$ to the right, and rotate is by 180° across the upper half plane, around the branch point $1 + i\delta|_{\xi=1}$. This contour deformation does not cross any branch points, either. This second choice of integration contour gives

$$\mathcal{I}_R^{(4)} = 2i \sin\left(\frac{1}{2} \pi \alpha' K_{R2} \cdot K_{R3}\right) \int_1^\infty d\xi \xi^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (\xi - 1)^{\frac{1}{2}\alpha' K_{R2} \cdot K_{R3}}. \quad (2.39)$$

To proceed further, we introduce the generalized Mandelstam variables,

$$s_L = -(K_{L1} + K_{L2})^2, \quad t_L = -(K_{L1} + K_{L3})^2, \quad u_L = -(K_{L1} + K_{L4})^2, \quad (2.40a)$$

$$s_R = -(K_{R1} + K_{R2})^2, \quad t_R = -(K_{R1} + K_{R3})^2, \quad u_R = -(K_{R1} + K_{R4})^2, \quad (2.40b)$$

³We use the convention $d^2 z_2 = dx d\tilde{y}$ in (2.31).

and define $\alpha_x \equiv -\frac{1}{4}\alpha'x - 1$. Note that $\alpha_{s_L} + \alpha_{t_L} + \alpha_{u_L} = 1$ and $\alpha_{s_R} + \alpha_{t_R} + \alpha_{u_R} = 1$. It follows that

$$\mathcal{I}_L^{(4)} = B(\alpha_{s_L}, \alpha_{u_L}), \quad (2.41a)$$

$$\begin{aligned} \mathcal{I}_R^{(4)} &= -2i \sin(\pi \alpha_{s_R}) B(\alpha_{s_R}, \alpha_{t_R}) \\ &= -2i \sin(\pi \alpha_{u_R}) B(\alpha_{u_R}, \alpha_{t_R}), \end{aligned} \quad (2.41b)$$

where $B(a, b)$ is the Euler beta function, with

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (2.42)$$

The second equality in (2.41b) can be shown explicitly by using the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \quad (2.43)$$

Therefore,

$$\mathcal{I}^{(4)} = B(\alpha_{u_L}, \alpha_{s_L}) \sin(\pi \alpha_{s_R}) B(\alpha_{s_R}, \alpha_{t_R}) \quad (2.44a)$$

$$= B(\alpha_{s_L}, \alpha_{u_L}) \sin(\pi \alpha_{u_R}) B(\alpha_{u_R}, \alpha_{t_R}). \quad (2.44b)$$

The same result in (2.44a) can be confirmed by evaluating $\mathcal{I}^{(4)}$ in (2.31) directly as in [18].⁴ Recall that, to derive this result, we fixed ζ while performing the integral over ξ . Alternatively, if we switch the order of integration by first performing the integral over ζ with ξ fixed, then

$$\mathcal{I}^{(4)} = B(\alpha_{u_R}, \alpha_{s_R}) \sin(\pi \alpha_{s_L}) B(\alpha_{s_L}, \alpha_{t_L}) \quad (2.45a)$$

$$= B(\alpha_{s_R}, \alpha_{u_R}) \sin(\pi \alpha_{u_L}) B(\alpha_{u_L}, \alpha_{t_L}). \quad (2.45b)$$

It is also straightforward to show that the expressions in (2.44) and (2.45) are equivalent by repetitively using (2.43). For example, applying (2.43) to the beta functions in (2.44a) gives

$$\begin{aligned} &B(\alpha_{u_L}, \alpha_{s_L}) \sin(\pi \alpha_{s_R}) B(\alpha_{s_R}, \alpha_{t_R}) \\ &= \frac{\sin(\pi \alpha_{s_R}) \sin(\pi \alpha_{t_L}) \sin(\pi \alpha_{u_R})}{\sin(\pi \alpha_{s_L}) \sin(\pi \alpha_{t_R}) \sin(\pi \alpha_{u_L})} B(\alpha_{u_R}, \alpha_{s_R}) \sin(\pi \alpha_{s_L}) B(\alpha_{s_L}, \alpha_{t_L}). \end{aligned} \quad (2.46)$$

From (2.19), we find

$$\sin\left(\frac{1}{2}\alpha'\pi K_{Ri} \cdot K_{Rj}\right) = (-1)^{n_i w_j + n_j w_i} \sin\left(\frac{1}{2}\alpha'\pi K_{Li} \cdot K_{Lj}\right), \quad (2.47)$$

which implies

$$\begin{aligned} &B(\alpha_{u_L}, \alpha_{s_L}) \sin(\pi \alpha_{s_R}) B(\alpha_{s_R}, \alpha_{t_R}) \\ &= (-1)^{n_1(w_2+w_3+w_4) + (n_2+n_3+n_4)w_1} B(\alpha_{u_R}, \alpha_{s_R}) \sin(\pi \alpha_{s_L}) B(\alpha_{s_L}, \alpha_{t_L}) \\ &= (-1)^{-2n_1 w_1} B(\alpha_{u_R}, \alpha_{s_R}) \sin(\pi \alpha_{s_L}) B(\alpha_{s_L}, \alpha_{t_L}) \\ &= B(\alpha_{u_R}, \alpha_{s_R}) \sin(\pi \alpha_{s_L}) B(\alpha_{s_L}, \alpha_{t_L}). \end{aligned} \quad (2.48)$$

This shows explicitly that (2.44a) and (2.45a) are equivalent.

⁴Note that the integral in (2.31) is defined to be over the complex plane with $d^2z_2 = dx d\tilde{y}$, which is different from the convention used in [18]. This leads to an extra factor 1/2 in our result.

3 Open string amplitude and KLT relation

The decomposition of the four closed string tachyon amplitude into the left-moving and right-moving parts in (2.36) is already in form the same as the standard KLT relation that relates closed string amplitudes to open string amplitudes. The exceptions here include the cocycle factors and that $\mathcal{I}_L^{(4)}$ and $\mathcal{I}_R^{(4)}$ depend on K_{Li} and K_{Ri} , respectively. The kinematic data K_{Li} and K_{Ri} are different from each other when the winding number w_i is nonzero. As we shall see the expressions of $\mathcal{I}_{L,R}^{(4)}$ in (2.37)–(2.39) correspond to open string amplitudes. This will be the case by introducing appropriate D-brane configurations, which we discuss in detail below.

3.1 Winding open strings

Consider a stack of D-branes that extend in all the noncompactified directions and are localized in the compactified X -direction. We require that the q -th brane is located at

$$x_q = x_0 + qL \pmod{2\pi R}, \quad q \in \mathbb{Z}, \quad (3.1)$$

in the compactified circle, such that the q -th and $(q+1)$ -th branes are separated by L . See figure 2 for an illustration, where $L = 2\pi\tilde{R}$ is fixed.

We now take the worldsheet to be a strip, with $\tau \in \mathbb{R}$ and $\sigma \in (0, \pi)$. Consider open strings that satisfy the Dirichlet boundary condition in the compactified X -direction and the Neumann boundary condition in directions along the D-branes. We fix the ends of a given open string at two of the D-branes — say, the q -th and $(q+n)$ -th D-brane, with $n \in \mathbb{Z}$ — separated by a distance $|n|L$.⁵ The compactified worldsheet field $X(\tau, \sigma)$ takes on the following mode expansion on the strip:

$$X(\tau, \sigma) = x_q + \frac{1}{\pi} (nL + 2\pi R w) \sigma - \sqrt{2\alpha'} \sum_{m \neq 0} \frac{\alpha_m}{m} e^{-m\tau} \sin(m\sigma). \quad (3.2)$$

Here, x_q is the location of the q -th D-brane and $x_{q+n} = x_q + nL$ is the location of the $(q+n)$ -th D-brane. The winding number w denotes how many additional loops the open string winds around the compactified circle. Note that, when $x_{q+n} = x_q$, the open string ends on the same D-brane. The worldsheet field $X(\tau, \sigma)$ in (3.2) satisfies the Dirichlet boundary conditions

$$X(\tau, 0) = x_q, \quad X(\tau, \pi) = x_{q+n}. \quad (3.3)$$

Under the change of coordinates introduced in (2.5),

$$z = e^{\tau+i\sigma}, \quad \bar{z} = e^{\tau-i\sigma}, \quad (3.4)$$

⁵Note that $|n|L$ can be larger than the circumference of the compactified circle, in which case the physical distance between the two D-branes along the circle can be smaller than $|n|L$.

the worldsheet is mapped to the upper half plane with $X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$ and

$$X_L(z) = x_L - \frac{i}{2\pi}(nL + 2\pi R w) \ln z - i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m}{m} \frac{1}{z^m}, \quad (3.5a)$$

$$X_R(\bar{z}) = x_R + \frac{i}{2\pi}(nL + 2\pi R w) \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m}{m} \frac{1}{\bar{z}^m}, \quad (3.5b)$$

with $x_q = x_L + x_R$. The boundary of the worldsheet is at $z = \bar{z}$.

The OPE between two X 's on the disk that satisfy the Dirichlet boundary conditions is

$$X(z_1) X(z_2) \sim -\frac{\alpha'}{2} \left(\ln |z_1 - z_2|^2 - \ln |z_1 - \bar{z}_2|^2 \right). \quad (3.6)$$

It then follows that the OPEs between X_L 's and X_R 's are

$$X_L(z_1) X_L(z_2) \sim -\frac{\alpha'}{2} \ln z_{12}, \quad X_L(z_1) X_R(\bar{z}_2) \sim \frac{\alpha'}{2} \ln(z_1 - \bar{z}_2), \quad (3.7a)$$

$$X_R(\bar{z}_1) X_R(\bar{z}_2) \sim -\frac{\alpha'}{2} \ln \bar{z}_{12}, \quad X_R(\bar{z}_1) X_L(z_2) \sim \frac{\alpha'}{2} \ln(\bar{z}_1 - z_2). \quad (3.7b)$$

The OPEs on the boundary can be readily derived by setting $z_i = \bar{z}_i = y_i$; the OPE in (3.6) vanishes on the boundary.

The total winding number of the open string can be non-integer, given by

$$W = \frac{1}{2\pi R} \oint_{\mathcal{C}_o} (dz \partial_z X_L + d\bar{z} \partial_{\bar{z}} X_R) = \frac{nL}{2\pi R} + w, \quad (3.8)$$

where the contour \mathcal{C} for an open string vertex operator inserted at y on the real axis traverses counterclockwise along a small semi-circle centered at y in the upper half plane. We refer to $nL/(2\pi R)$ in (3.8) as the *fractional* winding number of the open string, which means that the open string winds over a fraction of the compactified circle. From (3.8) we define the winding number operator

$$\hat{W} = \frac{1}{2\pi R} \oint_{\mathcal{C}_o} (dz \partial_z X + d\bar{z} \partial_{\bar{z}} X). \quad (3.9)$$

Similarly, the momentum operator in the compactified direction is

$$\hat{p} = \frac{1}{2\pi\alpha'} \oint_{\mathcal{C}_o} (dz \partial_z X_L - d\bar{z} \partial_{\bar{z}} X_R). \quad (3.10)$$

We are interested in the open string tachyon vertex operator that corresponds to the eigenstate of \hat{W} with the eigenvalue W in (3.8). Such a vertex operator takes the form

$$\mathcal{V}_o(y) = g_o : e^{iQ_L \cdot \mathbb{Y}_L(y) + iQ_R \cdot \mathbb{Y}_R(y)} : , \quad (3.11)$$

where $y \in \mathbb{R}$ and

$$\mathbb{Y}_{L,R}^M \equiv \mathbb{X}_{L,R}^M \Big|_{z=\bar{z}}, \quad Q_L^M = (k^\mu, WR/\alpha'), \quad (3.12a)$$

$$Q_R^M = (k^\mu, -WR/\alpha'). \quad (3.12b)$$

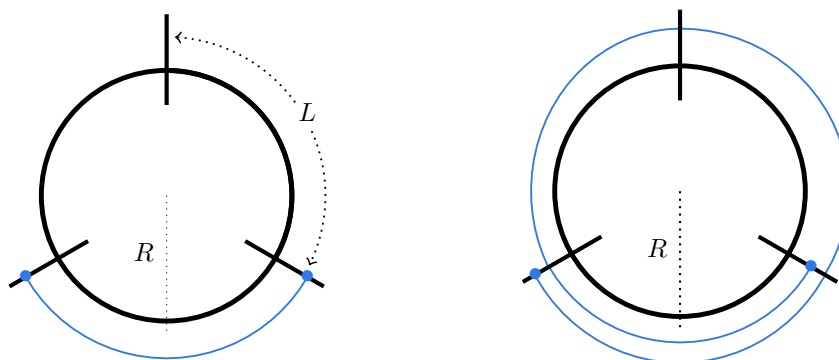


Figure 4. The circle of radius R represents the compactified X -direction and the ticks represent a series of D-branes transverse to the compactified circle, equidistantly positioned along X at $x_q = x_0 + qL$, $q = 0, 1, 2, \dots$, as shown in figure 2. The diagrams above depict the special case when $2\pi R/L = 3$, which is rational. As a result, maximally, there are only three transverse branes in the circle. Note that these diagrams also represent the D-brane configuration when L takes other values, for example, $L = 4\pi R/3$. An open string, shown in blue, can stretch between two D-branes as in the left diagram, or as in the right diagram by making an integer number of wrappings around the compactified direction. Generically, when $2\pi R/L$ is irrational, there is no maximal number of transverse branes in the circle.

The vertex operator \mathcal{V}_0 corresponds to an eigenstate of \hat{W} with eigenvalue W and of \hat{p} with eigenvalue $p = 0$, consistent with that open strings carry zero momentum in the direction satisfying the Dirichlet boundary condition. We recall that $\mathcal{X}^M = (X^\mu, X)$, with X^μ satisfying the Neumann boundary condition and X satisfying the Dirichlet boundary condition. The BRST invariance of \mathcal{V}_0 requires

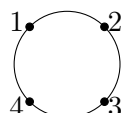
$$-k^2 = \frac{W^2 R^2}{\alpha'^2} - \frac{1}{\alpha'}, \tag{3.13}$$

which gives the on-shell condition.

As a final remark, we note that, in the degenerate case when $2\pi R/L$ is rational, there can be D-branes coinciding with each other in the compactified direction. See figure 4 for a special case with $2\pi R/L = 3$. When $2\pi R/L$ is irrational, none of the D-branes will coincide after compactification and the branes densely fill the circle.

3.2 Four open string tachyon amplitudes

We are ready to compute open string scattering amplitudes with the open strings ending on the D-brane configuration introduced in section 3.1. We are interested in the following open string amplitude of four open string tachyons inserted along the real axis in the complex plane, with the ends of each open string attached to the D-branes:

$$\mathcal{A}_0(1, 2, 3, 4) = e^{-\chi\Phi_0} \left\langle \prod_{i=1}^4 :c(\mathbb{Y}_i) \mathcal{V}_0(\mathbb{Y}_i): \right\rangle_{D_2} \tag{3.14}$$


The Euler characteristic $\chi = 1$ for the disk and we have specified the permutation. We apply the $\text{PSL}(2, \mathbb{R})$ symmetry to fix $y_1 = 0$, $y_3 = 1$, and $y_4 = \infty$. Then,

$$\mathcal{A}_o(1, 2, 3, 4) \propto \frac{i}{\alpha'} (2\pi)^{25} \delta^{(25)}(k_1 + \dots + k_4) \delta_{W_1+\dots+W_4, 0} \mathcal{M}_o(1, 2, 3, 4). \quad (3.15)$$

Here,

$$\mathcal{M}_o(1, 2, 3, 4) = \int_0^1 dy_2 |y_2|^{2\alpha' K_1 \cdot K_2} |1 - y_2|^{2\alpha' K_2 \cdot K_3}, \quad (3.16)$$

with

$$K_i^M = (k_i^\mu, W_i R/\alpha'). \quad (3.17)$$

We have dropped a proportionality constant that depends on the coupling constant g_o . In general, there can also be color factors from Chan-Paton charges, which we drop when the KLT relation is concerned. The total winding number W_i defined in (3.8) is

$$W_i = \frac{n_i L}{2\pi R} + w_i, \quad n_i, w_i \in \mathbb{Z}. \quad (3.18)$$

The conservation law of W_i is imposed by the Kronecker symbol in (3.15), which gives

$$\frac{L}{2\pi R} \sum_i n_i + \sum_i w_i = 0. \quad (3.19)$$

In the case when $L/(2\pi R)$ is irrational, (3.19) further implies

$$\sum_i n_i = \sum_i w_i = 0. \quad (3.20)$$

In the degenerate case when $L/(2\pi R)$ is rational, it is still possible to choose the definitions of n_i and w_i such that the split of the conservation law (3.19) in (3.20) is satisfied. We therefore require that (3.20) hold in the rest of the paper.

Note that the amplitude in (3.15) involves only open strings, which can only join or split on a D-brane. See figure 5 for an illustration. The total number of D-branes that are involved in the scattering process is given by $n_+ + 1$, where n_+ is the sum of $n_i > 0$ associated with the incoming closed strings. See figure 1 and 2. In contrast, scatterings between open strings that do not occur on the same D-brane necessarily involve closed strings. One such example is given in figure 6. These scattering processes with asymptotic closed string states are not captured by the open string amplitudes that we compute here and are irrelevant in the KLT relation that we consider in this paper.

Intriguingly, if we take $L = 2\pi\tilde{R}$, with $\tilde{R} = \alpha'/R$ the radius of the T-dual circle over which the X -direction is compactified, we find that K_i^M in (3.17) becomes

$$K_i^M = \left(k_i^\mu, \frac{n_i L}{2\pi\alpha'} + \frac{w_i R}{\alpha'} \right) = \left(k_i^\mu, \frac{n_i}{R} + \frac{w_i R}{\alpha'} \right), \quad (3.21)$$

which is identical to K_{Li} defined for closed strings in (2.10a), and n_i can be interpreted as the KK number for the closed string tachyon.⁶ For the open string amplitude in (3.16) to

⁶If w_i in K_i^M is identified with the negative of the closed string winding number, K_i in (3.21) is mapped to K_{Ri} defined in (2.10b).

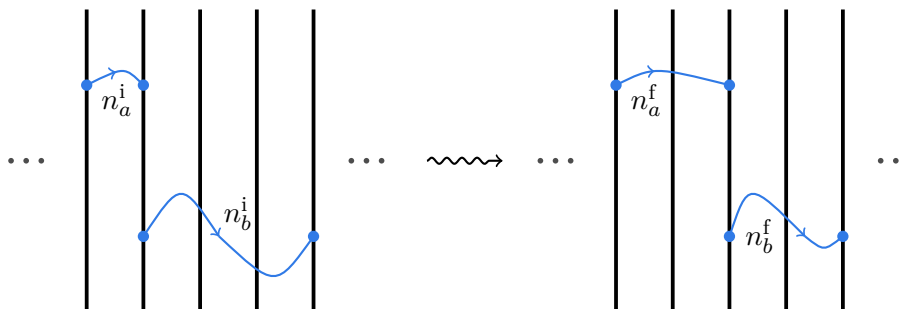


Figure 5. The vertical lines represent D-branes separated by the same distance L , and the blue wavy lines represent fundamental strings with their ends residing on different D-branes. The picture on the left represents the initial state and the picture on the right represents the final state of the scattering process. During the scattering process, the two strings join into one at the second D-brane from the left; then, the single intermediate string splits at the third D-brane into two. This picture captures the conservation law $n_1 + n_2 + n_3 + n_4 = 0$, with $n_1 = 1$, $n_2 = 3$, and $n_3 = n_4 = -2$. The incoming strings on the left carry the initial quantum numbers $n_a^i = n_1$ and $n_b^i = n_2$. The outgoing strings on the right carry the quantum numbers $n_a^f = -n_3$ and $n_b^f = -n_4$. The amplitude for this type of scattering process is given by (3.14). The total number of D-branes involved in the scattering process is equal to $n_1 + n_2 + 1 = 5$.

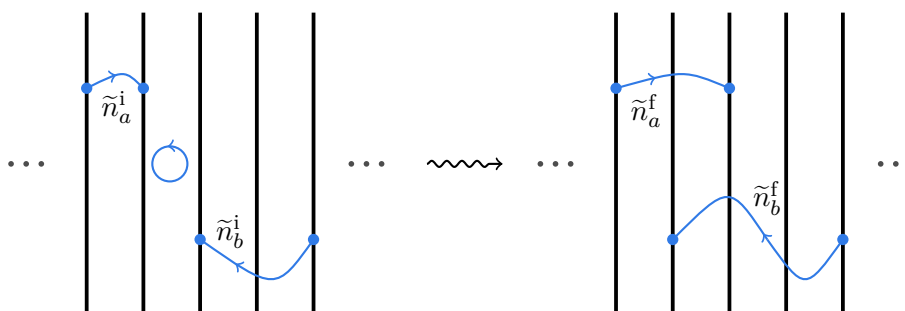


Figure 6. Scattering between two open strings that do not end on the same D-brane necessarily involve asymptotic closed string states. The figure with closed strings realizes the same conservation law for n_i in figure 5 as follows. The incoming strings on the left carry the initial quantum numbers $\tilde{n}_a^i = n_1$ and $\tilde{n}_b^i = n_3$. The outgoing strings on the right carry the final quantum numbers $\tilde{n}_a^f = -n_4$ and $\tilde{n}_b^f = -n_2$. Nevertheless, the scattering process considered in this figure requires inserting an external closed string vertex operator, and is thus *not* computed by (3.14).

match the expression in (2.37a), we have to make the following rescalings of α' , R , and L :

$$\alpha' \rightarrow \frac{\alpha'}{4}, \quad R \rightarrow \frac{R}{4}, \quad L \rightarrow \frac{L}{4} = \frac{\pi\alpha'}{2R}. \tag{3.22}$$

Here, the rescaling of R and L essentially uniformly rescales the geometry of the system. Under the rescalings in (3.22), the open string amplitude (3.16) becomes

$$\mathcal{M}_L(1, 2, 3, 4) \equiv \mathcal{M}_o(1, 2, 3, 4)_{\alpha' \rightarrow \frac{\alpha'}{4}, R \rightarrow \frac{R}{4}, L \rightarrow \frac{L}{4}} = \mathcal{I}_L^{(4)}, \tag{3.23}$$

where $\mathcal{I}_L^{(4)}$ is given in (2.37a).

Similarly, for different permutations of vertex operators and appropriate choices of which D-branes the open strings end at, we find the following open string amplitude interpretation for the quantities in (2.38) and (2.39),

$$\mathcal{I}_R^{(4)} = 2i \sin\left(\frac{1}{2} \pi \alpha' K_{R1} \cdot K_{R2}\right) \mathcal{M}_R(1, 3, 4, 2) \quad \begin{array}{c} \bullet 1 \\ \circ \\ \bullet 3 \\ \bullet 2 \\ \circ \\ \bullet 4 \end{array} \quad (3.24a)$$

$$= 2i \sin\left(\frac{1}{2} \pi \alpha' K_{R2} \cdot K_{R3}\right) \mathcal{M}_R(1, 3, 2, 4) \quad \begin{array}{c} \bullet 1 \\ \circ \\ \bullet 3 \\ \bullet 4 \\ \circ \\ \bullet 2 \end{array} \quad (3.24b)$$

We have defined

$$\mathcal{M}_R(i_1, i_2, i_3, i_4) \equiv \mathcal{M}_o(i_1, i_2, i_3, i_4)_{\alpha' \rightarrow \frac{\alpha'}{4}, R \rightarrow \frac{R}{4}, L \rightarrow \frac{L}{4}, w_i \rightarrow -w_i} \quad (3.25)$$

The second identity in (3.24b) comes from the open string relation

$$\sin(2\pi \alpha' P_1 \cdot P_2) \mathcal{M}_o(1, 3, 4, 2) = \sin(2\pi \alpha' P_2 \cdot P_3) \mathcal{M}_o(1, 3, 2, 4), \quad (3.26)$$

which is the stringy generalization [2, 3] of the Bern-Carrasco-Johansson (BCJ) relation in field theories [15]. However, in the case of nonzero winding, the field theory limit $\alpha' \rightarrow 0$ does not converge, as winding states do not have a conventional QFT interpretation.

3.3 Four-point KLT relation

We are finally ready to write down the first KLT relation for winding strings. Plugging (3.23) and (3.24a) into (2.30) and (2.36), we find the following KLT relation between closed string amplitude \mathcal{M}_c and open string amplitudes \mathcal{M}_L and \mathcal{M}_R :

$$\begin{aligned} \mathcal{M}_c(1, 2, 3, 4) &= \\ &= -\frac{1}{\alpha'} C(1, 2, 3, 4) \sin\left(\frac{1}{2} \pi \alpha' K_{R1} \cdot K_{R2}\right) \mathcal{M}_L(1, 2, 3, 4) \mathcal{M}_R(2, 1, 3, 4) \end{aligned} \quad (3.27a)$$

$$= -\frac{1}{\alpha'} C(1, 2, 3, 4) \sin\left(\frac{1}{2} \pi \alpha' K_{R2} \cdot K_{R3}\right) \mathcal{M}_L(1, 2, 3, 4) \mathcal{M}_R(1, 3, 2, 4), \quad (3.27b)$$

where C denotes contribution from the cocycle factors, with

$$C(i_1, i_2, i_3, i_4) \equiv \exp\left(i\pi \sum_{\substack{p, q=1 \\ p < q}}^4 n_{i_p} w_{i_q}\right). \quad (3.28)$$

Here, (3.27a) matches (2.44a) and (3.27b) matches (2.44b). We now show explicitly that (3.27) is invariant under switching “L” and “R”. First, note that the closed string amplitude is unchanged upon permuting the vertex operators. Starting with (3.27a) and performing the permutation $(1, 2, 3, 4) \rightarrow (3, 2, 4, 1)$, we have

$$\begin{aligned} \mathcal{M}_c(1, 2, 3, 4) &= \mathcal{M}_c(3, 2, 4, 1) \\ &= -\frac{1}{\alpha'} C(3, 2, 4, 1) \sin\left(\frac{1}{2} \pi \alpha' K_{R3} \cdot K_{R2}\right) \mathcal{M}_L(3, 2, 4, 1) \mathcal{M}_R(2, 3, 4, 1). \end{aligned} \quad (3.29)$$

Using

$$C(3, 2, 4, 1) = (-1)^{n_3 w_2 - \tilde{n}_2 w_3} C(1, 2, 3, 4), \quad (3.30)$$

and applying (2.47), we find

$$\begin{aligned} \mathcal{M}_c(1, 2, 3, 4) &= -\frac{1}{\alpha'} C(1, 2, 3, 4) \sin\left(\frac{1}{2} \pi \alpha' K_{L2} \cdot K_{L3}\right) \mathcal{M}_R(1, 2, 3, 4) \mathcal{M}_L(1, 3, 2, 4), \end{aligned} \quad (3.31)$$

which is (3.27b) but with “L” and “R” switched, and it matches (2.45b). Similarly, we can show that (3.27b) is equal to

$$-\frac{1}{\alpha'} C(1, 2, 3, 4) \sin\left(\frac{1}{2} \pi \alpha' K_{L1} \cdot K_{L2}\right) \mathcal{M}_R(1, 2, 3, 4) \mathcal{M}_L(2, 1, 3, 4), \quad (3.32)$$

which is (3.27a) but with “L” and “R” switched, and it matches (2.45a).

From the above four tachyon amplitudes, we make the following general observations of the KLT relation for winding strings, which are also valid for amplitudes involving massless and massive vertex operators. The KLT relation in (3.27) is for string amplitudes in spacetime with a spatial direction compactified over a circle of radius R . The KLT relation maps the closed string amplitude to a sum of the products of a pair of open string amplitudes with the open strings ending on an array of D-branes transverse to the compactified direction. The D-branes are positioned equidistantly along the spatial circle, with the consecutive ones separated by $L = 2\pi\tilde{R}$, where $\tilde{R} = \alpha'/R$ is the T-dual radius. For a closed string that carries winding number w_i and KK number n_i ,⁷ the corresponding open string wraps w_i times around the compactified circle. In addition, excluding the w_i times that it wraps around the circle, the open string is stretched between the q -th and $(q + n_i)$ -th branes.⁸ Using (3.8), we find that the closed string momentum n_i/R along the compactified direction is mapped to the fractional winding number (defined in (3.8))

$$\frac{L}{2\pi R} n_i = \frac{\tilde{R}}{R} n_i = \frac{\alpha'}{R^2} n_i \quad (3.33)$$

of the open strings. The number of D-branes that are involved in the scattering between open strings is $n_+ + 1$, with n_+ the sum of $n_i > 0$ associated with the incoming closed strings. See figure 1–3 for illustrations.

There is an equivalent interpretation for the open string amplitudes in the T-dual frame, where the open strings satisfy Neumann boundary conditions in the compactified direction. In this case, the open strings can carry nonzero momenta but only zero winding.

⁷For tachyons we have $n_i w_i = 0$, which implies that at least one of n_i and w_i is zero. However, when other vertex operators are considered, the quantity $n_i w_i$ is not necessarily zero.

⁸Since the series of D-branes is wrapped around the compactified circle, the shortest separation along the circle between the ends of the open string might span over fewer than $n_i - 1$ D-branes. For example, in the diagram on the right in figure 4, if $w_i = 0$, the open string starts at the q -th brane and ends at the $(q + 4)$ -th brane in the series, which implies that $n_i = 4$. However, the shortest separation between the ends of the open string along the compactified circle is just between two D-branes.

The KK number of a closed string is mapped to the KK number of the associated open string, while the winding number of a closed string is mapped to a constant gauge field, which gives rise to a shift in the open string momentum: it gives rise to fractional momentum. The gauge field A is associated with a $U(n_+ + 1)$ Chan-Paton charge, where $n_+ + 1$ is the number of D-branes. Since we take A to be constant, one can diagonalize it to be

$$A = \frac{1}{2\pi\tilde{R}} \text{diag}(\theta_1, \dots, \theta_{n_++1}), \quad \theta_q = \frac{qL}{2\pi\tilde{R}}, \quad (3.34)$$

breaking the gauge group to $U(1)^{n_++1}$. Note that θ_q is T-dual to the location of the q -th D-brane. The field A is pure gauge, but a Wilson loop that winds around the compactified circle of dual radius \tilde{R} gains a phase factor of the form

$$\exp\left(i \oint dX A_q\right) = e^{i\theta_q}, \quad (3.35)$$

leading to a shift in the momentum \tilde{n}_i/\tilde{R} along the compactified X -direction. For a string in the Chan-Paton state $|q + \tilde{w}_i, q\rangle$, its canonical momentum in X is

$$\tilde{K}_i^M = \left(k_i^\mu, \frac{\tilde{n}_i + \theta_{q+\tilde{w}_i} - \theta_q}{\tilde{R}}\right) = \left(k_i^\mu, \frac{\tilde{n}_i}{\tilde{R}} + \frac{\tilde{w}_i\tilde{R}}{\alpha'}\right). \quad (3.36)$$

Here, $\tilde{n}_i = w_i$ and $\tilde{w}_i = n_i$, with w_i (n_i) the winding (momentum) number of the corresponding closed string.

The above observations hold for any higher point amplitudes among general vertex operators discussed in section 2.2. We prove the general KLT relation for winding strings in the next section. We also note that all such constructions straightforwardly generalize to the Narain compactification, in which more than one spatial direction is compactified. In this more case, we need to consider Dp -branes that are transverse to multiple compactified directions.

4 KLT relation for higher-point amplitudes

In this section, we derive the KLT relation for higher-point amplitudes, involving winding closed string vertex operators that take the general form as discussed in section 2.2.

4.1 Higher-point closed string amplitude

We start with an \mathcal{N} -point tree-level closed string amplitude, which, similar to the four-point amplitude in (2.27), takes the following general form:

$$\mathcal{A}_c^{(\mathcal{N})} \propto (2\pi)^{25} \delta^{(25)}\left(\sum_{i=1}^{\mathcal{N}} k_i\right) \delta_{\sum_{i=1}^{\mathcal{N}} n_i, 0} \delta_{\sum_{i=1}^{\mathcal{N}} w_i, 0} (i\mathcal{M}_c^{(\mathcal{N})}), \quad (4.1)$$

where

$$\begin{aligned} & \mathcal{M}_c(1, \dots, \mathcal{N}) \\ &= \frac{1}{\alpha'^{\mathcal{N}-3}} |z_1 - z_{\mathcal{N}-1}|^2 |z_1 - z_{\mathcal{N}}|^2 |z_{\mathcal{N}-1} - z_{\mathcal{N}}|^2 \\ & \times \left(\prod_{i=2}^{\mathcal{N}-2} \int_{\mathbb{C}} d^2 z_i \right) F(z_1, \dots, z_{\mathcal{N}}; \bar{z}_1, \dots, \bar{z}_{\mathcal{N}}) \prod_{\substack{j,k=1 \\ j < k}}^{\mathcal{N}} e^{i\pi w_j n_k} z_{jk}^{\frac{1}{2}\alpha' K_{Lj} \cdot K_{Lk}} \bar{z}_{jk}^{\frac{1}{2}\alpha' K_{Rj} \cdot K_{Rk}}. \end{aligned} \quad (4.2)$$

We defined $z_{ij} \equiv z_i - z_j$. Here, the factor F comes from contracting the derivative terms in the vertex operators that for example take the form of (2.21) and (2.24). The factor F is single valued. Fixing $z_1 = 0$, $z_{\mathcal{N}-1} = 1$, and $z_{\mathcal{N}} = \infty$, we find

$$\begin{aligned} & \mathcal{M}_c(1, \dots, \mathcal{N}) \\ &= \frac{1}{\alpha'^{\mathcal{N}-3}} C(1, \dots, \mathcal{N}) \left(\prod_{i=2}^{\mathcal{N}-2} \int_{\mathbb{C}} d^2 z_i \right) F(z_2, \dots, z_{\mathcal{N}-2}; \bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}) \\ & \times \prod_{\substack{i,j,k=2 \\ j < k}}^{\mathcal{N}-2} z_i^{\frac{1}{2}\alpha' K_{L1} \cdot K_{Li}} (1 - z_i)^{\frac{1}{2}\alpha' K_{L(\mathcal{N}-1)} \cdot K_{Li}} z_{kj}^{\frac{1}{2}\alpha' K_{Lj} \cdot K_{Lk}} \times \\ & \bar{z}_i^{\frac{1}{2}\alpha' K_{R1} \cdot K_{Ri}} (1 - \bar{z}_i)^{\frac{1}{2}\alpha' K_{R(\mathcal{N}-1)} \cdot K_{Ri}} \bar{z}_{kj}^{\frac{1}{2}\alpha' K_{Rj} \cdot K_{Rk}}, \end{aligned} \quad (4.3)$$

where the cocycle factor $C(1, \dots, \mathcal{N})$ is defined by

$$C(i_1, \dots, i_{\mathcal{N}}) \equiv \exp \left(i\pi \sum_{\substack{p,q=1 \\ p < q}}^{\mathcal{N}} n_{i_p} w_{i_q} \right). \quad (4.4)$$

We have abbreviated $F(z_1, \dots, z_{\mathcal{N}}; \bar{z}_1, \dots, \bar{z}_{\mathcal{N}})|_{z_1=0, z_{\mathcal{N}-1}=1, z_{\mathcal{N}}=\infty}$ by dropping the arguments z_1 , $z_{\mathcal{N}-1}$, and $z_{\mathcal{N}}$, as well as their complex conjugates.

We write schematically the i -th closed string vertex operators as

$$\begin{aligned} & \sum_{\substack{k_1+\dots+k_p=N \\ \ell_1+\dots+\ell_q=\tilde{N}}} \epsilon_{I_1 \dots I_p J_1 \dots J_q}^{(i)} \left[e^{\frac{1}{2} i\pi R w_i (\hat{p}_L + \hat{p}_R)} \right]_{\mathcal{C}_i} : \partial^{k_1} \mathbb{X}_{Li}^{I_1} \dots \partial^{k_p} \mathbb{X}_{Li}^{I_p} \bar{\partial}^{\ell_1} \mathbb{X}_{Ri}^{J_1} \dots \bar{\partial}^{\ell_q} \mathbb{X}_{Ri}^{J_q} \\ & \times e^{iK_{Li} \cdot \mathbb{X}_{Li}(z) + iK_{Ri} \cdot \mathbb{X}_{Ri}(\bar{z})}. \end{aligned} \quad (4.5)$$

With appropriate choices of the polarization tensors $\epsilon^{(i)}$, this general operator reduces to the vertex operators considered in (2.21) and (2.24). In general, the tensors $\epsilon^{(i)}$ satisfy various conditions by imposing BRST invariance on the vertex operator. The dispersion relation and the level matching condition for the state corresponding to the vertex operator (4.5) are in form the same as the expressions in (2.23). In the following calculation, we employ the standard trick in [1] and rewrite the polarization tensor as

$$\epsilon_{I_1 \dots I_p J_1 \dots J_q}^{(i)} \rightarrow \epsilon_{I_1 \dots I_p}^{Li} \epsilon_{J_1 \dots J_q}^{Ri}. \quad (4.6)$$

Under this replacement, the F -factor in (4.3) decomposes as

$$F(z_2, \dots, z_{\mathcal{N}-2}; \bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}) \rightarrow F_L(z_2, \dots, z_{\mathcal{N}-2}) F_R(\bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}), \quad (4.7)$$

where F_L (F_R) contains polarization factor ϵ^{Li} (ϵ^{Ri}). The new factors $F_{L,R}$ will be interpreted as factors in open string amplitudes coming from contracting derivative terms in open string vertex operators. Applying the replacement (4.7) to (4.3), we find

$$\begin{aligned}
 & \mathcal{M}_c(1, \dots, \mathcal{N}) \\
 & \rightarrow \frac{1}{\alpha'^{\mathcal{N}-3}} C(1, \dots, \mathcal{N}) \left(\prod_{i=2}^{\mathcal{N}-2} \int_{\mathbb{C}} d^2 z_i \right) F_L(z_2, \dots, z_{\mathcal{N}-2}) F_R(\bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}) \\
 & \quad \times \prod_{\substack{i,j,k=2 \\ j < k}}^{\mathcal{N}-2} z_i^{\frac{1}{2}\alpha' K_{L1} \cdot K_{Li}} (1 - z_i)^{\frac{1}{2}\alpha' K_{L(\mathcal{N}-1)} \cdot K_{Li}} z_{kj}^{\frac{1}{2}\alpha' K_{Lj} \cdot K_{Lk}} \times \\
 & \quad \quad \quad \bar{z}_i^{\frac{1}{2}\alpha' K_{R1} \cdot K_{Ri}} (1 - \bar{z}_i)^{\frac{1}{2}\alpha' K_{R(\mathcal{N}-1)} \cdot K_{Ri}} \bar{z}_{kj}^{\frac{1}{2}\alpha' K_{Rj} \cdot K_{Rk}}.
 \end{aligned} \tag{4.8}$$

Note that $F_{L,R}$ are single-valued functions that do not contribute any branch points. Collecting terms multilinear in ϵ^{Li} and ϵ^{Ri} and replacing their product with $\epsilon^{(i)}$, we recover the original amplitude (4.3). In the following, we will take \mathcal{M}_c to denote the expression in (4.8), with the splitting of the polarization factor taken into account implicitly.

4.2 Splitting the worldsheet integrals

It is a standard exercise to perform the integrals over the complex plane such that it splits into two sets of contour integrals for the left- and right-movers, respectively [1], bearing caveats concerning the cocycle factor. For completeness, we will review the complex integral and articulate the differences.

First, make the change of variables with $z_i = x_i + i \tilde{y}_i$ for real variables x_i and \tilde{y}_i . Then, we promote the integrals over \tilde{y}_i to be in the complex plane, with the branch points at $i x_i$, $-i(1 - x_i)$, and $\tilde{y}_j + i(x_i - x_j)$. Deform the contour for the \tilde{y}_i -integral that runs along the real axis by rotating it counterclockwise by $\frac{\pi}{2} - \epsilon$, $\epsilon \rightarrow 0^+$. This is done by a second change of variables,

$$\tilde{y}_i = \exp\left[i\left(\frac{\pi}{2} - \epsilon\right)\right] y_i \approx i y_i + \epsilon y_i. \tag{4.9}$$

We require that such deformations of the contours are done simultaneously for all the \tilde{y}_i -integrals, in order to ensure that none of the branch points are crossed. Define

$$\xi_i \equiv x_i + y_i, \quad \zeta_i \equiv x_i - y_i, \quad \delta_i \equiv \epsilon y_i = \frac{1}{2} \epsilon (\xi_i - \zeta_i), \tag{4.10}$$

in terms of which we rewrite (4.8) as

$$\mathcal{M}_c(1, \dots, \mathcal{N}) = \frac{1}{\alpha'^{\mathcal{N}-3}} C(1, \dots, \mathcal{N}) \left(\prod_{i=2}^{\mathcal{N}-2} \int_{\mathbb{R}} d\zeta_i \int_{\mathbb{R}} d\xi_i \right) O_L O_R, \tag{4.11}$$

where

$$\begin{aligned}
 O_L &= F_L \prod_{j=2}^{\mathcal{N}-2} (\zeta_j + i \delta_j)^{\frac{1}{2} \alpha' K_{L1} \cdot K_{Lj}} (1 - \zeta_j - i \delta_j)^{\frac{1}{2} \alpha' K_{L(\mathcal{N}-1)} \cdot K_{Lj}} \\
 &\quad \times \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} [\zeta_\ell - \zeta_k + i (\delta_\ell - \delta_k)]^{\frac{1}{2} \alpha' K_{Lk} \cdot K_{L\ell}}, \tag{4.12a}
 \end{aligned}$$

$$\begin{aligned}
 O_R &= F_R \prod_{j=2}^{\mathcal{N}-2} (\xi_j - i \delta_j)^{\frac{1}{2} \alpha' K_{R1} \cdot K_{Rj}} (1 - \xi_j + i \delta_j)^{\frac{1}{2} \alpha' K_{R(\mathcal{N}-1)} \cdot K_{Rj}} \\
 &\quad \times \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} [\xi_\ell - \xi_k - i (\delta_\ell - \delta_k)]^{\frac{1}{2} \alpha' K_{Rk} \cdot K_{R\ell}}. \tag{4.12b}
 \end{aligned}$$

We now perform the integrals over ξ_i for fixed ζ_i , $i = 2, \dots, \mathcal{N} - 2$. First, we define the permutation $\sigma(2, \dots, \mathcal{N} - 2)$ of the particle numbers such that the fixed values of ζ_i 's are ordered as

$$\zeta_{\sigma(2)} < \zeta_{\sigma(3)} < \dots < \zeta_{\sigma(\mathcal{N}-2)}. \tag{4.13}$$

In the case when $\zeta_{\sigma(2)} \in (-\infty, 0)$, the branch points for the $\xi_{\sigma(2)}$ -integral are at

$$i \delta_{\sigma(2)} \Big|_{\xi_{\sigma(2)}=0} = \frac{i}{2} \epsilon(-\zeta_{\sigma(2)}), \tag{4.14a}$$

$$1 + i \delta_{\sigma(2)} \Big|_{\xi_{\sigma(2)}=1} = 1 + \frac{i}{2} \epsilon(1 - \zeta_{\sigma(2)}), \tag{4.14b}$$

$$\xi_{\sigma(i)} + i (\delta_{\sigma(2)} - \delta_{\sigma(i)}) \Big|_{\xi_{\sigma(2)}=\xi_{\sigma(i)}} = \xi_{\sigma(i)} + \frac{i}{2} \epsilon(\zeta_{\sigma(i)} - \zeta_{\sigma(2)}), \quad i > 2. \tag{4.14c}$$

All these branch points reside in the upper half plane. Completing the contour in the lower half plane, we find that the contribution from $\zeta_{\sigma(2)} \in (-\infty, 0)$ to the amplitude \mathcal{M}_c is zero. Similarly, when $\zeta_{\sigma(\mathcal{N}-2)} \in (1, \infty)$, all the branch points for the $\xi_{\sigma(\mathcal{N}-2)}$ -integral are in the lower half plane. Therefore, the contribution from $\zeta_{\sigma(\mathcal{N}-2)} \in (1, \infty)$ to \mathcal{M}_c is also zero. Therefore, the integrals in (4.11) are only nonzero when

$$0 < \zeta_{\sigma(2)} < \dots < \zeta_{\sigma(\mathcal{N}-2)} < 1, \tag{4.15}$$

which partitions up the associated integration region from 0 to 1. As a result, (4.11) becomes

$$\begin{aligned}
 \mathcal{M}_c(1, \dots, \mathcal{N}) &= \mathcal{M}_c(1, \sigma(2), \dots, \mathcal{N} - 2, \mathcal{N} - 1, \mathcal{N}) \\
 &= \left(\frac{i}{2\alpha'} \right)^{\mathcal{N}-3} C(1, \sigma(2), \dots, \mathcal{N} - 2, \mathcal{N} - 1, \mathcal{N}) \int_{\Delta} \left(\prod_{i=2}^{\mathcal{N}-2} d\zeta_i \int_{\mathbb{R}} d\xi_i \right) O_L O_R, \tag{4.16}
 \end{aligned}$$

with the integration domain

$$\Delta = \{\zeta_2, \dots, \zeta_{\mathcal{N}-2} | 0 < \zeta_{\sigma(2)} < \dots < \zeta_{\sigma(\mathcal{N}-2)} < 1\}, \tag{4.17}$$

and

$$\begin{aligned}
 O_L &= F_L \prod_{j=2}^{\mathcal{N}-2} \zeta_j^{\frac{1}{2}\alpha' K_{L1} \cdot K_{Lj}} (1 - \zeta_j)^{\frac{1}{2}\alpha' K_{L(\mathcal{N}-1)} \cdot K_{Lj}} \\
 &\quad \times \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} (\zeta_{\sigma(\ell)} - \zeta_{\sigma(k)})^{\frac{1}{2}\alpha' K_{L\sigma(k)} \cdot K_{L\sigma(\ell)}}, \tag{4.18a}
 \end{aligned}$$

$$\begin{aligned}
 O_R &= F_R \prod_{j=2}^{\mathcal{N}-2} (\xi_j - i\delta_j)^{\frac{1}{2}\alpha' K_{R1} \cdot K_{Rj}} (1 - \xi_j + i\delta_j)^{\frac{1}{2}\alpha' K_{R(\mathcal{N}-1)} \cdot K_{Rj}} \\
 &\quad \times \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} [\xi_{\sigma(\ell)} - \xi_{\sigma(k)} - i(\delta_{\sigma(\ell)} - \delta_{\sigma(k)})]^{\frac{1}{2}\alpha' K_{R\sigma(k)} \cdot K_{R\sigma(\ell)}}. \tag{4.18b}
 \end{aligned}$$

When $k < \ell$, for a fixed $\xi_{\sigma(\ell)}$, the $\xi_{\sigma(k)}$ -integral has a branch point at

$$\xi_{\sigma(\ell)} + \frac{i}{2} \epsilon (\zeta_{\sigma(\ell)} - \zeta_{\sigma(k)}), \tag{4.19}$$

which resides in the upper half plane if the $\xi_{\sigma(k)}$ -contour traverses the real axis. This implies that, in order to avoid branch points, we need to shift the contour associated with the $\xi_{\sigma(\ell)}$ -integral to be slightly above the contour associated with the $\xi_{\sigma(k)}$ -integral in the complex plane [4].

We now elaborate on the factors

$$(\zeta_{\sigma(\ell)} - \zeta_{\sigma(k)})^{\frac{1}{2}\alpha' K_{L\sigma(k)} \cdot K_{L\sigma(\ell)}}, \quad k < \ell, \tag{4.20}$$

that appear in (4.18a). When $\sigma(\ell) < \sigma(k)$, this factor ultimately comes from

$$(z_{\sigma(k)} - z_{\sigma(\ell)})^{\frac{1}{2}\alpha' K_{L\sigma(k)} \cdot K_{L\sigma(\ell)}} \tag{4.21}$$

in (4.8). To pass from (4.21) to (4.20), an extra phase factor that is a function of K_{Li} appears. The same reordering for \bar{z}_i 's in (4.8) also introduces a similar phase factor, which is a function of K_{Ri} . In the case when there is zero winding, we have $K_{Li} = K_{Ri}$ and there is no extra phase factor introduced after simultaneously reordering z_i 's and \bar{z}_i 's. This is, however, not the case when winding modes are present: the extra phase factors from reordering z_i 's and \bar{z}_i 's do *not* exactly cancel when $K_{Li} \neq K_{Ri}$. The remainder phase factor can be absorbed into the cocycle factor $C(1, \dots, \mathcal{N})$ in (4.11), which turns the cocycle factor into $C(1, \sigma(2), \dots, \mathcal{N}-2), \mathcal{N}-1, \mathcal{N})$ in (4.16), where the arguments are permuted. The above observation is summarized by the following identity:

$$\begin{aligned}
 &C(1, \dots, \mathcal{N}) \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} (z_\ell - z_k)^{\frac{1}{2}\alpha' K_{Lk} \cdot K_{L\ell}} (\bar{z}_\ell - \bar{z}_k)^{\frac{1}{2}\alpha' K_{Rk} \cdot K_{R\ell}} \\
 &= C(1, \sigma(2), \dots, \mathcal{N}-2), \mathcal{N}-1, \mathcal{N}) \\
 &\quad \times \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} (z_{\sigma(\ell)} - z_{\sigma(k)})^{\frac{1}{2}\alpha' K_{L\sigma(k)} \cdot K_{L\sigma(\ell)}} (\bar{z}_{\sigma(\ell)} - \bar{z}_{\sigma(k)})^{\frac{1}{2}\alpha' K_{R\sigma(k)} \cdot K_{R\sigma(\ell)}}. \tag{4.22}
 \end{aligned}$$

4.3 Higher-point KLT relation

Now, the integrals over ζ_i and ξ_i in (4.16) can be performed in the standard way as in [1, 3, 4], generalizing the same procedure for four-point amplitudes in section 2.3 to \mathcal{N} -point amplitudes. We find the following \mathcal{N} -point KLT relation:

$$\begin{aligned}
 \mathcal{M}_c(1, \dots, \mathcal{N}) &= (-1)^{\mathcal{N}-3} \sum_{\sigma, \beta, \gamma} C(1, \sigma(2), \dots, \mathcal{N}-2), \mathcal{N}-1, \mathcal{N}) \\
 &\quad \times \mathcal{S}_R[\gamma \circ \sigma(2), \dots, \ell-1 \mid \sigma(2), \dots, \ell-1]_{K_{R1}} \\
 &\quad \times \mathcal{S}_R[\sigma(\ell), \dots, \mathcal{N}-2 \mid \beta \circ \sigma(\ell), \dots, \mathcal{N}-2]_{K_{R(\mathcal{N}-1)}} \\
 &\quad \times \mathcal{M}_L(1, \sigma(2), \dots, \mathcal{N}-2), \mathcal{N}-1, \mathcal{N}) \\
 &\quad \times \mathcal{M}_R(\gamma \circ \sigma(2), \dots, \ell-1), 1, \mathcal{N}-1, \beta \circ \sigma(\ell), \dots, \mathcal{N}-2), \mathcal{N}).
 \end{aligned} \tag{4.23}$$

Note that β permutes $\mathcal{N} - \ell - 1$ indices and γ permutes $\ell - 2$ indices. Here, we defined the open string amplitude \mathcal{M}_L after the rescaling (3.22) as

$$\begin{aligned}
 \mathcal{M}_L(1, \sigma(2), \dots, \mathcal{N}-2), \mathcal{N}-1, \mathcal{N}) &= \int_{\Delta} F_L \prod_{i=2}^{\mathcal{N}-2} d\zeta_i |\zeta_i|^{\frac{1}{2}\alpha' K_{L1} \cdot K_{Li}} |1 - \zeta_i|^{\frac{1}{2}\alpha' K_{L(\mathcal{N}-1)} \cdot K_{Li}} \\
 &\quad \times \prod_{\substack{j, k=2 \\ j < k}}^{\mathcal{N}-2} |\zeta_{\sigma(k)} - \zeta_{\sigma(j)}|^{\frac{1}{2}\alpha' K_{L\sigma(j)} \cdot K_{L\sigma(k)}},
 \end{aligned} \tag{4.24}$$

where the domain Δ is defined in (4.17). The i -th open string involved in the scattering amplitude is labeled by the quantum numbers (n_i, w_i) . See (3.23) for an example. Open string amplitudes with a different cyclic ordering are defined accordingly by permuting the integration domain and the indices in (4.24). The open string amplitude \mathcal{M}_R is defined analogously by replacing the subscript ‘‘L’’ with ‘‘R’’, with the i -th open string involved in the scattering amplitude labeled by the quantum numbers $(n_i, -w_i)$. See (3.25) for an example. The momentum kernel \mathcal{S}_R is defined as [19]

$$\begin{aligned}
 &\mathcal{S}_R[i_1, \dots, i_k \mid j_1, \dots, j_k]_P \\
 &= (\alpha')^{-k} \prod_{t=1}^k \sin \left[\frac{1}{2} \pi \alpha' \left(P \cdot K_{Ri_t} + \sum_{q=t+1}^k \theta(i_t, i_q) K_{Ri_t} \cdot K_{Ri_q} \right) \right],
 \end{aligned} \tag{4.25}$$

which takes into account the phase factors associated to crossing a branch cut. Here, i_1, \dots, i_k and j_1, \dots, j_k are elements in $\{1, \dots, k\}$. Moreover, if the ordering of i_t and i_q is the opposite in the ordered sets (i_1, \dots, i_k) and (j_1, \dots, j_k) , we set $\theta(i_t, i_q) = 1$; if the ordering is the same, we set $\theta(i_t, i_q) = 0$. The KLT relation in (4.23) shows how the \mathcal{N} -point amplitude for closed strings with windings can be factorized as a product of open string amplitudes. The open strings satisfy the Dirichlet boundary condition in the compactified direction, with their ends positioned on an array of D-branes transverse to the compactified circle, as we have detailed in section 3.

Note that (4.23) is independent of $\ell \in \{3, \dots, \mathcal{N} - 2\}$. This is reminiscent of the four-point amplitude that we discussed earlier in section 2.3, which has two equivalent expressions in (2.38) and (2.39), depending on whether the integration contour for the ξ -integral is deformed to the left or right. To derive the \mathcal{N} -point amplitude (4.23), we have deformed $\ell - 2$ contours associated with the $\xi_{\sigma(k)}$ -integrals, $k = 2, \dots, \ell - 1$, and the remaining $\mathcal{N} - \ell - 1$ contours have been deformed to the right, without crossing any branch cuts. From the open string amplitude perspective, the existence of different ways of writing the KLT relation is due to the open string generalization of the Kleiss-Kuijff and BCJ relations [2–4].⁹ The KLT relation can also be cast in equivalent forms other than (4.23). When $\ell = 2$, all the contours for the ξ_i -integral are deformed to the right, yielding

$$\begin{aligned} \mathcal{M}_c(1, \dots, \mathcal{N}) &= (-1)^{\mathcal{N}-3} \sum_{\rho, \sigma} C(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\ &\quad \times \mathcal{S}_R[\sigma(2, \dots, \mathcal{N} - 2) | \rho(2, \dots, \mathcal{N} - 2)]_{K_{R(\mathcal{N}-1)}} \\ &\quad \times \mathcal{M}_L(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\ &\quad \times \mathcal{M}_R(1, \mathcal{N} - 1, \rho(2, \dots, \mathcal{N} - 2), \mathcal{N}). \end{aligned} \tag{4.26}$$

Here, ρ is the same as σ that permutes the ordered set $(2, \dots, \mathcal{N} - 2)$. When $\ell = \mathcal{N} - 1$, all the contours for the ξ_i -integrals are deformed to the left, where

$$\begin{aligned} \mathcal{M}_c(1, \dots, \mathcal{N}) &= (-1)^{\mathcal{N}-3} \sum_{\rho, \sigma} C(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\ &\quad \times \mathcal{S}_R[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{R1}} \\ &\quad \times \mathcal{M}_L(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\ &\quad \times \mathcal{M}_R(\rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1, \mathcal{N}). \end{aligned} \tag{4.27}$$

The expressions in (4.26) and (4.27) can be thought as two limiting cases of (4.23).¹⁰

To show explicitly that these expressions are invariant under swapping the subscripts “L” and “R”, we start with the expression in (4.26) and permute the closed string vertex

⁹Also see [11] for generalizations to higher genus Riemann surfaces using twisted homology.

¹⁰There is also a different representation of KLT relations as noted in [1], which can be obtained by evaluating the integrals in (4.11) directly without performing any further deformation of the integration contours. This representation has a simple combinatorial interpretation [20]. In the winding case, this representation has the benefit of making the resulting KLT relation manifestly symmetric with respect to the subscripts “L” and “R”. However, this KLT relation contains $[(\mathcal{N} - 1)!/2]^2$ terms. In contrast, the KLT relation (4.23) is more useful in practice since it only contains $(\mathcal{N} - 3)!(\ell - 2)!(\mathcal{N} - \ell - 1)!$ terms, with $2 \leq \ell \leq \mathcal{N} - 1$, which are far fewer than $[(\mathcal{N} - 1)!/2]^2$ terms when $\mathcal{N} > 3$ [4, 20].

operators as

$$\begin{aligned}
 \mathcal{M}_c(1, \dots, \mathcal{N}) &= \mathcal{M}_c(\mathcal{N}, 2, \dots, \mathcal{N} - 2, 1, \mathcal{N} - 1) \\
 &= (-1)^{\mathcal{N}-3} \sum_{\rho, \sigma} C(\mathcal{N}, \rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1) \\
 &\quad \times \mathcal{S}_R[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{R1}} \\
 &\quad \times \mathcal{M}_L(\mathcal{N}, \rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1) \\
 &\quad \times \mathcal{M}_R(\mathcal{N}, 1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1) \\
 &= (-1)^{\mathcal{N}-3} \sum_{\rho, \sigma} C(\mathcal{N}, \rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1) \\
 &\quad \times \mathcal{S}_R[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{R1}} \\
 &\quad \times \mathcal{M}_R(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\
 &\quad \times \mathcal{M}_L(\rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1, \mathcal{N}). \tag{4.28}
 \end{aligned}$$

We have swapped the dummy permutation symbol ρ and σ to get the second equality above. Furthermore, note that

$$\begin{aligned}
 &C(\mathcal{N}, \rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1) \mathcal{S}_R[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{R1}} \\
 &= C(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \mathcal{S}_L[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{L1}}, \tag{4.29}
 \end{aligned}$$

where

$$\begin{aligned}
 &\mathcal{S}_L[i_1, \dots, i_k | j_1, \dots, j_k]_P \\
 &\equiv (\alpha')^{-k} \prod_{t=1}^k \sin \left[\frac{1}{2} \pi \alpha' \left(P \cdot K_{Li_t} + \sum_{q>t}^k \theta(i_t, i_q) K_{Li_t} \cdot K_{Li_q} \right) \right]. \tag{4.30}
 \end{aligned}$$

Plugging (4.29) back into (4.28), we find

$$\begin{aligned}
 \mathcal{M}_c(1, \dots, \mathcal{N}) &= (-1)^{\mathcal{N}-3} \sum_{\rho, \sigma} C(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\
 &\quad \times \mathcal{S}_L[\rho(2, \dots, \mathcal{N} - 2) | \sigma(2, \dots, \mathcal{N} - 2)]_{K_{L1}} \\
 &\quad \times \mathcal{M}_R(1, \sigma(2, \dots, \mathcal{N} - 2), \mathcal{N} - 1, \mathcal{N}) \\
 &\quad \times \mathcal{M}_L(\rho(2, \dots, \mathcal{N} - 2), 1, \mathcal{N} - 1, \mathcal{N}), \tag{4.31}
 \end{aligned}$$

which is exactly the expression in (4.27) but with the subscripts “L” and “R” swapped. Another way to understand this invariance under switching “L” and “R” is by observing that (4.27) comes from integrating $\{\xi_i\}$ before $\{\zeta_i\}$ in (4.11), and that (4.31) comes from integrating $\{\zeta_i\}$ before $\{\xi_i\}$ in (4.11). Therefore, the invariance under switching “L” and “R” is the consequence of the invariance under commuting the orders of performing different integrals. Similarly, starting with the expression from switching “L” and “R” in (4.27) and then going through the analogous procedure in (4.28)–(4.31), we arrive at (4.26) but with the subscripts “L” and “R” swapped. It also follows immediately that the more general KLT relation in (4.23) has to hold after switching “L” and “R”.

5 Conclusions and outlooks

In this paper we extended the KLT relation to winding strings in a toroidal compactification of string theory. We showed that the string amplitudes of winding closed strings factorize into quadratic products of amplitudes for open strings ending on an array of D-branes transverse to the compact directions. The winding number w of a closed string is mapped to an integer winding number of an open string wrapping around the compactified circle; while the momentum of a closed string is mapped to the fractional winding number that encodes how many D-branes the open string traverses, excluding the w times that the open string winds around the full circle. The general form of the KLT relation is given in (4.23). In appendix A, we use intersection theory to write the KLT relations in a succinct form in (A.26).

In the case of nonzero winding, there is no field theory limit by simply sending the Regge slope α' to zero. However, there still is a nonsingular $\alpha' \rightarrow 0$ limit for winding strings in the presence of a near critical Kalb-Ramond field. This type of limit was originally applied to open strings on spacetime filling D-branes, which led to noncommutative open string (NCOS) theory [21, 22]. This NCOS limit was then realized for winding closed strings [23–25] as well as winding open strings with Dirichlet boundary conditions [26, 27], leading to self-contained nonrelativistic closed and open string theory, where the string spectra enjoy a Galilean-invariant dispersion relation. It has also been shown that NCOS is T-dual to nonrelativistic open string theory [28]. Recently, there has been growing interest in nonrelativistic string theory, leading to a plethora of applications to nonrelativistic gravity and field theory (see, e.g., [27, 29–37]). Amplitudes for nonrelativistic closed strings have been studied in [24], while amplitudes for nonrelativistic open strings and the associated KLT relation are still in progress [38]. This endeavor may also enrich the current S-matrix program to include nonrelativistic field theories.

The KLT relation for winding string amplitudes explored in this paper opens up new possibilities of generalizing other modern techniques developed for field-theoretical amplitudes to the study of winding string states. The program of mapping out structures of winding string amplitudes may bring us new insights into string theory and quantum gravity in general.

Acknowledgments

It is a pleasure to thank Freddy Cachazo and Sebastian Mizera for reading a draft and their comments. This research is supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities.

A KLT and twisted period relation

In this appendix, we review the modern interpretation of the KLT relation using intersection theory. We will see that the KLT relation for winding strings given in section 4.3 is indeed underlay by the twisted Riemann's period relation [39] (which generalizes Riemann's bilinear relation), and the inverse KLT kernel receives the geometrical interpretation as an intersection number between cycles in the twisted homology, defined on the moduli space. This is a direct generalization of the results in [9, 40] for the standard KLT relation, which we follow closely in the following discussion. For the mathematical part of the following discussion, we mostly follow [41, 42].

A.1 Twisted cocycles and closed string amplitudes

We start with the closed string side of the KLT relation. Recall the integrals for closed string amplitudes in (4.8), and its split form in (4.16), which we choose to separate into the multi-valued function given by the Koba-Nielsen factor,

$$U(z_2, \dots, z_{\mathcal{N}-2}) = \prod_{\substack{i,j=1 \\ i < j}}^{\mathcal{N}-1} z_{ji}^{\frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}}, \quad (\text{A.1})$$

and the following factors that are single valued:

$$\begin{aligned} f_L(z_2, \dots, z_{\mathcal{N}-2}) &= F_L(z_2, \dots, z_{\mathcal{N}-2}) \prod_{\substack{i,j=1 \\ i < j}}^{\mathcal{N}-1} z_{ji}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj} - \frac{1}{2}\alpha' K_{Ri} \cdot K_{Rj}} \\ &= F_L(z_2, \dots, z_{\mathcal{N}-2}) \prod_{\substack{i,j=1 \\ i < j}}^{\mathcal{N}-1} z_{ji}^{n_i w_j + n_j w_i}, \quad n_i, w_i \in \mathbb{Z}; \end{aligned} \quad (\text{A.2a})$$

$$f_R(\bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}) = F_R(\bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}). \quad (\text{A.2b})$$

Note that $z_1 = 0$ and $z_{\mathcal{N}-1} = 1$ are fixed. Using the above notation, the closed string amplitude (4.8) can be written as

$$\begin{aligned} \mathcal{M}_c(1, \dots, \mathcal{N}) &= C(1, \dots, \mathcal{N}) \left(\prod_{i=2}^{\mathcal{N}-2} \int_{\mathbb{C}} d^2 z_i \right) f_L(z_2, \dots, z_{\mathcal{N}-2}) f_R(\bar{z}_2, \dots, \bar{z}_{\mathcal{N}-2}) \\ &\quad \times |U(z_2, \dots, z_{\mathcal{N}-2})|^2. \end{aligned} \quad (\text{A.3})$$

This closed string amplitude is reminiscent of the intersection number between cocycles in the twisted cohomology, which we brief in the following.

We start with the definition of the twisted cohomology [43] on the moduli space. The singular loci of the Koba-Nielsen factor (A.1) are

$$z_i = 0, \quad z_i - 1 = 0, \quad z_j - z_i = 0, \quad 2 \leq i < j \leq \mathcal{N} - 2. \quad (\text{A.4})$$

We define a divisor D as the union of the hyperplanes in (A.4). Then, the closed string moduli space is $X = \mathbb{C}^{\mathcal{N}-3} \setminus D$. Consider the twisted logarithmic one-form of the multi-valued Koba-Nielsen factor (A.1) [9],

$$\omega \equiv d \log U = \frac{\alpha'}{2} \sum_{i=2}^{\mathcal{N}-2} E_i dz_i, \quad E_i \equiv \sum_{\substack{j=1 \\ j \neq i}}^{\mathcal{N}-1} \frac{K_{Ri} \cdot K_{Rj}}{z_{ij}}, \quad (\text{A.5})$$

and the covariant derivative $\nabla_\omega = d + \omega \wedge$ on X . We require that $\frac{\alpha'}{2} K_{Ri} \cdot K_{Rj} \in \mathbb{R} \setminus \mathbb{Z}$. Here, E_i denotes the scattering equation [5, 44]. With respect to ∇_ω , we define the $(\mathcal{N} - 3)$ -th twisted de Rham cohomology

$$H^{\mathcal{N}-3}(X, \nabla_\omega) = \frac{\text{Ker}(\nabla_\omega : \Omega^{\mathcal{N}-3}(X) \rightarrow 0)}{\text{Im}(\nabla_\omega : \Omega^{\mathcal{N}-4}(X) \rightarrow \Omega^{\mathcal{N}-3}(X))}, \quad (\text{A.6})$$

where $\Omega^k(X)$ is the space of smooth k -forms on X .¹¹ Consider the single-valued $(\mathcal{N} - 3)$ -forms on X ,

$$\varphi_L(z_2, \dots, z_{\mathcal{N}-2}) = f_L(z_2, \dots, z_{\mathcal{N}-2}) dz_2 \wedge \dots \wedge dz_{\mathcal{N}-2}, \quad (\text{A.7a})$$

$$\varphi_R(z_2, \dots, z_{\mathcal{N}-2}) = f_R(z_2, \dots, z_{\mathcal{N}-2}) dz_2 \wedge \dots \wedge dz_{\mathcal{N}-2}, \quad (\text{A.7b})$$

which are cocycles in $H^{\mathcal{N}-3}(X, \nabla_\omega)$. We now consider a pairing between $H^{\mathcal{N}-3}(X, \nabla_\omega)$ and its dual $H_c^{\mathcal{N}-3}(X, \nabla_{\omega^\vee})$. It is important that the dual cohomology is with compact support, and hence the subscript “c” [43]. We keep this implicit in the following expressions. These two cohomologies are dual to each other under the pairing

$$\langle \varphi_L, \varphi_L^\vee \rangle = \int_X (U U^\vee) \varphi_L \wedge \varphi_L^\vee. \quad (\text{A.8})$$

Defining $\omega^\vee \equiv \bar{\omega}$, $U^\vee \equiv \bar{U}$, and $\varphi_L^\vee \equiv \bar{\varphi}_R$, where the barred quantities are complex conjugations, we find

$$\langle \varphi_L, \bar{\varphi}_R \rangle = \int_X |U|^2 \varphi_L \wedge \bar{\varphi}_R. \quad (\text{A.9})$$

In terms of (A.9), the closed string amplitude (A.3) becomes

$$\mathcal{M}_c(1, \dots, \mathcal{N}) = C(1, \dots, \mathcal{N}) \langle \varphi_L, \bar{\varphi}_R \rangle. \quad (\text{A.10})$$

Note that this is different from the pairing of two k -th twisted cohomology groups that gives rise to the intersection number of k -th twisted cocycles, which requires a different choice of the dual quantities, namely with $\omega^\vee = -\omega$ and $U^\vee = U^{-1}$ [43]. The duality that we are interested in here, which maps ω and U to their conjugations, has been studied in [9, 42].

¹¹A common basis for the twisted cohomology takes a logarithmic form, which is related to the Parke-Taylor factor [9, 45].

A.2 Loaded cycles and open string amplitudes

Now we move on to the open string side. In terms of the Koba-Nielsen factor U in (A.1) and the differential form φ_L in (A.7a), the open string amplitude \mathcal{M}_L introduced in (4.24) can be cast in the form

$$\Theta(\beta(1, \dots, \mathcal{N}) | 1, \dots, \mathcal{N}) \mathcal{M}_L(\beta) = \int_{\Delta(\beta)} U_{\Delta}(\beta) \varphi_L(z_2, \dots, z_{\mathcal{N}-2}), \quad (\text{A.11})$$

where permutation β fixes the cyclic ordering $(\beta(1), \dots, \beta(\mathcal{N}))$. We introduced the Θ -factor,

$$\Theta(i_1, \dots, i_{\mathcal{N}} | j_1, \dots, j_{\mathcal{N}}), \quad (\text{A.12})$$

that contributes a factor $(-1)^{n_i w_j + n_j w_i}$ if the ordering of i and j is the opposite in the ordered sets $(i_1, \dots, i_{\mathcal{N}})$ and $(j_1, \dots, j_{\mathcal{N}})$. In the case when

$$(\beta(1), \dots, \beta(\mathcal{N})) = (1, \sigma(2), \dots, \sigma(\mathcal{N}-2), \mathcal{N}-1, \mathcal{N}-2), \quad (\text{A.13})$$

the open string amplitude in (A.11) becomes (4.24). To be concrete, we explain different ingredients in (A.11) using the example (A.13). The integration domain,

$$\Delta(1, \sigma(2), \dots, \sigma(\mathcal{N}-2), \mathcal{N}-1, \mathcal{N}-2), \quad (\text{A.14})$$

represents the $(\mathcal{N}-3)$ -dimensional region $0 < z_{\sigma(2)} < \dots < z_{\sigma(\mathcal{N}-2)} < 1$. The closure of this integration domain is an $(\mathcal{N}-3)$ -simplex in a smooth triangulation of the real section of the moduli space X which we refer to as $X(\mathbb{R})$. Since there exist more than two coalescing punctures, a minimal blowup of the moduli space is required [42]. The blowup of a simplex is an associahedron $K_{\mathcal{N}-1}$ [46, 47]. The multi-valued Koba-Nielsen factor U has its branch fixed with respect to the simplex $\Delta(\beta)$ in (A.11), giving rise to quantity U_{Δ} that is single-valued in $\Delta(\beta)$. In the concrete example (A.13), we have

$$\begin{aligned} & U_{\Delta}(1, \sigma(2), \dots, \sigma(\mathcal{N}-2), \mathcal{N}-1, \mathcal{N}) \\ &= \prod_{i=2}^{\mathcal{N}-2} z_{\sigma(i)}^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R\sigma(i)}} (1 - z_{\sigma(i)})^{\frac{1}{2}\alpha' K_{R(\mathcal{N}-1)} \cdot K_{R\sigma(i)}} \prod_{\substack{k, \ell=2 \\ k < \ell}}^{\mathcal{N}-2} (z_{\sigma(\ell)} - z_{\sigma(k)})^{\frac{1}{2}\alpha' K_{R\sigma(k)} \cdot K_{R\sigma(\ell)}}. \end{aligned} \quad (\text{A.15})$$

In general, for a given permutation β in $\Delta(\beta)$, there corresponds a unique associahedron, for which a particular branch of the Koba-Nielsen factor is chosen [9].

The above pairing between the simplex $\Delta(\beta)$ and the single-valued branch choice $U(\beta)$ motivates the definition of a *loaded simplex* [48], $\Gamma(\beta) = \Delta(\beta) \otimes U(\beta)$. Using the loaded simplices we define the $(\mathcal{N}-3)$ -th twisted homology group $H_{\mathcal{N}-3}(X, \mathcal{L}_{\omega})$, with coefficients in the sheaf \mathcal{L}_{ω} consisting of local solutions to $d\psi = \omega \psi$, where ψ is a locally holomorphic function that has the formal solution $\psi \propto U$ [41, 43]. Then, the loading U in a loaded cycle $\Delta \otimes U$ is a section of \mathcal{L}_{ω} on Δ . The open string amplitude (A.11) is now defined as a pairing between the twisted homology group $H_{\mathcal{N}-3}(X, \mathcal{L}_{\omega})$ and the twisted cohomology group $H^{\mathcal{N}-3}(X, \nabla_{\omega})$,

$$\langle \Delta(\beta) \otimes U(\beta), \varphi_L \rangle = \int_{\Delta(\beta)} U_{\Delta}(\beta) \varphi_L. \quad (\text{A.16})$$

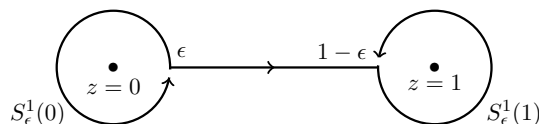


Figure 7. Regularization of the path $[0, 1]$. $S_\epsilon^1(z)$ is a circle centered at z with radius ϵ .

Here, $\Delta(\beta) \otimes U(\beta)$ is a loaded cycle¹² that is an element of $H_{\mathcal{N}-3}(X, \mathcal{L}_\omega)$, defined on the non-compact manifold X .

Furthermore, we define the locally finite homology group $H_{\mathcal{N}-3}^{\text{lf}}(X, \mathcal{L}_{\bar{\omega}})$ [48], and introduce a pairing between $H_{\mathcal{N}-3}^{\text{lf}}(X, \mathcal{L}_{\bar{\omega}})$ and the cohomology group $H_c^{\mathcal{N}-3}(X, \nabla_{\bar{\omega}})$ with a compact support. This pairing represents the open string amplitude \mathcal{M}_R , with

$$\langle \Delta(\beta) \otimes \bar{U}(\beta), \bar{\varphi}_R \rangle = \int_{\Delta(\beta)} \bar{U}_{\Delta}(\beta) \bar{\varphi}_R = \mathcal{M}_R(\beta), \quad (\text{A.17})$$

where $\Delta(\beta) \otimes \bar{U}(\beta)$ is a loaded cycle in $H_{\mathcal{N}-3}^{\text{lf}}(X, \mathcal{L}_{\bar{\omega}})$.

A.3 Intersection number and twisted period relation

Finally, we consider a pairing between $H_{\mathcal{N}-3}(X, \mathcal{L}_\omega)$ and the locally finite homology group $H_{\mathcal{N}-3}^{\text{lf}}(X, \mathcal{L}_{\bar{\omega}})$. This pairing gives rise to a geometric interpretation for the inverse KLT kernel as an intersection number between two loaded cycles [9]. There exists an isomorphism map [41],

$$\text{reg}: H_{\mathcal{N}-3}^{\text{lf}}(X, \mathcal{L}_{\bar{\omega}}) \rightarrow H_{\mathcal{N}-3}(X, \mathcal{L}_{\bar{\omega}}). \quad (\text{A.18})$$

The map “reg” can be realized by a regularization of the loaded cycles. For example, when $\mathcal{N} = 4$, and when the tachyonic vertex operators are concerned, we encounter the integral

$$\int_{0 < z_2 < 1} U(1, 2, 3, 4) \varphi_L(z_2), \quad (\text{A.19})$$

with

$$U(1, 2, 3, 4) = z^{\frac{1}{2}\alpha' K_{R1} \cdot K_{R2}} (1 - z)^{\frac{1}{2}\alpha' K_{L2} \cdot K_{R3}}, \quad \varphi_L(z_2) = f_L(z) dz. \quad (\text{A.20})$$

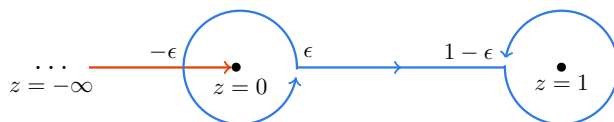
This integral is equal to the open string amplitude (2.37a) up to the Θ -factor as specified in (A.11). Moreover, (A.19) only converges when both the exponents (that we have assumed to be non-integers) in (A.20) are larger than -1 . The regularization of the integration domain $(0, 1)$ is the Pochhammer contour, which gives rise to the regularization $\Delta(1, 2, 3, 4)$ of the path $[0, 1]$ as in figure 7. We have

$$\Gamma(1, 2, 3, 4) = [0, 1] \otimes U(1, 2, 3, 4) \in H_1^{\text{lf}}(X, \mathcal{L}_\omega), \quad (\text{A.21a})$$

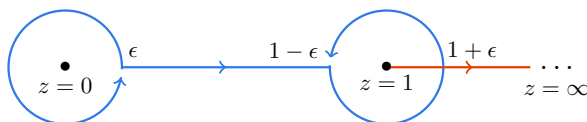
$$\text{reg } \Gamma(1, 2, 3, 4) = \Delta(1, 2, 3, 4) \otimes U(1, 2, 3, 4) \in H_1(X, \mathcal{L}_\omega), \quad (\text{A.21b})$$

which can be generalized to arbitrary \mathcal{N} . See [9, 41, 42] for further details.

¹²We follow [49] and use the terminology “loaded cycle” instead of the usual name “twisted cycle”.



(a) Intersection between $\text{reg}\Gamma(1, 2, 3, 4)$ (in blue) and $\Gamma(2, 1, 3, 4)$ (in red) at $z = -\epsilon$. Using (A.22), the intersection number is computed to be $[2i \sin(\frac{1}{2}\pi\alpha' K_{R1} \cdot K_{R2})]^{-1}$, whose inverse matches the KLT kernel in (3.27a).



(b) Intersection between $\text{reg}\Gamma(1, 2, 3, 4)$ (in blue) and $\Gamma(1, 3, 2, 4)$ (in red) at $z = 1 + \epsilon$. Using (A.22), the intersection number is computed to be $[2i \sin(\frac{1}{2}\pi\alpha' K_{R2} \cdot K_{R3})]^{-1}$, whose inverse matches the KLT kernel in (3.27b).

Figure 8. Intersection numbers between different loaded cycles.

The intersection number between the two loaded cycles is defined with respect to the intersection points between the paths Γ and $\text{reg}\Gamma$. In general, we define the intersection number as the following pairing between $\text{reg}\Gamma(\beta) \in H_{\mathcal{N}-3}^{\text{l.f.}}(X, \mathcal{L}_\omega)$ and $\Gamma(\gamma) \in H_{\mathcal{N}-3}^{\text{l.f.}}(X, \mathcal{L}_{\bar{\omega}})$ [42]:

$$\langle \text{reg}\Gamma(\beta), \Gamma(\gamma) \rangle = \sum_{p \in \Delta \cap \Delta'} \text{I}_p(\Delta, \Delta') \frac{U_\Delta \bar{U}_{\Delta'}}{|U|^2} \Big|_p, \quad (\text{A.22})$$

where we defined

$$\text{reg}\Gamma(\beta) = \Delta \otimes U_\Delta, \quad \Gamma(\gamma) = \Delta' \otimes \bar{U}_{\Delta'}. \quad (\text{A.23})$$

Here, Δ and Δ' are $\mathcal{N} - 3$ simplices and I_p denotes the topological intersection number at the intersecting point p , which only depends on the relative orientations of Δ and Δ' at p . It is shown in [9] that (A.22) precisely gives the inverse of the KLT kernel that is a function of K_{Ri} . For example, see figure 8 for intersection numbers that give rise to the inverse kernels in the KLT relations from (3.27). See [9] for detailed calculation. This intersection number between two loaded cycles also receives the physical interpretation as the α' -corrected bi-adjoint scalar amplitudes [8].

Assembling the ingredients introduced above, we now rewrite the KLT relation for winding strings as a twisted Riemann's period relation,

$$\langle \varphi_L, \bar{\varphi}_R \rangle = \sum_{\Gamma'(\beta), \Gamma(\gamma)} \langle \varphi_L, \Gamma'(\beta) \rangle m^{-1}(\beta|\gamma) \langle \Gamma(\gamma), \bar{\varphi}_R \rangle, \quad (\text{A.24})$$

where $m^{-1}(\beta|\gamma)$ is the inverse of the intersection matrix $m(\beta|\gamma) \equiv \langle \Gamma'(\beta), \Gamma(\gamma) \rangle$. Also note that

$$\varphi_L \in H^{\mathcal{N}-3}(X, \nabla_\omega), \quad \Gamma' \in H_{\mathcal{N}-3}(X, \mathcal{L}_\omega), \quad (\text{A.25a})$$

$$\varphi_R \in H_c^{\mathcal{N}-3}(X, \nabla_{\bar{\omega}}), \quad \Gamma \in H_{\mathcal{N}-3}^{\text{l.f.}}(X, \mathcal{L}_{\bar{\omega}}). \quad (\text{A.25b})$$

Here, the sum over Γ' and Γ are taken over generic bases of the loaded cycles in $H_{\mathcal{N}-3}(X, \mathcal{L}_\omega)$ and $H_{\mathcal{N}-3}^{1,f}(X, \mathcal{L}_{\bar{\omega}})$, respectively. In the case of zero winding, (A.24) reduces to the one discussed in [9, 42]. Taking the relations to closed and open string amplitudes (A.10), (A.11), and (A.17) into account, we find that (A.24) becomes

$$\mathcal{M}_c = \sum_{\beta, \gamma} C(\beta) \mathcal{M}_L(\beta) m^{-1}(\beta|\gamma) \mathcal{M}_R(\gamma), \tag{A.26}$$

which succinctly summarizes the KLT relations given in (4.23), (4.26), and (4.27).¹³ If we start with the multi-valued function,

$$U(z_2, \dots, z_{\mathcal{N}-2}) = \prod_{\substack{i, j=1 \\ i < j}}^{\mathcal{N}-1} z_{ji}^{\frac{1}{2}\alpha' K_{Li} \cdot K_{Lj}}, \tag{A.27}$$

instead of the expression for U in (A.1) that has been considered through this appendix, we will land on the twisted Riemann's period relation that represents the same KLT relations but with “L” and “R” swapped (which of course swaps K_{Li} and K_{Ri}).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1 [[INSPIRE](#)].
- [2] N.E.J. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Minimal Basis for Gauge Theory Amplitudes*, *Phys. Rev. Lett.* **103** (2009) 161602 [[arXiv:0907.1425](#)] [[INSPIRE](#)].
- [3] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, [arXiv:0907.2211](#) [[INSPIRE](#)].
- [4] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [[arXiv:1010.3933](#)] [[INSPIRE](#)].
- [5] F. Cachazo, S. He and E.Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev. D* **90** (2014) 065001 [[arXiv:1306.6575](#)] [[INSPIRE](#)].
- [6] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*, *Nucl. Phys. B* **873** (2013) 419 [[arXiv:1106.2645](#)] [[INSPIRE](#)].
- [7] S. He and O. Schlotterer, *New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level*, *Phys. Rev. Lett.* **118** (2017) 161601 [[arXiv:1612.00417](#)] [[INSPIRE](#)].
- [8] S. Mizera, *Inverse of the String Theory KLT Kernel*, *JHEP* **06** (2017) 084 [[arXiv:1610.04230](#)] [[INSPIRE](#)].

¹³The KLT relation (A.26) also include the type with $\text{reg } \Gamma(\beta) = \Gamma'(\beta)$. In the case of $\mathcal{N} = 4$, this involves the intersection between $\text{reg } \Gamma(1, 2, 3, 4)$ and $\Gamma(1, 2, 3, 4)$, instead of $\Gamma(2, 1, 3, 4)$ or $\Gamma(1, 3, 2, 4)$ in figure 8. In general, such KLT relations can be obtained from the KLT relations discussed in the bulk of the paper by using the stringy Kleiss-Kuijf and BCJ relations in [2, 3].

- [9] S. Mizera, *Combinatorics and Topology of Kawai-Lewellen-Tye Relations*, *JHEP* **08** (2017) 097 [[arXiv:1706.08527](#)] [[INSPIRE](#)].
- [10] S. Mizera, *Scattering Amplitudes from Intersection Theory*, *Phys. Rev. Lett.* **120** (2018) 141602 [[arXiv:1711.00469](#)] [[INSPIRE](#)].
- [11] E. Casali, S. Mizera and P. Tourkine, *Monodromy relations from twisted homology*, *JHEP* **12** (2019) 087 [[arXiv:1910.08514](#)] [[INSPIRE](#)].
- [12] Z. Bern, L.J. Dixon, M. Perelstein and J.S. Rozowsky, *Multileg one loop gravity amplitudes from gauge theory*, *Nucl. Phys. B* **546** (1999) 423 [[hep-th/9811140](#)] [[INSPIRE](#)].
- [13] F. Cachazo, S. He and E.Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149 [[arXiv:1412.3479](#)] [[INSPIRE](#)].
- [14] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [[arXiv:1004.0476](#)] [[INSPIRE](#)].
- [15] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](#)] [[INSPIRE](#)].
- [16] J.J.M. Carrasco and H. Johansson, *Five-Point Amplitudes in $N = 4$ Super-Yang-Mills Theory and $N = 8$ Supergravity*, *Phys. Rev. D* **85** (2012) 025006 [[arXiv:1106.4711](#)] [[INSPIRE](#)].
- [17] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2007) [[DOI](#)] [[INSPIRE](#)].
- [18] R. Blumenhagen, D. Lüüst and S. Theisen, *Basic concepts of string theory*, Springer Science & Business Media (2012) [[DOI](#)].
- [19] N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *Gravity and Yang-Mills Amplitude Relations*, *Phys. Rev. D* **82** (2010) 107702 [[arXiv:1005.4367](#)] [[INSPIRE](#)].
- [20] R. Britto, S. Mizera, C. Rodriguez and O. Schlotterer, *Coaction and double-copy properties of configuration-space integrals at genus zero*, *JHEP* **05** (2021) 053 [[arXiv:2102.06206](#)] [[INSPIRE](#)].
- [21] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *JHEP* **06** (2000) 021 [[hep-th/0005040](#)] [[INSPIRE](#)].
- [22] R. Gopakumar, J.M. Maldacena, S. Minwalla and A. Strominger, *S duality and noncommutative gauge theory*, *JHEP* **06** (2000) 036 [[hep-th/0005048](#)] [[INSPIRE](#)].
- [23] I.R. Klebanov and J.M. Maldacena, *(1 + 1)-dimensional NCOS and its $U(N)$ gauge theory dual*, *Int. J. Mod. Phys. A* **16** (2001) 922 [[hep-th/0006085](#)] [[INSPIRE](#)].
- [24] J. Gomis and H. Ooguri, *Nonrelativistic closed string theory*, *J. Math. Phys.* **42** (2001) 3127 [[hep-th/0009181](#)] [[INSPIRE](#)].
- [25] U.H. Danielsson, A. Guijosa and M. Kruczenski, *IIA/B, wound and wrapped*, *JHEP* **10** (2000) 020 [[hep-th/0009182](#)] [[INSPIRE](#)].
- [26] U.H. Danielsson, A. Guijosa and M. Kruczenski, *Newtonian gravitons and D-brane collective coordinates in wound string theory*, *JHEP* **03** (2001) 041 [[hep-th/0012183](#)] [[INSPIRE](#)].
- [27] J. Gomis, Z. Yan and M. Yu, *Nonrelativistic Open String and Yang-Mills Theory*, *JHEP* **03** (2021) 269 [[arXiv:2007.01886](#)] [[INSPIRE](#)].

- [28] J. Gomis, Z. Yan and M. Yu, *T-duality in Nonrelativistic Open String Theory*, *JHEP* **02** (2021) 087 [[arXiv:2008.05493](#)] [[INSPIRE](#)].
- [29] R. Andringa, E. Bergshoeff, J. Gomis and M. de Roo, ‘Stringy’ Newton-Cartan Gravity, *Class. Quant. Grav.* **29** (2012) 235020 [[arXiv:1206.5176](#)] [[INSPIRE](#)].
- [30] E.A. Bergshoeff, J. Gomis, J. Rosseel, C. Şimşek and Z. Yan, *String Theory and String Newton-Cartan Geometry*, *J. Phys. A* **53** (2020) 014001 [[arXiv:1907.10668](#)] [[INSPIRE](#)].
- [31] E. Bergshoeff, J. Gomis and Z. Yan, *Nonrelativistic String Theory and T-duality*, *JHEP* **11** (2018) 133 [[arXiv:1806.06071](#)] [[INSPIRE](#)].
- [32] J. Gomis, J. Oh and Z. Yan, *Nonrelativistic String Theory in Background Fields*, *JHEP* **10** (2019) 101 [[arXiv:1905.07315](#)] [[INSPIRE](#)].
- [33] A.D. Gallegos, U. Gürsoy and N. Zinnato, *Torsional Newton Cartan gravity from non-relativistic strings*, *JHEP* **09** (2020) 172 [[arXiv:1906.01607](#)] [[INSPIRE](#)].
- [34] T. Harmark, J. Hartong, L. Menculini, N.A. Obers and Z. Yan, *Strings with Non-Relativistic Conformal Symmetry and Limits of the AdS/CFT Correspondence*, *JHEP* **11** (2018) 190 [[arXiv:1810.05560](#)] [[INSPIRE](#)].
- [35] T. Harmark, J. Hartong, L. Menculini, N.A. Obers and G. Oling, *Relating non-relativistic string theories*, *JHEP* **11** (2019) 071 [[arXiv:1907.01663](#)] [[INSPIRE](#)].
- [36] A.D. Gallegos, U. Gürsoy, S. Verma and N. Zinnato, *Non-Riemannian gravity actions from double field theory*, [arXiv:2012.07765](#) [[INSPIRE](#)].
- [37] E.A. Bergshoeff, J. Lahnsteiner, L. Romano, J. Rosseel and C. Şimşek, *A Non-Relativistic Limit of NS-NS Gravity*, [arXiv:2102.06974](#) [[INSPIRE](#)].
- [38] J. Gomis, Z. Yan and M. Yu, *Amplitudes in nonrelativistic open string theory*, work in progress (2021).
- [39] K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann’s period relations I*, *Nagoya Math. J.* **139** (1995) 67.
- [40] D. Gaiotto and J. Lamy-Poirier, *Irregular Singularities in the H_3^+ WZW Model*, [arXiv:1301.5342](#) [[INSPIRE](#)].
- [41] K. Aomoto and M. Kita, *Theory of hypergeometric functions*, Springer Monographs in Mathematics, Springer Japan (2011) [[DOI](#)].
- [42] K. Mimachi and M. Yoshida, *Intersection numbers of twisted cycles and the correlation functions of the conformal field theory. 2.*, *Commun. Math. Phys.* **234** (2003) 339 [[math/0208097](#)] [[INSPIRE](#)].
- [43] K. Matsumoto, *Intersection numbers for logarithmic k-forms*, *Osaka J. Math.* **35** (1998) 873.
- [44] D. Fairlie and D. Roberts, *Dual models without tachyons — a new approach*, unpublished Durham preprint PRINT-72-2440 **1972** (1972).
- [45] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude II. Amplitude and Hypergeometric Function Structure*, *Nucl. Phys. B* **873** (2013) 461 [[arXiv:1106.2646](#)] [[INSPIRE](#)].
- [46] D. Tamari, *Monoïdes préordonnés et chaînes de Malcev*, *Bull. Soc. Math. Fr.* **82** (1954) 53.
- [47] J.D. Stasheff, *Homotopy associativity of H-spaces. I*, *Trans. Am. Math. Soc.* **108** (1963) 275.
- [48] M. Kita and M. Yoshida, *Intersection theory for twisted cycles*, *Math. Nachr.* **166** (1994) 287.

- [49] M. Yoshida, *Hypergeometric functions, my love: Modular interpretations of configuration spaces*, vol. 32, Springer Science & Business Media (2013) [[DOI](#)].