On Bethe vectors in $\mathfrak{gl}_3$-invariant integrable models

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Abstract: We consider quantum integrable models solvable by the nested algebraic Bethe ansatz and possessing $\mathfrak{gl}_3$-invariant R-matrix. We study a new recently proposed approach to construct on-shell Bethe vectors of these models. We prove that the vectors constructed by this method are semi-on-shell Bethe vectors for arbitrary values of Bethe parameters. They thus do become on-shell vectors provided the system of Bethe equations is fulfilled.

Contents

1 Introduction

Recently, a new method to construct Bethe vectors in $\mathfrak{gl}_N$-invariant quantum spin chains was proposed in [1]. In the present paper we study this method by the nested algebraic Bethe ansatz (NABA) in the case of quantum integrable models with $\mathfrak{gl}_3$-invariant R-matrix.

There exist several ways to study quantum integrable models with a high rank of symmetry. A nested version of the Bethe ansatz [2] was proposed in [3–5]. In the context of the Quantum Inverse Scattering Method (QISM) [6–9], an algebraic version of this method (NABA) was developed in [10–12]. One more approach based on the qKZ equation and Jackson integrals was proposed in [13–15] and generalized to the superalgebras in [16]. We should also mention a method to construct Bethe vectors via certain projection of Drinfel'd currents, that was developed in a series of works [17–21]. The Separation of Variables (SoV) method [22, 23] was applied to the study of $\mathfrak{gl}_3$-invariant quantum spin chains in [24].

The main task of the methods listed above is to construct the eigenfunctions of the quantum Hamiltonians. Traditionally they are called *on-shell Bethe vectors*. In distinction of the $\mathfrak{gl}_2$ based models, a form of these eigenfunctions for the models with higher rank of symmetry is quite involved. This is due to the fact that these models describe physical systems with several types of particles. Respectively, one has to consider several creation operators corresponding to each type of excitations.

For instance, within the framework of QISM, we deal with a quantum monodromy matrix $T(u)$, whose trace plays the role of generating functional of the integrals of motion. The upper-triangular entries of the monodromy matrix $T_{ij}(u)$ with $i < j$ are creation operators, and a physical space of states can be generated by successive action of these operators on a referent state $|0\rangle$. In the case of the $\mathfrak{gl}_2$ based models, there exits only one creation operator $T_{12}(u)$. Respectively, the eigenvectors of the quantum Hamiltonians have the form of products of these operators acting onto a referent state $|0\rangle$. However, already in the case of the $\mathfrak{gl}_3$ based models, we deal with three creation operators, and the form of on-shell Bethe vectors immediately becomes much more complex (see e.g. [25] and (2.14) for explicit formulas).

It was observed in [1] that an operator used for constructing the SoV basis of the $\mathfrak{gl}_2$-invariant spin chain can be also used for generating the basis of the on-shell Bethe vectors. It was conjectured in [1] that a similar effect might take place in the spin chains with higher rank of symmetry. In particular, in the $\mathfrak{gl}_3$-invariant spin chain one should consider an operator[1]

$$B^g(u) = T_{23}(u)T_{12}(u-i)T_{23}(u) - T_{23}(u)T_{22}(u-i)T_{13}(u)$$
$$+ T_{13}(u)T_{11}(u-i)T_{23}(u) - T_{13}(u)T_{21}(u-i)T_{13}(u) \qquad (1.1)$$

for constructing the SoV basis [24]. Here $T_{ij}(u)$ are entries of a twisted monodromy matrix (see section 3 for more details). Then, in complete analogy with the case of $\mathfrak{gl}_2$ based models, on-shell Bethe vectors can be presented as a successive action of $B^g(u_i)$ onto the referent state

$$B^g(u_1)\dots B^g(u_a)|0\rangle. \qquad (1.2)$$

This conjecture was justified by the computer calculation, however, an analytical proof is lacking so far. The goal of this paper is to find such the proof.

Our proof of representation (1.2) is given within the framework of NABA. We show that representation (1.2) for on-shell Bethe vectors holds not only for spin chains, but for

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[1]In [1] this operator was denoted as $B^{\text{good}}(u)$. We find this notation too heavy and reduce it to $B^g(u)$.

a more wide class of integrable models possessing $\mathfrak{gl}_3$-invariant R-matrix. In particular, we do not use the SoV method.

The paper is organized as follows. We recall basic notions of NABA in section 2. There we also give a standard description of Bethe vectors within this method. Section 3 is devoted to special NABA-solvable models that usually are applied to the systems of physical interest. The main results of our paper are gathered in section 4. There we give explicit representation of the states (1.2) in terms of the monodromy matrix entries acting on the pseudovacuum vector. We also describe a relationship between the states (1.2) and the Bethe vectors obtained by the standard NABA approach. In the rest of the paper we give the proofs of the results of section 4. We identify the state (1.2) with a Bethe vector in section 5. In section 6 we compute the action of the operator $B^g(u)$ on a generic Bethe vector. Finally, in section 7 we express the state (1.2) in terms of the monodromy matrix entries acting on the pseudovacuum vector. Several auxiliary identities for rational functions are gathered in appendix A. Appendix B contains a proof of connection between two types of Bethe vectors considered in the paper. Finally, the formulas of the action of the monodromy matrix entries onto the Bethe vectors are given in appendix C.

2 Basic notions of NABA

We consider quantum integrable models solvable by NABA and possessing the $\mathfrak{gl}_3$-invariant R-matrix

$$R(u,v) = \mathbf{I} \otimes \mathbf{I} + g(u,v)\mathbf{P}, \qquad g(u,v) = \frac{c}{u-v}. \tag{2.1}$$

Here $\mathbf{I}$ is the identity matrix in $\mathbf{C}^3$, $\mathbf{P}$ is the permutation matrix in $\mathbf{C}^3 \otimes \mathbf{C}^3$, and c is a constant.[2]

The monodromy matrix $T(u)$ is a 3×3 matrix with operator-valued entries $T_{ij}(u)$ acting in a Hilbert space $\mathcal{H}$. Their commutation relations are give by an RTT-relation

$$R(u,v)\big(T(u) \otimes \mathbf{I}\big)\big(\mathbf{I} \otimes T(v)\big) = \big(\mathbf{I} \otimes T(v)\big)\big(T(u) \otimes \mathbf{I}\big)R(u,v). \tag{2.2}$$

It follows from (2.2) that an operator

$$\mathcal{T}(u) = \operatorname{tr} T(u) = \sum_{i=1}^{3} T_{ii}(u) \tag{2.3}$$

has the following property: $[\mathcal{T}(u), \mathcal{T}(v)] = 0$ for arbitrary u and v. This operator is called a transfer matrix. It plays the role of a generating functional of the integrals of motion of a quantum model under consideration. One of the main tasks of NABA is to find eigenvectors of this operator.

If a 3×3 c-number matrix K is such that $[R(x,y), K \otimes K] = 0$, then the matrices $KT(u)$ and $T(u)K$ also satisfy the RTT-relation (2.2). A peculiarity of the R-matrix (2.1) is that $[R(x,y), K \otimes K] = 0$ holds for arbitrary $K \in \mathfrak{gl}_3$. In particular, if K is invertible,

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[2]To compare our presentation with the results of [1] one should set $c = i$.

then one can consider a transformation $T(u) \to T^{(K)}(u) = KT(u)K^{-1}$. Obviously, this transformation preserves the transfer matrix.

Besides the monodromy matrix $T(u)$, we also will consider a matrix $\widehat{T}(u)$ that is closely associated to a quantum comatrix [26, 27]. First, we introduce quantum minors

$$t^{j_1,j_2}_{k_1,k_2}(u) = T_{j_1,k_1}(u)T_{j_2,k_2}(u-c) - T_{j_2,k_1}(u)T_{j_1,k_2}(u-c). \qquad (2.4)$$

The entries of the quantum comatrix $\widetilde{T}_{jk}(u)$ then are given by

$$\widetilde{T}_{jk}(u) = (-1)^{j+k} t^{\bar{k}}_{\bar{j}}(u), \qquad (2.5)$$

where $\bar{j} = \{1, 2, 3\} \setminus j$. The quantum comatrix plays the role of the inverse monodromy matrix due to

$$\widetilde{T}(u-c)T(u) = \text{qdet}\, T(u)\, \mathbf{I}, \qquad (2.6)$$

where $\text{qdet}\, T(u)$ is a quantum determinant of $T(u)$ [26–29].

The matrix $\widehat{T}(u)$ is defined as the transposition of $\widetilde{T}(u)$ with respect to the secondary diagonal:

$$\widehat{T}_{jk}(u) = \widetilde{T}_{4-k,4-j}(u). \qquad (2.7)$$

It is known [25–27, 30] that a mapping $\phi : T(u) \mapsto \widehat{T}(u)$ is an automorphism of the RTT-algebra (2.2). Thus, the matrix $\widehat{T}(u)$ satisfies the RTT-relation with the same R-matrix (2.1).

Using the matrix $\widehat{T}(u)$ we can write down the operator $B^g(u)$ (1.1) in a more compact form

$$B^g(u) = T_{23}(u)\widehat{T}_{13}(u) - T_{13}(u)\widehat{T}_{12}(u). \qquad (2.8)$$

Similar representation for B^g was used in [24].

2.1 Notation

Besides the function $g(u, v)$ we also introduce two new functions

$$f(u, v) = 1 + g(u, v) = \frac{u - v + c}{u - v}, \qquad h(u, v) = \frac{f(u, v)}{g(u, v)} = \frac{u - v + c}{c}. \qquad (2.9)$$

The following obvious properties of the functions introduced above are useful:

$$g(u, v) = -g(v, u), \quad h(u-c, v) = \frac{1}{g(u, v)}, \quad g(u+c, v) = \frac{1}{h(u, v)}, \quad f(u-c, v) = \frac{1}{f(v, u)}. \qquad (2.10)$$

Before giving a description of the Bethe vectors we formulate a convention on the notation. We denote sets of variables by a bar: $\bar{u}$, $\bar{v}$, and so on. Individual elements of the sets are denoted by subscripts: u_j, v_k, and so on. Notation $\bar{u} + c$ means that the constant c is added to all the elements of the set $\bar{u}$. Subsets of variables are denoted by roman indices: $\bar{u}_{\text{I}}$, $\bar{u}_{\text{II}}$, $\bar{v}_{\text{ii}}$, and so on. In particular, we consider partitions of sets into subsets. Then the notation $\bar{u} \Rightarrow \{\bar{u}_{\text{I}}, \bar{u}_{\text{II}}\}$ means that the set $\bar{u}$ is divided into two disjoint subsets $\bar{u}_{\text{I}}$ and $\bar{u}_{\text{II}}$. The order of the elements in each subset is not essential. A special notation $\bar{u}_j$ is used for subsets complementary to the element u_j, that is, $\bar{u}_j = \bar{u} \setminus u_j$, $\bar{v}_k = \bar{v} \setminus v_k$ and so on.

In order to avoid too cumbersome formulas we use shorthand notation for products of functions depending on one or two variables. Namely, if the functions g, f, and h depend on sets of variables, this means that one should take the product over the corresponding set. For example,

$$h(\bar{u}, v) = \prod_{u_j \in \bar{u}} h(u_j, v); \quad g(z_i, \bar{z}_i) = \prod_{\substack{z_j \in \bar{z} \\ z_j \neq z_i}} g(z_i, z_j); \quad f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) = \prod_{u_j \in \bar{u}_{\mathrm{II}}} \prod_{u_k \in \bar{u}_{\mathrm{I}}} f(u_j, u_k).$$
(2.11)

In the last equation of (2.11) the set $\bar{u}$ is divided into two subsets $\bar{u}_{\mathrm{I}}$, $\bar{u}_{\mathrm{II}}$, and the double product is taken with respect to all u_k belonging to $\bar{u}_{\mathrm{I}}$ and all u_j belonging to $\bar{u}_{\mathrm{II}}$. We use the same prescription for the products of commuting operators and their vacuum eigenvalues λ_i (see (2.13))

$$\lambda_i(\bar{u}) = \prod_{u_j \in \bar{u}} \lambda_i(u_j); \quad T_{ij}(\bar{v}_{\mathrm{I}}) = \prod_{v_k \in \bar{v}_{\mathrm{I}}} T_{ij}(v_k).$$
(2.12)

By definition, any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty.

2.2 Bethe vectors

Now we pass to the description of Bethe vectors. They belong to a Hilbert space $\mathcal{H}$, in which the operators $T_{ij}(u)$ act. We assume that this space contains a referent state (pseudovacuum vector) $|0\rangle$ such that

$$\begin{aligned} T_{jj}(u)|0\rangle &= \lambda_j(u)|0\rangle, \\ T_{ij}(u)|0\rangle &= 0, \qquad i > j, \end{aligned}$$
(2.13)

where $\lambda_j(u)$ are some scalar functions. Generically, they are free functional parameters.

The action of the operators $T_{ij}(u)$ with $i < j$ onto pseudovacuum is free. Within the framework of NABA, it is assumed that successive action of these operators onto $|0\rangle$ generates vectors of the space $\mathcal{H}$. Bethe vectors are special polynomials in $T_{ij}(u)$ with $i < j$ acting on $|0\rangle$. Their explicit form will be given later. Here we would like to mention that in the models with $\mathfrak{gl}_3$-invariant R-matrix Bethe vectors depend on two sets of complex parameters $\bar{u} = \{u_1, \ldots, u_a\}$ and $\bar{v} = \{v_1, \ldots, v_b\}$ called Bethe parameters. We denote these vectors by $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$, where a and b respectively are the cardinalities of the sets $\bar{u}$ and $\bar{v}$. A characteristic property of the Bethe vectors is that they become eigenvectors of the transfer matrix $\mathcal{T}(z) = \mathrm{tr}\, T(z)$ provided $\bar{u}$ and $\bar{v}$ enjoy ceratin constraint. In this case they are called on-shell Bethe vectors. Otherwise, if $\bar{u}$ and $\bar{v}$ are generic complex numbers, then the corresponding vector is called *off-shell Bethe vector*.

In physical models, vectors of the space $\mathcal{H}$ describe states with quasiparticles (excitations) of two different types (colors). We say that a state has coloring $\{a, b\}$, if it contains a quasiparticles of the color 1 and b quasiparticles of the color 2. The vector $|0\rangle$ has zero coloring. The operator T_{12} is the creation operator of quasiparticles of the first color, while the operator T_{23} creates quasiparticles of the second color. The operator T_{13} creates one quasiparticle of the first color and one quasiparticle of the second color. The diagonal

operators T_{ii} are neutral, the matrix elements T_{ij} with $i > j$ play the role of annihilation operators. Generally, there are no restrictions on the coloring $\{a, b\}$, thus, the parameters a and b are arbitrary non-negative integers. In specific models, some restrictions may appear.

Different methods to construct Bethe vectors were developed in [12, 15–17]. Several equivalent explicit representations were found in [25]. One of this representations reads

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum_{n=0}^{\min(a,b)} \sum_{\#\bar{u}_{\mathrm{I}}=\#\bar{v}_{\mathrm{I}}=n} \frac{K_n(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})f(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})}{\lambda_2(\bar{v}_{\mathrm{II}})\lambda_2(\bar{u})g(\bar{v}, \bar{u})} T_{13}(\bar{u}_{\mathrm{I}})T_{12}(\bar{u}_{\mathrm{II}})T_{23}(\bar{v}_{\mathrm{II}})|0\rangle. \quad (2.14)$$

Recall that here we use the shorthand notation (2.11), (2.12) for the products of the operators T_{ij} and the functions λ_2, f, and g. The sum in (2.14) is taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$ such that $\#\bar{u}_{\mathrm{I}} = \#\bar{v}_{\mathrm{I}} = n$, where $n = 0, 1, \ldots, \min(a, b)$. It is easy to see that each term of this sum has a fixed coloring $\{a, b\}$, and thus, Bethe vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ has coloring that coincides with the cardinalities of the Bethe parameters. We would like to stress that generically there is no any restriction on the cardinalities of the Bethe parameters $\bar{u}$ and $\bar{v}$. In particular, one might have $a < b$, that is $\#\bar{u} < \#\bar{v}$.

The function $K_n(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})$ in (2.14) is a partition function of the six-vertex model with domain wall boundary condition (DWPF) [31, 32]. It depends on two sets of variables $\bar{v}$ and $\bar{u}$; the subscript shows that $\#\bar{v} = \#\bar{u} = n$. The function K_n has the following determinant representation [32]:

$$K_n(\bar{v}|\bar{u}) = h(\bar{v}, \bar{u}) \left(\prod_{j<k}^{n} g(v_j, v_k)g(u_k, u_j) \right) \det_n \left(\frac{g(v_j, u_k)}{h(v_j, u_k)} \right). \quad (2.15)$$

Some properties of K_n are gathered in appendix A.

Observe that the normalization in (2.14) differs from the normalization of Bethe vectors used in [25]. The present normalization is chosen so that the Bethe vector does not have singularities for $v_j = u_k$ and $v_j - c = u_k$.

We also consider Bethe vectors $\widehat{\mathbb{B}}_{a,b}(\bar{u}; \bar{v})$ which correspond to the monodromy matrix $\widehat{T}(u)$. They have the form

$$\widehat{\mathbb{B}}_{a,b}(\bar{u}; \bar{v}) = \sum_{n=0}^{\min(a,b)} \sum_{\#\bar{u}_{\mathrm{I}}=\#\bar{v}_{\mathrm{I}}=n} \frac{K_n(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})f(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})}{\hat{\lambda}_2(\bar{v}_{\mathrm{II}})\hat{\lambda}_2(\bar{u})g(\bar{v}, \bar{u})} \widehat{T}_{13}(\bar{u}_{\mathrm{I}})\widehat{T}_{12}(\bar{u}_{\mathrm{II}})\widehat{T}_{23}(\bar{v}_{\mathrm{II}})|0\rangle, \quad (2.16)$$

where $\hat{\lambda}_2(z) = \lambda_1(z)\lambda_3(z - c)$.

The automorphism $T(u) \mapsto \widehat{T}(u)$ generates a connection between the Bethe vectors $\mathbb{B}$ and $\widehat{\mathbb{B}}$:

$$\widehat{\mathbb{B}}_{b,a}(\bar{v} + c; \bar{u}) = (-1)^{a+b+ab} \frac{\lambda_2(\bar{u})\lambda_2(\bar{v})}{\lambda_1(\bar{u})\lambda_3(\bar{v})} \mathbb{B}_{a,b}(\bar{u}; \bar{v}). \quad (2.17)$$

The proof is given in appendix B.

We have mentioned already that a generic Bethe vector becomes an on-shell Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ satisfy a special constraint. This constraint is known as

a system of Bethe equations and has the following form:

$$\frac{\lambda_1(u_j)}{\lambda_2(u_j)} = \frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)} f(\bar{v}, u_j), \qquad j = 1, \dots, a,$$

$$\frac{\lambda_2(v_k)}{\lambda_3(v_k)} = \frac{f(v_k, \bar{v}_k)}{f(\bar{v}_k, v_k)} \frac{1}{f(v_k, \bar{u})}, \qquad k = 1, \dots, b. \qquad (2.18)$$

If the system (2.18) is fulfilled, then

$$\mathcal{T}(z)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \tau(z|\bar{u}, \bar{v})\, \mathbb{B}_{a,b}(\bar{u}; \bar{v}), \qquad (2.19)$$

where

$$\tau(z) \equiv \tau(z|\bar{u}, \bar{v}) = \lambda_1(z)f(\bar{u}, z) + \lambda_2(z)f(z, \bar{u})f(\bar{v}, z) + \lambda_3(z)f(z, \bar{v}). \qquad (2.20)$$

Below we will need the action formulas of the operators $T_{ij}(z)$ and $\widehat{T}_{ij}(z)$ on the generic Bethe vectors. They were obtained in [25]. We give the list of necessary formulas in appendix C.

3 Special NABA-solvable models

At the first sight, a method to construct on-shell Bethe vectors by means of the operator $B^g(u)$ (1.1) contradicts to the content of the previous section. Indeed, according to the general scheme, the on-shell Bethe vector depends on *two* sets of variables subject to the equations (2.18). At the same time, vector (1.2) depends on only one set of variables. The solution of this contradiction lies in the fact that in some models there is a kind of hierarchy between the variables $\bar{u}$ and $\bar{v}$: the set $\bar{u}$ plays a basic role, while the variables $\bar{v}$ are auxiliary. In particular, the system of Bethe equations can be reformulated as a constraint on the Bethe parameters $\bar{u}$ only (see (3.4) below).

This class of models includes the XXX SU(3)-invariant Heisenberg chain, for which the operator $B^g(u)$ was originally constructed in [1]. A characteristic property of these models is that only the operators $T_{12}(u)$ and $T_{13}(u)$ are true creation operators, while $T_{23}(u)|0\rangle = 0$. In spite of these models are a particular case of the models considered above, they find a wide application in physics.[3]

Consider a monodromy matrix $T^0(u)$ such that $T_{23}^0(u)|0\rangle = 0$. This condition immediately implies a restriction on the vacuum eigenvalues $\lambda_j(u)$. Indeed, it follows from the RTT-relation that

$$[T_{32}^0(u), T_{23}^0(v)] = g(u, v)\big(T_{22}^0(v)T_{33}^0(u) - T_{22}^0(u)T_{33}^0(v)\big). \qquad (3.1)$$

Acting with this equation onto $|0\rangle$ we obtain

$$0 = \big(\lambda_2(v)\lambda_3(u) - \lambda_2(u)\lambda_3(v)\big)|0\rangle, \qquad (3.2)$$

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[3]One can also consider models, in which $T_{12}(u)|0\rangle = 0$, while $T_{23}(u)$ and $T_{13}(u)$ are true creation operators. This case is equivalent to the one considered in this paper, due to an automorphism of the RTT-algebra (2.2) with respect to the replacement $T_{ij}(u) \to T_{4-j,4-i}(-u)$.

leading to $\lambda_2(u) = \kappa\lambda_3(u)$, where κ is a constant. Without loss of generality we can set $\lambda_2(z) = \kappa$ and $\lambda_3(z) = 1$. At the same time, the vacuum eigenvalue $\lambda_1(z)$ still remains a free functional parameter. Below we omit the subscript and denote it $\lambda_1(z) = \lambda(z)$.

Bethe equations (2.18) take the form

$$\lambda(u_j) = \kappa\frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)}f(\bar{v}, u_j), \qquad j = 1, \ldots, a, \tag{3.3a}$$

$$\kappa f(v_k, \bar{u}) = \frac{f(v_k, \bar{v}_k)}{f(\bar{v}_k, v_k)}, \qquad k = 1, \ldots, b. \tag{3.3b}$$

One can show (see e.g. [33]) that this system implies

$$\det_a\left(\delta_{jk} + \alpha\lambda(u_j)\frac{f(\bar{u}_j, u_j)}{h(u_k, u_j)}\right) = (1 + \alpha)^b(1 + \alpha\kappa)^{a-b}. \tag{3.4}$$

Here α is a complex number. Equation (3.4) should be valid for an arbitrary value of this parameter. As both sides of (3.4) are polynomials in α of degree a, this condition is equivalent to a set of a equations for a variables $\bar{u} = \{u_1, \ldots, u_a\}$ (the free terms in both sides obviously are equal to 1). We see that the set of auxiliary variables $\bar{v}$ is eliminated.

According to the coloring prescriptions, quasiparticles of the second color now can be created by the action of the operator $T_{13}^0(u)$ only. Since this operator simultaneously creates a quasiparticle of the first color, we conclude that the coloring of any state in these models has a property $b \le a$. In particular, Bethe vectors $\mathbb{B}_{a,b}^0(\bar{u}; \bar{v})$ for such the monodromy matrix possess this property. Their explicit form also simplifies:

$$\mathbb{B}_{a,b}^0(\bar{u}; \bar{v}) = \sum_{\#\bar{u}_{\mathrm{I}}=b} \frac{K_b(\bar{v}|\bar{u}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})}{\kappa^a g(\bar{v}, \bar{u})}T_{13}^0(\bar{u}_{\mathrm{I}})T_{12}^0(\bar{u}_{\mathrm{II}})|0\rangle. \tag{3.5}$$

In distinction of (2.14), here the sum is taken over partitions of the set $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ such that $\#\bar{u}_{\mathrm{I}} = b$, while the set $\bar{v}$ is not divided into subsets. We see that a generic off-shell Bethe vector $\mathbb{B}_{a,b}^0(\bar{u}; \bar{v})$ still depends on the set of auxiliary Bethe parameters $\bar{v}$. We will show, however, that the auxiliary parameters can be eliminated from on-shell Bethe vectors, as it was done for the system of Bethe equations.

Thus, for the models with the monodromy matrix $T^0(u)$, one can actually restrict himself with a one set of the Bethe parameters only. However, if we substitute the operators $T_{ij}^0(u)$ into equation (1.1) for $B^g(u)$, then we see that $B^g(u)|0\rangle = 0$. This is due to the fact that $T_{23}^0(u)|0\rangle = 0$. Thus, the operator (1.1) cannot be used as a creation operator in these models.

A nontrivial action of $B^g(u)$ onto the pseudovacuum vector can be provided by an appropriate twist transformation

$$T(u) = KT^0(u)K^{-1}. \tag{3.6}$$

In paper [1], a generic twist matrix K was considered. We restrict ourselves with a 'minimal' twist, which provides a condition $T_{23}(u)|0\rangle \ne 0$, but does not change the action of other operators T_{ij} onto $|0\rangle$. Let

$$K = I + \frac{\beta}{1 - \kappa}E_{23}, \tag{3.7}$$

where $\beta \neq 0$ is a complex number and E_{23} is an elementary unit matrix $(E_{23})_{ij} = \delta_{i2}\delta_{j3}$. It is easy to see that the matrix $T(u)$ has the same vacuum eigenvalues $\lambda_1(z) = \lambda(z)$, $\lambda_2(z) = \kappa$, and $\lambda_3(z) = 1$. However, now we have $T_{23}(u)|0\rangle = \beta|0\rangle$ provided $\kappa \neq 1$.

Of course, the twist matrix (3.7) is not the only matrix, ensuring the condition $T_{23}(u)|0\rangle \neq 0$. We discuss more general twists in Conclusion.

4 Main results

We are now in position to formulate our main results.

Proposition 4.1. *Let the vacuum eigenvalues of the monodromy matrix $T(u)$ be given be equations*

$$T_{11}(u)|0\rangle = \lambda(u)|0\rangle, \qquad T_{22}(u)|0\rangle = \kappa|0\rangle, \qquad T_{33}(u)|0\rangle = |0\rangle, \qquad (4.1)$$

and $T_{23}(u)|0\rangle = \beta|0\rangle$. Let $\bar{u}$ and $\bar{v}$ be two sets of complex numbers such that $\#\bar{u} = a$, $\#\bar{v} = b$, and the constraint (3.3a) is fulfilled. Then Bethe vector $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$ has the following representation:

$$\mathbb{B}_{a,b}(\bar{u}, \bar{v}) = \sum_{n=0}^{a} \frac{\beta^{b-n}}{\kappa^{a+b} g(\bar{v}, \bar{u})} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}})$$

$$\times\, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}})|0\rangle. \qquad (4.2)$$

Here the sum is taken over partitions of the set $\bar{u}$ into three subsets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}\}$. The cardinalities of the subsets are shown explicitly by the subscripts of the sum symbol in (4.2).

The proof of proposition 4.1 is based on the explicit representation for the Bethe vectors (2.14). This is done in section 5. Here we give several comments on this proposition.

The condition (3.3a) is a part of Bethe equations, therefore, the corresponding Bethe vector can be called a *semi-on-shell Bethe vector* [34]. The constraint (3.3a) is a system of a equations for $a + b$ variables. In particular, if[4] $b \geq a$, then we can consider (3.3a) as the system of equations for the parameters v_k, $k = 1, \ldots, b$. At the same time the parameters $\bar{u}$ remain generic complex numbers, and one can easily show that the system is solvable. Furthermore, it follows from representation (4.2) that if v_k, $k = 1, \ldots, b$ and v'_k, $k = 1, \ldots, b'$ are two different solutions to the system (3.3a), then

$$\kappa^{b'-b} g(\bar{v}', \bar{u}) \mathbb{B}_{a,b'}(\bar{u}, \bar{v}') = \beta^{b'-b} g(\bar{v}, \bar{u}) \mathbb{B}_{a,b}(\bar{u}, \bar{v}). \qquad (4.3)$$

This property is due to the very specific action of the operator $T_{23}(z)$ onto the pseudovacuum vector: $T_{23}(z)|0\rangle = \beta|0\rangle$. Thus, two semi-on-shell Bethe vectors with different sets of the Bethe parameters $\bar{v}$ and $\bar{v}'$ actually are proportional to each other. In fact, for an appropriate normalization, semi-on-shell Bethe vectors (4.2) do not depend on the parameters of the set $\bar{v}$.

Proposition 4.1 implies that on-shell Bethe vectors also have representation (4.2). In this case the parameters $\bar{u}$ and $\bar{v}$ enjoy the additional set of equations (3.3b). We see, however, that the condition (3.3a) is already sufficient to eliminate the parameters $\bar{v}$ from the representation for the Bethe vector. They are only included in the normalization factor.

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[4]Recall that due to $T_{23}(z)|0\rangle \neq 0$ we have no restriction $b \leq a$.

4.1 Multiple action of the operator B^g

Now we give an explicit representation for the multiple action of the operator B^g onto pseudovacuum vector $|0\rangle$. It was shown in [1] that $[B^g(u), B^g(v)] = 0$ for arbitrary u and v. Thus, given a set $\bar{u} = \{u_1, \ldots, u_a\}$, then the notation

$$B^g(\bar{u}) = \prod_{j=1}^{a} B^g(u_j) \tag{4.4}$$

is well defined.

Proposition 4.2. *Let the vacuum eigenvalues of the monodromy matrix $T(u)$ be as in proposition 4.1 and $T_{23}(u)|0\rangle = \beta|0\rangle$. Let a set $\bar{u}$ consist of generic complex numbers and $\#\bar{u} = a$. Then*

$$B^g(\bar{u})|0\rangle = \sum_{n=0}^{a} \beta^{2a-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}})$$
$$\times T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}})|0\rangle. \tag{4.5}$$

Here the sum over partitions of $\bar{u}$ is taken as in proposition 4.1.

This proposition gives the result of multiple action of the operator B^g onto $|0\rangle$ in terms of multiple actions of the creation operators T_{12} and T_{13}. The proof of proposition 4.2 is given in section 7.

Comparing (4.5) and (4.2) we immediately arrive at

Corollary 4.1. *Under the conditions of propositions 4.1 and 4.2*

$$B^g(\bar{u})|0\rangle = \beta^{2a-b} \kappa^{a+b} g(\bar{v}, \bar{u}) \mathbb{B}_{a,b}(\bar{u}, \bar{v}). \tag{4.6}$$

Thus, if the Bethe parameters $\bar{u}$ and $\bar{v}$ satisfy Bethe equations (3.3a), (3.3b), then the vector $B^g(\bar{u})|0\rangle$ is on-shell Bethe vector, as it is proportional to the on-shell Bethe vector $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$. One can also consider the vector $B^g(\bar{u})|0\rangle$ for generic complex $\bar{u}$. Equation (4.6) remains true in this case, if the set $\bar{v}$ satisfies the system (3.3a). Due to the property (4.3) one can always provide the solvability of this system for generic complex $\bar{u}$.

5 Proof of proposition 4.1

We begin with an explicit form of Bethe vectors corresponding to the twisted monodromy matrix $T(u)$ (3.6). This form follows from the general representation (2.14), where one should take into account the condition $T_{23}(u)|0\rangle = \beta|0\rangle$. Then

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum_{n=0}^{\min(a,b)} \sum_{\#\bar{u}_{\mathrm{I}}=\#\bar{v}_{\mathrm{I}}=n} \frac{\beta^{b-n} K_n(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) f(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})}{\kappa^{a+b-n} g(\bar{v}, \bar{u})} T_{13}(\bar{u}_{\mathrm{I}}) T_{12}(\bar{u}_{\mathrm{II}})|0\rangle. \tag{5.1}$$

Here, like in (2.14), the sum is taken over partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$. The subscripts of the sums show that the partitions satisfy restrictions $\#\bar{u}_{\mathrm{I}} = \#\bar{v}_{\mathrm{I}} = n$, where $n = 0, 1, \ldots, \min(a, b)$.

The sum over partitions $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$ can be transformed into a sum over additional partitions of the subset $\bar{u}_{\mathrm{I}}$ via (A.2), in which one should set $\bar{x} = \bar{v}$ and $\bar{y} = \bar{u}_{\mathrm{I}}$. Then

$$\sum_{\#\bar{v}_{\mathrm{I}}=n} K_n(\bar{v}_{\mathrm{I}}|\bar{u}_{\mathrm{I}})f(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}}) = \sum_{s=0}^{n} \sum_{\#\bar{u}_{\mathrm{i}}=s} (-1)^{n-s} f(\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{ii}}) f(\bar{v}, \bar{u}_{\mathrm{i}}). \tag{5.2}$$

Here in the lhs, the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$ so that $\#\bar{v}_{\mathrm{I}} = n$. In the rhs, the sum is taken over all possible partitions $\bar{u}_{\mathrm{I}} \Rightarrow \{\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{ii}}\}$. Substituting this into (5.1) we find

$$\mathbb{B}_{a,b}(\bar{u}, \bar{v}) = \sum_{n=0}^{a} \frac{\beta^{b-n}}{\kappa^{a+b-n}g(\bar{v}, \bar{u})} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{i}}=s \\ \#\bar{u}_{\mathrm{ii}}=n-s}} (-1)^{n-s} f(\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{ii}}) f(\bar{v}, \bar{u}_{\mathrm{i}}) f(\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{II}}) f(\bar{u}_{\mathrm{ii}}, \bar{u}_{\mathrm{II}})$$

$$\times T_{13}(\bar{u}_{\mathrm{i}}) T_{13}(\bar{u}_{\mathrm{ii}}) T_{12}(\bar{u}_{\mathrm{II}})|0\rangle. \tag{5.3}$$

In (5.3), the sum is taken over partitions of the set $\bar{u}$ into three subsets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{ii}}, \bar{u}_{\mathrm{II}}\}$. The cardinalities of subsets are shown explicitly by the subscripts of the sum.

Note that we have replaced the upper summation limit $\min(a, b)$ with a in the sum over n. If $a \leq b$, then this replacement certainly is possible. If $a > b$, then all the terms in the sum over n with $n > b$ vanish due to proposition A.2. Indeed, due to this proposition the sum in the r.h.s. of (5.2) gives a determinant (A.14). The latter vanishes for $n > b$. Thus, if $a > b$, then the sum in (5.3) actually breaks at $n = b$.

Suppose that $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$ is a semi-on-shell Bethe vector whose Bethe parameters satisfy the condition (3.3a). Then taking the product of equations (3.3a) over subset $\bar{u}_{\mathrm{i}}$ we find

$$f(\bar{v}, \bar{u}_{\mathrm{i}}) = \kappa^{-s} \lambda(\bar{u}_{\mathrm{i}}) \frac{f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{i}}) f(\bar{u}_{\mathrm{ii}}, \bar{u}_{\mathrm{i}})}{f(\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{ii}}) f(\bar{u}_{\mathrm{i}}, \bar{u}_{\mathrm{II}})}. \tag{5.4}$$

Substituting this into (5.3) we arrive at

$$\mathbb{B}_{a,b}(\bar{u}, \bar{v}) = \sum_{n=0}^{a} \frac{\beta^{b-n}}{\kappa^{a+b}g(\bar{v}, \bar{u})} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{i}}=s \\ \#\bar{u}_{\mathrm{ii}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{i}}) f(\bar{u}_{\mathrm{ii}}, \bar{u}_{\mathrm{i}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{i}}) f(\bar{u}_{\mathrm{ii}}, \bar{u}_{\mathrm{II}})$$

$$\times T_{13}(\bar{u}_{\mathrm{i}}) T_{13}(\bar{u}_{\mathrm{ii}}) T_{12}(\bar{u}_{\mathrm{II}})|0\rangle. \tag{5.5}$$

This representation coincides with (4.2) up to the labels of the subsets. Thus, proposition 4.1 is proved.

6 Action of $B^g(z)$ on Bethe vectors

We use induction over a in order to prove proposition 4.2. However, before doing this, we find the action of the operator $B^g(z)$ on an arbitrary Bethe vector $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$. This will give us a necessary tool for the proof.

Below, for some time, we do not use restrictions $T_{23}(z)|0\rangle = \beta|0\rangle$, $\lambda_2(z) = \kappa$, and $\lambda_3(z) = 1$. Instead, we consider the most general case of the monodromy matrix. In order to avoid new notation, we still denote this monodromy matrix by $T(z)$. However, we do not assume that the action of $T_{23}(z)$ has some peculiarity, nor do we impose any restrictions on the eigenvalues $\lambda_j(z)$. We simply consider the action of the operator $B^g(z)$ (2.8) on an arbitrary Bethe vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ using (2.17) and action formulas (C.2)–(C.4). We also replace the expression for $B^g(z)$ (2.8) by $T_{23}(z_2)\widehat{T}_{13}(z_1) - T_{13}(z_2)\widehat{T}_{12}(z_1)$ and consider the limit $z_k \to z$ ($k = 1, 2$) in the end of the calculations. Then we specify the obtained result to the semi-on-shell Bethe vectors described in section 4.

6.1 Action of $T_{23}(z_2)\widehat{T}_{13}(z_1)$

In this section we study the action of $T_{23}(z_2)\widehat{T}_{13}(z_1)$:

$$\Lambda_1 = T_{23}(z_2)\widehat{T}_{13}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v}). \tag{6.1}$$

Using (2.17) we have

$$\Lambda_1 = (-1)^{a+b+ab}\frac{\lambda_1(\bar{u})\lambda_3(\bar{v})}{\lambda_2(\bar{u})\lambda_2(\bar{v})}T_{23}(z_2)\widehat{T}_{13}(z_1)\widehat{\mathbb{B}}_{b,a}(\bar{v} + c; \bar{u}). \tag{6.2}$$

Then due to (C.2) we obtain

$$\Lambda_1 = \frac{(-1)^{a+b+ab}\lambda_1(\bar{u})\lambda_3(\bar{v})\hat{\lambda}_2(z_1)}{h(z_1, \bar{v} + c)h(\bar{u}, z_1)\lambda_2(\bar{u})\lambda_2(\bar{v})}T_{23}(z_2)\widehat{\mathbb{B}}_{b+1,a+1}(\{\bar{v} + c, z_1\}; \{\bar{u}, z_1\}). \tag{6.3}$$

Turning back from $\widehat{\mathbb{B}}$ to $\mathbb{B}$ and using $\hat{\lambda}_2(z) = \lambda_1(z)\lambda_3(z - c)$ we arrive at

$$\Lambda_1 = \frac{(-1)^{a+b+1}\lambda_2(z_1)\lambda_2(z_1 - c)g(z_1, \bar{v})}{h(\bar{u}, z_1)}T_{23}(z_2)\mathbb{B}_{a+1,b+1}(\{\bar{u}, z_1\}; \{\bar{v}, z_1 - c\}). \tag{6.4}$$

It remains to act with $T_{23}(z_2)$ onto the obtained vector via (C.4):

$$\begin{aligned}\Lambda_1 &= \frac{(-1)^{a+b+1}\lambda_2(z_1)\lambda_2(z_2)\lambda_2(z_1 - c)g(z_1, \bar{v})g(z_1, z_2)}{h(\bar{u}, z_1)h(z_2, \bar{u})h(z_2, z_1)h(\bar{v}, z_2)}\\ &\quad \times \sum_{\#\bar{\eta}_{\mathrm{I}}=1}\frac{f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})h(\bar{v}, \bar{\eta}_{\mathrm{I}})h(z_2, \bar{\eta}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}}, z_2)g(z_1, \bar{\eta}_{\mathrm{I}})}\mathbb{B}_{a+1,b+2}(\bar{\eta}_{\mathrm{II}}; \{\bar{v}, z_1 - c, z_2\}),\end{aligned} \tag{6.5}$$

where $\bar{\eta} = \{\bar{u}, z_1, z_2\}$ and the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ so that $\#\bar{\eta}_{\mathrm{I}} = 1$.

We see that $\bar{\eta}_{\mathrm{I}} \neq z_1$ due to the function $g(z_1, \bar{\eta}_{\mathrm{I}})$ in the denominator of (6.5). Thus, either $\bar{\eta}_{\mathrm{I}} = z_2$ or $\bar{\eta}_{\mathrm{I}} = u_j$, where $j = 1, \ldots, a$. Respectively, we can present Λ_1 in the following form

$$\Lambda_1 = \Lambda_1^{(0)} + \sum_{j=1}^{a}\Lambda_1^{(j)}. \tag{6.6}$$

Here $\Lambda_1^{(0)}$ corresponds to the case $\bar{\eta}_{\mathrm{I}} = z_2$:

$$\Lambda_1^{(0)} = \frac{\lambda_2(z_1)\lambda_2(z_2)\lambda_2(z_1 - c)g(\bar{v}, z_1)g(\bar{u}, z_2)g(z_1, z_2)}{h(\bar{u}, z_1)}\mathbb{B}_{a+1,b+2}(\{\bar{u}, z_1\}; \{\bar{v}, z_1 - c, z_2\}). \tag{6.7}$$

The contributions $\Lambda_1^{(j)}$ correspond to the case $\bar{\eta}_{\text{I}} = u_j$ and have the form:

$$\Lambda_1^{(j)} = \frac{(-1)^{a+b+1}\lambda_2(z_1)\lambda_2(z_2)\lambda_2(z_1-c)g(z_1,\bar{v})g(z_1,z_2)}{h(\bar{u},z_1)h(z_2,\bar{u})h(z_2,z_1)h(\bar{v},z_2)}$$
$$\times f(u_j,\bar{u}_j)h(u_j,z_1)h(\bar{v},u_j)f(z_2,u_j)\mathbb{B}_{a+1,b+2}(\{\bar{u}_j,z_1,z_2\};\{\bar{v},z_1-c,z_2\}). \qquad (6.8)$$

Due to (C.2) we can present the vector $\mathbb{B}_{a+1,b+2}(\{\bar{u}_j,z_1,z_2\};\{\bar{v},z_1-c,z_2\})$ as a result of the $T_{13}(z_2)$ action:

$$\mathbb{B}_{a+1,b+2}(\{\bar{u}_j,z_1,z_2\};\{\bar{v},z_1-c,z_2\}) = \frac{h(z_2,\bar{u}_j)h(z_2,z_1)h(\bar{v},z_2)}{\lambda_2(z_2)g(z_1,z_2)}$$
$$\times T_{13}(z_2)\mathbb{B}_{a,b+1}(\{\bar{u}_j,z_1\};\{\bar{v},z_1-c\}). \qquad (6.9)$$

Then

$$\Lambda_1^{(j)} = \frac{(-1)^{a+b+1}\lambda_2(z_1)\lambda_2(z_1-c)g(z_1,\bar{v})}{h(\bar{u},z_1)}h(u_j,z_1)g(z_2,u_j)f(u_j,\bar{u}_j)h(\bar{v},u_j)$$
$$\times T_{13}(z_2)\mathbb{B}_{a,b+1}(\{\bar{u}_j,z_1\};\{\bar{v},z_1-c\}). \qquad (6.10)$$

Observe that here we can take the limit $z_1 = z_2 = z$:

$$\Lambda_1^{(j)}\Big|_{z_1=z_2=z} = \frac{(-1)^a\lambda_2(z)\lambda_2(z-c)g(\bar{v},z)}{h(\bar{u},z)}f(u_j,z)f(u_j,\bar{u}_j)h(\bar{v},u_j)$$
$$\times T_{13}(z)\mathbb{B}_{a,b+1}(\{\bar{u}_j,z\};\{\bar{v},z-c\}). \qquad (6.11)$$

6.2 Action of $T_{13}(z_2)\widehat{T}_{12}(z_1)$

Now we study the action of $T_{13}(z_2)\widehat{T}_{12}(z_1)$:

$$\Lambda_2 = T_{13}(z_2)\widehat{T}_{12}(z_1)\mathbb{B}_{a,b}(\bar{u};\bar{v}). \qquad (6.12)$$

Using again (2.17) we have

$$\Lambda_2 = (-1)^{a+b+ab}\frac{\lambda_1(\bar{u})\lambda_3(\bar{v})}{\lambda_2(\bar{u})\lambda_2(\bar{v})}T_{13}(z_2)\widehat{T}_{12}(z_1)\widehat{\mathbb{B}}_{b,a}(\bar{v}+c;\bar{u}), \qquad (6.13)$$

and due to (C.3) we obtain

$$\Lambda_2 = \frac{(-1)^{a+b+ab}\hat{\lambda}_2(z_1)\lambda_1(\bar{u})\lambda_3(\bar{v})g(z_1,\bar{v})}{\lambda_2(\bar{u})\lambda_2(\bar{v})h(\bar{u},z_1)}T_{13}(z_2)$$
$$\times \sum_{\#\bar{\xi}_{\text{I}}=1}\frac{f(\bar{\xi}_{\text{II}},\bar{\xi}_{\text{I}})h(\bar{\xi}_{\text{I}},z_1)}{g(\bar{\xi}_{\text{I}},\bar{v})h(z_1,\bar{\xi}_{\text{I}})}\widehat{\mathbb{B}}_{b+1,a}(\{\bar{v}+c,z_1\};\bar{\xi}_{\text{II}}), \qquad (6.14)$$

where $\bar{\xi} = \{\bar{u},z_1\}$ and the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\text{I}},\bar{\xi}_{\text{II}}\}$ so that $\#\bar{\xi}_{\text{I}} = 1$. Turning back to the vector $\mathbb{B}$ we find

$$\Lambda_2 = \frac{(-1)^{a+1}\lambda_2(z_1)\lambda_2(z_1-c)g(z_1,\bar{v})}{h(\bar{u},z_1)}T_{13}(z_2)$$
$$\times \sum_{\#\bar{\xi}_{\text{I}}=1}\frac{\lambda_1(\bar{\xi}_{\text{I}})f(\bar{\xi}_{\text{II}},\bar{\xi}_{\text{I}})h(\bar{\xi}_{\text{I}},z_1)}{\lambda_2(\bar{\xi}_{\text{I}})g(\bar{\xi}_{\text{I}},\bar{v})h(z_1,\bar{\xi}_{\text{I}})}\mathbb{B}_{a,b+1}(\bar{\xi}_{\text{II}};\{\bar{v},z_1-c\}). \qquad (6.15)$$

There is no problem to compute the action of $T_{13}(z_2)$, however, we do not do this. Instead we present the obtained result in the form similar to (6.6)

$$\Lambda_2 = \Lambda_2^{(0)} + \sum_{j=1}^{a} \Lambda_2^{(j)}. \tag{6.16}$$

Here $\Lambda_2^{(0)}$ corresponds to the partition $\bar{\xi}_{\mathrm{I}} = z_1$:

$$\Lambda_2^{(0)} = (-1)^{a+1}\lambda_1(z_1)\lambda_2(z_1 - c)g(\bar{u}, z_1)T_{13}(z_2)\mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z_1 - c\}). \tag{6.17}$$

Observe that here we can take the limit $z_1 = z_2 = z$:

$$\Lambda_2^{(0)}\Big|_{z_1=z_2=z} = (-1)^{a+1}\lambda_1(z)\lambda_2(z - c)g(\bar{u}, z)T_{13}(z)\mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z - c\}). \tag{6.18}$$

The contributions $\Lambda_2^{(j)}$ correspond to the partitions $\bar{\xi}_{\mathrm{I}} = u_j$ and have the following form:

$$\begin{aligned}\Lambda_2^{(j)} &= \frac{(-1)^a\lambda_2(z_1)\lambda_2(z_1 - c)g(z_1, \bar{v})}{h(\bar{u}, z_1)}T_{13}(z_2) \\ &\times \frac{\lambda_1(u_j)f(\bar{u}_j, u_j)f(u_j, z_1)}{\lambda_2(u_j)g(u_j, \bar{v})}\mathbb{B}_{a,b+1}(\{\bar{u}_j, z_1\}; \{\bar{v}, z_1 - c\}).\end{aligned} \tag{6.19}$$

Here we also can take the limit

$$\begin{aligned}\Lambda_2^{(j)}\Big|_{z_1=z_2=z} &= \frac{(-1)^a\lambda_2(z)\lambda_2(z - c)g(z, \bar{v})}{h(\bar{u}, z)}T_{13}(z) \\ &\times \frac{\lambda_1(u_j)f(\bar{u}_j, u_j)f(u_j, z)}{\lambda_2(u_j)g(u_j, \bar{v})}\mathbb{B}_{a,b+1}(\{\bar{u}_j, z\}; \{\bar{v}, z - c\}).\end{aligned} \tag{6.20}$$

6.3 Action of $B^g(z)$ on semi-on-shell Bethe vectors

Consider the difference of the contributions $\Lambda_1^{(j)}$ and $\Lambda_2^{(j)}$ at $z_1 = z_2 = z$. Using (6.11) and (6.20) we find

$$\begin{aligned}\left(\Lambda_1^{(j)} - \Lambda_2^{(j)}\right)\Big|_{z_1=z_2=z} &= (-1)^a\lambda_2(z)\lambda_2(z - c)\frac{f(u_j, z)g(\bar{v}, z)}{g(\bar{v}, u_j)h(\bar{u}, z)} \\ &\times \left\{f(u_j, \bar{u}_j)f(\bar{v}, u_j) - \frac{\lambda_1(u_j)}{\lambda_2(u_j)}f(\bar{u}_j, u_j)\right\}T_{13}(z)\mathbb{B}_{a,b+1}(\{\bar{u}_j, z\}; \{\bar{v}, z - c\}).\end{aligned} \tag{6.21}$$

If $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ is a semi-on-shell Bethe vector such that

$$\frac{\lambda_1(u_j)}{\lambda_2(u_j)}f(\bar{u}_j, u_j) = f(u_j, \bar{u}_j)f(\bar{v}, u_j), \tag{6.22}$$

then this difference vanishes. In particular, if we impose the constraint (3.3a) (setting $\lambda_1(u_j) = \lambda(u_j)$ and $\lambda_2(u_j) = \kappa$), then the contributions $\Lambda_1^{(j)}$ and $\Lambda_2^{(j)}$ cancel each other. It is remarkable, however, that the cancellation of these terms takes place in the most general case of the semi-on-shell Bethe vectors, for which $\bar{\lambda}_1(z)$ and $\lambda_2(z)$ are free functional parameters.

7 Proof of proposition 4.2

Now we are able to prove proposition 4.2 via induction over a. For this, we specify the action formulas of section 6 to the case $\lambda_1(z) = \lambda(z)$ and $\lambda_2(z) = \kappa$.

7.1 Inductive basis

Consider the action of $B^g(z)$ onto $|0\rangle = \mathbb{B}_{0,0}(\emptyset; \emptyset)$. Then due to (6.7), (6.18) we have

$$\Lambda_1^{(0)} = \kappa^3 g(z_1, z_2)\, \mathbb{B}_{1,2}(z_1; \{z_1 - c, z_2\})\Big|_{z_1=z_2=z},$$
$$\Lambda_2^{(0)} = -\kappa\lambda(z) T_{13}(z)\mathbb{B}_{0,1}(\emptyset; z - c). \tag{7.1}$$

Using (5.3) we easily find $\mathbb{B}_{0,1}(\emptyset; z - c) = \beta\kappa^{-1}|0\rangle$. In the case $a = 1$, $b = 2$, equation (5.3) gives

$$\mathbb{B}_{1,2}(u; \{v_1, v_2\}) = \frac{\beta^2}{\kappa^3 g(\bar{v}, u)} \left(T_{12}(u)|0\rangle + \frac{\kappa}{\beta}(f(\bar{v}, u) - 1) T_{13}(u)|0\rangle \right). \tag{7.2}$$

Setting here $u = z_1$, $v_1 = z_1 - c$, and $v_2 = z_2$ we obtain

$$\mathbb{B}_{1,2}(z_1; \{z_1 - c, z_2\}) = \frac{\beta^2}{\kappa^3 g(z_1, z_2)} \left(T_{12}(z_1)|0\rangle - \frac{\kappa}{\beta} T_{13}(z_1)|0\rangle \right), \tag{7.3}$$

leading to

$$\Lambda_1^{(0)} = \beta^2 T_{12}(z)|0\rangle - \kappa\beta T_{13}(z)|0\rangle. \tag{7.4}$$

Thus, we arrive at

$$B^g(z)|0\rangle = \Lambda_1^{(0)} - \Lambda_2^{(0)} = \beta^2 T_{12}(z)|0\rangle + \beta(\lambda(z) - \kappa) T_{13}(z)|0\rangle. \tag{7.5}$$

It is easy to see that representation (4.5) gives the same result for $a = 1$:

$$B^g(z)|0\rangle = \sum_{n=0}^{1} \beta^{2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{z}_{\mathrm{I}}=s \\ \#\bar{z}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{z}_{\mathrm{I}}) T_{13}(\bar{z}_{\mathrm{I}}) T_{13}(\bar{z}_{\mathrm{II}}) T_{12}(\bar{z}_{\mathrm{III}})|0\rangle. \tag{7.6}$$

Here the sum is taken over partitions of the set $\bar{z}$ (consisting on one element z) into three subsets $\bar{z}_{\mathrm{I}}$, $\bar{z}_{\mathrm{II}}$, and $\bar{z}_{\mathrm{III}}$. Clearly, two of these subsets are empty. Because of this reason we did not write the product of the f-functions in (7.6) (see (4.5)), as these products are taken at least over one empty set. Setting successively in (7.6) $\bar{z}_{\mathrm{I}} = z$, $\bar{z}_{\mathrm{II}} = z$, and $\bar{z}_{\mathrm{III}} = z$ we obtain three contributions coinciding with (7.5). Thus, the induction basis is checked.

It is interesting to write down this result in terms of the entries of the original monodromy matrix $T^0(u)$. Using (3.6) and (3.7) we find

$$T_{13}(u) = T_{13}^0(u) - \frac{\beta}{1 - \kappa} T_{12}^0(u), \qquad T_{12}(u) = T_{12}^0(u). \tag{7.7}$$

Then replacing z with u in representation (7.5) we obtain

$$B^g(u)|0\rangle = \beta(\lambda(u) - \kappa) T_{13}^0(u)|0\rangle + \beta^2 \left(\frac{1 - \lambda(u)}{1 - \kappa} \right) T_{12}^0(u)|0\rangle. \tag{7.8}$$

The monodromy matrix $T^0(u)$ has two on-shell Bethe vectors in the case $a = 1$: $B^0_{1,0}(u, \emptyset)$ and $B^0_{1,1}(u, v)$. In the first case, there is only one Bethe equation $\lambda(u) = \kappa$, and hence, (7.8) yields

$$B^g(u)|0\rangle = \beta^2 T^0_{12}(u)|0\rangle. \tag{7.9}$$

In the second case we have a system of two Bethe equations

$$\lambda(u) = \kappa f(v, u), \qquad \kappa f(v, u) = 1, \tag{7.10}$$

what implies $\lambda(u) = 1$. Then (7.8) yields

$$B^g(u)|0\rangle = \beta(1 - \kappa)T^0_{13}(u)|0\rangle. \tag{7.11}$$

Both vectors $T^0_{12}(u)|0\rangle$ and $T^0_{13}(u)|0\rangle$ indeed are on-shell Bethe vectors respectively for $\lambda(u) = \kappa$ and $\lambda(u) = 1$. Thus, the action of $B^g(u)$ onto the pseudovacuum vector does give the on-shell Bethe vectors, if u is a root of Bethe equations.

7.2 Inductive step

We assume that (4.5) holds for some $a \geq 1$. Then due to corollary 4.1 the action $B^g(\bar{u})|0\rangle$ with $\bar{u} = \{u_1, \ldots, u_a\}$ is proportional to the semi-on-shell Bethe vector $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$ (4.2), where the set $\bar{v}$ enjoys the constraint (3.3a). Hence, we have

$$B^g(z)B^g(\bar{u})|0\rangle = \beta^{2a-b}\kappa^{a+b}g(\bar{v}, \bar{u})B^g(z)\mathbb{B}_{a,b}(\bar{u}, \bar{v}). \tag{7.12}$$

The action of $B^g(z)$ onto $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$ is given by the terms $\Lambda^{(0)}_2$ (6.18) and $\Lambda^{(0)}_1$ (6.7) (in the limit $z_1 = z_2 = z$). Thus,

$$B^g(z)B^g(\bar{u})|0\rangle = \beta^{2a-b}\kappa^{a+b}g(\bar{v}, \bar{u})\left(\Lambda^{(0)}_1 - \Lambda^{(0)}_2\right)\Big|_{z_1=z_2=z}. \tag{7.13}$$

Now we should set $\lambda_1(z) = \lambda(z)$ and $\lambda_2(z) = \kappa$ in (6.7) and (6.18) for $\Lambda^{(0)}_k$ and substitute these expressions into (7.13). We obtain

$$B^g(z)B^g(\bar{u})|0\rangle = M_1 + M_2, \tag{7.14}$$

where

$$M_1 = \beta^{2a-b}\kappa^{a+b+1}g(\bar{v}, \bar{u})\lambda(z)g(z, \bar{u})T_{13}(z)\mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z - c\}), \tag{7.15}$$

and

$$M_2 = \beta^{2a-b}\kappa^{a+b+3}g(\bar{v}, \bar{u})\frac{g(\bar{v}, z_1)g(\bar{u}, z_2)g(z_1, z_2)}{h(\bar{u}, z_1)}\,\mathbb{B}_{a+1,b+2}(\{\bar{u}, z_1\}; \{\bar{v}, z_1 - c, z_2\})\Big|_{z_1=z_2=z}. \tag{7.16}$$

It remains to substitute explicit expression (5.3) for the Bethe vectors $\mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z - c\})$ and $\mathbb{B}_{a+1,b+2}(\{\bar{u}, z_1\}; \{\bar{v}, z_1 - c, z_2\})$ into (7.15) and (7.16). However, before doing this, it is convenient to describe an expected form of the result.

We expect that multiple action $B^g(z)B^g(\bar{u})|0\rangle$ is given by (4.5), in which one should replace $\bar{u}$ with $\bar{\eta} = \{\bar{u}, z\}$ and $a \to a+1$. That is,

$$B^g(z)B^g(\bar{u})|0\rangle = \sum_{n=0}^{a+1} \beta^{2a+2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{\eta}_{\mathrm{I}}=s \\ \#\bar{\eta}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{\eta}_{\mathrm{I}}) f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}}) f(\bar{\eta}_{\mathrm{III}}, \bar{\eta}_{\mathrm{I}}) f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}})$$
$$\times T_{13}(\bar{\eta}_{\mathrm{I}}) T_{13}(\bar{\eta}_{\mathrm{II}}) T_{12}(\bar{\eta}_{\mathrm{III}})|0\rangle. \qquad (7.17)$$

Let us give more details on the expected form of the result (7.17).

There are three possibilities in the sum over partitions in the r.h.s. of (7.17): $z \in \bar{\eta}_{\mathrm{I}}$; $z \in \bar{\eta}_{\mathrm{II}}$; $z \in \bar{\eta}_{\mathrm{III}}$. Respectively, there are three contributions

$$B^g(z)B^g(\bar{u})|0\rangle = W_1 + W_2 + W_3. \qquad (7.18)$$

In the first case we have

$$W_1 = \lambda(z) T_{13}(z) \sum_{n=0}^{a} \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}})$$
$$\times f(\bar{u}_{\mathrm{II}}, z) f(\bar{u}_{\mathrm{III}}, z) T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}})|0\rangle. \qquad (7.19)$$

Indeed, we can set $\bar{\eta}_{\mathrm{I}} = \{z, \bar{u}_{\mathrm{I}}\}$, $\bar{\eta}_{\mathrm{II}} = \bar{u}_{\mathrm{II}}$, and $\bar{\eta}_{\mathrm{III}} = \bar{u}_{\mathrm{III}}$. The set $\bar{\eta}_{\mathrm{I}}$ is not empty, thus, $s \in [1, \ldots, n]$. This implies $n \in [1, \ldots, a+1]$. Shifting $n \to n+1$ and $s \to s+1$ in (7.17) we arrive at (7.19).

In the second case $\bar{\eta}_{\mathrm{II}} = \{z, \bar{u}_{\mathrm{II}}\}$, $\bar{\eta}_{\mathrm{I}} = \bar{u}_{\mathrm{I}}$, and $\bar{\eta}_{\mathrm{III}} = \bar{u}_{\mathrm{III}}$. The set $\bar{\eta}_{\mathrm{II}}$ is not empty, thus, $s \in [0, \ldots, n-1]$. We also have $n \in [1, \ldots, a+1]$, because the union $\{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ is not empty. Shifting $n \to n+1$ in (7.17) we arrive at

$$W_2 = -\kappa T_{13}(z) \sum_{n=0}^{a} \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}})$$
$$\times f(z, \bar{u}_{\mathrm{I}}) f(z, \bar{u}_{\mathrm{III}}) T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}})|0\rangle. \qquad (7.20)$$

Finally, in the third case $\bar{\eta}_{\mathrm{I}} = \bar{u}_{\mathrm{I}}$, $\bar{\eta}_{\mathrm{II}} = \bar{u}_{\mathrm{II}}$, and $\bar{\eta}_{\mathrm{III}} = \{z, \bar{u}_{\mathrm{III}}\}$. The set $\bar{\eta}_{\mathrm{III}}$ is not empty, thus, $n \in [0, \ldots, a]$. We obtain

$$W_3 = \sum_{n=0}^{a} \beta^{2a+2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}})$$
$$\times f(z, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, z) T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) T_{12}(z)|0\rangle. \qquad (7.21)$$

Thus, our goal is to check that equations (7.14)–(7.16) give all three contributions W_j, $j = 1, 2, 3$.

7.2.1 Contribution W_1

Consider the term M_1. Using (5.3) for $\mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z - c\})$ we obtain

$$M_1 = \lambda(z) f(\bar{u}, z) T_{13}(z) \sum_{n=0}^{a} \kappa^n \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-1)^{n-s}$$
$$\times f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) \frac{f(\bar{v}, \bar{u}_{\mathrm{I}})}{f(\bar{u}_{\mathrm{I}}, z)} f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{III}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) |0\rangle. \quad (7.22)$$

Taking into account (3.3a) we arrive at

$$M_1 = \lambda(z) T_{13}(z) \sum_{n=0}^{a} \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} f(\bar{u}_{\mathrm{II}}, z) f(\bar{u}_{\mathrm{III}}, z)$$
$$\times \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) |0\rangle. \quad (7.23)$$

We see that this is exactly W_1 (7.19).

7.2.2 Contributions W_2 and W_3

Consider the term M_2. Using (5.3) for $\mathbb{B}_{a+1,b+2}(\{\bar{u}, z_1\}; \{\bar{v}, z_1 - c, z_2\})$ and setting $\{z_1, \bar{u}\} = \bar{\eta}$ we obtain

$$M_2 = \sum_{n=0}^{a+1} \kappa^n \beta^{2a+2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{\eta}_{\mathrm{I}}=s \\ \#\bar{\eta}_{\mathrm{II}}=n-s}} (-1)^{n-s} \frac{f(\bar{v}, \bar{\eta}_{\mathrm{I}}) f(z_2, \bar{\eta}_{\mathrm{I}})}{f(\bar{\eta}_{\mathrm{I}}, z_1)}$$
$$\times f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}) f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{III}}) f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}}) \, T_{13}(\bar{\eta}_{\mathrm{I}}) T_{13}(\bar{\eta}_{\mathrm{II}}) T_{12}(\bar{\eta}_{\mathrm{III}}) |0\rangle. \quad (7.24)$$

We see that $z_1 \notin \bar{\eta}_{\mathrm{I}}$, otherwise $1/f(\bar{\eta}_{\mathrm{I}}, z_1) = 0$. Thus, either $z_1 \in \bar{\eta}_{\mathrm{II}}$ or $z_1 \in \bar{\eta}_{\mathrm{III}}$. Respectively, M_2 consists of two contributions: $M_2^{(1)}$ corresponding to the case $z_1 \in \bar{\eta}_{\mathrm{II}}$ and $M_2^{(2)}$ corresponding to the case $z_1 \in \bar{\eta}_{\mathrm{III}}$.

Let $z_1 \in \bar{\eta}_{\mathrm{II}}$. Then we can set $\bar{\eta}_{\mathrm{I}} = \bar{u}_{\mathrm{I}}$, $\bar{\eta}_{\mathrm{III}} = \bar{u}_{\mathrm{III}}$, and $\bar{\eta}_{\mathrm{II}} = \{z_1, \bar{u}_{\mathrm{II}}\}$. We also have $n - s > 0$, and thus, $s \in [0, \ldots, n-1]$ and $n \in [1, \ldots, a+1]$. Shifting $n \to n+1$ and setting $z_1 = z_2 = z$ we obtain

$$M_2^{(1)} = -\kappa T_{13}(z) \sum_{n=0}^{a} \kappa^n \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-1)^{n-s} f(z, \bar{u}_{\mathrm{I}}) f(z, \bar{u}_{\mathrm{III}})$$
$$\times f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) f(\bar{v}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{III}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) |0\rangle. \quad (7.25)$$

Using (3.3a) we arrive at

$$M_2^{(1)} = -\kappa T_{13}(z) \sum_{n=0}^{a} \beta^{2a+1-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} f(z, \bar{u}_{\mathrm{I}}) f(z, \bar{u}_{\mathrm{III}})$$
$$\times \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) |0\rangle, \quad (7.26)$$

and we see that this is exactly W_2 (7.20).

Let now $z_1 \in \bar{\eta}_{\mathrm{III}}$. Then we can set $\bar{\eta}_{\mathrm{I}} = \bar{u}_{\mathrm{I}}$, $\bar{\eta}_{\mathrm{II}} = \bar{u}_{\mathrm{II}}$, and $\bar{\eta}_{\mathrm{III}} = \{z_1, \bar{u}_{\mathrm{III}}\}$. We also have $n \in [0, \ldots, a]$. Setting $z_1 = z_2 = z$ we obtain

$$
M_2^{(2)} = \sum_{n=0}^{a} \kappa^n \beta^{2a+2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-1)^{n-s} f(z, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, z)
$$
$$
\times f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) f(\bar{v}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{III}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) T_{12}(z) |0\rangle. \tag{7.27}
$$

Using (3.3a) we arrive at

$$
M_2^{(2)} = \sum_{n=0}^{a} \beta^{2a+2-n} \sum_{s=0}^{n} \sum_{\substack{\#\bar{u}_{\mathrm{I}}=s \\ \#\bar{u}_{\mathrm{II}}=n-s}} (-\kappa)^{n-s} f(z, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, z)
$$
$$
\times \lambda(\bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{III}}, \bar{u}_{\mathrm{I}}) f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}) \, T_{13}(\bar{u}_{\mathrm{I}}) T_{13}(\bar{u}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{III}}) T_{12}(z) |0\rangle. \tag{7.28}
$$

We see that this is exactly W_3 (7.21). Thus, the multiple action $B^g(z) B^g(\bar{u}) |0\rangle$ is given by the formulas (7.19)–(7.21), leading to (7.17). Hence, proposition 4.2 is proved.

Conclusion. In this paper we have proved one of conjectures of [1]. Namely we have shown that the successive action of the operator B^g (1.1) on the pseudovacuum vector generates on-shell Bethe vectors in $\mathfrak{gl}_3$-invariant models, provided the arguments of these operators satisfy Bethe equations. Furthermore, if the arguments of the B^g operators are generic complex numbers, then the successive action of B^g gives a semi-on-shell Bethe vector. This property holds not only for $\mathfrak{gl}_3$-invariant spin chains, but for a wider class of models, for instance, for the two-component generalization of the Lieb-Liniger model [3, 35–37]. At the same time, we would like to emphasize that the operator B^g can not be used to construct on-shell Bethe vectors in generic NABA-solvable models. The restriction $T_{23}^0(u)|0\rangle = 0$ is crucial. On the other hand, the existence of this restriction was clear from the outset, since within the framework of the new approach Bethe vectors depend only on one set of variables by construction, rather than two sets, as is the case of the Bethe vectors of the general form.

In this paper, we considered the minimal twist (3.7). A general twist K^{gen} can be treated as further twisting of the matrix $T(u)$. It is quite natural to expect that the effect of the general twist must be similar to what one has in the case of $\mathfrak{gl}_2$ based models [38]. Namely, we saw that for the minimal twist, the multiple action of B^g was equivalent to the *one* semi-on-shell Bethe vector $\mathbb{B}_{a,b}(\bar{u}, \bar{v})$. Most probably, that the multiple action of B^g in the case of the general twist is equivalent to a linear combination of semi-on-shell Bethe vectors with different sets of the Bethe parameters. However, as soon as we impose Bethe equations, only one term in this linear combination should survive. The proof of this property in the case of $\mathfrak{gl}_2$-invariant models is very simple (see [38]). However, a generalization of this proof to the models with $\mathfrak{gl}_3$-invariant R-matrix meats certain technical difficulties. Therefore, we did not consider the case of the general twist.

Despite the fact that we have proved the property of $B^g(u)$ to generate on-shell Bethe vectors, we still do not have a clear understanding of why this is happening. In this

context, the most intrigues looks the cancellation of 'unwanted' terms (6.21). Recall that this cancellation takes place for a *general* semi-on-shell Bethe vector. We do not need to assume any specific form of $\lambda_j(u)$ and specific action of $T_{23}(u)$ onto $|0\rangle$. Perhaps this is due to some hidden structure of the operator $B^g(u)$, which is not yet clear. It would be very interesting to find this structure.

Finally it is worth mentioning that a generalization of the operator $B^g(u)$ to the $\mathfrak{gl}_N$-invariant spin chains ($N > 3$) was also proposed in [1]. It was conjectured that this operator also generates on-shell Bethe vectors, similarly to the $\mathfrak{gl}_3$ case. Basing on the results of this paper we can assume that the successive action of $B^g(u)$ is equivalent to a semi-on-shell Bethe vector of a certain $\mathfrak{gl}_N$-invariant integrable model. However, the method that we used in this paper hardly can be applied to the case $N > 3$, as it becomes very bulky.

Acknowledgments

A Properties of DWPF

The DWPF $K_n(\bar{x}|\bar{y})$ defined by (2.15) is a rational function of $\bar{x}$ and $\bar{y}$. It is symmetric over $\bar{x}$ and symmetric over $\bar{y}$. If $x_j \to \infty$ (or $y_j \to \infty$) and all other variables are fixed, then $K_n(\bar{x}|\bar{y}) \to 0$. This function has simple poles at $x_j = y_k$, $j, k = 1, \ldots, n$. The residues in these poles can be expressed in terms of K_{n-1}. Due to the symmetry of K_n over $\bar{x}$ and over $\bar{y}$, it is enough to consider the residue at $x_n = y_n$:

$$K_n(\bar{x}|\bar{y})\Big|_{x_n \to y_n} = g(x_n, y_n)f(\bar{x}_n, x_n)f(y_n, \bar{y}_n)K_{n-1}(\bar{x}_n|\bar{y}_n) + reg, \qquad (A.1)$$

where *reg* means regular part.

The properties listed above, together with the initial condition $K_1(x|y) = g(x, y)$ fix the function $K_n(\bar{x}|\bar{y})$ unambiguously [31, 32].

Proposition A.1. *Let $\#\bar{x} = m$ and $\#\bar{y} = n$ so that $m \geq n$. Then*

$$\sum_{\#\bar{x}_{\mathrm{I}}=n} K_n(\bar{x}_{\mathrm{I}}|\bar{y})f(\bar{x}_{\mathrm{II}}, \bar{x}_{\mathrm{I}}) = \sum_{k=0}^{n} \sum_{\#\bar{y}_{\mathrm{I}}=k} (-1)^{n-k} f(\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{II}})f(\bar{x}, \bar{y}_{\mathrm{I}}). \qquad (A.2)$$

Here in the lhs, the sum is taken over partitions $\bar{x} \Rightarrow \{\bar{x}_{\mathrm{I}}, \bar{x}_{\mathrm{II}}\}$ so that $\#\bar{x}_{\mathrm{I}} = n$. In the rhs, the sum is taken over all possible partitions $\bar{y} \Rightarrow \{\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{II}}\}$.

Proof. We use induction over n. For $n = 1$, equation (A.2) takes the form

$$\sum_{j=1}^{m} g(x_j, y) f(\bar{x}_j, x_j) = f(\bar{x}, y) - 1. \tag{A.3}$$

Obviously, the l.h.s. of (A.3) is partial fraction decomposition of the r.h.s. Thus, identity (A.2) is valid for $n = 1$ and arbitrary $m \geq 1$.

Assume that (A.2) holds for some $n - 1$ and arbitrary $m \geq n - 1$. Let

$$H_{n,m}^{\ell}(\bar{x}; \bar{y}) = \sum_{\#\bar{x}_{\mathrm{I}} = n} K_n(\bar{x}_{\mathrm{I}} | \bar{y}) f(\bar{x}_{\mathrm{II}}, \bar{x}_{\mathrm{I}}),$$
$$H_{n,m}^{r}(\bar{x}; \bar{y}) = \sum_{k=0}^{n} \sum_{\#\bar{y}_{\mathrm{I}} = k} (-1)^{n-k} f(\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{II}}) f(\bar{x}, \bar{y}_{\mathrm{I}}). \tag{A.4}$$

Consider properties of $H_{n,m}^{\ell}$ and $H_{n,m}^{r}$ as functions of y_n at other variables fixed. Both functions are rational functions of y_n. Due to the properties of $K_n(\bar{x}_{\mathrm{I}} | \bar{y})$, the function $H_{n,m}^{\ell}(\bar{x}; \bar{y})$ vanishes as $y_n \to \infty$. Let us show that $H_{n,m}^{r}(\bar{x}; \bar{y})$ has the same property. We use the fact that for arbitrary finite z the functions $f(z, y_n)$ and $f(y_n, z)$ go to 1 as $y_n \to \infty$.

Clearly, we have either $y_n \in \bar{y}_{\mathrm{I}}$ or $y_n \in \bar{y}_{\mathrm{II}}$ in the sum over partitions over $\bar{y}$. Consider the first case. Then $k > 0$ and we can set $\bar{y}_{\mathrm{I}} = \{y_n, \bar{y}_{\mathrm{i}}\}$. We obtain

$$\lim_{y_n \to \infty} \sum_{k=1}^{n} \sum_{\#\bar{y}_{\mathrm{i}} = k-1} (-1)^{n-k} f(y_n, \bar{y}_{\mathrm{II}}) f(\bar{y}_{\mathrm{i}}, \bar{y}_{\mathrm{II}}) f(\bar{x}, y_n) f(\bar{x}, \bar{y}_{\mathrm{i}}) = \sum_{k=0}^{n-1} \sum_{\#\bar{y}_{\mathrm{i}} = k} (-1)^{n-k-1} f(\bar{y}_{\mathrm{i}}, \bar{y}_{\mathrm{II}}) f(\bar{x}, \bar{y}_{\mathrm{i}}). \tag{A.5}$$

In the second case $k < n$ and we can set $\bar{y}_{\mathrm{II}} = \{y_n, \bar{y}_{\mathrm{ii}}\}$. We obtain

$$\lim_{y_n \to \infty} \sum_{k=0}^{n-1} \sum_{\#\bar{y}_{\mathrm{I}} = k} (-1)^{n-k} f(\bar{y}_{\mathrm{I}}, y_n) f(\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{ii}}) f(\bar{x}, \bar{y}_{\mathrm{I}}) = \sum_{k=0}^{n-1} \sum_{\#\bar{y}_{\mathrm{I}} = k} (-1)^{n-k} f(\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{ii}}) f(\bar{x}, \bar{y}_{\mathrm{I}}). \tag{A.6}$$

Relabeling $\bar{y}_{\mathrm{i}} \to \bar{y}_{\mathrm{I}}$ in (A.5) and $\bar{y}_{\mathrm{ii}} \to \bar{y}_{\mathrm{II}}$ in (A.6) we see that the obtained sums over partitions cancel each other. Thus $H_{n,m}^{r}(\bar{x}; \bar{y}) \to 0$ as $y_n \to \infty$.

It remains to compare the residues of two rational functions in the poles $y_n = x_j$, $j = 1, \ldots, m$. Let $y_n \to x_j$ in the function $H_{n,m}^{\ell}(\bar{x}; \bar{y})$. The pole occurs if and only if $x_j \in \bar{x}_{\mathrm{I}}$. Setting $\bar{x}_{\mathrm{I}} = \{x_j, \bar{x}_{\mathrm{I}'}\}$ and using (A.1) we find

$$H_{n,m}^{\ell}(\bar{x}; \bar{y})\Big|_{y_n \to x_j} = \sum_{\#\bar{x}_{\mathrm{I}'} = n-1} g(x_j, y_n) f(\bar{x}_{\mathrm{I}'}, x_j) f(y_n, \bar{y}_n) K_{n-1}(\bar{x}_{\mathrm{I}'} | \bar{y}_n) f(\bar{x}_{\mathrm{II}}, \bar{x}_{\mathrm{I}'}) f(\bar{x}_{\mathrm{II}}, x_j) + reg, \tag{A.7}$$

where reg means regular part. Obviously $f(\bar{x}_{\mathrm{I}'}, x_j) f(\bar{x}_{\mathrm{II}}, x_j) = f(\bar{x}_j, x_j)$. Hence,

$$H_{n,m}^{\ell}(\bar{x}; \bar{y})\Big|_{y_n \to x_j} = g(x_j, y_n) f(\bar{x}_j, x_j) f(y_n, \bar{y}_n) \sum_{\#\bar{x}_{\mathrm{I}'} = n-1} K_{n-1}(\bar{x}_{\mathrm{I}'} | \bar{y}_n) f(\bar{x}_{\mathrm{II}}, \bar{x}_{\mathrm{I}'}) + reg. \tag{A.8}$$

The remaining sum over partitions gives $H_{n-1,m-1}^{\ell}(\bar{x}_j; \bar{y}_n)$, and we finally arrive at

$$H_{n,m}^{\ell}(\bar{x}; \bar{y})\Big|_{y_n \to x_j} = g(x_j, y_n) f(\bar{x}_j, x_j) f(y_n, \bar{y}_n) H_{n-1,m-1}^{\ell}(\bar{x}_j; \bar{y}_n) + reg. \tag{A.9}$$

Consider now the behavior of $H_{n,m}^r(\bar{x};\bar{y})$ at $y_n \to x_j$. The pole occurs if and only if $y_n \in \bar{y}_{\mathrm{I}}$. Setting $\bar{y}_{\mathrm{I}} = \{y_n, \bar{y}_{\mathrm{I}'}\}$ we obtain

$$H_{n,m}^r(\bar{x};\bar{y})\Big|_{y_n \to x_j} = \sum_{k=1}^{n} \sum_{\#\bar{y}_{\mathrm{I}'}=k-1} (-1)^{n-k} f(y_n, \bar{y}_{\mathrm{II}}) f(\bar{y}_{\mathrm{I}'}, \bar{y}_{\mathrm{II}})$$

$$\times f(\bar{x}_j, \bar{y}_{\mathrm{I}'}) g(x_j, y_n) f(\bar{x}_j, x_j) f(y_n, \bar{y}_{\mathrm{I}'}) + reg. \qquad (A.10)$$

Using $f(y_n, \bar{y}_{\mathrm{I}'}) f(y_n, \bar{y}_{\mathrm{II}}) = f(y_n, \bar{y}_n)$ and changing $k \to k+1$ we find

$$H_{n,m}^r(\bar{x};\bar{y})\Big|_{y_n \to x_j} = g(x_j, y_n) f(\bar{x}_j, x_j) f(y_n, \bar{y}_n) \sum_{k=0}^{n-1} \sum_{\#\bar{y}_{\mathrm{I}'}=k} (-1)^{n-1-k} f(\bar{y}_{\mathrm{I}'}, \bar{y}_{\mathrm{II}}) f(\bar{x}_j, \bar{y}_{\mathrm{I}'}) + reg.$$

$$(A.11)$$

The remaining sum over partitions gives $H_{n-1,m-1}^r(\bar{x}_j; \bar{y}_n)$, and we finally arrive at

$$H_{n,m}^r(\bar{x};\bar{y})\Big|_{y_n \to x_j} = g(x_j, y_n) f(\bar{x}_j, x_j) f(y_n, \bar{y}_n) H_{n-1,m-1}^r(\bar{x}_j; \bar{y}_n) + reg. \qquad (A.12)$$

Due to the induction assumption $H_{n-1,m-1}^r(\bar{x}_j; \bar{y}_n) = H_{n-1,m-1}^\ell(\bar{x}_j; \bar{y}_n)$. Hence, the residues of $H_{n,m}^r(\bar{x};\bar{y})$ and $H_{n,m}^\ell(\bar{x};\bar{y})$ in the poles at $y_n = x_j$ coincide. Since both functions vanish at $y_n \to \infty$ we conclude that $H_{n,m}^r(\bar{x};\bar{y}) = H_{n,m}^\ell(\bar{x};\bar{y})$. $\qquad\square$

Proposition A.2. *Let $\#\bar{x} = m$ and $\#\bar{y} = n$. Then*

$$\sum_{k=0}^{n} \sum_{\#\bar{y}_{\mathrm{I}}=k} (-1)^{n-k} f(\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{II}}) f(\bar{x}, \bar{y}_{\mathrm{I}}) = \det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_j)}{h(y_j, y_k)} - \delta_{jk} \right). \qquad (A.13)$$

Here the sum is taken over all possible partitions $\bar{y} \Rightarrow \{\bar{y}_{\mathrm{I}}, \bar{y}_{\mathrm{II}}\}$. If $m < n$, then

$$\det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_j)}{h(y_j, y_k)} - \delta_{jk} \right) = 0. \qquad (A.14)$$

Proof. Expanding the determinant in the r.h.s. of (A.13) over diagonal minors we find

$$\det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_j)}{h(y_j, y_k)} - \delta_{jk} \right)$$

$$= (-1)^n + \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \le j_1 < \cdots < j_s \le n} \left(\prod_{p=1}^{s} f(y_{j_p}, \bar{y}_{j_p}) f(\bar{x}, y_{j_p}) \right) \det_s \frac{1}{h(y_{j_i}, y_{j_k})}. \qquad (A.15)$$

The determinant in the r.h.s. is the Cauchy determinant, hence,

$$\det_s \frac{1}{h(y_{j_i}, y_{j_k})} = \prod_{\substack{p,q=1 \\ p\ne q}}^{s} \frac{1}{f(y_{j_p}, y_{j_q})}. \qquad (A.16)$$

Thus, we obtain

$$\det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_j)}{h(y_j, y_k)} - \delta_{jk} \right)$$

$$= (-1)^n + \sum_{s=1}^{n} (-1)^{n-s} \sum_{1 \le j_1 < \cdots < j_s \le n} \left(\prod_{p=1}^{s} f(y_{j_p}, \bar{y}_{j_p}) f(\bar{x}, y_{j_p}) \right) \prod_{\substack{p,q=1 \\ p\ne q}}^{s} \frac{1}{f(y_{j_p}, y_{j_q})}. \qquad (A.17)$$

This is exactly the sum over partitions in the l.h.s. of (A.13).

Let now $m < n$. Obviously,

$$\det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_j)}{h(y_j, y_k)} - \delta_{jk} \right) = \det_n \left(\frac{f(y_j, \bar{y}_j) f(\bar{x}, y_k)}{h(y_j, y_k)} - \delta_{jk} \right), \quad (A.18)$$

because both matrices are related by a similarity transformation. It is easy to see that the matrix in the r.h.s. of (A.18) has an eigenvector with zero eigenvalue:

$$\sum_{k=1}^n \frac{f(y_j, \bar{y}_j) f(\bar{x}, y_k)}{h(y_j, y_k)} \nu_k - \nu_j = 0, \quad (A.19)$$

where

$$\nu_k = \frac{g(y_k, \bar{y}_k)}{g(\bar{x}, y_k)}. \quad (A.20)$$

Indeed, consider a function

$$\frac{1}{h(z, \bar{y}) g(\bar{x}, z)} = \frac{c^{n-m} \prod_{p=1}^m (x_p - z)}{\prod_{q=1}^n (z - y_q + c)}. \quad (A.21)$$

Due to the condition $m < n$ this function vanishes as $z \to \infty$. Hence, it has the following partial fraction decomposition

$$\frac{c^{n-m} \prod_{p=1}^m (x_p - z)}{\prod_{q=1}^n (z - y_q + c)} = \sum_{k=1}^n \frac{c^{n-m} \prod_{p=1}^m (x_p - y_k + c)}{(z - y_k + c) \prod_{q=1, q \neq k}^n (y_k - y_q)} = \sum_{k=1}^n \frac{g(y_k, \bar{y}_k) h(\bar{x}, y_k)}{h(z, y_k)}. \quad (A.22)$$

Setting here $z = y_j$ we arrive at

$$\sum_{k=1}^n \frac{g(y_k, \bar{y}_k) h(\bar{x}, y_k)}{h(y_j, y_k)} - \frac{1}{h(y_j, \bar{y}) g(\bar{x}, y_j)} = 0. \quad (A.23)$$

On the other hand, substituting ν_j from (A.20) into (A.19) we immediately obtain the l.h.s. of (A.23). $\qquad \square$

B Proof of the connection between two types of Bethe vectors

The proof of (2.17) is based on the double induction, first on a, and then on b.

B.1 First step of induction

We first assume that $b = 0$. Then (2.17) takes the form

$$\widehat{\mathbb{B}}_{0,a}(\emptyset; \bar{u}) = (-1)^a \frac{\lambda_2(\bar{u})}{\lambda_1(\bar{u})} \mathbb{B}_{a,0}(\bar{u}; \emptyset). \quad (B.1)$$

For $a = 0$, (B.1) turns into a trivial identity: $|0\rangle = |0\rangle$. It is easy to see that (B.1) also holds for $a = 1$:

$$\widehat{\mathbb{B}}_{0,1}(\emptyset; u) = \frac{\widehat{T}_{23}(u)|0\rangle}{\hat{\lambda}_2(u)} = -\frac{T_{12}(u)|0\rangle}{\lambda_1(u)} = -\frac{\lambda_2(u)}{\lambda_1(u)} \mathbb{B}_{1,0}(u; \emptyset), \quad (B.2)$$

where we used (C.8) for $\widehat{T}_{23}(u)$.

Assume now that (B.1) holds for some $a \geq 1$. Then we have for $\#\bar{u} = a$

$$\widehat{\mathbb{B}}_{0,a+1}(\emptyset; \{\bar{u}, z\}) = \widehat{T}_{23}(z) \frac{\widehat{T}_{23}(\bar{u})|0\rangle}{\hat{\lambda}_2(z)\hat{\lambda}_2(\bar{u})} = (-1)^a \widehat{T}_{23}(z) \frac{T_{12}(\bar{u})|0\rangle}{\hat{\lambda}_2(z)\lambda_1(\bar{u})}. \tag{B.3}$$

Substituting here $\widehat{T}_{23}(z)$ from (C.8) we find

$$\widehat{\mathbb{B}}_{0,a+1}(\emptyset; \{\bar{u}, z\}) = (-1)^a \big(T_{13}(z)T_{32}(z-c) - T_{12}(z)T_{33}(z-c)\big) \frac{T_{12}(\bar{u})|0\rangle}{\hat{\lambda}_2(z)\lambda_1(\bar{u})}. \tag{B.4}$$

To calculate the obtained action we use commutation relations of the monodromy matrix entries. The RTT-relation (2.2) implies

$$[T_{ij}(u), T_{kl}(v)] = g(u,v)\big(T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v)\big). \tag{B.5}$$

In particular, we have

$$T_{32}(u)T_{12}(v) = T_{12}(v)T_{32}(u)f(u,v) - T_{12}(u)T_{32}(v)g(u,v). \tag{B.6}$$

We see that permuting the operators T_{32} and T_{12} we obtain the annihilation operator T_{32} on the right. Eventually, this operator approaches the vector $|0\rangle$ and annihilates it. Thus, the contribution from the term $T_{13}(z)T_{32}(z-c)$ vanishes.

The commutation relations (B.5) also imply

$$T_{33}(u)T_{12}(v) = T_{12}(v)T_{33}(u) + g(u,v)\big(T_{13}(v)T_{32}(u) - T_{13}(u)T_{32}(v)\big). \tag{B.7}$$

We see that when the operator T_{33} is permuted with the operator T_{12}, it either commutes or generates the operator T_{32}. As we have already seen, the latter annihilates the state $T_{12}(\bar{u})|0\rangle$. Thus, the operator $T_{33}(z-c)$ acts on the state $T_{12}(\bar{u})|0\rangle$ as

$$T_{33}(z-c)T_{12}(\bar{u})|0\rangle = \lambda_3(z-c)T_{12}(\bar{u})|0\rangle. \tag{B.8}$$

Substituting this into (B.4), we arrive at

$$\widehat{\mathbb{B}}_{0,a+1}(\emptyset; \{\bar{u}, z\}) = (-1)^{a+1}\lambda_3(z-c)\frac{T_{12}(z)T_{12}(\bar{u})|0\rangle}{\lambda_1(\bar{u})\hat{\lambda}_2(z)} = (-1)^{a+1}\frac{\lambda_2(\bar{u})\lambda_2(z)}{\lambda_1(\bar{u})\lambda_1(z)}\mathbb{B}_{a+1,0}(\{\bar{u}, z\}; \emptyset), \tag{B.9}$$

what completes the first step of the induction. Thus, equation (2.17) holds for $b=0$ and a arbitrary non-negative.

B.2 Second step of induction

We pass to the second step of induction. This time we use a recursion for the Bethe vectors $\widehat{\mathbb{B}}$ [25]

$$\hat{\lambda}_2(z)g(\bar{u}, z)\widehat{\mathbb{B}}_{b+1,a}(\{\bar{v}+c, z\}; \bar{u}) = \widehat{T}_{12}(z)\widehat{\mathbb{B}}_{b,a}(\bar{v}+c; \bar{u})$$
$$+ \sum_{j=1}^{a} g(u_j, z)\frac{f(\bar{u}_j, u_j)}{g(u_j, \bar{v})}\widehat{T}_{13}(z)\widehat{\mathbb{B}}_{b,a-1}(\bar{v}+c; \bar{u}_j). \tag{B.10}$$

This recursion allows us to uniquely construct the Bethe vector $\widehat{\mathbb{B}}_{b+1,a}$, knowing the Bethe vectors[5] $\widehat{\mathbb{B}}_{b,a}$ and $\widehat{\mathbb{B}}_{b,a-1}$.

Assume that (2.17) holds for some $b \geq 0$ and a arbitrary. Then we can replace the Bethe vectors $\widehat{\mathbb{B}}$ by $\mathbb{B}$ in the r.h.s. of (B.10). We obtain

$$\hat{\lambda}_2(z)g(\bar{u},z)\widehat{\mathbb{B}}_{b+1,a}(\{\bar{v}+c,z\};\bar{u}) = (-1)^{a+b+ab}\frac{\lambda_2(\bar{u})\lambda_2(\bar{v})}{\lambda_1(\bar{u})\lambda_3(\bar{v})}\left\{\widehat{T}_{12}(z)\,\mathbb{B}_{a,b}(\bar{u};\bar{v})\right.$$
$$\left. +(-1)^{b+1}\sum_{j=1}^{a}\frac{\lambda_1(u_j)g(u_j,z)f(\bar{u}_j,u_j)}{\lambda_2(u_j)g(u_j,\bar{v})}\widehat{T}_{13}(z)\mathbb{B}_{a-1,b}(\bar{u}_j;\bar{v})\right\}. \tag{B.11}$$

We should compute the action of the operator $\widehat{T}_{12}(z)$ on $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ and the action of the operator $\widehat{T}_{13}(z)$ on $\mathbb{B}_{a-1,b}(\bar{u}_j;\bar{v})$. This is done in sections C.2.2 and C.2.1 respectively. The results have the following form:

$$\widehat{T}_{13}(z)\mathbb{B}_{a-1,b}(\bar{u}_j;\bar{v}) = (-1)^{a+b}\lambda_2(z)\lambda_2(z-c)\frac{g(z,\bar{v})}{h(\bar{u}_j,z)}\mathbb{B}_{a,b+1}(\{\bar{u}_j,z\};\{\bar{v},z-c\}), \tag{B.12}$$

and

$$\widehat{T}_{12}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{a+1}\lambda_2(z-c)\left\{\lambda_1(z)g(\bar{u},z)\mathbb{B}_{a,b+1}(\bar{u};\{\bar{v},z-c\})\right.$$
$$\left. +\lambda_2(z)g(\bar{v},z)\sum_{j=1}^{a}\frac{\lambda_1(u_j)g(z,u_j)f(\bar{u}_j,u_j)}{\lambda_2(u_j)g(\bar{v},u_j)h(\bar{u}_j,z)}\mathbb{B}_{a,b+1}(\{\bar{u}_j,z\};\{\bar{v},z-c\})\right\}. \tag{B.13}$$

Substituting these formulas into (B.11) we immediately arrive at

$$\hat{\lambda}_2(z)\widehat{\mathbb{B}}_{b+1,a}(\{\bar{v}+c,z\};\bar{u}) = (-1)^{1+b+ab}\frac{\lambda_2(\bar{u})\lambda_2(\bar{v})}{\lambda_1(\bar{u})\lambda_3(\bar{v})}\lambda_1(z)\lambda_2(z-c)\mathbb{B}_{a,b+1}(\bar{u};\{\bar{v},z-c\}). \tag{B.14}$$

Finally, using $\hat{\lambda}_2(z) = \lambda_1(z)\lambda_3(z-c)$ we obtain

$$\widehat{\mathbb{B}}_{b+1,a}(\{\bar{v}+c,z\};\bar{u}) = (-1)^{a+(b+1)+a(b+1)}\frac{\lambda_2(\bar{u})\lambda_2(\bar{v})\lambda_2(z-c)}{\lambda_1(\bar{u})\lambda_3(\bar{v})\lambda_3(z-c)}\mathbb{B}_{a,b+1}(\bar{u};\{\bar{v},z-c\}). \tag{B.15}$$

This completes the second step of the induction.

C Action formulas

C.1 Actions of the operators T_{ij} on Bethe vectors $\mathbb{B}_{a,b}$

In this section we give a list of formulas for the actions of the operators $T_{ij}(z)$ on the Bethe vectors $\mathbb{B}_{a,b}(\bar{u};\bar{v})$. These formulas were obtained in [25]. Here they are adopted to the new normalization of the Bethe vectors. In all action formulas $\bar{\eta} = \{z,\bar{u}\}$ and $\bar{\xi} = \{z,\bar{v}\}$. We also set

$$\Lambda(z) = \frac{\lambda_2(z)}{h(\bar{v},z)h(z,\bar{u})}. \tag{C.1}$$

---
[5]We set by definition $\widehat{\mathbb{B}}_{b,-1} = 0$.

- Action of $T_{13}(z)$:

$$T_{13}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \Lambda(z)\mathbb{B}_{a+1,b+1}(\bar{\eta};\bar{\xi}). \tag{C.2}$$

- Action of $T_{12}(z)$:

$$T_{12}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \Lambda(z)\sum_{\#\bar{\xi}_{\mathrm{I}}=1}\frac{f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta})}{h(z,\bar{\xi}_{\mathrm{I}})}\mathbb{B}_{a+1,b}(\bar{\eta};\bar{\xi}_{\mathrm{II}}). \tag{C.3}$$

The sum is taken over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}}\}$ so that $\#\bar{\xi}_{\mathrm{I}}=1$.

- Action of $T_{23}(z)$:

$$T_{23}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \Lambda(z)\sum_{\#\bar{\eta}_{\mathrm{I}}=1}\frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})h(\bar{\xi},\bar{\eta}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},z)}\mathbb{B}_{a,b+1}(\bar{\eta}_{\mathrm{II}};\bar{\xi}). \tag{C.4}$$

The sum is taken over partitions $\bar{\eta}\Rightarrow\{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}}\}$ so that $\#\bar{\eta}_{\mathrm{I}}=1$.

- Action of $T_{22}(z)$:

$$T_{22}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v})=\Lambda(z)\sum_{\#\bar{\xi}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{I}}=1}\frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta})h(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})}{h(z,\bar{\xi}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{I}},z)}\mathbb{B}_{a,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}). \tag{C.5}$$

The sum is taken over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}}\}$ and $\bar{\eta}\Rightarrow\{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}}\}$ so that $\#\bar{\xi}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{I}}=1$.

- Action of $T_{11}(z)$:

$$T_{11}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v})=\Lambda(z)\sum_{\#\bar{\xi}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{I}}=1}\frac{\lambda_1(\bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})}{\lambda_2(\bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})h(z,\bar{\xi}_{\mathrm{I}})}\mathbb{B}_{a,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}). \tag{C.6}$$

The sum is taken over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}}\}$ and $\bar{\eta}\Rightarrow\{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}}\}$ so that $\#\bar{\xi}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{I}}=1$.

- Action of $T_{21}(z)$:

$$\begin{aligned}T_{21}(z)\mathbb{B}_{a-1,b}(\bar{u};\bar{v})=\Lambda(z)\sum_{\substack{\#\bar{\eta}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{II}}=1\\\#\bar{\xi}_{\mathrm{I}}=1}}\frac{\lambda_1(\bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}},\bar{\eta}_{\mathrm{III}})f(\bar{\eta}_{\mathrm{III}},\bar{\eta}_{\mathrm{I}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})}{\lambda_2(\bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{II}},z)h(z,\bar{\xi}_{\mathrm{I}})}\\\times h(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{III}})h(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{II}})\mathbb{B}_{a-1,b}(\bar{\eta}_{\mathrm{III}};\bar{\xi}_{\mathrm{II}}). \tag{C.7}\end{aligned}$$

The sum is taken over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}}\}$ and $\bar{\eta}\Rightarrow\{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}},\bar{\eta}_{\mathrm{III}}\}$ so that $\#\bar{\xi}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{II}}=1$.

The actions of $\widehat{T}_{ij}$ onto $\widehat{\mathbb{B}}_{a,b}(\bar{u};\bar{v})$ are given by the same formulas, where we should put hats for the operators, the vacuum eigenvalues $\lambda_k(z)$, and the Bethe vectors.

C.2 Actions of the operators $\widehat{T}_{ij}$ on Bethe vectors $\mathbb{B}_{a,b}$

The action formulas (C.2)–(C.7) allow us to derive the actions of the operators $\widehat{T}_{ij}$ onto the Bethe vectors $\mathbb{B}_{a,b}(\bar{u};\bar{v})$. For this we should express $\widehat{T}_{ij}$ in terms of the original entries T_{ij} via (2.5)–(2.7). In particular, we have

$$
\begin{aligned}
\widehat{T}_{12}(z) &= T_{21}(z)T_{13}(z-c) - T_{23}(z)T_{11}(z-c), \\
\widehat{T}_{23}(z) &= T_{13}(z)T_{32}(z-c) - T_{12}(z)T_{33}(z-c), \\
\widehat{T}_{13}(z) &= T_{12}(z)T_{23}(z-c) - T_{13}(z)T_{22}(z-c).
\end{aligned}
\tag{C.8}
$$

Then the actions of $\widehat{T}_{ij}$ onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ can be obtained via successive application of the formulas (C.2)–(C.7). Below we give some details of this derivation for the action of $\widehat{T}_{13}$ and $\widehat{T}_{12}$.

C.2.1 Action of $\widehat{T}_{13}$

The operator $\hat{T}_{13}(z)$ is given by the last equation (C.8). It is convenient to consider the following combination

$$
T_{12}(x)T_{23}(y) - T_{13}(x)T_{22}(y)
\tag{C.9}
$$

and set $x = z$, $y = z - c$ in the end. Such the replacement of $\widehat{T}_{13}(z)$ allows us to avoid singular expressions in the intermediate computations.

Applying successively, first (C.5) and (C.2), and then (C.4) and (C.3) we obtain

$$
T_{13}(x)T_{22}(y)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \frac{\Lambda(x,y)}{h(y,x)} \sum_{\substack{\#\bar{\xi}_{\mathrm{I}}=1 \\ \#\bar{\eta}_{\mathrm{I}}=1}} \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})\, h(\bar{\xi}_{\mathrm{I}},\bar{\eta})h(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})}{h(y,\bar{\xi}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{I}},y)f(x,\bar{\xi}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{I}},x)}\mathbb{B}_{a+1,b+1}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}),
\tag{C.10}
$$

and

$$
T_{12}(x)T_{23}(y)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \frac{\Lambda(x,y)}{h(y,x)} \sum_{\substack{\#\bar{\xi}_{\mathrm{I}}=1 \\ \#\bar{\eta}_{\mathrm{I}}=1}} \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta})h(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},y)h(x,\bar{\xi}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{I}},x)}\mathbb{B}_{a+1,b+1}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}).
\tag{C.11}
$$

Here $\bar{\eta} = \{\bar{u},x,y\}$ and $\bar{\xi} = \{\bar{v},x,y\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}}\}$ so that $\#\bar{\eta}_{\mathrm{I}} = \#\bar{\xi}_{\mathrm{I}} = 1$. Here we also introduced

$$
\Lambda(x,y) = \frac{\lambda_2(x)\lambda_2(y)}{h(x,y)h(\bar{v},x)h(\bar{v},y)h(x,\bar{u})h(y,\bar{u})}.
\tag{C.12}
$$

Taking the difference of (C.10) and (C.11) we arrive at

$$
\widehat{T}_{13}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \Lambda(x,y) \sum_{\substack{\#\bar{\xi}_{\mathrm{I}}=1 \\ \#\bar{\eta}_{\mathrm{I}}=1}} \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}},\bar{\eta})h(\bar{\xi}_{\mathrm{II}},\bar{\eta}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},y)h(x,\bar{\xi}_{\mathrm{I}})h(y,\bar{\xi}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{I}},x)}\mathbb{B}_{a+1,b+1}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}})\Big|_{\substack{x=z \\ y=z-c}}.
\tag{C.13}
$$

Now we should consider several cases. First of all, we see that $\bar{\eta}_{\mathrm{I}} \neq x$, because otherwise the factor $1/f(\bar{\eta}_{\mathrm{I}}, x)$ in (C.13) is equal to zero. Thus, either $\bar{\eta}_{\mathrm{I}} = y$ or $\bar{\eta}_{\mathrm{I}} = u_j$, $j = 1, \ldots, a$. Consider the first case $\bar{\eta}_{\mathrm{I}} = y$ and denote this contribution by G. Then

$$G = \Lambda(x,y)f(y,\bar{u})h(\bar{v},y)h(x,y) \sum_{\#\bar{\xi}_{\mathrm{I}}=1} \frac{f(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}}, x)h(\bar{\xi}_{\mathrm{I}}, \bar{u})}{h(x, \bar{\xi}_{\mathrm{I}})h(y, \bar{\xi}_{\mathrm{I}})}\mathbb{B}_{a+1,b+1}(\{\bar{u},x\}; \bar{\xi}_{\mathrm{II}})\Big|_{\substack{x=z \\ y=z-c}} . \tag{C.14}$$

This case respectively should be divided into subcases.

- $\bar{\xi}_{\mathrm{I}} = x$, hence, $\bar{\xi}_{\mathrm{II}} = \{\bar{v}, y\}$. Then, substituting (C.12) in (C.14) we find

$$G_1 = (-1)^{a+b+1}\lambda_2(z)\lambda_2(z-c)\frac{g(z,\bar{v})}{h(\bar{u},z)}\mathbb{B}_{a+1,b+1}(\{\bar{u},z\}; \{\bar{v},z-c\}). \tag{C.15}$$

- $\bar{\xi}_{\mathrm{I}} = y$, hence, $\bar{\xi}_{\mathrm{II}} = \{\bar{v}, x\}$. Then $h(\bar{\xi}_{\mathrm{I}}, x) = h(y, x) \to 0$, as $x \to z$ and $y \to z - c$. Thus, this contribution vanishes.

- $\bar{\xi}_{\mathrm{I}} = v_j$, $j = 1, \ldots, b$, hence, $\bar{\xi}_{\mathrm{II}} = \{\bar{v}_j, x, y\}$. Then

$$G_1^{(2)} = \frac{\lambda_2(x)\lambda_2(y)g(y,\bar{u})f(\bar{v}_j,v_j)g(x,v_j)g(y,v_j)h(v_j,\bar{u})}{h(\bar{v}_j,x)h(x,\bar{u})}\mathbb{B}_{a+1,b+1}(\{\bar{u},x\}; \{\bar{v}_j,x,y\})\Big|_{\substack{x=z \\ y=z-c}} . \tag{C.16}$$

In this case the Bethe vector $\mathbb{B}_{a,b}(\{\bar{u},x\}; \{\bar{v}_j,x,y\})$ vanishes in the limit $x = z$ and $y = z - c$. Indeed, we have due to (C.2)

$$\mathbb{B}_{a+1,b+1}(\{\bar{u},x\}; \{\bar{v}_j,x,y\}) = \frac{1}{\lambda_2(x)}h(y,x)h(\bar{v}_j,x)h(x,\bar{u})T_{13}(x)\mathbb{B}_{a,b}(\bar{u}; \{\bar{v}_j,y\}), \tag{C.17}$$

and the r.h.s. of (C.17) vanishes, because $h(z-c,z) = 0$.

Similarly, one should consider the case $\bar{\eta}_{\mathrm{I}} = u_j$, $j = 1, \ldots, a$. The analysis of this case shows that all the corresponding contributions vanish. Thus, the action of $\widehat{T}_{13}(z)$ on the Bethe vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ is given by (C.15):

$$\widehat{T}_{13}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{a+b+1}\lambda_2(z)\lambda_2(z-c)\frac{g(z,\bar{v})}{h(\bar{u},z)}\mathbb{B}_{a+1,b+1}(\{\bar{u},z\}; \{\bar{v},z-c\}). \tag{C.18}$$

C.2.2 Action of $\widehat{T}_{12}$

The action of $\widehat{T}_{12}(z)$ can be considered exactly in the same manner. Using (C.8) and the action formulas (C.2)–(C.7) we obtain

$$\widehat{T}_{12}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \Lambda(x,y) \sum_{\substack{\#\bar{\eta}_{\mathrm{I}}=\#\bar{\eta}_{\mathrm{II}}=1 \\ \#\bar{\xi}_{\mathrm{I}}=1}} \frac{\lambda_1(\bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}})f(\bar{\eta}_{\mathrm{III}}, \bar{\eta}_{\mathrm{I}})f(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{\lambda_2(\bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{II}}, x)h(x, \bar{\xi}_{\mathrm{I}})h(y, \bar{\xi}_{\mathrm{I}})}$$

$$\times h(\bar{\xi}, \bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_{\mathrm{III}})\mathbb{B}_{a,b+1}(\bar{\eta}_{\mathrm{III}}; \bar{\xi}_{\mathrm{II}})\Big|_{\substack{x=z \\ y=z-c}} . \tag{C.19}$$

Here $\bar{\eta} = \{\bar{u}, x, y\}$ and $\bar{\xi} = \{\bar{v}, x, y\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ so that $\#\bar{\eta}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{II}} = \#\bar{\xi}_{\mathrm{I}} = 1$.

Again one should consider several cases. The analysis shows that non-vanishing contributions arise if and only if $\bar{\xi}_{\mathrm{I}} = x$ and $\bar{\eta}_{\mathrm{II}} = y$. Then

$$
\begin{aligned}
\widehat{T}_{12}(z)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = {}& \frac{\lambda_2(x)\lambda_2(y)g(y, \bar{u})g(\bar{v}, x)}{h(x, \bar{u})} \\
& \times \sum_{\#\bar{\eta}_{\mathrm{I}}=1} \frac{\lambda_1(\bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{III}}, \bar{\eta}_{\mathrm{I}})h(x, \bar{\eta}_{\mathrm{III}})}{\lambda_2(\bar{\eta}_{\mathrm{I}})g(\bar{v}, \bar{\eta}_{\mathrm{I}})g(y, \bar{\eta}_{\mathrm{I}})}\mathbb{B}_{a,b+1}(\bar{\eta}_{\mathrm{III}}; \{\bar{v}, y\})\Big|_{\substack{x=z \\ y=z-c}},
\end{aligned}
\tag{C.20}
$$

where $\bar{\eta} = \{\bar{u}, x\}$ and the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{III}}\}$ so that $\#\bar{\eta}_{\mathrm{I}} = 1$. Then we should consider two cases. First, we can set $\bar{\eta}_{\mathrm{I}} = x$ and $\bar{\eta}_{\mathrm{III}} = \bar{u}$. Then we obtain the first term in (B.13). The second case is $\bar{\eta}_{\mathrm{I}} = u_j$ and $\bar{\eta}_{\mathrm{III}} = \{\bar{u}_j, x\}$, $j = 1, \ldots, a$. Then we obtain the second term in (B.13).

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